(I)

D51621

First Semester MA-Degree Examination, November 2018.

(CUCSS-PG)

ECO 1C 04 - Quantitative Methods for Economic Analysis - I

(2015 Syllabus Year)

Time: 3 hrs Maximum: 36 Weightage.

Part A

1. Evaluate $\begin{vmatrix} -1 & 2 & -3 \\ 2 & -3 & -1 \\ -3 & -1 & 2 \end{vmatrix}$

$= (-1)\begin{vmatrix} -3 & -1 \\ -1 & 2 \end{vmatrix} - (2)\begin{vmatrix} 2 & -1 \\ -3 & 2 \end{vmatrix} + (-3)\begin{vmatrix} 2 & -3 \\ -3 & -1 \end{vmatrix}$

$= (-1)[(-3\times 2) - (-1\times -1)] - 2[(2\times 2) - (-1\times -3)] + (-3)[(2\times -1) - (-3\times -3)]$

$= (-1)[-6 - 1] - 2[4 - 3] - 3[-2 - 6]$

$= -1\times(-7) - 2\times(1) - 3\times(-8)$

$= 7 - 2 + 24$

$= 7 + 24 - 2$

$= 31 - 2$

$= 29$

2. Find the rank of $\begin{bmatrix} 4 & 1 & 2 \\ -3 & 2 & 4 \\ 8 & -1 & -2 \end{bmatrix}$.

Let $A = \begin{bmatrix} 4 & 1 & 2 \\ -3 & 2 & 4 \\ 8 & -1 & -2 \end{bmatrix}$

$|A| = \begin{vmatrix} 4 & 1 & 2 \\ -3 & 2 & 4 \\ 8 & -1 & -2 \end{vmatrix} = 4\begin{vmatrix} 2 & 4 \\ -1 & -2 \end{vmatrix} - 1\begin{vmatrix} -3 & 4 \\ 8 & -2 \end{vmatrix} + 2\begin{vmatrix} -3 & 2 \\ 8 & -1 \end{vmatrix}$

$= 4[-4 - (-4)] - 1[(-3\times -2) - (4\times 8)] + 2[(-3\times -1) - (2\times 8)]$

$= 4[-4 + 4] - [6 - 32] + 2[3 - 16]$

$= 4\times 0 - (-26) + 2(-13)$

$r(A) \neq 3$.

$r(A) < 3$.

Consider Evaluate next Lower order (order = 2) minors.

$$\begin{vmatrix} 4 & 1 \\ 3 & -2 \end{vmatrix} = -8 - 3 = -11 \neq 0$$

$\therefore r(A) = \underline{\underline{2}}$.

rank of the given matrix = 2//.

(3). Show that the characteristic equation of the square matrix

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix} \text{ is } \lambda^3 - 3\lambda^2 - 7\lambda - 11 = 0.$$

Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$.

then $A - \lambda I = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 1 & 2 \\ 3 & 1-\lambda & 1 \\ 2 & 3 & 1-\lambda \end{bmatrix}$$

Characteristic equation is

$$|A - \lambda I| = 0.$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 2 \\ 3 & 1-\lambda & 1 \\ 2 & 3 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 3 & 1-\lambda \end{vmatrix} - (1)\begin{vmatrix} 3 & 1 \\ 2 & 1-\lambda \end{vmatrix} + 2\begin{vmatrix} 3 & 1-\lambda \\ 2 & 3 \end{vmatrix}$$

$$= (1-\lambda)[(1-\lambda)(1-\lambda) - (1\times 3)] - 1[3(1-\lambda) - (1\times 2)] + 2[(3\times 3) - (1-\lambda)2]$$

$$= (1-\lambda)[(1-\lambda)^2 - 3] - [3(1-\lambda) - 2] + 2[9 - 2(1-\lambda)]$$

$$= (1-\lambda)[1-2\lambda+\lambda^2-3] - [3-3\lambda-2] + 2[9-2+2\lambda]$$

$$= (1-\lambda)[\lambda^2-2\lambda+1-3] - [-3\lambda+3-2] + 2[2\lambda+9-2]$$

$$= (1-\lambda)[\lambda^2-2\lambda-2] - [-3\lambda+1] + 2[2\lambda+7]$$

$$= (1-\lambda)[\lambda^2-2\lambda-2] + 3\lambda - 1 + 4\lambda + 14$$

$$= 1(\lambda^2-2\lambda-2) - \lambda(\lambda^2-2\lambda-2) + 3\lambda - 1 + 4\lambda + 14$$

$$= \lambda^2-2\lambda-2-\lambda^3+2\lambda^2+2\lambda+3\lambda-1+4\lambda+14$$

$$= -\lambda^3+\lambda^2+2\lambda^2-2\lambda+2\lambda+3\lambda+4\lambda-2-1+14$$

$$= -\lambda^3+3\lambda^2+\cancel{0}\lambda+7\lambda+14-3$$

$$= -\lambda^3+3\lambda^2+7\lambda+\cancel{0}\lambda+14-3$$

$$= -\lambda^3+3\lambda^2+7\lambda+11$$

ie $|A-\lambda I| = -\lambda^3+3\lambda^2+7\lambda+11$

$|A-\lambda I| = 0$

$\Rightarrow -\lambda^3+3\lambda^2+7\lambda+11 = 0$

$\Rightarrow -(-\lambda^3+3\lambda^2+7\lambda+11) = -0 = 0$

$\Rightarrow \lambda^3-3\lambda^2-7\lambda-11 = 0.$

ie $|A-\lambda I| = 0 \Rightarrow$

$\lambda^3-3\lambda^2-7\lambda-11 = 0.$

(4) Let an exponential function be $y = a^x$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{d}{dx}a^x = a^x \log_e a$$

(5) If the total cost of making x litres of an acid is $T = -30 + 80x^{1/2}$ rupees, Find the number of units at which the marginal cost is Rs 1.25.

Given $TC = -30 + 80x^{1/2}$

$= -30 + 80\sqrt{x}$

$MC = \frac{d}{dx}(TC)$

$= \frac{d}{dx}(-30 + 80\sqrt{x})$

$= \frac{d}{dx}(-30) + \frac{d}{dx}(80\sqrt{x})$

$= 0 + 80\frac{d}{dx}(\sqrt{x})$

$= 0 + 80\frac{d}{dx}(x^{1/2})$

$= 80 \, \frac{1}{2} x^{\frac{1}{2}-1}$

$= 80 \frac{1}{2} x^{-1/2}$ $\quad \left(\frac{1}{2} - 1 = -\frac{1}{2}\right)$

$= 80 \frac{1}{2x^{\frac{1}{2}}}$

$= \frac{80}{2} \times \frac{1}{x^{\frac{1}{2}}}$

$= \frac{40}{\sqrt{x}}$ ||

If $MC = 1.25$, then $\frac{40}{\sqrt{x}} = 1.25$

$\therefore 40 = 1.25\sqrt{x}$

$\therefore \frac{40}{1.25} = \sqrt{x}$

ie $x = \left(\frac{40}{1.25}\right)^2 = 32^2$

$= \underline{1024}$.

(6) Find the total derivative of $u = x^2y^3 + x^3y^2$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy.$$

$$\frac{\partial u}{\partial x} = 2xy^3 + 3x^2y^2 \quad (y \text{ constant})$$

$$\frac{\partial u}{\partial y} = x^2 3y^2 + x^3 2y \quad (x \text{ constant})$$

$$\therefore du = (2xy^3 + 3x^2y^2)\,dx + (x^2 3y^2 + x^3 2y)\,dy$$

(7) The cost for a monopolist firm producing x mobile phones per week is given to be $4x^2 - 80x + 500$ rupees. To have a minimum cost, how many units should be produced per week?

$$\text{Cost/week} = 4x^2 - 80x + 500$$
$$= TC$$

$$\frac{d}{dx}(TC) = 4 \times 2x - 80 \times 1 + 0$$
$$= 8x - 80$$

$$\frac{d}{dx} TC = 0$$
$$\Rightarrow 8x - 80 = 0$$
$$\Rightarrow 8x = 80$$
$$\Rightarrow x = \frac{80}{8} = 10.$$

$\therefore$ To minimize cost, 10 units should be produced per week.

(8) Integrate $\dfrac{3x^3 - 5x^2 + 6x - 8}{x}$ with respect to x.

$$\int \frac{3x^3 - 5x^2 + 6x - 8}{x} dx = \int \frac{3x^3}{x} dx - \int \frac{5x^2}{x} dx + \int \frac{6x}{x} dx - \int \frac{8}{x} dx$$
$$= \int 3x^2 dx - \int 5x\, dx + \int 6\, dx - \int \frac{8}{x} dx$$
$$= 3\frac{x^3}{3} - 5\frac{x^2}{2} + 6x - 8\log x + C$$
$$= x^3 - \frac{5}{2}x^2 + 6x - 8\log x + C.$$

(9) Find $\int \frac{x-5}{x^2-10x+11}\,dx$

Let $u = x^2-10x+11$,

then $\frac{du}{dx} = 2x-10 = 2(x-5)$

$\therefore \quad du = 2(x-5)dx$

i.e $\quad \frac{du}{2} = (x-5)dx$

$$\therefore \int \frac{x-5}{x^2-10x+11}\,dx = \int \frac{1}{u}\frac{du}{2} = \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}\log u + C$$

$$= \frac{1}{2}\log(x^2-10x+11) + C$$

(10) Give the axiomatic definition of probability.

Let 'S' be the sample space of a random experiment. Let A be an event of the random experiment so that A is a subset of S. Then we can associate a real number P(A) to the event A. This number of P(A) will be called probability of A if it satisfies the following three axioms.

Axiom 1: P(A) is a real number such that $P(A) \geq 0$ for every A subset of S

Axiom 2: $P(S) = 1$, where 's' is the sample space.

Axiom 3: $P(A \cup B) = P(A) + P(B)$ where A and B are two non intersecting subset of S, ie $A \cap B = \phi$.

(11) State the addition theorem of probability.

Ans

(a) Addition rule for mutually exclusive events.

If A and B are two mutually exclusive events, then

$P(A \cup B) = P(A) + P(B)$

(b) Addition rule for any two events (need not be mutually exclusive)

If A and B are any two events, then

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

(12) If $E(x) = 2.5$, find $E(3X+7)$

$$E(3X+7) = 3EX + E(7)$$
$$= 3E(X) + 7$$
$$= 3\times 2.5 + 7$$
$$= 7.5 + 7$$
$$= 14.5$$

Part B

(Answer any eight questions. Weightage 2 for each question)

(13) If $A = \begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -4 \\ 3 & -2 & 3 \end{bmatrix}$, Show that $\frac{1}{2}(A - A^T)$ is skew-symmetric.

Ans

$$A = \begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -4 \\ 3 & -2 & 3 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 3 & -2 \\ -1 & -4 & 3 \end{bmatrix}$$

$$A - A^T = \begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -4 \\ 3 & -2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 3 \\ 1 & 3 & -2 \\ -1 & -4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3 & -4 \\ -3 & 0 & -2 \\ 4 & 2 & 0 \end{bmatrix}$$

$$\frac{1}{2}(A - A^T) = \frac{1}{2}\begin{bmatrix} 0 & 3 & -4 \\ -3 & 0 & -2 \\ 4 & 2 & 0 \end{bmatrix} = \begin{bmatrix} \frac{0}{2} & \frac{3}{2} & \frac{-4}{2} \\ \frac{-3}{2} & \frac{0}{2} & \frac{-2}{2} \\ \frac{4}{2} & \frac{2}{2} & \frac{0}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{3}{2} & -2 \\ -\frac{3}{2} & 0 & -1 \\ \ldots & 1 & 0 \end{bmatrix}$$

ie $\frac{1}{2}(A - A^T) = \begin{bmatrix} 0 & 1.5 & -2 \\ -1.5 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

Let $B = \frac{1}{2}(A - A^T)$

then $B^T = \begin{bmatrix} 0 & -1.5 & 2 \\ 1.5 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix}$

$\therefore -B^T = -\begin{bmatrix} 0 & -1.5 & 2 \\ 1.5 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1.5 & -2 \\ -1.5 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

$= B$

ie $B = -B^T$

therefore $B = \frac{1}{2}(A - A^T)$ is skew symmetric.

(14) $A = \begin{bmatrix} 2 & 3 & 6 \\ 3 & -6 & 2 \\ -6 & -2 & 3 \end{bmatrix}$, find A^TA. Hence or otherwise evaluate A^{-1}.

What is the peculiarity of the matrix A?

Ans $A^T = \begin{bmatrix} 2 & 3 & -6 \\ 3 & -6 & -2 \\ 6 & 2 & 3 \end{bmatrix}$

$A^TA = \begin{bmatrix} 2 & 3 & -6 \\ 3 & -6 & -2 \\ 6 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 6 \\ 3 & -6 & 2 \\ -6 & -2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 4+9+36 & 6-18+12 & 12+6-18 \\ 6-18+12 & 9+36+4 & 18-12-6 \\ 12+6-18 & 18-12-6 & 36+4+9 \end{bmatrix}$

$= \begin{bmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{bmatrix}$

$$= 49\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$= 49I$, where I is the identity matrix of order 3.

i.e, $A^TA = 49I$

$\therefore \frac{1}{49}(A^TA) = I$

i.e, $\left(\frac{1}{7}A^T\right)\left(\frac{1}{7}A\right) = I$

i.e $\left(\frac{1}{7}A\right)^T\left(\frac{1}{7}A\right) = I$ — (1) $\left(\text{since } \frac{1}{7}A^T = \left(\frac{1}{7}A\right)^T\right)$

We have if A^{-1} is the inverse of A, then

$$A^{-1}A = I$$

In (1) $\left(\frac{1}{7}A\right)^T\left(\frac{1}{7}A\right) = I$

Therefore $\left(\frac{1}{7}A\right)^T$ is the inverse of $\frac{1}{7}A$

i.e, $\left(\frac{1}{7}A\right)^{-1} = \left(\frac{1}{7}A\right)^T$

i.e $7A^{-1} = \left(\frac{1}{7}A\right)^T$ $\left(\text{since } (cA)^{-1} = c^{-1}A^{-1}\right)$

i.e, $7A^{-1} = \frac{1}{7}A^T$

$$A^{-1} = \frac{1}{49}A^T$$

The inverse of A is $\frac{1}{49}A^T$

(15) A manufacturer produces three products A, B, C which are sold

in Delhi and Calcutta. Annual sales of these products are given below:

	Products		
	A	B	C
Delhi	5000	7500	15000
Calcutta.	9000	12000	8700

If the sale price of the products A, B, C per unit be Rs 20, Rs 30, Rs 40 respectively, calculate the total revenue in each centre by using matrices.

let $A = \begin{bmatrix} 5000 & 7500 & 15000 \\ 9000 & 12000 & 8700 \end{bmatrix}$, $B = \begin{bmatrix} 20 \\ 30 \\ 40 \end{bmatrix}$

(The sales matrix) (Price matrix)

Find AB,

$$AB = \begin{bmatrix} 5000 & 7500 & 15000 \\ 9000 & 12000 & 8700 \end{bmatrix} \begin{bmatrix} 20 \\ 30 \\ 40 \end{bmatrix}.$$

$$= \begin{bmatrix} 5000\times20 + 7500\times30 + 15000\times40 \\ 9000\times20 + 12000\times30 + 8700\times40 \end{bmatrix}$$

$$= \begin{bmatrix} 835000 \\ 888000 \end{bmatrix}$$

Total revenue of Delhi Centre = 83500

" Calcutta " = 888000

(16) Using the function $f(x,y) = x^2+y^2-2xy+8x+9y+3$, show that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

Ans

$$f(x,y) = x^2+y^2-2xy+8x+9y+3.$$

$$\frac{\partial f}{\partial x} = 2x+0-2\times(1)y+8\times1+9\times0+0$$

$$=2x-2y+8$$

$$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2x-2y+8)$$

$$= 0-2+0$$

$$= -2 \quad -(1)$$

$$\frac{\partial f}{\partial y} = 0+2y-2x\times(1)+8\times0+9\times1+0$$

$$= 2y-2x+9$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}(2y-2x+9)$$

$$= 0-2\times1+0$$

$$= -2 \quad -(2)$$

From (1) and (2).

$$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial^2 f}{\partial x\partial y}$$

(17) Given the production function $V = (\beta K^{-\rho}+\alpha L^{-\rho})^{-\frac{1}{\rho}}$, when V is the output, K is capital, L is labour, and α, β, ρ are constants. Find dV.

Ans

$$dV = \frac{\partial V}{\partial K}dK + \frac{\partial V}{\partial L}dL$$

$$\frac{\partial V}{\partial K} = -\frac{1}{\rho}(\beta K^{-\rho}+\alpha L^{-\rho})^{-\frac{1}{\rho}-1}\times(\beta\times(-\rho)K^{-\rho-1}+0)$$

$$= -\frac{1}{\rho}\times\beta\times-\rho\times K^{-\rho-1}(\beta K^{-\rho}+\alpha L^{-\rho})^{-\frac{1}{\rho}-1}$$

$$= \beta K^{-\rho-1}(\beta K^{-\rho}+\alpha L^{-\rho})^{-\frac{1}{\rho}-1}$$

$$\frac{\partial y}{\partial L} = -\frac{1}{\rho}\left(\beta k^{-\rho}+\alpha L^{-\rho}\right)^{-\frac{1}{\rho}-1} \times \left(0+\alpha x - \rho L^{-\rho-1}\right)$$

$$= -\frac{1}{\rho}\left(\beta k^{-\rho}+\alpha L^{-\rho}\right)^{-\frac{1}{\rho}-1} \alpha x - \rho x L^{-\rho-1}$$

$$= -\frac{1}{\rho} \times \alpha x - \rho \times L^{-\rho-1}\left(\beta k^{-\rho}+\alpha L^{-\rho}\right)^{-\frac{1}{\rho}-1}$$

$$= \alpha L^{-\rho-1}\left(\beta k^{-\rho}+\alpha L^{-\rho}\right)^{-\frac{1}{\rho}-1}$$

$$\therefore dv = \beta k^{-\rho-1}\left(\beta k^{-\rho}+\alpha L^{-\rho}\right)^{-\frac{1}{\rho}-1} dk + \alpha L^{-\rho-1}\left(\beta k^{-\rho}+\alpha L^{-\rho}\right)^{-\frac{1}{\rho}-1} dL$$

$$= \left(\beta k^{-\rho}+\alpha L^{-\rho}\right)^{-\frac{1}{\rho}-1}\left[\beta k^{-\rho-1} dk + \alpha L^{-\rho-1} dv\right]$$

(19) Find the maximum and minimum values of the function

$$y = 2x^3 - 3x^2 - 36x + 12.$$

$$\frac{dy}{dx} = 2 \times 3x^2 - 3 \times 2x - 36 \times 1 + 0$$

$$= 6x^2 - 6x - 36$$

$$\frac{dy}{dx} = 0 \Rightarrow 6x^2 - 6x - 36 = 0$$

$$\Rightarrow 6(x^2 - x - 6) = 0$$

$$\Rightarrow x^2 - x - 6 = 0$$

$$a = 1,\ b = -1,\ c = -6$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4\times1\times-6}}{2\times1}$$

$$= \frac{1 \pm \sqrt{1+24}}{2}$$

$$= \frac{1 \pm \sqrt{25}}{2}$$

$$= \frac{1 \pm 5}{2}$$

$$= \frac{1+5}{2} \text{ or } \frac{1-5}{2}$$

$$= \frac{6}{2} \text{ or } \frac{-4}{2}$$

$$= 3 \text{ or } -2$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(6x^2 - 6x - 36\right) \quad (\text{from } ①)$$

$$= 6 \times 2x - 6 \times 1 - 0$$

$$= 12x - 6.$$

at $x = 3$,

$$\frac{d^2y}{dx^2} = 12x - 6 = 12 \times 3 - 6$$

$$= 36 - 6$$

$$= 30 > 0$$

y is minimum at $x = 3$,

at $x = -2$

$$\frac{d^2y}{dx^2} = 12x - 6 = 12 \times -2 - 6$$

$$= -24 - 6$$

$$= -30 < 0$$

$\therefore$ at $x = -2$, y is maximum.

Minimum value of y

$$y = 2x^3 - 3x^2 - 36x + 12$$

$$= 2 \times (3)^3 - 3 \times (3)^2 - 36 \times (3) + 12$$

$$= 2 \times 27 - 3 \times 9 - 36 \times 3 + 12$$

$$= 54 - 27 - 108 + 12$$

$$= 54 + 12 - 27 - 108$$

$$= 66 - 135$$

$$= -69$$

Maximum value of y

$$y = 2x^3 - 3x^2 - 36x + 12$$

$$= 2 \times (-2)^3 - 3 \times (-2)^2 - 36 \times -2 + 12$$

$$= 2 \times 8 - 3 \times 4 + 72 + 12$$

$$= 16 - 12 + 72 + 12$$

$$= 16 + 72 + 12 - 12$$

$$= 88$$

min = -69, max = 88

20 Find the minimum of $u = 2x^2 - 6y^2$ under the condition that $x+2y=4$. What is the value of u?

Ans

Let $L = 2x^2 - 6y^2 + \lambda(x+2y-4)$

This function is called the Legrangian of the problem.

Find $\frac{\partial L}{\partial \lambda}$, $\frac{\partial L}{\partial x}$ and $\frac{\partial L}{\partial y}$ and equate to zero.

$$\frac{\partial L}{\partial \lambda} = \frac{\partial}{\partial \lambda}\left[2x^2 - 6y^2 + \lambda(x+2y-4)\right]$$

$$= 0 - 0 + 1 * (x+2y-4) \quad (\lambda \text{ - is the variable})$$

$$= x + 2y - 4$$

$$\therefore \frac{\partial L}{\partial \lambda} = 0 \Rightarrow x + 2y - 4 = 0. \quad \text{—— (1)}$$

$$\frac{\partial L}{\partial x} = \frac{\partial}{\partial x}\left[2x^2 - 6y^2 + \lambda(x+2y-4)\right]$$

$$= 2 \times 2x - 0 + \lambda(1+0-0).$$

$$= 4x + \lambda$$

$$\frac{\partial L}{\partial x} = 0 \Rightarrow 4x + \lambda = 0 \quad \text{—— (2)}$$

$$\frac{\partial L}{\partial y} = \frac{\partial}{\partial y}\left[2x^2 - 6y^2 + \lambda(x+2y-4)\right]$$

$$= 0 - 6 \times 2y + \lambda(0 + 2\times 1 - 0)$$

$$= -12y + \lambda(2)$$

$$= -12y + 2\lambda.$$

$$\frac{\partial L}{\partial y} = 0 \Rightarrow -12y + 2\lambda = 0. \quad \text{—— (3)}$$

The equations (1), (2) and (3) are

$$x + 2y - 4 = 0 \quad —(1)$$
$$4x + \lambda = 0 \quad —(2)$$
$$-12y + 2\lambda = 0 \quad (3)$$

(1) can be writt as.

$$x + 2y = 4 \quad —(A)$$

$$(2) \times 2 = 2(4x + \lambda) = 2 \times 0$$
$$\Rightarrow 8x + 2\lambda = 0 \quad —(2)$$

(2) − (3) ⇒
$$8x + 2\lambda = 0 \; -$$
$$-12y + 2\lambda = 0$$

$$8x + 12y = 0 \quad —(B)$$

$$(A) \times 8 \Rightarrow 8(x + 2y) = 8 \times 4$$
$$\Rightarrow 8x + 16y = 32 \quad —(A)^*$$

$(A)^* - (B) \Rightarrow$
$$8x + 16y = 32 \; -$$
$$8x + 12y = 0$$

$$0 + 4y = 32$$
ie $4y = 32$
$$\therefore y = \frac{32}{4} = 8$$

Substitute $y = 8$ in (A), then

$$x + 2 \times 8 = 4$$
ie $x + 16 = 4$
ie $x = 4 - 16 = -12$.

$\therefore x = -12, y = 8$

$$\therefore \min \text{ of } u = 2 \times (-12)^2 - 6 \times 8^2$$
$$= 2 \times 144 - 6 \times 64 = 288 - 384$$
$$= -96$$

21) Evaluate $\int x^3 \log x \, dx$

$$\int x^3 \log x \, dx = \log x \int x^3 dx - \int \left(\frac{d}{dx}(\log x) \int x^3 dx\right) dx$$

$$= \log x \left(\frac{x^4}{4}\right) - \int \frac{1}{x} \frac{x^4}{4} \, dx$$

$$= \frac{x^4}{4} \log x - \int \frac{x^3}{4} \, dx$$

$$= \frac{x^4}{4} \log x - \frac{1}{4} \frac{x^4}{4} + C$$

$$= \frac{x^4}{4} \log x - \frac{x^4}{16} + C$$

22).

8B, 8G	5
Group.	Committee

(i) 3 Boys & 2 Girls.

total = 16

p(committee consists of 3 Boys & 2 Girls)

$$= \frac{8C_3 \times 8C_2}{16C_5}.$$

(ii) p(committee consists of 'atleast one girl')

$= 1 - p($'no girl'$)$

$$= 1 - \frac{8C_5}{16C_5}$$

(no girl ⇒ committee of 5 boys)

(~~[illegible]~~).

23 rains — earn 1000/- per day

fair — earn 800/- per day.

$p(\text{rain}) = 0.4$

$p(\text{fair}) = 1 - p(\text{rain}) = 1 - .4$
$= .6$

Let X - be the earning per day of the taxi driver, then

X takes values 1000/- and 800/- with

corresponding probabilities .4 and .6

$\therefore EX = 1000 \times .4 + 800 \times .6$

$= 400 + 480$

$= 880$

Rs 880/-

24 $f(x) = 6x(1-x); \quad 0 \le x \le 1.$

moments about mean, μ_r

$\mu_r = E(X - \text{mean})^r \quad r = 1, 2, \ldots$

mean, $EX = \int_x x\, f(x)\, dx = \int_0^1 x\, 6x(1-x)\, dx$

$= 6\int_0^1 x^2(1-x)\, dx$

$= 6 \int_0^1 (x^2 - x^3)\, dx$

$= 6\int_0^1 x^2 dx - 6\int_0^1 x^3 dx$

$= 6\left[\frac{x^3}{3}\right]_0^1 - 6\left[\frac{x^4}{4}\right]_0^1$

$= 6\left[\frac{1^3}{3}-0\right] - 6\left[\frac{1^4}{4}-0\right]$

$= 6\times\frac{1}{3} - 6\times\frac{1}{4}$

$= \frac{6}{3} - \frac{6}{4}$

$= 2 - \frac{3}{2}$

$= \frac{4-3}{2}$

$= \frac{1}{2}$ //

$\mu_2 = E(x-\text{mean})^2$

$= E(X-EX)^2 = EX^2 - (EX)^2$

$=$ variance

$EX^2 = \int_0^1 x^2 f(x)\,dx$

$= \int_0^1 x^2 6x(1-x)\,dx$

$= 6\int_0^1 x^2 x(1-x)\,dx$

$= 6\int_0^1 x^3(1-x)\,dx$

$= 6\int_0^1 x^3 - 6\int_0^1 x^4\,dx$

$= 6\left[\frac{x^4}{4}\right]_0^1 - 6\left[\frac{x^5}{5}\right]_0^1$

$= 6\left[\frac{1^4}{4}-0\right] - 6\left[\frac{1^5}{5}-0\right]$

$= 6\left[\frac{1}{4}\right] - 6\left[\frac{1}{5}\right]$

$= \frac{6}{4} - \frac{6}{5}$

$= 6\left[\frac{1}{4}-\frac{1}{5}\right]$

$= 6\left[\frac{5-4}{20}\right]$

$= 6\times\frac{1}{20}$

$= \frac{3}{10}$ //

$\therefore \mu_2 =$ variance $= \frac{3}{10} - \left(\frac{1}{2}\right)^2 = \frac{3}{10} - \frac{1}{4} = \frac{12-10}{40} = \frac{2}{40} = \frac{1}{20}$ //

$\therefore \mu_1 = 0$, $\mu_2 =$ variance $= \frac{1}{20}$, mean $= \frac{1}{2}$ //

Part C

25.

$$x+y+z=9$$
$$2x+5y+7z=52$$
$$2x+y-z=0$$

By cramer's Rule

$$x=\frac{|A_1|}{|A|},\ y=\frac{|A_2|}{|A|} \text{ and } z=\frac{|A_3|}{|A|}$$

where $A=\begin{bmatrix}1&1&1\\2&5&7\\2&1&-1\end{bmatrix}$, $A_1=\begin{bmatrix}9&1&1\\52&5&7\\0&1&-1\end{bmatrix}$, $A_2=\begin{bmatrix}1&9&1\\2&52&7\\2&0&-1\end{bmatrix}$

$A_3=\begin{bmatrix}1&1&9\\2&5&52\\2&1&0\end{bmatrix}$

$$|A| = 1\begin{vmatrix}5&7\\1&-1\end{vmatrix}-1\begin{vmatrix}2&7\\2&-1\end{vmatrix}+1\begin{vmatrix}2&5\\2&1\end{vmatrix}$$

$=1(5\times-1-1\times7)-1(2\times-1-2\times7)+1(2\times1-2\times5)$

$=1(-5-7)-(-2-14)+(2-10)$

$=-12-(-16)+-8$

$=-12+16-8$

$=16-12-8$

$=16-20$

$=\underline{-4}.$

$$|A_1| = \begin{vmatrix}9&1&1\\52&5&7\\0&1&-1\end{vmatrix}=9\begin{vmatrix}5&7\\1&-1\end{vmatrix}-1\begin{vmatrix}52&7\\0&-1\end{vmatrix}+1\begin{vmatrix}52&5\\0&1\end{vmatrix}$$

$=9(-5-7)-(-52-0)+(52-0)$

$=9\times-12+52+52$

$=-108+104$

$=-4$//

$$|A_2| = \begin{vmatrix} 1 & 9 & 1 \\ 2 & 52 & 7 \\ 2 & 0 & -1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 52 & 7 \\ 0 & -1 \end{vmatrix} - 9\begin{vmatrix} 2 & 7 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 2 & 52 \\ 2 & 0 \end{vmatrix}$$

$$= 1(-52-0) - 9(-2-14) + (0-104)$$

$$= -52 + 9\times16 + -104$$

$$= 144 - 52 - 104$$

$$= \underline{\underline{-12}}.$$

$$|A_3| = \begin{vmatrix} 1 & 1 & 9 \\ 2 & 5 & 52 \\ 2 & 1 & 0 \end{vmatrix}$$

$$= 1\begin{vmatrix} 5 & 52 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 2 & 52 \\ 2 & 0 \end{vmatrix} + 9\begin{vmatrix} 2 & 5 \\ 2 & 1 \end{vmatrix}$$

$$= 1[0-52] - [0-104] + 9[2-10]$$

$$= -52 + 104 + 9\times-8$$

$$= -52 + 104 - 72$$

$$= 104 - 52 - 72$$

$$= \underline{\underline{-20}}.$$

$$\therefore x = \frac{|A_1|}{|A|} = \frac{-4}{-4} = 1, \quad y = \frac{|A_2|}{|A|} = \frac{-12}{-4} = 3, \quad z = \frac{-20}{-4} = 5$$

$$\underline{\underline{x = 1,\ y = 3,\ z = 5}}$$

26

(a) $y = e^x \log x$

$$\frac{dy}{dx} = \frac{d}{dx}(e^x \log x) = e^x \frac{1}{x} + \log x\, e^x$$

$$\frac{d^2y}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{e^x}{x} + \log x\, e^x\right)$$

$$= \frac{x\frac{d}{dx}e^x - e^x\frac{d}{dx}x}{x^2} + \log x \frac{d}{dx}e^x + e^x\frac{d}{dx}\log x$$

$$= \frac{x e^x - e^x \times 1}{x^2} + \log x\, e^x + e^x \frac{1}{x}$$

$$= \frac{x e^x}{x^2} - \frac{e^x}{x^2} + e^x \log x + \frac{e^x}{x}$$

$$= \frac{e^x}{x} - \frac{e^x}{x^2} + e^x\log x + \frac{e^x}{x}$$

$$= \frac{2e^x}{x} + e^x\log x - \frac{e^x}{x^2}$$

$\therefore$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - x(x+1)y$$

$$= x^2\left[\frac{2e^x}{x} + e^x\log x - \frac{e^x}{x^2}\right] + x\left[\frac{e^x}{x} + \log x\, e^x\right]$$

$$-x(x+1)(e^x\log x)$$

$$= x^2\frac{2e^x}{x} + x^2e^x\log x - x^2\frac{e^x}{x^2} + x\frac{e^x}{x} + x\log x\, e^x$$

$$-(x^2+x)e^x\log x$$

$$= x2e^x + x^2e^x\log x - e^x + e^x + xe^x\log x$$

$$- x^2e^x\log x - xe^x\log x$$

$$= 2xe^x + x^2e^x\log x - x^2e^x\log x - e^x + e^x + xe^x\log x - xe^x\log x$$

$$= 2xe^x + 0 + 0 + 0 = 2xe^x.$$

(b) $u = x^2 + y^2$, $x = t^3 + 3$.

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy.$$

$$\frac{\partial u}{\partial x} = 2x + 0 = 2x,$$

$$\frac{\partial u}{\partial y} = 0 + 2y$$

$$\therefore du = 2x\,dx + 2y\,dy.$$

since $x = t^3 + 3$,

$$\frac{dx}{dt} = 3t^2 + 0 = 3t^2$$

$$\therefore dx = 3t^2 dt$$

$$\therefore du = 2x\,3t^2 dt + 2y\,dy.$$

Note: Incomplete question (since $y(t)$ missing).

(27) (a) $R(x) = \frac{x(75-x)}{3} - \left(\frac{x^2}{25} + 3x + 100\right)$

$$R'(x) = \frac{dR(x)}{dx} = \frac{d}{dx}\left(\frac{x(75-x)}{3}\right) - \frac{d}{dx}\left(\frac{x^2}{25} + 3x + 100\right)$$

$$= \frac{75 - 2x}{3} - \frac{2x}{25} - 3$$

$R'(x) = 0$

$$\Rightarrow \frac{75-2x}{3} - \frac{2x}{25} - 3 = 0$$

$$\Rightarrow 25 - \frac{2}{3}x - \frac{2}{25}x - 3 = 0$$

$$\Rightarrow 22 = \frac{2}{3}x - \frac{2}{25}x$$

$$= x\left(\frac{2}{3} - \frac{2}{25}\right)$$

$$= x(50-6)/75$$

$$\Rightarrow x\left(\frac{44}{75}\right) = 22$$

$$\Rightarrow x = \frac{22\times 75}{44}$$

$$\Rightarrow x = \frac{75}{2} = 37.5$$

$\therefore \quad \left(R''(x) = -\frac{2}{3} - \frac{2}{25} < 0.\right)$

Net revenue will be maximum at $x = 37.5$

(b) $\quad y = (2-x)^2 + (3-x)^2 + (4-x)^2,$

$$\frac{dy}{dx} = \frac{d}{dx}\left[(2-x)^2 + (3-x)^2 + (4-x)^2\right]$$

$$= 2(2-x)\times -1 + 2(3-x)\times -1 + 2(4-x)\times -1$$

$$= -2\left[2-x + 3-x + 4-x\right]$$

$$= -2\left[2+3+4-3x\right]$$

$$\frac{dy}{dx} = 0 \Rightarrow -2\left[2+3+4-3x\right] = 0$$

$$\Rightarrow 2+3+4-3x = \frac{0}{-2} = 0$$

ie $\quad 2+3+4-3x = 0$

ie $\quad 2+3+4 = 3x$

ie $\quad \dfrac{2+3+4}{3} = x$

ie $\quad x =$ mean of $2, 3, 4$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[-2(2+3+4-3x)\right]$$

$$= -2\frac{d}{dx}(9-3x)$$

$$= -2\times -3$$

$$= 6 > 0.$$

$\therefore$ at $x = \frac{2+3+4}{3}$ y is minimum

28) let

A – the event of defective output

B_1 – event of output produced by machine I

B_2 – " machine II

B_3 – " " machine III

$$P(B_1) = \frac{3000}{3000+2500+4500} = 0.3$$

$$P(B_2) = \frac{2500}{10000} = .25$$

$$P(B_3) = \frac{4500}{1000} = .45$$

Given $P(A|B_1) = \frac{1}{100} = .01$

(1% of output produced by Machine I is defective)

$$P(A|B_2) = \frac{1.2}{100} = .012$$

(1.2% of output produced by Machine II is defective)

$$P(A|B_3) = \frac{2}{100} = .02$$

(2% of output produced by Machine III is defective)

$\therefore$ $P(B_1) = .3$, $P(B_2) = .25$, $P(B_3) = .45$

$P(A|B_1) = .01$, $P(A|B_2) = .012$, $P(A|B_3) = .02$

By Baye's Theorem

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)}$$

(a) Machine I

$$P(B_1|A) = \frac{P(B_1)P(A|B_1)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)}$$

$$= \frac{0.3 \times .01}{0.3 \times 0.01 + 0.25 \times 0.012 + 0.45 \times 0.02}$$

$$= \frac{.003}{.015} = \frac{1}{5} = 0.2$$

(b) Machine II

$$P(B_2|A) = \frac{P(B_2)P(A|B_2)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)}$$

$$= \frac{0.25 \times .012}{0.3 \times .01 + .25 \times .012 + .45 \times .02}$$

$$= \frac{.003}{.015} = \frac{1}{5} = .0.2$$

$$= 0.2$$

(c) Machine III

$$P(B_3|A) = \frac{P(B_3)P(A|B_3)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)}$$

$$= \frac{0.45 \times 0.02}{0.3 \times .01 + .25 \times .012 + .45 \times .02} = \frac{.009}{.015} = 3/5 = .6$$