Module 4

Estimation Theory

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(Stastical estimation is concerned with the methods by which population characteristics are estimated from sample. The true value of a population parameter is an unknown constant that can be correctly ascertained only by an exhaustive study of the population. However it is ordinarly too expensive or it is infeasible to enumerate complete population to obtain the required information. (Therefore we estimate those parameters of the population through sample. This is stastical estimation.

With respect to estimating a parameter the following two types of estimates are possible.

1- Point estimate (In point estimate we are finding of a single value for the parameter)

2- Interval estimates. =(b/w 2 values) (finding an interval for the parameter).

Estimator & Estimates

A sample statistic (function) that is used to estimate the population parameter is called an estimator.

For eg :- sample mean is an estimator of the population mean.

An estimate is a specific observed value of a statistic. To find an estimate, we select an sample & compute the value of the estimator from that sample.

For eg :- If sample mean is the estimator, the particular value of the sample mean obtained from the sample is the estimate and the population mean is the parameter.

Criteria for a good estimator
(Desirable properties)

The following are some of characteristics which should be satisfied by a good estimator.

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1) An estimator should be unbiased (unbiasedness)

2) " " " consistent (consistency)

3) " " " efficient (efficiency)

4) " " " sufficient (sufficiency)

Unbiasedness

A statistic 't' is used to be an unbiased estimator of population parameter 'θ' if $\boxed{E(t) = \theta}$.

For eg (1) If 't' follows a sampling distribution and mean of that distribution is the value of the parameter 'θ', then 't' is the unbiased estimator of θ.

eg: 2) Sample mean ($\bar{x}$) is the unbiased estimator of the population mean (μ)

ie; $E\bar{x} = \mu$,

$\bar{x}$ is the sample mean and μ is the population mean

$\bar{x} \sim N(\mu, \sigma^2/n)$
$E(\bar{x}) = \mu$
$V(\bar{x}) = \sigma^2/n$

Consistency

The estimator ~~function~~ = $t_n = t(x_1, \ldots, x_n)$ of parameter θ is called consistent. If "t_n converges to θ in probability", ie, for $\varepsilon > 0$

$$\lim_{n \to \infty} P[|t_n - \theta| \le \varepsilon] = 1$$

Eg:- Sample mean is a consistent estimator of population mean since for large values of n, sample tends to population mean.

Efficiency

Let t_1 & t_2 are two unbiased estimators of a parameter θ. If $V(t_1)$ is less then $V(t_2)$. Then t_1 said to be more efficient than t_2.

$V(t_1) < V(t_2)$ then t_1 is more efficient.

Sufficiency

A statistic 't' is said to be sufficient estimator of parameter θ, if it contains all the information in the sample, regarding the parameter. In other words, a (sufficient) statistic utilizes all the information that a given sample can furnish about the parameter.

Note: A sufficient estimator is most efficient. If an efficient estimator exists, it is always consistent estimator. It may or may not be unbiased.

Eg 1- Sample means is sufficient estimator of population mean.

Factorization Theorem

Let $x_1, x_2 \dots x_n$ be a random of size n from a population with density functions $f(x; \theta)$ where, θ denotes the parameter. Then a statistic $t = f(x_1 \dots x_n)$ is sufficient if and only if (iff) probability density function of $x_1, x_2 \dots x_n$ is capable of being expressed in the form,

$$L(x_1, x_2 \dots x_n; \theta) = L_1(t; \theta)\, L_2(x_1, x_2 \dots x_n)$$

where the functions,

$L_2(x_1 \dots x_n)$ is independent of θ

Note, $L(x_1, x_2 \dots x_n; \theta) = \prod_{i=1}^{n} f(x_i; \theta)$

Eg: $x \sim P(\lambda)$,

$$L(x_1, x_2 \dots x_n; \lambda) = \prod_{i=1}^{n} f(x_i; \lambda) = \prod_{i=1}^{n} \left(\frac{e^{-\lambda} \lambda^{x_i}}{x_i!}\right)$$

product

$$= \frac{e^{-n\lambda} \lambda^{\sum_{i=1}^{n} x_i}}{\prod x_i!}$$

$x_1! x_2! \dots x_n!$

$\bar{x} = \frac{x_1 + \dots + x_n}{n} = \frac{\sum x_i}{n}$

$n\bar{x} = \sum x_i$

$$= e^{-n\lambda} \lambda^{n\bar{x}} \left(\frac{1}{\prod x_i!}\right)$$

$$= L_1(\lambda; \bar{x})\, L_2(x_1 \dots x_n)$$

$\bar{x}$ is ~~statistic~~ sufficient estimator of λ.

Interval estimation

Let $x_1, x_2 \dots x_n$ be a random sample of size n from a population with density function $f(x; \theta)$, θ being the parameter.

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Interval estimation is a statistical technique which consist in the determination of 2 constants t_1 and t_2 such that

$$\boxed{P(t_1 < \theta < t_2) = 1-\alpha}$$

alpha

where, α is the level of significance. The interval (t_1, t_2) is called $(0<\alpha<1)$ $100(1-\alpha)\%$ confidence interval for θ.

The interval (t_1, t_2) is a random interval. Hence in general it will be different in different samples. The limits t_1 and t_2 are called 'confidence limits'. $1-\alpha$ is called 'confidence coefficients' of the interval (t_1, t_2).

Statistical Hypothesis

A statistical hypothesis may be defined as a tentative conclusion logically drawn concerning the parameter or the form of the distribution of the population.

For eg:- the assumption "the sample is drawn from a normal population with mean 40 of SD = 10 is a hypothesis.

Tests of Hypothesis (statistical tests)

statistical test of hypothesis is a process or procedure under which a statistical hypothesis is laid down and it is accepted or rejected on the basis of a random sample drawn from the population. The tests conducted to accept or reject the hypothesis are known as statistical test of hypothesis.

Commonly used tests are Z-test (standard normal), t-test, χ^2-test F-test.

1- Null and Alternative hypothesis :

The hypothesis to be tested usually referred to as the "null hypothesis" and is denoted by the symbol 'H0'. ~~The null~~ The hypothesis is a proportion of zero differences.

Any hypothesis other than null hypothesis is called an alternative hypothesis.

So when the null hypothesis is rejected we accepted other hypothesis known as alternative hypothesis. Alternative hypothesis is denoted by 'H1'.

for eg:- when ~~are~~ we want to test ~~the~~ whether the population is 65 or not. The null hypothesis is "population mean is 65" and alternative hypothesis is "population mean is not 65".

2- Simple & Composite hypothesis :

If a hypothesis is concerning the population completely such as the functional form and the parameter, it is called simple hypothesis.

If a hypothesis is not simple, then it is a composite hypothesis.

Ex: population is normal with mean 25 and SD = 10" is a simple hypothesis.

While the hypothesis "population follows normal distribution with mean = 25" is a composite hypothesis

3- Parametric and non-parametric hypothesis :

A hypothesis which specifies only the parameters of the probability density function is called a parametric hypothesis.

If a hypothesis specifies only the form of the density function of the population, it is called a non-parametric hypothesis.

Ex: the hypothesis "mean of the population is 25" is parametric while the hypothesis "population is normal" is non-parametric.

Type I & Type II errors

Rejecting a null hypothesis ~~to~~ Ho when it is actually true is called type I error or error of 1st kind.

Accepting a null hypothesis Ho when it is false (H1 is true) is called type II error or error of IInd kind.

Definitions : We define, the ~~population~~ probabilities α and β as follows;

$$\alpha = P(\text{type I error}) = P(\text{Rejecting } H_0 \text{ given } H_0 \text{ is true})$$

$$= P(\text{rejecting } H_0 \mid H_0)$$

$$\beta = P(\text{Type II error})$$

$$= P(\text{accepting } H_0 \text{ given } H_1 \text{ is true})$$

$= P(\text{accepting } H_0 / H_1)$

Test statistic

In a testing procedure, an appropriate function of the sample values is chosen and the decision either to accept or reject the H_0 is taken based on the value of this function. This function is called "test statistic" or "test criterion".

Critical Region

In a test procedure we calculate test statistic on which we take our decision. The range of variation of this statistic is divided into two regions, acceptance region or rejection region. If the computed value of the test statistic falls in the rejection region we reject the null hypothesis. The rejection region is also known as critical region.

Critical value

The value of the test statistic which separates the rejection region from the acceptance region is called critical value.

Level of significance

The probability with which we may reject a null hypothesis, when it is true, is called the level of significance or probability of type I error. The level of significance is the risk, a ~~satisfaction~~ statistician running in his decisions. The level of significance is denoted by α.

Then $\alpha = P(\text{rejecting } H_0 / H_0)$
$= P(\text{type I error})$

Power of a test

Probability for rejecting the null hypothesis when the alternative hypothesis is true is called power of a test.

$\therefore$ Power of a test $= 1 - P(\text{type II error})$
$= 1 - \beta$

Most powerfull Test

The test based on the most powerful critical region is called most powerfull test.

Note :

In testing of hypothesis, we can ~~so~~ think of a no. of critical regions each having same level of ~~substance~~ significance. Of all these critical regions that which has least type II error is called best critical region (B C R).