Module 3

OPTIMIZATION and INTEGRATION

Unconstrained optimisation

1. Find the maximum and minimum of the function

$$f(x) = x^3 - 6x^2 + 9x + 1$$

An:

$$f'(x) = \frac{d}{dx}(x^3 - 6x^2 + 9x + 1)$$

$$= 3x^2 - 12x + 9$$

$$f''(x) = \frac{d(f'x)}{dx} = \frac{d}{dx}(3x^2 - 12x + 9)$$

$$= 3(2x) - 12\times1 + 0$$

$$= 6x - 12$$

let $f'(x) = 0$

$$3x^2 - 12x + 9 = 0$$

$$\cancel{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{}}$$

$$x^2 - 4x + 3 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{-4^2 - 4\times1\times3}}{2\times1}$$

$$= \frac{4 \pm \sqrt{16-12}}{2}$$

$$= \frac{4+\sqrt{4}}{2} \quad \text{or} \quad \frac{4-\sqrt{4}}{2}$$

$$= \frac{4+2}{2} \quad \text{or} \quad \frac{4-2}{2}$$

$$= 6/2 \quad \quad 2/2$$

$$= 3 \quad \text{or} \quad 1$$

$f''(x) = 6x - 12$

$f''(3) = 6 \times 3 - 12 = 6$

$f''(1) = 6 \times 1 - 12 = \cancel{-8}\ \ -6$

The function is minimum at $x = 3$

the function is maximum at $x = 1$

2. $x^3 - 6x + 4$. Find optimisation ?

Ans: $f'(x) = \frac{d}{dx}(x^3 - 6x + 4)$

$= 3x^2 - 6$

Let $f'(x) = 0$

$3x^2 - 6 = 0$

$x^2 - 2 = 0$

$x^2 = 2$

$x = \pm\sqrt{2}$

$f''(x) = \frac{d}{dx}(3x^2 - 6)$

$= 6x$

$x = +\sqrt{2}$

$f(x) = 6\sqrt{2} > 0$ $\cancel{8.4}$

$x = -\sqrt{2}$

$f(x) = 6 - \sqrt{2} < 0$ $\cancel{8.4}$ $-6\sqrt{2} < 0$

The $f(x)$ is minimum at $x = \sqrt{2}$

The $f(x)$ is maximum at $x = -\sqrt{2}$

3. $f(x) = x^3 - 3x^2 + 3x + 2$ find optimisation

Ans: $f'(x) = \frac{d}{dx}(x^3 - 3x^2 + 3x + 2)$

$= 3x^2 - 3(2x) + 3 \times 1 + 0$

$= 3x^2 - 6x + 3$

Let $f'(x) = 0$

$3x^2 - 6x + 3 = 0$

$x^2 - 2x + 1 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{-2^2 - 4 \times 1 \times 1}}{2 \times 1}$$

$$= \frac{2 \pm \sqrt{4-4}}{2} = \frac{2+0}{2} = 1$$

Scanned by CamScanner

$$\frac{2-0}{2} = 1$$

$$f''(x) = \frac{d}{dx}\left(3x^2-6x+3\right)$$

$$= 6x-6$$

$$f''(1) = 6\times1-6 = 0$$

4. Find maximum and minimum of $y = \cancel{2x^3-3}$

$y = 2x^3-3x^2-12x+4.$

Ans

$$f'(x) = \frac{d}{dx}\left(2x^3-3x^2-12x+4\right)$$

$$= 6x^2-6x-12$$

Let $f'(x) = 0$

$$6x^2-6x-12 = 0$$

$$x^2-x-2 = 0$$

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$= \frac{1\pm\sqrt{1^2-4\times1\times-2}}{2\times1}$$

$$= \frac{1\pm\sqrt{1+8}}{2} \quad \frac{1+3}{2} = \frac{4}{2} = 2$$

$$= \frac{1-3}{2} = \frac{-2}{2} = -1$$

$$f''(x) = \frac{d}{dx}\left(6x^2-6x-12\right)$$

$$= 12x-6$$

$f(x) = 2$

$$\cancel{12x} \quad f(2) = 12\times2-6 =$$

$$= 24-6 = 18$$

$f(x) = -1$

$$f(-1) = 12\times-1-6 = -12-6 = -18$$

The function is minimum at $x = 2$

and function is maximum at $x = -1$

$$f(x) = 2x^3-3x^2-12x+4$$

maximum value

$$f(-1) = 2\times(-1)^3-3(-1)^2-12\times-1+4$$

$$= -2+3+12+4$$

$$= \cancel{0}17$$

minimum value,

$$f(2) = 2\times(2^3)+3(2)^2-12(2)+4$$

$$= 16-12-24+4$$

$$= \cancel{-24} \quad -16$$

Constrained optimisation

1. Maximise $5-(x_1-2)^2-2(x_2-1)^2$ subject to

$x_1+4x_2=3$

Let, $L(x_1, x_2, \lambda) =$

$5(x-2)^2-2(x_2-1)^2+\lambda(x_1+4x_2-3)$

This function is called Lagrangian of the problem.

$\frac{\partial L}{\partial x_1}, \frac{\partial L}{\partial x_2}, \frac{\partial L}{\partial \lambda}$ and equate to zero.

$$\frac{\partial L}{\partial x_1} = \frac{\partial}{\partial x_1}\left(5-(x_1-2)^2-2(x_2-1)^2+\lambda(x_1+4x_2-3)\right)$$

$$= 0-2(x_1-2)-2\times 0+\lambda(1+0+0)$$

$$= -2x_1+4+\lambda$$

$$\frac{\partial L}{\partial x_1} = -2x_1+4+\lambda = 0 \quad -(1)$$

$\frac{d}{dx}(x_1-2)^2$

$2(x_1-2)\times\frac{d}{dx}(x_1-2)$

$2(x_1-2)(1-0)$

$=2(x_1-2)$

$$\frac{\partial L}{\partial x_2} = 0-0-2\times 2(x_2-1)+\lambda(0+4\times 1-0)$$

$$=-4(x_2-1)+\cancel{2}\cdot 4\lambda$$

$$=-4x_2+4+4\lambda$$

$$\frac{\partial L}{\partial x_2} = -4x_2+4+4\lambda = 0 \quad -(2)$$

$$\frac{\partial L}{\partial x_1} = -2x_1+4+\lambda = 0$$

$$\lambda = 2x_1-4 \quad -(1)$$

$$\frac{\partial L}{\partial x_2} = -4x_2+4+4\lambda = 0$$

$$4\lambda = 4x_2-4 \quad -(2)$$

$$\lambda = \frac{4x_2-4}{4}$$

$$\lambda = x_2-1$$

$\frac{d}{dx}(x_2-1)^2$

$2(x_2-1)\times\frac{d}{dx}(x_2-1)$

$2(x_2-1)\,1\text{-}$

$$\frac{\partial L}{\partial \lambda} = 0\cancel{-2(x_1-2)}-2\times\cancel{2(x_2-1)+0}$$

$$=\cancel{-2x_1+4-4x_2+4}$$

$$=\cancel{-2x_1-4x_2+8}$$

$$=0-0-0+1(x_1+4x_2-3)$$

$$= x_1+4x_2-3 \quad -(3)$$

$$\frac{\partial L}{\partial \lambda} \Rightarrow x_1+4x_2-3=0 \quad -(3)$$

$$x_1+4x_2=3 \quad -(3)$$

(1) × (2) $2x_1-4=\lambda$

$4x_2-4=4\lambda$

(1) $\Rightarrow$ $2x_1+0x_2-4=\lambda$

(2) $\Rightarrow 0x_1 + 4x_2 - 4 = 4\lambda$

(1) $\times 4$ $\quad 8x_1 + 0x_2 - 16 - 4\lambda$ $\quad$ — (A)

(2) $\quad \cancel{0}x_1 + 4x_2 - 4 = 4\lambda$ $\quad$ — (B)

(A) − (B) $= 8x_1 - 4x_2 - 12 = 0$

ie $8x_1 - 4x_2 - 12 = 0$

$8x_1 - 4x_2 = 12$

$2x_1 - x_2 = 3$ $\quad$ — (C)

Solve, (3) & (C),

$\Rightarrow x_1 + 4x_2 = 3$ $\quad$ — (3)

$2x_1 - x_2 = 3$ $\quad$ — (C)

(3) $\times 2$ $\quad 2x_1 + 8x_2 = 6$

$2x_1 - x_2 = 3$

(3) − (C) $\quad 0 + 9x_2 = 3$

$9x_2 = 3$

$x_2 = \frac{3}{9} = 1/3$ //

substitute $x_2 = 1/3$ in (1) eqt

$2x_1 + 8 \times 1/3 = 6$

$2x_1 + \frac{8}{3} = 6$

$2x_1 = 6 - 8/3$

$2x = \frac{18-8}{3} = \frac{10}{3}$

$x_1 = \frac{(10/3)}{2} = 10/3 \times \frac{1}{2} = \frac{10}{6}$

$= 5/3$ //

Substitute x_1 & x_2 in eqt (1)

$\cancel{\lambda = 2x_1 - 4}$

$\cancel{= 2 \times 5/3 - 4}$

$\cancel{= \frac{10}{3} - 4}$

$\lambda = \frac{10-12}{3} = -2/3$

$\frac{1}{3} - 1 = \frac{1-3}{3} = -2/3$

$5 - (5/3 - 2)^2 - 2(1/3 - 1)^2$

$= 5 - (\frac{25}{9} - 4) - 2(1/9 - 1)$

2. Minimise $2x_1^2 + x_2^2$ subject to $x_1 + x_2 = 1$

Ans: Let $(x_1, x_2, \lambda) =$

$f = 2x_1^2 + x_2^2 + \lambda(x_1 + x_2 - 1)$

$\frac{\partial L}{\partial x_1} = \frac{\partial}{\partial x}\left(2x_1^2 + x_2^2 + \lambda(x_1 + x_2 - 1)\right)$

$= 2 \times 2x_1 + 0 + \lambda(1 + 0 - 0)$

$= 4x_1 + \lambda$

$\Rightarrow 4x_1 + \lambda = 0$

$\lambda = -4x_1 \quad - (1)$

$\frac{\partial L}{\partial x_2} = 0 + 2x_2 + \lambda(0 + 1 - 0)$

$= 2x_2 + \lambda$

$\Rightarrow 2x_2 + \lambda = 0$

$\lambda = -2x_2 \quad - (2)$

$\frac{\partial L}{\partial \lambda} = x_1 + x_2 - 1$

$\Rightarrow x_1 + x_2 - 1 = 0$

$x_1 + x_2 = 1 \quad - (3)$

$-4x_1 = \lambda$

$-2x_2 = \lambda$

$(1) \Rightarrow -4x_1 + 0x_2 = \lambda$

$(2) \Rightarrow 0x_1 - 2x_2 = \lambda$

$(1) - (2) = -4x_1 + 2x_2 = 0 \quad - (A)$

$(3) \Rightarrow x_1 + x_2 = 1$

$(A) \Rightarrow -4x_1 + 2x_2 = 0$

$(3) \times 2 = 2x_1 + 2x_2 = 2$

$-4x_1 + 2x_2 = 0$

$(3) - (A) \quad 6x_1 - 0 = 2$

$6x_1 = 2$

$x_1 = 2/6 = 1/3$

Substitute $x_1 = 2/6$ in (3) rd eqt

$2/6 + x_2 = 1$

$x_2 = 1 - 2/6$

$= \frac{6-2}{6} = 4/6 = 2/3$

$\cancel{\lambda = -4 \times 2/6} = \cancel{-8/6} = \cancel{-4/3}$

$x_1 = 1/3$ $\quad x_2 = 2/3$

3. Find the minimum of $U = 6x^2 + y^2$ subject to constraint that $4x - y = 1$. What is the value of U:

$$L = 6x^2 + y^2 + \lambda(4x - y - 1)$$

$$\frac{\partial L}{\partial x} = \frac{\partial}{\partial x}\left(6x^2 + y^2 + \lambda(4x - y - 1)\right)$$

$$= 12x + 0 + \lambda(4\times1) - 0 - 0)$$

$$= 12x + 4\lambda$$

$$\Rightarrow 12x + 4\lambda = 0$$

$$4\lambda = -12x \qquad -(1)$$

$$\frac{\partial L}{\partial y} = 0 + 2y + \lambda(0 - 1 - 0)$$

$$= 2y - \lambda$$

$$\Rightarrow 2y - \lambda = 0$$

$$2y = \lambda \qquad -(2)$$

$$\frac{\partial L}{\partial \lambda} = 4x - y - 1 \qquad -(3)$$

(1) & (2) $\quad -12x = 4\lambda$

$$2y = \lambda$$

(2) $\times 4$ $\quad 8y = 4\lambda$

$$\cancel{-12x = 4\lambda}$$

$$\cancel{8y = 4\lambda}$$

$\cancel{(1) - (2)}$ $= \quad -12x + 0y = 4\lambda$

(1) - (2) $\quad 0x + 8y = 4\lambda$

$$-12x - 8y = 0 \qquad -(A)$$

$$4x - y = 1 \qquad -(3)$$

$$-12x - 8y = 0$$

(3) $\times$ (8) $= \quad 32x - 8y = 8$

$$-12x - 8y = 0$$

(3) - (A) $\quad 44x \quad 0 = 8$

$$44x = 1$$

$$x = \frac{8}{44} = \frac{2}{11}$$ //

substitute in eqt (1)

$$4\lambda = -12 \times \frac{2}{11} = \frac{-24}{11}$$

$$\lambda = \frac{-24}{11} \times \frac{1}{4} = \frac{-24}{44} = \frac{-6}{11}$$

$$2y = \frac{6}{11}$$

$$y = \frac{-6}{11} \times \frac{1}{2} = \frac{12}{22} = \frac{-6}{22}$$

$$= \frac{-3}{11}$$ //

$-3/11$

$2/11$

$2/11$

Methods of Integration

$\frac{dy}{dx} x^n = nx^{n-1}$

$\int nx^{n-1} dx = x^n$

$\frac{x^n}{n} = \int x^{n-1} dx$

when $n = 3$

$\frac{x^3}{3} = \int x^{3-1} dx$

when $n = 4$

$\frac{x^4}{4} = \int x^{4-1} dx$

$$\boxed{\begin{gathered} \frac{d}{dx} x^n = nx^{n-1} \\ \int x^n dx = \frac{x^{n+1}}{n+1} \end{gathered}}$$

$\sin x = \cos x$

$\int \cos x = \sin x$

$\int e^x = e^x$

$\cos x = \sin x$

$\int \sin x = -\cos x$

eg:- Find the integrale of the followings ?

1) x^3

Ans. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$\int x^3 dx = \frac{x^{3+1}}{3+1} = \frac{x^4}{4} + c$

$\frac{d}{dx} \frac{x^4}{4} + c$

$= \frac{d}{dx} \frac{x^4}{4} + \frac{d}{dx} c$

$= \frac{1}{4} 4x^3 + 0$

$= x^3$

$\frac{d}{dx} \left(\frac{x^4}{4} + c \right)$

$= \frac{d}{dx} \frac{x^4}{4} \times \frac{d}{dx} c$

$= \frac{1}{4} 4x^3 + 0$

$= x^3$

2) x^5

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$\int x^n dx = \frac{x^{5+1}}{5+1} = \frac{x^6}{6} + c$

3) x

$\int x^n dx = \frac{x^{n+1}}{n}$

$\int x^1 dx = \frac{x^{1+1}}{n+1} = \frac{x^2}{2} + c$

4) 1.

$x^0 = 1$

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$\int x^0 dx = \frac{x^{0+1}}{0+1} + c = \frac{x^1}{1} + c$

$$\boxed{\int k f(x)\,dx = k\int f(x)\,dx.}$$

1) eg: $2x^3$

$$\int 2x^3 dx = 2\int x^3 dx$$
$$= 2\times\frac{x^4}{4}+c$$
$$= \frac{x^4}{2}+c$$

2) $2e^x$

$$\int 2e^x dx = 2\int e^x dx$$
$$= 2\times e^x + c$$
$$= 2e^x + c$$

3) 10

$$\int 10\,dx = \int 10\times 1\,dx.$$
$$= 10\int 1\,dx$$
$$= 10x + c$$

$f(x)$	integration
x^n	$\frac{x^{n+1}}{n+1}+c$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$
e^x	$e^x + c$
1	$x + c$
$\frac{1}{x}$	$\log x + c$

4) $1/x$

$$\int 1/x\,dx = \int 1/x\,dx$$
$$= \log x + c$$

HW

→ Find $\int \sqrt{x}\,dx$

$$= \cancel{\int \sqrt{x}\,dx} = \frac{1}{2\sqrt{x}} + c$$

→ Find $\int \frac{dx}{2\sqrt{2x^3}}$

Ans 1 $\int \sqrt{x}\, dx = \int x^{1/2}\, dx$

$= \dfrac{x^{1/2+1}}{1/2+1} + c$

$= \dfrac{x^{3/2}}{3/2} + c$

$= \frac{2}{3} x^{3/2} + c$

$= \frac{2}{3} x\sqrt{x} + c$

$\frac{2}{3} x^{1/2+1} \sqrt{x} + c$

$\frac{2}{3} x^{1/2} x^{1} + c$

$\frac{2}{3} \sqrt{x}\, x + c$

$\frac{2}{3} x \sqrt{x} + c$

Ans → $\int \dfrac{dx}{2\sqrt{2x^3}} = \int \dfrac{dx}{2\sqrt{2x^3}} \int dx \dfrac{1}{2\sqrt{2x^3}}\, dx + c$

$= \cancel{dx} \int \dfrac{1}{2\sqrt{2}\sqrt{x^3}} + dx$

$= \dfrac{1}{2\sqrt{2}(1/2)^3} \int \dfrac{1}{2\sqrt{2}\, x^{1/2(3)}}\, dx$

$= \int \dfrac{1}{2\sqrt{2}\, x^{3/2}}\, dx$

$= \int \dfrac{1}{2\, 2^{1/2} x^{3/2}}\, dx$

$= \int \frac{1}{2}\, 2^{-1/2}\, x^{-3/2}\, dx$

$= \frac{1}{2} \int 2^{-1}\, 2^{-1/2}\, x^{-3/2}\, dx$

$= \int 2^{-1+-1/2}\, x^{-3/2}\, dx$

$= \int 2^{-3/2}\, x^{-3/2}\, dx$

$= 2^{-3/2} \int x^{-3/2}$ $\qquad \dfrac{-3+2}{2}$

$= 2^{-3/2} \times \dfrac{x^{-3/2+1}}{-3/2+1} + c$

$= \dfrac{2\, x^{-1/2}}{(-1/2)} + c$

$= \cancel{2 \times 2\, x^{-1/2} + c}$

$= \cancel{-4 x^{-1/2} + c}$

$= 2^{-3/2} \dfrac{x^{-1/2}}{-1/2} + c$

$= 2^{-3/2}\, \cancel{-2 \times x^{-1/2}} + c$

→ $\int (x^3 - 4x^2 + x)\, dx$

$\boxed{\int [f_1(x) + f_2(x)\, dx = \int f_1(x)\, dx + \int f_2(x)\, dx}$

Ans:- $\int (x^3 - 4x^2 + x)\, dx = \int x^3 dx - \int 4x^2 dx + \int x\, dx$

$= \dfrac{x^4}{4} - 4\dfrac{x^3}{3} + \dfrac{x^2}{2} + c$

$= \dfrac{x^4}{4} - \frac{4}{3} x^3 + \frac{x^2}{2} + c$

? Find $\int (3x)^{-7} dx$

Ans: $\int 3^{-7} x^{-7} dx$

$= 3^{-7} \int x^{-7} dx$

$= 3^{-7} \frac{x^{-7+1}}{-7+1} + c$

$= 3^{-7} \frac{x^{-6}}{-6} + c$

? Integrate $5x^{-1/2}$

Ans: $\int 5x^{-1/2} dx = 5 \int x^{-1/2} dx$ $\quad \frac{-1+2}{2}$

$= 5 \frac{x^{-1/2+1}}{-1/2+1} + c$

$= 5 \frac{x^{1/2}}{1/2} + c$

$= 10\sqrt{x} + c$

? $\int (ax^5 + 9/x - 5)\, dx$

$= \int ax^5 dx + \int 9/x\, dx - \int 5\, dx$

$\int 1 = x$

$\int 5 dx$

$= 5 \int 1$

$= 5x$

$= a \int x^5 dx + a \int 1/x\, dx - \int 5\, dx$

$= a \frac{x^6}{6} + a \log x - 5x + c$

? $\int 6 \sin x\, dx$

$= 6 \int \sin x\, dx$

$= 6(-\cos x) + c$

$= -6\cos x + c$

? $\int \frac{x^4 - 8}{x} dx$

$\int \frac{x^4 - 8}{x} dx = \int \frac{x^4}{x} - \frac{8}{x} dx$

$= \int \frac{x^4}{x} dx - \int \frac{8}{x} dx$

$= \int \frac{x^4}{x} dx - 8 \int 1/x\, dx$

$= \cancel{\frac{x^5}{x}} \quad \cancel{\frac{x^5}{}} \quad \cancel{\int \frac{1}{x} \cdot x^4 dx - 8 \int \frac{1}{x} dx}$

$= \cancel{\log x} \frac{x^5}{5} - \cancel{8 \log x + c}$

$= \int x^3 dx - 8 \int \frac{1}{x} dx$

$= \frac{x^4}{4} - 8 \log x + c$

? $\int x(1+x)(1-x)\,dx$

Ans: $\int x(1+x)(1-x)\,dx = \int x(1-x+x-x^2)\,dx$

$= \int (x-x^2+x^2-x^3)\,dx$

$= \int \cancel{x\,dx - \int x^2 dx + \int x^2 dx - \int x^3 dx}$

$= \int x - x^3\,dx$

$= \frac{x^2}{2} - \frac{x^4}{4} + c$

? $\int 13e^x\,dx$

$\int 13e^x\,dx = 13\int e^x\,dx$

$= 13e^x + c$

? $\int \left(5e^x + \frac{3}{x^2}\right) dx$

$\int \left(5e^x + 3/x^2\right) dx = 5\int e^x\,dx + \int 3/x^2\,dx$

$= 5\int e^x dx + 3\int 1/x^2\,dx$

$= 5\int e^x dx + 3\int x^{-2}\,dx$

$= 5e^x + 3\frac{x^{-1}}{-1} + c$

HW ? $\int 2x(x^2+1)\,dx$

$\int 2x(x^2+1)\,dx = \int (2x^3+2x)\,dx$

$= \int 2x^3 dx + \int 2x\,dx$

$= 2\int x^3 dx + 2\int x\,dx$

$= \cancel{2}\frac{x^4}{\cancel{4}} + \cancel{2}\frac{x^2}{\cancel{2}} + c$

$= \frac{x^4}{2} + x^2 + c$

? $\int \left(\frac{4x^2+2+\sqrt{x}}{x^2}\right) dx$

$\int \frac{4x^2+2+\sqrt{x}}{x^2}\,dx = \int \frac{4x^2}{x^2}\,dx + \int \frac{2}{x^2}\,dx + \int \frac{\sqrt{x}}{x^2}\,dx$

$= \int 4\,dx + 2\int \frac{1}{x^2}\,dx + \int \frac{x^{1/2}}{x^2}\,dx$

$= 4x + 2\int x^{-2}\,dx + \int x^{1/2} x^{-2}\,dx$

$= 4x + 2\frac{x^{-1}}{-1} + \int x^{-3/2}\,dx$

$= 4x + \frac{2x^{-1}}{-1} + \frac{x^{-3/2+1}}{-3/2+1} + c$

$= 4x + 2\cdot\frac{x^{-1}}{-1} + \frac{x^{-1/2}}{-1/2} + c$

Integration by Parts

integration of a product of the function u & v

$$\int uv\,dx = u\int v\,dx - \int\left[\frac{d}{dx}(u)\int v\,dx\right]dx$$

1. $\int x\sin x\,dx$

$$\int x\sin x\,dx = \cancel{\sin x}$$
$$= x\int \sin x\,dx - \int\frac{d}{dx}(x)\int \sin x\,dx\,]dx$$
$$= x(-\cos x)\int \cancel{-\int \sin x\,dx}^{\cos x}]dx$$
$$= \cancel{x(\cos x) x} - \cos x$$
$$= \cancel{-x\cos x +}$$
$$= -x\cos x - \int -1\cos x\,dx$$
$$= -x\cos x + \int \cos x\,dx$$
$$= -x\cos x + \sin x + c$$

2. $\int x^2 \log x\,dx$

$$\cancel{\int x^2 \log x\,dx} = \cancel{x\int \log x\,dx}$$
$$\int \log x\, x^2\,dx = \log x\int x^2\,dx - \int\left(\frac{d}{dx}(\log x)\int x^2\,dx\right)dx$$
$$= \log x\frac{x^3}{\cancel{4}} - \int \frac{1}{x}\frac{x^3}{3}\,dx$$
$$= \cancel{\log x\left(\frac{x^3}{3}\right) - \log x\frac{x^3}{3}dx}$$
$$= \log x\left(\frac{x^3}{3}\right) - \int\frac{x^2}{3}\,dx$$
$$= \frac{x^3}{3}\log x - \int\frac{1}{3}\int x^2\,dx$$
$$= \frac{x^3}{3}\log x - \frac{1}{3}\left(\frac{x^3}{3}\right)dx$$
$$= \frac{x^3}{3}\log x - \frac{x^3}{9} + c$$

? $\int \log x\,dx$

$$= \int 1\times\log x\,dx$$
$$= \log x\int 1\,dx - \int\left[\frac{d}{dx}(\log x)\int 1\,dx\right]dx$$
$$= \log x(x) - \int \frac{1}{x}(x))\,dx$$
$$= \cancel{\log x\, x}\ x\log x - \int \frac{x}{x}\,dx$$
$$= x\log x - \int 1\,dx$$
$$= x\log x - x + c$$

? $\int e^x\cos x\,dx$

$$\int e^x\cos x\,dx = e^x\int \cos x\,dx - \int\left(\frac{d}{dx}e^x\int\cos x\,dx\right)dx$$
$$= e^x(\sin x) - \int e^x\sin x\,dx$$
$$= \cancel{e^x\sin x - \int e^x s}$$
$$= e^x\sin x - e^x\int\sin x\,dx - \int\frac{d}{dx}e^x\int\sin x\,dx$$

$$= e^x \sin x - e^x(-\cos x) + \int e^x(-\cos x)\,dx$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx$$

$\int e^x \cos x\,dx \quad = e^x \sin x + e^x \cos x - \int e^x \cos x\,dx$

ie $\int e^x \cos x\,dx + \int e^x \cos x\,dx = e^x \sin x + e^x \cos x$

$$= e^x(\sin x + \cos x)$$

$2\left(\int e^x \cos x\,dx\right) = e^x(\sin x + \cos x)$

$$e^x \cos x\,dx = \frac{e^x(\sin x + \cos x)}{2} + c$$

? $\cancel{\int e^{2x}dx}$

<u>Substitution Method</u>

1. Find $\int e^{mx}dx$

Let $mx = u$

$$\frac{du}{dx} = \frac{d}{dx}(mx)$$

$$= m$$

Therefore, $du = m\,dx$

$$\therefore\ dx = \frac{du}{m}$$

$$= \int e^u \frac{du}{m}$$

$$= \frac{1}{m}\int e^u du$$

$$= \frac{1}{m}e^u + c$$

$$= \frac{1}{m}e^{mx} + c$$

2. Find $\int \sqrt{5x+3}\ dx$

Ans:- $\int \sqrt{5x+3} = \cancel{\int (5x+3)^{1/2}}\,dx$

Let $\sqrt{5x+3} = u$

$$\frac{du}{dx} = \frac{d}{dx}\sqrt{5x+3}$$

$$= \cancel{\frac{d}{dx}(5x+3)^{1/2}}$$

$$\frac{du}{dx} = 5$$

Therefore; $dx = \frac{du}{5}$

$$= \int \sqrt{u}\,\frac{du}{5}$$

$$= \frac{1}{5}\int \sqrt{u}\,du$$

$$= \frac{1}{5}\int u^{1/2}\,du$$

$$= \frac{1}{5} \times \frac{u^{3/2}}{3/2}\ \cancel{du} + c$$

$\frac{1}{2}+\frac{1}{1}$

$\frac{1+2}{2}$

$= \frac{1}{5} \frac{(5x+3)^{3/2}}{3/2} + c$

$= \frac{1}{5}(5x+3)^{3/2} \times \frac{2}{3} + c$

$= \frac{2}{15}(5x+3)^{3/2} + c$

3. $\int \frac{1}{(5x+3)^2} dx$

$\int \frac{1}{(5x+3)^2} dx$

let $u = 5x+3$

$\frac{du}{dx} = \frac{d}{dx}(5x+3)$

$= 5$

$dx = \frac{du}{5}$

$\therefore = \int \frac{1}{u^2} \frac{du}{5}$

$= \frac{1}{5} \int \frac{1}{u^2} du$

$= \frac{1}{5} \int u^{-2} du$

$= \frac{1}{5} \frac{u^{-1}}{-1} + c$

$= \frac{1}{5} \frac{(5x+3)^{-1}}{-1} + c$

$= \frac{-(5x+3)^{-1}}{5} + c$

4. $\int e^{\cos x} \sin x \, dx$

let $\cos x = u$

$\frac{du}{dx} = \frac{d}{dx} \cos x$

$= -\sin x$

$dx = \frac{du}{-\sin x}$

$\therefore = \int e^{u} \cancel{\sin x} \frac{du}{\cancel{-\sin x}}$

$= -\int e^{u} du$

$= -e^{u} + c$

$= -e^{\cos x} + c$

$du = -\sin x$

$dx = \frac{du}{..}$

Definit integal

Eg 1.- $\int x^3 dx = \left[\frac{x^4}{4}\right]_0^1$

$= \frac{1^4}{4} - \frac{0^4}{4}$

$= \frac{1}{4} - 0$

$= 1/4$