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PROBABILITY DISTRIBUTION

Discrete and Continous Random variable

A random variable may be discrete or continous. A random variable is said to be discreate if it assumes only specified values in an interval. When X takes values 1, 2, 3, 4, 5, 6 ... it is a discreate variable.

A random variable is said to be continous if it can assume any value in a given interval. when x ~~takes~~ any value in a given interval ~~x~~, ~~x~~ (a, b) it is a continous variable on that interval.

Continous

Probability function of a continous random variable (Probability Density function)

Let x be a continous random variable. Let the possible range of values of x be ~~to~~ interval (a, b). Consider a small interval within interval (a, b) say (x, x+dx). where, x is any value of X.

Let, probability for X lie lying b/w x and $x+dx$ is $P(x \le X \le x+dx)$. be denoted by $f(x)dx$.

Then $f(x)$ is called probability density function of X.

Properties

1) $f(x) \ge 0$, $\forall x$

2) $\int_{b}^{a} f(x)\,dx = 1$

1- Show that $f(x) = e^{-x}$, $x > 0$ is a probability density function.

Ans: 1) $e^{-x} \ge 0$

2) $\int f(x)\,dx = \int_0^{\infty} e^{-x}\,dx$

$$\boxed{\int e^{mx} = \frac{e^{mx}}{m}}$$

$$\boxed{e^{0} = 1 \quad e^{\infty} = \infty \quad e^{-\infty} = 0}$$

$$= \left[\frac{e^{-x}}{-1}\right]_0^{\infty}$$

$$= \frac{e^{-\infty}}{-1} - \frac{e^{-0}}{-1}$$

$$= \frac{0}{-1} - \frac{1}{-1}$$

$$= 0 - -1$$

$$= +1 = 1$$

2. If Prove that, $f(x) = \frac{12}{37}(10-x)x^2$ is a probability density function for $0 \le x \le 1$.

Ans:- $f(x) = \frac{12}{37}(10-x)x^2$, $0 < x < 1$

$$\int_x f(x)\,dx = \int_0^1 \frac{12}{37}(10-x)\,x^2\,dx$$

$$= \frac{12}{37}\int_0^1 (10-x)\,x^2\,dx$$

$$= \frac{12}{37}\int_0^1 (10x^2 - x^3)\,dx$$

$$= \frac{12}{37}\left[\int_0^1 10x^2\,dx - \int_0^1 x^3\,dx\right]$$

$$= \frac{12}{37}\left[10\left[\frac{x^3}{3}\right]_0^1 - \left[\frac{x^4}{4}\right]_0^1\right]$$

$$= \frac{12}{37}\left\{10\left(\frac{1^3}{3} - \frac{0^3}{3}\right) - \left(\frac{1^4}{4} - \frac{0^4}{4}\right)\right\}$$

$$= \frac{12}{37}\left\{10\left(\frac{1}{3} - 0\right) - \frac{1}{4}\right\}$$

$$= \frac{12}{37}\left(10 \times \frac{1}{3} - \frac{1}{4}\right)$$

$$= \frac{12}{37}\left(\frac{10}{3} - \frac{1}{4}\right)$$

$$= \frac{12}{37}\left(\frac{40-3}{12}\right) =$$

$$= \frac{12}{37} \times \frac{37}{12}$$

$$= 1$$

Discrete probability distribution

① Binominal distribution :

A random variable 'X' is said to follow binominal distribution with parameter n & P. If its probability func's,

$$P(x) = f(x) = \begin{cases} \binom{n}{x} p^x q^{n-x}, & x = 0, 1, 2 \dots, n \\ & 0 < P < 1 \\ 0, \text{ otherwise} \end{cases}$$

$q = 1-p$
$q + P = 1$

Binominal expansion

$$\boxed{(a+b)^n = a^n + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \dots + b^n}$$

$X \sim B(n, P)$ for eg: $X \sim B(10, \frac{1}{4})$

$$f(x) = \binom{10}{x} \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{10-x}, \quad x = 0, 1, 2, 3, 4, 5, 6, \dots 10.$$

Total Probability,

$$\sum_{x=0}^{n} P(x) = P(0) + P(1) + P(2) + P(n)$$

$$= \binom{n}{0} p^0 q^{n-0} + \binom{n}{1} p^1 q^{n-1} + \binom{n}{2} p^2 q^{n-2} + \dots \dots \binom{n}{n} p^n q^{n-n}$$

$$= q^n + \binom{n}{1} p^1 q^{n-1} + \binom{n}{2} p^2 q^{n-2} + p^n$$

$$= (q+p)^n$$

$$= 1^n = 1$$

Where, P = Probability of success

q = Probability of failure

n = no. of trails.

→ In the probability function of the binominal distribution 'X' is the no. of success out of 'n' trials.

- 'P' represent probability for success in a ~~sig~~ single trial.

Binominal distribution can be applied :

1) The random experiment has 2 outcomes, wich can be called "success" & "failure".

2) Probability for success in a single trial remains constant from trial to trial of the experiment.

3) The experiment is repeated finite no. of times.

4) Trials are independent.

1- 4 coins are tossed simultaneously. What is the probability of getting 2 Heads.

Let, n = no. of coins tossed = 4

p = Probability of getting head in a single trial / toss = $\frac{1}{2}$

The distribution follows binomial with $n = 4$ & $p = \frac{1}{2}$. Therefore,

$$P(x) = \binom{n}{x} p^x q^{n-x}$$

$$= \binom{4}{x}\left(\frac{1}{2}\right)^x \left(1-\frac{1}{2}\right)^{4-x}, \quad x = 0, 1, 2, 3, 4$$

$$= \binom{4}{x} \frac{1}{2}^x \frac{1}{2}^{4-x}$$

P(getting two heads) = $P(x = 2)$

$= P(2)$

$= \binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{4-2}$

$^4C_2 = \frac{4\times 3}{1\times 2} = \frac{12}{2} = 6$

$= 6 \cdot \frac{1}{4} \left(\frac{1}{2}\right)^2$

$= 6 \times \frac{1}{4} \times \frac{1}{4}$

$= \frac{6}{16} = \frac{3}{4}$ //

2- Probability that a batsman score a century in a cricket match is $\frac{1}{3}$. Find the probability that out of 5 matches he may score century, if;

1) exactly 2 matches.

2) No matches.

Ans:- Let n = no. of matches = 5

p = Probability of scoring century = $\frac{1}{3}$

The distribution follows binomial with $n = 5$ & $p = \frac{1}{3}$. Therefore,

$$P(x) = \binom{5}{x}\left(\frac{1}{3}\right)^x\left(1-\frac{1}{3}\right)^{5-x}, \quad x = 0, 1, 2, 3, 4, 5.$$

$$= \binom{5}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{5-x}$$

1) P(getting century in 2 matches) = $P(x = 2)$

$= P(2)$

$= \binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^{5-2}$

$= 10 \times \frac{1}{9} \times \left(\frac{2}{3}\right)^3$

$= 10 \times \frac{1}{9} \times \frac{8}{27}$

$= 10 \times \frac{4}{81} = \frac{40}{81}$

$= 10 \times \frac{1}{9} \times \frac{8}{27}$

$= 10 \times \frac{8}{243}$

$= \frac{80}{243}$ //

$\frac{5 \times 4}{1 \times 2} = 10$

2) P(getting century in no matches) = $P(X=0)$

$= P(0)$

$= \binom{5}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^{5-0}$

$= \cancel{5 \times 0}$

$= \cancel{1 \times \frac{1}{3}} \times 1$

$= 1 \times 1 \times \frac{32}{243}$

$= \frac{32}{243}$ //

Mean & Variance of Binomial Distribution

$$\boxed{\begin{array}{l} \text{Mean} = np \\ \text{Variance} = npq \end{array}}$$

1. For a Binomial distribution mean = 4 & variance equal to 12/9. write down the probability distribution ?

Ans:- $P(x) = \binom{n}{x} p^x q^{n-x}$, $x = 0, 1, 2, 3, 4 \ldots n$

given, mean = 4

variance = 12/9

$np = 4$

$npq = \frac{12}{9}$

$4q = \frac{12}{9}$

$q = \frac{\left(\frac{12}{9}\right)}{4}$

$q = \frac{12}{9} \times \frac{1}{4}$

$= \frac{3}{9} = \frac{1}{3}$ //

ie $q = \frac{1}{3}$

$1 - p = \frac{1}{3}$

$1 - \frac{1}{3} = p$

$\frac{2}{3} = p$

$p = \frac{2}{3}$ //

$$\boxed{\begin{array}{l} nC_0 = 1 \\ nC_1 = n \end{array}}$$

since $np = 4$

$n \times \frac{2}{3} = 4$

$n = \frac{4}{(2/3)} = 4 \times \frac{3}{2}$

$= \frac{12}{2} = 6$

$P(x) = \binom{n}{x} p^x q^{n-x}, \quad x = 0, 1, 2, 3, 4, 5, 6$

$P(0) = \binom{6}{0} (2/3)^0 (1/3)^{6-0}$

$= 1 \times 1 \times \frac{1}{729}$

$= 1/729$

$P(1) = \binom{6}{1} (2/3)^1 (1/3)^{6-1}$

$= 6 \times 2/3 \, (1/3)^5$

$= 6 \times 2/3 \times 1/243$

$= 12/729$

$P(2) = \binom{6}{2} (2/3)^2 (1/3)^{6-2}$

$= 15 \times 4/9 \times (1/3)^4$

$= 15 \times 4/9 \times 1/81$

$= \frac{60}{729}$

$P(3) = \binom{6}{3} (2/3)^3 (1/3)^{6-3}$

$= 20 \times \frac{8}{27} \times (1/3)^3$

$= 20 \times 8/27 \times 1/27$

$= 160/729$

$P(4) = \binom{6}{4} (2/3)^4 (1/3)^{6-4}$

$= 15 \times 16/81 \times (1/3)^2$

$= 15 \times 16/81 \times 1/9$

$= 240/729$

$P(5) = \binom{6}{5} \times (2/3)^5 (1/3)^{6-5}$

$= 6 \times \frac{32}{243} \times 1/3$

$= 192/729 = \frac{192}{729}$

$\frac{6 \times 5 \times 4}{1 \times 2 \times 3} = \frac{120}{6}$

$\frac{6 \times 5}{1 \times 2}$

$\frac{6 \times 5 \times 4 \times 3}{1 \times 2 \times 3 \times 4} = \frac{15}{4}$

$\frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \times 4 \times 5 \times 6}$

$\boxed{\begin{array}{l} 6C_0 = 6C_6 \\ 6C_2 = 6C_4 \\ 6C_1 = 6C_5 \end{array}}$

$$P(6) = \binom{6}{6}\left(\frac{2}{3}\right)^{6}\left(\frac{1}{3}\right)^{6-6}$$

$$= 1 \times \frac{64}{729} \times 1$$

$$= \frac{64}{729}$$

$$\boxed{E(x) > V(x)}$$

→ variance of x is less than expectation of x.

2. The mean and variance of a binomial distribution are 2.5 and 1.875 respectively. Obtain the Binomial probability distribution.

Ans:

Given, mean, $np = 2.5$

and variance, $npq = 1.875$,

$npq = 1.875$

$2.5q = 1.875$

$q = \frac{1.875}{2.5} = 0.75$

ie, $q = 0.75$

$1-p = 0.75$

$1 - 0.75 = p$

$0.25 = p$

$p = 0.25$

Since, $np = 2.5$

$n \times 0.25 = 2.5$

$n = \frac{2.5}{.25}$

$= \frac{0.25 \times 10}{0.25}$

$= 10$

Therefore, $P(x) = \binom{n}{x} p^x q^{n-x}$

$x = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$

$$P(x) = \binom{10}{x} 0.25^{x} (0.75)^{10-x}$$

3. If the mean & variance of a binomial distribution 4 and 2 respectively. Find the probability of

1) exactly 2 successes.
2) less than 2 successes
3) more than 6 successes
4) Atleast a success

Ans: Given, mean $np = 4$

& variance $npq = 2$

$npq = 2$

$4q = 2$

$q = \frac{2}{4}$

ie $q = 2/4$

$1-p = 2/4$

$1 - 2/4 = p$

$2/4 = p$

$p = 2/4$

Therefore, $np = 4$

$n \times 2/4 = 4$

$n = \frac{4}{(2/4)} = 4 \times 4/2$

$= 16/2 = 8$

$P(x) = \binom{n}{x} p^x q^{n-x}$

$P(x) = \binom{8}{x}(1/2)^x (1/2)^{8-x}$ $x = 1, 2, 3, 4, 5, 6, 7, 8.$

1) Probability of 2 successes.

$P(2 \text{ successes}) = P(x = 2)$

$= P(2)$

$= \binom{8}{2}(1/2)^2 (1/2)^{8-2}$

$= 28 \times (\frac{1}{2})^2 \times (1/2)^6$

$= 28 \times \frac{1}{4} \times \frac{1}{64}$

$= \frac{28}{256} = \frac{7}{64}$

$\frac{8 \times 7}{1 \times 2}$

$= 10$

256

$7/64$

2) Probability of less than 2 success.

$P(\text{less than 2 success}) = P(x = 0 \text{ or } x = 1)$

$= P(0) + P(1)$

$P(0) = \binom{8}{0}(0.5)^0 (1/2)^8$

$= 1 \times (\frac{1}{2})^8$

$= (1/2)^8 = 1/256$

$P(1) = \binom{8}{1}(1/2)^1 \times (1/2)^{8-1}$

$= \binom{8}{1}(1/2)^1 \times (1/2)^7$

$= 8 \times (0.5)^8 = 8/256$

$P(<2) = (1/2)^8 + 8 \times (1/2)^8 = 9/256$

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3) More than 6 successes.

P(more than 6 success) = $P(x \geq 7$ or $P(x \geq 8)$

$= P(x=7) + P(x=8)$

$P(x=7) = \binom{8}{7}(1/2)^7 (1/2)^{8-7}$

$= \binom{8}{7}(1/2)^7 (1/2)^1$

$= 8(1/2)^8$

$P(x \geq 8) = \binom{8}{8}(1/2)^8 (1/2)^{8-8}$

$= 1 \times (1/2)^8$

$P(x > 6) = 8 \times (1/2)^8 + (1/2)^8$

4) At least 2 successes) = $P(x > 2)$

$= 1 - (P(x=0$ or $x=1))$

$= 1 - (P(x=0) + x=1))$

$= 1 - ((1/2)^8 + 8(1/2)^8)$

$= 1 - \left(\frac{1}{256} + 8 \times \frac{1}{256}\right)$

$= 1 - \left(\frac{1}{256} + \frac{8}{256}\right)$

$= 1 - \frac{9}{256}$

$= \frac{256-9}{256}$

$= \frac{247}{256}$

4. In 256 sets of 10 tosses of a coin, in how many cases one may expects 8 heads and 4 tails?

$N \cdot P(x=8) = ?$

$N = 256$

$n = 12$

$P = 1/2$ (coins probability is head or tail)

$P(\text{getting 8 heads}) = \binom{12}{8}(1/2)^8 (1/2)^{12-8}$

$= \binom{12}{8}(1/2)^8 (1/2)^4$

$= 495 \times (1/2)^{12}$

$= \frac{495}{2^{12}}$

$= \frac{495}{64 \times 64} = \frac{495}{4096}$

$= 0.1208.$

$\binom{12}{8} = \frac{12 \cdot 12 \times 11 \times}{1 \times 2 \times 3 \times}$

Expected number of sets of 12 tosses giving 8 heads equal to $256 \times P(x=8)$

$$256 \times 0.1208 = \underline{\underline{30.93}}$$

$\rightarrow$ $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3}$ $\quad \beta_2 = \dfrac{\mu_4}{\mu_2^2}$

$\gamma_1 = \sqrt{\beta_1}$ $\quad \therefore \gamma_2 = \beta_2 - 3$

Central moments

$\mu_1 = 0$

$\mu_2 = E(x - Ex)^2$

$= V(x) = npq$

$\mu_3 = E(x - Ex)^3$

Raw moments

$\mu_1' = Ex = np$

$\mu_2' = Ex^2$

$\mu_3' = Ex^3$

$\beta_1 = \mu_3^2$

Skewness and Kurtosis in Binomial Distribution

1- Skewness

$*$ $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3} = \dfrac{(q-p)^2}{npq}$

$\mu_3 = npq(q-p)$

$\mu_2 = npq$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}$$

$$= \frac{[npq(q-p)]^2}{(npq)^3}$$

$$= \frac{n^2p^2q^2(q-p)^2}{n^3p^3q^3} = \frac{(q-p)^2}{npq}$$

$\sqrt{x^2} = x$

$$\gamma_1 = \sqrt{\beta_1} = \sqrt{\frac{(q-p)^2}{npq}} = \frac{(q-p)}{\sqrt{npq}}$$

A binomial distribution is positively skewed if $q > p$, symmetric if $q = p$ and negatively skewed if $q < p$.

$$\frac{q-p}{\sqrt{npq}} = 0 \Rightarrow q = p$$

$\rightarrow$ $\gamma_1 = 0$ distribution is symmetric $q = p$, $p = 1/2$

$\rightarrow$ +vely skewed $\gamma_1 > 0$, $q > p$, $q - p > 0$.

$\rightarrow$ -vely skewed $\gamma_1 < 0$, $q < p$, $q - p < 0$

2- **Kurtosis**

$$\beta_2 = 3 + \frac{1-6pq}{npq}$$

$$\gamma_2 = \beta_2 - 3 = \frac{1-6pq}{npq}$$

→ A distribution is lepto kurtic if $\gamma_2 > 0$. and meso kurtic if $\gamma_2 = 0$ and platy kurtic if $\gamma_2 < 0$.

→ A binomial distribution is lepto kurtic if $pq > 1/6$, meso kurtic if $pq = 1/6$ and platy kurtic if $pq < 1/6$

→ $\gamma_2 = \frac{1-6pq}{npq} = 0$

$1 - 6pq = 0 \times npq$

$1 - 6pq = 0$

$1 = 6pq$

$\underline{1/6 = pq}$ ⇒ meso kurtic binomial distribution

Fitting of Binomial Distribution

By fitting a binomial distribution we mean determined the expected or theoretical binomial frequencies against the given observed frequencies. The theoretical frequencies are obtained by multiplying the binomial probabilities by the total frequency.

If $f(x)$ denotes the binomial frequency function it is denoted by,

$$\boxed{\begin{aligned} f(x) &= N\,P(x) \\ &= N\binom{n}{x} p^x q^{n-x} \\ & \quad x = 0, 1, 2 \ldots n \end{aligned}}$$

To calculate the binomial probabilities we require n & p. n can be determined from the values of x in the data. If p is not available in the data, find the mean of frequency distribution and equate it to np, mean of the binomial distribution.

1- If the litters of 4 ~~male~~ mice. The no. of litters with contained zero, 1, 2, 3, 4 ~~total~~ females were noted. The figures are given the table below:-

No. of females mice	0	1	2	3	4	total
No. of litters	8	32	34	24	5	103

If chance of obtaining female in a single trial assumed constant, estimate this constant of unknown probability. Find also the expected frequencies?

Ans:
Mean $= \bar{x} = \frac{\sum x_i f_i}{\sum f}$.

x	observed frequency (f)	xf	$P(x)$	$N.P(x)$ $103\,P(x)$
0	8	0	0.0813	8.375
1	32	32	0.283	29.149
2	34	68	0.371	38.123
3	24	72	0.216	22.248
4	5	20	0.0471	4.857
		192		103

$\frac{\sum x_i f_i}{N} = n$

Mean $= \bar{x} = \frac{192}{103}$

$= 1.864$

Here $n = 4$.

Therefore $np = 1.864$

ie $p = \frac{1.864}{4}$

$p = 0.466$

Therefore, $q = 1 - p$

$= 1 - 0.466 = 0.534$

$P(x) = \binom{n}{x} p^x q^{n-x}$

$P(x) = \binom{4}{x}(0.466)^x(0.534)^{n-x}$, $x = 0, 1, 2, 3, 4$

$P(0) = \binom{4}{0}(0.466)^0(0.534)^{4-0}$

$= 1 \times 1 \times (0.534)^4$

$= \underline{\underline{0.0813}}$

$P(1) = \binom{4}{1}(0.466)^1(0.534)^{4-1}$

$= 4 \times 0.466 \times (0.534)^3$

$= 4 \times 0.466 \times 0.152$

$= \underline{\underline{0.283}}$

$P(2) = \binom{4}{2}(0.466)^2(0.534)^{4-2}$

$= 6 \times 0.217 \times (0.534)^2$

$= 6 \times 0.217 \times 0.285$

$= \underline{\underline{0.371}}$

$P(3) = \binom{4}{3}(0.466)^3(0.534)^{4-3}$

$= 4 \times 0.101 \times 0.534$

$= \underline{\underline{0.216}}$

$$P(4) = \binom{4}{4}(0.466)^4(0.534)^{4-4}$$

$= 1 \times 0.0471 \times 1$

$= 0.0471$

Thus, Expected frequency:

$0.813 \times 103 = 8.375 \quad = 9$

$0.283 \times 103 = 29.149 \quad = 29$

$0.371 \times 103 = 38.123 \quad = 38$

$0.216 \times 103 = 22.248 \quad = 22$

$0.0471 \times 103 = 4.857 \quad = 5$

$\underline{\underline{103.}}$

2. The screws produced by certain machines were checked by examines sample. The following table shows the distribution of 128 sample according to the no. of defective items. they contains

No. of defect	0	1	2	3	4	5	6	7
No. of Samples	7	6	19	35	30	23	7	1

Fit a binomial distribution to find the mean and variance of the distribution?

$$\text{Mean} = \bar{x} = \frac{\Sigma x_i f_i}{\Sigma f}$$

x	observed frequency(f)	xf	$P(x)$	$N.P(x)$
0	7	0	0.00981	1.2533
1	6	6	0.064	8.192
2	19	38	0.1805	23.104
3	35	105	0.2813	36.0064
4	30	120	0.261	33.408
5	23	115	0.147	18.816
6	7	42	0.045	5.76
7	1	7	0.00613	0.7846
	128	433		

$$\text{Mean } \bar{x} = \frac{433}{128}$$

$= \cancel{3.6083}\ \cancel{3.82}\ \underline{3.382}$

Here $n = 7$

Therefore $np_1 = \cancel{3.6083}\ 3.382$

ie, $p = \frac{\cancel{3.6083}}{7}\ \frac{3.382}{7}$

$= \cancel{0.515}$

$= 0.4831$

Therefore $q = 1-p$

$= 1 - 0.483$

$= 0.517$

$P(x) = \binom{n}{x} p^x q^{n-x}$, $x = 0,1,2,3,4,5,7$

$P(x) = \binom{7}{x}(0.483)^x (0.517)^{n-x}$

$P(0) = \binom{7}{0}(0.483)^0 (0.517)^{7-0}$

$= 1 \times 1 \times 0.00987$

$= 0.00987$

$P(1) = \binom{7}{1}(0.483)^1 (0.517)^{7-1}$

$= 7 \times 0.483 \times 0.01909$

$= 0.064$

$\frac{7\times 6}{1\times 2} = \frac{42}{2} = 21$

$\frac{7\times6\times5\times4}{1\times2\times3\times4}$ 6

$P(2) = \binom{7}{2}(0.483)^2 (0.517)^{7-2}$

$= 21 \times 0.233 \times 0.0369$

$= 0.1805$

$P(3) = \binom{7}{3}(0.483)^3 (0.517)^{7-3}$

$= 35 \times 0.1126 \times 0.0714$

$= 0.2813$

$P(4) = \binom{7}{4}(0.483)^4 (0.517)^{7-4}$

$= 35 \times 0.054 \times 0.138$

$= 0.261$

$P(5) = \binom{7}{5}(0.483)^5 (0.517)^{7-5}$

$= 21 \times 0.0262 \times 0.067$

$= 0.147$

$P(6) = \binom{7}{6}(0.483)^6 (0.517)^{7-6}$

$= 7 \times 0.0126 \times 0.517$

$= 0.045$

$P(7) = \binom{7}{7}(0.483)^7 (0.517)^{7-7}$

$= 1 \times 0.006131 \times 1$

$= 0.00613$

$npq = 3.382 \times 0.~~483~~ 0.517$

variance $= 1.748$

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Thus, expected frequency,

$0.00987 \times 128 = 1.2633 = 1$

$0.064 \times 128 = 8.192 = 8$

$0.1805 \times 128 = 23.104 = 23$

$0.2813 \times 128 = 36.0064 = 36$

$0.261 \times 128 = 33.408 = 33$

$0.147 \times 128 = 18.816 = 19$

$0.045 \times 128 = 5.76 = 6$

$0.00613 \times 128 = 0.7846 = 1$

HW

The following data show the number seeds germinating out of 5 in damp filter for 80 sets of seeds. Fit a binomial distribution of this data and find the expected frequencies.

x :	0	1	2	3	4	5
f :	6	20	28	12	8	6

Ans. Since 'p' is not given, find mean of the given data.

x	f	fx	P(x)	N. P(x)
0	6	0	0.0575	4.6 = 5
1	20	20	0.2216	17.7306 = 18
2	28	56	0.3408	27.27 = 27
3	12	36	0.2627	21.017 = 21
4	8	32	0.1011	8.088 = 8
5	6	30	0.0155	1.246 = 1
	80	174		80

$$\text{Mean} = \frac{\Sigma x_i f_i}{\Sigma f}$$

$$\bar{x} = \frac{174}{80} = 2.175$$

ie, $np = 2.175$

$n = 5$

$\therefore P = \frac{2.175}{5} = 0.435$

and $q = 1 - P = 0.565$

$\therefore$ Binomial distribution is,

$$P(x) = \binom{n}{x} p^x q^{n-x}$$

$$P(x) = \binom{5}{x} (0.435)^x (0.565)^{5-x}$$

for $x = 0, 1, 2, 3, 4, 5$

where $N = 80$

$$P(0) = \binom{5}{0}(0.435)^{0}(0.565)^{5-0}$$

$$= 1 \times 1 \times 0.0575$$

$$= 0.0575$$

Expected frequency $= 0.0575 \times 80$

$= 4.6 = 5$

$$P(1) = \binom{5}{1} \times (0.435)^{1} \times (0.565)^{5-1}$$

$$= 5 \times 0.435 \times 0.1019$$

$$= 0.2216$$

Expected frequency $= 0.2216 \times 80$

$= 17.7306 = 18$

$$P(2) = \binom{5}{2} \times (0.435)^{2} \times (0.565)^{5-2}$$

$$= 10 \times 0.189 \times 0.1803$$

$$= 0.3408$$

Expected frequency $= 0.3408 \times 80$

$= 27.27 = 27$

$$P(3) = \binom{5}{3}(0.435)^{3} \times (0.565)^{5-3}$$

$$= 10 \times 0.0823 \times 0.319$$

$$= 0.2627$$

Expected frequency $= 0.2627 \times 80$

$= 21.017 = 21$

$$P(4) = \binom{5}{4}(0.435)^{4} \times (0.565)^{5-4}$$

$$= 5 \times 0.0358 \times 0.565$$

$$= 0.1011$$

Expected frequency $= 0.1011 \times 80 =$

$= 8.088 = 8$

$$P(5) = \binom{5}{5} \times (0.435)^{5} \times (0.565)^{5-5}$$

$$= 1 \times 0.0155 \times 1$$

$$= 0.0155$$

Expected frequency $= 0.0155 \times 80$

$= 1.246 = 1.$

Poisson Distribution

A discrete random variable X is said to follow poisson distribution if its probability density function is,

$$P(x) = \begin{cases} \dfrac{e^{-\lambda} \lambda^{x}}{x!} , & x = 0, 1, 2 \ldots \quad (\lambda > 0) \\ 0, & \text{otherwise} \end{cases}$$

$x \sim P(\lambda)$

Poisson distribution as a limiting case of binomial distribution. Poisson distribution may be obtain as a limiting case of binomial distribution under the following conditions :-

1 - No. of trails is very large ($n \to \infty$).

2 - The probability of success for each trial is very small. ($P \to 0$).

3 - np is finite (says λ).

→ Practical situation use poisson distribution can be drawn. ~~to coun~~

1- To count the no. of defects per unit of a manufacturing product.

2- To cont the no. of accidents taking place in a day on a sity.

3- To count the no. of casnalities due to rare deseases.

Q. If 3% of electric bulbes manufactured by a company are defective. Find the probability that in a sample of 100 bulbes, exactly 5 bulbes are defective.

Ans: Here, $n = 100$

$p \to$ probability of defective bulb $= 3/100 = 0.03$

Since, p is small and n is large (>30).

We may assume poisson distribution.

Therefore $\lambda = np$

$= 100 \times 0.03$

$= 3$

$$P(x) = \frac{e^{-3}\, 3^{x}}{x!}$$

$$P(5 \text{ defective bulb}) = \frac{e^{-3}\, 3^{5}}{5!}$$

$$= \frac{0.0498 \times 243}{120}$$

$$= \frac{12.1014}{120}$$

$$= \underline{\underline{0.100845}}$$

Mean and Variance

$E\{x\} = \lambda$

$v(x) = \lambda$

$S.D = \sqrt{\lambda}$

Q. It is known from past experience that in a certain plant there are on the average 4 industrial accidents per month. Find the probability that in a given year will be less than 3 accidents. Assume poisson distribution.?

Here, $\lambda = 4$

$np = 4$

Then

we may assume poisson distribution

$$P(x) = \frac{e^{-4}\, 4^{x}}{x!}$$

$P(\text{Less than 3 accidents}) = P(x=0) + P(x=1) + P(x=2)$

$$P(0) = \frac{e^{-4}\, 4^{0}}{0!} = \frac{0.01832 \times 1}{0!}$$

$$= \underline{\underline{0.01832}}$$

$$P(1) = \frac{e^{-4}\, 4^{1}}{1!} = \frac{0.01832 \times 4}{1}$$

$$= \underline{\underline{0.07328}}$$

$$P(2) = \frac{e^{-4}\, 4^{2}}{2!} = \frac{0.01832 \times 16}{2}$$

$$= \frac{0.29312}{2} = \underline{\underline{0.14656}}$$

$P(\text{Less than 3 accident}) = 0.01832 + 0.07328 + 0.14656$
$= \underline{0.23816}$

$e^{-4} = 0.01832$

Q. In a town 10 accidents took place in a span of 100 days. Assume that the number of accidents follows ~~poi~~ Poisson, find the probability that there will be 3 or more accident in a day.

Ans: Average no. of trials $= \frac{10}{100} = 0.1$

Therefore, $\lambda = 0.1$

$$P(x) = \frac{e^{-0.1} \times 0.1^{x}}{x!}$$

$$P(\text{no. of accidents} \geq 3) = \cancel{P(x=} \ 1 - P(X \leq 2)$$

$$= 1 - \left(P(0) + P(1) + P(2)\right)$$

$$P(0) = \frac{e^{-0.1} \times 0.1^{0}}{0!}$$

$$= \frac{0.9048 \times 1}{1}$$

$$= 0.9048$$

$$P(1) = \frac{e^{-0.1} \times 0.1^{1}}{1!}$$

$$= \frac{0.9048 \times 0.1}{1}$$

$$= 0.09048$$

$$P(2) = \frac{e^{-0.1} \times (0.1)^2}{2!}$$

$$= \frac{0.9048 \times 0.01}{2}$$

$$= \frac{0.009048}{2}$$

$$= 0.004524$$

$$P(\text{no. of accident} \geq 3) = 1 - \left[0.9048 + 0.09048 + 0.004524\right]$$

$$= 1 - \cancel{1.04052} \ 0.9998$$

$$= 0.00028$$

Fitting of Poisson Distribution

If ~~poiss~~ we want to fit a poisson distribution to a frequency distribution, we have calculate the mean of given distribution and take it as λ. Once λ is known the poisson distribution is obtained by putting that value of λ in $\frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$. The expected (theoretical) frequencies can be obtained, by putting $x = 0, 1, 2, \ldots n$ in $N \times \frac{e^{-\lambda} \lambda^x}{x!}$

Q. Fit a Poisson distribution to the following data & calculate the theoretical frequencies.

x :	0	1	2	3	4
f :	123	59	14	3	1

Ans.

x	f	xf	P(x)	N(Px)
0	123	0	0.6065	121
1	59	59	0.30325	61
2	14	28	0.0758	15
3	3	9	0.0126	3
4	1	4	0.001579	0
	200	100		200

N = 200

$\therefore$ Mean $\bar{x} = \frac{\Sigma xf}{\Sigma f} = \frac{100}{200} = 0.5$

Therefore $\lambda = 0.5$

$P(x) = \frac{e^{-0.5} \times 0.5^x}{x!}$, $x = 0, 1, 2, 3, 4$

$P(0) = \frac{e^{-0.5} \times 0.5^0}{0!}$ $0! = 1$

$= \frac{0.6065 \times 1}{1}$

$= 0.6065$

$P(1) = \frac{e^{-0.5} \times 0.5^1}{1!}$ $1! = 1$

$= \frac{0.6065 \times 0.5}{1}$

$= 0.30325$

$P(2) = \frac{e^{-0.5} \times 0.5^2}{2!}$ $2! = 2$

$= \frac{0.6065 \times 0.25}{2}$

$= \frac{0.1516}{2} = 0.0758$

$P(3) = \frac{e^{-0.5} \times 0.5^3}{3!}$ $3! = 6$

$= \frac{0.6065 \times 0.125}{6}$

$= \frac{0.0758}{6} = 0.0126$

$P(4) = \frac{e^{-0.5} \times 0.5^4}{4!}$ $4! = 24$

$= \frac{0.6065 \times 0.0625}{24}$

21

$$= \frac{0.0379}{24}$$

$$= 0.001579$$

$N \times P(0) = 200 \times 0.6065 = 121.3 \approx 121$

$N \times P(1) = 200 \times 0.30325 = 60.65 \approx 61$

$N \times P(2) = 200 \times 0.0758 = 15.16 \approx 15$

$N \times P(3) = 200 \times 0.0126 = 2.52 \approx 3$

$N \times P(4) = 200 \times 0.001579 = 0.3158 \approx 0$

$\overline{200}$

Q. A systematic sample of 100 pages was taken from the dictionary and the observed frequency distribution of foreign words per page was found to be ~~was~~ follows:

No. of foreign words per page (x)	0	1	2	3	4	5	6
Frequency	48	27	12	7	4	1	1

Calculate the expected frequencies using poisson distribution. Also compute the variance of fitting distribution?

Ans.

X	f	xf	P(x)	N(P(x))
0	48	0	0.3716	37.16 = 37
1	27	27	0.3678	36.78 = 37
2	12	24	0.1821	18.21 = 18
3	7	21	0.0600	6.008 = 6
4	4	16	0.0148	1.487 = 2
5	1	5	0.0029	0.29 = 0
6	1	6	0.0004	0.48 = 0
	100	99		100

$$\text{Mean} = \frac{\Sigma xf}{\Sigma f} = \frac{99}{100}$$

$$= 0.99$$

ie. $\lambda = 0.99$

$$P(x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= \frac{e^{-0.99}\, 0.99^x}{x!}, \quad x = 0, 1, 2, 3, 4, 5, 6$$

$$P(0) = \frac{e^{-0.99} \times 0.99^0}{0!}$$

$$= \frac{0.3716 \times 1}{1}$$

$$= 0.3716$$

$e^{-0.99} = e^{-.90} * e^{-.09}$

$e^{-.90} = 0.04066 *$

$e^{-0.09} = 0.9139$

$e^{-0.99} = \overline{0.3716}$

$$P(1) = \frac{e^{-0.99} \times 0.99^{1}}{1!}$$

$$= \frac{0.3716 \times 0.99}{1}$$

$$= 0.3678$$

$$P(2) = \frac{0.3716 \times 0.99^{2}}{2!}$$

$$= \frac{0.3716 \times 0.9801}{2}$$

$$= \frac{0.3642}{2} = 0.1821$$

$$P(3) = \frac{0.3716 \times 0.99^{3}}{3!}$$

$$= \frac{0.3716 \times 0.9702}{6}$$

$$= \frac{0.3605}{6} = 0.06008$$

$$P(4) = \frac{0.3716 \times 0.99^{4}}{4!}$$

$$= \frac{0.3716 \times 0.9605}{24}$$

$$= \frac{0.3569}{24} = 0.01487$$

$$P(5) = \frac{0.3716 \times 0.99^{5}}{5!}$$

$$= \frac{0.3716 \times 0.9509}{120} \qquad \boxed{5! = 120}$$

$$= \frac{0.3533}{120} = 0.0029$$

$$P(6) = \frac{0.3716 \times 0.99^{6}}{6!} =$$

$$= \frac{0.3716 \times 0.9414}{720}$$

$$= \frac{0.3498}{720} = 0.0048$$

$N \times P(0) = 100 \times 0.3716 = 37.16 \approx 37$

$N \times P(1) = 100 \times 0.3678 = 36.78 \approx 37$

$N \times P(2) = 100 \times 0.1821 = 18.21 \approx 18$

$N \times P(3) = 100 \times 0.06008 = 6.008 \approx 6$

$N \times P(4) = 100 \times 0.01487 = 1.487 \approx 2$

$N \times P(5) = 100 \times 0.0029 = 0.29 \approx 0$

$N \times P(6) = 100 \times 0.0048 = 0.48 \approx 0$

$\overline{100}$

Expected frequencies:

x	0	1	2	3	4	5	6
Ex	37	37	18	6	2	0	0

Q. Following mistakes per page where observed in a book

No. of mistakes : 0 1 2 3 4

No. of pages : 211 90 19 5 0

Fit a Poisson distribution to the above table?

Ans:-

x	f	xf	$P(x)$	$NP(x)$
0	211	0	0.64404	209.313 = 209
1	90	90	0.2833	920.0725 = 92
2	19	38	0.06234	20.2605 = 21
3	5	15	0.00914	2.9703 = 3
4	0	0	0.001	0.325 = 0
	325	143		325

Mean $= \frac{\Sigma xf}{\Sigma f}$

$= \frac{143}{325} = \cancel{0.409}\ 0.44$

ie; $\lambda = \cancel{0.409}\ 0.44$

$P(x) = \frac{e^{-\lambda} \lambda^x}{x!}$

$= \frac{e^{-0.44}\, 0.44^x}{x!}$

$P(0) = \frac{e^{-0.44} \times 0.44^0}{0!}$

$= \frac{0.64403 \times 1}{1}$

$= 0.64404$

$P(1) = \frac{e^{-0.44} \times 0.44^1}{1!}$

$= \frac{0.64404 \times 0.44}{1}$

$= 0.2833$

$P(2) = \frac{e^{-0.44} \times 0.44^2}{2!}$

$= \frac{0.64404 \times 0.1936}{2}$

$= \frac{0.1246}{2} = 0.06234$

$P(3) = \cancel{0.6}\ \frac{e^{-0.44} \times 0.44^3}{3!}$

$= \frac{0.64404 \times 0.0852}{6}$

$= \frac{0.05486}{6} = 0.00914$ //

$P(4) = \frac{e^{-0.44} \times 0.44^4}{4!}$

$= \frac{0.64404 \times 0.0374}{24}$

$= \frac{0.02413}{24} = 0.001$ //

~~Pd~~

$N \times P(0) = 325 \times 0.64404$

$= 209.313 = 209$ //

$N \times P(1) = 325 \times 0.2833 = 92.0725 = 92$ //

$N \times P(2) = 325 \times 0.06234 = 20.2605 = 20$ //

$N \times P(3) = 325 \times 0.00914 = 2.9703 = 3$ //

$N \times P(4) = 325 \times 0.001 = 0.325 = 0$ //

Q. In a random variable expose poisson distribution such that, ~~P(x) = 1~~ = $P(x=1) = P(x=2)$.

Find $P(x=0)$.

Ans:- Let, λ is the parameter,

Then, $P(x) = \frac{e^{-\lambda} \lambda^x}{x!}$

Given, $P(x=1) = P(x=2)$

$\therefore \frac{e^{-\lambda} \lambda^1}{1!} = \frac{e^{-\lambda} \lambda^2}{2!}$

$\frac{\lambda^1}{1!} = \frac{\lambda^2}{2!}$

$\frac{\lambda}{1} = \frac{\lambda^2}{2}$

$\frac{1}{1} = \frac{\lambda}{2}$

i.e., $\lambda = 2$

$P(0) = \frac{e^{-2} \times 2^0}{0!} = \frac{0.1353 \times 1}{1}$

$= 0.1353$

$\frac{\lambda^2}{2} \times \frac{1}{\lambda}$

$= \frac{\lambda}{2} = 1$

$\lambda = 2$

$\frac{x}{3} = 5$

$x = 3 \times 5$

Moments, skewness & kurtosis

$X \sim P(\lambda)$.

$\mu_2 = \text{variance} = \lambda$.

$\mu_3 = \lambda$

$\mu_4 = 3\lambda^2 + \lambda$.

Skewness

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\lambda^2}{\lambda^3} = \frac{1}{\lambda}$$

$$\boxed{\gamma_1 = \sqrt{\beta_1} = \frac{1}{\sqrt{\lambda}}}$$

Note: since $\lambda > 0$, Poisson distribution is a positively skewed distribution.

Kurtosis

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\lambda^2 + \lambda}{\lambda^2} = 3 + \frac{1}{\lambda}.$$

$$\gamma_2 = \beta_2 - 3 = 3 + \frac{1}{\lambda} - 3$$

$$= \frac{1}{\lambda}$$

Note: since $\lambda > 0$, Poisson distribution is leptokurtic.