Module 2

Applications of differencial Calculas

$(a+b)^2 = a^2+2ab+$

Slope is tangent; tangent = degree

$\sin\theta = \frac{b}{c}$

$\cos\theta = \frac{a}{c}$

$\tan\theta = \frac{b}{a}$

Suppose, y is a qty, that depends on another qty x. Then y is function of x. and the $y = f(x)$. If x changes from x_1 to x_2, the change in x is

$\Delta x = x_2 - x_1$ and the corresponding change in y is. $\Delta y = f(x_2) - f(x_1)$

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$ is called the

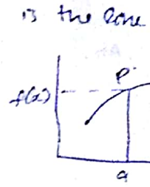
average rate of change of y. with respect to x over the interval x_1, x_2. ~~Let~~

Letting, Δx approaches to zero. The limit of this average rate of change is called instantaneous rate of change of y with respect to x at $x = x_1$.

$$\text{Rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\lim_{x_2 \to x_1} \frac{f(x_2) - (f(x_1))}{x_2 - x_1}$$

The tangent line to the curve $y = f(x)$ at the point $P(a, f(x))$ is the line through the point P with slope,



$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Derivative

Derivative of a function f at a number a.

denoted by, $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

or $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

1) Find the differencial of x^2.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f(x) = x^2$

$f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$

$f(x+h) - f(x) = x^2 + 2xh + h^2 - x^2$

$= 2xh + h^2$

$$\frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2}{h}$$

$= \cancel{h(2}\; 2x + h$

$$\lim_{h \to 0} 2x + h = 2x + 0 = 2x$$

? $f(x) = x^2 - 8x + 9$

Ans.- $\lim_{h \to 0} \frac{f(x+h) - f(x)}{f(x)}$

$f(x) = x^2 - 8x + 9$

$f(x+h) = (x+h)^2 - 8(x+h) + 9$

$= x^2 + 2xh + h^2 - 8x + 8h + 9$

$f(x+h) - f(x) = x^2 \cancel{- 8x + 9}$ $8x - 8h + 9 - (x^2 - 8x + 9)$

$= x^2 + 2xh + h^2 - \cancel{x^2 - 8x + 8h + 9}$

$= 2xh + h^2 - \cancel{8x} + \cancel{9} - 8h$

$\frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2 - 8h \cancel{- 8x + 8h + 9}}{h}$

$2x + h - \cancel{8x} + 8 \cancel{+ 9h}$

$= 2x + h - 8$

$\lim_{h \to 0} 2x + h - \cancel{8x} + 8 - 9h = 2x - 8$

Derivatives

1- $\boxed{\frac{d}{dx} x^n = n x^{(n-1)}}$ eg: $\frac{d}{dx} x^2 = 2x^{2-1} = 2x$

eg: $\frac{d}{dx} 4x^3 = 4 \frac{d}{dx} x^3$

$= 4 \times 3 x^{3-1} = 4 \times 3x^2$

$= 12x^2$

$n = \sqrt{x} = x^{1/2}$

$\frac{d}{dx} \sqrt{x} = \frac{d}{dx} x^{1/2} = \frac{1}{2} x^{1/2 - 1} = \frac{1}{2} x^{-1/2}$

$= \frac{1}{2} \times \frac{1}{x^{1/2}}$

$= \frac{1}{2} \times \frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}$

$\boxed{\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}}$

eg:- $\frac{1}{x} = x^{-1}$

$\frac{d}{dx} \frac{1}{x} = \frac{d}{dx} x^{-1} = -1 x^{-1-1} = -1 x^{-2}$

$\boxed{\frac{1}{x} = -1 \times \frac{1}{x^2} = \frac{-1}{x^2}}$

→ $x = \frac{d}{dx}(x) = x^1 = 1x^{1-1} = x^0 = 1$

→ $\frac{d}{dx} x^2 = 2x$

→ $\frac{d}{dx} \frac{1}{x^2} = -2x^{-2-1} = -2x^{-3} = \frac{-2}{x^3}$

→ $\boxed{\frac{d}{dx}(\sin x) = \cos x}$

x^{1}

$\frac{1}{x}$

→ $\frac{d}{dx} \cos x = -\sin x$

→ $\frac{d}{dx} \log x = \frac{1}{x}$

→ $\frac{d}{dx} e^x = e^x$

→ $\frac{d}{dx} 2^x = 2^x \log 2$ → $\frac{d}{dx} a^x = a^x \log a$

? $\frac{d}{dx}$ → $\frac{d}{dx} \tan x = \sec^2 x$

Product Rule.

$$\boxed{\frac{d}{dx} (f(x))(g(x)) = f(x) \frac{d}{dx} g(x) + g(x) \frac{d}{dx} f(x)}$$

? $x^2 \sin x$

$$\frac{d}{dx} x^2 \sin x = x^2 \frac{d}{dx} \sin x + \sin x \frac{d}{dx} x^2$$

$$= x^2(\cos x) + \sin x(2x)$$

$$= x^2 \cos x + 2x \sin x$$

? $e^x \cos x$

$$\frac{d}{dx} e^x \cos x = e^x \frac{d}{dx} \cos x + \cos x \frac{d}{dx} e^x$$

$$= e^x(-\sin x) + \cos x\, e^x$$

$$= e^x(-\sin x) + e^x \cos x$$

$$= -\sin x\, e^x + e^x \cos x$$

$$= e^x(\cos x - \sin x)$$

Tangent The tangent line to the curve $y = f(x)$ at the point

Division Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \frac{d}{dx} f(x) - f(x) \frac{d}{dx}(g x)}{(g x)^2}$$

? $\frac{x+3}{x^2}$

$$\frac{d}{dx}\left(\frac{x+3}{x^2}\right) = \frac{x^2 \frac{d}{dx}(x+3) - x+3 \frac{d}{dx} x^2}{(x^2)^2}$$

$$= \frac{x^2(1+0) - (x+3)(2x)}{x^4}$$

$$= \frac{x^2 - 2x^2 + 6x}{x^4}$$

$$= \frac{-x^2 - 6x}{x^4} = \frac{-x^2}{x^4} - \frac{6x}{x^4}$$

$$= -\frac{1}{x^2} - \frac{6}{x^3}$$

7. $x^3 \log x$

Ans. $\frac{d}{dx}(x^3\log x) = x^3\frac{d}{dx}\log x + \log x\frac{d}{dx}x^3$

$= x^3\left(\frac{1}{x}\right) + \log x\,(3x^2)$

$= \frac{x^3}{x} + \log x\,(3x^2)$

$= x^2 + \log x\,(3x^2)$

Derivative of function of function

Let $f(x)$ and $g(x)$ are two functions

$\frac{d}{dx}f(g(x)) = f'(g(x))\,\frac{d}{dx}g(x)$

Eg. $(x+2)^3$

$\frac{d}{dx}(x+2)^3 = 3(x+2)^{3-1} \times \frac{d}{dx}(x+2)$

$= 3(x+2)^2 \times 1 + 0$

$= 3(x+2)^2 \times 1$

$= 3(x+2)^2$

2. $(2x+3)^2$

Ans. $\frac{d}{dx}(2x+3)^2 = 2(2x+3)^{2-1} \times \frac{d}{dx}(2x+3)$

$= 2(2x+3) \times 2+0$

$= 4(2x+3)$

$= 8x+12$

3. $(x^3+2x^2-3)^4$

Ans. $\frac{d}{dx}(x^3+2x^2-3)^4 = 4(x^3+2x^2-3)^3 \times \frac{d}{dx}(x^3+2x^2-3)$

$= 4(x^3+2x^2-3)^3 \times 3x^2+4x$

4. $\sin(\cos x)$

$\frac{d}{dx}(\sin(\cos x)) = \cos(\cos x) \times \frac{d}{dx}(\cos(x))$

$= \cos(\cos x) \times -\sin x$

$= -\cos(\cos x)\sin x$

5. $\frac{1}{\sqrt{x^2+1}}$

$\frac{d}{dx}\left(\frac{1}{\sqrt{x^2+1}}\right) = \frac{1}{(x^2+1)^{1/2}}$

$= (x^2+1)^{-1/2}$

$\frac{d}{dx}(x^2+1)^{-1/2} = -\frac{1}{2}(x^2+1)^{-1/2-1} \times \frac{d}{dx}(x^2+1)$

$$= -\tfrac{1}{2}(x^2+1^{-3/2}) \times 2x+0$$

$$= -\tfrac{1}{2}\times 2x\,(x^2+1)^{-3/2}\}$$

$$= -x\,(x^2+1)^{-3/2}\}$$

$$= \frac{-x}{(x^2+1)^{3/2}}$$

$$= \frac{\cancel{-x}}{\cancel{(x^2+1)(x+1)}^{3/2}} = \frac{-x}{(x^2+1)\sqrt{x^2+1}}$$

? If $y = 10x^3+5x^2$ find

→ dy/dx

→ d^2y/dx^2

→ show that $x^2\,d^2y/dx^2 - 4x\,dy/dx + 6y = 0$

Ans:- $y = 10x^3+5x^2$

$$dy/dx\,(10x^3+5x^2) = dy/dx\,10x^3 + dy/dx\,5x^2$$

$$= \underline{\underline{30x^2+10x}}$$

$$d^2y/dx^2\,(30x^2+10x) = dy/dx\,30x^2 + dy/dx\,10x$$

$$= \underline{\underline{60x+10}}$$

$$x^2\,d^2y/dx^2 - 4x\,dy/dx + 6y = 0$$

$$= x^2(60x+10) - 4x(30x^2+10x) + 6\cancel{y=0}\,(10x^3+5x^2)$$

$$= 60x^3+10x^2 - 120x^3 - 40x^2 + \cancel{6y=0}$$

$$\cancel{6(10x^3+5x^2)} + 60x^3+30x^2$$

$$= 120x^3-120x^3-40x^2+40x^2 = 0$$

? $y = 3+e^{1/2x}$

Prove that $dy/dx + \tfrac{1}{2}y = \tfrac{3}{2}$

$$\boxed{\begin{array}{l} e^x = e^x \\ e^{mx} = me^{mx} \end{array}}$$

Ans:- $y = 3+e^{1/2x}$

$$dy/dx\left(\frac{d}{dx}\,3+e^{-1/2x}\right) = \frac{d}{dx}(3) + d/dx\,(e^{-1/2x})$$

$$= \cancel{0 + -\tfrac{1}{2}x\,e^{-1/2x}}$$

$$= \cancel{\tfrac{1}{2}}\;0 + -\tfrac{1}{2}e^{-1/2x}$$

So $\cancel{dy/dx + \tfrac{1}{2}y = \tfrac{3}{2}}$

$$= \cancel{-\tfrac{1}{2}e^{-1/2x}} + \tfrac{1}{2}y$$

$$= -\tfrac{1}{2}\cancel{e}^{-1/2}\,x\,\cancel{e}$$

$$= -\tfrac{1}{2}\cancel{e}^{-1/2}$$

$$= -\tfrac{1}{2}\,\frac{1}{e^{1/2}}$$

$$-\cancel{\tfrac{1}{2}}\times\frac{1}{\cancel{\sqrt{x}}} = -\frac{\cancel{1}}{2\sqrt{x}}$$

$$dy/dx + \tfrac{1}{2}y = \tfrac{3}{2}$$

$$= -\tfrac{1}{2}\times\frac{1}{e^{1/2x}} + \tfrac{1}{2}(3+e^{-1/2x})$$

$$= \frac{-1}{2e^{1/2x}} + \tfrac{3}{2} + \tfrac{1}{2}\,1/\cancel{e^x}\,e^{-1/2x}$$

$= 0 + 3/2 = 3/2$

Partial and Total differenciation

Eg:- Let $U = 5x - 6y + 8$

Then, the partial derivative of U with respect to x $= \frac{\partial U}{\partial x} = 5 + 0 + 0$

$= 5$

the partial derivative of U with respect to y,

B, $\frac{\partial U}{\partial y} = 0 + 6 + 0 = -6$

1. Find the partial derivative of z,

$z = 4x^2 + 4xy + y^2$

An: $\frac{\partial U}{\partial x}(4x^2 + 4xy + y^2) = 8x + 4y + 0 = 8x + 4y$

$\frac{\partial U}{\partial y}(4x^2 + 4xy + y^2) = 0 + 4x + 2y$

$= 4x + 2y$

2. $z = x^3 e^{2y}$ find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

$\frac{\partial z}{\partial x}(x^3 e^{2y}) = \cancel{\frac{d}{dx}(x^3 \frac{d}{dx}(e^{2y})} + e^{2y}\frac{d}{dx}x^3$

$= \cancel{x^3 \cdot 2e^{2y}} + e^{2y}(3x^2)$

$= 3x^2 e^{2y}$

$\frac{\partial z}{\partial y}(x^3 e^{2y}) = x^3 2e^{2y} = 2x^3 e^{2y}$

Partial derivative of second order

Eg:- $z = x^3 e^{2y}$

$\frac{\partial z}{\partial x} = 3x^2 e^{2y}$

$\frac{\partial z}{\partial y} = x^3 2e^{2y}$

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(3x^2 e^{2y})$

$= 6x e^{2y}$

$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}(x^3 2e^{2y})$

$= \cancel{x^3 2 \times 2ye^{2y}}$

$= 2x^3 2e^{2y} = 4x^3 e^{2y}$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(2x^3 e^{2y}\right) = 2 \times 3x^2 e^{2y}$$

$$= \underline{\underline{6x^2 e^{2y}}}$$

$f_{xy} = f_{yx}$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(3x^2 e^{2y}\right)$$

$$= 3x^2 2 e^{2y}$$

$$= \underline{\underline{6x^2 e^{2y}}}$$

Total differentiation

Let $z = f(x, y)$

$\frac{\partial z}{\partial x}$ will give the change in 'z' when there is a small change in x holding 'y' constant.

$\frac{\partial z}{\partial y}$ give the change in 'z' when there is a small change in y keeping 'x' as constant.

If, suppose that x changes by an amount dx, then $\frac{\partial z}{\partial x} dx$ be the change in 'z' by 'x'.

Similiarly, $\frac{\partial z}{\partial y} dy$ be the change in 'z' by 'y'.

Therefore total change,

$$\boxed{dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy}$$

or $f_x dx + f_y dy$

Eg:- $z = 3x^2 + xy - 2y^3$ find the total derivative of z.

Ans. $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(3x^2 + xy - 2y^3\right)$$

$$= \underline{\underline{6x + y}}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(3x^2 + xy - 2y^3\right)$$

$$= 0 + x - 6y^2$$

$$= \underline{x - 6y^2}$$

$$\underline{\underline{dz = (6x + y)(dx) + (x - 6y^2)(dy)}}$$

2. Find the total derivative of u with respect to t

If $u = x^2 + xy + y^3$, $x = t^3$, $y = t^3 + t^2$

Ans:- $u = x^2 + xy + y^3$

~~$u = (t^3)^2 + t^3(t^3 + t^2) + (t^3 + t^2)^3$~~

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + xy + y^3\right)$$

$$= \underline{\underline{2x + y}}$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial z}\left(x^2 + xy + y^3\right)$$

$$= 0 + x + 3y^2 = x + 3y^2$$

$$dz = (2x+y)\,dx + x + 3y^2\,(dy)$$

$$\frac{dx}{dt} = \frac{d}{dt}(t^3)$$

$$dx = \underline{3t^2(dt)}$$

$$\frac{dy}{dt} = \frac{d}{dt}(t^3 + t^2)$$

$$dy = \underline{3t^2 + 2t(dt)}$$

$$dz = (2x+y)(3t^2 dt) + (x+3y^2)(3t^2 - 2t)\,dt$$

$$\left[(2x+y)3t^2 + (x+3y^2)(3t^2 - 2t)\right] dt$$

Elasticity

"Elasticity measures the percentage change in the dependent variable in response to percentage change in an independent variable."

$$e_s = \frac{\%\text{ change in dependent variable}}{\%\text{ change in independent variable}}.$$

Price elasticity of demand

A mkt dd curve is a function of in which qty demanded is depended on produced price. The price elasticity of demand is defined as the percentage change in qty demanded resulting from 1% percentage change in price.

$$e_s = \frac{\frac{\Delta Q}{Q}}{\frac{\Delta P}{P}} \qquad \frac{\frac{\Delta Q}{Q}}{\frac{\Delta P}{P}} = \frac{\Delta Q}{Q} \times \frac{P}{\Delta P} = \frac{\Delta Q}{\Delta P} \times \frac{P}{Q}$$

where Q, is qty demanded

P is price

Point Elasticity

If price changes in a very small amount we can compute the point elasticity of demand.

$$= \lim_{\Delta P \to 0} \frac{\Delta Q}{\Delta P} \times \frac{P}{Q}$$

$$= \frac{P}{Q} \lim_{\Delta P \to 0} \frac{\Delta Q}{\Delta P}$$

$$= \frac{P}{Q} \times \frac{dQ}{dP}$$

1. If the demand law is, $P = \frac{10}{(x+1)^2}$ find the elasticity of demand in terms of x

If the qty x is 4 units find the elasticity of dd.

Ans. e_s of dd $= \frac{P}{Q} \times \frac{dQ}{dP}$

$= \frac{p}{x} \times \frac{dx}{dP}$

$= \frac{10}{(x+1)^2} \times \frac{dx}{dP}$

$\frac{dx}{dp} = \frac{1}{\frac{dP}{dx}}$

$\frac{dP}{dx} = \frac{d}{dx}\left(\frac{10}{(x+1)^2}\right)$

$= \frac{\cancel{(x+1)^2 \frac{d}{dx}(10) - 10\frac{d}{dx}((x+1)^2)}}{\cancel{((x+1)^2)^2}}$

$\cancel{(x+1)^2 \times 0 - 10(}$

$= 10 \frac{d}{dx}\left(\frac{1}{(x+1)^2}\right)$

$= 10 \frac{d}{dx}\left((x+1)^{-2}\right)$

$= 10\left[-2(x+1)^{-2-1} \frac{d}{dx}(x+1)\right]$

$= 10\left[-2(x+1)^{-3}(1)\right]$

$= -20(x+1)^{-3}$

$\frac{dP}{dx} = \frac{-20}{(x+1)^3}$ //

$\frac{dx}{dP} = \frac{1}{\left[\frac{-20}{(x+1)^3}\right]}$

$= \frac{-(x+1)^3}{20}$ //

$e_s = \frac{P}{(x)} \times \frac{-(x+1)^3}{20}$

$e_s = \frac{dx}{dP} \times \frac{P}{x}$

$= \frac{-(x+1)^3}{20} \times \frac{10}{x(x+1)^2}$

$= \frac{-(x+1)}{2x}$

$P/x = \left(\frac{10}{(x+1)^2}\right)/x$

$= \frac{10}{(x+1)^2} \times \frac{1}{x}$

$= \frac{10}{x(x+1)^2}$ //

$x^{1/2}$

$x^{1/2} = \sqrt{x}$

x^{-2}

$1/x^2$

If the qty $x = 4$

$e_d = \frac{-x+1}{2x} = \frac{-4+1}{2 \times 4} = -3/8$ //

? $y = x^2 \log x$, find $\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{d}{dx}(x^2 \log x)$

$$= x^2 \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(x^2)$$

$$= x^2 (1/x + \log x (2x)$$

$$= x^2/x + \log x (2x)$$

$$= x + 2x \log x$$

$$= x + x(2 + \log)$$

$$= \underline{\underline{x + 2\log}}$$

$10x\ 2x$

$x(10x2)$

$20x$

3. Given a production function $P = KL^{\alpha}C^{\beta}$ where, P is the pdt, L labour, C is capital and k, α, β are constant. Find dp.

$$z = f(x, y)$$

$$\frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$\phi = f(L, C)$$

$$dP = \frac{\partial P}{\partial L} dL + \frac{\partial P}{\partial C} dk$$

$$\frac{\partial P}{\partial L} = \frac{\partial}{\partial L}(KL^{\alpha}C^{\beta})$$

$$= K\alpha L^{\alpha-1} C^{\beta}$$

$$= K\alpha L^{\beta} C^{\beta}$$

$\alpha + \beta = 1$

$\beta = \alpha - 1$

$$\frac{\partial}{\partial C} = \frac{\partial}{\partial C}(KL^{\alpha}C^{\beta}) = KL^{\alpha}\beta C^{\beta-1}$$

$$= \beta C^{\beta-1} = \cancel{K\beta C^{\alpha}}\ KL^{\alpha}\beta C^{\alpha}$$

$$\cancel{dP = K\alpha L^{\alpha-1}}$$

$$\underline{\underline{dP = K\alpha L^{\alpha-1} C^{\beta} dL + KL^{\alpha}\beta C^{\beta-1} dC}}$$