Ministry of Higher Education and
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Bernstein Sequence

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Chapter 2

Bernstein Sequence

2.1 Bernstein Sequence

Definition 2.1.1

Suppose that $f(t) \in C[0,1]$. The n-th order Bernstein operators are defined as

$$B_n(f(t);x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)$$

Theorem 2.1.1

The functions $b_{n,k}(x)$ have the following properties

1. $\sum_{k=0}^{n} b_{n,k}(x) = 1$
2. $\sum_{k=0}^{n} k b_{n,k}(x) = nx$
3. $\sum_{k=0}^{n} k^2 b_{n,k}(x) = n(n-1)x^2 + nx$
4. $\sum_{k=0}^{n} k^3 b_{n,k}(x) = n(n-1)(n-2)x^3 + 3n(n-1)x^2 + nx$

***Proof*:**
We have

1.

$$\sum_{k=0}^{n} b_{n,k}(x) = \sum_{k=0}^{n} x^k (1-x)^{n-k} = (x+1-x)^n = 1$$

2.

$$\begin{aligned}
\sum_{k=0}^{n} k b_{n,k}(x) &= \sum_{k=0}^{n} k \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} \\
&= 0 + \sum_{k=1}^{n} k \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} \\
&= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} x^k (1-x)^{n-k} \\
&= \sum_{k=0}^{n-1} \frac{n(n-1)!}{k!(n-1-k)!} x^{k+1} (1-x)^{n-1-k}
\end{aligned}$$

$$= nx\sum_{k=0}^{n-1} b_{n-1,k}(x) = nx$$

3.

$$\begin{aligned}
\sum_{k=0}^{n} k^2 b_{n,k}(x) &= \sum_{k=0}^{n} k^2 \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} \\
&= 0 + \sum_{k=1}^{n} k^2 \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} \\
&= \sum_{k=1}^{n} k \frac{n!}{(k-1)!(n-k)!} x^k (1-x)^{n-k} \\
&= \sum_{k=0}^{n-1} (k+1) \frac{n(n-1)!}{k!(n-1-k)!} x^{k+1} (1-x)^{n-1-k} \\
&= nx \sum_{k=0}^{n-1} (k+1) b_{n-1,k}(x) \\
&= nx \left\{ \sum_{k=0}^{n-1} k b_{n-1,k}(x) + \sum_{k=0}^{n-1} b_{n-1,k}(x) \right\} \\
&= nx \Big\{ (n-1)x + 1 \Big\} \\
&= n(n-1)x^2 + nx
\end{aligned}$$

4.

$$\begin{aligned}
\sum_{k=0}^{n} k^3 b_{n,k}(x) &= \sum_{k=0}^{n} k^3 \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} \\
&= 0 + \sum_{k=1}^{n} k^3 \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} \\
&= \sum_{k=1}^{n} k^2 \frac{n!}{(k-1)!(n-k)!} x^k (1-x)^{n-k} \\
&= \sum_{k=0}^{n-1} (k+1)^2 \frac{n(n-1)!}{k!(n-1-k)!} x^{k+1} (1-x)^{n-1-k}
\end{aligned}$$

$$
\begin{aligned}
&= nx\sum_{k=0}^{n-1}(k+1)^2 b_{n-1,k}(x)\\
&= nx\sum_{k=0}^{n-1}(k^2+2k+1) b_{n-1,k}(x)\\
&= nx\Big[(n-1)(n-2)x^2+(n-1)x+2\big[(n-1)x\big]+1\Big]\\
&= n(n-1)(n-2)x^3+(1+2)n(n-1)x^2+nx\\
&= n(n-1)(n-2)x^3+3n(n-1)x^2+nx
\end{aligned}
$$

Theorem 2.1.2 [Korovkin's Theorem]

Suppose that $f(t)$ is continuous function on $[a,b]$ or on $[0,\infty)$ and $B_n(f;x)$ satisfying the conditions

1. $B_n(1;x) \to 1$ *as* $n \to \infty$
2. $B_n(t;x) \to x$ *as* $n \to \infty$
3. $B_n(t^2;x) \to x^2$ *as* $n \to \infty$

Then $B_n(f;x) \to f(x)$ *as* $n \to \infty$

***Proof*:**
We have

1. $B_n(1;x) = \sum_{k=0}^{n} b_{n,k}(x)\cdot 1 = 1 \to 1$ as $n \to \infty$.
2. $B_n(t;x) = \sum_{k=0}^{n} b_{n,k}(x)\cdot\left(\dfrac{k}{n}\right) = \dfrac{1}{n}\cdot nx = x \to x$ as $n \to \infty$.
3. $B_n(t^2;x) = \sum_{k=0}^{n} b_{n,k}(x)\cdot\left(\dfrac{k^2}{n^2}\right) = \dfrac{1}{n^2}\cdot[n(n-1)x^2+nx] \to x^2$ as $n \to \infty$.

Therefore $B_n(f;x) \to f(x)$ as $n \to \infty$. $\square$

Example

Find an approximation polynomial of degree 3 for the function $\sin t \in C[0,1]$.

Solution:

The approximation by Bernstein sequence gives an approximation polynomial of degree n, so we use Bernstein polynomials for this.

$$\begin{aligned} B_3(\sin t; x) &= \sum_{k=0}^{3} b_{3,k}(x) \sin\left(\frac{k}{3}\right) \\ &= b_{3,0}(x) \sin\left(\frac{0}{3}\right) + b_{3,1}(x) \sin\left(\frac{1}{3}\right) + b_{3,2}(x) \sin\left(\frac{2}{3}\right) + b_{3,3}(x) \sin\left(\frac{3}{3}\right) \end{aligned}$$

2.2 The m-th Order Moment

Definition 2.2.1

We define the m-th order moment for Bernstein sequence $B_n(f;x)$ as follows

$$T_{n,m}(x) = B_n((t-x)^m; x) = \sum_{k=0}^{n} b_{n,k}(x) \left(\frac{k}{n} - x\right)^m$$

Theorem 2.2.1

We have

1. $T_{n,0}(x) = 1$
2. $T_{n,1}(x) = 0$
3. $T_{n,2}(x) = \dfrac{x(1-x)}{n}$

***Proof*:**

1.
$$T_{n,0}(x) = B_n((t-x)^0; x) = B_n(1; x) = 1$$

2.
$$T_{n,1}(x) = B_n((t-x)^1; x)$$

$$
\begin{aligned}
&= B_n(t - x; x) \\
&= B_n(t; x) - B_n(x; x) \\
&= x - xB_n(1; x) = x - x = 0
\end{aligned}
$$

3.

$$
\begin{aligned}
T_{n,2}(x) &= B_n((t - x)^2; x) \\
&= B_n(t^2 - 2xt + x^2; x) \\
&= B_n((t^2; x) - 2xB_n(t; x) + x^2 B_n(1; x) \\
&= \frac{1}{n^2}\Big[n(n - 1)x^2 + nx\Big] - 2x \cdot x + x^2 \\
&= \frac{x(1 - x)}{n}
\end{aligned}
$$