Formation of Partial Differential Equations

Introduction

An equation involving partial differential coefficients of a function of two or more variables is known as a partial differential equation. If a partial differential equation contains n^{th} and lower order derivatives, it is said to be of n^{th} order PDE. The degree of such equation is the greatest exponent of the highest order. Further such equation will be called linear if, it is of 1^{st} degree in the dependent variable and its partial derivatives (i.e. powers or products of the dependent variable and its partial derivatives must be absent). An equation which is not linear is called a non-linear differential equation.

In case of two independent variables we usually assume them to be x and y and z to be dependent on x and y. If there are n-independent variables we take them to be $x_1, x_2, x_3, \cdots\cdots, x_n$ and z is then regarded as the dependent variable.

Definition:- A PDE in a single unknown u is an equation involving u and its partial derivatives. All such equations can be written as $F(u, u_1, u_2) = 0$

Example 1:

Let $u = u(t, x)$, then $\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial x^2} + u = t$, is a 2^{nd} order linear PDE.

We say this is a linear PDE with constant coefficients because u and its derivatives appear linearly (i.e. first power only) and multiplied only by constants.

Formation of Partial Differential Equations

Partial differential equations can be formed by the elimination of arbitrary constants or arbitrary functions.

If we have $z = f(x, y)$ then we have the following symbolic representation of the partial derivatives

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s, \quad \frac{\partial^2 z}{\partial y^2} = t,$$

Sometimes these partial derivatives are also denoted by making use of suffixes.

Thus we write

$$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y} \quad u_{yy} = \frac{\partial^2 u}{\partial y^2}.$$

Rule 1: Formation of PDE by eliminating arbitrary constants.

Consider an equation

$$F(x, y, z, a, b) = 0 \qquad (1.1)$$

where a and b are arbitrary constants. Let z be regarded as function of two independent variables x and y.

Differentiating (1.1) partially with respect to x and y, we get

$$\frac{\partial F}{\partial x}+\mathrm{p}\frac{\partial F}{\partial z}=0 \quad \text{and} \quad \frac{\partial F}{\partial y}+\mathrm{q}\frac{\partial F}{\partial z}=0 \qquad \text{...(1.2)}$$

Eliminating two constants a and b from the three equation of (1.1) and (1.2), we will obtain an equation of the form

$$f(x,y,z,p,q)=0 \qquad \text{... (1.3)}$$

which is partial differential equation of the first order.

In a similar manner it can be shown that if there are more arbitrary constants than the number of independent variables, the above procedure of elimination will give rise to partial differential equations of higher order than the first.

Exercise 1: Find a partial differential equation by eliminating a and b from the equation

$$z = ax + by + a^2 + b^2$$

Solution: Given $z = ax + by + a^2 + b^2$ (1)

Differentiating (1) partially with respect to x and y, we get

$$\frac{\partial z}{\partial x} = a \text{ and } \frac{\partial z}{\partial y} = b$$

Substituting these values of a and b in (1), we get

$$z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + \left(\frac{\partial z}{\partial y}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

which is the required partial differential equation.

Exercise 2: Form the partial differential equation by eliminating a and b from the equation

$$z = (x^2 + a)(y^2 + b).$$

Solution: Given $z = (x^2 + a)(y^2 + b)$ (1)

Differentiating (1) partially with respect to x and y, we get

$$\frac{\partial z}{\partial x} = 2x(y^2 + b) \text{ or } (y^2 + b) = \frac{1}{2x}\frac{\partial z}{\partial x} \qquad (2)$$

and

$$\frac{\partial z}{\partial y} = 2y(x^2 + a) \text{ or } (x^2 + a) = \frac{1}{2y}\frac{\partial z}{\partial y} \qquad (3)$$

Substituting the values of $(y^2 + b)$ and $(x^2 + a)$ from (2) and (3) in (1), we have

$$z = \frac{1}{2y}\frac{\partial z}{\partial y}\cdot\frac{1}{2x}\frac{\partial z}{\partial x}$$

$$\text{or } 4xyz = \left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial z}{\partial x}\right)$$

which is the required partial differential equation.

Exercise 3: Eliminate arbitrary constants from $z = (x-a)^2 + (y-b)^2$ to form the partial differential equation.

Solution: Given $z = (x-a)^2 + (y-b)^2$ (1)

Differentiating (1) partially with respect to x and y, we get

$$\frac{\partial z}{\partial x} = 2(x-a) \text{ and } \frac{\partial z}{\partial y} = 2(y-b)$$

Squaring and adding these equations, we have

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4(x-a)^2 + 4(y-b)^2$$
$$= 4[(x-a)^2 + (y-b)^2]$$
$$= 4z$$

which is the required partial differential equation.

Exercise 4: Eliminate arbitrary constants from $z = axe^y + \frac{1}{2}a^2e^{2y} + b$ to form the partial differential equation.

Solution: Given $z = axe^y + \frac{1}{2}a^2e^{2y} + b$ (1)

Differentiating (1) partially with respect to x and y, we get

$$\frac{\partial z}{\partial x} = ae^y \tag{2}$$

and

$$\frac{\partial z}{\partial y} = axe^y + \frac{1}{2}a^2e^{2y}\cdot 2$$
$$= axe^y + a^2e^{2y}$$
$$= x(ae^y) + (ae^y)^2 \tag{3}$$

Substituting the values of ae^y from (2) in (3), we get

$$\frac{\partial z}{\partial y} = x\left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial z}{\partial x}\right)^2$$

which is the required partial differential equation.

Rule 2: The PDE by the elimination of arbitrary functions $\emptyset$ from the equation

$$\emptyset(u, v) = 0,$$

where u and v are functions of x, y and z is

$$Pp + Qq = R,$$

Where

$$P = \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial y} \end{vmatrix},$$

$$Q = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial z} \end{vmatrix}$$

and

$$R = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial x} \end{vmatrix}.$$

Exercise 5: Form the partial differential equation by eliminating the arbitrary function $\emptyset$ from $\emptyset(x+y+z, x^2+y^2-z^2)=0$.

Solution: Given $\emptyset(x+y+z, x^2+y^2-z^2)=0$ (1)

Let $u = x+y+z$ and $v = x^2+y^2-z^2$, then

$$\frac{\partial u}{\partial x}=1,\quad \frac{\partial u}{\partial y}=1,\quad \frac{\partial u}{\partial z}=1 \text{ and } \frac{\partial v}{\partial x}=2x,\quad \frac{\partial v}{\partial y}=2y,\quad \frac{\partial v}{\partial z}=-2z \tag{2}$$

From (1), we have

$$\emptyset(u,v)=0 \tag{3}$$

We know the partial differential equation of (3) is

$$Pp+Qq=R \tag{4}$$

Where

$$P = \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ -2z & 2y \end{vmatrix}$$

$$= 2y+2z$$

$$= 2(y+z),$$

$$Q = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 2x & -2z \end{vmatrix}$$

$$= -2z - 2x$$

$$= -2(z + x)$$

and

$$R = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial x} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 2y & 2x \end{vmatrix}$$

$$= 2x - 2y$$

$$= 2(x - y)$$

Substituting the values of P, Q and R in (4), we have

$$2(y + z)p - 2(z + x)q = 2(x - y)$$

$$\Rightarrow (y + z)p - (z + x)q = (x - y)$$

which is the required partial differential equation.

Exercise 6: Form the partial differential equation by eliminating the arbitrary function $\emptyset$ from $\emptyset\,(x^2 + y^2 + z^2, z^2 - 2xy\,) = 0$.

Solution: Given $\emptyset(x^2 + y^2 + z^2, z^2 - 2xy\,) = 0$ (1)

Let $u = x^2 + y^2 + z^2$ and $v = z^2 - 2xy$, then

$\frac{\partial u}{\partial x} = 2x,\ \ \frac{\partial u}{\partial y} = 2y,\ \ \frac{\partial u}{\partial z} = 2z$ and $\frac{\partial v}{\partial x} = -2y,\ \frac{\partial v}{\partial y} = -2x,\ \ \frac{\partial v}{\partial z} = 2z$ (2)

From (1), we have

$$\emptyset(u, v) = 0 \qquad (3)$$

We know the partial differential equation of (3) is

$$Pp + Qq = R \qquad (4)$$

Where

$$P = \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 2z & 2y \\ 2z & -2x \end{vmatrix}$$
$$= -4zx - 4zy$$
$$= -4z(x + y),$$

$$Q = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial z} \end{vmatrix}$$
$$= \begin{vmatrix} 2x & 2z \\ -2y & 2z \end{vmatrix}$$
$$= 4xz + 4zy$$
$$= 4z(x + y)$$

and

$$R = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial x} \end{vmatrix}$$
$$= \begin{vmatrix} 2y & 2x \\ -2x & -2y \end{vmatrix}$$
$$= -4y^2 + 4x^2$$
$$= 4(x^2 - y^2)$$

Substituting the values of P, Q and R in (4), we have

$$-4z(x + y)p + 4z(x + y)q = 4(x^2 - y^2)$$
$$\Rightarrow -zp + zq = (x - y)$$
$$\Rightarrow (p - q)z = y - x$$

which is the required partial differential equation.

H.W.

Formulate the PDE from

(1) $z = (x + a)(y + b)$ (2) $z = f(x + iy) + g(x - iy)$

(3) $z = ax + by + ab$ (4) $z = f(x + ay) + g(x - ay)$

Solution: (1) Given $z = (x + a)(y + b)$ (1)

Diff. partially (1), w. r. t., x and y

$\frac{\partial z}{\partial x} = p = (y + b)$ and $\frac{\partial z}{\partial y} = q = (x + a)$

Substituting the values of $(y + b)$ and $(x + a)$ in (1), we get

$z = pq$, which is the required PDE.

(2) Given

$$z = f(x + iy) + g(x - iy) \qquad (1)$$

Diff. partially with respect to x and y we get

$$\frac{\partial z}{\partial x} = p = f'(x + iy) + g'(x - iy)$$

$$\frac{\partial z}{\partial y} = q = if'(x + iy) - ig'(x - iy)$$

Again diff. partially with respect to x and y we get

$$\frac{\partial^2 z}{\partial x^2} = f''(x + iy) + g''(x - iy) \qquad (2)$$

$$\frac{\partial^2 z}{\partial y^2} = -f''(x + iy) - g''(x - iy) \qquad (3)$$

Adding (2) and (3), we get

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0,$$ which is the required PDE.

(3) Given

$$z = ax + by + ab \qquad (1)$$

Diff. partially w.r.t., x and y we get

$$\frac{\partial z}{\partial x} = a \qquad \frac{\partial z}{\partial y} = b$$

Using the value of a and b in the given equation we get

$$z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + \frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

or $$z = px + qy + pq$$

which is the required PDE.

(4) Same as (2).

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