Lecture-2

The Dot and Cross Product

Dot Product:

THE DOT OR SCALAR PRODUCT of two vectors **A** and **B**, denoted by $\mathbf{A}\cdot\mathbf{B}$ (read **A** dot **B**), is defined as the product of the magnitudes of **A** and **B** and the cosine of the angle θ between them. In symbols,

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta. \qquad 0 \leq \theta \leq \pi$$

The following laws are valid:

1. $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$ — Commutative Law for Dot Products
2. $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$ — Distributive Law
3. $m(\mathbf{A}\cdot\mathbf{B}) = (m\mathbf{A})\cdot\mathbf{B} = \mathbf{A}\cdot(m\mathbf{B}) = (\mathbf{A}\cdot\mathbf{B})m$, where m is a scalar.
4. $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1, \quad \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0$
5. If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$, then

$$\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$$
$$\mathbf{A}\cdot\mathbf{A} = A^2 = A_1^2 + A_2^2 + A_3^2$$
$$\mathbf{B}\cdot\mathbf{B} = B^2 = B_1^2 + B_2^2 + B_3^2$$

6. If $A \cdot B = 0$ and A and B are not null vectors, then A and B are perpendicular.

The Cross Product:

THE CROSS OR VECTOR PRODUCT of **A** and **B** is a vector $\mathbf{C} = \mathbf{A}\times\mathbf{B}$ (read **A** cross **B**). The magnitude of $\mathbf{A}\times\mathbf{B}$ is defined as the product of the magnitudes of **A** and **B** and the sine of the angle θ between them. The direction of the vector $\mathbf{C} = \mathbf{A}\times\mathbf{B}$ is perpendicular to the plane of **A** and **B** and such that **A**, **B** and **C** form a right-handed system. In symbols,

$$\mathbf{A}\times\mathbf{B} = AB\sin\theta\,\mathbf{u}. \qquad 0 \leq \theta \leq \pi$$

where $\mathbf{u}$ is a unit vector indicating the direction of $\mathbf{A}\times\mathbf{B}$.

The following laws are valid

1. $\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$ — (Commutative Law for Cross Products Fails.)
2. $\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$ — Distributive Law
3. $m(\mathbf{A}\times\mathbf{B}) = (m\mathbf{A})\times\mathbf{B} = \mathbf{A}\times(m\mathbf{B}) = (\mathbf{A}\times\mathbf{B})m$, where m is a scalar.
4. $\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$, $\quad \mathbf{i}\times\mathbf{j} = \mathbf{k}$, $\mathbf{j}\times\mathbf{k} = \mathbf{i}$, $\mathbf{k}\times\mathbf{i} = \mathbf{j}$
5. If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$, then

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

6. The magnitude of $\mathbf{A}\times\mathbf{B}$ is the same as the area of a parallelogram with sides $\mathbf{A}$ and $\mathbf{B}$.
7. If $\mathbf{A}\times\mathbf{B} = \mathbf{0}$, and $\mathbf{A}$ and $\mathbf{B}$ are not null vectors, then $\mathbf{A}$ and $\mathbf{B}$ are parallel.

Example 1:

If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$, prove that $\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$.

Proof:

$$\begin{aligned}
\mathbf{A}\cdot\mathbf{B} &= (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})\cdot(B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}) \\
&= A_1\mathbf{i}\cdot(B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}) + A_2\mathbf{j}\cdot(B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}) + A_3\mathbf{k}\cdot(B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}) \\
&= A_1B_1\mathbf{i}\cdot\mathbf{i} + A_1B_2\mathbf{i}\cdot\mathbf{j} + A_1B_3\mathbf{i}\cdot\mathbf{k} + A_2B_1\mathbf{j}\cdot\mathbf{i} + A_2B_2\mathbf{j}\cdot\mathbf{j} + A_2B_3\mathbf{j}\cdot\mathbf{k} + A_3B_1\mathbf{k}\cdot\mathbf{i} + A_3B_2\mathbf{k}\cdot\mathbf{j} + A_3B_3\mathbf{k}\cdot\mathbf{k} \\
&= A_1B_1 + A_2B_2 + A_3B_3
\end{aligned}$$

Example 2:

Find the angle between $\mathbf{A} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{B} = 6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.

Solution: We know

$$A\cdot B = AB\cos\theta$$
$$\Rightarrow \cos\theta = \frac{A\cdot B}{AB}$$

Now,

$A = \sqrt{(2)^2 + (2)^2 + (-1)^2} = 3$, $\quad B = \sqrt{(6)^2 + (-3)^2 + (2)^2} = 7$

$\mathbf{A}\cdot\mathbf{B} = (2)(6) + (2)(-3) + (-1)(2) = 12 - 6 - 2 = 4$

Therefore,

$\cos\theta = \dfrac{\mathbf{A}\cdot\mathbf{B}}{AB} = \dfrac{4}{(3)(7)} = \dfrac{4}{21} = 0.1905$ and $\theta = 79^\circ$ approximately.

Example 3:

Determine the value of a so that $\mathbf{A} = 2\mathbf{i} + a\mathbf{j} + \mathbf{k}$ and $\mathbf{B} = 4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ are perpendicular.

Solution: Since A and B are perpendicular, so we have

$$A\cdot B = 0$$
$$\Rightarrow (2i + aj + k)\cdot(4i - 2j - 2k) = 0$$
$$\Rightarrow 2\cdot 4 + a\cdot(-2) + 1\cdot(-2) = 0$$
$$\Rightarrow 8 - 2a - 2 = 0$$
$$\therefore a = 3.$$

Example 4:

Find the angles which the vector $\mathbf{A} = 3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$ makes with the coordinate axes.

Solution:

Let α, β, γ be the angles which $\mathbf{A}$ makes with the positive x, y, z axes respectively.

$$\mathbf{A}\cdot\mathbf{i} = (A)(1)\cos\alpha = \sqrt{(3)^2+(-6)^2+(2)^2}\cos\alpha = 7\cos\alpha$$

$$\mathbf{A}\cdot\mathbf{i} = (3\mathbf{i}-6\mathbf{j}+2\mathbf{k})\cdot\mathbf{i} = 3\mathbf{i}\cdot\mathbf{i} - 6\mathbf{j}\cdot\mathbf{i} + 2\mathbf{k}\cdot\mathbf{i} = 3$$

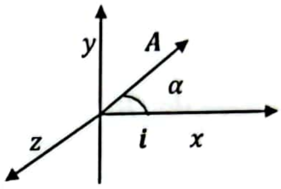
Then

$$7\cos a = 3$$

$$\Rightarrow \cos a = \frac{3}{7}$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{3}{7}\right)$$

$$\therefore \alpha = 64.6^{\circ} \text{ approximately.}$$



Similarly, $\cos\beta = -6/7$, $\beta = 149^{\circ}$ and $\cos\gamma = 2/7$, $\gamma = 73.4^{\circ}$.

Example 5:

Find the projection of the vector $\mathbf{A} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ on the vector $\mathbf{B} = 4\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$.

Solution:

A unit vector in the direction $\mathbf{B}$ is $\mathbf{b} = \dfrac{\mathbf{B}}{B} = \dfrac{4\mathbf{i}-4\mathbf{j}+7\mathbf{k}}{\sqrt{(4)^2+(-4)^2+(7)^2}} = \dfrac{4}{9}\mathbf{i} - \dfrac{4}{9}\mathbf{j} + \dfrac{7}{9}\mathbf{k}$.

Projection of $\mathbf{A}$ on the vector $\mathbf{B}$ $= \mathbf{A}\cdot\mathbf{b} = (\mathbf{i}-2\mathbf{j}+\mathbf{k})\cdot(\frac{4}{9}\mathbf{i} - \frac{4}{9}\mathbf{j} + \frac{7}{9}\mathbf{k})$

$$= (1)(\tfrac{4}{9}) + (-2)(-\tfrac{4}{9}) + (1)(\tfrac{7}{9}) = \frac{19}{9}.$$

Example 6:

If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$, prove that $\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$.

Proof:

$$\begin{aligned}
\mathbf{A}\times\mathbf{B} &= (A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k})\times(B_1\mathbf{i}+B_2\mathbf{j}+B_3\mathbf{k}) \\
&= A_1\mathbf{i}\times(B_1\mathbf{i}+B_2\mathbf{j}+B_3\mathbf{k}) + A_2\mathbf{j}\times(B_1\mathbf{i}+B_2\mathbf{j}+B_3\mathbf{k}) + A_3\mathbf{k}\times(B_1\mathbf{i}+B_2\mathbf{j}+B_3\mathbf{k}) \\
&= A_1B_1\mathbf{i}\times\mathbf{i} + A_1B_2\mathbf{i}\times\mathbf{j} + A_1B_3\mathbf{i}\times\mathbf{k} + A_2B_1\mathbf{j}\times\mathbf{i} + A_2B_2\mathbf{j}\times\mathbf{j} + A_2B_3\mathbf{j}\times\mathbf{k} + A_3B_1\mathbf{k}\times\mathbf{i} + A_3B_2\mathbf{k}\times\mathbf{j} + A_3B_3\mathbf{k}\times\mathbf{k} \\
&= (A_2B_3 - A_3B_2)\mathbf{i} + (A_3B_1 - A_1B_3)\mathbf{j} + (A_1B_2 - A_2B_1)\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}.
\end{aligned}$$

Example 7:

If $A = 2i - 3j - k$ and $B = i + 4j - 2k$, find (*a*) $A \times B$, (*b*) $B \times A$, (*c*) $(A + B) \times (A - B)$.

Solution:

(*a*) $$A \times B = (2i - 3j - k) \times (i + 4j - 2k) = \begin{vmatrix} i & j & k \\ 2 & -3 & -1 \\ 1 & 4 & -2 \end{vmatrix}$$

$$= i\begin{vmatrix} -3 & -1 \\ 4 & -2 \end{vmatrix} - j\begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} + k\begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix} = 10i + 3j + 11k$$

(*b*) $$B \times A = (i + 4j - 2k) \times (2i - 3j - k) = \begin{vmatrix} i & j & k \\ 1 & 4 & -2 \\ 2 & -3 & -1 \end{vmatrix}$$

$$= i\begin{vmatrix} 4 & -2 \\ -3 & -1 \end{vmatrix} - j\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} + k\begin{vmatrix} 1 & 4 \\ 2 & -3 \end{vmatrix} = -10i - 3j - 11k.$$

(*c*) $$A + B = (2i - 3j - k) + (i + 4j - 2k) = 3i + j - 3k$$

$$A - B = (2i - 3j - k) - (i + 4j - 2k) = i - 7j + k$$

Then $$(A + B) \times (A - B) = (3i + j - 3k) \times (i - 7j + k) = \begin{vmatrix} i & j & k \\ 3 & 1 & -3 \\ 1 & -7 & 1 \end{vmatrix}$$

$$= i\begin{vmatrix} 1 & -3 \\ -7 & 1 \end{vmatrix} - j\begin{vmatrix} 3 & -3 \\ 1 & 1 \end{vmatrix} + k\begin{vmatrix} 3 & 1 \\ 1 & -7 \end{vmatrix} = -20i - 6j - 22k.$$

Example 8:

If $A = 3i - j + 2k$, $B = 2i + j - k$, and $C = i - 2j + 2k$, find (*a*) $(A \times B) \times C$, (*b*) $A \times (B \times C)$.

Solution:

(*a*) $$A \times B = \begin{vmatrix} i & j & k \\ 3 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix} = -i + 7j + 5k.$$

Then $$(A \times B) \times C = (-i + 7j + 5k) \times (i - 2j + 2k) = \begin{vmatrix} i & j & k \\ -1 & 7 & 5 \\ 1 & -2 & 2 \end{vmatrix} = 24i + 7j - 5k.$$

(*b*) $$B \times C = \begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ 1 & -2 & 2 \end{vmatrix} = 0i - 5j - 5k = -5j - 5k.$$

Then $$A \times (B \times C) = (3i - j + 2k) \times (-5j - 5k) = \begin{vmatrix} i & j & k \\ 3 & -1 & 2 \\ 0 & -5 & -5 \end{vmatrix} = 15i + 15j - 15k.$$

Example 9:

Determine a unit vector perpendicular to the plane of $\mathbf{A} = 2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$ and $\mathbf{B} = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

Solution:

$\mathbf{A} \times \mathbf{B}$ is a vector perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$.

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix} = 15\mathbf{i} - 10\mathbf{j} + 30\mathbf{k}$$

A unit vector parallel to $\mathbf{A} \times \mathbf{B}$ is $\dfrac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|} = \dfrac{15\mathbf{i} - 10\mathbf{j} + 30\mathbf{k}}{\sqrt{(15)^2 + (-10)^2 + (30)^2}} = \dfrac{3}{7}\mathbf{i} - \dfrac{2}{7}\mathbf{j} + \dfrac{6}{7}\mathbf{k}$.

Example 10:

If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$, $\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}$ show that

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

Proof:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{A} \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$= (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}) \cdot \left[(B_2C_3 - B_3C_2)\mathbf{i} + (B_3C_1 - B_1C_3)\mathbf{j} + (B_1C_2 - B_2C_1)\mathbf{k}\right]$$

$$= A_1(B_2C_3 - B_3C_2) + A_2(B_3C_1 - B_1C_3) + A_3(B_1C_2 - B_2C_1) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

Example 11:

Evaluate $(2\mathbf{i} - 3\mathbf{j}) \cdot [(\mathbf{i} + \mathbf{j} - \mathbf{k}) \times (3\mathbf{i} - \mathbf{k})]$.

Solution: By the example 10, the result is

$$\begin{vmatrix} 2 & -3 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & -1 \end{vmatrix} = 4.$$