18-01-2022

Vector Integration :-

Q. If

$$\vec{R} = (u-u^2)\hat{i} + 2u^3\hat{j} - 3\hat{k}$$

find (a) $\int \vec{R}(u)\,du$ and

(b) $\int_2^1 \vec{R}(u)\,du$

a) Given,

$$\vec{R} = (u-u^2)\hat{i} + 2u^3\hat{j} - 3\hat{k}$$

$$\int \vec{R}(u)\,du = \int \left[(u-u^2)\hat{i} + 2u^3\hat{j} - 3\hat{k}\right] du$$

$$= \hat{i}\int (u-u^2)\,du + \hat{j}\int 2u^3\,du - \hat{k}\int 3\,du$$

$$= \hat{i}\left(\frac{u^2}{2} - \frac{u^3}{3} + c_1\right) + \left(2\frac{u^4}{4} + c_2\right) + \hat{k}\,(3u + c_3)$$

$$= \left(\frac{u^2}{2} - \frac{u^3}{3}\right)\hat{i} + c_1\hat{i} + \frac{u^4}{2}\hat{j} + c_2\hat{j} - 3u\hat{k} + c_3\hat{k}$$

$$= \left(\frac{u^2}{2} - \frac{u^3}{3}\right)\hat{i} + \frac{u^4}{2}\hat{j} - 3u\hat{k} + \vec{c}$$

Q. 2

If $\vec{A} = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$,

evaluate $\int_C \vec{A} \cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the following paths

$C: x = t, y = t^2, z = t^3$

Soln:

Given,

$$\vec{A} = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$$
$$= (3t^2+6t^2)\hat{i} - 14t^2 \cdot t^3\hat{j} + 20t \cdot t^6\hat{k}$$
$$= 9t^2\hat{i} - 14t^5\hat{j} + 20t^7\hat{k}$$

$$\therefore \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\therefore d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$
$$= d(t)\hat{i} + d(t^2)\hat{j} + d(t^3)\hat{k}$$
$$= dt\hat{i} + 2t\,dt\,\hat{j} + 3t\,dt\,\hat{k}$$

$\to \vec{A} \cdot d\vec{r} = (9t^2\hat{i} - 14t^5\hat{j} + 20t^7\hat{k}) \cdot$

$(dt\,\hat{i} + 2t\,dt\,\hat{j} + 3t^2dt\,\hat{k})$

$= 9t^2dt - 28t^6dt + 60t^9\,dt$

$\int_c \vec{A} \cdot d\vec{r} = \int_0^1 (9t^2dt - 28t^6dt + 60t^9dt)$

$= \left[9\frac{t^3}{3}\right]_0^1 - 28\left[\frac{t^7}{7}\right]_0^1 + 60\left[\frac{t^{10}}{10}\right]_0^1$

$= 3(1-0) - 4(1-0) + 6(1-0)$

$= 3 - 4 + 6$

$= 5$ Ans:

H.W

Example - 4, 5

Q-5

If $\phi = 2xyz^2$, $\vec{F} = xy\hat{i} - z\hat{j} + x^2\hat{k}$

and c is the curve $x = t^2$, $y = 2t$,

$z = t^3$ from $t=0$ to $t=1$, evaluate

the integrals (a) $\int_c \phi\, d\vec{r}$, (b) $\int_c \vec{F} \times d\vec{r}$

sol^n: Given,

$\phi = 2xyz^2$

$= 2t^2 . 2t . t^6$

$= 4t^9$

$\therefore \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$= t^2\hat{i} + 2t\hat{j} + t^3\hat{k}$

$\therefore d\vec{r} = (2t\hat{i} + 2\hat{j} + 3t^2\hat{k})dt$

(a) $\int_c \phi\, d\vec{r} = \int_0^1 4t^9 (2t\hat{i} + 2\hat{j} + 3t^2\hat{k})\,dt$

$= \hat{i}\int_0^1 8t^{10}\,dt + \hat{j}\int_0^1 8t^9\,dt + \hat{k}\int_0^1 12t^{11}\,dt$

$= \hat{i}\left[\frac{8t^{11}}{11}\right]_0^1 + \hat{j}\left[\frac{8t^{10}}{10}\right]_0^1 + \hat{k}\left[\frac{12t^{12}}{12}\right]_0^1$

$$= \hat{i}\left(\frac{8}{11} - 0\right) + \hat{j}\left(\frac{8}{10} - 0\right) + \hat{k}(1-0)$$

$$= \frac{8}{11}\hat{i} + \frac{8}{10}\hat{j} + \hat{k}$$ Ans:

(b) $\vec{F} = xy\hat{i} - z\hat{j} + x^2\hat{k}$

$$= 2t^3\hat{i} - t^3\hat{j} + t^4\hat{k}$$

$$\vec{F} \times d\vec{r} = (2t^3\hat{i} - t^3\hat{j} + t^4\hat{k}) \times (2t\,\hat{i} + 2\,\hat{j} + 3t^2\,\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t^3 & -t^3 & t^4 \\ 2t & 2 & 3t^2 \end{vmatrix} dt$$

$$= \left[\hat{i}(-3t^5 - 2t^4) - \hat{j}(6t^5 - 2t^5) + \hat{k}(4t^3 + 2t^4)\right] dt$$

$$\therefore \int_c \vec{F} \times d\vec{r} = \hat{i}\int_0^1 (-3t^5 - 2t^4)\,dt - \hat{j}\int_0^1 (6t^5 - 2t^5)\,dt$$

$$+ \hat{k}\int_0^1 (4t^3 + 2t^4)\,dt$$

$$= -\frac{9}{10}\hat{i} - \frac{2}{3}\hat{j} + \frac{7}{5}\hat{k}$$ **Ans!**