Pair of Straight line:

Q. Show that the homogeneous quadratic eq^n $ax^2 + 2hxy + by^2 = 0$ represents two straight lines.

Solve: Given that,

$$ax^2 + 2hxy + by^2 = 0 \quad \text{——— (1)}$$

If $b \neq 0$, dividing both sides of eq^n-(1) by x^2b

We have,

$$\frac{ax^2 + 2hxy + by^2}{x^2b} = 0$$

$$\Rightarrow \frac{a}{b} + \frac{2hy}{xb} + \cancel{b}.\frac{y^2}{x^2\cancel{b}} = 0$$

$$\Rightarrow \left(\frac{y}{x}\right)^2 + 2\frac{h}{b}.\left(\frac{y}{x}\right) + \frac{a}{b} = 0 \quad \text{——— (ii)}$$

Let,

m_1, m_2 be the roots of eq^n – (ii) in y/x

$$\therefore m_1 + m_2 = -\frac{2h}{b}$$

$$\therefore m_1 m_2 = \frac{a}{b}$$

Procedure!

$$ax^2 + bx + c = 0$$

$$m_1 + m_2 = -\frac{b}{a}$$

$$m_1 m_2 = ab$$

Eq^n-(ii) must be equivalent to

$$\left(\frac{y}{x}\right)^2 - \left(\frac{y}{x}\right)(m_1+m_2) + m_1 m_2 = 0$$

$$\Rightarrow \left(\frac{y}{x}\right)^2 - m_2\left(\frac{y}{x}\right) - m_1\left(y/x\right) + m_1 m_2 = 0$$

$$\Rightarrow \frac{y}{x}\left(\frac{y}{x} - m_2\right) - m_1\left(\frac{y}{x} - m_2\right) = 0$$

$$\therefore \left(\frac{y}{x} - m_2\right)\left(\frac{y}{x} - m_1\right) = 0$$

$$\therefore \frac{y}{x} - m_1 = 0 \qquad \Bigg| \qquad \frac{y}{x} - m_2 = 0$$

$$\therefore y - m_1 x = 0 \qquad \Bigg| \qquad \therefore y - m_2 x = 0$$

Then eq^n - (1) represents two straight lines.

General eq^n of Second degree :

The general eq^n of 2nd degree :

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

will represent :

(1) a pair of straight line :

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

(i) two parallel lines of straight line if $\Delta = 0$, $ab = h^2$

(ii) two perpendicular lines, if $\Delta = 0$, $a + b = 0$

2. a circle if $a = b$, $h = 0$
3. a parabola if $ab = h^2$, $\Delta \neq 0$
4. a ellipse if $ab - h^2$, $\Delta \neq 0$, $ab - h^2 > 0$
5. a hyperbola if $ab - h^2 < 0$, $\Delta \neq 0$
6. a rectangular hyperbola

The general ~~eq~~ eqⁿ of 2nd degree:

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$\Rightarrow abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

Point of intersection:

$$\alpha = \frac{hf - bg}{ab - h^2}$$

$$\beta = \frac{gh - af}{ab - h^2}$$

Angle : $\theta = \tan^{-1} \frac{\sqrt{2h^2 - ab}}{a+b}$

Bisectors of angle : $\frac{(x-\alpha)^2 - (y-\beta)^2}{a-b}$

$$= \frac{(x-\alpha)(y-\beta)}{h}$$

*** Q. Prove that $3y^2 - 8xy - 3x^2 - \cancel{2gx} - 29x + 3y - 18 = 0$ represents two straight lines. Find point of intersection, angle between them and eq^n of bisectors.

Solve:

Given,

$$3y^2 - 8xy - 3x^2 - 29x + 3y - 18 = 0 \quad \text{---}(1)$$

Comparing eq^n-① with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

We get,

$a = -3$ $\qquad$ $h = -4$

$b = 3$ $\qquad$ $g = -\frac{29}{2}$

$c = -18$ $\qquad$ $f = \frac{3}{2}$

Now,

$\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2$

$= (-3 \times 3 \times -18) + 2 \times \frac{3}{2} \times \left(-\frac{29}{2}\right) \times (-4)$

$\quad - (-3) \times \left(\frac{3}{2}\right)^2 - 3 \times \left(-\frac{29}{2}\right)^2 - (-18) \times (-4)^2$

$= 0$

Let, point of intersection be

$\therefore \alpha = \dfrac{hf - bg}{ab - h^2}$

$= \dfrac{\left(-4 \times \frac{3}{2}\right) + 3\left(-\frac{29}{2}\right)}{-3 \times 3 - (-4)^2}$

$= -\frac{3}{2}$

$$\beta = \frac{gh - bg}{ab - h^2}$$

$$= \frac{-\frac{29}{2} \times (-4) - \left(-3 \times \frac{3}{2}\right)}{-3 \times 3 - (-4)^2}$$

$$= -\frac{5}{2}$$

Let, θ be the angle between the lines,

$$\theta = \tan^{-1} \frac{\sqrt{2h^2 - ab}}{a+b}$$

$$= \tan^{-1} \frac{\sqrt{2 \times (-4)^2 - (-3 \times 3)}}{-3 + 3}$$

$$= 90^\circ$$

Eqn of bisectors:

$$\frac{(x-\alpha)^2-(y-\beta)^2}{a-b} = \frac{(x-\alpha)(y-\beta)}{h}$$

$$\Rightarrow \frac{\left\{x-\left(-\frac{3}{2}\right)\right\}^2-\left\{y-\left(-\frac{5}{2}\right)\right\}^2}{-3-3} = \frac{\left\{x-\left(-\frac{3}{2}\right)\right\}\left\{y-\left(-\frac{5}{2}\right)\right\}}{-4}$$

$$\Rightarrow \frac{\left(x+\frac{3}{2}\right)^2-\left(y+\frac{5}{2}\right)^2}{-6} = \frac{\left(x+\frac{3}{2}\right)\left(y+\frac{5}{2}\right)}{-4}$$

$$\Rightarrow \frac{x^2+2\cdot x\cdot\frac{3}{2}+\left(\frac{3}{2}\right)^2-\left\{y^2+2\cdot y\cdot\frac{5}{2}+\left(\frac{5}{2}\right)^2\right\}}{-6} = \frac{\left(x+\frac{3}{2}\right)\left(y+\frac{5}{2}\right)}{-4}$$

$$\Rightarrow \frac{x^2+6x+\frac{9}{4}-y^2-5y-\frac{25}{4}}{-6} = \frac{\left(xy+\frac{5}{2}x+\frac{3}{2}y+\frac{15}{4}\right)}{-4}$$

$$\Rightarrow -4x^2+6x\times(-4)-9+4y^2+20y+25$$

$$= -6xy-15x-9y-\frac{90}{4}$$

$\Rightarrow -4x^2 - 24x - 9 + 4y^2 + 20y + 25$

$= -6xy - 15x - 9y - \frac{90}{4}$

$\Rightarrow 4y^2 - 4x^2 - 9x + 29y + 6xy + \frac{77}{2} = 0$

$\Rightarrow 8y^2 - 8x^2 - 18x + 58y + 12xy + 77 = 0$

Q. Find the value of λ or k, So that the following eqn represent pair of straight line:

1. $\lambda x^2 + 4xy + y^2 - 4x - 2y - 3 = 0$
2. $6x^2 + xy + ky^2 - 11x + 43y - 35 = 0$
3. $x^2 - \lambda xy + 2y^2 + 3x - 5y + 2 = 0$

Solve:

1. Given that,

$\lambda x^2 + 4xy + y^2 - 4x - 2y - 3 = 0$ —(1)

Comparing eq^n (1) with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

We get,

$a = \lambda$ $\quad$ $g = -2$

$h = 2$ $\quad$ $f = -1$

$b = 1$ $\quad$ $c = -3$

Now,

$\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$-3\lambda + 2.(-1).(-2).2 - \lambda.(-1)^2 - (-2)^2 - (-3).2^2 = 0$

$\Rightarrow -3\lambda + 8 - \lambda - 4 + 12 = 0$

$\Rightarrow -4\lambda + 16 = 0$

$\therefore \lambda = 4$ (Ans)

CamScanner

Q. Prove that $2y^2 - xy - x^2 + y + 2x - 1 = 0$ represents two straight lines. find point of intersection, angle between them and eq^n of bisectors.

Solve:

Given that,

$$2y^2 - xy - x^2 + y + 2x - 1 = 0 \quad \text{—— (1)}$$

Comparing eq^n (1) with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$a = -1$ $\qquad$ $h = -\frac{1}{2}$

$b = 2$ $\qquad$ $g = 1$

$c = -1$ $\qquad$ $f = \frac{1}{2}$

Now,

$$\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2$$

$$= -1 \times 2 \times (-1) + 2 \cdot \frac{1}{2} \cdot 1 \cdot \left(-\frac{1}{2}\right) - (-1) \cdot \left(\frac{1}{2}\right)^2 - 2 \times 1^2 - (-1) \cdot \left(-\frac{1}{2}\right)^2$$

$$= 0$$

let, the point of intersection,

$$\alpha = \frac{hf - bg}{ab - h^2}$$

$$= \frac{-\frac{1}{2} \times \frac{1}{2} - 2 \times 1}{-1 \times 2 - (-\frac{1}{2})^2}$$

$$= 1$$

$$\beta = \frac{gh - af}{ab - h^2}$$

$$= \frac{1 \times (-\frac{1}{2}) - (-1) \times \frac{1}{2}}{-1 \times 2 - (-\frac{1}{2})^2}$$

$$= 0$$

let, θ be the angle between the lines,

$$\theta = \tan^{-1} \frac{\sqrt{2h^2 - ab}}{a+b}$$

$$= \tan^{-1} \frac{\sqrt{2 \times (-\frac{1}{2})^2 - (-1) \times 2}}{-1 + 2}$$

$$= \tan^{-1} \frac{\sqrt{10}}{2}$$

$$= 57.68°$$

Eqn of bisectors :

$$\frac{(x-\alpha)^2-(y-\beta)^2}{a-b}=\frac{(x-\alpha)(y-\beta)}{h}$$

$$\Rightarrow \frac{(x-1)^2-(y-0)^2}{-1-2}=\frac{(x-1)(y-0)}{-\frac{1}{2}}$$

$$\Rightarrow \frac{(x^2-2x+1)-y^2}{-3}=\frac{xy-y}{-\frac{1}{2}}$$

$$\Rightarrow \frac{1}{2}\left\{(x^2-2x+1)-y^2\right\}=3xy-3y$$

$$\Rightarrow \frac{1}{2}x^2-x+\frac{1}{2}-\frac{1}{2}y^2=3xy-3y$$

$$\Rightarrow \frac{1}{2}x^2-x+\frac{1}{2}-\frac{1}{2}y^2-3xy+3y=0$$

$$\Rightarrow x^2-2x+1-y^2-6xy+6y=0$$

$$\therefore x^2-y^2-6xy-2x+6y+1=0$$

(Ans)

Q. Reduce $x^2 + 12xy - 4y^2 - 6x + 4y + 9 = 0$

The standard form and identify it.

Solve:

Given,

$$x^2 + 12xy - 4y^2 - 6x + 4y + 9 = 0 \quad \text{--- (1)}$$

Comparing eq^n - (1) with,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

We get,

$a = 1$ $\quad$ $h = 6$

$b = -4$ $\quad$ $g = -3$

$c = 9$ $\quad$ $f = 2$

Now,

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

$$= 1 \times (-4) \times 9 + 2 \times 2 \times (-3) \times 6 - 1 \times 2^2$$

$$- (-4) \times (-3)^2 - 9 \times 6^2$$

$$= -400 \neq 0$$

$ab - h^2 = -40 < 0$

Thus, the given eqn represents a hyperbola.

Let, (α, β) be the centre of the conic

$$\therefore \alpha = \frac{hf - bg}{ab - h^2}$$

$$= \frac{6 \times 2 - (-1) \times (-3)}{1 \times (-4) - 6^2}$$

$$= 0$$

$$\therefore \beta = \frac{hg - af}{ab - h^2}$$

$$= \frac{6 \times (-3) - 1 \times 2}{1 \times (-4) - 6^2}$$

$$= \frac{1}{2}$$

$\therefore$ center is at $(0, \frac{1}{2})$

Now, transfer the origin to the center $(0, 1/2)$

Then, eq^n - ① becomes,

$$x^2 + 12xy - 4y^2 + c_1 = 0 \quad \text{------ (ii)}$$

Where,

$$c_1 = g\alpha + f\beta + c$$

$$= (-3) \times 0 + 2 \times \frac{1}{2} + 9$$

$$= 10$$

from eq^n (ii),

$$x^2 + 12xy - 4y^2 + 10 = 0 \quad \text{------ (iii)}$$

Now, transfer the axes such that remove the xy term, then

$$a_1 x^2 + b_1 y^2 + 10 = 0 \quad \text{——— (iv)}$$

Now, by theory of invariants.

We have,

$$a_1 + b_1 = a + b = -3$$

$$\therefore a_1 + b_1 = -3 \quad \text{——— (v)}$$

$$\therefore a_1 b_1 = ab - h^2 = -40$$

Now,

$$(a_1 - b_1)^2 = (a_1 + b_1)^2 - 4a_1 b_1$$

$$= (-3)^2 - 4 \times (-40)$$

$$= 9 + 160$$

$$= 169$$

$$\therefore a_1 - b_1 = 13 \quad \text{——— (vi)}$$

Adding (v) & (vi),

$$a_1 + b_1 = -3$$
$$a_1 - b_1 = 13$$
$$\overline{2a_1 \quad = 10}$$

$$\therefore a_1 = 5$$

From eq^n-(v),

$$\therefore b_1 = -8$$

From eq^n - (vi),

$$5x^2 - 8y^2 + 10 = 0$$

$$\Rightarrow 5x^2 - 8y^2 = -10$$

$$\therefore \frac{x^2}{-2} + \frac{y^2}{5/4} = 1$$

Which is the required standard eq^n.

Three Dimension

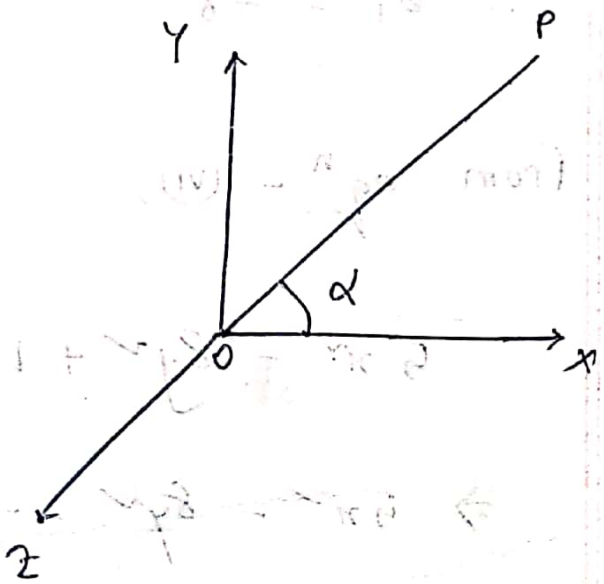
Direction Cosines of a line:

If a given line OP makes angles α, β, γ with the positive direction of x, y and z respectively then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the line OP and are generally denoted by l, m, n that is

$l = \cos\alpha$

$m = \cos\beta$

$n = \cos\gamma$



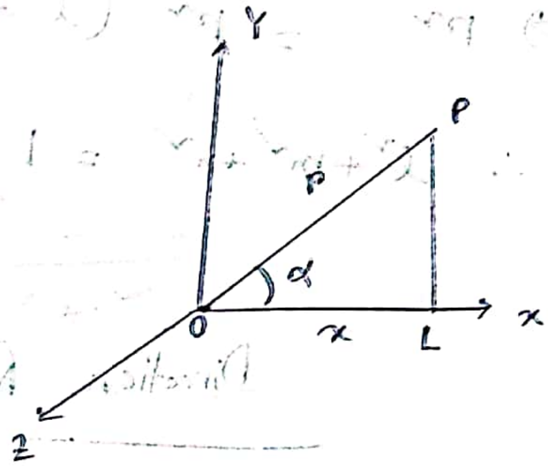
Q. Prove that $l^2+m^2+n^2=1$

Solve:

Let, op be drawn through the origin parallel to the given line so that l, m, n are the direction cosine of op.

Let (x, y, z) be the coordinates of any point P on the line.

Let, $OP = r$



Now, from right angled triangle OPL.

We have,

$$\frac{OL}{OP} = \cos\alpha$$

$$\Rightarrow \frac{x}{r} = \cos\alpha$$

$$\Rightarrow x = r\cos\alpha$$

$$\therefore x = lr$$

Similarly,

We can write

$$y = mr$$

$$z = nr$$

$$\therefore x^2 + y^2 + z^2 = r^2 (l^2 + m^2 + n^2)$$

$$\Rightarrow r^2 = r^2 (l^2 + m^2 + n^2)$$

$$\therefore l^2 + m^2 + n^2 = 1$$

Direction Ratios of Line

Any three numbers a, b, c which are proportional to the direction cosines l, m, n respectively of a given line are called the direction ratios of the given line.

$$\therefore \frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2+b^2+c^2}} = \pm\frac{1}{\sqrt{a^2+b^2+c^2}}$$

$$\therefore l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}$$

$$\therefore m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}$$

$$\therefore n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

Angle between two lines:

If (l_1, m_1, n_1) & (l_2, m_2, n_2) be the direction cosine of any two lines and θ be the angle between them, then

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

Condition of perpendicularity of two lines:

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

Condition of parallelism of two lines:

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} = 1$$

Q. Find the angle between the lines whose direction cosines are related by $l+m+n=0$ and $l^2+m^2-n^2=0$

Solve:

Given that,

$$l+m+n=0 \quad \text{——} \quad (1)$$

$$l^2+m^2-n^2=0 \quad \text{——} \quad (2)$$

from eqn - ①,

$$n=-(l+m) \quad \text{——} \quad (3)$$

using (3) in (2) we get,

$$l^2+m^2-\{-(l+m)\}^2=0$$

$$\Rightarrow l^2+m^2-(l+m)^2=0$$

$$\Rightarrow l^2+m^2-l^2-2ml-m^2=0$$

$$\Rightarrow -2ml=0$$

$$\Rightarrow ml=0$$

$$\therefore m=0 \quad \text{——} \quad (4)$$

$$\therefore l=0 \quad \text{——} \quad (5)$$

Now, from eq^n (1) & (5) we have,

$$l+m+n = 0 \text{ ——— } (1)$$

$$1.l + 0.m + 0.n = 0 \text{ ——— } (5)$$

$$\therefore \frac{l}{0-0} = \frac{m}{1-0} = \frac{n}{0-1}$$

$$\Rightarrow \frac{l}{0} = \frac{m}{1} = \frac{n}{-1} = \frac{1}{\sqrt{0^2+1^2+1^2}} = \frac{1}{\sqrt{2}}$$

$$\therefore l = 0, \quad m = \frac{1}{\sqrt{2}}, \quad n = -\frac{1}{\sqrt{2}}$$

Again, from eq^n (1) & (4)

$$l+m+n = 0$$

$$0.l + 1.m + 0.n = 0$$

$$\therefore \frac{l}{0-1} = \frac{m}{0-0} = \frac{n}{1-0}$$

$$\Rightarrow \frac{l}{-1} = \frac{m}{0} = \frac{n}{1} = \frac{1}{\sqrt{1^2+0^2+1^2}} = \frac{1}{\sqrt{2}}$$

$$\therefore l = -\frac{1}{\sqrt{2}}, \quad m = 0, \quad n = \frac{1}{\sqrt{2}}$$

Let, θ be the angle between the lines,

$$\therefore \cos\theta = \pm (l_1 l_2 + m_1 m_2 + n_1 n_2)$$

$$= \pm \frac{1}{2}$$

$$\therefore \cos\theta = \cos\frac{\pi}{3} \text{ or } \cos\frac{2\pi}{3}$$

$$\therefore \theta = \frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$

The plane:

General Equation of a plane : $ax + by + cz + d = 0$

Eq^n of a plane passing through (x_1, y_1, z_1):

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

Eq^n of a plane passing through (x_1, y_1, z_1) and (x_2, y_2, z_2).

$$(x - x_1)(y - y_2) - (x - x_2)(y - y_1)$$

$$= A\{(x - x_1)(z - z_1) - (x - x_2)(z - z_1)\}$$

Eq^n of a plane through 3 points :

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}$$

Q. Find the eq^n of the plane through $(2,3,1)$, $(1,1,3)$ and $(2,2,3)$.

Solve:

The eq^n of any plane through $(2,3,1)$ is :

$$a(x-2)+b(y-3)+c(z-1)=0 \quad \text{——— (1)}$$

Since eq^n (1) passes through $(1,1,3)$ and $(2,2,3)$ from eq^n (1),

$$a(1-2)+b(1-3)+c(3-1)=0$$

$$\Rightarrow -a-2b+2c=0$$

$$\therefore a+2b-2c=0 \quad \text{——— (ii)}$$

Again,

$$a(2-2)+b(2-3)+c(3-1)=0$$

$$\Rightarrow -b+2c=0$$

$$\therefore b-2c=0 \quad \text{——— (iii)}$$

Solving eq^n (ii) & (iii) by cross multiplication a, b, c we have,

$$\frac{a}{-4+2} = \frac{b}{\cancel{0-1}\ -2-0} = \frac{c}{1-0}$$

$$\Rightarrow \frac{a}{-2} = \frac{b}{-2} = \frac{c}{1}$$

Putting these values in eq^n (1) we have,

$$-2(x-2) + (-2)(y-3) + 1\cdot(z-1) = 0$$

$$\Rightarrow -2x+4-2y+6+z-1 = 0$$

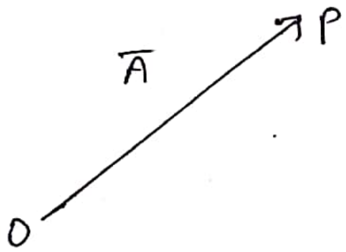
$$\Rightarrow 2x-2y-z+3 = 0$$

(Ans)

Vactors and Scalars

A vector is a quantity having both magnitude and direction.

Example: Displacement, velocity, force, acceleration etc



A salar is a quantity having magnitude but no direction. Ex: Mass, length, time, Temeperature etc.

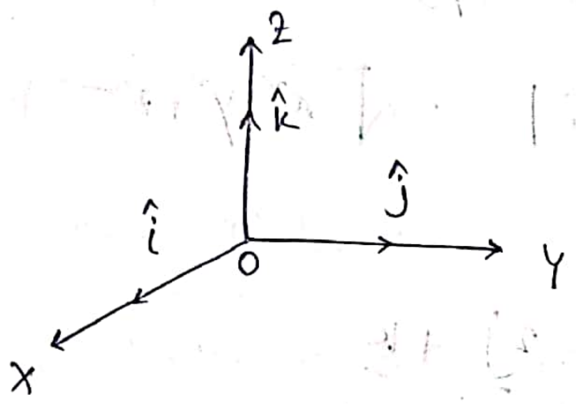
Unit Vactor: A unit vactor is a Vactor having limit magnitude.

If $\overline{A}$ is a vactor, then the unit vactor in the direction of $\overline{A}$ is

$$\hat{a} = \frac{\overline{A}}{|\overline{A}|}$$

Rectangular unit vectors:

A set of unit vectors in the directions the positive x, y, z axes and denoted $\hat{i}$, $\hat{j}$, $\hat{k}$



Components of a vector:

$$\overline{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$$

A_1, A_2, A_3 are called the rectangular components of $\overline{A}$ in the x, y, z directions

Magnitude of $\overline{A}$ is

$$A = \sqrt{A_1^2 + A_2^2 + A_3^2}$$

Position Vector: The position vector $\overline{r}$ from O to the point (x, y, z) is

$$\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\therefore r = |\overline{r}| = \sqrt{x^2+y^2+z^2}$$

Q. $\overrightarrow{r_1} = 3\underline{i} - 2\underline{j} + \underline{k}$

$\overrightarrow{r_2} = 2\underline{i} - 4\underline{j} - 3\underline{k}$

$\overrightarrow{r_3} = -\underline{i} + 2\underline{j} + 2\underline{k}$

1. $|\vec{r_1}| = \sqrt{3^2+(-2)^2+1^2} = \sqrt{14}$

2. $\overrightarrow{r_1} + \overrightarrow{r_2} + \overrightarrow{r_3}$

$= 3\underline{i} - 2\underline{j} + \underline{k} + 2\underline{i} - 4\underline{j} - 3\underline{k} - \underline{i} + 2\underline{j} + 2\underline{k}$

$= 4\underline{i} - 4\underline{j} + 0$

$= 4\underline{i} - 4\underline{j}$

(iii) $2\bar{r}_1 - 3\bar{r}_2 - 5\bar{r}_3$

$= 2(3\underline{i} - 2\underline{j} + \underline{k}) - 3(2\underline{i} - 4\underline{j} - 3\underline{k}) - 5(-\underline{i} + 2\underline{j} + 2\underline{k})$

$= 6\underline{i} - 4\underline{j} + 2\underline{k} - 6\underline{i} + 12\underline{j} + 9\underline{k} + 5\underline{i} - 10\underline{j} - 10\underline{k}$

$= 5\underline{i} - 4\underline{j}$

The Dot and Cross Product:

The dot or Scalar product of two vectors $\bar{A}$ and $\bar{B}$ denoted by $\bar{A}\cdot\bar{B}$ is defined by $\bar{A}\cdot\bar{B} = AB\cos\theta$, $0 \leq \theta \leq \pi$

Properties:

1. $\bar{A}\cdot\bar{B} = \bar{B}\cdot\bar{A}$
2. $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$
3. $m(\bar{A}\cdot\bar{B}) = (m\bar{A})\bar{B} = \bar{A}(m\bar{B}) = (\bar{A}\bar{B})m$
4. $\underline{i}\cdot\underline{i} = \underline{j}\cdot\underline{j} = \underline{k}\cdot\underline{k} = 1$

   $\underline{i}\cdot\underline{j} = \underline{j}\cdot\underline{k} = \underline{k}\cdot\underline{i} = 0$

5. $\bar{A} = A_1 \underline{i} + A_2 \underline{j} + A_3 \underline{k}$

$\bar{B} = B_1 \underline{i} + B_2 \underline{j} + B_3 \underline{k}$

$\bar{A} \cdot \bar{B} = A_1 B_1 + A_2 B_2 + A_3 B_2$

6. $\bar{A} \cdot \bar{B} = 0$ are perpendecular.

The cross or Vector product of two Vectors $\bar{A}$ and $\bar{B}$ denoted by $\bar{A} \times \bar{B}$ is defined by

$$\bar{A} \times \bar{B} = \eta AB \sin\theta \qquad 0 \leq \theta \leq \pi$$

Where η is the unit vector indicating the direction of $\bar{A} \times \bar{B}$.

Properties:

1. $\bar{A} \times \bar{B} = -\bar{B} \times \bar{A}$
2. $\bar{A} \times (\bar{B} + \bar{C}) = \bar{A} \times \bar{B} + \bar{A} \times \bar{C}$

3. $m(\bar{A} \times \bar{B}) = (m\bar{A}) \times \bar{B} = \bar{A} \times (m\bar{B}) = (\bar{A} \times \bar{B})m$

4. $\underline{i} \times \underline{i} = \underline{j} \times \underline{j} = \underline{k} \times \underline{k} = 0$

$\underline{i} \times \underline{j} = \underline{k}, \quad \underline{j} \times \underline{k} = \underline{i}, \quad \underline{k} \times \underline{i} = \underline{j}$

6. $\bar{A} = A_1 \underline{i} + A_2 \underline{j} + A_3 \underline{k}$

$\bar{B} = B_1 \underline{i} + B_2 \underline{j} + B_3 \underline{k}$

$$\bar{A} \times \bar{B} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$\bar{A} \times \bar{B} = 0$ are parallel.

Example:

8. Find the angle between $\overline{A} = 2\underline{i} + 2\underline{j} - \underline{k}$ $\overline{B} = 6\underline{i} - 3\underline{j} + 2\underline{k}$

Solve:

$$A = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

$$B = \sqrt{6^2 + (-3)^2 + 2^2} = 7$$

We know,

$$\overline{A} \cdot \overline{B} = AB \cos\theta$$

$$\Rightarrow (2\underline{i} + 2\underline{j} - \underline{k}) \cdot (6\underline{i} - 3\underline{j} + 2\underline{k}) = 3 \cdot 7 \cos\theta$$

$$\Rightarrow (12 - 6 - 2) = 21 \cos\theta$$

$$\Rightarrow \frac{4}{21} = \cos\theta$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{4}{21}\right)$$

$$\therefore \theta = 79.01^\circ$$

13. Find the projection of the vector $A = \underline{i} - 2\underline{j} + \underline{k}$ on the vector $B = 4\underline{i} - 4\underline{j} + 7\underline{k}$

Solve:

The projection of $\bar{A}$ on the $\bar{B} = \dfrac{\bar{A} \cdot \bar{B}}{|\bar{B}|}$

$$= \frac{(\underline{i} - 2\underline{j} + \underline{k})(4\underline{i} - 4\underline{j} + 7\underline{k})}{\sqrt{4^2 + (-4)^2 + 7^2}}$$

$$= \frac{19}{9} \quad \text{(Ans)}$$

28. If $\bar{A} = 2\underline{i} - 3\underline{j} - \underline{k}$ and $B = \underline{i} + 4\underline{j} - 2\underline{k}$

find (a) $\bar{A} \times \bar{B}$ (b) $\bar{B} \times \bar{A}$ (c) $(\bar{A} + \bar{B}) \times (\bar{A} - \bar{B})$

Solve:

(a) $\bar{A} \times \bar{B} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -3 & -1 \\ 1 & 4 & -2 \end{vmatrix}$

$$= \underline{i}(6+4) - \underline{j}(-4+1) + \underline{k}(8+3)$$

$$= 10\underline{i} + 3\underline{j} + 11\underline{k}$$

(b) $\bar{B} \times \bar{A} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 4 & -2 \\ 2 & -3 & \end{vmatrix}$

$= \underline{i}(-4-6) - \underline{j}(-1+4) + \underline{k}(-3-8)$

$= -10\underline{i} - 3\underline{j} + (-11)\underline{k}$

$= -10\underline{i} - 3\underline{j} - 11\underline{k}$

(c) $\bar{A} + \bar{B} = 2\underline{i} - 3\underline{j} - \underline{k} + \underline{i} + 4\underline{j} - 2\underline{k}$

$= 3\underline{i} + \underline{j} - 3\underline{k}$

$\bar{A} - \bar{B} = 2\underline{i} - 3\underline{j} - \underline{k} - \underline{i} - 4\underline{j} + 2\underline{k}$

$= \underline{i} - 7\underline{j} + \underline{k}$

$(\bar{A}+\bar{B}) \times (\bar{A}-\bar{B}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 1 & -3 \\ 1 & -7 & 1 \end{vmatrix}$

$(-21-1)$

$= \underline{i}(1-21) - \underline{j}(3+3) + \underline{k}(\cancel{\cdot 1+7})$

$= -20\underline{i} - 6\underline{j} - 22\underline{k}$

CamScanner

$$\boxed{\text{Procedure}}$$

$$A = A_1 i + A_2 j + A_3 k$$

$$B = B_1 i + B_2 j + B_3 k$$

$$C = C_1 i + C_2 j + C_3 k$$

$$A \cdot (B \times C) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

Q. Evaluate $(2i - 3j) \cdot [(i + j - k) \times (3j - k)]$

Solve:

$$(2i - 3j) \cdot [(i+j-k) \times (3j-k)] = \begin{vmatrix} 2 & -3 & 0 \\ 1 & 1 & -1 \\ 0 & 3 & -1 \end{vmatrix}$$

$$= 2(-1+3) - (-3)(0-3) + 0$$

$$= 4 - 9$$

$$= -5$$

Solve pdf → Exercise : 57, 58, 63, 82, 83

Gradient, Divergence, Curl

*. Define করো:

Gradient:

$$\nabla\varphi = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)\varphi$$

$$= \frac{\partial \varphi}{\partial x} i + \frac{\partial \varphi}{\partial y} j + \frac{\partial \varphi}{\partial z} k$$

Divergence:

$$\nabla \cdot V = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot \left(V_1 i + V_2 j + V_3 k\right)$$

$$= \frac{\partial}{\partial x} V_1 + \frac{\partial}{\partial y} V_2 + \frac{\partial}{\partial z} V_3$$

Curl:

$$\nabla \times V = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial k}\right) \times \left(V_1 i + V_2 j + V_3 k\right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

1Q. P

Solve

3Q.

Solve

(a)

φ

1Q. If $\varphi(x,y,z) = 3x^2y - y^3z^2$, find $\nabla\varphi$ at the point $(1,-2,-1)$.

Solve:

$$\nabla\varphi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(3x^2y - y^3z^2\right)$$

$$= 6xy\,\hat{i} + (3x^2 - 3y^2z^2)\,\hat{j} + (0 - 2zy^3\hat{k})$$

$$= 6xy\,\hat{i} + (3x^2 - 3y^2z^2)\,\hat{j} - 2zy^3\,\hat{k}$$

$$= 6\times1\times(-2)\,\hat{i} + \{3.1^2 - 3.(-2)^2\times(-1)^2\}\,\hat{j} - 2\times(-1)\times(-2)^3\,\hat{k}$$

$$= -12\hat{i} - 9\hat{j} - 16\hat{k}$$

3Q. Find $\nabla\varphi$ if (a) $\varphi = \ln|r|$, (b) $\varphi = \frac{1}{r}$

Solve:

(a) $r = x\hat{i} + y\hat{j} + z\hat{k}$

$$|r| = \sqrt{x^2+y^2+z^2}$$

$$\varphi = \ln(r)$$

$$= \ln\sqrt{x^2+y^2+z^2}$$

$$\therefore \varphi = \frac{1}{2}\ln(x^2+y^2+z^2)$$

$$\nabla\varphi = \frac{1}{2}\nabla\varphi$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(\ln(x^2+y^2+z^2)\right)$$

$$= \frac{1}{2}\left(\frac{2x}{x^2+y^2+z^2}\hat{i} + \frac{2y}{x^2+y^2+z^2}\hat{j} + \frac{2z}{x^2+y^2+z^2}\right)$$

(b) $$\nabla\varphi = \nabla\frac{1}{r}$$

$$= \nabla\cdot\frac{1}{\sqrt{x^2+y^2z^2}}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial z}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2}\hat{i} + \frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2}\hat{j}$$

$$+ \frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2}\hat{k}$$

$$= \left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2x\right\}\hat{i} + \left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2y\right\}\hat{j}$$

$$+ \left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2z\right\}\hat{k}$$

$$= \frac{-x\hat{i} - y\hat{j} - z\hat{k}}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{-r}{r^3} \quad (\text{Ans})$$

10 Q. Find the directional derivative of $\varphi = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction $2i - j - 2k$.

Solve:

$$\nabla\varphi = \nabla(x^2yz + 4xz^2)$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2yz + 4xz^2)$$

$$= (2xyz + 4z^2)\hat{i} + (x^2z + 0)\hat{j} + (x^2y + 8xz)\hat{k}$$

$$= \{2\times1\times(-2)\times(-1) + 4\times(-1)^2\}\hat{i} + (1^2\cdot(-1) + 0)\hat{j} + \{1^2\cdot(-2) + 8\cdot1\cdot(-1)\}\hat{k}$$

$$= (4+4)\hat{i} - \hat{j} - 10\hat{k}$$

$$= 8\hat{i} - \hat{j} - 10\hat{k}$$

The unit vector in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$ is

$$a = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{2^2 + (-1)^2 + (-2)^2}}$$

$$= \frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$$

Then the required directional derivative is,

$$\nabla\varphi \cdot a = (8\hat{i} - \hat{j} - 10\hat{k}) \cdot \left(\frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}\right)$$

$$= \frac{16}{3} + \frac{1}{3} + \frac{20}{3}$$

$$= \frac{37}{3}$$

Since this is positive, φ is increasing of this maximum.

15. If $A = x^2 z\hat{i} - 2y^3 z^2 \hat{j} + xy^2 z\hat{k}$ find $\nabla . A$ at the point $(1, -1, 1)$.

Solve:

$$\nabla . A = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) . \left(x^2 z\hat{i} - 2y^3 z^2 \hat{j} + xy^2 z\hat{k}\right)$$

$$= 2xz - 6y^2 z^2 + xy^2$$

$$= 2 \times 1 \times 1 - 6 \times (-1)^2 \times 1^2 + 1 \times (-1)^2$$

$$= 2 - 6 + 1$$

$$= -3$$

16. Given $\phi = 2x^3 y^2 z^4$

(a) Find $\nabla . \nabla \phi$ (or div grad ϕ)

(b) Show that $\nabla . \nabla \phi = \nabla^2 \phi$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ denotes the Laplacian operator.

Solve:

(a) $$\nabla\varphi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(2x^3y^2z^4\right)$$

$$= 6x^2y^2z^4\hat{i} + 4x^3yz^4\hat{j} + 8x^3y^2z^3\hat{k}$$

Then,

$$\nabla\cdot\nabla\varphi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(6x^2y^2z^4\hat{i} + 4x^3yz^4\hat{j} + 8x^3y^2z^3\hat{k}\right)$$

$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

(b) $$\nabla\cdot\nabla\varphi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(\frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial z}\right)$$

$$= \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\varphi$$

$$= \nabla^2\varphi$$

Example : 17, 23, 24, 37
Ex : 62, 73

22. Determine the constant a so that the vector $V = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$ is solenoidal.

Solve:

A vector $\vec{V}$ is solenoidal if its divergence is zero.

$$\vec{\nabla}\cdot\vec{V} = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left\{(x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}\right\} = 0$$

$$\Rightarrow 1 + 1 + a = 0$$

$$\Rightarrow a = -2$$

23. If $A = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4k$). find $\nabla \times A$ at the point $(1, -1, 1)$

Solve:

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cancel{xz^3} & \cancel{-2x^2yz} & \cancel{2yz^4} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$\cancel{= \hat{i}\left(\frac{\partial}{\partial y}2yz^4 + \frac{\partial}{\partial z}2x^2yz\right) - \hat{j}\left(\frac{\partial}{\partial x}\right.}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}2yz^4 + \frac{\partial}{\partial z}2x^2yz\right) - \hat{j}\left(\frac{\partial}{\partial x}2yz^4 - \frac{\partial}{\partial z}xz^3\right)$$
$$+ \hat{k}\left(\frac{\partial}{\partial x}(-2x^2yz) - \frac{\partial}{\partial y}xz^3\right)$$

$$= \hat{i}\left(2z^4 + 2x^2y\right) - \hat{j}\left(-3xz^2\right) + \hat{k}\left(-4xyz\right)$$

$$= \left(2z^4 + 2x^2y\right)\hat{i} + 3xz^2\,\hat{j} - 4xyz\,\hat{k}$$

$$= \left\{2.1^4 + 2.1^2.(-1)\right\}\hat{i} + 3.1.1^2\,\hat{j} - 4.1.(-1).1\,\hat{k}$$

$$= 3\hat{j} + 4\hat{k} \quad \text{at } (1,-1,1)$$

24. If $\vec{A} = x^2y\,\hat{i} - 2xz\,\hat{j} + 2yz\,\hat{k}$,

find Curl Curl $\vec{A}$.

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & -2xz & 2yz \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y} 2yz + \frac{\partial}{\partial z} 2xz\right) - \hat{j}\left(\frac{\partial}{\partial x} 2yz - \frac{\partial}{\partial z} x^2 y\right)$$

$$+ \hat{k}\left(\frac{\partial}{\partial x}(-2xz) - \frac{\partial}{\partial y}(x^2 y)\right)$$

$$= \hat{i}\,(2z + 2x) - \hat{j}\,(0) + \hat{k}\,(-2z - x^2)$$

$$= (2z + 2x)\,\hat{i} - (2z + x^2)\,\hat{k}$$

$\cancel{= 2x\hat{j}}$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2z+2x) & 0 & -(2z+2x^2) \end{vmatrix}$$

$$\cancel{= \left\{\frac{\partial}{\partial x}\left(-(2z+2x^2)\right) - 0\right\}\hat{i} - \hat{j}\left\{\frac{\partial}{\partial}\right.}$$

$$= \left\{\frac{\partial}{\partial y}\left(-2z-2x^2\right) - 0\right\}\hat{i} - \hat{j}\left\{\frac{\partial}{\partial x}\left(-2z-2x^2\right) - \frac{\partial}{\partial z}(2z+2x)\right\}$$

$$+ \hat{k}\left\{\frac{\partial}{\partial x}\times 0 - \frac{\partial}{\partial y}(2z+2x)\right\}$$

$$= 4x\,\hat{j} + 2\hat{j}$$

30. If $\overline{V} = \overline{\omega} \times \overline{r}$, prove $\omega = \frac{1}{2}$ curl V where w is a constant vector.

Soluc:

$$\text{curl } \overline{V} = \overline{\nabla} \times \overline{V} = \overline{\nabla} \times (\overline{\omega} \times \overline{r})$$

$$\overline{\nabla} \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \overline{\nabla} \times \left[\hat{i}\,(\omega_2 z - \omega_3 y) - \hat{j}\,(\omega_1 z - \omega_3 x) + \hat{k}\,(\omega_1 y - \omega_2 x)\right]$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & -(\omega_1 z - \omega_3 x) & (\omega_1 y - \omega_2 x) \end{vmatrix}$$

$$= \hat{i}\,(\omega_1 + \omega_1) - \hat{j}\,(-\omega_2 - \omega_2) + \hat{k}\,(\omega_3 + \omega_3)$$

$$= 2\omega_1 \hat{i} + 2\omega_2 \hat{j} + \omega_3 \hat{k}$$

$$= 2(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k})$$

$$= 2\omega$$

then,

$$\omega = \frac{1}{2} \bar{\nabla} \times \bar{V}$$

$$= \frac{1}{2} \text{ curl } V$$

(proved)

32.(a) A vector $\bar{V}$ is called irrotational if curl $\bar{V} = 0$

$$\bar{V} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k}$$

Find constant a, b, c, so that is irrotational.

Solve:

If curl $\bar{V} = 0$ then $\bar{V}$ is called irrotational.

$$\text{curl } \bar{V} = \bar{\nabla} \times \bar{V}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$= \hat{i}(c+1) - \hat{j}(4-a) + \hat{k}(b-2)$$

This equal zero when $a = 1$, $b = 2$, $c = -1$
and,

$$\bar{V} = (x + 2y + 4z)\,\hat{i} + (2x - 3y - z)\,\hat{j} + (4x - 1y + 2z)\,\hat{k}$$

37. If $\bar{A} = 2yz\,\hat{i} - x^2y\,\hat{j} + xz^2\,\hat{k}$

$$\bar{B} = x^2\,\hat{i} + yz\,\hat{j} - xy\,\hat{k}$$

$$\varphi = 2x^2yz^3$$

(a) $(A\cdot\bar{\nabla})\varphi$ (b) $A\cdot\nabla\varphi$ (c) $(B\cdot\nabla)\varphi$ (d) $(A\times\bar{\nabla})\varphi$
(e) $A\times\bar{\nabla}\varphi$

Solve:

(a) $$(A\cdot\bar{\nabla})\varphi = \left[\left(2yz\,\hat{i} - x^2y\,\hat{j} + xz^2\,\hat{k}\right)\cdot\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\right]\varphi$$

$$= \left[2yz\frac{\partial}{\partial x} - x^2y\frac{\partial}{\partial y} + xz^2\frac{\partial}{\partial z}\right]\varphi$$

$$= \left[2yz\frac{\partial}{\partial x} - x^2y\frac{\partial}{\partial y} + xz^2\frac{\partial}{\partial z}\right]\cdot(2x^2yz^3)$$

$$= \left[2yz\frac{\partial}{\partial x}(2x^2yz^3) - x^2y\frac{\partial}{\partial y}(2x^2yz^3) + xz^2\frac{\partial}{\partial z}(2x^2yz^3)\right]$$

$$= 2yz\cdot 4xyz^3 - x^2y\cdot 2x^2z^3 + xz^2\cdot 6x^2yz^2$$

$$= 8xy^2z^4 - 2x^4yz^3 + 6x^3yz^4$$

(b) $\bar{A} \cdot \bar{\nabla}\varphi$

$$= \left(2yz\,\hat{i} - x^2y\,\hat{j} + xz^2\,\hat{k}\right) \cdot \left\{\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot 2x^2yz^3\right\}$$

$$= \left(2yz\,\hat{i} - x^2y\,\hat{j} + xz^2\,\hat{k}\right) \cdot \left(4xyz^3\hat{i} + 2x^2z^3\,\hat{j} + 6x^2yz^2\hat{k}\right)$$

$$= 8xy^2z^4 - 2x^4yz^3 + 6x^3yz^4$$

(c) $(B \cdot \nabla)\,\varphi$

$$= \left\{\left(x^2\,\hat{i} + yz\,\hat{j} - xy\,\hat{k}\right)\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\right\}\left(2x^2yz^3\right)$$

$$= \left(x^2\frac{\partial}{\partial x} + yz\frac{\partial}{\partial y} - xy\frac{\partial}{\partial z}\right)\left(2x^2yz^3\right)$$

$$= x^2\frac{\partial}{\partial x}2x^2yz^3 + yz\frac{\partial}{\partial y}2x^2yz^3 - xy\frac{\partial}{\partial z}2x^2yz^3$$

$$= x^2 \cdot 4xyz^3 + yz \cdot 2x^2z^3 - xy \cdot 2x^2y$$

$$= 4x^3yz^3 + 2x^2yz^4 - 2x^3y^2$$

(d) $(\bar{A} \times \bar{\nabla})\varphi$

$$(\bar{A} \times \bar{\nabla})\varphi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2yz & -x^2y & xz^2 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} \cdot \varphi$$

$$= \left[\hat{i}\left(-x^2y\frac{\partial}{\partial z} - xz^2\frac{\partial}{\partial y}\right) - \hat{j}\left(2yz\frac{\partial}{\partial z} - xz^2\frac{\partial}{\partial x}\right) + \hat{k}\left(2yz\frac{\partial}{\partial y} + x^2y\frac{\partial}{\partial x}\right)\right]\varphi$$

$$= \left[\hat{i}\left(-x^2y\frac{\partial}{\partial z}\varphi - xz^2\frac{\partial}{\partial y}\varphi\right) - \hat{j}\left(2yz\frac{\partial}{\partial z}\varphi - xz^2\frac{\partial}{\partial x}\varphi\right) + \hat{k}\left(2yz\frac{\partial}{\partial y}\varphi + x^2y\frac{\partial}{\partial x}\varphi\right)\right]$$

$$= \left[\hat{i}\left(-x^2y\frac{\partial}{\partial z}(2x^2yz^3) - xz^2\frac{\partial}{\partial y}(2x^2yz^3)\right) - \hat{j}\left(2yz\frac{\partial}{\partial z}2x^2yz^3 - xz^2\frac{\partial}{\partial x}2x^2yz^3\right) + \hat{k}\left(2yz\frac{\partial}{\partial y}2x^2yz^3 + x^2y\frac{\partial}{\partial x}2x^2yz^3\right)\right]$$

$$= \left(-x^2y \cdot 6x^2yz^2 - xz^2 \cdot 2x^2z^3\right)\hat{i} - \hat{j}\left(2yz \cdot 6x^2yz^2 - xz^2 \cdot 4xyz^3\right) + \hat{k}\left(2yz \cdot 2x^2z^3 + x^2y \cdot 4xyz^3\right)$$

$$= -\left(6x^4y^2z^2 + 2x^3z^5\right)\hat{i} - \left(12x^2y^2z^3 - 4x^2yz^5\right)\hat{j} + \left(4x^2yz^4 + 4x^3y^2z^3\right)\hat{k}.$$

(Ans)

Vector Integration

Q. The acceleration of a particle at any time $t > 0$ is given by $a = \frac{dv}{dt} = 12\cos 2t\,\hat{i} - 8\sin 2t\,\hat{j} + 16t\,\hat{k}$
If the velocity v and displacement r are zero at $t = 0$ and r at any time.

Solve:

$$V = \hat{i}\int 12\cos 2t\,dt + \hat{j}\int -8\sin 2t + \hat{k}\int 16t\,dt$$

$$= \hat{i}\cdot 12\cdot\frac{\sin 2t}{2} + \hat{j}\left(-8\cdot\frac{-\cos 2t}{2}\right) + \hat{k}\cdot 16\,\frac{t^2}{2}$$

$$= 6\sin 2\hat{i}\,t + 4\cos 2t\,\hat{j} + 8t^2\hat{k} + c_1$$

Putting $v = 0$ when $t = 0$, we find,

$$0 = 6\cdot\sin 2\cdot 0\,\hat{i} + 4\cdot\cos 2\cdot 0\,\hat{j} + \cancel{8t^2\hat{k}}\ 8\cdot 0\,\hat{k}$$

$$= 0\,\hat{i} + 4\hat{j} + 16\cdot 0\,\hat{k} + c_1$$

$$= 4\hat{j} + c_1$$

$$\therefore c_1 = -4\hat{j}$$

$$\bar{v} = 6\sin 2t\,\hat{i} + (4\cos 2t - 4)\,\hat{j} + 8t^2\,\hat{k}$$

$$\bar{V} = \frac{d\bar{r}}{dt}$$

$$\Rightarrow d\bar{r} = \bar{V}\,dt$$

$$\Rightarrow \bar{r} = \int d\bar{r} = \int \bar{V}\,dt$$

$$\Rightarrow \bar{r} = \hat{i}\int 6\sin 2t\,dt + \hat{j}\int (4\cos 2t - 4)\,dt + \hat{k}\int 8t^2\,dt$$

$$= -3\cos 2t\,\hat{i} + (2\sin 2t - 4t)\,\hat{j} + \frac{8}{3}t^3\,\hat{k} + c_2$$

$t = 0,\ \bar{r} = 0$

$$\Rightarrow 0 = -3\hat{i} + c_2$$

$$\therefore c_2 = -3\hat{i}$$

$$\therefore \bar{r} = (3 - 3\cos 2t)\,\hat{i} + (2\sin 2t - 4t)\,\hat{j} + \frac{8}{3}t^3\,\hat{k}$$

Q. If $\bar{A} = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$ evaluate $\int_c \bar{A}\, d\bar{r}$. from $(0,0,0)$ to $(1,1,1)$ along the path.

$c: x = t,\ y = t^2,\ z = t^3$

Solve:

Let, $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$d\bar{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$$\int_c \bar{A}\, d\bar{r} = \int_c \left[(3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}\right](dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \int_c (3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz$$

$$= \int_{t=0}^{1} (3t^2+6t^2)\,dt - 14t^2.t^3.dt^2 + 20.t.t^3 dt^3$$

$$= \int_{t=0}^{1} [9t^2\,dt - 14t^5.2t\,dt + 20t^7.3t^2\,dt]$$

$$= \int_{t=0}^{1} \left[9t^2 dt - 28t^6 dt + 60t^9 dt \right]$$

$$= \left[9 \cdot \frac{t^3}{3} - 28 \cdot \frac{t^7}{7} + 60 \cdot \frac{t^{10}}{10} \right]_{t=0}^{1}$$

$$= \left[3t^3 - 4t^7 + 6t^{10} \right]_{t=0}^{1}$$

$$= 3 \cdot 1^3 - 4 \cdot 1^7 + 6 \cdot 1^9$$

$$= 3 - 4 + 6$$

$$= 5$$

6. $\varphi = 2xyz^2 = 2(t^2)\cdot(2t)\cdot(t^3)^2 = 4t^9$

$\bar{F} = xy\,\hat{i} - z\,\hat{j} + x^2\,\hat{k}$

$c:\; x = t^2,\; y = 2t,\; z = t^3$ from $t=0$ to $t=1$

$\bar{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$

$= t^2\,\hat{i} + 2t\,\hat{j} + t^3\,\hat{k}$

$\Rightarrow d\bar{r} = 2t\,\hat{i} + 2\,\hat{j} + 3t^2\,\hat{k}$ [Differentiation]

(i) $\cancel{\int_c \varphi\, d\bar{r} = 2xyz^2 \int_c (2t\,\hat{i} + 2\,\hat{j} + 3t^2\,\hat{k})}\, dt$

$= \cancel{2xyz^2 \int_c (2t\,\hat{i} + 2\,\hat{j} + 3t^2\,\hat{k})\, dt}$

$= \cancel{2xyz^2 \left(\cancel{2}\cdot\frac{t^2}{\cancel{2}}\,\hat{i} + 2t\,\hat{j} + 3\cdot\frac{t^3}{3}\,\hat{k}\right)}$

$= \cancel{2xyz^2}$

$=$

(i) $\int_c \varphi \, d\bar{r} = \int_c (2xyz^2) \left(2t\,\hat{i} + 2\hat{j} + 3t^2\,\hat{k}\right) dt$

$= \int_0^1 (4t^9) \cdot \left(2t\,\hat{i} + 2\hat{j} + 3t^2\,\hat{k}\right) dt$

$= \int_0^1 \left(8t^{10}\,\hat{i} + 8t^9\,\hat{j} + 12t^{11}\right) dt$

$= \left[\frac{8}{11}t^{11}\,\hat{i} + \frac{8}{10}t^{10}\,\hat{j} + \frac{12}{12}t^{12}\,\hat{k}\right]_0^1$

$= \frac{8}{11}\,\hat{i} + \frac{8}{10}\,\hat{j} + 1\,\hat{k}$

(ii) $E = xy\,\hat{i} - z\,\hat{j} + x^2\,\hat{k}$

$= 2t^3\,\hat{i} - t^3\,\hat{j} + t^4\,\hat{k}$

$$\bar{F} \times d\bar{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t^3 & -t^3 & t^4 \\ 2t & 2 & 3t^2 \end{vmatrix}$$

$$= \hat{i}\left(-3t^5 - 2t^4\right) - \hat{j}\left(6t^5 - 2t^5\right) + \hat{k}\left(4t^3 + 2t^4\right)$$

$$= \hat{i}\int_0^1 \left(-3t^5 - 2t^4\right) dt - \hat{j}\int_0^1 \left(6t^5 - 2t^5\right) dt + \hat{k}\int_0^1 \left(4t^3 + 2t^4\right) dt$$

$$= \left[\left(-3.\frac{t^6}{6} - 2.\frac{t^5}{5}\right)\hat{i} - \left(t - \frac{t^6}{3}\right)\hat{j} + \left(t^4 + \frac{2}{5}t^5\right)\hat{k}\right]_0^1$$

$$= \left[\left(-\frac{1}{2} - \frac{2}{5}\right)\hat{i} - \left(1 - \frac{1}{3}\right)\hat{j} + \left(1 + \frac{2}{5}\right)\hat{k}\right]$$

$$= -\frac{9}{10}\hat{i} - \frac{2}{3}\hat{j} + \frac{7}{5}\hat{k}$$

(Ans)