Gradient, Divergence and Curl

Q. Find the directional derivative of $Q = x^2yz + 4xz^2$ at $(-1, -2, -1)$ in the equation $2\hat{i} - \hat{j} - 2\hat{k}$

Sol^n: Given,

$$Q = x^2yz + 4xz^2$$

$$\therefore \vec{\nabla} Q = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k} \right)$$

$$(x^2yz + 4xz^2)$$

$$= \frac{\delta}{\delta x}(x^2yz+4xz^2)\hat{i} + \frac{\delta}{\delta y}(x^2yz+4xz^2)\hat{j} + \frac{\delta}{\delta z}(x^2yz+4xz^2)\hat{k}$$

$$= (2xyz+4z^2)\hat{i} + (x^2z\hat{j}) + (x^2y+8xz)\hat{k}$$

At the point $(1,-2,-1)$

$$\overline{A}Q = \{2\cdot 1\cdot(-2)\cdot(-1)+4(-1)^2\}\hat{i} + (1)^2\cdot(-2)\hat{j} + \{(1)^2\cdot(-2) + 8\cdot(1)\cdot(-1)\}\hat{k}$$

$$= 8\hat{i} - \hat{j} - 10\hat{k}$$

The unit vector in the direction of $2\hat{i}-\hat{j}-2\hat{k}$

$$\hat{a} = \frac{2\hat{i}-\hat{j}-2\hat{k}}{\sqrt{2^2+(-1)^2+(-2)^2}}$$

$= \frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$

$\therefore$ The required directional derivative is $\vec{\nabla}Q \cdot \hat{a}$

$$= (8\hat{i} - \hat{j} - 10\hat{k}) \cdot \left(\frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}\right)$$

$$= \frac{16}{3} + \frac{1}{3} + \frac{20}{3}$$

$$= \frac{37}{3} \quad \underline{\text{Ans:}}$$

Q:] Prove that $\nabla^2\left(\frac{1}{r}\right) = 0$

$\underline{\text{Sol}^n:}$

$$\nabla^2\left(\frac{1}{r}\right) = \left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\frac{\delta}{\delta x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\delta}{\delta x}(x^2+y^2+z^2)^{-1/2}$$

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x$$

$$= -x\,(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\delta}{\delta x^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)^2 = -x\frac{\delta}{\delta x}(x^2+y^2+z^2)^{-3/2}$$

$$+ (x^2+y^2+z^2)^{-3/2}\frac{\delta}{\delta x}(-x)$$

$$= -x(-3/2)(x^2+y^2+z^2)^{-5/2}.2x$$

$$-(x^2+y^2+z^2)^{-3/2}$$

$$= 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$

$$= \frac{3x^2}{(x^2+y^2+z^2)^{5/2}} - \frac{1}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{3x^2-x^2+y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$$

$$= \frac{2x^2-y^2+z^2}{(x^2+y^2+z^2)^{5/2}}$$

Similarly, $\frac{\delta^2}{\delta y^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$

$$=\frac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}$$

and $\frac{\delta^2}{\delta z^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$

$$=\frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\therefore\left(\frac{\delta^2}{\delta x^2}+\frac{\delta^2}{\delta y^2}+\frac{\delta^2}{\delta z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$=0$$

$\therefore \nabla^2\left(\frac{1}{r}\right)=0$ (Proved)

Q. Determine the constant a so that the vector $\vec{V}=(x+3y)\hat{i}+(y-2z)\hat{j}+(x+az)\hat{k}$ is solenoidal.

Sol^n:

Given,

$\vec{V} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$.

$\vec{\nabla}\cdot\vec{V} = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)$.
$[(x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}]$

$= \frac{\delta}{\delta x}(x+3y) + \frac{\delta}{\delta y}(y-2z) + \frac{\delta}{\delta z}(x+az)$

$= 1+1+a$

$= 2+a$

According to the question,

$\vec{\nabla}\cdot\vec{V} = 0$

$\Rightarrow 2+a = 0$

$\therefore a = -2$ (Ans:)

$= 4\hat{i} - 2\hat{j}$

chapter - 4

Gradient, Divergence and curl

$$\vec{\nabla} \equiv \frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}$$

Q. If, $Q(x,y,z) = 3x^2y - y^3z^2$, find $\vec{\nabla}\phi$ (or grad ϕ) at the point $(1,-2,-1)$

Solⁿ:

Given, $\phi(x,y,z) = 3x^2y - y^3z^2$

$$\vec{\nabla}\phi = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)(3x^2y - y^3z^2)$$

$$= \frac{\delta}{\delta x}(3x^2y - y^3z^2)\hat{i} + \frac{\delta}{\delta y}(3x^2 - y^3z^2)\hat{j} + \frac{\delta}{\delta z}(3x^2y - y^3z^2)\hat{k}$$

$$= 6xy\hat{i} + (3x^2 - 3y^2z^2)\hat{j} - 2y^3z\hat{k}$$

At the point $(1,-2,-1)$

$$\vec{\nabla}\phi = 6\cdot1(-2)\hat{i} + \{3\cdot1^2 - 3\cdot(-2)^2 + 1^2\}\hat{j} - 2(-2)^3(-1)\cdot\hat{k}$$

$$= -12\hat{i} - 9\hat{j} - 16\hat{k}$$

Q.2 If

$\vec{A} = x^2 z\hat{i} - 2y^3 z^2\hat{j} + xy^2 z\hat{k}$, find

$\vec{\nabla}.\vec{A}$ (or div $\vec{A}$) at the point

$(1, -1, 1)$

Solⁿ: Given,

$$\vec{A} = x^2 z\hat{i} - 2y^3 z^2\hat{j} + xy^2 z\hat{k}$$

$$\therefore \vec{\nabla}\vec{A} = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right).\left(x^2 z\hat{i} - 2y^3 z^2\hat{j} + xy^2 z\hat{k}\right)$$

$$= \frac{\delta}{\delta x}(x^2 z) + \frac{\delta}{\delta y}(-2y^3 z^2) + \frac{\delta}{\delta z}(xy^2 z)$$

$$= 2xz - 6y^2 z^2 + xy^2$$

At the point $(1, -1, 1)$

$$\therefore \vec{\nabla}.\vec{A} = 2.1.1 - 6.(-1)^2 + 1.(-1)^2$$

$$= 2 - 6 + 1$$

$$= -3 \quad \text{Ans:}$$

Q.3 If $\vec{A} = x^2 z\,\hat{i} - 2y^3 z^2\,\hat{j} + xy^2 z\,\hat{k}$,
find $\vec{\nabla}\times\vec{A}$ (or curl $\vec{A}$) at the point $(1, -1, 1)$

Sol^n; Given,

$$\vec{A} = x^2 z\,\hat{i} - 2y^3 z^2\,\hat{j} + xy^2 z\,\hat{k}$$

$$\therefore \vec{\nabla}\times\vec{A} = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\times\left(x^2 z\,\hat{i} - 2y^3 z^2\,\hat{j} + xy^2 z\,\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x^2 z & -2y^3 z^2 & xy^2 z \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\delta}{\delta y}(xy^2 z) - \frac{\delta}{\delta z}(-2y^3 z^2)\right\} - \hat{j}\left\{\frac{\delta}{\delta x}(xy^2 z) - \frac{\delta}{\delta z}(x^2 z)\right\} + \hat{k}\left\{\frac{\delta}{\delta x}(-2y^3 z^2) - \frac{\delta}{\delta y}(x^2 z)\right\}$$

$$= \hat{i}\,(2xyz + 4y^3z) - \hat{j}\,(y^2z - x^2) + \hat{k}\cdot 0$$

$$= (2xyz + 4y^3z)\,\hat{i} - (y^2z - x^2)\,\hat{j}$$

At the point $(1,-1,1)$,

$$\vec{\nabla} \times \vec{A} = (2\times 1\cdot(-1)\cdot 1 + 4\cdot(-1)^3\cdot 1)\,\hat{i}$$
$$- ((-1)^2\cdot 1 - 1^2)\,\hat{j}$$

$$= (-2-4)\,\hat{i} - (1-1)\,\hat{j}$$

$$= -6\,\hat{i}$$ Ans.