Homogeneous linear partial differential equations with constant coefficients and higher order

A linear partial differential equation with constant coefficients is called homogeneous if all it's derivatives are of the same order. The general form of such an equation is

$$A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \cdots + A_n \frac{\partial^n z}{\partial y^n} = f(x,y) \quad \ldots\ldots\ldots\ldots(1)$$

Where $A_0, A_1, \ldots, A_n$ are constant coefficients.

For example:

1. $3\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 y}{\partial y^2} = o$ homo. of order 2.
2. $2\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 5\frac{\partial^3 z}{\partial x \partial y^2} - 8\frac{\partial^3 y}{\partial y^3} = x + y$ homo. of order 3.

For convenience $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ will be denoted by D or D_x and D' or D_y respectively. Then (1) can be rewritten as:

$$(A_0 D_x^n + A_1 D_x^{n-1} D_{y+\cdots+} A_n D_y^n) z = f(x,y) \quad \ldots\ldots\ldots\ldots(2)$$

On the other hand, when all the derivatives in the given equation are not of the same order, then it is called a non-homogenous linear partial differential equation with constant coefficients.

In this section we propose to study the various methods of solving homogeneous linear partial differential equation with constant coefficients, namely (2).

Equation (2) may rewritten as:

$$\boxed{F(D_x, D_y) z = f(x,y)} \quad \ldots\ldots\ldots\ldots\ldots(3)$$

Where $F(D_x, D_y) = A_0 D_x^n + A_1 D_x^{n-1} D_{y+\cdots+} A_n D_y^n$

Equation (3) has a general solution when $f(x,y) = 0$

i.e $F(D_x, D_y) z = 0$

$$\rightarrow \boxed{(A_0 D_x^n + A_1 D_x^{n-1} D_{y+\cdots+} A_n D_y^n) z = 0.} \ldots\ldots\ldots(4)$$

❖ Now, we will find the general solution of (4)

Let $z = \emptyset(y + mx)$ be a solution of (4) where $\emptyset$ is an arbitrary function and m is a constant , then

$$D_x z = \emptyset'(y + mx).m$$
$$D_x^2 z = \emptyset''(y + mx).m^2$$
$$\vdots$$
$$D_x^n z = \emptyset^{(n)}(y + mx).m^n$$

$$D_y z = \emptyset'(y + mx)$$
$$D_y^2 z = \emptyset''(y + mx)$$
$$\vdots$$
$$D_y^n z = \emptyset^{(n)}(y + mx)$$

$$D_x D_y z = m\,\emptyset'(y + mx)$$
$$D_x^2 D_y z = m^2 \emptyset^{(3)}(y + mx)$$
$$\vdots$$
$$D_x^r D_y^s z = m^r \emptyset^{(r+s)}(y + mx)$$

$= m^r \emptyset^{(n)}(y + mx)$, where $r + s = n$

Substituting these values in (4) and simplifying, we get :

$(A_0 m^n + A_1 m^{n-1} + A_2 m^{n-2} + \dots + A_n)\emptyset^{(n)}(y + mx) = 0$ ...(5)

Which is true if m is a root of the equation

$A_0 m^n + A_1 m^{n-1} + A_2 m^{n-2} + \dots + A_n = 0$ ....(6)

The equation (6) is known as the (characteristic equation) or the (auxiliary equation(A.E.)) and is obtained by putting $D_x = m$ and $D_y = 1$ in $F(D_x, D_y)z = 0$, and it has $n$roots.

Let $m_1, m_2, \dots, m_n$ be n roots of A.E. (6). **Three cases arise:**

Case 1 when the roots are distinct.

If $m_1, m_2, \dots, m_n$ are n distinct roots of A.E. (6) then $\emptyset_1(y + m_1 x), \emptyset_2(y + m_2 x), \dots \dots \dots, \emptyset_n(y + m_n x)$ are the linear solution corresponding to them and since the sum of any linear solutions is a solution too than the general solution in this case is:

$z = \emptyset_1(y + m_1 x) + \emptyset_2(y + m_2 x) + \dots + \emptyset_n(y + m_n x)$ .....(7)

Ex.1: **Find the general solution of**

$$(D_x^3 + 2D_x^2 D_y - 5D_x D_y^2 - 6D_y^3)z = 0$$

Sol. The A.E. is $m^3 + 2m^2 - 5m - 6 = 0$

$$\rightarrow \quad (m+1)(m^2 + m - 6 = 0$$

$$\rightarrow \quad (m+1)(m+3)(m-2) = 0$$

$$m_1 = -1, \quad m_2 = -3, \quad m_3 = 2$$

Note that m_1, m_2 and m_3 are different roots, then the general solution is

$$z = \emptyset_1(y + m_1 x) + \emptyset_2(y + m_2 x) + \emptyset_3(y + m_3 x)$$

$$\rightarrow \quad z = \emptyset_1(y - x) + \emptyset_2(y - 3x) + \emptyset_3(y + 2x)$$

Where $\emptyset_1, \emptyset_2, \emptyset_3$ are arbitrary functions.

Ex.2: **Find the general solution of** $m^2 - a^2 = 0$ **where** a **is a real number.**

Sol. Given that $\quad m^2 - a^2 = 0 \rightarrow m^2 = a^2$

$$\rightarrow \quad m = \pm a \quad \text{different root}$$

$$m_1 = a \quad , m_2 = -a$$

The general solution is

$$z = \emptyset_1(y + ax) + \emptyset_2(y - ax)$$

Where $\emptyset_1, \emptyset_2$ are arbitrary functions.

Case 2 **when the roots are repeated.**

If the root m is repeated k times . i.e. $m_1 = m_2 = \cdots = m_k$,

then the corresponding solution is :

$$\boxed{z = \emptyset_1(y + m_1 x) + x\emptyset_2(y + m_1 x) + \cdots + x^{k-1}\emptyset_n(y + m_1 x)} \quad ..(8)$$

Where $\emptyset_1, \dots, \emptyset_k$ are arbitrary functions.

Note: If some of the roots $m_1, m_2, \dots, m_n$ are repeated and the other are not . i.e. $m_1 = m_2 = \cdots = m_k \neq m_{k+1} \neq \cdots \neq m_n$ then the general solution is :

$$z = \emptyset_1(y + m_1x) + x\emptyset_2(y + m_1x) + \cdots + x^{k-1}\emptyset_n(y + m_1x) + \emptyset_{k+1}(y + m_{k+1}x) + \cdots + \emptyset_n(y + m_nx) \quad \ldots\ldots\ldots\ldots(9)$$

Ex.3: Solve $(D_x^3 - D_x^2D_y - 8D_xD_y^2 + 12D_y^3)z = 0$

Sol. The A.E. is $\quad m^3 - m^2 - 8m + 12 = 0$

$$\rightarrow \quad (m - 2)(m - 2)(m + 3) = 0$$

$m_1 = m_2 = 2 \quad , \quad m_3 = -3$

Then, the general solution is

$$z = \emptyset_1(y + 2x) + x\emptyset_2(y + 2x) + \emptyset_3(y - 3x)$$

Where $\emptyset_1, \emptyset_2, \emptyset_3$ are arbitrary functions.

Case 3 when the roots are complex.

If one of the roots of the given equation is complex let be m_1 then the conjugate of m_1 is also a root, let be m_2, so the general solution is:

$$\boxed{z = \emptyset_1(y + m_1x) + \emptyset_2(y + m_2x) + \cdots + \emptyset_n(y + m_nx)}$$

Where $\emptyset_1, \ldots, \emptyset_n$ are arbitrary functions.

Example 4:

Solve $(D_x^2 + D_y^2)z = 0$

Sol. The A. E. is $\quad m^2 + 1 = 0 \quad \rightarrow \quad m = \pm i$

$$\therefore \quad m_1 = i \quad , \quad m_2 = -i$$

The general solution is

$$z = \emptyset_1(y + ix) + \emptyset_2(y - ix)$$

Where $\emptyset_1, \emptyset_2$ are arbitrary functions.

Example 5:

Solve $(D_x^2 - 2D_xD_y + 5D_y^2)z = 0$

Sol. The A. E. is $\quad m^2 - 2m + 5 = 0$

$$\rightarrow \quad m = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$$

$$\therefore \quad m_1 = 1 + 2i \quad , \quad m_2 = 1 - 2i$$

$$z = \emptyset_1(y + (1+2i)x) + \emptyset_2(y + (1-2i)x)$$

That is the general solution where $\emptyset_1 , \emptyset_2$ are arbitrary functions.

Methods for finding Particular Integral (P.I.)

Method for finding P.I. when $f(x, y)$ is of the form $\emptyset\,(ax + by)$

Working Rule for finding Particular integral when $f(x,y) = \phi(ax+by)$:

(i) when $F(a,b) \neq 0$, then

$$P.I. = \frac{1}{F(D_x, D_y)} \phi(ax+by)$$

$$= \frac{1}{F(a,b)} \iint \cdots \int f(v)\,dv\,dv \cdots dv,$$

where $v = ax+by$

(ii) when $F(a,b) = 0$, then

$$P.I. = \frac{1}{F(D_x, D_y)} \phi(ax+by)$$

$$= \frac{1}{(bD_x - aD_y)^n} \phi(ax+by)$$

$$= \frac{x^n}{b^n\, n!} \phi(ax+by)$$

Ex. 1 Solve $(D^2 + 3DD' + 2D'^2)z = x + y$

Sol. The Auxiliary equation of the given equation is

$$m^2 + 3m + 2 = 0$$

Solving

$$m = -1, -2$$

Therefore

C.F. $= \phi_1(y - x) + \phi_2(y - 2x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now

$$\text{P.I.} = \frac{1}{D^2 + 3DD' + 2D'^2}(x + y)$$

$$= \frac{1}{1^2 + 3\cdot1\cdot1 + 2\cdot1^2}\iint v\, dv\, dv \text{ where } v = (x + y)$$

$$= \frac{1}{6}\int \frac{v^2}{2}\, dv$$

$$= \frac{1}{6}\cdot\frac{v^3}{6}$$

$$= \frac{1}{36}(x + y)^3$$

Hence the required general solution is $z = C.F. + P.I.$

$$z = \phi_1(y - x) + \phi_2(y - 2x) + \frac{1}{36}(x + y)^3$$

Ex. 2. Solve $(2D^2 - 5DD' + 2D'^2)z = 24(y - x)$

Sol: Try yourself

Ex.3. Solve $(D^2 + 3DD' + 2D'^2)z = 2x + 3y$

Sol: Try yourself

Ex: Solve $(D_x^2 + 2D_xD_y + D_y^2)z = e^{2x+3y}$

Sol^n: $m^2 + 2m + 1 = 0$

$\Rightarrow m = -1, -1$

C.F. $= \phi_1(y - x) + x\phi_2(y - x)$

P.I. $= \frac{1}{D_x^2 + 2D_xD_y + D_y^2} e^{2x+3y}$ $\left[\begin{matrix} a = 2 \\ b = 3 \end{matrix}\right.$

$x = 2$, $y = 3$ $\Rightarrow$ $= \frac{1}{2^2 + 2\cdot2\cdot3 + 3^2}\iint e^v\, dv\, dv$, where $v = 2x + 3y$]

$= \frac{1}{25}\int e^v\, dv$

$$= \frac{1}{25} e^{\cdots}$$

$$= \frac{1}{25} e^{2x+3y}$$

$$z = C.F. + P.I.$$

$$= \phi_1(y-x) + x\,\phi_2(y-x) + \frac{1}{25} e^{2x+3y}$$

Ex: Solve $(4D_x^2 - 4D_xD_y + D_y^2)z = 16\log(x+2y)$

Sol^n:

$$4m^2 - 4m + 1 = 0$$

$$\Rightarrow m = \frac{1}{2}, \frac{1}{2}$$

$$C.F. = \phi_1\left(y + \tfrac{1}{2}x\right) + x\,\phi_2\left(y + \tfrac{1}{2}x\right)$$

$$P.I. = \frac{1}{4D_x^2 - 4D_xD_y + D_y^2}\, 16\log(x+2y)$$

$$= \frac{1}{(2D_x - D_y)^2}\, 16\log(x+2y)$$

$$= 16\,\frac{x^2}{2^2 \cdot 2!}\log(x+2y)$$

$$= 2x^2\log(x+2y)$$

$$\therefore z = C.F. + P.I.$$

$$\because \frac{\phi(ax+by)}{(bD_x - aD_y)^n}$$

$a = 1,\ b = 2,\ n = 2$

$$= \frac{x^n}{b^n\, n!}\,\phi(ax+by)$$

Ex: Solve $(D_x^2 - 2D_xD_y + D_y^2)z = \tan(y+x)$

Sol^n: A.E. is,

$m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2 = 0$

$\Rightarrow m = 1, 1$

$\therefore$ C.F $= \phi_1(y-x) + x\,\phi_2(y-x)$

$$P.I. = \frac{1}{D_x^2 - 2D_xD_y + D_y^2}\tan(y+x)$$

$$= \frac{1}{(D_x - D_y)^2}\tan(y+x)$$

$$= \frac{x^2}{1^2 \cdot 2!}\tan(y+x)$$

$$= \frac{x^2}{2}\tan(y+x)$$

$$\frac{1}{(bD_x - aD_y)^n}\phi(ax+by)$$

$a=1,\ b=1$
$n=2$

$$= \frac{x^n}{b^n\, n!}\phi(ax+by)$$

$\therefore$ The g.s. is

$z = C.F. + P.I.$