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Derangement

In combinatorial mathematics, a **derangement** is a permutation of the elements of a set, such that no element appears in its original position. In other words, derangement is a permutation that has no fixed points.

The number of derangements of a set of size n, usually written D_n, d_n, or $!n$, is called the "derangement number" or "de Montmort number". (These numbers are generalized to rencontres numbers.) The **subfactorial** function (not to be confused with the factorial $n!$) maps n to $!n$.[1] No standard notation for subfactorials is agreed upon; $n¡$ is sometimes used instead of $!n$.[2]

The problem of counting derangements was first considered by Pierre Raymond de Montmort[3] in 1708; he solved it in 1713, as did Nicholas Bernoulli at about the same time.

Contents

Example

Suppose that a professor gave a test to 4 students – A, B, C, and D – and wants to let them grade each other's tests. Of course, no student should grade his or her own test. How many ways could the professor hand the tests back to the students for grading, such that no student received his or her own test back? Out of 24 possible permutations (4!) for handing back the tests:

ABCD, **AB**DC, **A**CB**D**, **A**CDB, **A**DBC, **A**D**C**B,
BA**CD**, *BADC*, BCA**D**, *BCDA*, *BDAC*, BD**C**A,
CAB**D**, *CADB*, C**B**A**D**, C**B**DA, *CDAB*, *CDBA*,
DABC, DA**C**B, D**B**AC, D**BC**A, *DCAB*, *DCBA*.

there are only 9 derangements (shown in blue italics above). In every other permutation of this 4-member set, at least one student gets his or her own test back (shown in bold red).

Another version of the problem arises when we ask for the number of ways n letters, each addressed to a different person, can be placed in n pre-addressed envelopes so that no letter appears in the correctly addressed envelope.

Counting derangements

Suppose that there are n people who are numbered 1, 2, ..., n. Let there be n hats also numbered 1, 2, ..., n. We have to find the number of ways in which no one gets the hat having the same number as their number. Let us assume that the first person takes hat i. There are $n-1$ ways for the first person to make such a choice. There are now two possibilities, depending on whether or not person i takes hat 1 in return:

1. Person i does not take the hat 1. This case is equivalent to solving the problem with $n-1$ persons and $n-1$ hats: each of the remaining $n-1$ people has precisely 1 forbidden choice from among the remaining $n-1$ hats (i's forbidden choice is hat 1).
2. Person i takes the hat 1. Now the problem reduces to $n-2$ persons and $n-2$ hats.

From this, the following relation is derived:

$$!n=(n-1)(!(n-1)+!(n-2)).$$

where !n, known as the subfactorial, represents the number of derangements, with the starting values !0 = 1 and !1 = 0.

Also, the following formulae are known:[4]

$$!n=n!\sum_{i=0}^{n}\frac{(-1)^i}{i!},$$

$$!n=\left[\frac{n!}{e}\right]=\left\lfloor\frac{n!}{e}+\frac{1}{2}\right\rfloor,\quad n\geq 1$$

where $[x]$ is the nearest integer function and $\lfloor x\rfloor$ is the floor function.

$$!n=\left\lfloor(e+e^{-1})n!\right\rfloor-\lfloor en!\rfloor,\quad n\geq 2,$$

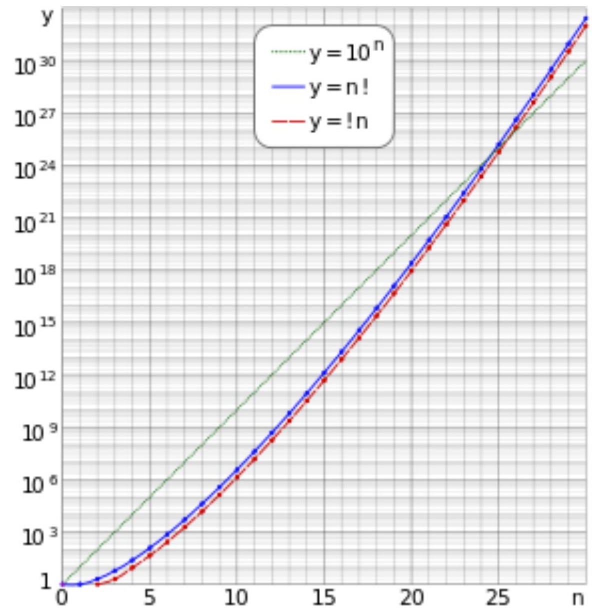
$$!n=n!-\sum_{i=1}^{n}\binom{n}{i}\cdot !(n-i),$$

The following recurrence relationship also holds:[5]

$$!n=n[!(n-1)]+(-1)^n$$

Starting with n = 0, the numbers of derangements of n are:

1, 0, 1, 2, 9, 44, 265, 1854, 14833, 133496, 1334961, 14684570, 176214841, 2290792932, ... (sequence A000166 in the OEIS).



Number of possible permutations and derangements of n elements. $n!$ (n factorial) is the number of n-permutations; $!n$ (n subfactorial) is the number of derangements — n-permutations where all of the n elements change their initial places.

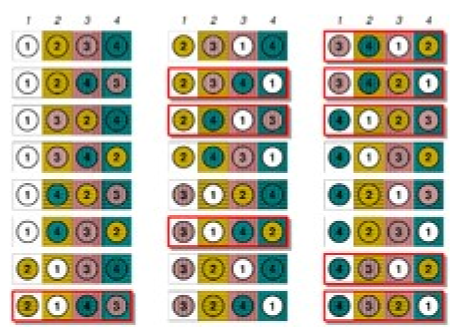
These numbers are also called **subfactorial** or **rencontres numbers**.

A well-known method of counting derangements uses the inclusion–exclusion principle: counting all arrangements, subtracting those that fix at least one element and permute the rest in any way, then adding back those that fix at least two elements, etc.

Table of values

n	Permutations, $n!$	Derangements, $!n$	
0	1 $=1\times10^{0}$	1 $=1\times10^{0}$	=
1	1 $=1\times10^{0}$	0	=
2	2 $=2\times10^{0}$	1 $=1\times10^{0}$	=
3	6 $=6\times10^{0}$	2 $=2\times10^{0}$	≈
4	24 $=2.4\times10^{1}$	9 $=9\times10^{0}$	=
5	120 $=1.20\times10^{2}$	44 $=4.4\times10^{1}$	≈
6	720 $=7.20\times10^{2}$	265 $=2.65\times10^{2}$	≈
7	5 040 $\approx5.04\times10^{3}$	1 854 $\approx1.85\times10^{3}$	≈
8	40 320 $\approx4.03\times10^{4}$	14 833 $\approx1.48\times10^{4}$	≈
9	362 880 $\approx3.63\times10^{5}$	133 496 $\approx1.33\times10^{5}$	≈
10	3 628 800 $\approx3.63\times10^{6}$	1 334 961 $\approx1.33\times10^{6}$	≈
11	39 916 800 $\approx3.99\times10^{7}$	14 684 570 $\approx1.47\times10^{7}$	≈
12	479 001 600 $\approx4.79\times10^{8}$	176 214 841 $\approx1.76\times10^{8}$	≈
13	6 227 020 800 $\approx6.23\times10^{9}$	2 290 792 932 $\approx2.29\times10^{9}$	≈
14	87 178 291 200 $\approx8.72\times10^{10}$	32 071 101 049 $\approx3.21\times10^{10}$	≈
15	1 307 674 368 000 $\approx1.31\times10^{12}$	481 066 515 734 $\approx4.81\times10^{11}$	≈
16	20 922 789 888 000 $\approx2.09\times10^{13}$	7 697 064 251 745 $\approx7.70\times10^{12}$	≈
17	355 687 428 096 000 $\approx3.56\times10^{14}$	130 850 092 279 664 $\approx1.31\times10^{14}$	≈
18	6 402 373 705 728 000 $\approx6.40\times10^{15}$	2 355 301 661 033 953 $\approx2.36\times10^{15}$	≈

19	121 645 100 408 832 000 ≈1.22×10^17	44 750 731 559 645 106 ≈4.48×10^16	≈
20	2 432 902 008 176 640 000 ≈2.43×10^18	895 014 631 192 902 121 ≈8.95×10^17	≈
21	51 090 942 171 709 440 000 ≈5.11×10^19	18 795 307 255 050 944 540 ≈1.88×10^19	≈
22	1 124 000 727 777 607 680 000 ≈1.12×10^21	413 496 759 611 120 779 881 ≈4.13×10^20	≈
23	25 852 016 738 884 976 640 000 ≈2.59×10^22	9 510 425 471 055 777 937 262 ≈9.51×10^21	≈
24	620 448 401 733 239 439 360 000 ≈6.20×10^23	228 250 211 305 338 670 494 289 ≈2.28×10^23	≈
25	15 511 210 043 330 985 984 000 000 ≈1.55×10^25	5 706 255 282 633 466 762 357 224 ≈5.71×10^24	≈
26	403 291 461 126 605 635 584 000 000 ≈4.03×10^26	148 362 637 348 470 135 821 287 825 ≈1.48×10^26	≈
27	10 888 869 450 418 352 160 768 000 000 ≈1.09×10^28	4 005 791 208 408 693 667 174 771 274 ≈4.01×10^27	≈
28	304 888 344 611 713 860 501 504 000 000 ≈3.05×10^29	112 162 153 835 443 422 680 893 595 673 ≈1.12×10^29	≈
29	8 841 761 993 739 701 954 543 616 000 000 ≈8.84×10^30	3 252 702 461 227 859 257 745 914 274 516 ≈3.25×10^30	≈
30	265 252 859 812 191 058 636 308 480 000 000 ≈2.65×10^32	97 581 073 836 835 777 732 377 428 235 481 ≈9.76×10^31	≈



The 9 derangements (from 24 permutations) are highlighted

$$!n = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots \pm \binom{n}{n}0! = n! + \sum_{i=1}^{n}(-1)^i\binom{n}{i}(n-i)!.$$

Factoring out $n!$ gives the formula above, $!n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$.

Limit of ratio of derangement to permutation as n approaches ∞

Using this recurrence, it can be shown that, in the limit,

$$\lim_{n\to\infty} \frac{!n}{n!} = \frac{1}{e} \approx 0.3679\ldots.$$

This is the limit of the probability $p_n = d_n/n!$ that a randomly selected permutation is a derangement. The probability converges to this limit extremely quickly as n increases, which is why d_n is the nearest integer to $n!/e$. The above semi-log graph shows that the derangement graph lags the permutation graph by an almost constant value.

More information about this calculation and the above limit may be found in the article on the statistics of random permutations.

Generalizations

The problème des rencontres asks how many permutations of a size-n set have exactly k fixed points.

Derangements are an example of the wider field of constrained permutations. For example, the *ménage problem* asks if n opposite-sex couples are seated man-woman-man-woman-... around a table, how many ways can they be seated so that nobody is seated next to his or her partner?

More formally, given sets A and S, and some sets U and V of surjections $A \to S$, we often wish to know the number of pairs of functions (f, g) such that f is in U and g is in V, and for all a in A, $f(a) \neq g(a)$; in other words, where for each f and g, there exists a derangement φ of S such that $f(a) = \varphi(g(a))$.

Another generalization is the following problem:

> *How many anagrams with no fixed letters of a given word are there?*

For instance, for a word made of only two different letters, say n letters A and m letters B, the answer is, of course, 1 or 0 according to whether $n = m$ or not, for the only way to form an anagram without fixed letters is to exchange all the A with B, which is possible if and only if $n = m$. In the general case, for a word with n_1 letters X_1, n_2 letters X_2, ..., n_r letters X_r it turns out (after a proper use of the inclusion-exclusion formula) that the answer has the form:

$$\int_0^\infty P_{n_1}(x) P_{n_2}(x) \cdots P_{n_r}(x) e^{-x}\, dx,$$

for a certain sequence of polynomials P_n, where P_n has degree n. But the above answer for the case $r = 2$ gives an orthogonality relation, whence the P_n's are the Laguerre polynomials (up to a sign that is easily decided).[6]

In particular, for the classical derangements

$$!n = \int_0^\infty (x-1)^n e^{-x} \, dx.$$

Computational complexity

It is NP-complete to determine whether a given permutation group (described by a given set of permutations that generate it) contains any derangements.[7]

References

1. The name "subfactorial" originates with William Allen Whitworth; see Cajori, Florian (2011), *A History of Mathematical Notations: Two Volumes in One* (https://books.google.com/books?id=gxrO8ZnMK_YC&pg=RA1-PA77), Cosimo, Inc., p. 77, ISBN 9781616405717.
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3. de Montmort, P. R. (1708). *Essay d'analyse sur les jeux de hazard*. Paris: Jacque Quillau. *Seconde Edition, Revue & augmentée de plusieurs Lettres*. Paris: Jacque Quillau. 1713.
4. Hassani, M. "Derangements and Applications." J. Integer Seq. 6, No. 03.1.2, 1–8, 2003
5. See the notes for (sequence A000166 in the OEIS).
6. Even, S.; J. Gillis (1976). "Derangements and Laguerre polynomials" (http://journals.cambridge.org/action/displayAbstract?fromPage=online&aid=2128316). *Mathematical Proceedings of the Cambridge Philosophical Society*. **79** (01): 135–143. doi:10.1017/S0305004100052154 (https://doi.org/10.1017%2FS0305004100052154). Retrieved 27 December 2011.
7. Lubiw, Anna (1981), "Some NP-complete problems similar to graph isomorphism", *SIAM Journal on Computing*, **10** (1): 11–21, doi:10.1137/0210002 (https://doi.org/10.1137%2F0210002), MR 0605600 (https://www.ams.org/mathscinet-getitem?mr=0605600). Babai, László (1995), "Automorphism groups, isomorphism, reconstruction", *Handbook of combinatorics, Vol. 1, 2* (http://people.cs.uchicago.edu/~laci/handbook/handbookchapter27.pdf) (PDF), Amsterdam: Elsevier, pp. 1447–1540, MR 1373683 (https://www.ams.org/mathscinet-getitem?mr=1373683), "A surprising result of Anna Lubiw asserts that the following problem is NP-complete: Does a given permutation group have a fixed-point-free element?".

External links

- Baez, John (2003). "Let's get deranged!" (http://math.ucr.edu/home/baez/qg-winter2004/derangement.pdf) (PDF).
- Bogart, Kenneth P.; Doyle, Peter G. (1985). "Non-sexist solution of the ménage problem" (http://www.math.dartmouth.edu/~doyle/docs/menage/menage/menage.html).
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