Lecture 5: Conditional Choice Probabilities (CCPs) Based Methods

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1 Introduction

- Finding a fixed point can be computationally demanding.

- Can we somehow skip computing a fixed point when estimating structural parameters?

- Two-step methods do this job.

- A class of two step methods have been proposed in the literature. They are powerful especially in estimating dynamic games.

- To fix the idea, let us discuss the static entry model of Seim (2006).

1.1 Example

1.1.1 Model

- Consider a large number of potential entrants: N_{max}

- Parameterization:

$$\Pi(N, S) = S \cdot (\alpha - \gamma(N-1)) - F$$

- (α, γ, F) are parameters and (N, S) are observed in each market

- ϵ_i is private information: Observed by firm i but not other firms or econometrician. Assume iid across players.

- Hence, firm i must form expectation about other firms entering

- Firm i enters if

$$E(\Pi(N,S)|a_i = enter) + \epsilon_i \geq 0$$

or (linearity allows to move through expectations operator)

$$S \cdot (\alpha - \gamma E(N-1|a_i = enter)) - F + \epsilon_i \geq 0$$

- Assume symmetry

- Let

$$\Pr(Entry) = p = G_\epsilon(E[\Pi(N,S)|a_i = enter])$$

- Hence, firm i enters if

$$S \cdot (\alpha - \gamma[p \cdot (N_{max} - 1)]) - F + \epsilon_i \geq 0$$

- A Bayesian-Nash Equilibrium is given by a fixed point:

$$p = G_\epsilon(S \cdot (\alpha - \gamma[p \cdot (N_{max} - 1)]) - F)$$

- If ϵ_{it} iid extreme value:

$$p = \frac{\exp(S \cdot (\alpha - \gamma[p \cdot (N_{max} - 1)]) - F)}{1 + \exp(S \cdot (\alpha - \gamma[p \cdot (N_{max} - 1)]) - F)}$$

- Likelihood contribution of market with N active firms and market demand S:

$$\ell(N, S, p) = \frac{N_{max}!}{(N_{max} - N)!N!} p^N (1 - p)^{(N_{max} - N)}$$

1.1.2 Estimation

- Nested fixed point algorithm:

1. Fix parameter vector $\theta = (\alpha, \gamma, F)$

2. Find fixed point in

$$p_m = G_\epsilon(S_m \cdot (\alpha - \gamma[p_m \cdot (N_{max} - 1)]) - F)$$

for every market $m = 1...., M$, yielding $p_m(\theta)$

3. Evaluate log-likelihood

$$L_M(\theta) = \frac{1}{M} \sum_{m=1}^{M} \log(\ell(N_m, S_m, p_m(\theta)))$$

4. Use numerical algorithm to find θ that maximizes $L_M(\theta)$

- Alternatively, we could form an moment estimator:

    1. Estimate probability of entry in each market

    $$\hat{p}_m = \frac{N_m}{N_{\mathsf{max}}}$$

    2. Moment condition:

    $$E[\hat{p}_m - G_\epsilon\left(S_m \cdot (\alpha - \gamma[\hat{p}_m \cdot (N_{\mathsf{max}} - 1)]) - F\right)|Z_m] = 0$$

    where Z_m is a vector of exogenous variables (e.g., S_m). The sample analogues is

    $$\frac{1}{M}\sum_{m=1}^{M} Z_m \otimes [\hat{p}_m - G_\epsilon(S_m \cdot (\alpha - \delta[\hat{p}_m \cdot (N_{\mathsf{max}} - 1)]) - F)] = 0$$

- Loosely speaking, two-step methods let the data tell us what agents did in the first stage, and choose structural parameters to explain why they did so in the second stage.

- Advantage of two-step estimators:
  - No need to solve for equilibrium for each trial of parameter vector
  - Avoid problem of multiple equilibria
    - Even with multiplicity, we can calculate the equilibrium probability in each market when N_{max} is large ("picked by the data") as in step 1 above (see social interaction literature, e.g., Brock and Durlauf, 2001)
    - When N_{max} is small, we need to pool different markets
    - Either focusing on unique outcomes or specifying an equilibrium selection rule is needed

2 CCP Methods in Dynamic Models

- Can we apply the same idea to our Rust-type model?

- Recall our Bellman equation

$$V(x_t, \varepsilon_t, \theta) = \max_{i \in D(x_t)} \{u(x_t, i, \theta) + \varepsilon_t(i) + \beta E[V(x_{t+1}, \varepsilon_{t+1}, \theta) | x_t, \varepsilon_t, i]\}$$

- Recall: under CI, the integrated value function formulation in our Rust-type model is

$$\begin{aligned} V(x_t, \theta) = \mathcal{P}_{x_t} & \left[u(x_t, 1, \theta) + \widetilde{\varepsilon}_1^{\mathcal{P}_{x_t}} + \beta E[V(x_{t+1}, \theta) | x_t, 1] \right] \\ & + (1 - \mathcal{P}_{x_t}) \left[u(x_t, 0, \theta) + \widetilde{\varepsilon}_0^{\mathcal{P}_{x_t}} + \beta E[V(x_{t+1}, \theta) | x_t, 0] \right] \end{aligned}$$

where $\widetilde{\varepsilon}_i^{\mathcal{P}_{x_t}} = E[\varepsilon_t(i) | i]$ and $\mathcal{P}_{x_t} \equiv \Pr(i_t = 1 | x_t, \theta)$.

- In matrix notation,

$$\mathbf{V} = \Pi^{\mathcal{P}} + \beta G^{\mathcal{P},p}\mathbf{V}$$

where the j-th element of Π is

$$\Pi_j^{\mathcal{P}} = \mathcal{P}_{x_t}(u(x_t, 1, \theta) + \tilde{\varepsilon}_1^{\mathcal{P}_{x_t}}) + (1 - \mathcal{P}_{x_t})(u(x_t, 0, \theta) + \tilde{\varepsilon}_0^{\mathcal{P}_{x_t}})$$

with $x_t = x_t\,(j)$, and (i, j) element of $G^{\mathcal{P},p}$ is

$$G_{ij}^{\mathcal{P},p} = \mathcal{P}_x p(x'|0, 1, \theta) + (1 - \mathcal{P}_x)p(x'|x, 0, \theta)$$

with $x = x_t(i)$ and $x' = x_t(j)$.

- We get

$$\mathbf{V} = (I - \beta G^{\mathcal{P},p})^{-1}\Pi^{\mathcal{P}}. \tag{1}$$

- Notice that RHS of equation (1) can be computed from $\mathcal{P}_{x_t} \equiv \Pr(i_t = 1|x_t, \theta)$ and $p(x_{t+1}|x_t, i_t, \theta)$.

- The basic idea of CCP-based methods is to estimate $\Pr(i_t = 1|x_t, \theta)$ and $p(x_{t+1}|x_t, i_t, \theta)$ directly from the data in the first stage, and then estimate structural parameters in the second stage.

- The first two-step method is proposed by Hotz and Miller (1993)

- Notice a big difference from the nested fixed point algorithm: we skip solving a fixed point problem.

- From now, let's assume we observe N individual over T time periods. We use n and t as a subscript to index individual and period, respectively: data $\{i_{nt}, x_{nt} : n = 1, ..., N; t = 1, ..., T\}$

- Typically, T is relatively short; asymptotics is in terms of N.

- Suppose x has a finite support. Then we can use

$$\widehat{\mathcal{P}}_x = \frac{\sum_{n=1}^{N}\sum_{t=1}^{T}\mathbf{1}(i_{nt}=1, x_{nt}=x)}{\sum_{n=1}^{N}\sum_{t=1}^{T}\mathbf{1}(x_{nt}=x)}$$

and

$$\widehat{p}(x'|x,i,\theta) = \frac{\sum_{n=1}^{N}\sum_{t=1}^{T-1}\mathbf{1}(x_{nt+1}=x', x_{nt}=x, i_{nt}=i)}{\sum_{n=1}^{N}\sum_{t=1}^{T-1}\mathbf{1}(x_{nt}=x, i_{nt}=i)}$$

Using these, we can compute

$$\widehat{\mathbf{V}} = (I - \beta G^{\widehat{\mathcal{P}},\widehat{p}})^{-1}\Pi^{\widehat{\mathcal{P}}}.$$

- Then,

$$\Pr(i_{nt}=1|x_{nt},\theta) = \Pr\left(\widehat{\bar{V}^1}(x_{nt},\theta) + \varepsilon_{1nt} \geq \widehat{\bar{V}^0}(x_{nt},\theta) + \varepsilon_{0nt}\right) \quad (2)$$

where

$$\widehat{\bar{V}^0}(x_{nt},\theta) = u(x_{nt},0,\theta) + \beta E[\widehat{V}(x_{nt+1},\theta)|x_{nt}, i_{nt}=0,\theta]$$
$$\widehat{\bar{V}^1}(x_{nt},\theta) = u(x_{nt},1,\theta) + \beta E[\widehat{V}(x_{nt+1},\theta)|x_{nt}, i_{nt}=1,\theta].$$

- We can use (2) to form the likelihood for the choices in the data.

- What we do not know are $u(x_{nt}, 0, \theta)$ and $u(x_{nt}, 1, \theta)$ only. The parameters in these functions are estimated by MLE.

- Identification of these structural parameters is discussed later. In general, the discount factor and the distribution of ε should be assumed known.

- Let's make things more formal and general. Now we have K alternatives, and let the choice be denoted by $d \in \{1, ..., K\}$.

- For simplicity, suppose $p(x_{t+1}|x_t, d_t, \theta)$ is known, so we focus on choice probabilities.

- Define

$$\begin{aligned}\Pr(d_{nt} = d | x_{nt}, \theta) &\equiv \Psi\left(d | x_{nt}; \mathcal{P}, \theta\right) \\ &= \Pr\left(d = \arg\max_{d' \in \{1,\ldots,K\}} \left\{ \widehat{\bar{V}^{d'}}\left(x_{nt}, \theta\right) + \varepsilon_{d'nt} \right\}\right)\end{aligned}$$

- Assume (a) the true parameters θ_0 is identified, and (b) the observations are independent across individuals and $\Pr\left(x_{nt} = x\right) > 0$ for all x in $\mathbf{X}$

- Pseudo likelihood function

$$Q_N\left(\theta, \mathcal{P}\right) = \frac{1}{N} \sum_{n=1}^{N} \sum_{t=1}^{T} \ln \Psi\left(d_{nt} | x_{nt}; \mathcal{P}, \theta\right).$$

- The MLE for the nested-fixed point algorithm is

$$\hat{\theta}_{MLE} = \arg\max_{\theta \in \Theta} Q_N\left(\theta, \mathcal{P}\right) \text{ s.t. } \mathcal{P} = \Psi\left(\theta, \mathcal{P}\right).$$

- Finding this is costly because for any θ, one has to find $\mathcal{P}$ as a solution to the fixed point problem.

2.1 Pseudo Maximum Likelihood Estimation

- Suppose that we know the population probabilities $\mathcal{P}^0$ and consider

$$\hat{\theta} \equiv \arg\max_{\theta \in \Theta} Q_N\left(\theta, \mathcal{P}^0\right).$$

- Under standard regularity conditions, this estimator is root-N consistent and asymptotically normal.

- A feasible alternative is

$$\hat{\theta}_{2S} \equiv \arg\max_{\theta \in \Theta} Q_N(\theta, \hat{\mathcal{P}}^0)$$

where $\hat{\mathcal{P}}^0$ is agent's choice probabilities estimated non-parametrically.

Result Suppose (i) Assumptions (a)-(b) hold, (ii) $\Psi(\theta, \mathcal{P})$ is twice continuously differentiable, (iii) Θ is a compact set, (iv) $\theta^0 \in int(\Theta)$, and (v) let $\hat{\mathcal{P}}^0$ be an estimator of $\mathcal{P}^0$ such that $\sqrt{N}(\hat{\mathcal{P}}^0 - \mathcal{P}^0) \to_d N(0, \Sigma)$. Then, $\sqrt{N}(\hat{\theta}_{2S} - \theta^0) \to_d N(0, V_{2S})$, where

$$V_{2S} = \Omega_{\theta\theta}^{-1} + \Omega_{\theta\theta}^{-1}\Omega_{\theta\mathcal{P}}\Sigma\Omega_{\theta\mathcal{P}}'\Omega_{\theta\theta}^{-1}$$

and $\Omega_{\theta\theta} \equiv E\left(\{\nabla_\theta s_n\}\{\nabla_\theta s_n\}'\right)$, with $s_n \equiv \sum_t \ln\Psi\left(d_{nt}|x_{nt}; \mathcal{P}^0, \theta^0\right)$ and $\Omega_{\theta\mathcal{P}} \equiv E\left(\{\nabla_\theta s_n\}\{\nabla_\mathcal{P} s_n\}'\right)$.

- Drawbacks of the two-step PML: the nonparametric estimator of $\mathcal{P}^0$ can be very imprecise; and it cannot deal with unobservable individual type.

2.2 Nested Pseudo Likelihood Method

- Remember our problem is

$$\hat{\theta} = \arg\max_{\theta \in \Theta} Q\left(\theta, \hat{\mathcal{P}}\right).$$

- A recursive extension of the two-step PML estimator: Aguirregabiria and Mira (2002, 2007)

- Take $\hat{\mathcal{P}}_0$, an initial guess of the vector of (possibly non-consistent) agent's choice probabilities.

- A sequence of estimators $\{\hat{\theta}_K : K \geq 1\}$ defined as

$$\hat{\theta}_K = \arg\max_{\theta \in \Theta} Q(\theta, \hat{\mathcal{P}}_{K-1}) \tag{3}$$

  and the probabilities $\{\hat{\mathcal{P}}_K : K \geq 1\}$ are obtained recursively as

$$\hat{\mathcal{P}}_K = \mathbf{\Psi}(\hat{\mathcal{P}}_{K-1}, \hat{\theta}_K). \tag{4}$$

- If this sequence converges, its limit $(\hat{\theta}, \hat{\mathcal{P}})$ satisfies the following two properties: $\hat{\theta}$ maximizes the pseudo likelihood $Q_N(\theta, \hat{\mathcal{P}})$ and $\hat{\mathcal{P}} = \mathbf{\Psi}(\hat{\mathcal{P}}, \hat{\theta})$. Call any pair $(\theta, \mathcal{P})$ that satisfies these properties a NPL fixed point.

- Let $\mathcal{Y}_N$ be the set of NPL fixed points. Then the NPL estimator is defined as

$$(\hat{\theta}_{NPL}, \hat{\mathcal{P}}_{NPL}) = \arg \max_{(\theta, \mathcal{P}) \in \mathcal{Y}_N} Q_N(\theta, \mathcal{P}).$$

- Aguirregabiria and Mira (2007) show that NPL is asymptotically more efficient than the infeasible PML. See also Pesendorfer and Schmidt-Dengler (2010) and Kasahara and Shimotsu (2012).

2.3 Forward Simulation

- Forward simulation is one attractive alternative.

- The Hotz-Miller (1993), Hotz-Miller-Sanders-Smith (1994) idea.

- Let us come back to the simple Rust-type model.

- Recall that under the Extreme value distribution assumption, the choice probability is

$$\Pr(i_t = 1 | x_t, \theta) = \frac{\exp(\bar{V}^1(x_t, \theta))}{\exp(\bar{V}^0(x_t, \theta)) + \exp(\bar{V}^1(x_t, \theta))}$$

which implies:

$$\frac{\Pr(i_t = 1 | x_t, \theta)}{\Pr(i_t = 0 | x_t, \theta)} = \frac{\exp(\bar{V}^1(x_t, \theta))}{\exp(\bar{V}^0(x_t, \theta))}$$

or taking logs:

$$\ln(\Pr(i_t = 1|x_t, \theta)) - \ln(\Pr(i_t = 0|x_t, \theta)) = \bar{V}^1(x_t, \theta) - \bar{V}^0(x_t, \theta)$$

- Hence the choice probabilities (which we can estimate from the data) give us an estimate of the difference in alternative-specific value functions.

- This result (the difference in alternative-specific value functions is uniquely pinned down as a function of CCPs) holds for a wider class of error distributions; "Hotz-Miller inversion".

- Let $\hat{\mathcal{P}}(i, x_t)$ be an estimate of choice probabilities and $\hat{p}(x_{t+1}|x_t, i_t)$ be an estimate of the transition probability.

- Algorithm:

1. Estimate
$$\widehat{\bar{V}^1}(x_t) - \widehat{\bar{V}^0}(x_t) = \ln(\hat{\mathcal{P}}(1, x_t)) - \ln(\hat{\mathcal{P}}(0, x_t))$$
(I took out θ to emphasize that this estimate does not depend on structural parameters; directly recovered from the data)

2. Draw S paths of T-length of random shocks $[(\varepsilon^s_{0,1}, \varepsilon^s_{1,1})$, $(\varepsilon^s_{0,2}, \varepsilon^s_{1,2})$, ..., $(\varepsilon^s_{0,T}, \varepsilon^s_{1,T})]$

3. Given shock $(\varepsilon_{0,t}, \varepsilon_{1,t})$ and x_t we can find optimal choice because $i_t = 1$ if
$$\widehat{\bar{V}^1}(x_t) - \widehat{\bar{V}^0}(x_t) > \varepsilon^s_{0,t} - \varepsilon^s_{1,t}$$
and $i_t = 0$, otherwise.

4. Simulate S paths $(\tilde{x}^s_t, \tilde{i}^s_t)^T_{t=1}$ and corresponding choice specific value func-

tions for given parameters (θ_1, R) :

$$\widetilde{\bar{V}^1}(x_t, \theta_1, R) = -R - c(0, \theta_1) + \frac{1}{S} \sum_{s=1}^{S} \sum_{t=1}^{T} \beta^t \left(u(\tilde{x}_t^s, \tilde{\imath}_t^s, \theta_1, R) + \varepsilon_{\tilde{\imath}_t^s, t}^s \right)$$

$$\widetilde{\bar{V}^0}(x_t, \theta_1, R) = -c(x_t, \theta_1) + \frac{1}{S} \sum_{s=1}^{S} \sum_{t=1}^{T} \beta^t \left(u(\tilde{x}_t^s, \tilde{\imath}_t^s, \theta_1, R) + \varepsilon_{\tilde{\imath}_t^s, t}^s \right)$$

and calculate moments:

$$\frac{1}{\# X} \sum_{x_t \in X} Z_t \otimes \left(\begin{array}{c} \left[\widetilde{\bar{V}^1}(x_t, \theta_1, R) - \widetilde{\bar{V}^0}(x_t, \theta_1, R) \right] \\ - \left[\widehat{\bar{V}^1}(x_t) - \widehat{\bar{V}^0}(x_t) \right] \end{array} \right)$$

5. Repeat step 4 until the above is as close to zero as possible (remember that steps 1-3 are fixed in the parameter search

- If we need to perform this forward simulation for any θ, the computation burden may be heavy

- One way to significantly reduce the computation burden is to assume

$$u(x, i, \theta) + \varepsilon_{it} = \psi(x, i, \varepsilon_{it}) \cdot \theta$$

where $\psi(x, i, \varepsilon_i)$ is a vector of basis functions which does NOT depend on θ

- Sometimes, this comes with no cost (no additional assumption). Consider again our Rust-type model:

$$u(x, i, \theta) = \begin{cases} -\theta_1 x & \text{if } i = 0 \\ -R & \text{if } i = 1 \end{cases}$$

- Then, for example, the continuation payoff in $\widetilde{\bar{V}^1}(x_t, \theta_1, R)$ can be written as

$$
\begin{aligned}
&\frac{1}{S}\sum_{s=1}^{S}\sum_{t=1}^{T}\beta^t\left(u(\tilde{x}_t^s, \tilde{i}_t^s, \theta_1, R) + \varepsilon_{\tilde{i}_t^s, t}^s\right) \\
&= \frac{1}{S}\sum_{s=1}^{S}\sum_{t=1}^{T}\beta^t\left[(-\theta_1\tilde{x}_t^s)\,\mathbf{1}(\tilde{i}_t^s = 0) + (-R)\,\mathbf{1}(\tilde{i}_t^s = 1) + \varepsilon_{\tilde{i}_t^s, t}^s\right] \\
&= -\theta_1\frac{1}{S}\sum_{s=1}^{S}\sum_{t=1}^{T}\beta^t\tilde{x}_t^s\mathbf{1}(\tilde{i}_t^s = 0) - R\frac{1}{S}\sum_{s=1}^{S}\sum_{t=1}^{T}\beta^t\mathbf{1}(\tilde{i}_t^s = 1) + \frac{1}{S}\sum_{s=1}^{S}\sum_{t=1}^{T}\beta^t\varepsilon_{\tilde{i}_t^s, t}^s \\
&= -\theta_1 A - RB + C
\end{aligned}
$$

- (A, B, C) do not depend on structural parameters, so for any strategy profile, given by $(\tilde{x}_t^s, \tilde{i}_t^s)_{t=1}^T$, one can use the forward simulation procedure only once to estimate (A, B, C) and then obtain $\widetilde{\bar{V}^1}(x_t, \theta_1, R)$ and $\widetilde{\bar{V}^0}(x_t, \theta_1, R)$ easily for any value of θ.

- Comments:

  - $V(x)$ should be computed only for $x \in \widetilde{\mathbf{X}}$, where $\widetilde{\mathbf{X}}$ is the set of states that are observed in the data and states that are reachable within one time period from each of the observed states. Why?

  - But you still have to know CCPs for all the states. Why?

  - In practice, we cannot compute an infinite sum. How do we choose T?

  - Not exact.

2.4 Absorbing States

Another alternative is to exploit *absorbing states*.

- To explain how absorbing states can ease CCP estimation in general, let us use a single agent optimal stopping problem; continue ($i = 0$) or stop ($i = 1$).

- The per-period payoff function is denoted by $u(x, i)$ and we use additively separable payoff shocks $\varepsilon = (\varepsilon_0, \varepsilon_1)$. The value function is

$$V(x, \varepsilon, \theta) = \max_{i \in \{0,1\}} \left\{ u(x, i, \theta) + \varepsilon_i + \beta E \left[V \left(x', \varepsilon', \theta \right) | x, \varepsilon, i \right] \right\}.$$

Under the conditional independence assumption, alternative-specific value functions (net of ε) are

$$\bar{V}^0(x, \theta) = u(x, 0, \theta) + \beta E \left[V(x', \varepsilon', \theta) | x, 0 \right]$$
$$\bar{V}^1(x, \theta) = u(x, 1, \theta) + \beta E \left[V(x', \varepsilon', \theta) | x, 1 \right]$$

- Using this, as in Rust (1987)'s model, the value is

$$V(x,\varepsilon,\theta) = \max_{i\in\{0,1\}} \left\{\bar{V}^0(x,\theta)+\varepsilon_0, \bar{V}^1(x,\theta)+\varepsilon_1\right\}.$$

and

$$\bar{V}^0(x,\theta) = u(x,0,\theta) + \beta E\left[\max_{i\in\{0,1\}} \left\{\bar{V}^0(x',\theta)+\varepsilon_0', \bar{V}^1(x',\theta)+\varepsilon_1'\right\} |x,0\right]$$

$$\bar{V}^1(x,\theta) = u(x,1,\theta) + \beta E\left[\max_{i\in\{0,1\}} \left\{\bar{V}^0(x',\theta)+\varepsilon_0', \bar{V}^1(x',\theta)+\varepsilon_1'\right\} |x,1\right]$$

- Assuming ε follows the type I extreme value distribution,

$$\bar{V}^0(x,\theta) = u(x,0,\theta) + \beta \sum_{x'\in\mathbf{X}} \left[\mu + \ln\left\{e^{\bar{V}^0(x',\theta)} + e^{\bar{V}^1(x',\theta)}\right\}\right] g(x',0,x,\theta) \tag{5}$$

$$\bar{V}^1(x,\theta) = u(x,1,\theta) + \beta \sum_{x'\in\mathbf{X}} \left[\mu + \ln\left\{e^{\bar{V}^0(x',\theta)} + e^{\bar{V}^1(x',\theta)}\right\}\right] g(x',1,x,\theta) \tag{6}$$

and the choice probabilities are

$$\text{Pr}\left(i=0|x,\theta\right) = \frac{e^{\bar{V}^0(x,\theta)}}{e^{\bar{V}^0(x,\theta)}+e^{\bar{V}^1(x,\theta)}} \tag{7}$$

$$\text{Pr}\left(i=1|x,\theta\right) = \frac{e^{\bar{V}^1(x,\theta)}}{e^{\bar{V}^0(x,\theta)}+e^{\bar{V}^1(x,\theta)}}. \tag{8}$$

- Now assume that $i=1$ is a terminating action (that leads to an absorbing state, such as exit), so $u(x,1,\theta)=u(x,\theta)+\phi$ and $u(x,0,\theta)=u(x,\theta)$. Then,

$$\bar{V}^1(x,\theta) = u\left(x,\theta\right)+\phi, \tag{9}$$

where ϕ is the value of the terminating action (e.g., exit value)

- Rearrange (8) to get

$$e^{\bar{V}^0(x,\theta)}+e^{\bar{V}^1(x,\theta)} = \frac{e^{\bar{V}^1(x,\theta)}}{\text{Pr}\left(i=1|x,\theta\right)}$$

and plug this in into (5) to have

$$\bar{V}^0(x,\theta) = u(x,\theta) + \beta \sum_{x' \in \mathbf{X}} \left[\mu + \bar{V}^1(x',\theta) - \ln \Pr\left(i = 1 | x', \theta\right)\right] g(x', 0, x, \theta) \tag{10}$$

- This expression itself does not look very helpful. But once you use (9), the RHS of (10) is written only in terms of functions of parameters (u and ϕ) and something that we can recover from the data ($Pr\,(i = 1|x', \theta)$ and g).

- In other words, in the presence of a terminating action, the one-period-ahead probability of observing such an action can be used to cut off the future dependence, and so the value can be computed.

- This is a special case of finite dependence in Altug and Miller (1998) and Arcidiacono and Miller (2011).

2.5 Practical Issues

- Two step methods are a powerful tool: one does not need to compute a fixed point even once to estimate parameters.

- First-stage estimates may be imprecise: the PML estimator of Aguirregabiria and Mira (2002, 2007) could be helpful.

- The first stage CCP should be consistently estimated. When is it inconsistent?

  - when a researcher pools different markets in the presence of multiple equilibria or unobserved heterogeneity: to test, see Otsu, Pesendorfer, and Takahashi (2016).

  - two-step method allowing for unobserved heterogeneity: See Arcidiacono and Miller (2011) and Otsu, Pesendorfer, Sasaki, and Takahashi (2019).

- Specification of the first stage is up to the researcher; “flexibility-feasibility trade-off”
  - In theory, first stage estimates should be non-parametric.
  - In practice, one may need to impose some functional form assumption.
- What to do with typical data (short T)?
- In non-stationary environments (or the industry in question is still in a transition process during the sample period), some state is rarely observed. We may have to rely on extrapolation.
- What if there are several states that are not observed in the data at all?
  - Interpolation?
  - Extrapolation? See Hu, Sasaki and Takahashi (2016).