Lectures 7: Estimation of Dynamic Games

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1 Introduction

- Importance of dynamics in the game context is clear. Empirical work includes:

  - Repeated auction with capacity constraints (Jofre-Bonet and Pesendorfer, 2003)
  - Prices below average variable costs in the commercial aircraft industry, due learning-by-doing (Benkard, 2004)
  - Preemption of rivals' technology adoption (Schmidt-Dengler, 2006)
  - Sunk costs of entry and capacity adjustments (Ryan, 2012)
  - Demand uncertainty (Collard-Wexler, 2013)

- Main difficulties are curse of dimensionality and multiplicity of equilibrium.

2 Model

- Based on Pesendorfer and Schmidt-Dengler (2008)

- Discrete time, $t = 1, 2, ..., \infty$

- N firms compete repeatedly

- Finite set of $K + 1$ actions: $a_i^t \in \mathbf{A}_i = \{0, 1, \ldots, K\}$. Let $\mathbf{a}^t = (a_1^t, ..., a_N^t) \in \mathbf{A} = \times_{i=1}^N \mathbf{A}_i$ denote an action profile.

- Finite state space: player i has $s_i^t \in \mathbf{S}_i = \{0, 1, ..., L\}$. Use $\mathbf{s}^t = (s_1^t, ..., s_N^t) \in \mathbf{S} = \times_{i=1}^N \mathbf{S}_i$.

- m_s: cardinality of the state space, m_a: cardinality of the action space

Remark Most industry models have entry and exit. For example, in Ericson and Pakes (1995)'s model, n refers to the number of currently active firms in the market. Every time period, n_e potential firms make an entry decision. If potential firms decided not to enter the market, they die and are replaced with a new set of potential firms next period (i.e., they are "short-lived"). In our example, the maximum number of active firms is N. If we say $s_i^t = 0$ refers to "out of market", then we can interpret our model as a version of entry-exit models.

- A simple example would be $K = L = 1$, where actions are {entry,exit} and the state is {active, inactive}.

- Period t events:

  1. Publicly observed state $\mathbf{s}^t$

  2. Privately observed profitability shock $\varepsilon_i^t \in \mathbb{R}^K$ (observed by individual players, but not by rivals or econometrician)

3. Actions are simultaneously chosen

4. State $\mathbf{s}$ evolves

- State transition probability $g\left(\mathbf{a}^t, \mathbf{s}^t, \mathbf{s}^{t+1}\right)$

- ε_i^t conditionally independent: Drawn from some distribution $F(\varepsilon|\mathbf{s}^t)$, strictly monotone and continuous

- *Period payoff*

$$\bar{\pi}_i\left(\mathbf{a}^t, \mathbf{s}^t, \varepsilon_i^t\right) = \pi_i\left(\mathbf{a}^t, \mathbf{s}^t\right) + \sum_{k=1}^{K} \varepsilon_i^{tk} \cdot \mathbf{1}\left(a_i^t = k\right)$$

and use $\mathbf{\Pi}_i$ to denote the $(m_a \cdot m_s) \times 1$ dimensional period payoff vector.

- *Game payoff* is the discounted sum of future payoffs:

$$\mathbb{E}\left\{\sum_{t=0}^{\infty}\beta^t\bar{\pi}_i\left(\mathbf{a}^t,\mathbf{s}^t,\varepsilon_i^t\right)\middle|\,\mathbf{s}^0,\varepsilon_i^0\right\}$$

- Pure Markovian strategies $a_i(\varepsilon_i^t;s^t)$:

$$a_i(\varepsilon_i^t;\mathbf{s}^t)=a_i(\varepsilon_i^{t'};\mathbf{s}^{t'}) \text{ if } (\varepsilon_i^t;\mathbf{s}^t)=(\varepsilon_i^{t'};\mathbf{s}^{t'})$$

- Beliefs: $\sigma_i(\mathbf{a}|s)$ is i's belief that $\mathbf{a}$ is chosen in state s

- Focus on stationary Markov strategies: drop t superscript

- Using beliefs, the discounted sum of future payoffs for player i :

$$W_i\left(\mathbf{s},\varepsilon_i;\sigma_i\right)=\max_{a_i\in\mathbf{A}_i}\sum_{\mathbf{a}_{-i}\in\mathbf{A}_{-i}}\sigma_i(\mathbf{a}_{-i}|\mathbf{s})\cdot$$
$$\left[\pi_i\left(\mathbf{a}_{-i},a_i,\mathbf{s}\right)+\sum_{k=1}^{K}\varepsilon_i^k\cdot\mathbf{1}\left(a_i=k\right)+\right.$$
$$\left.+\beta\mathbb{E}_\varepsilon\sum_{\mathbf{s}'\in\mathbf{S}}g\left(\mathbf{a},\mathbf{s},\mathbf{s}'\right)W_i(\mathbf{s}^{'},.;\sigma_i)\right]$$

- MPE: $(\mathbf{a},\sigma)=(a_1,\ldots,a_N,\sigma_1,\ldots,\sigma_N)$ is a Markov perfect equilibrium if

    1. for all i, a_i is a best response to $\mathbf{a}_{-i}$ given the beliefs σ_i at all states $\mathbf{s}\in\mathbf{S}$;

    2. all players use Markovian strategies;

    3. for all i the beliefs σ_i are consistent with the strategies $\mathbf{a}$.

- Ex ante (integrated) value function:

$$
\begin{aligned}
&V_i\left(\mathbf{s};\sigma_i\right) = \\
&\sum_{\mathbf{a}\in\mathbf{A}} \sigma_i(\mathbf{a}|\mathbf{s}) \left[\pi_i\left(\mathbf{a},\mathbf{s}\right) + \beta \sum_{\mathbf{s}'\in\mathbf{S}} g(\mathbf{a},\mathbf{s},\mathbf{s}')V_i(\mathbf{s}';\sigma_i)\right] \\
&+ \sum_{k=1}^{K} \mathbb{E}_{\varepsilon}\left[\varepsilon_i^k | a_i = k\right] \cdot \sigma_i\left(a_i = k|\mathbf{s}\right)
\end{aligned}
$$

- Finite state and action space: Matrix notation:

$$
\begin{aligned}
\mathbf{V}_i\left(\sigma_i\right) &= \sigma_i \mathbf{\Pi}_i + \mathbf{D}_i\left(\sigma_i\right) + \beta\sigma_i \mathbf{G}\mathbf{V}_i\left(\sigma_i\right) \\
&= \left[\mathbf{I}_s - \beta\sigma_i\mathbf{G}\right]^{-1}\left[\sigma_i\mathbf{\Pi}_i + \mathbf{D}_i\left(\sigma_i\right)\right].
\end{aligned}
\tag{1}
$$

- Explicit expression for value function

- Define continuation value u_i :

$$u_i(a_i;\sigma_i,\theta) = \sum_{\mathbf{a}_{-i}\in\mathbf{A}_{-i}} \sigma_i(\mathbf{a}_{-i}|\mathbf{s})\cdot[\pi_i(\mathbf{a}_{-i},a_i,\mathbf{s}) \\ + \beta\sum_{\mathbf{s}'\in\mathbf{S}} g(\mathbf{a}_{-i},a_i,\mathbf{s},\mathbf{s}')\underbrace{V_i(\mathbf{s}';\sigma_i)}_{\text{Value function}}]$$

- This is $\bar{V}^1$ and $\bar{V}^0$ in our discussion on Rust (1987).

- Optimality of action a_i given the beliefs σ_i

$$u_i(a_i;\sigma_i,\theta)+\varepsilon_i^{a_i} \geq u_i(a_i';\sigma_i,\theta)+\varepsilon_i^{a_i'} \text{ for all } a_i'\in\mathbf{A}_i$$

- Optimal choice probabilities

$$p(a_i|\mathbf{s},\sigma_i) = \mathbf{\Psi}_i(a_i,\mathbf{s},\sigma_i;\theta) \\ = \int\prod_{k\neq a_i}\mathbf{1}\begin{pmatrix} u_i(a_i;\sigma_i,\theta)-u_i(k;\sigma_i,\theta) \\ \geq \varepsilon_i^k-\varepsilon_i^{a_i}\end{pmatrix}dF$$

- In matrix notation $(N \cdot K \cdot m_s)$ equations

$$\mathbf{p} = \mathbf{\Psi}\left(\sigma; \theta\right)$$

- Equilibrium: consistent beliefs

$$\mathbf{p} = \mathbf{\Psi}\left(\mathbf{p}; \theta\right) \tag{2}$$

- *Proposition: $\mathbf{p}$ satisfying equation (2) is a necessary and sufficient equilibrium condition* (also other representations of equilibrium, e.g. linear in payoff parameters)

- *Theorem: A Markov equilibrium exists* (fixed point problem)

- For continuity of $\mathbf{\Psi}$, continuity of F and private information is important.

2.1 Example Nonexistence of Symmetric Pure Strategy Equilibria

- From Doraszelski and Satterthwaite (2010)

- War of attrition with a common and constant exit value ϕ (no ε in our model)

- Two firms start an exit game, and no (re)entry allowed. State $s = (s_1, s_2)$ where $s_i = 1$ means i stays in and $s_i = 2$ means i is out. $\pi(s_i, s_{-i})$ is i's profit.

- Pick ϕ such that

$$\frac{\beta\pi(1,1)}{1-\beta} < \phi < \frac{\beta\pi(1,2)}{1-\beta}.$$

- Given firm 2's decision $a(1,1) \in \{0,1\}$, firm 1's values are

$$V(1,2) = \sup_{\widetilde{a}(1,2)\in\{0,1\}} \pi(1,2) + (1-\widetilde{a}(1,2))\,\phi + \widetilde{a}(1,2)\,\beta V(1,2)$$

$$\begin{aligned} V(1,1) = & \sup_{\widetilde{a}(1,1)\in\{0,1\}} \pi(1,1) + (1-\widetilde{a}(1,1))\,\phi \\ & + \widetilde{a}(1,1)\,\beta\,\{a(1,1)\,V(1,1) + (1-a(1,1))\,V(1,2)\} \end{aligned}$$

- Firm 1's optimal exit decision $\widetilde{a}(1,2)$ and $\widetilde{a}(1,1)$ satisfy

$$\widetilde{a}(1,2) = \mathbf{1}\{\phi \leq \beta V(1,2)\} \tag{3}$$

$$\widetilde{a}(1,1) = \mathbf{1}\{\phi \leq \beta\,\{a(1,1)\,V(1,1) + (1-a(1,1))\,V(1,2)\}\}. \tag{4}$$

- Consider a symmetric equilibrium in pure exit strategies (why?).

- Suppose $a(1,2) = 0$. Then $V(1,2) = \pi(1,2) + \phi$ and (3) implies $\phi \geq \beta(\pi(1,2) + \phi)$, contradiction.

- Suppose $a(1,1) = 1$. Then $V(1,1) = \frac{\pi(1,1)}{1-\beta}$ and (4) implies $\phi \leq \frac{\beta\pi(1,1)}{1-\beta}$, contradiction.

- Finally, $a(1,2) = 1$ and $a(1,1) = 0$. Then $V(1,2) = \frac{\pi(1,2)}{1-\beta}$ and (4) implies $\phi \geq \beta V(1,2)$, contradiction.

2.2 Example with Random Shocks

- Same setting as above, except that scrap value is $\phi + \epsilon\theta_i$ where ϕ is a constant as before and $\theta_i \sim F(\cdot)$ with $\mathbb{E}(\theta) = 0$.

- θ is privately observed and ϵ is a constant that measures the importance of incomplete information.

- Bellman equation of firm 1

$$
\begin{aligned}
V(1,2) &= \sup_{\widetilde{\sigma}(1,2)\in[0,1]} \pi(1,2) + (1-\widetilde{\sigma}(1,2))\phi \\
&\quad + \epsilon \int_{\theta > F^{-1}(\widetilde{\sigma}(1,2))} \theta dF(\theta) + \widetilde{\sigma}(1,2)\beta V(1,2) \\
V(1,1) &= \sup_{\widetilde{\sigma}(1,1)\in[0,1]} \pi(1,1) + (1-\widetilde{\sigma}(1,1))\phi \\
&\quad + \widetilde{\sigma}(1,1)\beta\{\sigma(1,1)V(1,1) + (1-\sigma(1,1))V(1,2)\}.
\end{aligned}
$$

- Optimal exit decisions of firm 1 are given by

$$
\begin{aligned}
\widetilde{\sigma}(1,2) &= F\left(\frac{\beta V(1,2)-\phi}{\epsilon}\right) \\
\widetilde{\sigma}(1,1) &= F\left(\frac{\beta\{\sigma(1,1)V(1,1)+(1-\sigma(1,1))V(1,2)\}-\phi}{\epsilon}\right)
\end{aligned}
$$

- In a symmetric equilibrium, $\widetilde{\sigma}(s_1, s_2) = \sigma(s_2, s_1)$, yielding a system of four equations with four unknowns: $V(1,2)$, $V(1,1)$, $\sigma(1,2)$, and $\sigma(1,1)$.

2.3 Multiple Equilibria

- Multiplicity of equilibria is a well-known feature inherent to games

- Formally, multiplicity exists if we have $\mathbf{p}^1$ and $\mathbf{p}^2$ such that

$$\mathbf{p}^1 = \mathbf{\Psi}\left(\mathbf{p}^1; \theta\right) \text{ and } \mathbf{p}^2 = \mathbf{\Psi}\left(\mathbf{p}^2; \theta\right).$$

- Generically a finite number of equilibria

- Example: five equilibria in Pesendorfer and Schmidt-Dengler (2008)
    - two players, binary actions $\{0, 1\}$ and binary states $\{0, 1\}$
    - F : standard normal, $\beta = 0.9$.
    - The state transition law is given by $s_i^{t+1} = a_i^t$.

- Period payoffs are symmetric and are parametrized as follows:

$$\pi\left(a_i, a_j, s_i\right) = \begin{cases} 0 & \text{if } a_i = 0;\ s_i = 0 \\ 0.1 & \text{if } a_i = 0;\ s_i = 1 \\ \pi^1 - 0.2 & \text{if } a_i = 1;\ a_j = 0;\ s_i = 0 \\ \pi^2 - 0.2 & \text{if } a_i = 1;\ a_j = 1;\ s_i = 0 \\ \pi^1 & \text{if } a_i = 1;\ a_j = 0;\ s_i = 1 \\ \pi^2 & \text{if } a_i = 1;\ a_j = 1;\ s_i = 1 \end{cases}$$

where $\pi^1 = 1.2$; and $\pi^2 = -1.2$.

- Multiplicity: for $(s_1, s_2) = ((0,0), (0,1), (1,0), (1,1))$

1. $\sigma(a_1 = 0|s_1, s_2) = (0.27, 0.39, 0.20, 0.25)'$, $\sigma(a_2 = 0|s_2, s_1) = (0.72, 0.78, 0.58, 0.71)'$

2. $\sigma(a_1 = 0|s_1, s_2) = (0.38, 0.69, 0.17, 0.39)'$, $\sigma(a_2 = 0|s_2, s_1) = (0.47, 0.70, 0.16, 0.42)'$

3. $\sigma(a_1 = 0|s_1, s_2) = (0.42, 0.70, 0.16, 0.41)'$, $\sigma(a_2 = 0|s_2, s_1) = (0.42, 0.70, 0.16, 0.41)'$

- Besanko, Doraszelski, Kryukov, and Satterthwaite (2010) found up to 9 equilibria

- Multiplicity comes from two sources: simultaneity and future expectation

  1. If values are fixed (or $\beta = 0$), our game is similar to a repetition of a static entry game. In the stage game, best response functions can cross more than once $\rightarrow$ Multiple equilibria

  2. Based on the expectation, each player chooses the best alternative. Then, this behavior should be consistent with the expectation that other players have. It may be possible to construct more than one set of beliefs such that this optimality and consistency are satisfied.

- A common misunderstanding is "a symmetric equilibrium is unique". This is not correct. For an example, see Doraszelski and Satterthwaite (2010).

- Imposing symmetry *may* be able to exclude the first source of multiplicity. This is not always the case, depending on the nature of strategic interaction; e.g., Brock and Durlauf (2001).

- Several authors introduce a continuous time framework to dynamic Markov games to alleviate the computational burden (Judd and Doraszelski, 2011; Arcidiacono, Bayer, Blevins, and Ellickson, 2015). The basic idea is that no more than one player changes its state at the same time.

- A common misunderstanding is "equilibrium is unique in continuous time models". This is not correct. A continuous time model definitely breaks the first source, but not the second one.

3 Solving the Model

- To apply this framework, we should be able to solve the model numerically.

- "Solving" means finding values V and policies p such that

    1. given policies p, the values V solve the Bellman equations

    2. given value functions V, the policies p satisfy the optimality conditions

- Remember that we can solve value functions explicitly as

$$\mathbf{V}_i\left(\sigma_i\right) = \left[\mathbf{I}_s - \beta\sigma_i\mathbf{G}\right]^{-1}\left[\sigma_i\mathbf{\Pi}_i + \mathbf{D}_i\left(\sigma_i\right)\right], \tag{5}$$

and optimal policies are

$$\begin{aligned} p(a_i|\mathbf{s},\sigma_i) &= \mathbf{\Psi}_i\left(a_i,\mathbf{s},\sigma_i;\theta\right) \\ &= \int \prod_{k\neq a_i} \mathbf{1}\left(\begin{array}{c} u_i(a_i;\sigma_i,\theta) - u_i(k;\sigma_i,\theta) \\ \geq \varepsilon_i^k - \varepsilon_i^{a_i} \end{array} \right) dF, \end{aligned} \tag{6}$$

where continuation value u_i :

$$u_i(a_i;\sigma_i,\theta) = \sum_{\mathbf{a}_{-i}\in\mathbf{A}_{-i}} \sigma_i(\mathbf{a}_{-i}|\mathbf{s})\cdot[\pi_i\left(\mathbf{a}_{-i},a_i,\mathbf{s}\right) + \beta\sum_{\mathbf{s}'\in S} g(\mathbf{a}_{-i},a_i,\mathbf{s},\mathbf{s}')V_i(\mathbf{s}';\sigma_i)] \tag{7}$$

Algorithm

Step 1 Set initial values and policies for all states in $\mathbf{S}$.

$V^0(\mathbf{s}) = \frac{\pi(\mathbf{a},\mathbf{s})}{1-\beta}$ for some arbitrary $\mathbf{a}$, and $\sigma_i^0(a|\mathbf{s}) = \frac{1}{K+1}$ for all i and a

- Note that if you have some information that gives you better initial conditions, you would use them

Step 2 Update values V^1 using $\sigma_i^0(a|\mathbf{s})$ and (5). Alternatively, we can update values using the original formula:

$$\mathbf{V}_i^1(\sigma_i) = \sigma_i^0 \mathbf{\Pi}_i + \mathbf{D}_i\left(\sigma_i^0\right) + \beta \sigma_i^0 \mathbf{G} \mathbf{V}_i^0$$

Step 3 Compute $u_i(a_i; \sigma_i^0, \theta)$ using (7) and $\mathbf{V}_i^1(\sigma_i)$.

Step 4 Update σ_i using (6):

$$\sigma_i^1(a|\mathbf{s}) = \mathbf{\Psi}_i\left(a_i, \mathbf{s}, \sigma_i^0; \theta\right).$$

Step 5 Repeat steps 2 to 4 until the changes in the values and policies from one iteration to the next are sufficiently small.

- Remarks

  - For any policies, value functions are unique, as is obvious from (5)

  - For any values, policies are uniquely computed from (6)

  - But this is NOT a contraction mapping (neither uniqueness nor convergence is guaranteed)

  - The above iteration often fails convergence

  - If it does not converge,

    1. try different iteration schemes, or

    2. use some non-linear solver (what is the dimension of the equation system?)

  - This is a major difference from single-agent DP problems

3.1 Computational Burden

- Computational burden is the product of

  (a) the number of points evaluated at each iteration (m_s)

  (b) the time per point evaluated; and

  (c) the number of iterations.

- For (a), the size of state space is $(L+1)^N$. If there are M state variables for each player and each of M variables can take $L+1$ different values, then $m_s = \left((L+1)^M\right)^N$.

- Imposing symmetry helps. We often assume "payoffs and strategies are symmetric and anonymous"

- Asymmetries between firms are then captured by the state variables only.

- This allows us to focus on a proper subset S^o of S :

$$S^o = \left\{ (s_i, c) : s_i \in S_i,\ c = (c_1, ..., c_L),\ c_l \in \mathcal{Z}^+,\ \sum c_l = N - 1 \right\}.$$

- The cardinality of the symmetric state space equals

$$\binom{N + L - 1}{L} (L + 1).$$

  For example, $L = 6$, $N = 20$. Then, we have $m_s = 1{,}239{,}700$.

- (b) consists of computing the relevant elements of $\{V(\cdot)\}$ and then solving for optimal policies given $\{V(\cdot)\}$. In the above example, 177,100 states need to be summed over to obtain the continuation values for a given firm.

- We do not know much about (c). It depends on the application. But it can easily be a large number

- Several methods have been proposed to alleviate the computational burden.

- Roughly speaking, there are two ways to do so:

  1. Change the structure (assumptions) of the model

     (a) Continuous time models; e.g., Doraszelski and Judd (2011), Arcidiacono, Bayer, Blevins, and Ellickson (2015)

     (b) Focusing on a particular set of strategies; e.g., Abbring and Campbell (2010), Abbring, Campbell, and Yang (2010)

  2. Approximate an MPE instead of finding an exact solution

     (a) Stochastic approximation algorithm by Pakes and McGuire (2001)

     (b) Oblivious equilibrium by Weintraub, Benkard, and Van Roy (2008)

4 Identification

- Model primitives $(\mathbf{\Pi}_1, \ldots, \mathbf{\Pi}_N, F, \beta, g)$

  Proposition: Suppose (F, β) *are given. At most* $K \cdot m_s \cdot N$ *parameters can be identified.*

- Non-identification concern

  Gets worse with multiple players

- Suppose interested in payoff parameters, fix (F, β)

- For each player, need to impose $(m_a \cdot m_s - K \cdot m_s)$ restrictions

$$\mathbf{R}_i \cdot \mathbf{\Pi}_i = \mathbf{r}_i$$

The alternative representation: Let $\overline{\varepsilon}_i^{a_i}(\mathbf{s})$ be the type that is indifferent between actions a_i and 0 in state s:

$$\sum_{\mathbf{a}_{-i}\in\mathbf{A}_{-i}} p(\mathbf{a}_{-i}|\mathbf{s})\cdot\left[\begin{array}{c}\pi_i\left(\mathbf{a}_{-i},a_i,\mathbf{s}\right)+\\ \beta\sum_{\mathbf{s}'\in\mathbf{S}}g(\mathbf{a}_{-i},a_i,\mathbf{s},\mathbf{s}')V_i(\mathbf{s}';p)\end{array}\right]$$
$$+\,\overline{\varepsilon}_i^{a_i}(\mathbf{s})$$
$$=\sum_{\mathbf{a}_{-i}\in\mathbf{A}_{-i}} p(\mathbf{a}_{-i}|\mathbf{s})\cdot\left[\begin{array}{c}\pi_i\left(\mathbf{a}_{-i},0,\mathbf{s}\right)+\\ \beta\sum_{\mathbf{s}'\in\mathbf{S}}g(\mathbf{a}_{-i},0,\mathbf{s},\mathbf{s}')V_i(\mathbf{s}';p)\end{array}\right]$$

Type $\left(\varepsilon_i^{a_i},\varepsilon_i^{a_i'}\right)$ will prefer action a_i over action a_i' in state $\mathbf{s}$ if

$$\varepsilon_i^{a_i}-\overline{\varepsilon}_i^{a_i}(\mathbf{s})>\varepsilon_i^{a_i'}-\overline{\varepsilon}_i^{a_i'}(\mathbf{s})$$

Equilibrium decision rule for player i with type ε_i in state $\mathbf{s}$ becomes

$$a_i(\varepsilon_i,\mathbf{s})=\begin{cases} k & \begin{array}{c}\text{if } \varepsilon_i^k>\overline{\varepsilon}_i^k(\mathbf{s}) \text{ and for all}\\ k'\neq k\text{: } \varepsilon_i^k-\varepsilon_i^{k'}>\overline{\varepsilon}_i^k(s)-\overline{\varepsilon}_i^{k'}(\mathbf{s});\end{array}\\ 0 & \text{if } \varepsilon_i^k<\overline{\varepsilon}_i^k(s) \text{ for all } k.\end{cases}$$

Choice probability is given by:

$$p(a_i = k|s) = \Pr\left(\begin{array}{c} \varepsilon_i^k > \overline{\varepsilon}_i^k(s) \text{ and for all} \\ k' \neq k\text{: } \varepsilon_i^k - \varepsilon_i^{k'} > \overline{\varepsilon}_i^k(s) - \overline{\varepsilon}_i^{k'}(s) \end{array}\right)$$

for all $i \in \mathbf{N}, k \in A_i, s \in S$.

- One-to-one relationship between choice probabilities and indifferent types based on above equation (Hotz and Miller (1993), Pesendorfer and Schmidt-Dengler (2003)).

- Any vector of choice probabilities implies a unique vector of indifferent types and vice versa.

- Indifferent types correspond to "choice specific values" in Hotz-Miller

- Let $\overline{\varepsilon}_i = \left[\overline{\varepsilon}_i^k(\mathbf{s})\right]_{s\in\mathbf{S},k=1,\ldots K}$ be the $(m_s \cdot K) \times 1$ dimensional vector of indifferent types.

- $\mathbf{P}$ denotes $m_s \times (m_a \cdot m_s)$ matrix with choice probability $p(\mathbf{a}|\mathbf{s})$ in row $\mathbf{s}$ column $(\mathbf{a}_{-i}, \mathbf{s})$ and zeros in row $\mathbf{s}$ column $(\mathbf{a}, \mathbf{s}^{'})$ with $\mathbf{s}^{'} \neq \mathbf{s}$.

- $\mathbf{P}_{-i}$ is the $m_s \times ((K+1)^{N-1} \cdot m_s)$ matrix with choice probability $p(\mathbf{a}_{-i}|\mathbf{s})$ in row $\mathbf{s}$ column $(\mathbf{a}_{-i}, \mathbf{s})$, and zeros in row $\mathbf{s}$ column $(\mathbf{a}_{-i}, \mathbf{s}^{'})$ with $\mathbf{s}^{'} \neq \mathbf{s}$

- The $(K \cdot m_s) \times (m_a \cdot m_s)$ dimensional matrix $\mathbf{P}_i(\overline{\varepsilon})$ is:

$$\mathbf{P}_i(\overline{\varepsilon}) = \begin{array}{c} a_i = \\ \overbrace{\begin{array}{ccccc} 0 & 1 & 2 & \ldots & K \end{array}} \\ \begin{bmatrix} -\mathbf{P}_{-i} & \mathbf{P}_{-i} & 0 & \ldots & 0 \\ -\mathbf{P}_{-i} & 0 & \mathbf{P}_{-i} & \ldots & 0 \\ \vdots & 0 & 0 & \ddots & 0 \\ -\mathbf{P}_{-i} & 0 & 0 & \ldots & \mathbf{P}_{-i} \end{bmatrix} \end{array}$$

- The indifference equations for player i as:

$$\mathbf{P}_i(\overline{\varepsilon})\,\mathbf{\Pi}_i + \beta \mathbf{P}_i(\overline{\varepsilon})\,\mathbf{G}\mathbf{V}_i + \overline{\varepsilon}_i = 0$$

- Now substitute for the Value function and equilibrium beliefs $\mathbf{P}$ for σ_i, we obtain $m_s \cdot K$ equations for player i:

$$\left[\mathbf{P}_i(\overline{\varepsilon}) + \beta \mathbf{P}_i(\overline{\varepsilon}) \mathbf{G} \left[\mathbf{I}_s - \beta \mathbf{P}_i(\overline{\varepsilon}) \mathbf{G}\right]^{-1} \mathbf{P}_i(\overline{\varepsilon})\right] \mathbf{\Pi}_i$$
$$+ \beta \mathbf{P}_i(\overline{\varepsilon}) \mathbf{G} \left[\mathbf{I}_s - \beta \mathbf{P}_i(\overline{\varepsilon}) \mathbf{G}\right]^{-1} \mathbf{D}_i(\overline{\varepsilon}) + \overline{\varepsilon}_i = 0$$

- Ex ante expected payoff shock can be uniquely written in terms of indifferent types, as:

$$D_i(\mathbf{s}) = \sum_{k=1}^{K} \int_{\overline{\varepsilon}_i^k(s)}^{\infty} \varepsilon^k \prod_{k' \geq 1, k' \neq k} F(\varepsilon^k + \overline{\varepsilon}^{k'} - \overline{\varepsilon}^k) f(\varepsilon^k) d\varepsilon^k.$$

- Let

$$\mathbf{X}_i = \left[\mathbf{P}_i(\overline{\varepsilon}) + \beta \mathbf{P}_i(\overline{\varepsilon}) \mathbf{G} \left[\mathbf{I}_s - \beta \mathbf{P}_i(\overline{\varepsilon}) \mathbf{G}\right]^{-1} \mathbf{P}_i(\overline{\varepsilon})\right]$$

and

$$\mathbf{Y}_i = \beta \mathbf{P}_i(\overline{\varepsilon}) \mathbf{G} \left[\mathbf{I}_s - \beta \mathbf{P}_i(\overline{\varepsilon}) \mathbf{G}\right]^{-1} \mathbf{D}_i(\overline{\varepsilon}) + \overline{\varepsilon}_i$$

and the unknown parameters are in $\mathbf{\Pi}_i$.

- Can thus re-write equation system *(2)* as an equation system linear in $\mathbf{\Pi}_i$:

$$\begin{bmatrix} \mathbf{X}_i \\ \mathbf{R}_i \end{bmatrix} \mathbf{\Pi}_i + \begin{bmatrix} \mathbf{Y}_i \\ \mathbf{r}_i \end{bmatrix} = 0.$$

Proposition: Suppose (F, β) *are given. If* $rank \begin{bmatrix} \mathbf{X}_i \\ \mathbf{R}_i \end{bmatrix} = (m_a \cdot m_s)$*, then* Π_i *is exactly identified.*

- Two restrictions that work in a number of settings:

$$\begin{aligned} \pi_i (a_i, \mathbf{a}_{-i}, s_i, \mathbf{s}_{-i}) &= \pi_i \left(a_i, \mathbf{a}_{-i}, s_i, \mathbf{s}'_{-i}\right) \\ &\quad \forall\ \mathbf{a} \in \mathbf{A}, (s_i, \mathbf{s}_{-i}), \left(s_i, \mathbf{s}'_{-i}\right) \in \mathbf{S} \\ \pi_i(0, \mathbf{a}_{-i}, s_i) &= r_i(\mathbf{a}_{-i}, s_i) \\ &\quad \forall\ \mathbf{a}_{-i} \in \mathbf{A}_{-i}, s_i \in \mathbf{S}_i \end{aligned}$$

- The first restriction fixes $m_a \cdot (m_s - L)$ payoffs
- The second restriction fixes $(K + 1)^{N-1} \cdot L$ payoffs

- With these two restrictions, if $L \geq K + \mathbf{1}$, we have fixed enough restrictions
- Need to satisfy rank restrictions as well
- When estimating a model, one has to make sure that the model is identified
- Sometimes, normalization is not innocuous for counterfactual purposes (Aguirregabiria and Suzuki, 2013)

5 Estimation

- As before, we could consider two classes of estimation methods:

    1. Full solution approach (nested fixed point approach)

    2. Two step methods (CCP based methods)

- First let us discuss several difficulties that the full solution approach faces.

- These difficulties clearly show the potentials of two step methods.

5.1 Full Solution Approach

- Ideally, we want a single time series so that data are generated by a single path of play. We assume the Markovian assumption, so players make the same choices over time for a given vector of state variables.

- In reality, type of data you typically have is a number of repetitions of the game (M markets) played by n firms over T years, with n and T being relatively small. In such a case, one has to impose the assumption "the data has been generated by only one Markov Perfect equilibrium"; e.g., Aguirregabiria and Mira (2007) and Ryan (2012). For the sake of argument, assume this is the case.

- Remember that

$$\mathbf{\Psi}_i\left(a_i, \mathbf{s}, \sigma_i; \theta\right) = \int \prod_{k \neq a_i} \mathbf{1}\begin{pmatrix} u_i(a_i; \sigma_i, \theta) - u_i(k; \sigma_i, \theta) \\ \geq \varepsilon_i^k - \varepsilon_i^{a_i} \end{pmatrix} dF,$$

- Assume that the probability density function g is estimated by maximizing the partial likelihood function $\sum_{m=1}^{M} \sum_{t=1}^{T_m - 1} \ln g\left(\mathbf{s}_m^{t+1} | \mathbf{a}_m^t, \mathbf{s}_m^t\right)$.

- (F, β, g) are given. The pseudo-likelihood function

$$Q\left(\theta, \sigma\right) = \sum_{m=1}^{M} \sum_{t=1}^{T_m} \sum_{i=1}^{N} \ln \mathbf{\Psi}_i\left(a_{imt}, \mathbf{s}_m^t, \sigma_i; \theta\right).$$

- Then, MLE is defined as

$$\hat{\theta}_{MLE} = \arg\max_{\theta \in \Theta} \left\{ \sup_{\mathbf{p} \in (0,1)^{N \times K \times m_s}} Q(\theta, \mathbf{p}) \text{ subject to : } \mathbf{p} = \mathbf{\Psi}(\mathbf{p}; \theta) \right\}. \tag{8}$$

- Implementing this is extremely difficult. For any given θ, one has to find all $\mathbf{p}$ that satisfies $\mathbf{p} = \mathbf{\Psi}(\mathbf{p}; \theta)$. Then, one has to find $\mathbf{p}$ among them that maximizes $Q(\theta, \mathbf{p})$. Remember that even though we imposed the assumption of "single equilibrium being played in the data", we still have to find which one was played.

- Finding all equilibria is difficult; see Besanko, Doraszelski, Kryukov, and Satterthwaite (2010).

- To be worse, the number of equilibria depends on the parameters and realization of other covariates.

- To be even worse, what if we relax the single equilibrium assumption?

- As was shown above, computing Markov Perfect equilibria is computationally demanding. Even if an equilibrium is unique, finding the MLE is very challenging, as one needs to solve a model many times in the parameter search. When it takes you 1 hour to compute your model once, how can you solve the model 20,000 times?

- (8) implies the potential of 2 step methods. If we can *directly* observe $\mathbf{p}$ from the data, then our estimator becomes

$$\hat{\theta} = \arg\max_{\theta \in \Theta} Q\left(\theta, \hat{\mathbf{p}}\right).$$

5.2 Two Step Methods

- As we discussed above, the computational burden and multiplicity are main obstacles when estimating dynamic games.

- If the same equilibrium is played in the data, multiplicity of equilibria is not an issue for estimation purpose. Why not?

- If only one equilibrium is played in the data, and the econometrician observes all the state in the state space, players' policy functions in equilibrium can be estimated nonparametrically.

  $\longrightarrow$ Conditional choice probabilities are recovered as a function of observed state variables

- Given these estimates, we can write a player's decision problem as a single agent problem. Then, we can relatively easily obtain its best response function.

- Estimator of structural parameters maximizes a criterion function based on these best responses.

- Loosely speaking, two-step methods let the data tell us what players did in the first stage, and choose structural parameters to explain why they did so in the second stage.

- Multiplicity of equilibria is an issue when performing counterfactual analyses. Why?

- Of course, the first stage CCP should be consistently estimated. When is it inconsistent?

  1. When a researcher pools different markets in the presence of multiple equilibria

     $\rightarrow$ Can this "unique equilibrium in the data" assumption be tested? See Otsu, Pesendorfer, and Takahashi (2016).

2. When there are unobservable state variables that are correlated with observable state variables

   $\rightarrow$ Weakness of two step methods. See Arcidiacono and Miller (2011).

5.2.1 Other Practical Issues

- What to do with typical data (short T)?

- Specification of the first stage is up to the researcher; "flexibility-feasibility trade-off"

  - In theory, first stage estimates are non-parametric

  - In practice, one may need to impose some functional form assumption

- In non-stationary environments (or the industry in question is still in a transition process during the sample period), some state is rarely observed. Therefore, in this case, the small sample bias of two-step methods is more significant

- What if there are several states that are not observed in the data at all?

  - Interpolation?

  - Extrapolation? Hu, Sasaki, and Takahashi (2016)

5.2.2 Estimation: First stage

- *Auxiliary parameter* estimators $(\widehat{\mathbf{p}}_T, \widehat{\mathbf{g}}_T)$ with

$$(\widehat{\mathbf{p}}_T, \widehat{\mathbf{g}}_T) \longrightarrow (\mathbf{p}(\theta_0), \mathbf{g}(\theta_0)) \quad \text{a.s.}$$

$$T^{1/2}((\widehat{\mathbf{p}}_T, \widehat{\mathbf{g}}_T) - (\mathbf{p}(\theta_0), \mathbf{g}(\theta_0))) \xrightarrow{d} \mathcal{N}(0, \boldsymbol{\Sigma}(\theta_0))$$

- Multinomial, ML yields

$$\widehat{p}(k,i,\mathbf{s}) = \frac{n_{ki\mathbf{s}}}{\sum_{l\in\mathbf{A}_i} n_{li\mathbf{s}}}$$

$$\widehat{g}(\mathbf{a},\mathbf{s},\mathbf{s}^{'}) = \frac{n_{\mathbf{ass}'}}{\sum_{\mathbf{s}''\in\mathbf{s}} n_{\mathbf{ass}''}}$$

5.2.3 Estimation: second stage

- Use the equilibrium equation system

$$\widehat{\mathbf{p}}_T - \mathbf{\Psi}\left(\widehat{\mathbf{p}}_T,\widehat{\mathbf{g}}_T,\theta\right) = 0$$

- $\widetilde{\theta}_T\left(\mathbf{W}_T\right)$ solves the problem

$$\min_{\theta\in\Theta}\left[\widehat{\mathbf{p}}_T - \mathbf{\Psi}\left(\widehat{\mathbf{p}}_T,\widehat{\mathbf{g}}_T,\theta\right)\right]'\mathbf{W}_T\left[\widehat{\mathbf{p}}_T - \mathbf{\Psi}\left(\widehat{\mathbf{p}}_T,\widehat{\mathbf{g}}_T,\theta\right)\right]$$

- A class of estimators

  Each estimator corresponds to a sequence $\mathbf{W}_T$

- Asymptotic properties?

$$T^{1/2}\left(\widetilde{\theta}_T\left(\mathbf{W}_T\right)-\theta_0\right)\xrightarrow{d}\mathcal{N}\left(0,\mathbf{\Omega}\left(\theta_0\right)\right)\qquad \textit{as } T\longrightarrow\infty,$$

  where

$$\begin{aligned}\mathbf{\Omega}\left(\theta_0\right)=\left(\nabla_\theta\mathbf{\Psi}'\mathbf{W}_0\nabla_{\theta'}\mathbf{\Psi}\right)^{-1}\nabla_\theta\mathbf{\Psi}'\mathbf{W}_0\cdot\\ \cdot\left[(\mathbf{I}:\mathbf{0})-\nabla_{(p,g)'}\mathbf{\Psi}\right]\mathbf{\Sigma}\left[(\mathbf{I}:\mathbf{0})-\nabla_{(p,g)'}\mathbf{\Psi}\right]'\cdot\\ \cdot\,\mathbf{W}_0\nabla_{\theta'}\mathbf{\Psi}\left(\nabla_\theta\mathbf{\Psi}'\mathbf{W}_0\nabla_{\theta'}\mathbf{\Psi}\right)^{-1}\end{aligned}$$

  and $\mathbf{\Sigma}=\mathbf{Var}(\hat{\mathbf{p}})$

- Optimal Weight Matrix

$$\begin{aligned}\mathbf{W}_0^*=(\left[(\mathbf{I}:\mathbf{0})-\nabla_{(p,g)'}\mathbf{\Psi}\right]\cdot\\ \cdot\,\mathbf{\Sigma}\left[(\mathbf{I}:\mathbf{0})-\nabla_{(p,g)'}\mathbf{\Psi}\right]')^{-1}.\end{aligned}$$

Asymptotic variance becomes

$$\mathbf{\Omega}\left(\mathbf{W}^{*}\right) = (\nabla_{\theta}\mathbf{\Psi}^{'}(\left[(\mathbf{I}:\mathbf{0}) - \nabla_{(p,g)'}\mathbf{\Psi}\right]\cdot$$
$$\mathbf{\Sigma}\left[(\mathbf{I}:\mathbf{0}) - \nabla_{(p,g)'}\mathbf{\Psi}\right]^{'})^{-1}\nabla_{\theta'}\mathbf{\Psi})^{-1}$$

- Derivative of equilibrium conditions w.r.t. $\mathbf{p},\mathbf{g}$:

$$\left[(\mathbf{I}:\mathbf{0}) - \nabla_{(p,g)'}\mathbf{\Psi}\right]$$

- Contrast with single agent problem

Examples

(i) Moment Estimator

- Moment condition, Hotz and Miller (1993)

$$E\left[Z \otimes \left[\alpha_{is} - \mathbf{\Psi}_{is}\left(\widehat{\mathbf{p}}_T, \widehat{\mathbf{g}}_T, \theta\right)\right]\right] = 0$$

where α_{is} is the $K \times \mathbf{1}$ dimensional r.v. of choices and Z is an instrument vector

- If $Z^t = Z^{is}$ for all $t \in \mathbf{T}_{is}$

$$\begin{aligned} & \frac{1}{NT} \sum_{i \in \mathbf{N}, s \in \mathbf{S}} \sum_{t \in \mathbf{T}_{is}} Z^t \otimes \left[\alpha^t - \mathbf{\Psi}_{is}\left(\widehat{\mathbf{p}}_T, \widehat{\mathbf{g}}_T, \theta\right)\right] \\ = & \frac{1}{NT} \sum_{i \in \mathbf{N}, s \in \mathbf{S}} n_{is} \left[Z^{is} \otimes \left[\widehat{\mathbf{p}}_{is} - \mathbf{\Psi}_{is}\left(\widehat{\mathbf{p}}_T, \widehat{\mathbf{g}}_T, \theta\right)\right]\right]. \end{aligned}$$

- Current and lagged states as instruments are used in Hotz and Miller (1993), Slade (1998) and Aguirregabiria (1999)

- Current and lagged states are not optimal instruments

- Optimal moment estimator

    1. A moment condition is considered for every agent and at every state

    2. The weight matrix $\mathbf{W}_0^*$ is used

- Enlarged set of moments differs from Hotz and Miller (1993)

(ii) Pseudo ML

Aguirregabiria and Mira (2002, 2007)

- Conditional on the transition prob. estimates $\widehat{\mathbf{g}}_T$

- First step: $\widehat{\mathbf{p}}_T$

- Likelihood

$$\ell = \sum_{s \in \mathbf{S}} \sum_{i \in \mathbf{N}} \sum_{k \in \mathbf{A}_i} n_{kis} \log \mathbf{\Psi}_{kis} \left(\widehat{\mathbf{p}}_T, \widehat{\mathbf{g}}_T, \theta \right)$$

- FOC

$$\frac{\partial \ell}{\partial \theta} = \left(\nabla_\theta \mathbf{\Psi}' \right) \mathbf{\Sigma}_p^{-1}(\mathbf{\Psi}) \left[\widehat{\mathbf{p}} - \mathbf{\Psi} \left(\widehat{\mathbf{p}}_T, \widehat{\mathbf{g}}_T, \theta \right) \right].$$

where $\widehat{\mathbf{p}}$ is the sample frequency estimator
(possibly 2 distinct auxiliary estimators $\widehat{\mathbf{p}}$ and $\widehat{\mathbf{p}}_T$)

- Weight matrix is $\mathbf{\Sigma}_p^{-1}$

- Optimal (partial) weight matrix should be
$$\left(\left[\mathbf{I} - \nabla_p \mathbf{\Psi} \right] \mathbf{\Sigma}_p \left[\mathbf{I} - \nabla_{p'} \mathbf{\Psi} \right]' \right)^{-1}$$

- Aguirregabiria and Mira (2002) show that $\nabla_p \mathbf{\Psi}$ vanishes in single agent problems, hence efficient

$\nabla_p \mathbf{\Psi}$ does not vanish in games

5.2.4 Unobserved Heterogeneity

- In applications, unobservable (to the econometrician) variables play an important role.

- Aguirregabiria and Mira (2007) account for unobservable market-level heterogeneity.

- Assume each market has a pair of time-invariant variables $(\bar{s}_m, \omega_m)$, where $\bar{s}_m$ is observed by the econometrician, while ω_m is not. ω_m has a discrete and finite support $\Omega = \left\{\omega^1, ..., \omega^J\right\}$. ω_m follows (iid) pmf $\varphi_j(\bar{s}_m) \equiv \Pr\left(\omega_m = \omega^j | \bar{s}_m\right)$ and we assume

$$\Pr\left(\mathbf{s}_{m,t+1} | \mathbf{a}_{mt}, \mathbf{s}_{mt}, \omega_m\right) = g\left(\mathbf{s}_{m,t+1} | \mathbf{a}_{mt}, \mathbf{s}_{mt}\right).$$

- Now θ includes the parameters in the conditional distributions of ω.

- Non-parametric identification of finite mixture models in dynamic discrete choice: Kasahara and Shimotsu (2009)

- "One equilibrium in the data" is rewritten as: Only one equilibrium is played in the data conditional on market type $(\bar{s}_m, \omega_m)$.

- Observationally equivalent markets can have different choice probabilities because of different values of ω.

- Log likelihood

$$
\begin{aligned}
\ln \Pr(\text{data}|\theta, \mathbf{P}) &= \sum_{m=1}^{M} \ln \Pr(\widetilde{\mathbf{a}}_m, \widetilde{\mathbf{s}}_m | \theta, \mathbf{P}) \\
&= \sum_{m=1}^{M} \ln \left(\sum_{j=1}^{J} \varphi_j(\bar{s}_m) \Pr\left(\widetilde{\mathbf{a}}_m, \widetilde{\mathbf{s}}_m | \omega^j; \theta, \mathbf{P}\right) \right)
\end{aligned}
$$

where $\widetilde{\mathbf{a}}_m = (\mathbf{a}_{1m}, ..., \mathbf{a}_{Tm})$ and $\widetilde{\mathbf{s}}_m = (\mathbf{s}_{1m}, ..., \mathbf{s}_{Tm})$. Rearranging gives

$$
\begin{aligned}
\ln \Pr(\text{data}|\theta, \mathbf{P}) = & \sum_{m=1}^{M} \ln \left(\sum_{j=1}^{J} \varphi_j (\bar{s}_m) \left(\prod_{t=1}^{T} \prod_{i=1}^{N} \Psi_i \left(a_{imt} | \mathbf{s}_{mt}, \omega^j; \mathbf{P}_j, \theta \right) \right. \right. \\
& \left. \times \Pr(\mathbf{s}_{m1}|\omega^j; \theta, \mathbf{P}) \right) \\
& + \sum_{m=1}^{M} \sum_{j=1}^{J} \ln g \left(\mathbf{s}_{mt} | \mathbf{a}_{m,t-1}, \mathbf{s}_{m,t-1}; \theta \right),
\end{aligned}
$$

where $\mathbf{P}_j \equiv \left\{ \Pr \left(\mathbf{a}_{mt} = \mathbf{a} | \mathbf{s}_{mt} = \mathbf{s}, \omega_m = \omega^j \right) : (\mathbf{a}, \mathbf{s}) \in \mathbf{A} \times \mathbf{S} \right\}$.

- Note that $g \left(s_{mt} | \mathbf{a}_{m,t-1}, \mathbf{s}_{m,t-1}; \theta \right)$ can be estimated directly from data, so now we focus on the first part.

- One problem is $\Pr(\mathbf{s}_{m1}|\omega^j) \neq \Pr(\mathbf{s}_{m1})$, which is called the initial condition problem. Why is this problematic?

- To deal with this problem, within the algorithm, we can compute the steady-state distribution of $\mathbf{s}_{m1}$.

- For CCPs $\hat{\mathbf{P}}_j$ (one for each market type), let $p^*(\mathbf{s}'|g, \hat{\mathbf{P}}_j)$ denote the steady-state distribution of $\mathbf{s}_{m1}$. We can find this as the unique solution to the system of linear equations,

$$p^*(\mathbf{s}'|g, \hat{\mathbf{P}}_j) = \sum_{\mathbf{s}\in\mathbf{S}} g^{\hat{P}_j}\left(\mathbf{s}'|\mathbf{s}\right) p^*(\mathbf{s}|g, \hat{\mathbf{P}}_j)$$

where

$$g^{\hat{P}_j}\left(\mathbf{s}'|\mathbf{s}\right) = \sum_{\mathbf{a}\in\mathbf{A}} \hat{\mathbf{P}}_j\left(\mathbf{a}|\mathbf{s}\right) g\left(\mathbf{a}, \mathbf{s}, \mathbf{s}'\right).$$

- Pseudo likelihood function is

$$Q_M\left(\theta, \mathbf{P}, g\right) = \frac{1}{M}\sum_{m=1}^{M} \ln\left(\sum_{j=1}^{J} \varphi_j\left(\bar{s}_m\right)\left(\prod_{t=1}^{T}\prod_{i=1}^{N} \Psi_i\left(a_{imt}|\mathbf{s}_{mt}, \omega^j; \mathbf{P}_j, \theta\right)\right)\right.$$
$$\left.\times\ p^*(\mathbf{s}_{m1}|g, \mathbf{P}_j)\right).$$

5.2.5 Possible Inconsistency of NPL

- Recall NPL: a sequence of estimators $\{\hat{\theta}_K : K \geq 1\}$ defined as

$$\hat{\theta}_K = \arg\max_{\theta \in \Theta} Q(\theta, \hat{\mathbf{p}}_{K-1}) \tag{9}$$

  and the probabilities $\{\hat{\mathbf{p}}_K : K \geq 1\}$ are obtained recursively as

$$\hat{\mathbf{p}}_K = \mathbf{\Psi}(\hat{\mathbf{p}}_{K-1}, \hat{\theta}_K). \tag{10}$$

- Let $\mathcal{Y}_M$ be the set of NPL fixed points. Then the NPL estimator is defined as

$$(\hat{\theta}_{NPL}, \hat{\mathbf{p}}_{NPL}) = \arg\max_{(\theta,\mathbf{p}) \in \mathcal{Y}_M} Q_M(\theta, \mathbf{p}).$$

- To guarantee consistency, one has to compare all NPL fixed points in the sample. This can be a daunting task, as the next example demonstrates.

- Based on Pesendorfer and Schmidt-Dengler (2010)

- Consider a game with two firms where each firm chooses whether to be active or not at each time period.

- If firm i is active and firm j is not, i receives ε_i^1. If firm i is active and so is firm j, then i receives $\theta + \varepsilon_i^1$ where $\theta < 0$. If firm i is not active, he receives ε_i^2. ε is private information.

- $\theta_0 \in [-10, -1]$.

- Assume the distribution of $\varepsilon_i^1 - \varepsilon_i^2$ is F_α defined as

$$F_\alpha(\varepsilon_i) = \begin{cases} 1 - \alpha + 2\alpha \left[\Phi\left(\frac{\varepsilon_i - 1 + \alpha}{\sigma}\right) - \frac{1}{2}\right] & [1-\alpha, \infty) \\ \varepsilon_i & [\alpha, 1-\alpha) \\ 2\alpha\Phi\left(\frac{\varepsilon_i - \alpha}{\sigma}\right) & [-\infty, \alpha) \end{cases}$$

- ε_i is uniform for the relevant range. By construction F_α approaches the uniform distribution in the limit when α vanishes.

- Let P^i denote the probability that firm i is active and let $\mathbf{P} = \left(P^1, P^2\right)$.

- Firm i is not active if and only if

$$\left(\theta + \varepsilon_i^1\right) P^j + \varepsilon_i^1 \left(1 - P^j\right) < \varepsilon_i^2,$$

  and thus the firm i's probability of being active is

$$\begin{aligned} P^i &= \mathbf{\Psi}\left(P^j, \theta\right) \\ &= 1 - F_\alpha\left(-\theta P^j\right). \end{aligned}$$

- Denote $\mathbf{\Psi}\left(\mathbf{P}, \theta\right) = \left(\mathbf{\Psi}\left(P^2, \theta\right), \mathbf{\Psi}\left(P^1, \theta\right)\right)$. An equilibrium solves $\mathbf{P} = \mathbf{\Psi}\left(\mathbf{P}, \theta\right)$.

- The symmetric equilibrium for α small is $P^1 = P^2 = \frac{1}{1-\theta}$.

- A NPL fixed point is a pair $(\hat{\theta}, \hat{\mathbf{P}})$ that satisfies

$$\hat{\theta} = \arg\max_{\theta \in \Theta} Q_M(\theta, \hat{\mathbf{P}}) \text{ and } \hat{\mathbf{P}} = \mathbf{\Psi}(\hat{\mathbf{P}}, \hat{\theta}),$$

and the NPL estimator is the fixed point that maximizes the pseudo likelihood.

Result (i) $\widetilde{\mathbf{P}}_M \to_P \mathbf{P}_0$. (ii) Suppose $\widetilde{\mathbf{P}}_M$ is the starting value of the NPL choice probability sequence. Then $\hat{\theta}_M^\infty \to_P -1$ for any $\theta_0 \in (-10, -1)$.

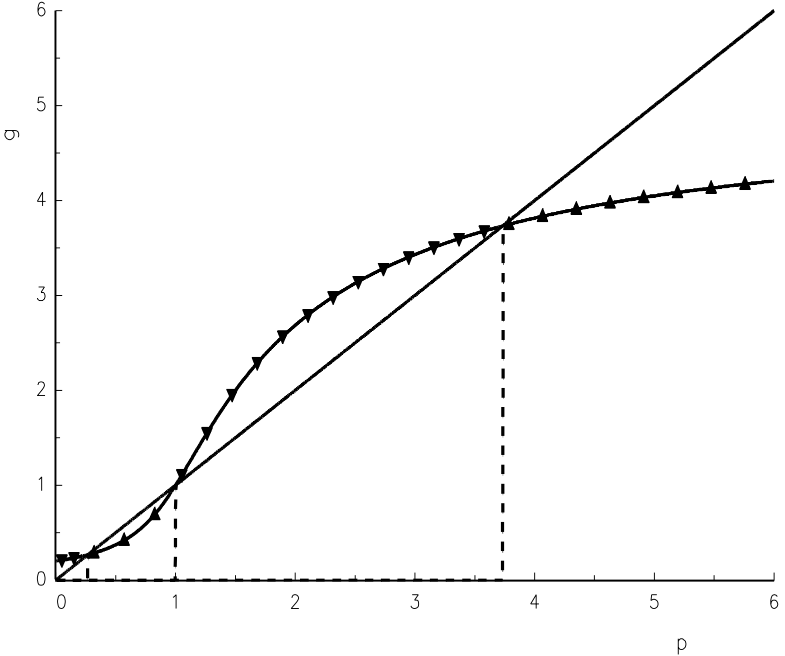
- Define $p_{K-1} = P_{K-1}^2 / P_{K-1}^1$. (9) and (10) imply the following difference equation:

$$p_K = g_M\left(p_{K-1}\right) = \frac{1 + h_M\left(p_{K-1}\right) / p_{K-1}}{1 + h_M\left(p_{K-1}\right)}$$

where

$$\begin{aligned} h_M(p) = & -\frac{2 - \widetilde{P}_M^1}{4} - \frac{2 - \widetilde{P}_M^2}{4} p \\ & + \frac{1}{4}\sqrt{[2 - \widetilde{P}_M^1 - (2 - \widetilde{P}_M^2)p]^2 + 4\widetilde{P}_M^1 \widetilde{P}_M^2 p}. \end{aligned}$$

- If $\theta_0 = -2$, there are three fixed points.



g
p
0
1
2
3
4
5
6

- The fixed point with $p = 1$ yields $\hat{\theta} = -2$, but is unstable. Other two fixed points yield $\hat{\theta} = -1$ and are stable.

- The NPL method converges to a stable fixed point with probability approaching 1 as M increases. With the frequency estimator, the probability that $\hat{p} = 1$ approaches zero as M increases

\- Kasahara and Shimotsu (2012) show that divergence may occur when the fixed point mapping does not have a local contraction property.

- They propose several alternative sequential estimation procedures.

- For example, they consider a class of mappings obtained as a log-linear combination of $\Psi(\mathbf{P}, \theta)$ and $\mathbf{P}$:

$$[\Lambda(\mathbf{P}, \theta)](\mathbf{a}|\mathbf{s}) \equiv \{\Psi(\mathbf{P}, \theta)(\mathbf{a}|\mathbf{s})\}^{\alpha} P(\mathbf{a}|\mathbf{s})^{1-\alpha}$$

- $\mathbf{P}$ is a fixed point of $\Psi(\mathbf{P}, \theta)$ if and only if it is a fixed point of $\Lambda(\mathbf{P}, \theta)$.

- Practical issue: how do we choose α?

5.3 The BBL algorithm

- A two-step method widely used in applications

- Use forward simulation proposed by Hotz, Miller, Sanders, and Smith (1994)

- Equilibrium conditions give a set of inequalities

- Choice variables can either be discrete or continuous

A.1 (Equilibrium Selection) The data are generated by a single Markov perfect equilibrium profile σ

- This may not hold particularly if one pools data from many markets

5.3.1 First Stage

Estimating Policy Functions and State Transitions

- First step is to estimate the policy functions σ and state transition probabilities $g\left(s' | s, a_i, a_{-i}\right)$

- If first stage estimation is too restrictive, policy functions may not be consistent with an equilibrium of the underlying model. On the other hand, if the state space is large, some restriction is necessary in practice

- One possibility is to fully exploit restrictions implied by theory; symmetry, constant returns, or monotonicity of optimal policies

- Under the static-dynamic breakdown, the demand and marginal cost parameters can be estimated using standard static methods

Discrete Choice

- Remember

$$\sigma_i\left(s, \varepsilon_i\right) = \arg\max_{a_i \in A} \left\{v_i^\sigma\left(a_i, s\right) + \varepsilon_i\left(a_i\right)\right\}.$$

- Put differently, firm i chooses an action a_i if and only if

$$v_i^\sigma\left(a_i, s\right) + \varepsilon_i\left(a_i\right) \geq v_i^\sigma\left(a_i', s\right) + \varepsilon_i\left(a_i'\right) \;\; \forall a_i' \in A_i.$$

- Hotz and Miller (1993) showed how to recover the difference in the choice-specific value functions using the difference in conditional choice probabilities. Clearly, the difference in the choice specific value functions is enough to estimate the policy rules σ.

- When the state space is large, a state-by-state inversion may be imprecise. One may want to flexibly parameterize the choice-specific value functions.

Continuous Choice

- Optimal policies have the form $a_i = \sigma_i(s, \varepsilon_i)$ where $a_i \in A_i \subset \mathbb{R}$ and ε is a scalar

A.2 (Monotone Choice) For each agent, $\widetilde{\Pi}_i(a, s, \varepsilon_i)$ has increasing differences in (a_i, ε_i)

- This implies firm i's optimal policy $\sigma_i(s, \varepsilon_i)$ is strictly increasing in ε_i

- Let $F_i(\bar{a}|s) = \mathsf{Pr}(a_i \leq \bar{a}|s)$. Thus,

$$
\begin{aligned}
F_i(\bar{a}|s) &= \mathsf{Pr}(a_i \leq \bar{a}|s) = \mathsf{Pr}(\sigma_i(s, \varepsilon_i) \leq \bar{a}|s) \\
&= \mathsf{Pr}\left(\varepsilon_i \leq \sigma^{-1}(s, \bar{a})\,|s\right) \\
&= G\left(\sigma^{-1}(s, \bar{a})\,|s\right).
\end{aligned}
$$

- Substitute $\bar{a} = \sigma_i(s, \varepsilon_i)$

$$F_i(\sigma_i(s, \varepsilon_i)|s) = G(\varepsilon_i|s)$$

$\Leftrightarrow$

$$\sigma_i(s, \varepsilon_i) = F_i^{-1}(G(\varepsilon_i|s)|s).$$

Thus, if one knows $F_i(\bar{a}|s)$ and G, the policy function at a given state x is recovered

- One may want to smooth the estimates of F_i across states and impose some theoretical restrictions on F_i

Estimating the Value Functions

- For a given Markov strategy σ

$$V_i^{\sigma}(s) = E\left[\sum_{t=1}^{\infty} \beta^{t-1}\, \widetilde{\Pi}_i\left(\sigma\left(s_t, \varepsilon_t\right), s_t, \varepsilon_{it}\right) \Big| \, s\right],$$

  where the expectation is over current and future values of the private shocks ε_t and states s

- Given first-stage estimate $\hat{f}$ of the transition probabilities, the value function $V_i^{\sigma}(s)$ can be estimated for any strategy profile σ and structural parameters θ

- Simulation procedure:

  Step 1. Draw ε_{i1} from G for each firm i

  Step 2. Calculate $a_{i1} = \sigma_i\left(s_1, \varepsilon_{i1}\right)$ for each i and the profit $\widetilde{\Pi}_i\left(a_1, s_1, \varepsilon_1\right)$

Step 3. Draw a new state s_2 using $\hat{g}\left(s_2|s_1, a_{i1}, a_{-i1}\right)$

Step 4. Repeat Steps 1-3 for T periods

Step 5. Average firm i's discounted sum of profits over many simulated paths of play. Let $\hat{V}_i^{\sigma}(s)$ denote an estimate of $V_i^{\sigma}(s)$

- One way to significantly reduce the computation burden is to assume

$$\widetilde{\Pi}_i\left(a, s, \varepsilon_i\right) = \psi\left(a, s, \varepsilon_i\right) \cdot \theta$$

where $\psi\left(a, s, \varepsilon_i\right)$ is a vector of basis functions. Then,

$$V_i^{\sigma}(s) = E\left[\sum_{t=1}^{\infty} \beta^{t-1} \psi\left(\sigma\left(s_t, \varepsilon_t\right), s_t, \varepsilon_{it}\right) \Big| \, s\right] \cdot \theta = W_i^{\sigma}(s) \cdot \theta.$$

- $W_i^{\sigma}(s)$ does not depend on structural parameters, so for any strategy profile σ, one can use the forward simulation procedure once to estimate each W_i^{σ} and then obtain V_i^{σ} easily for any value of θ

5.3.2 Second Stage

- We combine the firm's equilibrium policy functions, the state transitions, and the value functions to estimate the structural parameters

A.3 $\sigma_i(s, \varepsilon_i)$ and $g(s'|s, a_i, a_{-i})$ are parameterized by a finite parameter vector α, and there is a consistent estimator $\hat{\alpha}_M$

- σ^* is a Markov perfect equilibrium if and only if for all firms i, all state s, and all alternative Markov policies σ_i',

$$V_i^{\sigma_i^*, \sigma_{-i}^*}(s) \geq V_i^{\sigma_i', \sigma_{-i}^*}(s). \tag{11}$$

- Then, the set of parameters that rationalize the strategy profile σ^*

$$\Theta_0(\sigma^*, g) = \left\{\theta : \theta, \sigma^*, g \text{ satisfy (11) for all } s,\ i,\ \sigma_i'\right\}.$$

- Define

$$d\left(i, s, \sigma_i'; \theta, \alpha\right) = V_i^{\sigma_i^*, \sigma_{-i}^*}(s; \theta, \alpha) - V_i^{\sigma_i', \sigma_{-i}^*}(s; \theta, \alpha).$$

- Define

$$Q(\theta, \alpha) := \int (\min\left\{d\left(i, s, \sigma_i'; \theta, \alpha\right), 0\right\}^2) dH(z),$$

where $z = (i, s, \sigma_i') \in \mathcal{Z}$ and H is a distribution over the set $\mathcal{Z}$

- True parameters θ_0 satisfies $Q(\theta_0, \alpha_0) = 0$. Thus, the proposed estimator $\hat{\theta}$ minimizes the sample analogue of $Q(\theta, \alpha_0)$:

$$Q_M(\theta, \alpha) = \frac{1}{n_I} \sum_{k=1}^{n_I} (\min\left\{\hat{d}(z_k; \theta, \alpha), 0\right\})^2.$$

- In practice, the estimator may not be robust to choice of counterfactual policies

5.4 Application: Ryan (2012)

- Cost analysis of an environmental regulation is typically an engineering estimate of the expenditures on control and monitoring equipment necessary for compliance

- This static analysis ignores the dynamic effects of the regulation on entry, investment, and market power

- Shifts in the costs of entry and investment can lead to markets with few firms, which in turn may result in welfare loss due to concentration

- Ryan (2012) measures the welfare costs of the 1990 Clean Air Act Amendments on the US Portland cement industry, explicitly accounting for the dynamic effects resulting from a change in the cost structure

5.4.1 Model

- A game of capacity accumulation with entry and exit, based on Ericson and Pakes (1995)

- Infinite horizon and time is discrete

- In each period, (i) potential entrants draw investment and entry costs and incumbents draw investment/divestment costs and scrap value; (ii) all firms simultaneously make entry, exit, and investment decision; (iii) incumbent firms compete over quantities in the product market; and (iv) firms enter/exit, and investment mature

- Firms compete in quantities in a homogeneous goods product market. Demand is given by

$$\ln Q_m(\alpha) = \alpha_{0m} + \alpha_1 \ln P_m, \tag{12}$$

where $Q_m = \sum q_{mi}$ and P_m is the market price

- Cost of output q_i

$$C_i\left(q_i;\delta\right)=\delta_0+\delta_1 q_1+\delta_2 \mathbf{1}\left(q_i>vs_i\right)\left(q_i-vs_i\right)^2,$$

where s_i is capacity and $\left\{s_1,...,s_{\bar{N}}\right\}$ characterizes the market. Let $\bar{\pi}_i\left(s;\alpha,\delta\right)$ denote the profit earned in the capacity-constrained Cournot game. In estimation, add "post-1990 shifter" for each parameter

- Cost of capacity adjustment x_i

$$\Gamma\left(x_i;\gamma\right)=\mathbf{1}\left(x_i>0\right)\left(\gamma_{i1}+\gamma_2 x_i+\gamma_3 x_i^2\right)+\mathbf{1}\left(x_i<0\right)\left(\gamma_{i4}+\gamma_5 x_i+\gamma_6 x_i^2\right).$$

- Fixed costs

$$\mathbf{\Phi}_i\left(a_i;\kappa_i,\phi_i\right)=\begin{cases}-\kappa_i & \text{new entrant}\\ \phi_i & \text{exiting firm}\end{cases}$$

- Let $\epsilon_i=\left(\gamma_{i1},\gamma_{i4},\kappa_i,\phi_i\right)$ be a vector of private values

- Per-period payoff

$$\pi\left(s, a; \alpha, \delta, \epsilon_i\right) = \bar{\pi}_i\left(s; \alpha, \delta\right) - \Gamma\left(x_i; \gamma\right) + \Phi_i\left(a_i; \kappa_i, \phi_i\right). \tag{13}$$

- Values:

$$\begin{aligned}
V_i^{\sigma}\left(s, \epsilon_i\right) &= \bar{\pi}_i\left(s; \theta\right) \\
&+ \max\left\{\begin{array}{c}
\max_{x_i^*>0}\left[-\gamma_{i1} - \gamma_2 x_i^* - \gamma_3 x_i^{*2} + \beta \int E_{\epsilon_i} V_i^{\sigma}\left(s', \epsilon_i'\right) dP_{s'}\right], \\
\max_{x_i^*<0}\left[-\gamma_{i4} - \gamma_5 x_i^* - \gamma_6 x_i^{*2} + \beta \int E_{\epsilon_i} V_i^{\sigma}\left(s', \epsilon_i'\right) dP_{s'}\right], \\
+\beta \int E_{\epsilon_i} V_i^{\sigma}\left(s', \epsilon_i'\right) dP_{s'}, \phi_i
\end{array}\right\}
\end{aligned}$$

$$\begin{aligned}
&V_i^{e,\sigma}\left(s, \epsilon_i\right) \\
&= \max\left\{0, \max_{x_i^*>0}\left[-\gamma_{1i} - \gamma_2 x_i^* - \gamma_3 x_i^{*2} + \beta \int E V_i^{\sigma}\left(s, \epsilon_i\right) dP_{s'}\right] - \kappa_i\right\}
\end{aligned}$$

- Markov perfect Nash equilibrium requires for all s, ϵ_i, and σ' (corresponding inequalities hold for potential entrants)

$$V_i^{\sigma_i^*, \sigma_{-i}^*}\left(s, \epsilon_i\right) \geq V_i^{\sigma_i', \sigma_{-i}^*}\left(s, \epsilon_i\right).$$

5.4.2 Estimation

- Use the method proposed by Bajari, Benkard, and Levin (2007)

- Policy functions governing entry, exit, and investment along with the product market profit function are recovered in the first stage

- These functions are used in the second step and the restrictions of the MPE are imposed to recover the dynamic parameters governing the costs of capacity adjustment and exit

A.1 The same equilibrium is played in all markets

A.2 Firms assume that the regulatory environment is permanent

First Stage

- Estimate demand (12) using instrumental variables, such as gas prices, coal prices, electricity prices, and skilled labor wage rates

- Production costs are estimated in this stage too. For each guess of δ, solve for the vector of capacity-constrained Cournot quantities

- The estimator minimizes the sum of squared differences between the observed quantities and the predictions of the model

- For investment policy functions, assume first that firms have a target level of capacity, s_{it}^* which they adjust to when they make an investment:

$$\ln s_{it}^* = h_1 (s_{it}; \lambda_1) + h_2 \left(\sum\nolimits_{j \neq i} s_{jt}; \lambda_2 \right) + u_{it}^*.$$

Firms only adjust s_{it} to s^*_{it} when current capacity is out of the bound (s_l, s_h), which are given by

$$s_l = s^*_{it} - \exp\left(h_3(s_{it}; \lambda_3) + h_4\left(\sum_{j\neq i} s_{jt}; \lambda_4\right) + u_l\right)$$
$$s_h = s^*_{it} + \exp\left(h_3(s_{it}; \lambda_3) + h_4\left(\sum_{j\neq i} s_{jt}; \lambda_4\right) + u_h\right),$$

where $h_k(\cdot; \lambda_k)$ is a flexible function parameterized by λ_k. This specification can generate lumpy investment behavior

- Entry and exit policy functions are given by

$$\Pr(\chi_i = 1; s_i = 0, s) = \Phi\left(\psi_1 + \psi_2\left(\sum_{j\neq i} s_{jt}\right) + \psi_3 \mathbf{1}(t > 1990)\right)$$

and

$$\Pr(\chi_i = 1; s_i > 0, s) = \Phi\left(\psi_4 + \psi_5 s_{it} + \psi_6\left(\sum_{j\neq i} s_{jt}\right) + \psi_7 \mathbf{1}(t > 1990)\right),$$

where Φ is the CDF of the standard normal

Estimation - Second Stage

- First one needs to integrate out the private shocks in (13)

$$\begin{aligned}\pi(s,a;\alpha,\delta,\epsilon_i) = {} & \bar{\pi}_i(s;\alpha,\delta) + p_e(s)\,E(\phi_i|\text{exit}) \\ & + p_I(s)\left(E(\gamma_{i1}|x_i>0) + \gamma_2 x_i + \gamma_3 x_i^2\right) \\ & + p_D(s)\left(E(\gamma_{i4}|x_i<0) + \gamma_5 x_i + \gamma_6 x_i^2\right).\end{aligned}$$

where $p_I(s) = \Pr(\mathbf{1}(x_i > 0) = 1)$, $p_D(s) = \Pr(\mathbf{1}(x_i < 0) = 1)$, and $p_e(s) = \Pr(\text{exit})$

- Remember that all these conditional probabilities can be written as a function of $p_I(s)$, $p_D(s)$, and $p_e(s)$

- Ryan (2012) linearizes these probabilities using linear b-splines tensor products

- Define

$$d\left(\sigma_i';\theta\right) = V_i\left(s;\sigma_i^*,\sigma_{-i}^*,\theta\right) - V_i\left(s;\sigma_i',\sigma_{-i}^*,\theta\right)$$

  and

$$Q_n(\theta) = \frac{1}{n_k}\sum_{j=1}^{n_k} \min\left\{d\left(\sigma_i';\theta\right),0\right\}^2$$

  where n_k is the number of alternative policies. Then find the minimizer

- In practice, use the "linearity trick" we discussed last time; i.e., $V = W \cdot \theta$

- Add noise to the optimal policy functions to generate a set of inequalities

- Value function of a potential entrant is

$$V_i^e(s;\sigma,\theta,\varepsilon) = \max\left\{0, \max_{x_i}\left[-\kappa_i - \gamma_{1i} - \gamma_2 x_i - \gamma_3 x_i^2 + \beta E\left(V\left(s'\right)|s,\sigma\right)\right]\right\}$$

  where all terms except for κ have been obtained so far

- Assuming κ is distributed normally with mean μ_κ and variance σ_κ^2, we have

$$\Pr\left(\kappa_i + \gamma_{1i} \leq EV^e(s)\right) = \Phi\left(EV^e(s); \mu_\kappa + \mu_\gamma, \sigma_\kappa^2 + \sigma_\gamma^2\right).$$

- Drawing $s = \{1, ..., NS\}$ random states of the industry, search for parameters of this distribution which match the observed probabilities of entry as well as possible:

$$\min_{\{\mu_\kappa, \sigma_\kappa^2\}} \frac{1}{NS} \sum_{i=1}^{NS} \left[\Pr\left(\text{entry}|s_i\right) - \Phi\left(EV^e(s); \mu_\kappa + \mu_\gamma, \sigma_\kappa^2 + \sigma_\gamma^2\right)\right]^2.$$

5.4.3 Data

- Portland cement industry from 1980 to 1999

- Market level supply and demand data

  1. Quantity and price
  2. Plants in market
  3. Instruments (skilled wage, coal price, electricity price, natural gas price)
  4. Population

- Plant-level data: quantity, capacity and investment

- Local market

5.4.4 Results

- Amendments led to a marked increase in the expected entry costs

- Overall welfare has decreased between \$820M and \$3.2B, due to an increase in the average sunk cost of entry

- Costs of production have not changed significantly after the regulations, meaning that the welfare effect on producers depends critically on whether or not the firm is an incumbent

- Incumbent firms benefit from increased market power due to reduced competition.

- A static analysis thus gives the wrong sign for the welfare costs of the Amendments on incumbent firms while understating overall welfare costs by at least \$300M.