Lecture 10: Computation of Heterogeneous-Agent Economies

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1 Introduction

- We discuss computation of a simple heterogeneous-agent economy

- Remember our discussion on computing dynamic oligopoly models where

  - For any belief, we solve the value functions
  - For any value functions, we compute optimal choice probabilities, which will coincide with beliefs in MPE

- Roughly speaking, in computing heterogeneous-agent models, aggregation that corresponds to this second step is much more complicated and time consuming: computation of the distribution of the individual state variables

- Closely follow Chapter 7 of Heer and Maussner (2009)

2 A Simple Heterogeneous-Agent Model

- We assume there is no aggregate uncertainty

- We augment the standard Ramsey model

- While we allow households to be heterogeneous, we assume firms are all identical

- We focus on stationary equilibrium

2.1 Household

- The economy consists of a continuum of agents of total mass equal to one

- Households maximize their intertemporal utility

$$E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

where $\beta < 1$ is the discount factor.

- Each time period, an agent has (ϵ_t, a_t) where $\epsilon_t \in \{e, u\}$ is the employment status (e for employed and u for unemployed) and a_t is wealth

- the agent's instantaneous utility function is twice continuously differentiable, increasing and concave in consumption c_t:

$$u\left(c_t\right) = \frac{c_t^{1-\eta}}{1-\eta}, \quad \eta > 0$$

where η denotes the coefficient of relative risk aversion

- Labor supply is inelastic and normalized to one

- If the agent is employed ($\epsilon = e$) in period t, he earns gross wage w_t

- If the agent is unemployed ($\epsilon = u$) in period t, he receives b_t

- We assume $(1 - \tau)\, w_t > b_t$ where τ denotes the income tax rate

- The transition matrix of individual state is given by

$$\pi\left(\epsilon' | \epsilon\right) = \text{Pr}\left\{\epsilon_{t+1} = \epsilon' | \epsilon_t = \epsilon\right\} = \begin{pmatrix} p_{uu} & p_{ue} \\ p_{eu} & p_{ee} \end{pmatrix}. \tag{1}$$

- The transition of employment status is exogenous

- The household faces the following budget constraint

$$a_{t+1} = \begin{cases} (1 + (1 - \tau)\, r_t)\, a_t + (1 - \tau)\, w_t - c_t & \text{if } \epsilon = e \\ (1 + (1 - \tau)\, r_t)\, a_t + b_t - c_t & \text{if } \epsilon = u \end{cases} \tag{2}$$

where r_t denotes the interest rate in period t

- Each agent smoothes his consumption $\{c_t\}_{t=0}^{\infty}$ by holding the asset a

- Impose the asset constraint $a \geq a_{\min}$, where $a_{\min} \leq 0$

- The Lagrangean for the household:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \{\beta^t [u\,(c_t) + \lambda_t (\mathbf{1}_{\epsilon_t = u} b_t + (1 + (1-\tau)\, r_t)\, a_t \\ + \mathbf{1}_{\epsilon_t = e} (1-\tau) w_t - a_{t+1} - c_t)]\}$$

and take the first-order conditions with respect to c_t and a_{t+1}

$$\frac{u'\,(c_t)}{\beta} = E_t [u'\,(c_{t+1})\,(1 + (1-\tau)\, r_{t+1})]$$

- The solution is given by the policy function $c(\epsilon_t, a_t)$, which also gives next-period asset holdings $a_{t+1} = a'(\epsilon_t, a_t)$

- The policy function itself does not have time subscript

2.2 Production and Government

- Firms are owned by the households and maximize profits

- Output Y_t is produced using capital K_t and labor N_t:

$$Y_t = N_t^{1-\alpha} K_t^{\alpha}$$

- In competitive factor markets, factor prices are given by the value of marginal product:

$$r_t = \alpha \left(\frac{N_t}{K_t} \right)^{1-\alpha} - \delta \tag{3}$$

$$w_t = (1 - \alpha) \left(\frac{N_t}{K_t} \right)^{\alpha} \tag{4}$$

where δ is the depreciation rate of capital

- The government budget is assumed to balance in every period:

$$B_t = T_t$$

where B_t is the sum of unemployment compensation and T_t denotes government revenues

2.3 Stationary Equilibrium

- In a stationary equilibrium,

  - the aggregate variables and the factor prices are constant
  - the distribution of assets is constant for both the employed and unemployed agents (denoted by $F(e,a)$ and $F(u,a)$, with density of $f(e,a)$ and $f(u,a)$, respectively)
  - the numbers of employed and unemployed agents are constant
  - individual agents change their wealth and employment status over time

- Individual state is given by $(\epsilon, a) \in \mathcal{X} = \{e, u\} \times [a_{\mathsf{min}}, \infty)$

- The value function $V(\epsilon, a)$ satisfies the following Bellman equation:

$$V(\epsilon, a) = \max_{c,a'}[u(c) + \beta E\left\{V(\epsilon', a')|\epsilon\right\}] \tag{5}$$

with the budget constraint (24), the government policy $\{b, \tau\}$, and the stochastic process of ϵ

- Is the state space bounded?

Def A stationary equilibrium for a given government policy is a collection of $V(\epsilon, a)$, $c(\epsilon_t, a_t)$, $a'(\epsilon_t, a_t)$, $f(e, a)$, $f(u, a)$, $\{w, r\}$, and a vector of aggregates K, N, C, T, and B such that

1. aggregate variables are given by

$$
\begin{aligned}
K &= \sum_{\epsilon \in \{e,u\}} \int_{a_{\min}}^{\infty} a f(\epsilon, a)\, da \\
N &= \int_{a_{\min}}^{\infty} f(e, a)\, da \\
C &= \sum_{\epsilon \in \{e,u\}} \int_{a_{\min}}^{\infty} c(\epsilon, a) f(\epsilon, a)\, da \\
T &= \tau (wN + rK) \\
B &= (1 - N)\, b
\end{aligned}
$$

2. $c(\epsilon_t, a_t)$ and $a'(\epsilon_t, a_t)$ solve the Bellman equation in (17)

3. factor prices are given by (3) and (4)

4. the goods market clear:

$$N^{1-\alpha}K^{\alpha} + (1-\delta)K = C + K' = C + K$$

5. the government budget is balanced: $T = B$

6. the distribution of the individual state variable (ϵ, a) is stationary:

$$F\left(\epsilon', a'\right) = \sum_{\epsilon \in \{e,u\}} \pi(\epsilon'|\epsilon) F\left(\epsilon, a'^{-1}\left(\epsilon, a'\right)\right) \tag{6}$$

for all $(\epsilon', a') \in \mathcal{X}$ (when is a invertible?)

3 Computing the Stationary Equilibrium

3.1 Computation Algorithm

- The computation of the solution to the above problem consists of two basic steps

    1. the computation of the policy function $c(\epsilon_t, a_t)$ and $a'(\epsilon_t, a_t)$ for a given triplet $\{K, N, \tau\}$

    2. the computation of the invariant distribution: compute the distribution of the individual state variables, aggregate the individual state variables, and impose the aggregate consistency conditions

Algorithm 1. Compute the stationary employment N

2. Make initial guesses of the aggregate capital stock K and the tax rate τ

3. Compute the wage rate w and the interest rate r

4. Compute the household's decision functions

5. Compute the stationary distribution of assets for the employed and unemployed agents

6. Compute the capital stock K and taxes T that solve the aggregate consistency conditions

7. Compute the tax rate τ that solves the government budget

8. Update K and τ and return to Step 2 if necessary

- In Step 1, note that in our example, N_t only depends on the number of employed in the previous period N_{t-1} :

$$N_t = p_{ue}\left(1 - N_{t-1}\right) + p_{ee} N_{t-1}$$

- We already know how to do Step 4

- In Step 5, three methods to compute $F(\epsilon, a)$ are considered; (1) compute the distribution function on a discrete number of grid points over the assets, (2) use Monte-Carlo simulations by constructing a sample of households and tracking them over time, and (3) assume a specific functional form of the distribution function

3.2 Discretization of the Distribution Function

- Since ϵ is a binary variable in our example, we only discretize the asset level a into m grid points $\rightarrow$ the state variable (ϵ, a) can only take a discrete number of values $2m$

- Computing the Markov transition matrix of size $2m$ by $2m$ could be difficult

- We consider two iterative methods, one uses (6), while the other computes invariant density function

3.2.1 Computing Invariant Distribution Function

Algorithm 1. Place a grid on the asset space $\mathcal{A} = \{a_1 = a_{\text{min}}, a_2, ..., a_m = a_{\text{max}}\}$ such that the grid is finer than the one used to compute the optimal decision rules

2. Choose an initial piecewise distribution function $F_0(\epsilon = e, a)$ and $F_0(\epsilon = u, a)$ over the grid. The vectors have m rows each

3. Compute the inverse of the decision rule $a'(\epsilon, a)$

4. Iterate on

$$F_{i+1}\left(\epsilon', a'\right) = \sum_{\epsilon \in \{e,u\}} \pi(\epsilon'|\epsilon) F_i\left(\epsilon, a'^{-1}\left(\epsilon, a'\right)\right) \tag{7}$$

5. Iterate until F converges

- In Step 3, a' may not be invertible when $a' = a_{\text{min}}$ (why?)

- For this reason, define $a'^{-1}(\epsilon, a_{\text{min}})$ as the maximum a such that $a'(\epsilon, a) = a_{\text{min}}$

- Note that $a'(\epsilon, a)$ is stored for only a finite number of values $n < m$. Thus, we use linear interpolation for the computation of $a'(\epsilon, a)$ for $a_j < a < a_{j+1}$

- In Step 4, we impose two conditions:

  1. If $a'^{-1}\left(\epsilon, a_j\right) < a_{\text{min}}$, $F\left(\epsilon, a_j\right) = 0$
  2. If $a'^{-1}\left(\epsilon, a_j\right) \geq a_{\text{max}}$, $F\left(\epsilon, a_j\right) = g\left(\epsilon\right)$, where $g\left(\epsilon\right)$ denotes the ergodic distribution of the employment transition matrix

- To deal with round-off errors in computing $F_{i+1}(\epsilon', a')$, we normalize the number of all agents equal to one and multiply $F_{i+1}(e, a')$ and $F_{i+1}(u, a')$ by $g(e)/F_{i+1}(e, a_{\mathsf{max}})$ and $g(u)/F_{i+1}(u, a_{\mathsf{max}})$

- One problem here is that in (7), $a_0 = a'^{-1}\left(\epsilon, a_j\right)$, $j = 1, ..., m$ does not need to be a grid point. We use linear interpolation for the computation of $F(\epsilon, a)$ for $a_j < a < a_{j+1}$

- When calculating the aggregate capital K in Step 6, we also have to deal with this "discrete problem". Assume that the distribution of wealth a is uniform in any interval $[a_{j-1}, a_j]$

- Under this assumption, we have

$$\begin{aligned}
\int_{a_{j-1}}^{a_j} a f(\epsilon, a) da &= \int_{a_{j-1}}^{a_j} \frac{F\left(\epsilon, a_j\right) - F\left(\epsilon, a_{j-1}\right)}{a_j - a_{j-1}} da \\
&= \frac{1}{2}\left[\frac{a^2\left(F\left(\epsilon, a_j\right) - F\left(\epsilon, a_{j-1}\right)\right)}{a_j - a_{j-1}}\right]_{a_{j-1}}^{a_j} \\
&= \frac{1}{2}(\left(F\left(\epsilon, a_j\right) - F\left(\epsilon, a_{j-1}\right)\right)\left(a_j + a_{j-1}\right)
\end{aligned}$$

- Then, the aggregate capital is

$$\begin{aligned}
K &= \sum_{\epsilon \in \{e,u\}} \int_{a_{\text{min}}}^{\infty} a f\left(\epsilon, a\right) da \\
&\approx \sum_{\epsilon \in \{e,u\}} \left(\sum_{j=2}^{m} (\left(F\left(\epsilon, a_j\right) - F\left(\epsilon, a_{j-1}\right)\right) \frac{\left(a_j + a_{j-1}\right)}{2} + F\left(\epsilon, a_1\right) a_1\right)
\end{aligned}$$

- If the capital stock is close to the one in the previous iteration, we stop

- We can use a low accuracy for iterations (7) and value function iterations initially. We keep increasing the accuracy as the algorithm converges to the aggregate capital stock

- Note that this algorithm may not converge (we need to do trial and error)

- The convergence of the mean of the distribution during the final iteration over the capital stock is very slow. The convergence of the higher moments is even slower

3.2.2 Computing Invariant Density Function

Algorithm 1. Place a grid on the asset space $\mathcal{A} = \{a_1 = a_{\text{min}}, a_2, ..., a_m = a_{\text{max}}\}$ such that the grid is finer than the one used to compute the optimal decision rules

2. Choose an initial piecewise distribution function $f_0\left(\epsilon = e, a\right)$ and $f_0\left(\epsilon = u, a\right)$ over the grid. The vectors have m rows each

3. Set $f_{i+1}(\epsilon, a) = 0$ for all ϵ and a.

   (a) For every $a \in \mathcal{A}$, $\epsilon \in \{e, u\}$, compute the optimal next-period wealth $a_{j-1} \leq a' = a'(\epsilon, a) < a_j$

   (b) For all $a' \in \mathcal{A}$ and $\epsilon' \in \{e, u\}$ compute

$$f_{i+1}\left(\epsilon', a_{j-1}\right) = \sum_{\epsilon = e,u} \sum_{\substack{a \in \mathcal{A} \\ a_{j-1} \leq a'(\epsilon,a) < a_j}} \pi\left(\epsilon' | \epsilon\right) \frac{a_j - a'}{a_j - a_{j-1}} f_i\left(\epsilon, a\right)$$

$$f_{i+1}\left(\epsilon', a_j\right) = \sum_{\epsilon = e,u} \sum_{\substack{a \in \mathcal{A} \\ a_{j-1} \leq a'(\epsilon,a) < a_j}} \pi\left(\epsilon' | \epsilon\right) \frac{a' - a_{j-1}}{a_j - a_{j-1}} f_i\left(\epsilon, a\right)$$

4. Iterate until f converges

- This takes less time than the invariant distribution function iterations. It is because we do not have to compute the inverse of the policy function in the invariant density function iterations

3.3 Monte-Carlo Simulation

- We choose a large sample of households and track their behavior over time

- The household is subject to an employment shock that follows the Markov process (13)

Algorithm 1. Choose a sample size NS

2. Initialize the sample. Each household $i = 1, ..., NS$ is assigned an initial wealth level a_0^i and employment status ϵ_0^i

3. Compute the next-period wealth level $a'\left(\epsilon^i, a^i\right)$ for all $i = 1, ..., NS$

4. Use a random number generator to obtain $\epsilon^{i'}$ for all $i = 1, ..., NS$

5. Compute a set of statistics from this sample. We choose the mean and the standard deviation of a and ϵ

6. Iterate until the distributional statistics converge

- Since we use random numbers in Step 4, the number of employed agents does not need to be equal to the number of employed agents in the ergodic distribution $Ng(e)$

- Thus, we adjust the share of employed and unemployed agents in each iteration

- For example, if the number of employed agents in any iteration is higher than the ergodic number, we select a set of employed agents randomly and change their employment status to 'unemployed' until two numbers are equal

- Monte-Carlo simulation takes more time than the previous two, and the simulation distribution is less smooth

3.4 Function Approximation

- We approximate the distribution function by a flexible functional form with a finite number of coefficients

- Here we use the class of exponential functions for the n^{th} order approximation of the wealth holdings of the agents with employment status $\epsilon \in \{e, u\}$:

$$F(\epsilon, a) = 0 \quad \text{for } a < a_{\text{min}}$$
$$F(\epsilon, a) = \rho_0^{\epsilon} \int_{-\infty}^{a} e^{\rho_1^{\epsilon} x^1 + \cdots + \rho_n^{\epsilon} x^n} dx \quad \text{for } a \geq a_{\text{min}}$$

- Note that this allows for a positive number of agents at the borrowing constraint $a = a_{\text{min}}$

- For the exponential family, the first n moments capture the same information as the n+1 coefficients ρ_i

- Assume we focus on the first two moments μ^ϵ and $(\sigma^\epsilon)^2$ of the wealth distribution of the employed and the unemployed, respectively

- To find the values $\rho^\epsilon = \left(\rho_0^\epsilon, \rho_1^\epsilon, \rho_2^\epsilon\right)$, $\epsilon \in \{e, u\}$ that correspond to μ^ϵ and $(\sigma^\epsilon)^2$, we should solve

$$g(\epsilon) = \rho_0^\epsilon \int_{-\infty}^{a_{\max}} e^{\rho_1^\epsilon a + \rho_2^\epsilon a^2} da \tag{8}$$

$$\mu^\epsilon = \rho_0^\epsilon \int_{-\infty}^{a_{\max}} \max(a, a_{\min}) e^{\rho_1^\epsilon a + \rho_2^\epsilon a^2} da \tag{9}$$

$$(\sigma^\epsilon)^2 = \rho_0^\epsilon \int_{-\infty}^{a_{\max}} (\max(a, a_{\min}) - \mu^\epsilon)^2 e^{\rho_1^\epsilon a + \rho_2^\epsilon a^2} da \tag{10}$$

Algorithm 1. choose initial moments μ^ϵ and $(\sigma^\epsilon)^2$ for the wealth distribution for $\epsilon \in \{e, u\}$ and compute the corresponding parameters ρ^ϵ of the exponential distribution by solving the non-linear equation problem (8)-(10)

2. Compute the moments of the next-period wealth distribution for the employed and unemployed agents, respectively: for example, for $\epsilon = e$,

$$\mu^{e'} = \pi\left(e|e\right)\rho_0^e \int_{-\infty}^{a_{\mathsf{max}}} \mathsf{max}(a'(a,e), a_{\mathsf{min}}) e^{\rho_1^e a + \rho_2^e a^2} da$$
$$+ \pi\left(e|u\right)\rho_0^u \int_{-\infty}^{a_{\mathsf{max}}} \mathsf{max}(a'(a,u), a_{\mathsf{min}}) e^{\rho_1^u a + \rho_2^u a^2} da$$

and

$$\left(\sigma^{e\prime}\right)^2 = \pi\left(e|e\right)\rho_0^e \int_{-\infty}^{a_{\mathsf{max}}} (\mathsf{max}(a'(a,e), a_{\mathsf{min}}) - \mu^e)^2 e^{\rho_1^e a + \rho_2^e a^2} da$$
$$+ \pi\left(e|u\right)\rho_0^u \int_{-\infty}^{a_{\mathsf{max}}} (\mathsf{max}(a'(a,u), a_{\mathsf{min}}) - \mu^u)^2 e^{\rho_1^u a + \rho_2^u a^2} da$$

and compute the parameters of the distribution function ρ^ϵ, $\epsilon \in \{e, u\}$, corresponding to the computed next-period moments μ' and σ'^2

3. Iterate until the moments μ^ϵ and σ^ϵ converge

- In Step 2, we can apply Gauss-Chebyshev quadrature

- Approximation by this algorithm may be poor (especially if f is not similar to the parametrized function)

- Remarks

  1. Try density function iterations first

  2. If the functional form of the density function is similar to the one of a parametrized function, we may want to try the function approximation algorithm

  3. Monte Carlo simulations are easy to implement

  4. Discretization methods work better for our example, but Monte-Carlo simulation may be come better if the state space is of higher dimensions

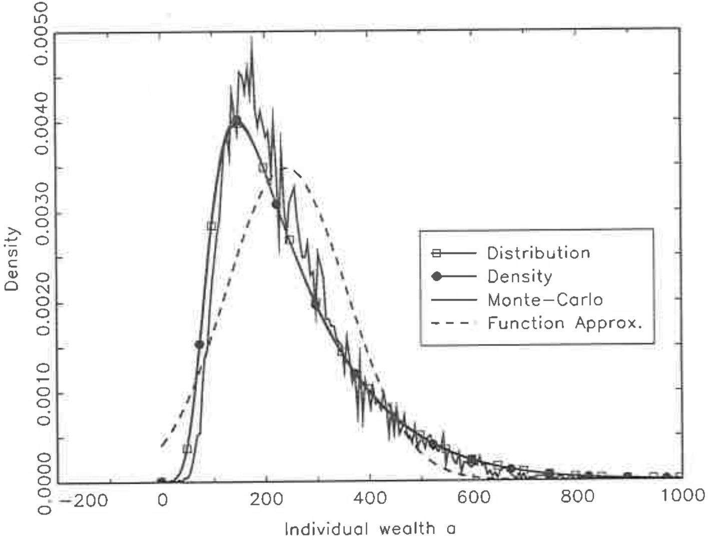

Figure 7.4: Invariant Density Function of Wealth for the Employed Worker

4 Application: Huggett (1993)

- Huggett (1993) tries to explain the "unusually" low risk-free rate in the data by introducing incomplete insurance (credit constraint)

- A simple exchange economy without production

- The endowment set $\mathcal{E} \in \{e_h, e_l\}$ where e_h is the earnings during employment and e_l is the earnings during unemployment, follows the first order Markov process $\pi(e'|e)$

- The agent maximizes

$$E_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

where

$$u(c_t) = \frac{c^{1-\eta}}{1-\eta}$$

with $\frac{1}{\eta}$ measuring the intertemporal elasticity of substitution

- Agents may hold a single asset $a \in \mathcal{A}$; a credit balance of a units is worth a units of consumption goods

- To obtain a credit balance of a' goods next period, $a'q$ goods have to be paid this period (we can interpret $r = 1/q - 1$)

- Agents cannot run a credit balance below $\bar{a} < 0$ (credit constraint)

- The budget constraint of the household is

$$c + a'q = a + e, \quad \text{where } a' \geq \bar{a} \tag{11}$$

- The value function is

$$v(e,a;q) = \max_{c,a'} u(c) + \beta \sum_{e'} \pi(e'|e) v(e',a';q)$$

subject to (11)

Def A stationary equilibrium for the exchange economy is a vector $(c(e,a)$, $a'(e,a)$, q, $F(e,a))$ satisfying

1. $c(e,a)$ and $a'(e,a)$ are optimal decision rules given q

2. Market clear:

$$\sum_e \left(\int_{\bar{a}}^{\infty} c(e,a) f(e,a) da + c(e,\bar{a}) F(e,\bar{a}) \right)$$
$$= \sum_e \left(\int_{\bar{a}}^{\infty} e f(e,a) da + e F(e,\bar{a}) \right)$$

and

$$\sum_e \left(\int_{\bar{a}}^{\infty} a'(e,a) f(e,a) da + a'(e,\bar{a}) F(e,\bar{a}) \right) = 0$$

3. $F(e, a)$ is a stationary distribution:

$$F(e', a') = \pi(e'|e_h)F(e_h, a_h) + \pi(e'|e_l)F(e_l, a_l)$$

for all $a' \in \mathcal{A}$ and $e' \in \{e_l, e_h\}$ and with $a' = a'(e_h, a_h) = a'(e_l, a_l)$

Algorithm 1. Make an initial guess of the interest rate r and compute the policy functions

2. Compute the stationary equilibrium and the equilibrium average asset holdings

3. Update the interest rate and return to Step 1 if necessary

- Computation of the stationary equilibrium is almost identical to our previous example

  - In the previous example, the equilibrium interest rate can be computed from the marginal product of capital

- In the present exchange economy, we have to guess the equilibrium price of next-period capital q which clears the credit market

Modification 1. Make two initial guesses of the interest rate (r_1, r_2)

2. Compute the average asset holdings for the two cases a_1 and a_2

3. Use the secant method: given two points (a_s, r_s) and (a_{s+1}, r_{s+1}), get

$$r_{s+2} = r_{s+1} - \frac{r_{s+1} - r_s}{a_{s+1} - a_s} a_{s+1}$$

- Under the "plausible" parameter values, the model can explain that the empirically observed risk-free rate of return is lower than the one found in standard representative-agent models

5 Dynamics of A Heterogeneous-Agent Model

- We discuss transition dynamics

- Closely follow Chapter 8 of Heer and Maussner (2009)

- Initially, we keep the assumption that there is no aggregate uncertainty

- Households maximize their intertemporal utility

$$E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

subject to

$$a_{t+1} = \begin{cases} \left(1 + \left(1 - \tau_t\right) r_t\right) a_t + \left(1 - \tau_t\right) w_t - c_t & \text{if } \epsilon = e \\ \left(1 + \left(1 - \tau_t\right) r_t\right) a_t + b_t - c_t & \text{if } \epsilon = u \end{cases} \tag{12}$$

- The transition matrix of individual state is given by

$$\pi\left(\epsilon'|\epsilon\right) = \mathsf{Pr}\left\{\epsilon_{t+1} = \epsilon'|\epsilon_t = \epsilon\right\} = \begin{pmatrix} p_{uu} & p_{ue} \\ p_{eu} & p_{ee} \end{pmatrix}. \tag{13}$$

- The aggregate consistency condition

$$K = \sum_{\epsilon \in \{e,u\}} \int_{a_{\mathsf{min}}}^{\infty} a_t f_t\left(\epsilon_t, a_t\right) da_t \tag{14}$$

  where f_t denotes the density function corresponding to the distribution functions F_t

- Outside the stationary equilibrium, the income tax rate τ_t is no longer time-invariant

- The dynamics of the distribution are described by the distribution of the individual state variable (ϵ, a):

$$F_{t+1}(\epsilon_{t+1}, a_{t+1}) = \sum_{\epsilon_t \in \{e,u\}} \pi(\epsilon_{t+1}|\epsilon_t) F_t\left(\epsilon_t, a_{t+1}^{-1}(\epsilon_t, a_{t+1})\right) \equiv G(F_t) \tag{15}$$

- The factor pries are determined by

$$r_t = \alpha \left(\frac{N_t}{K_t}\right)^{1-\alpha} - \delta$$
$$w_t = (1-\alpha)\left(\frac{K_t}{N_t}\right)^{\alpha}$$

- The first-order conditions with respect to c_t and a_{t+1}

$$u'(c_t) = \beta E_t[u'(c_{t+1})(1 + (1-\tau_{t+1}) r_{t+1})] \tag{16}$$

- For a given vector of $(K_t, N_t, w_t, r_t, \tau_t, F_t)$, the solution to (16) gives a time-invariant function $a'(\epsilon, a, F)$

- The difference from the case of stationary equilibrium is that F is included as an argument

- K_t and r_t are not constant, so we need information on K_t in the argument

- We condition on $F_t(\epsilon_t, a_t)$, instead of K_t

- Why? First, K_t can be calculated from $F_t(\epsilon_t, a_t)$ (see equation (14)).

- Second, information given by the total capital, K_t is not sufficient

  - Remember that the RHS of (16) includes r_{t+1}
  - In the stationary economy, we simply have $r = r_{t+1}$
  - In a representative-agent case, $K_{t+1} = a_{t+1}$

- However, in a transition dynamics with heterogeneous agents, to infer K_{t+1}, each agent has to know how all the other households in the economy decide and how much they save

- Thus, $F_t(\epsilon_t, a_t)$ is also an argument of the policy function

- Why isn't N an additional argument for the policy function?

- The household maximizes the value function

$$V(\epsilon, a, F) = \max_{c,a'}[u(c) + \beta E \left\{ V(\epsilon', a', F') | \epsilon, F \right\}] \tag{17}$$

subject to (24)

- To compute the transition dynamics for this type of model, we need to approximate the dynamics of the distribution (15)

1. Using "partial information": assume that households do not use all the information at hand

2. A shooting method which is only applicable to models with aggregate certainty

- Both approaches assume that the stationary equilibrium is stable and that the distribution function converges to the invariant distribution function

5.1 Partial Information

- We assume that agents perceive the dynamics of the distribution $F' = G(F)$ in a simplified way

- Agents characterize the distribution F by I statistics $m = (m_1, ..., m_I)$

- Consider the simple case where agents only use the first moment m_1 (for discussion, see Krusell and Smith, 1998)

- If most agents have approximately the same savings rate, the omission of higher moments is justified

- Accordingly, we assume that agents perceive the law of motion for m as follows:

$$m' = H_I(m) \tag{18}$$

c.f. Inclusive value sufficiency, used by, for example, Gowrisankaran and Rysman (2011)

- Given this law of motion and the initial value of m, the maximization problem is

$$V(\epsilon, a, m) = \max_{c, a'} [u(c) + \beta E \left\{ V(\epsilon', a', m') | \epsilon, m \right\}]$$

subject to (24), the government policy $\{b, \tau\}$, (13), and (18)

- Using a replacement ratio $\zeta \equiv b/(1-\tau)w$,

$$T = \tau N^{1-\alpha} K^{\alpha} = B = (1-N)\, b$$

$$b = \zeta\,(1-\tau)\, w = \zeta\,(1-\tau)\,(1-\alpha)\left(\frac{K}{N}\right)^{-\alpha}$$

- Finally, we need to approximate the law of motion for the moments m of the distribution. Assume

$$\ln K' = \gamma_0 + \gamma_1 \ln K \tag{19}$$

- Given the function H_I, we can solve the consumer's problem, which gives optimal policy functions

- With F_0 with mean K_0, we can simulate the behavior of the economy over time (the law of motion for K)

- We can compare the law of motion with the form (19)

- Algorithm

  1. Choose the initial distribution of assets F_0 with mean K_0
  2. Choose the order I of moments m
  3. Guess a parameterized functional form for H_I and choose initial parameters of H_I
  4. Solve the consumer's optimization problem and compute policies
  5. Simulate the dynamics of the distribution using (15)
  6. Use the time path for the distribution to estimate the law of motion for the moments m (γ_0 and γ_1)
  7. Iterate until the parameters of H_I converge
  8. Test the goodness of fit for H_I. If the fit is satisfying, stop, otherwise increase I or choose a different functional form for H_I

- This time, we have to choose grids for K : $\mathcal{K} = \{K_1, ..., K_{n_K}\}$, as well as $\mathcal{A} = \{a_1, ..., a_n\}$

- How to choose F_0 and K_0? If you are interested in simulating the effect of a policy change (say an increase in b), then you use the invariant distribution and K^* from the stationary equilibrium

- For Step 4, given value functions in iteration l, the value function of the employed agent for iteration $l + 1$ is

$$V_{l+1}(e, a_i, K_j) = \max_{a' \in A} u(c) + \beta \left\{ p_{ee} V_l(e, a', e^{\gamma_0 + \gamma_1 \ln K_j}) + p_{eu} V_l(u, a', e^{\gamma_0 + \gamma_1 \ln$$

with

$$c = (1 + (1 - \tau) r \left(K_j \right)) a_i + (1 - \tau) w \left(K_j \right) - a'$$

and the value function for the unemployed agent is

$$V_{l+1}(u, a_i, K_j) = \max_{a' \in A} u(c) + \beta \left\{ p_{ue} V_l(e, a', e^{\gamma_0 + \gamma_1 \ln K_j}) + p_{uu} V_l(u, a', e^{\gamma_0 + \gamma_1$$

with

$$c = (1 + (1 - \tau)\, r\left(K_j\right))a_i + b - a'$$

- The value function is computed for every aggregate capital stock $K_j \in \mathcal{K}$ and $a_i \in \mathcal{A}$

  - The outer loop of the iteration is over the capital stock K_j

  - Given $K = K_j$,we can compute K' using (19) as well as (w, r, b, τ)

  - For given (w, r, b, τ, K'), we can compute $V_{l+1}(e, a_i, K_j)$ and $V_{l+1}(u, a_i, K_j)$ at every grid point $a = a_i$

- Notice that the fact that we "know" K' in the RHS of the Bellman equation helps a lot (Why?)

- In Step 5, given F_0, we compute the dynamics of the distribution function (F_1) using (15) and the agent's savings function

- The optimal policy functions off grid points (a, K) are computed using linear interpolation

- For Step 6, we use OLS to compute γ_0 and γ_1

  - As we get closer to the stationary value of K, we only have observation points (K, K') where K and K' are almost identical

  - Thus, we use only a subset of $\{K\}_{t=0}^{T}$, say $T = 1,000$

5.2 A Shooting Method

- The method in the previous section may not be feasible if the number of arguments in the value function rises (e.g., endogenous N, multiple financial assets, etc)

- We consider another method for computation of the transition path that only considers the individual variables as arguments of the value function

- For the computation, we assume that the stationary equilibrium is reached in finite time, say after T periods

- We know the distribution of wealth at $t = 1$, so the aggregate capital stock and the factor prices in period $t = 1$ and $t = T$

- In order to compute the policy functions during the transition, we need to know the time path of the factor prices (equivalently, the time path of the aggregate capital stock)

- Intuitively, we start with an initial guess of the time path of the factor prices, compute policy functions, and then compute the implied time path of the factor prices. If the implied time path and the initial guess are different, we update and iterate

- Algorithm

1. Choose the number of transition periods T

2. Compute the stationary distribution $\widetilde{F}$ of the new stationary equilibrium. Initialize the first-period distribution function F^1

3. Guess a time path for the factor prices r and w, unemployment compensation b, and the income tax rate τ that balances the budget. The values of these variables in both periods $t = 1$ and $t = T$ are implied by the initial and stationary distribution, respectively

4. Compute the optimal decision functions using the guess for the interest rate r, the wage income w, the tax rate τ and the unemployment compensation b. Iterate backwards in time, $t = T - 1, ..., 1$

5. Simulate the dynamics of the distribution with the help of the optimal policy functions and the initial distribution for the transition from $t = 1$ to $t = T$

6. Compute the time path for the interest rate r, the wage w, unemployment compensation b, and the income tax rate τ, and return to Step 3, if necessary

7. Compare the simulated distribution F^T with the stationary distribution function $\widetilde{F}$. If the fit is poor, increase the number of transition periods T

- In Step 4, we compute policy functions backward

  - For $t = T - 1, ..., 1$, we can compute $c_t(\epsilon_t, a_t)$ and $a_{t+1}(\epsilon_t, a_t)$ for given policy functions $c_{t+1}(\epsilon_{t+1}, a_{t+1})$ and $a_{t+2}(\epsilon_{t+1}, a_{t+1})$ from the Euler equation

  - Or, we can compute the optimal policy functions with value function iteration from the Bellman equation

$$V_t(\epsilon_t, a_t) = \max_{c_t, a_{t+1}} [u(c_t) + \beta E_t \{V_{t+1}(\epsilon_{t+1}, a_{t+1}) | \epsilon_t\}]$$

- Remember that policy and value functions for T is known, so we can compute these for $t = T - 1$ and repeat backwards

\- There are several ways to update the time path:

1. Use the parameterization as in (19)

2. Use an initial sequence $\{K_t\}_{t=0}^{\infty}$ with an updating rule

$$K_t^l = K_t^{l-1} + \phi\left(K_t^{l-1} - K_t^l\right), \;\; t = 2, T - 1$$

3. Use a non-linear solution method to find the sequence $\{K_t\}_{t=0}^{T}$ that implies the same sequence for the simulated model

6 Aggregate Uncertainty

6.1 Introducing Aggregate Uncertainty

- Stochastic technology level Z_t with a Markov process with transition matrix $\Gamma_Z(Z'|Z)$

- The production function is

$$Y_t = Z_t N_t^{1-\alpha} K_t^{\alpha}$$

- Factor prices are given by

$$r_t = \alpha Z_t \left(\frac{N_t}{K_t}\right)^{1-\alpha} - \delta$$
$$w_t = (1-\alpha)\, Z_t \left(\frac{K_t}{N_t}\right)^{\alpha}$$

- The joint process of the two shocks is $\Gamma\left(Z', \epsilon' | Z, \epsilon\right)$

- Z can take Z_g and Z_b with $Z_g > Z_b$: from an agent's viewpoint, the joint processes on (Z, ϵ) are Markov chains with 4 states

- Summary of the model:

  - The household maximizes

$$V(\epsilon, a, Z, F) = \max_{c, a'}[u(c) + \beta E\left\{V(\epsilon', a', Z', F')|\epsilon, Z, F\right\}] \tag{20}$$

with the budget constraint

$$a' = \begin{cases} (1 + (1-\tau)\, r)\, a + (1-\tau)\, w - c & \text{if } \epsilon = e \\ (1 + (1-\tau)\, r)\, a + b - c & \text{if } \epsilon = u \end{cases}$$

$$a \geq a_{\text{min}}$$

and

$$\Gamma \left(Z', \epsilon' | Z, \epsilon\right) = \Pr\left\{Z_{t+1} = Z',\ \epsilon_{t+1} = \epsilon' | Z_t = Z,\ \epsilon_t = \epsilon\right\}$$

$$= \begin{pmatrix} p_{Z_g e Z_g e} & p_{Z_g e Z_g u} & p_{Z_g e Z_b e} & p_{Z_g e Z_b u} \\ p_{Z_g u Z_g e} & p_{Z_g u Z_g u} & p_{Z_g u Z_b e} & p_{Z_g u Z_b u} \\ p_{Z_b e Z_g e} & p_{Z_b e Z_g u} & p_{Z_b e Z_b e} & p_{Z_b e Z_b u} \\ p_{Z_b u Z_g e} & p_{Z_b u Z_g u} & p_{Z_b u Z_b e} & p_{Z_b u Z_b u} \end{pmatrix}$$

- The distribution of the individual states (ϵ, a) for given aggregate state variables (Z, K) in period t is denoted by $F(\epsilon, a; Z, K)$

- The dynamics of the distribution of individual states is

$$F'\left(\epsilon', a'; Z', K'\right) = \sum_{\epsilon \in \{e,u\}} \Gamma\left(Z', \epsilon' | Z, \epsilon\right) F\left(\epsilon, a; Z, K\right)$$

where $a = a'^{-1}\left(\epsilon, a'; Z, K\right)$ is the inverse of the policy function $a'(\epsilon, a; Z, K)$ and

$$K' = \sum_{\epsilon \in \{e,u\}} \int_a a' f\left(\epsilon, a; Z, K\right) da$$

- Factor prices are

$$r = \alpha Z \left(\frac{N}{K}\right)^{1-\alpha} - \delta$$
$$w = (1-\alpha) Z \left(\frac{K}{N}\right)^{\alpha}$$

- Aggregate variables

$$
\begin{aligned}
K &= \sum_{\epsilon \in \{e,u\}} \int_a a f\left(\epsilon, a; Z, K\right) da \\
N &= \int_a f\left(e, a; Z, K\right) da \\
C &= \sum_{\epsilon \in \{e,u\}} \int_a c\left(\epsilon, a; Z, K\right) f\left(\epsilon, a; Z, K\right) da \\
T &= \tau\left(wN + rK\right) \\
B &= \int_a b f\left(u, a; Z, K\right) da
\end{aligned}
$$

- The government policy is characterized by a constant replacement ratio $\zeta = b/(1-\tau)w$ and a balanced budget: $T = B$

- Major differences from the one without aggregate uncertainty

  1. The employment levels fluctuate

- Given an employment distribution in period t, the next-period employment distribution depends on Z'

- Since the factor prices are functions of both aggregate capital K_t and employment N_t, the households need to predict the law of motion for both state variables

- Let us suppose that the unemployment rate takes only two values u_g and u_b with $u_g < u_b$

- This implies the following restriction on Γ :
$$u_Z \frac{p_{ZuZ'u}}{p_{ZZ'}} + (1 - u_Z) \frac{p_{ZeZ'u}}{p_{ZZ'}} = u_{Z'}$$
for $Z, Z' \in \{Z_g, Z_b\}$. This implies unemployment is u_g and u_b if $Z' = Z_g$ and $Z' = Z_b$, respectively

2. The value function will depend on Z

- The household problem is

$$V(\epsilon, a, Z, F) = \max_{c,a'} [u(c) + \beta E \left\{ V(\epsilon', a', Z', F') | \epsilon, Z, F \right\}]$$

- The household is assumed to be boundedly rational and to use only the first I moments m in order to predict the law of motion for the distribution F with $m_1 = K$:

$$V(\epsilon, a, Z, m) = \max_{c,a'} [u(c) + \beta E \left\{ V(\epsilon', a', Z', m') | \epsilon, Z, m \right\}]$$

3. The distribution of wealth is not stationary

- The distribution of capital changes over time

- The law of motion of the aggregate capital stock depends on the productivity level Z;

$$m' = H_I(m, Z)$$

- We assume

$$\ln K' = \begin{cases} \gamma_{0g} + \gamma_{1g} \ln K & \text{if } Z = Z_g \\ \gamma_{0b} + \gamma_{1b} \ln K & \text{if } Z = Z_b \end{cases}$$

- Since the aggregate productivity is a stochastic variable, we can only simulate the dynamics of the economy

\- Algorithm

1. Compute aggregate next-period employment N as a function of current productivity Z : $N = N(Z)$

2. Choose the order I of moments m

3. Guess a parameterized functional form for H_I and choose initial parameters of H_I

4. Solve the consumer's optimization problem and compute $V(\epsilon, a, Z, m)$

5. Simulate the dynamics of the distribution function

6. Use the time path for the distribution to estimate the law of motion for the moments m

7. Iterate until the parameters of H_I converge

8. If the fit for H_I is satisfying, stop, otherwise increase I or choose a different functional form for H_I

6.2 Markov Chain Approximation of AR(1) Process

- In the data/reality, an AR(1) process is probably a better representation of the evolution of productivity shocks than a Markov chain

- Approximate AR(1) processes using a Markov chain

- Consider the process

$$Z_{t+1} = \rho Z_t + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma_\epsilon^2)$$

- The unconditional mean and variance of this process are 0 and $\sigma_Z^2 = \sigma_\epsilon^2 \left(1 - \rho^2\right)$

- Consider equidistant grid points $[z_1, ..., z_m]$ where $z_m = \lambda \sigma_Z$ and $z_1 = -z_m$

- Let dz denote half of the distance between two consecutive grid points

- Then,

$$\Pr(z_j - dz \leq z \leq z_j + dz) = \Phi\left(z_j + dz; \rho z_i, \sigma^2\right) - \Phi\left(z_j - dz; \rho z_i, \sigma^2\right)$$

- The probability to switch from state z_i to state z_j for $j = 2, ..., m-1$ is

$$p_{ij} = \int_{\frac{z_j - \rho z_i - dz}{\sigma_\epsilon}}^{\frac{z_j - \rho z_i + dz}{\sigma_\epsilon}} \phi(x)\, dx$$

  and

$$\sum_i p_{i1} = \int_{-\infty}^{z_1 + dz} \phi(x)\, dx$$

  and

$$p_{im} = 1 - \sum_{j=1}^{m-1} p_{ij}$$

- Algorithm

  1. Compute the discrete approximation of the realizations: Let ρ and σ_ϵ denote the autoregressive parameter and the standard deviation of innovations, respectively. Select the size of the grid by choosing $\lambda > 0$ so that $z_1 = -\lambda\sigma_\epsilon/\sqrt{1-\rho^2}$. Choose the number of grid points m. Put $step = -2z_1/(m-1)$ and for $i = 1, 2, ..., m$ compute $z_i = z_1 + (i-1)step$

2. Compute the transition matrix $P = (p_{ij})$: For $i = 1, 2, ..., m$ put

$$p_{i1} = \Phi\left(\frac{z_1 - \rho z_i}{\sigma_\epsilon} + \frac{step}{2\sigma_\epsilon}\right)$$
$$p_{ij} = \Phi\left(\frac{z_j - \rho z_i}{\sigma_\epsilon} + \frac{step}{2\sigma_\epsilon}\right) - \left(\frac{z_j - \rho z_i}{\sigma_\epsilon} - \frac{step}{2\sigma_\epsilon}\right), \quad j = 2, 3, ..., m-1$$
$$p_{im} = 1 - \sum_{j=1}^{m-1} p_{ij}$$

- Tauchen (1986) reports the results of Monte Carlo experiments that show that choosing $m = 9$ and $\lambda = 3$ gives an adequate representation of the underlying AR(1)-process

7 Application: Lee and Wolpin (2006)

- Lee and Wolpin (2006) develop and estimate a multisector multioccupation competitive equilibrium model of the U.S. labor market with both idiosyncratic shocks to preferences, and skill and aggregate production shocks

- This paper tries to explain the major shift in the industrial structure of the U.S. economy toward the production of services and the constancy of the hourly wage within the service sector relative to the wage within the goods sector

- This paper is one example of heterogeneous-agent models with aggregate uncertainty

- We focus on the computational algorithm

7.1 Model Setup

- Factor and product markets are competitive

- Consider an open economy (small country assumption) where the real rental price of capital and real product prices are exogenous; set internationally and taken as given

- The production side is simplified:

  - Production technologies in the two sectors are specified at the aggregate level (not firm level)

  - It is static: capital can flow between sectors costlessly (no adjustment cost)

- The worker-consumer side is modeled at the micro level as a dynamic optimization problem

- Ignore savings behavior

- Labor supply decisions are independent of relative product prices, so we do not have to specify a stochastic process for the evolution of the relative price of goods to services (Why not?)

- Two sector economy, the goods-producing sector (G) and the service sector (R), each producing output (Y)

- Use three skill categories of workers (W, P, B) and homogeneous capital K

- Production at time t, valued at the sector's period t real price (p)

$$
\begin{aligned}
p_t^j Y_t^j &= p_t^j \zeta_t^j F^j \left(S_t^{jW}, S_t^{jP}, S_t^{jB}, K_t^j\right) \\
&= z_t^j \Bigg\{ a_{1t}^j \left(S_t^{jP}\right)^{\sigma^j} + a_{2t}^j \left(S_t^{jB}\right)^{\sigma^j} \\
&+ \left(1 - a_{1t}^j - a_{2t}^j\right) \left[[\lambda_t^j \left(S_t^{jW}\right)^{v^j} + (1 - \lambda_t^j) \left(K_t^j\right)^{v^j} \right]^{\sigma^j / v^j} \Bigg\}^{1/\sigma^j}
\end{aligned} \tag{21}
$$

for $j = G, R$

- The elasticity of substitution between capital and white-collar skill is $1/(1 - v^j)$

- The elasticity of substitution between the composite capital-white-collar skill input and the other skill categories is $1/(1 - \sigma^j)$

- Sector-specific real productivity shocks $z_t = p_t^j \zeta_t^j$ follows

$$\log z_{t+1}^j - \log z_t^j = \phi_0^j + \sum_{k=G,R} \phi_k^j \left(\log z_t^k - \log z_{t-1}^k \right) + \epsilon_{t+1}^j, \quad j = G, R \tag{22}$$

  where ϵ_{t+1}^j follows joint normal with σ_{jk}^z $j,k = G, R$ being elements of the variance-covariance matrix

- For $j = G, R$, and $k = 1, 2, 3$, α_{kt}^j follow

$$\alpha_{kt}^j = \begin{cases} \alpha_{k0}^j & \text{if } t < 1960 \\ \alpha_{k0}^j + \alpha_{k1}^j (t - 1960) & \text{if } 1980 \geq t \geq 1960 \\ [\alpha_{k0}^j + 20\alpha_{k1}^j] + \alpha_{k2}^j (t - 1980) & \text{if } 2000 \geq t \geq 1980 \end{cases}$$

- At each age $a \in \{16, ..., 65\}$, an individual of type h at time t chooses among eight alternatives: $d_{hat}^j = 1$ if alternative j is chosen and 0 otherwise

- Sector-occupation categories are $\{GW = 1, GP = 2, GB = 3, RW = 4, RP = 5, RB = 6\}$, and "attend school" and "stay home"

- The flow utility at each age a for an individual of type h is

$$U_a^h = \sum_{k=1}^{6} \gamma_k d_a^k + \gamma_{7h} d_a^7 + (\gamma_{80h} + \gamma_{81} n_{05,a}) d_a^8 + \gamma_9 d_a^8 d_{a-1}^8$$

$$+\gamma_{10} d_a^7 (1 - d_{a-1}^7) I(E_a < 12) + \gamma_{11} d_a^7 (1 - d_{a-1}^7) I(E_a \geq 12) + u(c_a^G, c_a^R) \tag{23}$$

- γ_k for $k = 1, ..., 6$ are included because wage differentials alone do not provide a good fit to the choice distribution

- Let γ_k type-specific for $k = 7, 8$ to capture the strong degree of persistence in the choice of the schooling and home alternatives

- Include the number of small kids, $n_{05,a}$, which follows a specified transition function exogenously

- Include an indicator of whether the individual was at home in the previous period

- Let $\gamma_{kha} = \gamma_{kh} + \epsilon_{ka}$ $k = 7, 8$ where ϵ_{ka} follows the joint normal with a variance-covariance matrix, characterized by σ^{ϵ}_{jk}, $j, k = 7, 8$

- γ_{10} and γ_{11} capture a psychic cost of reentering high school and college, respectively

- The way $u(c^G_a, c^R_a)$ enters is important; the labor allocation decision is independent of the relative price of goods to services

- The individual faces the budget constraint:

$$\begin{aligned}\sum_{j=G,R} p^j_t c^j_a = \sum_{k=1}^{6} w^k_{at} d^k_a - [\beta_1 I(E_a \geq 12) + \beta_2 I(E_a \geq 16)] d^7_a \\ - \sum_{k=1}^{8} \sum_{j=1}^{6} \delta_{jk} d^j_a d^k_{a-1} \end{aligned} \tag{24}$$

- β_1: the cost of college attendance
- β_2: the additional cost of graduate school attendance
- δ_{jk} : direct cost of switching from any of the eight alternatives to any of the six employment alternatives

- An individual receives a wage offer in each period from each sector and in each occupation. The wage offer consists of
  - a sector-occupation-specific competitively determined skill rental prices (r) and
  - the amount of sector-occupation-specific skill units possessed by the individual (s)
- Skill units are produced through education and work experience

- The wage function is

$$\begin{aligned}
&\log w^j_{hat} \\
&= \log r^j_t + \log s^j_{ha} \\
&= \log r^j_t + \omega^j_{0h} + \omega^j_1 E_a + \left(\sum_{k=1}^{6} \omega^{jk}_2 X^k_a \right)^{\omega^j_3} - \omega^j_4 I(a > 40)(a - 40) + \eta^j_a
\end{aligned} \tag{25}$$

  - ω^j_{0h} : type specific skill endowments at age 16
  - $\left(\sum_{k=1}^{6} \omega^{jk}_2 X^k_a \right)^{\omega^j_3}$: sector-occupation-specific composite work experience
  - A linear age effect is introduced upon reaching age 40 to improve the model fit
  - η^j_a : shocks to skill

- Years of education and work experience evolve

$$
\begin{aligned}
E_a &= E_{a-1} + d^7_{a-1} \\
X^j_a &= X^j_{a-1} + d^j_{a-1}, \;\; j = 1, ..., 6
\end{aligned}
$$

- When a worker moves to a different sector-occupation, he incurs not only a direct mobility cost (δ) but also incurs a loss because the accumulated work experience in the origin sector produces less composite work experience in the destination sector

- Type depends on gender and initial completed schooling (years of schooling is 10 or more); 4 types in total

- The probability distribution is $\pi_h = \Pr(h = j | E_{16}, sex)$ for $j = 1, ..., 4$

7.2 Market Clearing and Rational Expectation Equilibrium

- The economy consists of overlapping generations of individuals age 16-65

- Each individual chooses 8 alternatives every time period to maximize the discounted sum of future (remaining) utilities subject to age, (23), (24), and (25)

- The individual's value is

$$V_a\left(\Omega_{at}\right) = \max_{\{d_{at}^j\}} \sum_{\tau=a}^{A} E\left[\rho^{\tau-a} U_\tau | \Omega_{at}\right]$$

  - ρ : discount factor
  - Ω_{at} : the information set
  - A : retirement age

- Agents in the economy form a common forecast of the distribution of future skill rental prices

- Aggregate skill supplied to each sector-occupation is the sum of the skill units of the individuals who choose that alternative:

$$S_t^j = \sum_{a=16}^{65} \sum_{n=1}^{N_{at}} s_{nat}^j d_{nat}^j \left(r_t^1, ..., r_t^6\right) \tag{26}$$

where N_{at} is the number of individuals who are age a at time t

- The aggregate supply of capital is perfectly elastic, which equals aggregate demand:

$$\bar{K}_t = K_t^G + K_t^R$$

- Given the static nature of the demand side, aggregate skill demand is determined such that the value of marginal product of each aggregate skill equals to the corresponding skill rental price:

$$\frac{\partial p_t^G Y_t^G \left(S_t^1, S_t^2, S_t^3, K_t^G\right)}{\partial S_t^j} = r_t^j \qquad (j = 1, 2, 3)$$

$$\frac{\partial p_t^R Y_t^R \left(S_t^4, S_t^5, S_t^6, K_t^R\right)}{\partial S_t^j} = r_t^j \quad (j = 4, 5, 6)$$

- In the same way, the aggregate demand for capital is determined by the same condition ($VMPK = r^k$):

$$\frac{\partial p_t^G Y_t^G \left(S_t^1, S_t^2, S_t^3, K_t^G\right)}{\partial K_t^G} = \frac{\partial p_t^R Y_t^R \left(S_t^4, S_t^5, S_t^6, K_t^R\right)}{\partial K_t^R} = r_t^K$$

- In a rational expectations equilibrium, aggregate skill supplies and demands are equal in all sector-occupations

$$[S_t^j]_{Demand} - [S_t^j]_{Supply} = e_t^j\left(r_t^1, ..., r_t^6; \widetilde{Z}_t, \widetilde{r}_t^K, \widetilde{\Omega}_t, \Theta\right) = 0, \quad j = 1, ..., 6 \tag{27}$$

- $\widetilde{Z}_t$: current and past productivity shocks
- $\widetilde{r}_t^K$: the vector of the current and past capital rental prices
- $\widetilde{\Omega}_t$: the state space of all agents in the economy
- Θ : the set of model parameters

- Assume that the solution to (27) for the growth rate of equilibrium skill rental prices are approximated by

$$\begin{aligned}\log r_{t+1}^j - \log r_t^j = \eta_0^j + \sum_{k=1}^{6} \eta_k^j \left[\log r_t^j - \log r_{t-1}^j\right] \\ + \eta_7^j \left[\log z_{t+1}^G - \log z_t^G\right] + \eta_8^j \left[\log z_{t+1}^R - \log z_t^R\right]\end{aligned} \tag{28}$$

- This says the contemporaneous growth rate of sector-specific productivity shocks and a one-period lag in the growth rate of skill rental prices summarize the histories of aggregate shocks and the state space distribution of the agents in the economy

  - This is a combination of Krusell and Smith (1998), which use moments of the aggregate distribution of the state space elements in the forecasting rule as an approximation to the rational expectations equilibrium, and Altug and Miller (1998), which use a Markov process for the forecasting rule of the equilibrium price in their model

  - This implies that we are agnostic as to what individuals know about future exogenous variables (e.g., technological change, the rental price of capital, relative product prices)

7.3 Solution Algorithm

- Given parameters for (21), (23), (24), and (25), a discount factor ρ, and observed sequences of output in each sector and of the rental prices of capital, the algorithm proceeds as follows:

- Algorithm

1. Choose a set of parameters in (22) and (28)

2. Solve the optimization problem for each cohort that exists from $t = 1, ..., T$

3. Guess an initial set of values for period one rental prices, $(r_1^j)^0$ for $j = 1, ..., 6$. Simulate agents' choices by drawing from the distribution of the idiosyncratic shocks to preferences and skills. Calculate aggregate skill levels in each sector-occupation using (26)

4. Obtaining $(p_t^G Y_t^G, p_t^R Y_t^R, r_t^K)$ from the data, the following four equations gives us $\left(z_t^G, z_t^R, K_t^G, K_t^R\right)$.

$$\begin{aligned}
p_t^G Y_t^G &= z_t^G F^G\left(S_t^{GW}, S_t^{GP}, S_t^{GB}, K_t^G\right) \\
p_t^R Y_t^R &= z_t^R F^R\left(S_t^{RW}, S_t^{RP}, S_t^{RB}, K_t^R\right) \\
r_t^K &= \frac{\partial p_t^G Y_t^G\left(S_t^1, S_t^2, S_t^3, K_t^G\right)}{\partial K_t^G} \\
r_t^K &= \frac{\partial p_t^R Y_t^R\left(S_t^4, S_t^5, S_t^6, K_t^R\right)}{\partial K_t^R}
\end{aligned}$$

5. With these aggregate variables, update skill rental prices according to

$$r_t^j = \frac{\partial p_t^G Y_t^G\left(S_t^1, S_t^2, S_t^3, K_t^G\right)}{\partial S_t^j} \quad (j = 1, 2, 3)$$

$$r_t^j = \frac{\partial p_t^R Y_t^R\left(S_t^4, S_t^5, S_t^6, K_t^R\right)}{\partial S_t^j} \quad (j = 4, 5, 6)$$

and denote them by $(r_1^j)^1$ for $j = 1, ..., 6$. Repeat Step 3 and Step 4 until skill rental prices and aggregate shocks converge to $(r_1^j)^*$ and $(z_1^j)^*$

6. Guess an initial set of values for period two rental prices, say $(r_2^j)^0 = (r_1^j)^*$ for $j = 1, ..., 6$. Repeat steps 3 and 4 for $t = 2$ to obtain $(r_2^j)^*$ and $(z_2^j)^*$

7. Repeat step 6 for $t = 3, ..., T$

8. Using the calculated series of equilibrium skill rental prices and aggregate shocks, estimate (22) and (28)

9. Using these estimates, repeat steps 2-8 until the series of skill rental prices and aggregate shocks converge

- About step 2, the value function can be written as the maximum over alternative-specific value functions

$$V_a(\Omega_{at}) = \max_j [V_a^j(\Omega_{at})]$$

$$V_a^j(\Omega_{at}) = \begin{cases} U_a^j(\Omega_{at}) + \rho EV\left(\Omega_{a+1,t+1} | d_{at}^j = 1, \Omega_{at}\right) & \text{for } a < 65 \\ U_a^j(\Omega_{at}) & \text{for } a = 65 \end{cases}$$

- To solve the DP problem, follow Keane and Wolpin (1994)

  - In each iteration, for given V_a^j, approximate EV by crude Monte Carlo integration. Take S draws of shocks (both idiosyncratic shocks and aggregate shocks), calculate the maximum of the value functions over the eight alternatives for each draw, and average the maximum over the S draws.

  - When the state space is large, simulating EV for every single point in the space is costly (remember the numerical integration mentioned above is multivariate integrations!)

- Thus, EV is calculated at a subset of the state points and their values are used to fit a linear-in-parameters regression approximation in the state variables

- Why didn't we have this problem in Rust (1987)?

- To implement the solution algorithm, assume the economy begins in 1860, while the periods that the model is fitted to actual data is 1968-2000

  - Assign arbitrary values for the state space to each person age 16-65 in 1860, zero experience for each occupation, and 8 years of schooling

  - Assume that the capital real rental price, cohort size, real output in the two sectors and the fertility process between 1860 and 1900 are the same as in 1900

  - For the capital real rental price, 1925 is the first available year, so assume it was constant between 1860 and 1925

- The cohort size is observed in 1900 for the first time, so assume it was constant between 1860 and 1900

- Output by sector is available starting from 1947, so extrapolate the series backward to 1900

- Assume that the equilibrium skill rental price process (28) governs the choices made by all individuals age 16-65 through the year 2050. This assumption is necessary to solve the optimization problems for individuals age 16-65 as of the year 2000

7.4 Estimation/Calibration

- The above solution algorithm is nested into the estimation algorithm

- Estimation by the simulated method of moments (a weighted average distance between sample moments and simulated moments is minimized with respect to the model's parameters)

- Combine the following data sources to calculate empirical moments

    1. CPS over the period 1968-2001 and NLSY79 over the period 1979-1993 for information on life cycle employment and schooling choices and wages

    2. Various U.S. Censuses from 1910-1960 for information on the age 16 schooling distributions over time and on the preschool children process

    3. BEA data on sectoral capital stocks and output from 1947-2000

- The result shows that the direct mobility cost ranges between 50 and 75% of average annual earnings

- Counterfactual experiments indicate:

  - If all else had remained the same at their 1960 levels except for supply-side factors, cohort size, and fertility, the service-sector employment share would have been the same

  - If only demand factors, production technology, and product and capital prices changed since 1960, the service-sector share of employment would have increased by 27 percentage points