Contents

Chapter 1

Introduction

1.1 Why Quantum Computing

1.1.1 Why to consider quantum computing at all?

The development of classical computers is still making enormous progress and no end of that seems to be in sight. More over, the design of quantum computers seems to be very questionable and almost surely enormously expensive. All this is true. However, there are at least four very good reasons for exploring quantum computing as much as possible.

- Quantum computing is a challenge
- Quantum computing seems to be a must and actually our destiny
- Quantum computing is a potential
- Finally, the development of quantum computing is a drive and gives new impetus to explore in more detail and from new points of view concepts, potentials, laws and limitations of the quantum world and to improve our knowledge of the natural world

1.1.2 Can quantum computers do what classical ones can not?

The answer depends on the point of view. It can be YES. Indeed, the simplest example is generation of random numbers. Quantum algorithms can generate truly random numbers. Deterministic algorithms can generate only pseudo-random numbers. Other examples come from the simulation of quantum phenomena. On the other hand, the answer can be also NO. A classical computer can produce truly random numbers when attached to a proper physical source.

1.1.3 Where lie the diferences between the classical and quantum information processing?

Classical information can be read, transcribed (into any medium), duplicated at will, transmitted and broadcasted. Quantum information, on the other hand, cannot be in general read or duplicated without being disturbed, but it can be "teleported"

In classical randomized computing, a computer always selects one of the possible computation paths, according to a source of randomnes, and "what-could-happen-but-did-not" has no influence what so ever on the out come of the computation. On the other hand, in quantum computing, exponentially many computational paths can be taken simultaneously in a single piece of hardware and in a special quantum way and "what-could-happen-but-did-not" can really matter

1.1.4 Can quantum computers solve some practically important problems much more eficiently?

Yes. For example, integer factorization can be done in polynomial time on quantum computers what seems to be impossible on classical computers. Searching in unordered database can be done provably with less queries on quantum computer.

1.1.5 Where does the power of quantum computing come from?

quantum computation offers enormous parallelism. The size of the computational state space is exponential in the physical size of the system and the energy available. A quantum bit can be in any of a potentially infinite number of states and quantum systems can be simultaneously in superposition of exponentially many of the basis states. A linear number of operations can create an exponentially large superposition of states and, in parallel, an exponentially large number of operations can be performed in one step.

1.2 From Randomized to Quantum Computation

A comparison of probabilistic Turing

machines (PTM), with quantum Turing machines (QTM) will allow us to see, in an easy and transparent way similarities and diferences

between these two basic models of classical and quantum computing. In this way we can also demonstrate the advantages and problems quantum computing has.

There are good reasons to start our introduction to quantum computing by comparing probabilistic and quantum Turing machines. Probabilistic Turing machines represent nowadays the most important model of classical computing. Polynomial time computation on probabilistic Turing machines stands for a formal equivalent of "feasibility" in classical computing. In addition, similarly to classical Turing machines, quantum Turing machines were historically the first really fully quantum and powerful model of quantum computing .

1.2.1 Probabilistic Turing machines

Formally, a (one-tape) **probabilistic Turing machine**, on a finite set Q of states and the finite alphabet Σ, is given by a transition function

$$\delta : \Sigma \times Q \times \Sigma \times Q \times \{\leftarrow, \downarrow, \rightarrow\} \rightarrow [0, 1]$$

assigning to each possible transition a probability in such away that for each configuration c_0 and all its successor-coniguration $c_0 \ldots c_K$ the following local probability condition is satisfied : If $p_i, 1 \leq i \leq k$, is the probability, assigned by δ of the transition from c_0 to c_i then :

$$\sum_{i=1}^{k} p_i = 1$$

On the base of the transition function δ of a PTM, M we can assign probabilities to all edges, to all nodes and also to all conigurations of each level of any coniguration tree of T.

1.2.2 Quantum Turing machines

Formally, a (one-tape) **quantum Turing machine** with a finite set Q of states and the finite alphabet Σ, is given by a transition function

$$\delta : \Sigma \times Q \times \Sigma \times Q \times \{\leftarrow, \downarrow, \rightarrow\} \rightarrow C_{[0,1]}$$

assigning a so-called **amplitude** (or **probability amplitude**) a complex number the absolute value of which is in the interval $[0, 1]$-to each transition in suchaway that for each configuration c_0 and all its successor congurations $c_1, \ldots, c_k$ the following **local probability condition** is satised : if α_i is the amplitude assigned to the transition from c_0 to the conguration c_i, then :

$$\sum_{i=1}^{k} |\alpha_i|^2 = 1$$

and therefore $|\alpha_i|^2$ can be seen (and will be seen) as a probability of transition from c_0 to c_i-However, as discussed later, this is not the only condition a transition function of a QTM has to satisfy.

The transition function of a QTM can be used to assign amplitudes (not probabilities) also to all edges, nodes and all configurations of the same level of a configuration tree. The amplitude assigned to an edge is given directly by δ. The amplitude assigned to a node is the product of the amplitudes assigned to all edges on the path from the root to that node, assuming again that the amplitude 1 is assigned to the root

1.2.3 Difference between a PTM and a QTM

In the case of a PTM, at each particular computation a single path through the configuration tree has to be chosen, and we could watch (though not ifluence) the path being taken. The result would be obtained with the probability attached to the final configuration. On the other hand, a QTM always follows all paths of the configuration tree simultaneously! Since the number of nodes at the levels of a configuration tree can grow exponentially, this means that a QTM can, simultaneously, take an exponentially large number of paths and can be, at particular steps of computation in a superposition of exponentially many configurations (with respect to the number of computational steps) at the same time! In addition, the computational evolution of a QTM, and of any quantum computation, is fully determined by its unitary matrix and it is deterministic.

Moreover, there is no way "to watch" the computations of a QTM. We could "put it into a box and let it run" but we cannot watch it, At the end of the computation we can try to observe (measure) the result

1.3 Hilbert Space Basic

Hilbert space is a mathematical framework suitable for describing the concepts, principles, processes and laws of quantum mechanics Pure states of quantum systems are considered to be vectors of a Hilbert space. One can say that to each isolated quantum system corresponds a Hilbert space Some even go farther by claiming that there is no reality on the quantum level; such a reality emerges only in the case of a measurement, and what we know about the quantum level are only computational procedures, expressed in terms of Hilbert space concepts, to compute evolutions of quantum systems and probabilities of the measurement outcomes.

1.3.1 inner-product space H

An **inner-product space** H is a complex vector space, equipped with an inner product $\langle\cdot|\cdot\rangle : H \times H \to C$ satisfying the following axioms for any vectors ϕ, $\psi, \phi_1, \phi_2 \in H$, and any $c_1, c_2 \in C$

$$\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$$

$$\langle\psi|\psi\rangle \leq 0 \;\; and \;\; \langle\psi|\psi\rangle = 0 \;\; if\, and\, only\, if \;\; \psi = 0$$

$$\langle\psi|c_1\phi_1 + c_2\phi_2\rangle = c_1 \langle\psi|\phi_1\rangle + c_2 \langle\psi|\phi_2\rangle$$

The inner product introduces on H the norm (length)

$$||\psi|| = \sqrt{\langle\psi|\psi\rangle}$$

and the metric (Euclidean distance)

$$dist(\phi, \psi) = ||\phi - \psi||$$

This allows us to introduce on H a metric topology and such concepts as continuity.

1.4 Tensor products in Hilbert spaces

If aquantum system S is composed of two quantum subsystems S_1 and S_2 and to them correspond Hilbert spaces H, H_1 and H_2, then H is the so-called **tensor-product** of H_1 and H_2, written as :

$$H = H_1 \otimes H_2$$

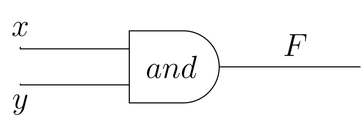
and this means that vectors of H are **tensor-products**, defined below of vectors from H_1 and H_2.

The tensor-product of vectors $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_n)$, notation $x \otimes y$, is an mn-dimensional vector with elements

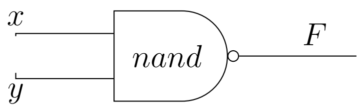
$$(x_1y_1, \ldots, x_1y_n, x_2y_1, \ldots, x_2y_n, \ldots, x_my_1, \ldots, x_my_n)$$

1.5 Reversible gates

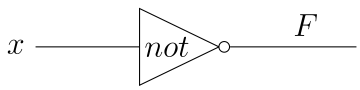
It is well known that two Boolean operations, or two gates, for instance NOT and AND, are sufficient to design Boolean circuits for any Boolean function $f \in B_n^m = \{g : g \mid \{0,1\}^n \to \{0,1\}^m\}$. Even, a single NAND gate, is universal in the sense that it alone is sufficient to design a circuit for any Boolean function



A	B	A and B
0	0	0
0	1	0
1	0	0
1	1	1



A	B	A nand B
0	0	1
0	1	0
1	0	0
1	1	0



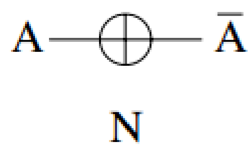
A	not A
0	1
1	0

Unfortunately, both AND and NAND are irreversible Boolean operations By that we mean that from the output value(s) of the gate one can not determine unambiguously the input values; information gets irreversibly lost "during the gate operation".

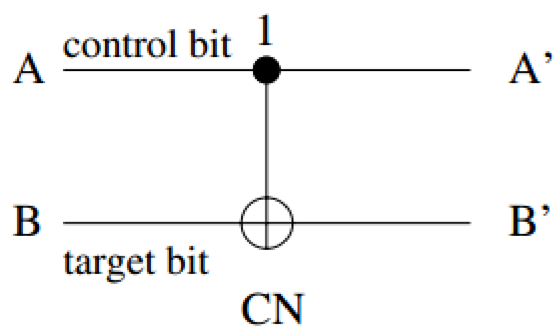
We talk about a reversible gate, Boolean function, operation or computation, as one with always enough information in the outputs to deduce the inputs unambiguously. Such operations, gates and computations are crucial for quantum computing because of the reversibility of the evolution in quantum physics. Three reversible gates have turned out to be of special importance : the usual NOT gate (N), CONTROL NOT gate (CN or CNOT or XOR), CONTROL CONTROL NOT gate (CCN or CCNOT) , also called Toffoli gate In the CN gate $A' = A$, i.e the input A gets through unchanged .

The filled circle on the first wire represents a control in the following sense : if $A = 0$ then $\otimes$ on the second wire just lets the signal B get through and therefore $B' = B$. If $A = 1$, then $\otimes$ on the second wire acts as a NOT gate and $B' = \bar{B}$.

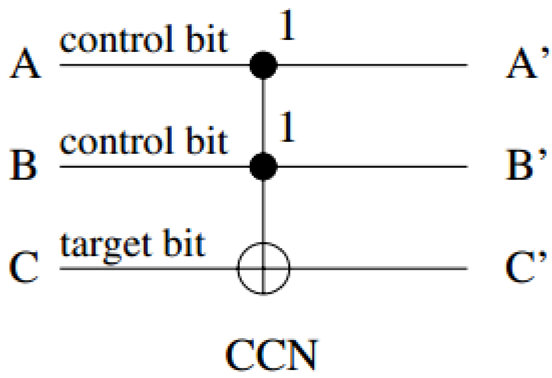
In the CCN gate $A' = A$ and $B' = B$. The $\otimes$ on the last wire acts as a NOT gate but only if $A = B = 1$



A	not A
0	1
1	0



A	B	A'	B'
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0



A	B	C	A'	B'	C'
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	0

Chapter 2

Elements

2.1 Introduction

The basic elements of quantum computing are easy to identify : quantum bits, quantum registers, quantum gates and quantum networks. However, at this point ananalogy with classical computing ends . Quantum bits, registers, gates and networks are very different , have other properties and larger power than their classical counterparts .

A quantum bit can be in any state within an infinite set of states. A quantum register of n quantum bits can be, at the same time, in any of the infinitely many superpositions of 2^n basis states. The parallelism a quantum register can exhibit is striking. The key new feature is that a quantum register can be in an entangled state. On one side, entangled states with their non-locality features are a hallmark of quantum mechanics. On the other side, quantum entanglement is an important resource of quantum information processing . There is a larger variety of quantum gates than of classical gates There are already infinitely many one-input/output quantum gates. In addition almost any two-input/output quantum gate is universal. A simple two-input/output gate together with one input rotation gates form a set of universal gates .

The aim of the chapter is to learn :

1. the basic concepts concerning quantum bits and registers
2. the concept of quantum entanglement and the examples of its power
3. the basic examples of quantum gates and of quantum circuits

4. some examples of universal quantum gates and a method to show universality of quantum gates

5. the basic quantum arithmetical circuits

6. the concept of quantum superoperator circuits

2.2 Quantum Bits and Registers

Four key problems of quantum information processing are: how to represent, how to store, how to transmit and how to manipulate quantum information. Two key elements to deal with these problems are quantum bits and quantum registers.

2.2.1 Qubits

Let S be a two dimensional quantum system with two orthonormal states, denoted $|0\rangle$ and $|1\rangle$, that can be considered as forming a natural, or standard, or preferred, basis of S.

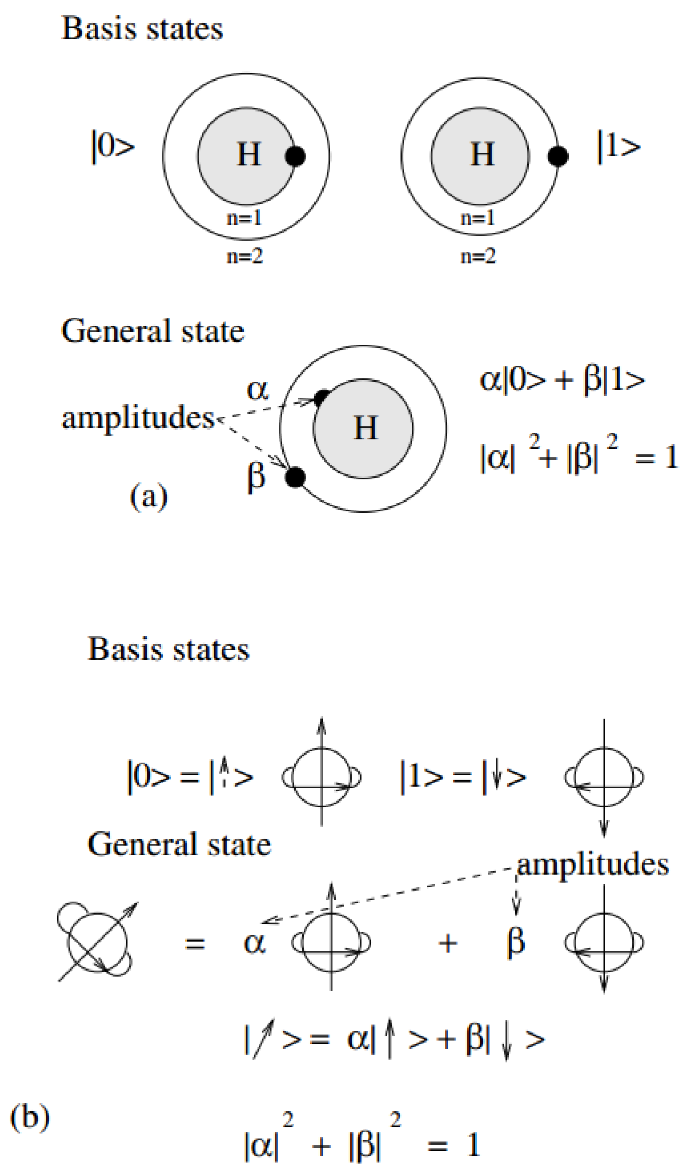
Definition of Qubit

A qubit (quantum bit) is a quantum state

$$|\phi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

where $\alpha, \beta \in C$ and $|\alpha|^2 + |\beta|^2 = 1$

2.2.2 Qubit representations by energy levels of an electron



There are many ways to realize qubits, there are many interesting/important two dimensional quantum systems known in physics. For example, by the polarization of a photon or by the ground (n=1) and excited (n=2) states of an electron in the hydrogen atom. One of the most often and best-explored two-level quantum systems is that of spin-$\frac{1}{2}$ particles with two basis states: spin-up (notation $|\uparrow\rangle$ or $|0\rangle$) and spin-down (notation$|\downarrow\rangle$ or $|1\rangle$)

From the implementation point of view the most promising candidates for qubits are so far photons, trapped ions and spins of atomic nuclei.

States $|0\rangle$ and $|1\rangle$ of a qubit can be seen, and are often referred to, as representing classical states (bits). The main difference between classical bits and qubits is that while a classical bit can be set up only to one of the two states, namely 0 or 1 a qubit can take any quantum linear superposition of $|0\rangle$ and $|1\rangle$, i.e: in principle can be in any of uncountably many states. This means that a large, even infinite, amount of information could potentially be encoded in amplitudes of a single qubit by appropriately choosing α and β

One way to represent states of qubits geometrically is as points on the surface of a unit Riemann sphere, where North and South poles correspond to the basis states (that correspond to bits). Qubits can be represented also by points on a Bloch-sphere , using the spherical coordinate system . This representation is based on the fact that any qubit can be represented as $\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$

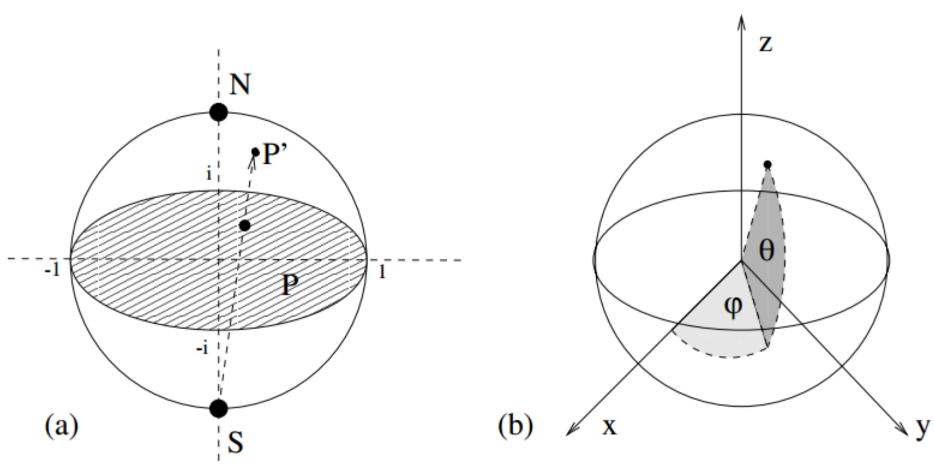


Figure 2.2: Representation of qubits on Riemann and Bloch spheres

2.3 Qubit evolution

Any quantum evolution of a qubit, or any quantum operation on a qubit is described, as already mentioned by a unitary matrix :

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

which transforms any qubit state $\alpha|0\rangle + \beta|1\rangle$ into the state $(a\alpha + b\beta)|0\rangle + (c\alpha + d\beta)|1\rangle$

2.4 Two-qubit registers

A tensor product of two qubits is called a 2-qubit quantum register,in 2-qubit quantom registers the state $|11\rangle$ is called singleton

It is usual to represent states of the standard basis in one of the following forms :

$$|0\rangle = |00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad |1\rangle = |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|2\rangle = |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad |3\rangle = |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

A general state of a 2-qubit quantum register has the form

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

where $|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$

2.5 Quantom registers

2.6 Quantom gates

The gates for basic Boolean reversible operations NOT, CNOT, and CCNOT can be described also using a notation for registers as follows :

$$\begin{aligned}
\textbf{NOT} &: |a\rangle \rightarrow |\bar{a}\rangle \\
\textbf{XOR} = \textbf{CNOT} &: |a, b\rangle \rightarrow |a, a \oplus b\rangle \\
\textbf{CCNOT} &: |a, b, c\rangle \rightarrow |a, b, (a \wedge b) \oplus c\rangle
\end{aligned}$$