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Intensity Transformation and Spatial Filtering

This chapter will deal with different image processing techniques when an image is present in spatial domain. General representation of spatial domain image transformation is given as:

$$g(x,y) = T[f(x,y)] \quad \mathbf{(3-1)}$$

Here are few basic concepts that are associated with spatial domain processing.

- $Intensity\ thresholding$: For the simplest case of binary thresholding, we decide that pixel values of lower than some specified threshold be changed to zero and higher values are changed to L-1 where L is the number of intensity levels.
- $Contrast\ stretching$: If k is intensity threshold, the process that darkens the intensity levels that are below k and brightens the pixels which are above threshold is called the contrast stretching
- $Image\ enhancement$: It is the process of manipulating the image pixel values such that the information in resulting image is clearer than the original.

3.1 Basic Image Transformations

There are three types of basic image transformation:

- Linear Transformation
- Non-Linear Transformation
    - Logarithmic Transformation
    - Power Law Transformation

3.1.1 Linear Transformation

In this transformation, the intensity value of a pixel is changed using some linear operation. This change may be increase or decrease in intensity value of an image. For example, to enhance the visibility of an image, intensity value of each pixel can be scaled up to some defined level. Linearly upscaled image shown in figure 3.1-1 have better visual information as compared to original image shown in figure 3.1-1 A. Similarly, linear intensity scaling can enhance the quality and color in color images.

$$s = r \pm i \quad \mathbf{(3-2)}$$

Negative of an image is obtained by subtracting each pixel intensity value from maximum intensity value that can be in the image. In other words, each intensity value represents the intensity level. If there are L distinct intensity levels in an image and r is the input pixel intensity value then output pixel intensity value can be calculated as:

$$s = L - 1 - r \quad \mathbf{(3-3)}$$

In normal gray level image, the intensity values range from $0 - 255$ $(0 - L - 1)$. If a pixel has intensity value 222 then its negative will be 255-222=33.

Python is very versatile and feature rich language. It supports both Single Instruction Single Data (SISD) and single instruction multiple data (SIMD) operations. SIMD operations are also called $vector\ operation$. For example, we can replace lines 8-12 in program P3-01 with single statement.

```
Img2=(L-1)-img1
```

This single subtraction operation will be applied on all pixels of `img1` and result will be assigned to `img2` on corresponding positions. There are some useful applications of image negative. In gray scale images, image negatives are used to enhance the gray level details which is present in darker areas. Specially if dark areas are dominant in image, information can't be seen directly. Taking image negative can enhance the contrast of image darker regions resulting in better visualization. For example, in image containing blood vessels, vessels can better be seen in negative image. Similarly, color image negative can provide information which is not clear in standard color space.

3.1.2 Log Transformation

In this transformation, log function is applied on intensity value of each pixel of input image. Equation 3-3 shows the process.

$$s = c * log(1 + r) \qquad \mathbf{(3-4)}$$

Log of large values results in small values. For example, if $K = [1,2,4,8,16,32,64,128]$, taking its $\log_2 K$ produces values that lie in shorter range i.e. $[0, 1, 2, 3, 4, 5, 6, 7]$. It can be observed that log transformation has compressed the intensity values. The log function has the important characteristic that it compresses the dynamic range of pixel values. The term *dynamic range* refers to the ratio of max and min intensity values. There are the imaging modalities that can create images with pixel intensity values beyond the standard range of 0-255. In that case, log transformation can play its role to bring larger values in standard range. There are two major application areas of log transformation:

- Expanding the dark pixels and compressing the corresponding bright pixels
- Compressing the dynamic range

3.1.3 Power law (Gamma) Transformation

Power law transformation is given by

$$s = cr^{\gamma} \qquad where \qquad c, \gamma \geq 0 \quad \mathbf{(3-5)}$$

Where c and γ are some positive constants. If we consider c=1, we can easily describe the behavior of γ constant. It is a non-linear operation and like log transformation, gamma transformation also compresses and expands the intensity values. A gamma value $\gamma < 1$ is sometimes called an encoding gamma, and the process of encoding with this compressive power-law nonlinearity is called ***gamma compression***; conversely a gamma value $\gamma > 1$ is called a decoding gamma and the application of the expansive power-law nonlinearity is called ***gamma expansion***. Following graph represents the compression behavior for different values of gamma parameter.

As the value of γ goes lower, low intnsity levels are compressed more and high intensity levels are stretched more. In case of γ value going higher than one, low intensity values are spread more and high intensity values are compressed more. When applying gamma transformation, one should be very careful about choosing the values of parameters. It would be good exercise, if you do some experimentation with the code used to generate this graph.

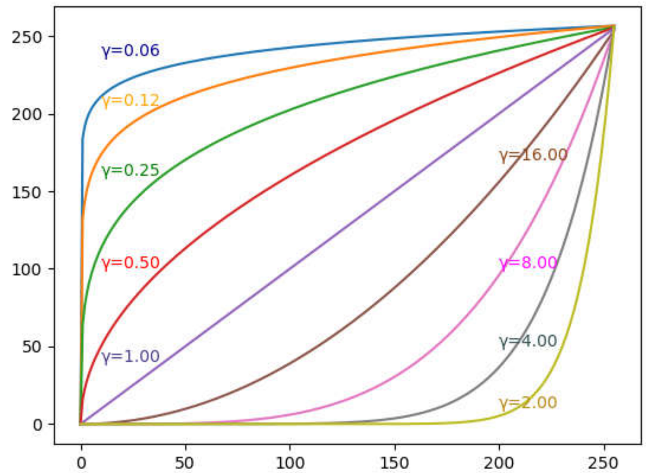
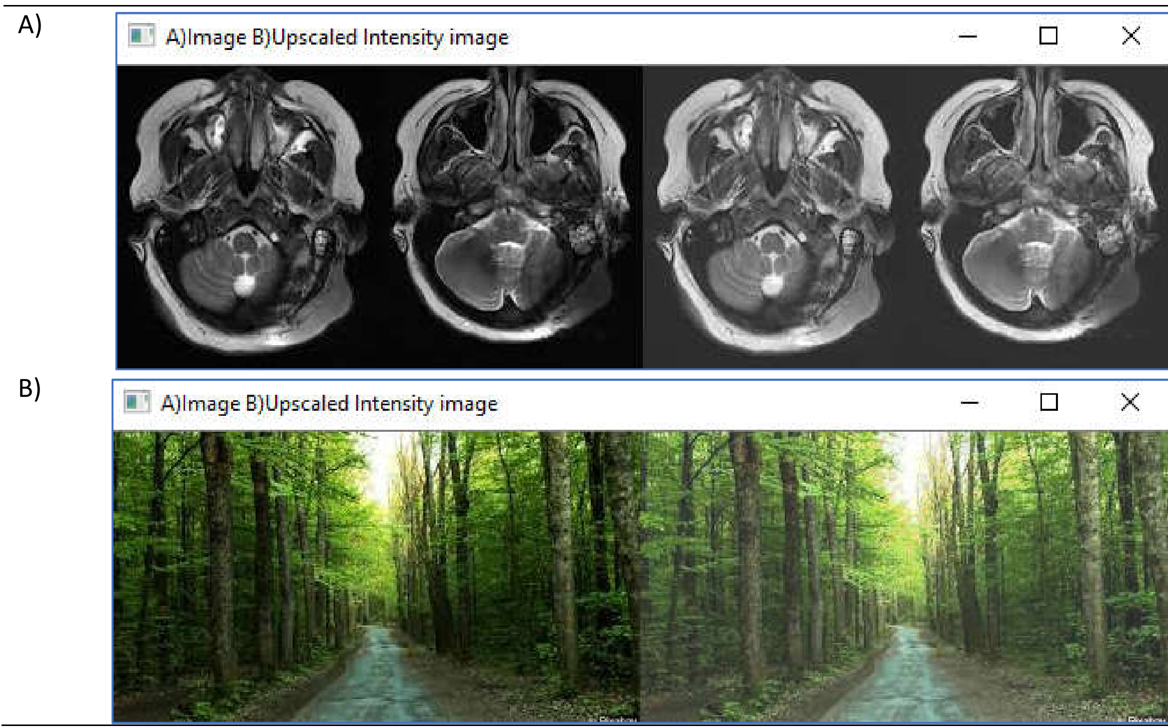


Figure ***Error! No text of specified style in document.****-1: Linear and Non-Linear Transformation*

When working with transformations in OpenCV, you must be careful about image data types and their intensity ranges. Considering the intensity resolution, there are two modes to work with images 1) operate on raw intensity values and 2) work with normalized intensity values. Operations till now, we have seen, are based on raw intensity values where pixel intensity ranges from 0-255. However, some of the operations require to normalize the pixel intensity value in the range 0-1 like gamma transformation. This can be easily done by dividing image each pixel with the maximum intensity value found in that image i.e.

$$norm(img) = \frac{img(i)}{max(img)} \qquad \forall\, i\, \in L-1 \qquad \mathbf{(3-6)}$$



C)


A)Image B)Negative

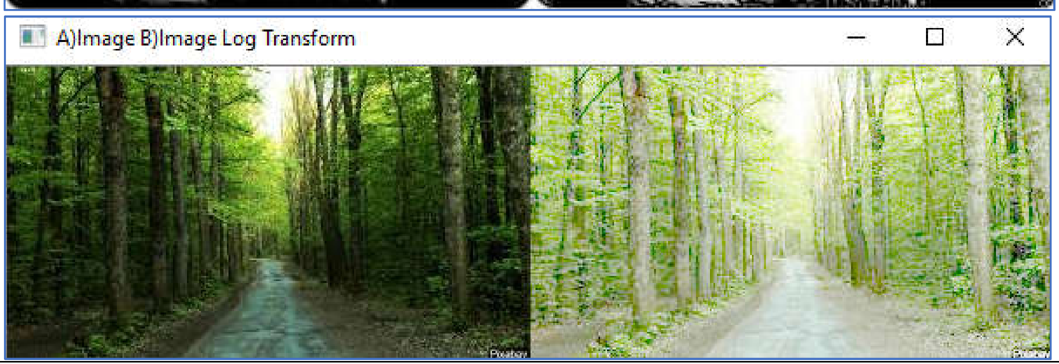
D)


A)Image B)Negative

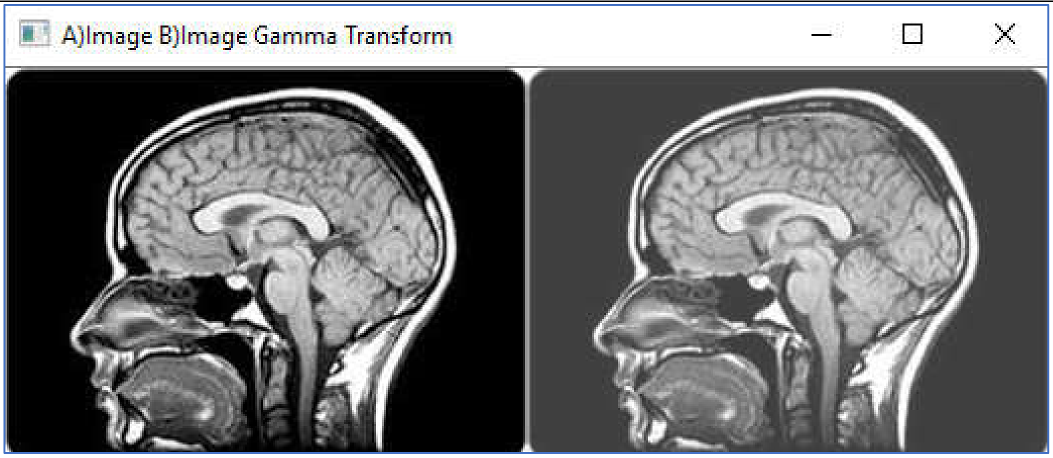
E)


A)Image B)Image Log Transform

F)


A)Image B)Image Log Transform

G)



H)

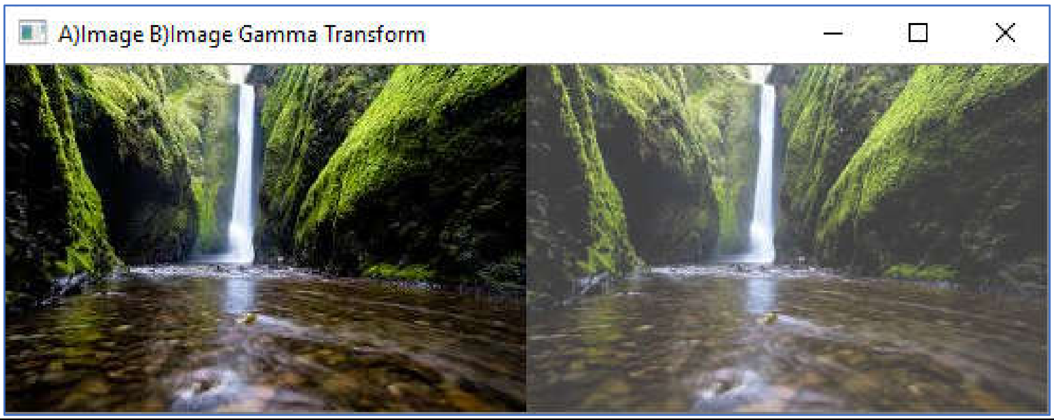


Figure 3-1: A-B) Linear Image Transformation in Grayscale and Color Images. C-D) Grayscale and Color Image Negatives E-F) Log Transformation on Grayscale and Color Images G-H) Gamma Transformation on Grayscale and color images.

Reference Programs
***Program P3-01**: Grayscale linear intensity transformation*
***Program P3-02**: Color linear intensity transformation*
***Program P3-03**: Grayscale image negative*
***Program P3-04**: Color image negative*
***Program P3-05**: Grayscale image log transformation*
***Program P3-06**: Color image log transformation*
***Program P3-07**: Drawing graph of gamma transformation*
***Program P3-08**: Grayscale image gamma transformation*
***Program P3-09**: Color image gamma transformation*

3.2 Piecewise Linear Transformation

Transformations discussed in last section operate on whole image. There may be the situations where we need to process different parts of an image differently by applying different type of transformations on difference pieces of image.

3.2.1 Piecewise Linear Thresholding (PLT)

An image contains pixel that have intensity range $0 \rightarrow L-1$. Simplest of such transform is to define a threshold or boundary T that divide all intensity levels to two parts i.e. $0 \leq p_1 \leq T \;\; and \; T+1 \leq p_2 \leq L-1$. For example, we may decide T=127 and change all intensity levels below this threshold to zero and all intensity levels above this threshold to 255.

$$s = T(r) = \begin{cases} r = 0 & if \; r \leq T \\ r = 255 & if \; r \geq T+1 \end{cases} \qquad \mathbf{(3-7)}$$

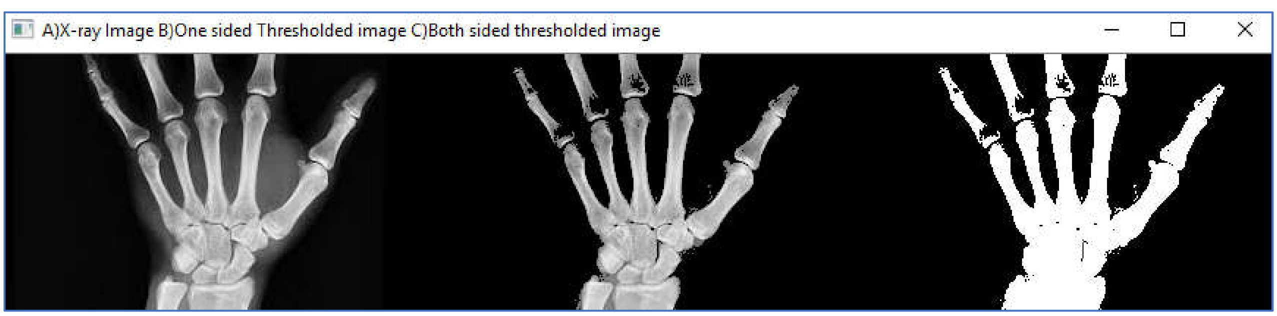


Figure Error! No text of specified style in document.**-2: Application of multi-level intensity thresholding – T=105**

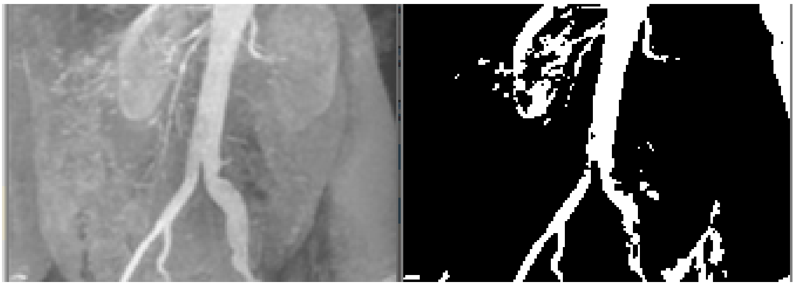
3.2.2 Intensity Level Slicing

There are applications in which it is of interest to highlight a specific range of intensities in an image. Some of these applications include enhancing features in satellite imagery, such as masses of water, and enhancing flaws in X-ray images. If we decide two thresholds T_1 and T_2 where

- Intensity values below T_1 are transformed using some function $T_1(r)$,
- Intensity values from T_1+1 to T_2 are transformed using function $T_2(r)$, and

- Intensity values above T_2, are transformed using function $T_3(r)$.

The values of T_1 and T_2 can be either found manually or using some sophisticated algorithm. This process is known as multi-thresholding.



3.2.3 Contrast Stretching

Contrast can be defined as the number of distinct colors in an image. In a grayscale image that contain intensity values from 0-255, it can be said that such image contains 256 colors ($L_{max} = 255$). Contrast can also be defined as difference between highest and lowest intensity values. If L is intensity value then

$$contrast = L_{highest} - L_{lowest} \quad \mathbf{(3-8)}$$

A grayscale image may contain contrast value below L_{max}, this allows us to use all gray colors in image by expanding the intensity distribution known as **contrast stretching**. Here is simple formula for contrast stretching:

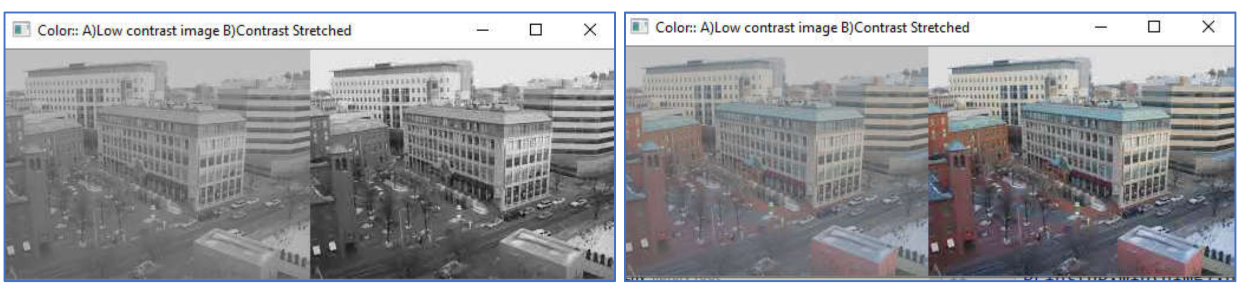
$$s_i = (r_i - r_{min}) * \frac{L_{max} - L_{min}}{r_{max} - r_{min}} + L_{min} \quad \mathbf{(3-9)}$$

Where

$s_{i,}\, r_i = Intensity\ value\ of\ ith\ output\ and\ input\ pixel\ respectivelly$

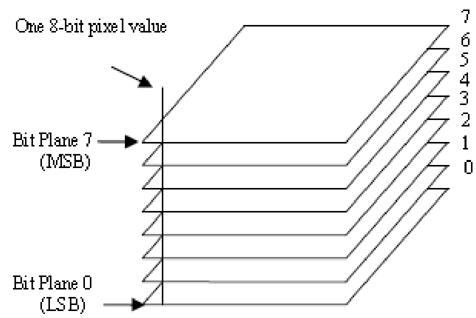
$r_{min}, r_{max} = Lowest\ and\ highest\ intensity\ value\ in\ input\ image$

$L_{min}, L_{max} = Lowest\ and\ highest\ intensity\ value\ of\ target\ contrast\ ranage$



3.2.4 Bit-Plane Slicing

Pixel values are represented by unsigned integers which are composed of bits. For example, a grayscale image is represented by 8-bit unsigned integer. Every bit in this representation of pixel intensity contains some information. Instead of slicing on intensity levels, we can slice the pixels on their bit levels. We can consider 8-bit grayscale image as a plane that consists of bits. For example, lowest plane contains 0^{th} bit of each pixel intensity, next higher plane contains 1^{st} bit and the topmost plane contains 8^{th} bit as shown in figure:



we can extract the 8^{th} bit of all pixels and form an image similarly we can extract all 7^{th} bits and form an image.

An image formed using nth bit from each pixel is called the nth bit plane. In case of images with 8-bit intensity values, there will be 8-bit planes.

Example: Bit-plane calculation

Nth bit plane is extracted by taking logical and with nth bit of each pixel. Let if binary value is 1101 1110 and we want to get 3^{rd} bit plane, we will compute as follows:

$$1101\ 1110$$

$$\frac{0000\ 0100}{0000\ 0100}\ \&$$

Let there is an image consisting of 4 pixels and their values are as under

Decimal Intensity values	50	101	222	255
Binary values	0011 0010	0110 0101	1101 1110	1111 1111
1^{st} Bit plane	0000 0000	0000 0001	0000 0000	0000 0001
2^{nd} Bit plane	0000 0010	0000 0000	0000 0010	0000 0010
…	…	…	…	…
8^{th} bit plane	0000 0000	0000 0000	1000 0000	1000 0000

We can see that most significant bit contains highest value of information and least significant bit contains least information in the image.

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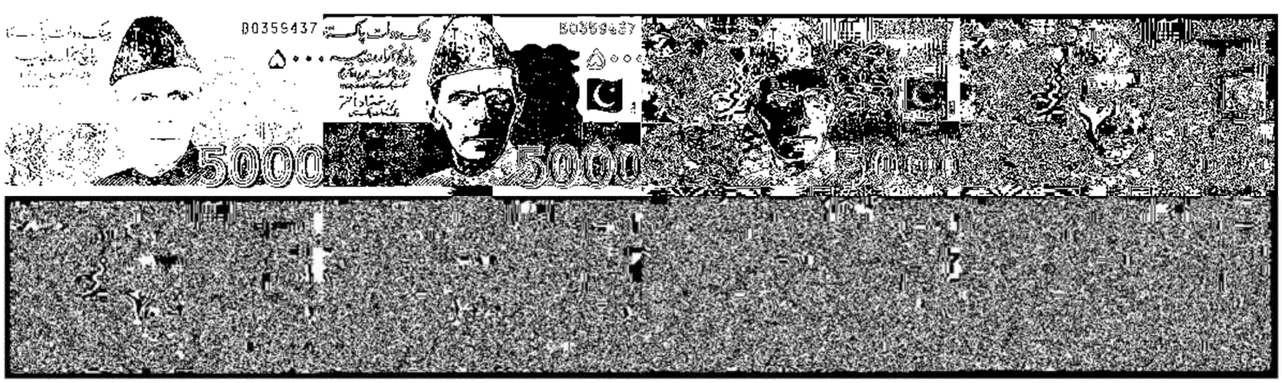


*Figure **Error! No text of specified style in document.**-3: Bit-planes. Image intensities are enhanced for reader visualization. When you write your code, you may not be able to visualize lower bit-planes as lower bits tends to be darker.*

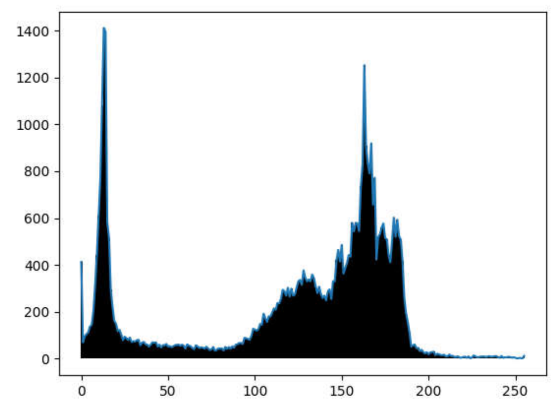
Reference Programs
***Program P3-10**: Single level image thresholding*
***Program P3-11**: Contrast stretching*
***Program P3-12**: Multi-level thresholding*
***Program P3-13**: Bit-plane slicing*

3.3 Histogram Processing

A histogram is a graphical representation that the number of pixels that lie on each intensity level in other words an image histogram shows the intensity distribution. Let for a given image

- L is the number of intensity levels ranging from $0 \rightarrow L-1$.
- M is the number of rows and N is the number of columns. Total number of pixels in the image are MN.
- There are n_k pixels for k intensity level. N_k is also known as k^{th} bin in image.
- For each intensity level there is one histogram bin and for $L-1$ levels there are $n_0, n_1, n_2, \dots., n_{L-1}$ histogram bins.

A histogram made with such data is called the $un-normalized$ histogram. Here is program to calculate histogram of an input image.



A histogram can be normalized in order to keep its values in the range of 0-1. The normalization process produces the probability density function (pdf) of each intensity level. Histogram graph produced by un-normalized and normalized bins will be same with only difference that in un-normalized graph each vertical (black) line represents the density of corresponding intensity level whereas in normalized graph it is probability density.

$$p_r(k) = \frac{n_k}{MN} \qquad (3-10)$$

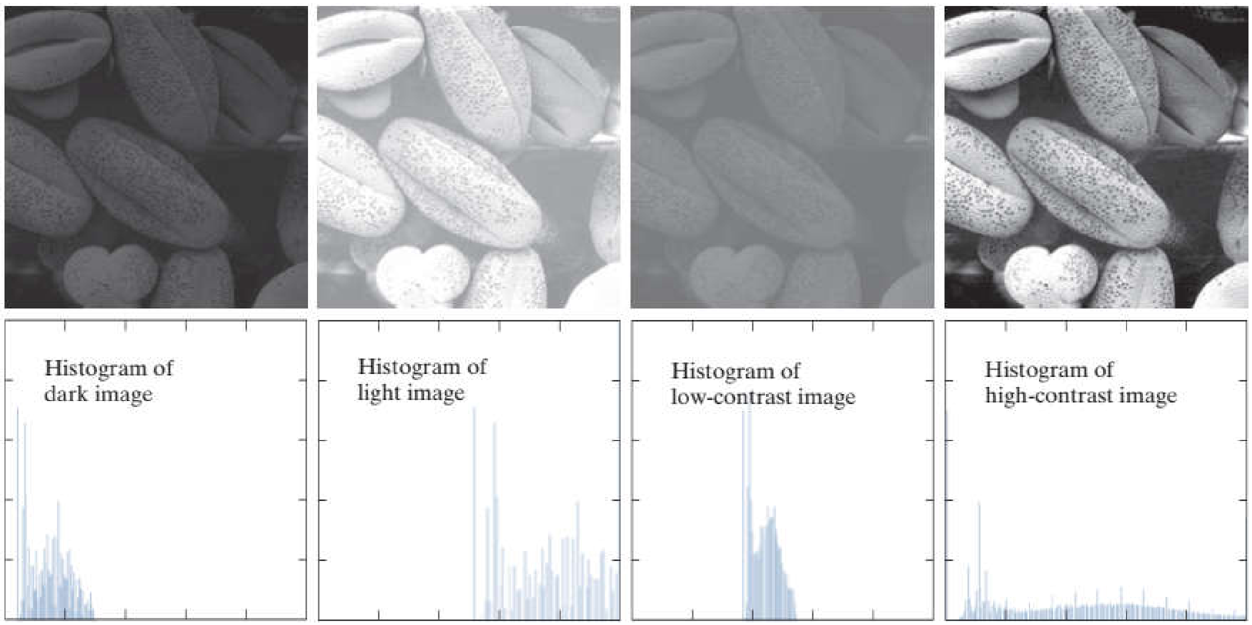
$$\sum_k p_r(k) = 1 \qquad (3-11)$$

Histogram shape is very much related to the appearance of image. Image quality and some characteristics can be described by looking at histogram. Here are some observations derived from histogram analysis:

- $Left\ shifted$: if a histogram is more on left and less on right side, such an image will be darker

- $Right\ shifted$: if a histogram is right shifted, most of the pixels will have brighter values and image will be more bright
- $Center\ located$: if intensity distribution is more located in center and less on both sides the image will be of low contrast
- $Equally\ distributed$: if histogram is equally distributed, image will have high contrast and better appearance.

Histogram spreading is a way to improve the quality of image. Following figures show different images and their corresponding histograms: <make own code for this shifting>



An image histogram could be processed partially or as a whole. ***Global histogram process*** refers to spreading the histogram of complete image but histogram spreading of a part of image is known as ***local histogram processing***. In following sections, we discuss both Global and Local histogram processing techniques.

3.3.1 Global Histogram equalization (Contrast Stretching)

A special way of contrast stretching is called the histogram equalization where compressed histogram is spread over all available intensity range. Let r be intensity of input image pixel where $r = 0,1,2,3, \dots, L-1$. Intensity transformation function is given by

$$s = T(r) \text{ where } \quad 0 \le r \le L-1 \qquad \mathbf{(3-12)}$$

Before explaining the histogram equalization, two concepts are required to understand.

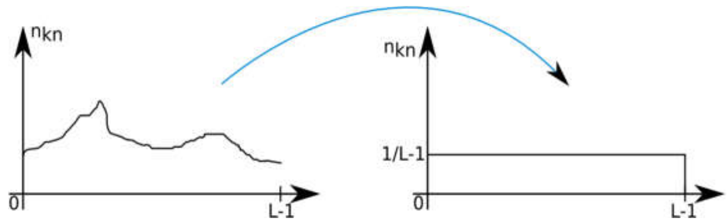
- $Probability\ density\ function\ (pdf)$: It is a function that tells the density (number of pixels) of particular intensity level i.e. probability.

$$pdf = p_k = \frac{n_k}{MN} = \frac{number\ of\ pixels\ with\ intensity\ k}{total\ number\ of\ pixels} \qquad \mathbf{(3-13)}$$

- $Cumulative\ distribution\ function\ (cdf)$: it is the function that tells total density till level k.

$$cdf = \sum_{i=0}^{k} p_i = \sum_{i=0}^{k} \frac{n_k}{MN} = \frac{number\ of\ pixels\ with\ intensity\ values\ from\ 0-k}{total\ number\ of\ pixels} \qquad \mathbf{(3-14)}$$

If $0 \leq k \leq L-1$, pdf of all k levels is computed and summed, it is equal to 1. Idea behind histogram equalization is very simple, change the non-uniform histogram to uniform histogram where both are equal to 1 when summed.



Cdf of input and output images equal to 1 if computed all intensity levels i.e. $L-1$. This can be written mathematically as:

$$s_{L-1} = (L-1)\sum_{0}^{L-1} p_i \ = 1 \qquad \mathbf{(3-15)}$$

Next, consider few cdf conditions

- The cdf of first level of output image must map to first value of pdf

$$s_0 = (L-1)\sum_{0}^{0} p_i \qquad \mathbf{(3-16)}$$

- And the output should be uniformly distributed over all range of available intensity levels so that same formula could be used to compute cdf for all intensity levels.

$$s_k = (L-1)\sum_{0}^{k} p_i \qquad \mathbf{(3-17)}$$

- The cdf computed for last intensity level must account for all intensities of input image as given in equation (3-15)

This is intuitive explanation of histogram equivalized distribution.

3.3.1.1 Mathematical Proof of Histogram Equalization

Now we provide mathematical basis for histogram equalization. We define two functions to understand our discussion.

- Monotonically increasing function is a function whose output is greater or equal to its previous output when input is given in ordered way i.e.

$$0 \leq T(r_k) \leq T(L-1) \qquad for\ \ 0 \leq k \leq L-1 \quad \mathbf{(3-18)}$$

- Strictly monotonically increasing function is a function whose output is always greater than its previous output when input is given in ordered way i.e.

$$0 < T(r_k) < T(L-1) \qquad for\ \ 0 \leq k \leq L-1 \quad \mathbf{(3-19)}$$

Probability density value for some intensity value k, in input image, is given by $p(k)$. If instead of writing pdf for particular level, we compute it for a range of intensity levels i.e. 2-4, can be written as $\Delta x = dx = k_2 - k_1$ in general form. Probability density for this range is given as:

$$p(x)d(x) \qquad \mathbf{(3-20)}$$

Similarly, for output image we can write:

$$p(y)d(y) \qquad \mathbf{(3-21)}$$

Equations (3-15), (3-16), (3-17), implies that

$$p(y)d(y) = p(x)d(x) \quad \mathbf{(3-22)}$$

Equation (3-22) says that number of pixels mapped from input image to output image are same. To understand the concept, you can start by considering $d(x) = L - 1$ and then keep on decreasing $d(x)$. In other words, when level-0 pixels i.e. n_0, are transformed to output image, output image has total pixels equal to n_0. When k level pixels n_k are transformed, total of transformed pixels are $n_1 + n_2 + .. + n_k$ and output image will have same number of pixels. Hence the equation holds.

As we have assumed that output image has uniform distribution and according to equation (3-15), cdf of total image is equal to 1. If $\Delta y = L - 1$ then $p(y) = 1$ $i.e.$ it is cdf of complete output image. Equation (3-22) now can be written as:

$$p(y)d(y) = p(x)d(x) \rightarrow (1)d(y) = p(x)d(x) \rightarrow d(y) = p(x)d(x) \quad \mathbf{(3-23)}$$

By rearranging the equation $\mathbf{(3-23)}$, we can write

$$\frac{d(y)}{d(x)} = p(x) \qquad or \qquad p(x) = \frac{d(y)}{d(x)} \quad \mathbf{(3-24)}$$

If we assume both x and y as continuous random variable, we can write the mapping function as:

$$y = f(x) = \int_0^x p(u)d(u) \qquad \mathbf{(3-25)}$$

$$and$$

$$y = f(L-1) = \int_0^{L-1} p(u)d(u) = P(L-1) - P(0) = 1 \quad \mathbf{(3-26)}$$

Equation 3-25 is cdf of input image that is used to produce output image. Equations 3-25 and 3-26 are used to show the relationship between differentiation and integration. As the gray levels in input image are discrete, we can convert the equations 3-25 and 3-26 as follows:

$$y' = f(x) = \sum_0^x p(u) \qquad \mathbf{(3-27)}$$

And

$$y' = f(L-1) = \sum_0^{L-1} p(u) \qquad \mathbf{(3-28)}$$

Now for input image, its $cdf = 1$ when intensity level ranges from $0 - L - 1$. We also know that cdf of output image is equal to 1 but we do not know the range of y. In order to keep same intensity range in input and output image, equations 3-26 and 3-28 suggest that it should be between $0 - L - 1$ instead of

$0 - 1$. To fulfil this requirement, we multiply equation 3-26 and 3-28 with L-1 which scales the intensity range in output image to $0 - L - 1$ and equation 3-28 becomes as follows:

$$y' = f(L-1) = (L-1)\sum_{0}^{L-1} p(u) \qquad \mathbf{(3-29)}$$

There is another condition which states that if equation 3-28 holds then

$$cdf(x) = T^{-1}(y)$$

It means that if there is some forward mapping from input image to output image with $L - 1$ levels, there should be a reverse mapping with same number of levels. Hence equation 3-29 is multiplied with $L - 1$.

Histogram Equalization Example

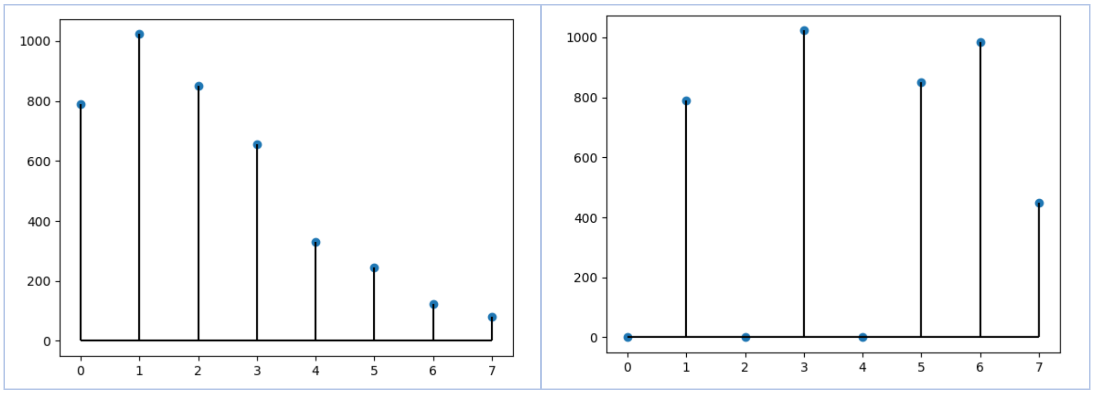
Let we have an image with following features:

- Intensity depth: $3 - bit$
- No. of intensity levels: $8\ (0 - 7)$
- Image spatial resolution: $64x64 \rightarrow\ 4096$
- Histogram equalization formula: $Equation\ \mathbf{(3-29)}$

Intensity level r_k	**Intensity density n_k**	**Probability density function $p_r(r_k) = \frac{n_k}{MN}$**	$s_k = T(r_k) = (L-1)\sum_0^k p_i$	$Round(s_k)$	n_{s_k}
$r_0 = 0$	790	$790/4096 \rightarrow 0.19$	$7 * (0.19) = 1.33$	1	790
$r_1 = 1$	1023	0.25	$7 * (0.19 + 0.25) = 3.08$	3	1023
$r_2 = 2$	850	0.21	$\mathbf{4.55}$	**5**	**850**
$r_3 = 3$	656	0.16	**5.67**	**6**	**656+329 =985**
$r_4 = 4$	329	0.08	**6.23**	**6**	
$r_5 = 5$	245	0.06	**6.65**	**7**	**245+122 +81 = 448**
$r_6 = 6$	122	0.03	**6.86**	**7**	
$r_7 = 7$	81	0.02	**7.00**	**7**	

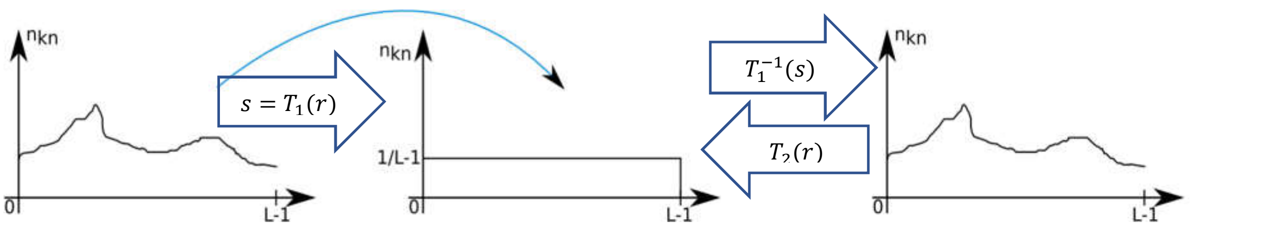
Now we create histograms of original and equalized images follows:

Before Equalization	**After Equalization**



3.3.1.2 Histogram Matching (Specification)

There are the situations where we need to transform the histogram of an input image that matches the histogram of some specified image. This process of matching histograms of two images is known as histogram matching or histogram specification.



Input Image Histogram	Equalized Histogram	Specified Histogram

Again, the idea behind histogram matching is simple. If you are given with two images A and B and you are asked to match the histogram of A with B. You will use the following algorithm:

1. Find and equalize the histogram of input image using some function T.
$$s = T(r) \qquad \mathbf{(3-30)}$$
2. Find and equalize the histogram of specified image using some function G.
$$z = G(y) \qquad \mathbf{(3-31)}$$
3. Now both histograms are equalized and are "equal" as both are uniform distributions. Applying G^{-1} on equalized histogram of specified image B will generate the specified image B.
$$T(r) = G(y) \qquad \mathbf{(3-32)}$$
4. Instead of applying this inverse function on equalized histogram of B, apply it on equalized histogram of A. It will produce an image that will have histogram similar to histogram of specified image B.
$$y = G^{-1}(s) \qquad \mathbf{(3-33)}$$

Now with this simple explanation of histogram matching, we discuss the mathematics behind this operation. Let s and r are the output and input intensities respectively then transformation is written as:

$$s_k = T(r_k) = (L-1)\sum_{i=0}^{k} p_i \qquad \mathbf{(3-34)}$$

As before, we suppose that $p_r(r)$ is pdf of input image that we want to match with $p_z(z)$, the reference pdf. Then we can write

$$s = T(r) \quad \mathbf{(3-35)}$$

$$z = G^{-1}(s) = G^{-1}(T(r)) \quad \mathbf{(3-36)}$$

We can write these equations more specifically as (histogram equalization of specified image):

$$G(z_q) = (L-1)\sum_{i=0}^{q} p_z(z_i) \quad \mathbf{(3-37)}$$

For a value of q that

$$G(z_q) = s_k \quad \mathbf{(3-38)} \text{ and}$$

$$z_q = G^{-1}(s_k) \quad \mathbf{(3-39)}$$

Considering these two relationships, we can re-design the histogram matching algorithm as follows:

1. Compute the histogram $p_r(r)$ of the input image
2. Equalize this histogram using equation $s_k = T(r_k) = (L-1)\sum_{i=0}^{k} p_r(r_i)$ and round the output to integer values
3. Compute the values of $G(z_q)$ by using the relationship $G(z_q) = (L-1)\sum_{i=0}^{q} p_z(z_i)$ where $p_z(z_i)$ is the values of specified histogram. Round the value of G.
4. For every value of s_k, use the stored values of G from step 3 such that $G(z_q)$ is closest to s_k. When more than one values of z_q give the same match, choose the smallest value by convention. Form the output image by mapping every equalized pixel s_k to corresponding z_q using step 4
5. Compute the output image pdf

Example: Histogram Matching

Let we have an image with following features:

- Intensity depth: $3-bit$
- No. of intensity levels: $8\ (0-7)$
- Image spatial resolution: $64x64 \rightarrow\ 4096$

Following tables shows the details of input and reference histograms

1	2	3	4	5	6	7	8	9
$\boldsymbol{r_k}$	n_k	$\boldsymbol{p_r k = \frac{n_k}{MN}}$	$s_k = T(r_k)$	$Round(\boldsymbol{s_k})$	$\boldsymbol{n_{s_k}}$	Specified $pdf = p_z(k)$	$\boldsymbol{G(z_q)}$ $=\boldsymbol{(L-1)\sum_0^k p_i}$	$\boldsymbol{G(z_q)}$
$\boldsymbol{r_0 = 0}$	790	**0.19**	1.33	**1**	790	0.00	0.00	0
$\boldsymbol{r_1 = 1}$	1023	**0.25**	3.08	**3**	1023	0.00	0.00	0
$\boldsymbol{r_2 = 2}$	850	**0.21**	$\mathbf{4.55}$	**5**	**850**	**0.00**	**0.00**	**0**
$\boldsymbol{r_3 = 3}$	656	**0.16**	**5.67**	**6**	**985**	**0.15**	**7*(0.15) =1.05**	**1**
$\boldsymbol{r_4 = 4}$	329	**0.08**	**6.23**	**6**		**0.20**	**2.45**	**2**
$\boldsymbol{r_5 = 5}$	245	**0.06**	**6.65**	**7**	**448**	**0.30**	**4.55**	**5**
$\boldsymbol{r_6 = 6}$	122	**0.03**	**6.86**	**7**		**0.20**	**5.95**	**6**
$\boldsymbol{r_7 = 7}$	81	**0.02**	**7.00**	**7**		**0.15**	**7.00**	**7**

1. Compute the pdf $p_r(r)$ for input image which is given in column 3 in above table

2. Equalize the histogram and find values that are given in column 5 in above table.
3. Compute the function $G(z_q)$ by considering the specified histogram as shown in column 8 and its rounded values are given in column 9
4. For every value of s_k, use the stored values of G from step 3 such that $G(z_q)$ is closest to s_k as shown in column 17
5. Output image pdf is shown in column 18.

10	11	12	13	14	15	16	17	18
r_k	n_k	s_k	n_{s_k}	Z_q	Specified pdf	$G(z_q)$	$n_z k$	$p_z(z_i)$
$r_0 = 0$	790	1	790	$Z_0 = 0$	0.00	0	0	0/4096=0
$r_1 = 1$	1024	3	1024	$Z_1 = 1$	0.00	0	0	0
$r_2 = 2$	850	5	850	$Z_2 = 2$	0.00	0	0	0
$r_3 = 3$	656	6	985	$Z_3 = 3$	0.15	1	790	0.19
$r_4 = 4$	329	6		$Z_4 = 4$	0.20	2	1024	0.25
$r_5 = 5$	245	7	448	$Z_5 = 5$	0.30	5	850	0.21
$r_6 = 6$	122	7		$Z_6 = 6$	0.20	6	985	0.24
$r_7 = 7$	81	7		$Z_7 = 7$	0.15	7	448	0.11

Histogram matching can only approximate the specified histogram. It is possible to use histogram matching to balance detector responses as a relative detector calibration technique. It can be used to normalize two images, when the images were acquired at the same local illumination (such as shadows) over the same location, but by different sensors, atmospheric conditions or global illumination. Exact histogram matching is the problem of finding a transformation for a discrete image so that its histogram exactly matches the specified histogram. Several techniques have been proposed for this. One simplistic approach converts the discrete-valued image into a continuous-valued image and adds small random values to each pixel so their values can be ranked without ties. However, this introduces noise to the output image. Because of this there may be holes or open spots in the output matched histogram. The concept of histogram matching can be extended to match multiple histograms.

3.3.1.3 Histogram Comparison

Histograms of two or more images can be compared in order to find the similarity between them. The process of comparing histogram finds distances between them and from these distances, a single score is obtained that is also named as distance i.e. $d(img1, img2)$. There are number of methods (algorithms) available that can be used to find the histogram-based distance of one image from another. Few of these algorithms are correlation, Chi-square and Intersection. If H_1 and H_2 are two histograms, using correlation the distance of two histograms can be found as:

$$d(H_1, H_2) = \frac{\sum_I [H_1(I) - \mathcal{H}_1(I)][H_2(I) - \mathcal{H}_2(I)]}{\sqrt{\sum_I [H_1(I) - \mathcal{H}_1(I)]^2 \sum_I [H_2(I) - \mathcal{H}_2(I)]^2}} \qquad (\mathbf{3} - 40)$$

Where

$$\mathcal{H}(I) = \frac{1}{N} \sum_j H_k(j)$$

$\mathcal{H}(I)$ is the average which is computed by summing densities of all bins and then dividing with total number of bins N.

Following figure shows different images and their similarity score. There are multiple uses of histogram comparison that may include image matching, image classification and searching similar images.

3.3.1.4 Local Histogram Processing (Local Histogram Equalization-LHE)

The methods we have studied till now are based on whole image histogram processing. Complete image enhancement fails to improve the dark regions in an image where some hidden information is present. This is because the number of pixels in such small dark regions are either small in number or have little difference with surrounding pixels. This shortcoming of global histogram processing can be removed by using information present in intensity distribution of reference pixel neighborhood. This can be achieved using adaptive histogram processing.

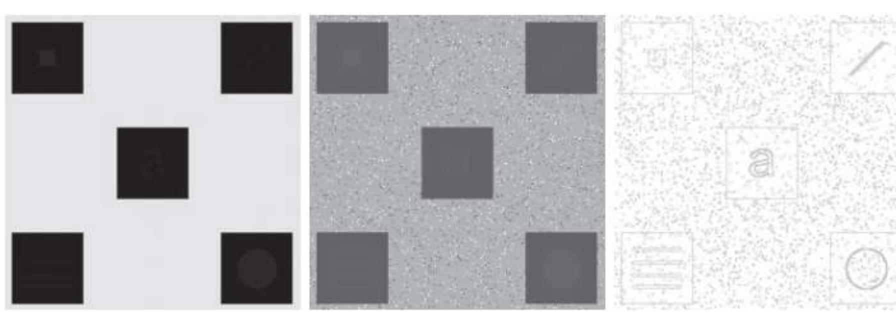


Figure Error! No text of specified style in document.**-4: A) Original Image, B) Global Histogram Equalization C) Local Histogram Processing**

The local histogram processing is done by defining a window size and moving it pixel to pixel in horizontal and vertical directions. At each location, the histogram of the points in the window is computed, and either a $histogram\ equalization$ or $histogram\ specification$ transformation function is obtained. This function is used to map the intensity of the pixel centered in the neighborhood. The center of window is then moved one pixel to right and the transformation is computed again. When the column-wise moving is complete then window center is placed again on first column of next row. This process is also known as ***adaptive histogram processing (AHE)***. It differs from ordinary histogram equalization in the respect that the adaptive method computes several histograms, each corresponding to a distinct region (window) of the image, and uses them to redistribute the lightness values of the image. It is therefore suitable for improving the local contrast and enhancing the definitions of edges in each region of an image.

However, AHE has a tendency to overamplify noise in relatively homogeneous regions of an image. A variant of adaptive histogram equalization called contrast limited adaptive histogram equalization (CLAHE) which attempts to prevent this by limiting the amplification however small artifacts remains in processed image. This is due to fact that if noise is present in local window of image, that is also amplified. The method is named as contrast limited as If any histogram bin is above the specified contrast limit (by default 40 in OpenCV), those pixels are clipped and distributed uniformly to other bins before applying histogram equalization.

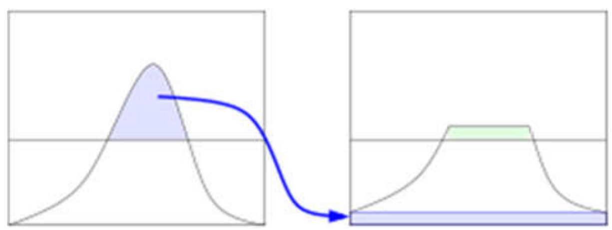


Figure Error! No text of specified style in document.**-5: Contrast bin clipping**

For AHE computation, a good approach is to use sliding window where the window slides one pixel at a time and incrementally update the histogram for each pixel.

Following figures show the application of CLAHE and enhancement of noise. As OpenCV uses 8x8 window, you can observe noise rectangles in processed image and it becomes more prominent when CLAHE is applied repeatedly.

3.3.1.5 Histogram statistics and image enhancement

Image enhancement can also be achieved by using statistics associated with an image. Most useful statistic measures are listed below:

Statistic	**Formula**
Mean	$m = \sum_{0}^{L-1} r_i p(r_i) \quad (\mathbf{3}-40)$
First moment	$\mu_1 = \sum_{0}^{L-1} (r_i - m) p(r_i) \quad (\mathbf{3}-41)$
Second Moment/ Variance $\boldsymbol{\sigma^2}$	$\mu_2 = \sum_{0}^{L-1} (r_i - m)^2 p(r_i) \quad (\mathbf{3}-42)$
Nth moment	$\mu_n = \sum_{0}^{L-1} (r_i - m)^n p(r_i) \quad (\mathbf{3}-43)$

Among these statistics, mean and variance are most useful that can be employed for contrast enhancement either at global or local level. When used on global level, m and $\sigma^2$are calculated for entire image and in case of local processing, these quantities are calculated for part of image (the window under consideration). If S_{xy} defines the neighborhood of window centered at pixel x, y. We can describe the m and σ^2 for this local image window as:

$$m_{s_{xy}} = \sum_{0}^{L-1} r_i p_{s_{xy}}(r_i) \quad (\mathbf{3}-44)$$

And variance is given as:

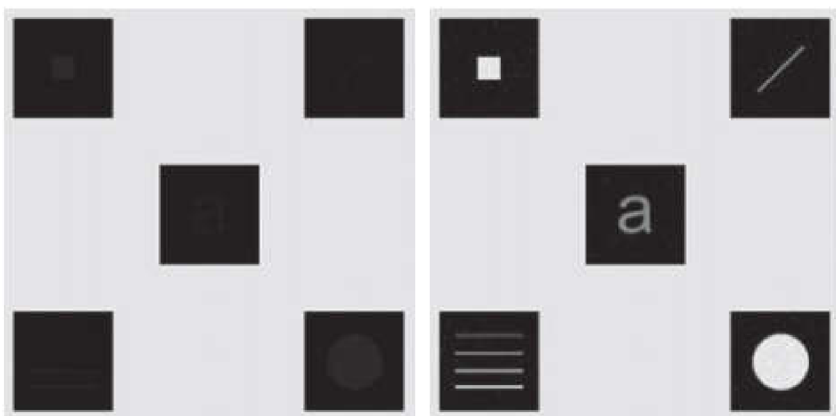
$$\mu_{s_{xy}} = \sum_{0}^{L-1} \left(r_i - m_{s_{xy}}\right)^2 p_{s_{xy}}(r_i) \quad (\mathbf{3}-45)$$

Example: Image enhancement using image statistics

The problem at hand is to enhance the low contrast detail in the dark areas of the image, while leaving the light background unchanged. A method used to determine whether an area is relatively light or dark at a point (x, y) is to compare the average local intensity. Image enhancement algorithm using statistical measures works as follows:

- Firstly, find all four statistics $m, m_{s_{xy}}, \sigma^2, \sigma^2_{s_{xy}}$.
- A pixel $p(x, y)$ is candidate for processing if $k_0 m \le m_{s_{xy}} \le k_1 m$ for some non-negative value of k where $k_0 \le k_1$. We choose k arbitrary. For example, if our focus is on areas that are darker than one-quarter of the mean intensity we would choose $k_0 = 0$ and $k_1 = 0.25$.
- A pixel is candidate for processing if $k_2\sigma \le \sigma_{s_{xy}} \le k_3\sigma$. For example, to enhance a dark area of low contrast, we might choose $k_2 = 0$ and $k_3 = 0.1$.
- A pixel that meets all the preceding conditions for local enhancement is processed by multiplying it by a specified constant, C, to increase (or decrease) the value of its intensity level relative to the rest of the image. Pixels that do not meet the enhancement conditions are not changed. Mathematically this procedure can be written as following formula

$$g(x,y) = f(x) = \begin{cases} cf(x,y), & if \quad k_0 m \le m_{s_{xy}} \le k_1 m \; \textbf{and} \; k_2\sigma \le \sigma_{s_{xy}} \le k_3\sigma \\ f(x,y), & otherwise \end{cases} \qquad (3-46)$$



Write a program to use image statistics for image enhancement

Reference Programs
***Program P3-14**: Calculating image histogram using python*
***Program P3-15**: Computing image histogram using OpenCV*
***Program P3-16**: Normalizing image histogram bins*
***Program P3-17**: Equalizing the histogram using python*
***Program P3-18**: Histogram equalization in OpenCV*
***Program P3-19**: Histogram Matching using book example*
***Program P3-20**: Histogram matching using images*
***Program P3-21**: Comparing image histograms*
***Program P3-22**: Distances based on histograms*
***Program P3-23**: Local adaptive histogram equalization*

3.4 The Spatial Filtering

If an image is represented in spatial domain (usually the raw image), we can perform different matrix-based operations in order to enhance the quality of image. Previously we have seen that a window of image pixels is selected and some operations are performed on them. We can extend this idea by introducing a new small matrix F which is operated with image to change image features. Although F is a matrix of values however, we use different terms to denote it. Here are some notations that are used with different image operations.

- $\boldsymbol{Filter}$: A special type of matrix that is operated with image. A filter is also known as $\boldsymbol{kernel}$, $\boldsymbol{mask}$, $\boldsymbol{template\ or\ window.}$ The term filter is taken from frequency domain where "filtering" refers to stopping, allowing or modifying some frequency components in signal (image is also a signal). We will discuss filter in details later on in this chapter. Usually the size of filter is smaller than image however there may be the situations where filter size is greater than image size.
- $\boldsymbol{Linear\ Filter}$: If the operation performed on image pixels is linear then operator matrix is called linear filter.
- $\boldsymbol{Non - Linear\ Filter}$: If the operator filter (matrix) performs the operation on image pixels that is non-linear then the filter is called non-linear filter.

3.4.1 The processing of spatial linear filtering

Operations like filtering frequency components can be achieved on image by devising "filter matrices" and then applying them on images. A spatial linear filter performs the $sum\ of\ products$ operation on image. Usually a kernel defines the neighborhood of some reference pixel in an image. To understand better, let we consider an image f of size 5x5 and kernel w of size 3x3 then general sum of products operation can be defined as:

Table 1: A) Filter Kernel w B) Image F

W(0,0)	W(0,1)	W(0,2)
W(1,0)	W(1,1)	W(1,2)
W(2,0)	W(2,1)	W(2,2)

F(0,0)	F(0,1)	F(0,2)	F(0,3)	F(0,4)
F(1,0)	F(1,1)	F(1,2)	F(1,3)	F(1,4)
F(2,0)	F(2,1)	F(2,2)	F(2,3)	F(2,4)
F(3,0)	F(3,1)	F(3,2)	F(3,3)	F(3,4)
F(4,0)	F(4,1)	F(4,2)	F(4,3)	F(4,4)

The filter maps over specific position on image which is shown by dark solid line. The operation is defined with reference to window center pixel and on overlapping items, it is denoted as:

$$g(x,y) = f(0,0)*w(0,0) + f(0,1)*w(0,1) + f(0,2)*w(0,2) +$$
$$f(1,0)*w(1,0) + f(1,1)*w(1,1) + f(1,2)*w(1,2) +$$
$$f(2,0)*w(2,0) + f(2,1)*w(2,1) + f(2,2)*w(2,2)$$
$$= \sum_{x=0}^{2}\sum_{y=0}^{2} f(x,y)*w(x,y) \qquad (3-47)$$

If we consider the center pixel as reference pixel then it coordinates will be (0,0). In this case, our matrices will be as follows:

Table 2: A) Filter Kernel w B) Image F

W(-1,-1)	W(0,-1)	W(1,-1)
W(-1,0)	W(0,0)	W(1,0)
W(-1,1)	W(0,1)	W(1,1)

F(-1,-1)	F(0,-1)	F(1,-1)	F(0,3)	F(0,4)
F(-1,0)	F(0,0)	F(1,0)	F(1,3)	F(1,4)
F(-1,1)	F(0,1)	F(1,1)	F(2,3)	F(2,4)
F(3,0)	F(3,1)	F(3,2)	F(3,3)	F(3,4)
F(4,0)	F(4,1)	F(4,2)	F(4,3)	F(4,4)

And resulting equation will be:

$$g(x,y) = f(-1,-1)*w(-1,-1) + f(-1,0)*w(-1,0) + f(-1,1)*w(-1,1) +$$
$$f(-1,0)*w(-1,0) + f(0,0)*w(0,0) + f(1,0)*w(1,0) +$$
$$f(-1,1)*w(-1,1) + f(0,1)*w(0,1) + f(1,1)*w(1,1)$$
$$= \sum_{x=-1}^{1}\sum_{y=-1}^{1} f(x,y)*w(x,y) \qquad (3-48)$$

The center of the kernel moves from pixel to pixel by varing coordinates x and y, generating the filtered image, g. As an example, for next processing, window slides to next location as shown:

W(-1,-1)	W(0,-1)	W(1,-1)
W(-1,0)	W(0,0)	W(1,0)
W(-1,1)	W(0,1)	W(1,1)

F(0,0)	F(-1,-1)	F(0,-1)	F(1,-1)	F(0,4)
F(1,0)	F(-1,0)	F(0,0)	F(1,0)	F(1,4)
F(2,0)	F(-1,1)	F(0,1)	F(1,1)	F(2,4)
F(3,0)	F(3,1)	F(3,2)	F(3,3)	F(3,4)
F(4,0)	F(4,1)	F(4,2)	F(4,3)	F(4,4)

This sliding process continues till the row ends and the process restarts from first pixel of next row. In this way all rows and columns in image are processed.

3.4.2 Spatial Correlation and Convolution

Spatial correlation is same process which has been described in previous section. Convolution is similar to correlation process except that in convolution, the filter is rotated by 180^0. We begin by explaining 1D correlation. The correlation and convolution for 1D signal are given as:

$$Correlation \rightarrow g(x) = w \star f = \sum_{-a}^{a} w(s)f(x+s) \qquad (3-49)$$

and

$$Convolution \rightarrow g(x) = w \ast f = \sum_{-a}^{a} w(s)f(x-s) \qquad (3-50)$$

Example: 1D Correlation and Convolution

Let we have kernel w of size 5 and signal (1D image) f of size 8 as follows:

$f = [2\ 1\ 3\ 1\ 0\ 1\ 2\ 1]$ and $w = [1\ 2\ 4\ 2\ 8]$

Correlation ($w \star f$)	Convolution ($w \ast f$)
[2 1 3 1 0 1 2 1] [1 2 4 2 8] We position the center of filter kernel at first element of image	[2 1 3 1 0 1 2 1] [8 2 4 2 1]
[0 0 2 1 3 1 0 1 2 1 0 0] [1 2 4 2 8] Pad zeros on both sides of signal to make operation valid. If we do not pad then there will be no value to which 1,2 will be multiplied. Padded values are shown as blue zeros.	[0 0 2 1 3 1 0 1 2 1 0 0] [8 2 4 2 1]
[0 0 2 1 3 1 0 1 2 1 0 0] [1 2 4 2 8] -------------------------------- 0*1+0*2+2*4+1*2+3*8 = 0+0+8+2+24 = 34	[0 0 2 1 3 1 0 1 2 1 0 0] [8 2 4 2 1] -------------------------------- 0*8+0*2+2*4+1*2+3*1 = 0+0+8+2+3 = 13
[0 0 2 1 3 1 0 1 2 1 0 0] [1 2 4 2 8] -------------------------------- 0*1+2*2+1*4+3*2+1*8 = 0+4+4+6+8 = 22	[0 0 2 1 3 1 0 1 2 1 0 0] [8 2 4 2 1] -------------------------------- 0*8+2*2+1*4+3*2+1*1 = 0+4+4+6+1 = 15
[0 0 2 1 3 1 0 1 2 1 0 0] [1 2 4 2 8] -------------------------------- 2*1+1*2+3*4+1*2+0*8 = 2+2+12+2+0 = 18	[0 0 2 1 3 1 0 1 2 1 0 0] [8 2 4 2 1] -------------------------------- 2*8+1*2+3*4+1*2+0*1 = 16+2+12+2+0 = 32
[0 0 2 1 3 1 0 1 2 1 0 0] [1 2 4 2 8] -------------------------------- 1*1+3*2+1*4+0*2+1*8 = 1+6+4+0+8 = 19	[0 0 2 1 3 1 0 1 2 1 0 0] [8 2 4 2 1] -------------------------------- 1*8+3*2+1*4+0*2+1*1 = 8+6+4+0+1 = 19
[0 0 2 1 3 1 0 1 2 1 0 0] [1 2 4 2 8] -------------------------------- 3*1+1*2+0*4+1*2+2*8 = 3+2+0+2+16 = 23	[0 0 2 1 3 1 0 1 2 1 0 0] [8 2 4 2 1] -------------------------------- 3*8+1*2+0*4+1*2+2*1 = 24+2+0+2+2 = 30
[0 0 2 1 3 1 0 1 2 1 0 0] [1 2 4 2 8] -------------------------------- 1*1+0*2+1*4+2*2+1*8 = 1+0+4+4+8 = 17	[0 0 2 1 3 1 0 1 2 1 0 0] [8 2 4 2 1] -------------------------------- 1*8+0*2+1*4+2*2+1*1 = 8+0+4+4+1 = 17
[0 0 2 1 3 1 0 1 2 1 0 0] [1 2 4 2 8]	[0 0 2 1 3 1 0 1 2 1 0 0] [8 2 4 2 1]

--------------------------------- 0*1+1*2+2*4+1*2+0*8 = 0+2+8+2+0 = 14	--------------------------------- 0*8+1*2+2*4+1*2+0*1 = 0+2+8+2+0 = 12
[0 0 2 1 3 1 0 1 2 1 0 0] [1 2 4 2 8] --------------------------------- 1*1+2*2+1*4+0*2+0*8 = 1+4+4+0+0 = 9 **We end the procedure when the center of filter kernel reaches the end point in image.**	[0 0 2 1 3 1 0 1 2 1 0 0] [8 2 4 2 1] --------------------------------- 1*8+2*2+1*4+0*2+0*1 = 8+4+4+0+0 = 16
$\boldsymbol{w ☆ f}$= [34 22 18 19 23 17 14 9]	$\boldsymbol{w \star f}$= [13 15 32 19 30 17 12 16]

Now we have good idea about convolution and correlation operations, we extend them to 2D. In this case both kernel and image will have 2D shape.

- $Correlation\ h \star f$:

$$g(x, y) = h ☆ f = \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s,t) * f(x+s, y+t) \qquad (3-51)$$

- $Convolution$:

$$g(x, y) = h \star f = \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s,t) * f(x-s, y-t) \qquad (3-52)$$

$\boldsymbol{Correlation}$ is the measure of the similarity of two signals. $\boldsymbol{Convolution}$ is used to find the effect of one signal (kernel) on another signal (image). Different image processing operations like smoothing can be obtained using convolution. Convolution filters are also known as $convolution\ masks$ or $convolution\ kernel$. The process is described with the help of following example.

Example: 2D Convolution

We write few steps of convolution here; the reader can work out rest of the steps. Let our image f and filter w are:

$$W = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix} \rightarrow w^{rotated} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \ (after\ 180^0\ roation)$$

And

$$f = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 2 & 3 & 1 & 1 & 2 \\ 1 & 4 & 2 & 2 & 0 \\ 3 & 2 & 3 & 3 & 0 \\ 1 & 0 & 0 & 2 & 1 \end{bmatrix} \quad \text{then}$$

1 $$\begin{bmatrix}1&2&0&1&2\\2&3&1&1&2\\1&4&2&2&0\\3&2&3&3&0\\1&0&0&2&1\end{bmatrix} \rightarrow \begin{bmatrix}0&0&0&0&0&0&0\\0&1&2&0&1&2&0\\0&2&3&1&1&2&0\\0&1&4&2&2&0&0\\0&3&2&3&3&0&0\\0&1&0&0&2&1&0\\0&0&0&0&0&0&0\end{bmatrix} (zero\ padding)$$

2 $$\begin{bmatrix}1&1&2\\2&1&1\\1&1&1\end{bmatrix} \star \begin{bmatrix}0&0&0&0&0&0&0\\0&\mathbf{1}&2&0&1&2&0\\0&2&3&1&1&2&0\\0&1&4&2&2&0&0\\0&3&2&3&3&0&0\\0&1&0&0&2&1&0\\0&0&0&0&0&0&0\end{bmatrix} (initial\ window\ placing)$$

3 $$= (0*1+0*1+0*2)+(0*2+1*1+2*1)+(0*1+2*1+3*1) = (0+3+5) = \mathbf{8}$$

4 $$\begin{bmatrix}1&1&2\\2&1&1\\1&1&1\end{bmatrix} \star \begin{bmatrix}0&0&0&0&0&0&0\\0&1&\mathbf{2}&0&1&2&0\\0&2&3&1&1&2&0\\0&1&4&2&2&0&0\\0&3&2&3&3&0&0\\0&1&0&0&2&1&0\\0&0&0&0&0&0&0\end{bmatrix} (window\ slides\ one\ pixel\ to\ right)$$

5 $$= (0*1+0*1+0*2)+(1*2+2*1+0*1)+(2*1+3*1+1*1) = 0+4+6 = \mathbf{10}$$

6 $$\begin{bmatrix}1&1&2\\2&1&1\\1&1&1\end{bmatrix} \star \begin{bmatrix}0&0&0&0&0&0&0\\0&1&2&\mathbf{0}&1&2&0\\0&2&3&1&1&2&0\\0&1&4&2&2&0&0\\0&3&2&3&3&0&0\\0&1&0&0&2&1&0\\0&0&0&0&0&0&0\end{bmatrix}$$

7 $$= (0*1+0*1+0*2)+(2*2+0*1+1*1)+(3*1+1*1+1*1) = 0+5+5 = \mathbf{10}$$

8 $$\begin{bmatrix}1&1&2\\2&1&1\\1&1&1\end{bmatrix} \star \begin{bmatrix}0&0&0&0&0&0&0\\0&1&2&0&\mathbf{1}&2&0\\0&2&3&1&1&2&0\\0&1&4&2&2&0&0\\0&3&2&3&3&0&0\\0&1&0&0&2&1&0\\0&0&0&0&0&0&0\end{bmatrix}$$

9 $$= (0*1+0*1+0*2)+(0*2+1*1+2*1)+(1*1+1*1+2*1) = 0+5+4 = \mathbf{7}$$

10 $$\begin{bmatrix}1&1&2\\2&1&1\\1&1&1\end{bmatrix} \star \begin{bmatrix}0&0&0&0&0&0&0\\0&1&2&0&1&\mathbf{2}&0\\0&2&3&1&1&2&0\\0&1&4&2&2&0&0\\0&3&2&3&3&0&0\\0&1&0&0&2&1&0\\0&0&0&0&0&0&0\end{bmatrix}$$

11 $$= (0*1+0*1+0*2)+(1*2+2*1+0*1)+(1*1+2*1+0*1) = 0+4+3 = \mathbf{7}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \star \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 \\ 0 & \mathbf{2} & 3 & 1 & 1 & 2 & 0 \\ 0 & 1 & 4 & 2 & 2 & 0 & 0 \\ 0 & 3 & 2 & 3 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= (0*1+1*1+2*2)+(0*2+2*1+3*1)+(0*1+1*1+4*1) = 5+5+5 = \mathbf{15}$$

The convolution result looks like following matrix.

$$\begin{bmatrix} 8 & 10 & 10 & 7 & 7 \\ 15 & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \end{bmatrix}$$

The reader is encouraged to perform the remaining computations. Here is the completed matrix.

$$\begin{bmatrix} 8 & 10 & 10 & 7 & 7 \\ 15 & 18 & 20 & 14 & 9 \\ 18 & 23 & 26 & 18 & 10 \\ 15 & 21 & 22 & 16 & 11 \\ 8 & 13 & 13 & 9 & 8 \end{bmatrix}$$

To find correlation, you can easily rotate a matrix by writing few lines of python code.

3.4.3 Boundary handling in Convolution

In the example given in last section, we have seen that when, using correlation, we have to compute the value of first pixel in an image, we need to pad the image with zeros. This is required in order to make element wise multiplications possible. This zero padding is known as $boundary\ handling$. As the correlation and convolution operations are similar, so we will explain different boundary handling methods using the term convolution. Before looking at details of different convolution methods, we need to define some parameters.

- $\boldsymbol{Kernel\ Size}$ $(\boldsymbol{k})$: it is the size or shape of kernel that is also referred as field of view. It is from the fact that at a time, kernel overlaps (views) a part of image. Larger the kernel size, larger the field of view. Generally larger kernel views (sees) coarse grained details of image whereas smaller kernel can look into fine grained details of image. Smallest kernel size in 1D is 3 and for 2D it is 3x3. As previously stated, kernel size is generally smaller than image size however it may be greater than image.
- $\boldsymbol{Stride}$ $(\boldsymbol{s})$: It defines the step size of kernel when traversing the image. Till now, we have seen that either in 1D or in 2D our kernel slide one pixel at a time. This sliding can be changed to 2 or more pixel jump. As an example, in 1D, stride 2 will be as follows:

$$\begin{matrix} \mathbf{0} & \mathbf{0} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \mathbf{0} & \mathbf{0} \\ 1 & 2 & 3 & 4 & 5 & . & . & . & . & . & . & . & . \end{matrix} \quad \rightarrow \quad \begin{matrix} \mathbf{0} & \mathbf{0} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \mathbf{0} & \mathbf{0} \\ . & . & 1 & 2 & 3 & 4 & 5 & . & . & . & . & . & . \end{matrix}$$

$$\begin{matrix} \mathbf{0} & \mathbf{0} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \mathbf{0} & \mathbf{0} \\ . & . & . & . & 1 & 2 & 3 & 4 & 5 & . & . & . & . \end{matrix} \quad \rightarrow \quad \begin{matrix} \mathbf{0} & \mathbf{0} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \mathbf{0} & \mathbf{0} \\ . & . & . & . & . & . & 1 & 2 & 3 & 4 & 5 & . & . \end{matrix}$$

- $\boldsymbol{Padding}$ $(\boldsymbol{p})$**:** Padding is the process of boundary filling which we have discussed in detail in last section
- $\boldsymbol{Channels}$ $(\boldsymbol{c})$**:** We have already seen that a grayscale image consists of only one channel and its shape is represented by two values i.e. rows and columns. A colored image consists of three channels: red, green and blue. Shape of a color image can be represented by three values i.e. rows, columns,

channels. For example, a color image with height 400 and width 600 will have shape [400,600,3]. Deep learning algorithms can generate output that may contain more than 100 channels.

These parameters are used to determine the size of image generated through convolution process. Let our kernel is $w = [5,4,3,2,1]$ and image is $f = [1,2,3,4,5,6,7,8,9]$. As it is required for convolution, we will rotate the kernel by 180^0 before using in calculations. There are three types of paddings

1. **Full Padding**: The term has been taken from the fact that we apply padding that is equal to $kernel - size\ - 1$ on both size of image (if working with 1D and on all four sides if working with 2D image). Padding is done in way such that last element of kernel maps first element of image.
   For example, in our case, 4 zeros will be padded on left and right side of image.

$$\begin{array}{ccccccccccccccccc} 0&0&0&0&1&2&3&4&5&6&7&8&9&0&0&0&0\\ 1&2&3&4&5&.&.&.&.&.&.&.&.&.&.&.&. \end{array} \; and \; \begin{array}{ccccccccccccccccc} 0&0&0&0&1&2&3&4&5&6&7&8&9&0&0&0&0\\ .&.&.&.&.&.&.&.&.&.&.&.&1&2&3&4&5 \end{array}$$

$$= [\,5\ \ 14\ \ 26\ \ 40\ \ 55\ \ 70\ \ 85\ \ 100\ \ 115\ \ 80\ \ 50\ \ 26\ \ 9]$$

$$O_{size} = \frac{f_{size} - w_{size} + 2p}{s} + 1 \qquad (3-53)$$

   In our case we have following parameters $f_{size} = 9, w_{size} = 5, p = 4, s = 1 \rightarrow \left[\frac{9-5+2*4}{1}\right] + 1 = 13$

2. **Half/Same Padding:** When same size padding is used, the output generated by correlation is of same size of input image. Size of output image can be calculated using equation (3-53).
   There are multiple variations of same size padding and are listed below:
   - **Zero Padding**: Image is padded with zero values in order to compute correlation. Following lines show the need of zero padding when kernel center is placed at first element of an image or at last element of the image. Padded zeros are shown in blue.

$$\begin{array}{ccccccccccccc} 0&0&1&2&3&4&5&6&7&8&9&0&0\\ 1&2&3&4&5&.&.&.&.&.&.&.&. \end{array} \; and \; \begin{array}{ccccccccccccc} 0&0&1&2&3&4&5&6&7&8&9&0&0\\ .&.&.&.&.&.&.&.&1&2&3&4&5 \end{array}$$

$$= [\,26\ \ 40\ \ 55\ \ 70\ \ 85\ \ 100\ \ 115\ \ 80\ \ 50]$$

   - **Border Replication**: In this method, instead of using zeros, left most value is repeated on left side and right most value is repeated on right side as shown below:

$$\begin{array}{ccccccccccccc} 1&1&1&2&3&4&5&6&7&8&9&9&9\\ 1&2&3&4&5&.&.&.&.&.&.&.&. \end{array} \; and \; \begin{array}{ccccccccccccc} 1&1&1&2&3&4&5&6&7&8&9&9&9\\ .&.&.&.&.&.&.&.&1&2&3&4&5 \end{array}$$

   - **Border Reflect**: This method reflects the image on both sides. The concept is shown below.

$$\begin{array}{ccccccccccccc} 2&1&1&2&3&4&5&6&7&8&9&9&8\\ 1&2&3&4&5&.&.&.&.&.&.&.&. \end{array} \; and \; \begin{array}{ccccccccccccc} 2&1&1&2&3&4&5&6&7&8&9&9&8\\ .&.&.&.&.&.&.&.&1&2&3&4&5 \end{array}$$

   - **Border Reflect 101**: This method is similar to Border Reflect method however image boundary element is skipped during boundary filling.

$$\begin{array}{ccccccccccccc} 3&2&1&2&3&4&5&6&7&8&9&8&7\\ 1&2&3&4&5&.&.&.&.&.&.&.&. \end{array} \; and \; \begin{array}{ccccccccccccc} 3&2&1&2&3&4&5&6&7&8&9&8&7\\ .&.&.&.&.&.&.&.&1&2&3&4&5 \end{array}$$

   - **Border Wrap**: In this scheme, it is assumed that left side of image is virtually connected with right side.

$$\begin{array}{ccccccccccccc} 8&9&1&2&3&4&5&6&7&8&9&1&2\\ 1&2&3&4&5&.&.&.&.&.&.&.&. \end{array} \; and \; \begin{array}{ccccccccccccc} 8&9&1&2&3&4&5&6&7&8&9&1&2\\ .&.&.&.&.&.&.&.&1&2&3&4&5 \end{array}$$

- **No Padding**: In this case, we assume that there is no padding or additional border to accommodate the kernel. As a result, you can not place kernel on first element of image. Kernel first element is placed on first image element and during kernel window sliding, no kernel element is allowed to go out of image, resulting correlation vector or matrix is of smaller size than original image. The size of new image is calculated using equation (3-53). For our example 1D kernel and image, result of convolution will be of size 5.

$$\begin{array}{ccccccccccccc} & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & & \\ . & . & 1 & 2 & 3 & 4 & 5 & . & . & . & . & . & . \end{array} \quad and \quad \begin{array}{ccccccccccccc} & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & & \\ . & . & . & . & . & . & 1 & 2 & 3 & 4 & 5 & . & . \end{array}$$

Every method has its own pros and cons. These convolutions can be calculated using $filter2D$ API of OpenCV. The concept of convolution can easily be extended to 2D, 3D and ND matrices or images.

3.4.4 Convolution and its types

Although convolution and correlation are similar operations however convolution is more versatile than correlation. Following table lists basic properties of correlation and convolution:

Property	Correlation	Convolution
Commutative	-	$f \star g = g \star f$
Associative	-	$f \star (g \star h) = f \star (g \star h)$
Distributive	$f ✰ (g + h) = (f ✰ g) + (f ✰ h)$	$f \star (g + h) = (f \star g) + (f \star h)$

Correlation and convolutions are used for different purposes but there are more applications of convolution due its commutative and associative properties. Different types of convolution operations have emerged due to introduction of neural networks in digital image processing by leveraging the basic convolution concept. Following section lists few specialized types of convolutions popular among deep learning community.

3.4.4.1 Dilated or Atrous Convolution

Dilated convolution introduces another parameter called dilation rate. We have used kernels in which values are consecutive without gap. We may define the kernels (2D) that have gap between their elements.

$$w_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 1 & . & 2 & . & 3 \\ . & . & . & . & . \\ 4 & . & 5 & . & 6 \\ . & . & . & . & . \\ 7 & . & 8 & . & 9 \end{bmatrix} \qquad w_3 = \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 5 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 7 & 0 & 8 & 0 & 9 \end{bmatrix}$$

Here w1 is compact of condensed kernel that has no gaps and will map 9 consecutive elements in the image. W2 is dilated kernel that has gaps between its elements. Such a kernel will map 5x5 window in an image. The gaps in kernel are usually converted to zeros in order to make multiplications valid operation. This dilation gives a larger field of view on same computational cost.

3.4.4.2 Deconvolution

Deconvolution is the inverse process of convolution. In normal convolution process if we use a window size of 3x3, 9 values from kernel and 9 from image are used to compute one value in output image. An inverse process of convolution will require one value to generate all 9 values of input image using same kernel.

3.4.4.3 Transposed Convolution

This type of convolution is also known as fractionally strided convolutions. Some authors use term "**deconvolution**" for transposed convolution that is confusing therefore we will use the term transposed convolution only. Transposed convolutions can be used to upscale image as a better alternative of interpolation. The simplest way to think about a transposed convolution on a given input is to imagine such an input as being the result of a direct convolution applied on some initial feature map. The transposed convolution can be then considered as the operation that allows to recover the shape. Let if we convolve our 1D image and kernel $w = [1,2,3,4,5]$ and $f = [1,2,3,4,5,6,7,8,9]$ then using valid size convolution, we get an image of size 5. It shows that we can use convolution to reduce the image size.

The $transpose\ convolution$ will be the convolution that gets the input image of size 5 and produces an image of size 9. It is not necessary that transposed convolution will generate same values as in original image. The size of the output of transposed convolution can be found using inverse of relevant equation i.e.

$$(O_{size} - 1) * s - 2p + w_{size} = f_{size} \qquad (3-54)$$

Equation (3-54) is rearrangement of equation (3-53). If consider O_{size} as input and f_{size} as output of transpose convolution and rename them accordingly, we can re-write the equation 3-54 as follows:

$$O_{size} = (f_{size} - 1) * s - 2p + w_{size} \qquad (\mathbf{3-55})$$

3.4.4.4 Separable Convolution – Convolution with multiple kernels

Sometime there may be the condition where we need to convolve an image with kernel-1 and then its result with 2nd kernel and so on. Such a situation for n-kernels can be written as:

$$w_n \star \ldots w_3 \star (w_2 \star (w_1 \star f)) \quad (\mathbf{3}-31)$$

We can simplify the operation by some mathematical work on equation $(\mathbf{3}-31)$. We use the commutative property of convolution in this mathematical workaround.

$$\boldsymbol{w} = w_1 \star w_2 \star \ldots . \star w_n \qquad (\mathbf{3}-32).$$

and

$$convolution = \boldsymbol{w} \star f \qquad (\mathbf{3}-33).$$

Separable Kernel Filters

A 2D function is said to be separable if it can be written as product two 1D function. A kernel is also a function and if that can be written as product of two kernels, it is also called the separable kernel.

$$G(x,y) = G_1(x)G_2(y) \qquad (\mathbf{3}-34).$$

If we represent two sub-kernels(u,v) as vectors of size $mx1$ and $nx1$ then these could be expressed as outer product of two sub-kernels.

$$G(x,y) = uv^T$$

Separable kernels provide computational advantage in two ways:

- A product of two 1D kernels is equal to convolution of these vectors.
- Associative property of convolution. We can process the kernels and images in any order and by selecting suitable processing order, we can reduce the number of basic operations (multiplication and addition).

$$\boldsymbol{w} \star f = (w_1 \star w_2) \star f = w_1 \star (w_2 \star f) = (w_1 \star f) \star w_2 \quad (\mathbf{3}-36).$$

If an image of size MN is convolved with a kernel of size mn, total number of operations (multiplications and additions) will be $MNmn$ where in case of separable kernel this becomes MNm and MNn or $MN(m+n)$. The computational advantage gained is:

$$MNmn: MN(m+n) \;\rightarrow\; mn: (m+n)$$

$$= \frac{mn}{m+n}$$

Example: Separable convolution

Let we have a kernel as follows:

$$w = G(x,y) = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \rightarrow G_1(x)G_2(y)$$

$$f(x,y) = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

$$f(x,y) \star G(x,y) = \begin{bmatrix} 19 & 21 & 13 \\ 22 & 29 & 24 \\ 11 & 23 & 25 \end{bmatrix}$$

By performing 2D convolution for each pixel of image, 9 multiplications and 8 additions are performed and for 9 image pixels, there will be 81 multiplications and 81 additions.

For separable kernel $G_1(x)$

$$f(x,y) \star G_1(x) = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 1 \\ 3 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix} \star \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0*1+2*2+3*1 & 0*1+2*2+1*1 & 0*1+1*2+2*1 \\ 2*1+3*2+0*1 & 2*1+1*2+2*1 & 1*1+2*2+4*1 \\ 3*1+0*2+0*1 & 1*1+2*2+0*1 & 2*1+4*2+0*1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 5 & 4 \\ 8 & 6 & 9 \\ 3 & 5 & 10 \end{bmatrix}$$

27 multiplications and additions are performed.

For separable kernel $G_2(x)$

$$[f(x,y) \star G_1(x)] \star G_2(x) = \begin{bmatrix} 7 & 5 & 4 \\ 8 & 6 & 9 \\ 3 & 5 & 10 \end{bmatrix} \star \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 7 & 5 & 4 & 0 \\ 0 & 8 & 6 & 9 & 0 \\ 0 & 3 & 5 & 10 & 0 \end{bmatrix} \star \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0*1+7*2+5*1 & 7*1+5*2+4*1 & 5*1+4*2+0*1 \\ 0*1+8*2+6*1 & 8*1+6*2+9*1 & 6*1+9*2+0*1 \\ 0*1+3*2+5*1 & 3*1+5*2+10*1 & 5*1+10*2+0*1 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 21 & 13 \\ 22 & 29 & 24 \\ 11 & 23 & 25 \end{bmatrix}$$

Similarly, for other sub-kernel 27 multiplications and additions are performed.

$$f(x,y) \star G(x,y) = MNmn = 81$$

$$[f(x,y) \star G_1(x)] \star G_2(x) = MNm + MNn = (27) + 27 = 54$$

$$MNm + MNn = MN(m+n) \ll MNmn$$

And Computational gain is $mn/(m+n) = 81/54$=1.5.

3.4.5 Convolution vs Correlation

Strength of convolution and correlation lies in two basic properties. These are $shift - invariant$ and $linear$. Shift invariant means that each operation is applied on every pixel and linearity means that value of each pixel is computed using linear combinations of its eight neighbors including itself.

$\boldsymbol{Correlation}$ is used to find the similarity between two images. $\boldsymbol{Auto\ correlation}$ is the process in which one image is correlated with itself but if two different images are correlated, it is known as $\boldsymbol{cross\ correlation}$. To quantify the similarity, sum of squares of difference (Euclidean distance) between filter w and part of image (f_{part}) mapped by filter is calculated. The difference is given by:

$$d(w, f_{part}) = \sum_{-N}^{N} [w(i) - f(x+i)]^2 \quad (\mathbf{3} - 55)$$

If we expand the equation (3-55), we get the following equation.

$$d(w, f_{part}) = \sum_{-N}^{N} w^2(i) + \sum_{-N}^{N} f^2(x+i) - 2\sum_{-N}^{N} w(i)f(x+i) \quad (\mathbf{3} - 56)$$

Equation 3-56 can be broken down into three parts:

- First part is sum of square of filter values
- Second part is sum of square of values of image part overlapped by filter
- Third part is twice of correlation of filter and image part.

By looking at part three of this equation (3-56), we can observe that by keeping all other things equal, as the similarity between two matrices increases, the third part of equation increases and resultantly the distance decreases. This observation gives us intuition to use correlation to match template with an image i.e. the places where the correlation (third part) is high, similarity between the two is high. There is one drawback of this approach that the image part where the image intensity is high, correlation will also be high (due to high intensity values).

In order to remove this drawback, normalized correlation is computed which is given by:

$$\boldsymbol{w} \star \boldsymbol{f}_{part} = \frac{\sum_{-N}^{N}[w(i)f(x+i)]}{\sqrt{\sum_{-N}^{N}[w(i)]^2 \sum_{-N}^{N}[f(x+i)]^2}} \quad (\mathbf{3} - 57)$$

This measure is invariant to scale i.e. if we scale the numerator by a factor of N, the denominator is also scaled by same factor. Normalized correlation is particularly useful when effects such as lightening, increases the brightness of an image by an unknown factor. It means that we can measure the similarity between two images that have different lightening effect by eliminating the lightening factor.

Example: Correlation and Normalized Correlation

Let we have filter $w = [3 \quad 7 \quad 5]$ and image $f = [3 \quad 2 \quad 4 \quad 1 \quad 3 \quad 8]$ and we want to find:

- Correlation

- Normalized Correlation
- Distance of kernel and image part

The correlation of w and f is computed to be

$$\boldsymbol{w} \star \boldsymbol{f} = [31\ 43\ 39\ 34\ 64\ 65]$$

To find the terms of equation (3-56), we assume that our filter maps with image at [3 2 4] then

$$\sum_{-1}^{1} w^2(i) = 9 + 25 + 49 = 83$$

$$\sum_{-1}^{1} f^2(i) = 9 + 4 + 16 = 29$$

$$2\sum_{-1}^{1} w(i)f(x+i) = 2*(3*3 + 7*2 + 5*4) = 2*(9 + 14 + 20) = 86$$

$$d\left(w, f_{part}\right) = 83 + 29 - 86 = 26$$

And normalized correlation is:

$$\sum_{-1}^{1} w(i)f(x+i) = 43$$

$$\sqrt{\sum_{-1}^{1} w^2(i) * \sum_{-1}^{1} f^2(i)} = \sqrt{83 * 29} = 49.061$$

$$\boldsymbol{w} \star \boldsymbol{f}_{part} = \frac{43}{49.061} = 0.876$$

Now we extend our example to see how areas of similarity are found.

Let $w = [3 \quad 7 \quad 5]$ and $f = [3 \quad 2 \quad 4 \quad 1 \quad 3 \quad 8 \quad 4 \quad 0 \quad 3 \quad 8 \quad 0 \quad 7 \quad 7 \quad 7 \quad 1 \quad 2]$ then

$$\begin{array}{cccccccccccccccccc} pixel\ index & \rightarrow & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ image & \rightarrow & 3 & 2 & 4 & 1 & 3 & 8 & 4 & 0 & 3 & 8 & 0 & 7 & 7 & 7 & 1 & 2 \\ correlation & \rightarrow & 31 & 43 & 39 & 34 & 64 & \mathbf{85} & 52 & 27 & 61 & 65 & 59 & \mathbf{84} & \mathbf{105} & 75 & 38 & 27 \end{array}$$

We get three high values. First, when we center filter at sixth position, high value 85 is achieved, 84 is achieved when filter is centered at 12th position and finally 105 if filter is centered at 13th position.

- First potential match:

$$\begin{array}{ccccc} filter & \rightarrow & 3 & 7 & 5 \\ image & \rightarrow & 3 & 8 & 4 \\ Correlation & \rightarrow & 64 & \mathbf{85} & 52 \end{array}$$

- Second potential match:

$$\begin{array}{ccccc} filter & \rightarrow & 3 & 7 & 5 \\ image & \rightarrow & 0 & 7 & 7 \\ Correlation & \rightarrow & 59 & \mathbf{84} & 105 \end{array}$$

- Third potential match:
$$\begin{array}{rcccc} filter & \rightarrow & 3 & 7 & 5 \\ image & \rightarrow & 7 & 7 & 7 \\ Correlation & \rightarrow & 84 & \mathbf{105} & 75 \end{array}$$

First match shows that filter values and corresponding image values are very close to each other hence this is correct point of similarity. Second match is not a good match but accidently its value is higher due to high intensity values in corresponding image region. Third match is not as good as the first one but its value is higher than the correlation value of better match. This is a problem with simple correlation that can be removed using normalized correlation. Now we find values of different correlation terms in order to find the normalized correlation.

- Normalized correlation for first match → $\boldsymbol{w} \star \boldsymbol{f}_{part} = \frac{\sum_{-1}^{1}[w(i)f(x+i)]}{\sqrt{\sum_{-1}^{1}[w(i)]^2 \sum_{-1}^{1}[f(x+i)]^2}} = \frac{85}{\sqrt{(83*89)}} = 0.99$
- Normalized correlation for first match → $\boldsymbol{w} \star \boldsymbol{f}_{part} = \frac{84}{\sqrt{(83*98)}} = 0.93$
- Normalized correlation for first match → $\boldsymbol{w} \star \boldsymbol{f}_{part} = \frac{105}{\sqrt{(83*147)}} = 0.95$

From these correlation values, we can see that first match is higher and at this point more similarity occurs.

An easy way is to use the Euclidian distance as given by following equation.

$$d(w, f_{part}) = \sqrt{\sum_{-1}^{1}[w(i) - f(x+i)]^2}$$

$$\begin{array}{rcccccccccccccccccc} pixel\ index & \rightarrow & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ image & \rightarrow & 3 & 2 & 4 & 1 & 3 & 8 & 4 & 0 & 3 & 8 & 0 & 7 & 7 & 7 & 1 & 2 \\ distances & \rightarrow & 6 & 5 & 5 & 6 & 8 & \mathbf{2} & 5 & 7 & 8 & 4 & 9 & 6 & 4 & 3 & 6 & 7 \end{array}$$

This method clearly shows that pixel at location 6 has minimum distance, hence this image part has maximum similarity with filter. If instead of filter, we use another image, the operation will give the point of similarity between two images.

Reference Programs

***Program P3-24**: 1D and 2D convolution in python*

***Program P3-25**: Convolution using OpenCV API*

***Program P3-26**: Correlation based distance calculation*

***Program P3-27**: Correlation and normalized correlation*

3.5 Spatial Domain Kernels

An image can be represented in different domains i.e. frequency domain. The filters constructed to process an image in any domain must have some specific properties. A filter designed for one domain may not be suitable to work in another domain. Therefore, their construction process varies from domain to domain.

3.5.1 Kernel Construction for Spatial Domain

Till now we have been using different kernels to operate on images without considering that how are they constructed and what features must be present in them. There are multiple ways to construct a filter:

1. Kernel construction using mathematical properties.
    - $\boldsymbol{Integration}$: A filter that works on neighborhood of a pixels and finds its value based on total effect of other pixels i.e. average filter that blurs the image.
    - $\boldsymbol{Differentiation}$: A filter that finds the local difference of a pixel from its neighborhood, can be used to sharpen the image.
2. Considering the shape of 2D function (when its distribution is plotted)
    - A Gaussian function can be used to weighted average filter
3. Using frequency response
    - We will discuss this type of filter in next chapter

3.5.2 The Smoothing (Low pass) filters

Smoothing or averaging filters are used to reduce the sharp changes in an image. The value of smoothing is determined by two things, size of kernel and value of its coefficients. For example, 5x5 kernel produces higher blur as compared to 3x3 filter. Similarly, a filter w_1 will have more smoothing effect than filter w_2

$$w_1 = \begin{bmatrix} 2 & 4 & 2 \\ 4 & 8 & 4 \\ 2 & 4 & 2 \end{bmatrix}, \qquad w_2 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \qquad (\mathbf{3}-37)$$

It can be applied in situations:

- $Noise\ reduction$: Noise can introduce sharp changes in image pixel intensities. In order to remove such sharp changes, an averaging filter can be used.
- $Reducing\ irrelevant\ details$: Smoothing can help in reducing unwanted details in given image. Hence it can be used in application where we want to prominent feature of given image. Usually details in a region that is smaller than kernel size is dumped.
- $False\ contour\ correction$: If insufficient intensity levels are used in an image, they introduce false lines, smoothing can be used to remove such false contours from image.
- $Image\ enhancement$: smoothing filter can be used in combination with other techniques to enhance the quality of an image.

Figure Error! No text of specified style in document.**-6: A) Image with salter and pepper noise B)Image after noise removal C) Good quality image D) Image with false contours**

3.5.3 Box filter kernels

It is simple, separable, low pass filter kernel whose all coefficients have same value, usually 1 as shown below:

$$box\ filter\ kernel = \frac{1}{C}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ where } 1 \leq C \qquad (\mathbf{3}-38)$$

Here C is some normalizing constant. The constant C plays two roles:

- The average value of an area of constant intensity would equal to the intensity in the filtered image
- It prevents introducing a bias during filtering. The sum of the pixels in the original and filtered images will be the same

Box filters are good for quick experimentation as they are simple and efficient to apply. There are few limitations that are associated with box filter kernels i.e. box filters are poor approximations to the blurring characteristics of camera lenses. These filters favor blurring along perpendicular directions.

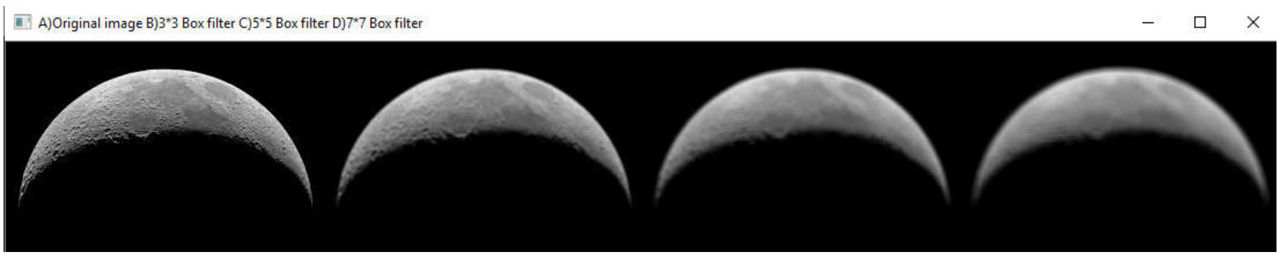


*Figure **Error! No text of specified style in document.**-7: A) Original Image and smoothing with B)3x3 box filter C)5x5 box filter D)7x7 filter*

3.5.4 Low pass Gaussian Filter Kernels

These filters are also known as $\boldsymbol{circularly\ symmetric}$ or $\boldsymbol{isotropic}$ because their response is independent of their orientation. They usually form the bell shape (Gaussian distribution) when plotted and given as:

$$w(s,t) = G(s,t) = Ke^{\frac{-(s^2+t^2)}{2\sigma^2}} = Ke^{\frac{-r}{2\sigma^2}} \qquad (\mathbf{3}-39)$$

It is observed that small Gaussian kernels have little effect on output image. Useful size of Gaussian kernel should be of size $\boldsymbol{6\sigma\ x\ 6\sigma}$. Usually Gaussian filter produces more uniform smoothing than box filter kernels.

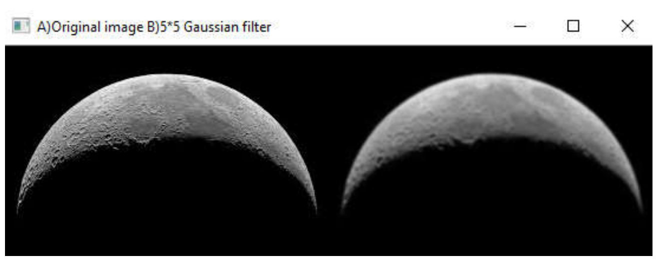


*Figure **Error! No text of specified style in document.**-8: Smoothing using 5x5 Gaussian kernel*

3.5.5 Non-Linear Filters

Non-linear spatial filters whose response is based on ordering (ranking) the pixels contained in the image region covered by the filter. Smoothing is achieved by replacing the value of the center pixel with the value determined by the ranking result.

3.5.5.1 Mean Filter

Median filters provide excellent noise reduction capabilities for certain types of random noise with less blurring as compared to box and gaussian filters. These filters are effective when salt-n-pepper noise (impulse noise) is to be reduced from image. **Median** is the center value of a sequence of values when they are arranged. If there is no single center value then two center values are summed and average to form the median. For example,

$$A = \{10, 15, 20, 20, \mathbf{20}, 20, 20, 25, 100\}$$
$$B = \{10, 15, 20, 20, \mathbf{20}, \mathbf{22}, 25, 26, 50, 100\}$$

In sequence A 20 is the center value but in case of sequence B, median value is $(20 + 22)/2 = 21$. Median filter works as follows (if we assume stride=1):

1. A window of size $\boldsymbol{mxn}$ is defined to scroll over the image
2. Image is padded with zeros by adding rows of zero values and columns of zero values in such a way that when window center element is placed on first pixel of picture, all elements of filter window match with either zero values or non-zero values.
3. Median filter is placed on first element of original image.
4. All image pixel values covered by median window are arranged in an order and center element of this arranged array is taken as output value for that pixel in output image.
5. Median window is slide to right by one column (stride 1). And perform the step 4.
6. If the window has reached to last element in image row, slide it back to first column and also slide window down by one row and continue the process.

Example: Non-Linear Filters

Let we consider the following image and 3x3 window for median filter:

$$f = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 2 & 3 & 1 & 1 & 2 \\ 1 & 4 & 2 & 2 & 0 \\ 3 & 2 & 3 & 3 & 0 \\ 1 & 0 & 0 & 2 & 1 \end{bmatrix} \qquad (\mathbf{3} - 40)$$

We list the picture elements covered by rectangle in ordered form $\{0,1,1,1,\mathbf{2},2,2,3,4\}$. Here 2 is our median. So, every pixel value in output image is replaced by median value of its corresponding window. Following figure shows few steps in this process:

$$\underline{Zero\ Padded\ Input\ Image\ } (\mathbf{3} - 41)$$

$$\begin{bmatrix} 0&0&0&0&0&0&0 \\ 0&1&2&0&1&2&0 \\ 0&2&3&1&1&2&0 \\ 0&1&4&2&2&0&0 \\ 0&3&2&3&3&0&0 \\ 0&1&0&0&2&1&0 \\ 0&0&0&0&0&0&0 \end{bmatrix} \rightarrow \begin{bmatrix} 0&0&0&0&0&0&0 \\ 0&1&2&0&1&2&0 \\ 0&2&3&1&1&2&0 \\ 0&1&4&2&2&0&0 \\ 0&3&2&3&3&0&0 \\ 0&1&0&0&2&1&0 \\ 0&0&0&0&0&0&0 \end{bmatrix} \rightarrow \begin{bmatrix} 0&0&0&0&0&0&0 \\ 0&1&2&0&1&2&0 \\ 0&2&3&1&1&2&0 \\ 0&1&4&2&2&0&0 \\ 0&3&2&3&3&0&0 \\ 0&1&0&0&2&1&0 \\ 0&0&0&0&0&0&0 \end{bmatrix} \rightarrow \begin{bmatrix} 0&0&0&0&0&0&0 \\ 0&1&2&0&1&2&0 \\ 0&2&3&1&1&2&0 \\ 0&1&4&2&2&0&0 \\ 0&3&2&3&3&0&0 \\ 0&1&0&0&2&1&0 \\ 0&0&0&0&0&0&0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 & 1 & x \\ x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \end{bmatrix}$$

$$\underline{Median\ filtered\ output\ image}$$

3.5.5.2 Min and Max Filter

Just like median filter, where we select center value in ordered list, in min-filter, we select minimum value from that ordered list as value of output pixel. And for max-filter, maximum value is selected. As an example, in right most matrix in equation $(3-40)$ value 0 will be output if min-filter is used and 2 will be out if max-filter is used.

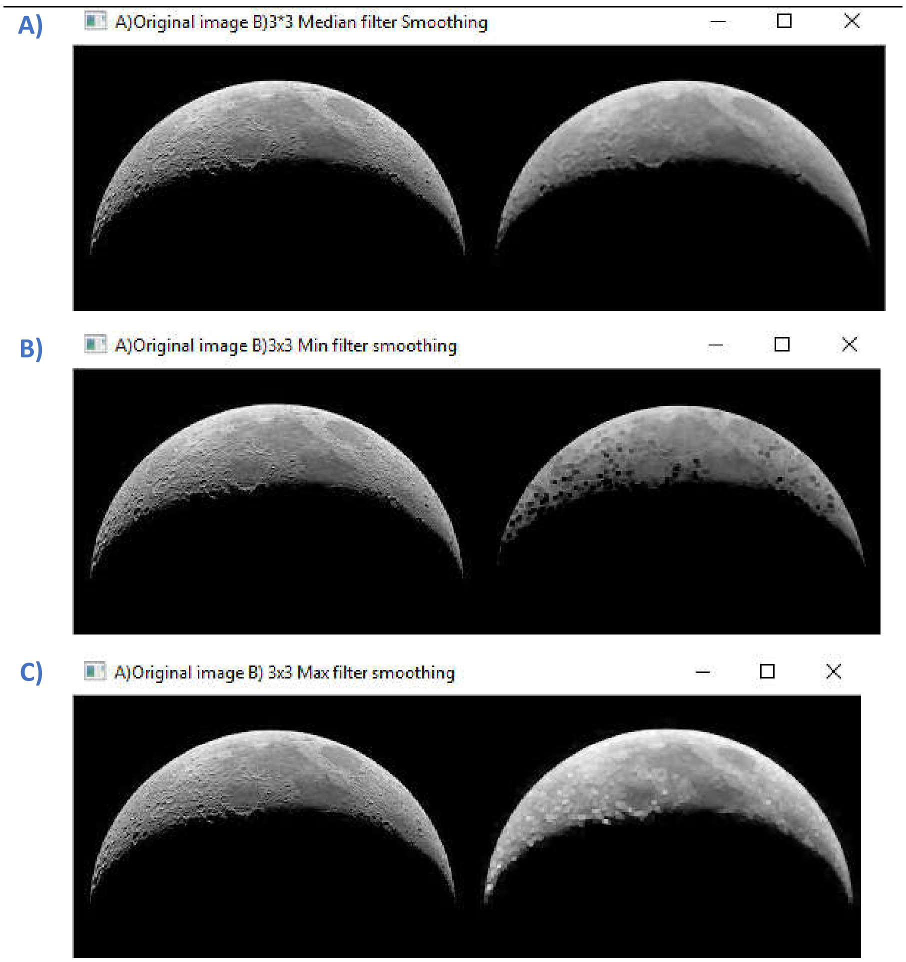


*Figure **Error! No text of specified style in document.**-9: Smoothing using Median, Min and Max filters*

You can observe that best smoothing result are obtained using median filter. Min filter introduces black dots in image whereas white dots are introduced by max filter on same positions as that of min filter black dots. The size of dot depends upon the size of smoothing filter used.

3.5.6 The Sharpening (High pass) filters

This filtering process high lights sharp changes in intensity. It is useful in applications where user need to detect such changes in an image. For example, change of tissue intensity in brain MRI. There are numerous applications of sharpening process in medical, industry, auto guided vehicles, agriculture, food quality measurement and military and the list is not exhaustive. Smoothing is an aggregate process that lowers the sharpness of an image. Sharpening is the inverse of aggregate and if smoothing is analogous to integration then sharpness is analogous to differentiation. For our understanding, we can explain the process of differentiation as:

"$\boldsymbol{A\ differentiation\ is\ the\ difference\ in\ dependent\ variable\ with\ respect\ to\ independent\ variable}$"

3.5.6.1 Differentiation and Its properties

Before discussing the sharpening techniques, let us have an overview of differentiation process and its properties. We define differentiation as:

$$if \qquad\qquad y = f(x) \qquad \text{then}$$

$$y' = \frac{dy}{dx} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad (\mathbf{3} - 41)$$

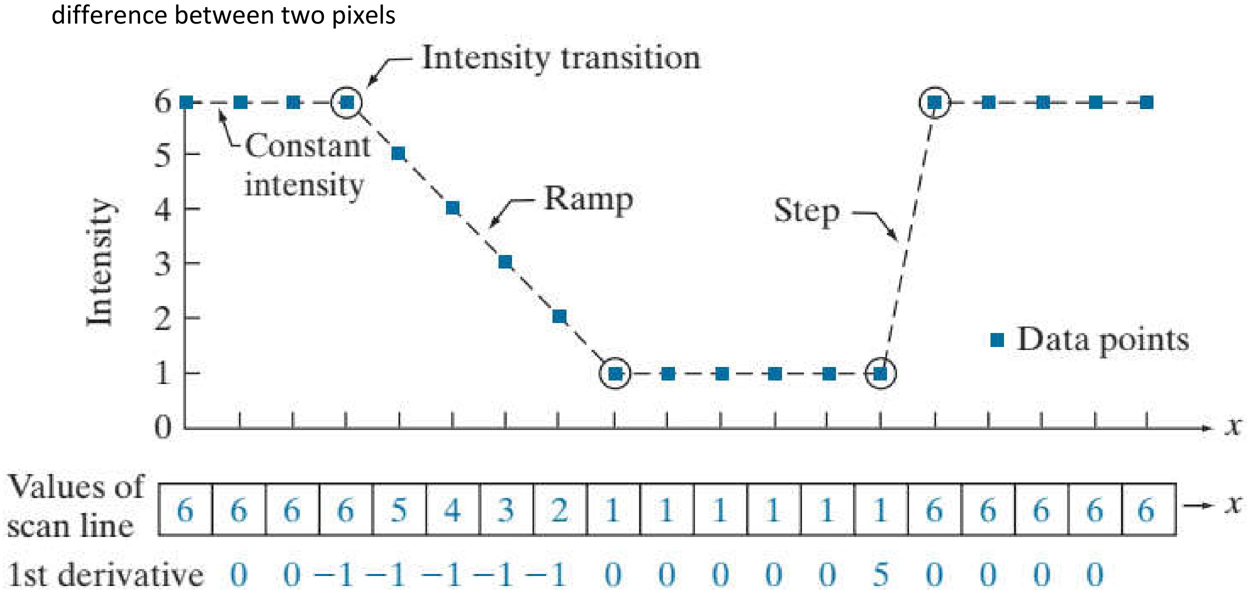
For continuous variables, we can define differentiation as "*it is the difference of dependent variable with respect to difference (h) in independent variable where h tends to be very small*". In case of digital image, the difference between two pixels is discrete and the difference between two consecutive pixels, the distance (Δx) will be 1 which simplifies the equation as follows:

$$\frac{dy}{dx} = f(x+1) - f(x) \quad (\mathbf{3} - 42)$$

This is called first order derivative as we find difference between two (consecutive) points. In sharpening process, we can use first order or higher order derivatives. Generally speaking, first order derivative finds coarse changes whereas with the increase in order of derivation, more fine details can be extracted.

$\boldsymbol{First\ \ order\ derivatives}$ have following properties:

1. Must be zero in areas of constant intensity i.e. there is zero difference between two pixels if they have same intensity value.
2. Must be non-zero on the onset of intensity step i.e. intensity value changes between two (consecutive) pixels.
3. Must be non-zero on intensity ramp i.e. if each pixel value changes along a line, there will be non-zero difference between two pixels



$\boldsymbol{Second\ order\ derivatives}$ have following properties:

1. Must be zero in areas of constant intensity
2. Must be nonzero at the onset and end of an intensity step or ramp
3. Must be zero along intensity ramps (where intensity changes at same rate)

The second order derivative (double difference) can be expressed mathematically as:

$$y'' = \frac{d^2y}{dx} = \frac{dy}{dx} - \frac{dy}{dx} = [f(x+1) - f(x)] - [f(x) - f(x-1)]$$

$$= f(x+1) - 2f(x) + f(x-1) \qquad (3-43)$$

For double difference, there must be three points i.e. $(x+1), (x), (x-1)$.

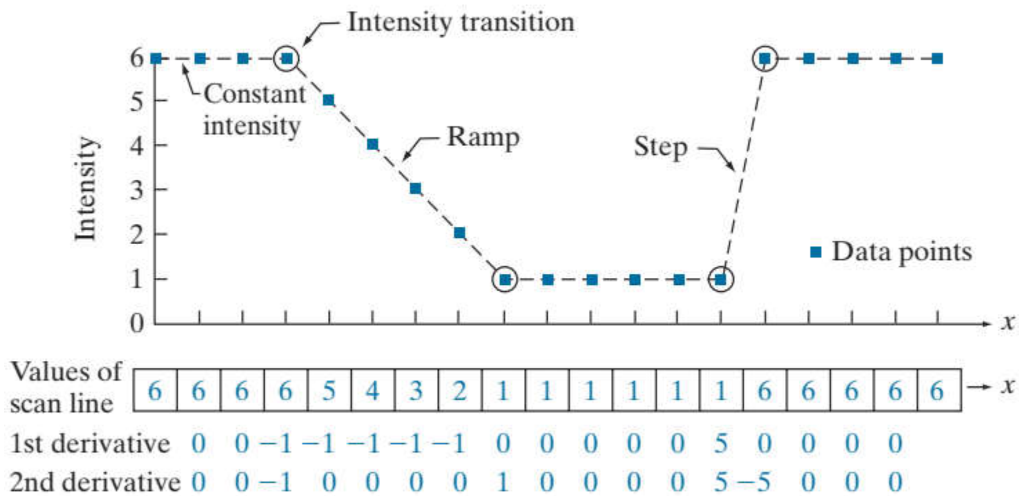
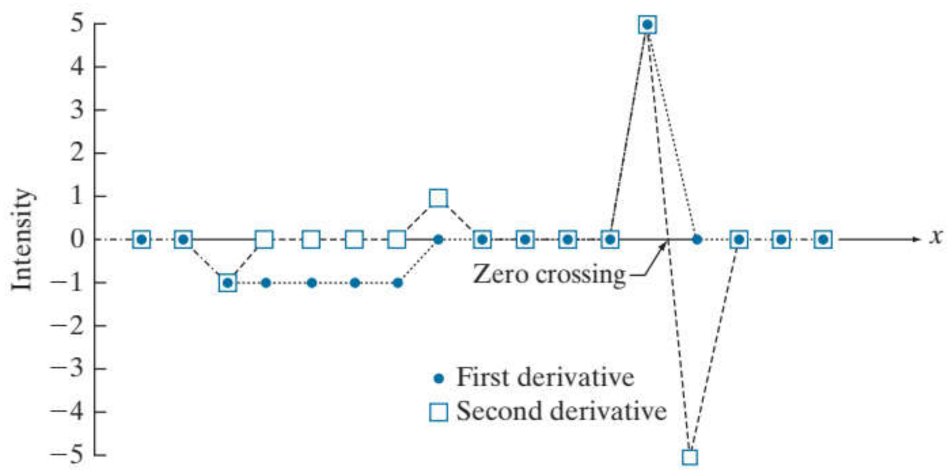


Figure Error! No text of specified style in document.**-10: Derivatives in different image regions**



This ***zero crossing*** property is quite useful for locating edges. In digital images, edges often form the ramp profile in which when an intensity change starts, it remains changing constantly for a longer duration as shown in figure 3-5. In this type of intensity profile, first order derivative exists from start of ramp to its end. All the ramp area forms the edge and a thicker edge is obtained.

On the other hand, in case of second order derivative, the derivative exists on start of ramp and at end of ramp. In between these two points, due to constant intensity change, 2^{nd} order derivative is zero and black line is formed. As a result, double edge is formed. Both of these edges will be one pixel thick (thinner than that of first order derivative edge). Hence double order difference is useful for

- Enhancing fine details in an image
- Second order derivative requires fewer operations than first order derivatives i.e. efficient computationally.

If we find first order difference or 2nd order difference in an image, the result is again a matrix of values known as **first order difference image** and **second order difference image**.

3.5.6.2 First derivative for image sharpening – The Gradient

First derivatives in image processing are implemented using the magnitude of the gradient. A gradient is a vector (collection of values). The $\boldsymbol{gradient}$ of an image f at coordinates (x, y) is defined as the two-dimensional column vector:

$$\nabla f = grad(f(x,y)) = \begin{bmatrix} g_x \\ g_y \end{bmatrix} = \begin{bmatrix} \frac{\sigma f}{\sigma x} \\ \frac{\sigma f}{\sigma y} \end{bmatrix} \qquad (\mathbf{3}-44)$$

The gradient has a very useful property which is used in optimization problems where we need to minimize some quantity of interest like loss or time. The property can be stated as:

$$\boldsymbol{The\ gradient\ points\ to\ the\ direction\ of\ greatest\ increase}$$

Keep following the change and you will find the minimum point. There exist different workarounds to avoid local minimum and search for global minimum. Equation $(\mathbf{3}-44)$ has two important properties:

- It points in the direction of the greatest rate of change of f at location (x, y). g_x points in x direction and tell that how much you have to move in x-direction. Similarly, g_y provides information that how much along y-direction we have to move to find the local minimum.
- The magnitude of vector ∇f is the value of rate of change in the direction of gradient vector. As magnitude is matrix of differences in both x and y direction then it also forms an image called $gradient\ image$.

$$M(x,y) = \|\nabla f\| = mag(\nabla f) = \sqrt{g_x^2 + g_y^2} \qquad (\mathbf{3}-45)$$

The magnitude can also be estimated using following formula:

$$M(x,y) = \|\nabla f\| = mag(\nabla f) = |g_x| + |g_y| \qquad (\mathbf{3}-46)$$

- As the components of ∇f are derivatives (differences), they are linear operations however its magnitude is not linear as it incorporates square and square root.
- Partial derivatives are not rotation invariant however $M(x,y)$ is rotation invariant.
- It is zero at local minimum and local maximum as both points have no single direction of increase however if we consider the direction with greatest increase, we can arbitrarily find non-zero derivative.

Example: Gradient Calculation

Let $f(x,y) = x^2 y$ then

a) Find $\nabla f(4,3)$
b) Find the derivation of f in the direction of $(1,2)$ at the point $(4,3)$

Solution:

a) $\nabla f(4{,}3)$

$$\frac{\partial f(x,y)}{\partial x} = 2xy = 2*4*3 = 24$$

$$\frac{\partial f}{\partial y} = x^2 = 4^2 = 16$$

Therefore, the gradient is: $$\nabla f(4{,}3) = 24i + 16j = (24{,}16)$$

a) Find the derivation of f in the direction of $(1{,}2)$ at the point $(3{,}2)$

Let $\boldsymbol{u} = u_1 i + u_2 j$ is the unit vector then the directional derivative in the direction of **u** at $(4{,}3)$ is:

$$D_u f(4{,}3) = \nabla f(4{,}3).\boldsymbol{u}$$

$$= (24i + 16j)(u_1 i + u_2 j) = (24u_1, 16u_2)$$

To find the directional derivative in the direction of the vector (1,2), we need to find a unit vector (direction) in the direction of this vector that is done by dividing it with its magnitude.

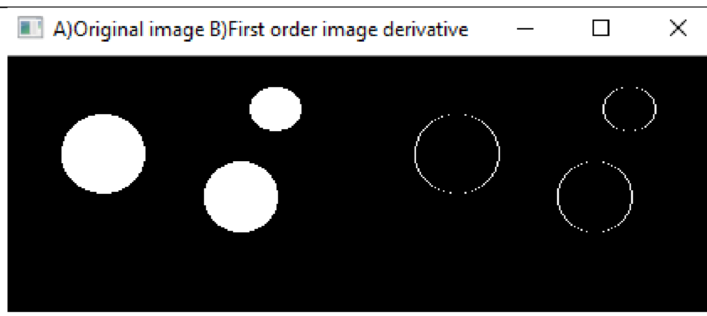
$$u = \frac{1{,}2}{\|(1{,}2)\|} = \frac{1{,}2}{\sqrt{1^2 + 2^2}} = \frac{1{,}2}{\sqrt{5}} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

Finding directional derivative of **u** in the direction of (1,2)

$$D_u f(4{,}3) = (24u_1, 16u_2) = \left(\frac{24}{\sqrt{5}}, \frac{32}{\sqrt{5}}\right)$$

$$= \frac{24}{\sqrt{5}} + \frac{32}{\sqrt{5}} = \frac{56}{\sqrt{5}}$$

A)



B)

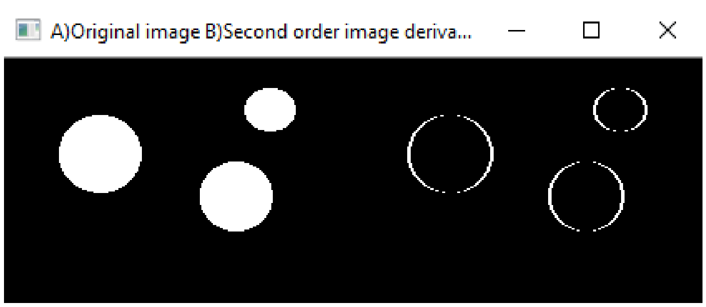


*Figure **Error! No text of specified style in document.**-11: First order and second order derivative of an image*

$\boldsymbol{Robert's\ cross\ gradient\ operators}$

Cross gradient operators can be defined g_x and g_y as follows (as defined by Roberts [1965]):

$$g_x = f(x+1, y+1) - f(x, y) \qquad (\mathbf{3}-47)$$

$$g_y = f(x, y+1) - f(x+1, y) \qquad (\mathbf{3}-48)$$

We can now easily find the magnitude by plugging the values of equations $(\mathbf{3}-47)\ and\ \ (\mathbf{3}-48)$ in equation (3-45). Equations $3-47$ and $3-48$ are also called the $\boldsymbol{Robert's\ cross\ gradient\ operators}$ and in matrix form they are listed as follows:

$$g_x = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad g_y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad (\mathbf{3}-49)$$

$\boldsymbol{Sobel's\ Gradient\ operators}$

Usually, we prefer to use kernels of odd sizes and the smallest of these is of size $3x3$. For a pixel located at some arbitrary location (x, y), the 3x3 filter can be expressed as:

$$w = \begin{bmatrix} f(x-1, y-1) & f(x, y-1) & f(x+1, y-1) \\ f(x-1, y) & f(x, y) & f(x+1, y) \\ f(x-1, y+1) & f(x, y+1) & f(x+1, y+1) \end{bmatrix} \quad (\mathbf{3}-50)$$

Approximation of g_x and g_y using w neighborhood centered at $f(x, y)$ are:

$$g_x = \frac{\partial f}{\partial x} = [f(x-1, y+1) + 2f(x, y+1) + f(x+1, y+1)] - [f(x-1, y-1) + 2f(x, y-1) + f(x+1, y-1)] \ (\mathbf{3}-51)$$

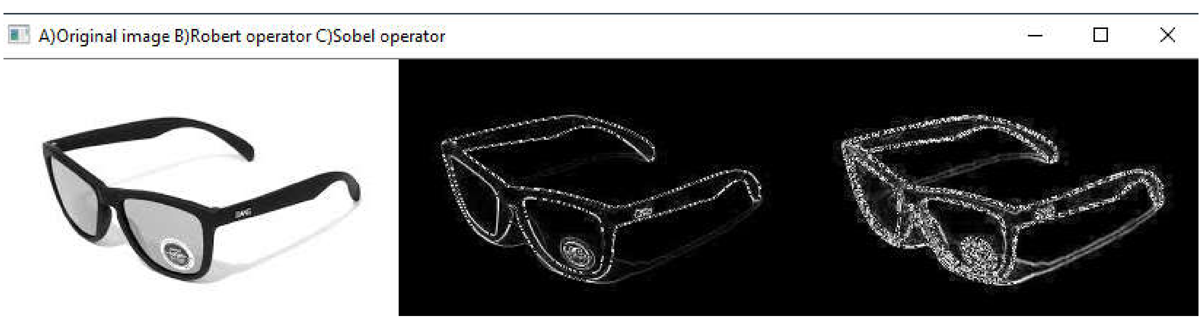
And

$$g_y = \frac{\partial f}{\partial y} = [f(x+1, y-1) + 2f(x+1, y) + f(x+1, y+1)] - [f(x-1, y-1) + 2f(x-1, y) + f(x-1, y+1)] \ (\mathbf{3}-52)$$

These equations are called the $\boldsymbol{Sobel's\ Gradient\ operators}$ and can be implemented as follows:

$$g_x = \begin{bmatrix} -1 & 2 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}, \qquad g_y = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \quad (\mathbf{3}-53)$$

It is to note that coefficients in all the kernels are equal to **zero**.



3.5.6.3 Second derivative for image sharpening – The Laplacian

An isotropic kernel is one that is independent of rotation. Simplest isotropic kernel (derivative operator) is known as Laplacian kernel. For an image $f(x,y)$, the Laplacian kernel is given as:

$$\nabla^2 f(x,y) = \frac{\partial^2 f}{\partial x} + \frac{\partial^2 f}{\partial y} \qquad (\mathbf{3}-54)$$

Equation $(\mathbf{3}-54)$ states that Laplacian could be found by taking double difference in an image, first along x-axis and then along y-axis. As defined earlier, for 2nd derivative (double difference), three points are needed. In case of 2D image, we can extend our equation $(\mathbf{3}-43)$ as:

$$\frac{\partial^2 f}{\partial x} = f(x+1,y) - 2f(x,y) + f(x-1,y) \qquad (\mathbf{3}-55)$$

And

$$\frac{\partial^2 f}{\partial y} = f(x,y+1) - 2f(x,y) + f(x,y-1) \qquad (\mathbf{3}-56)$$

And after rearranging of above two equations:

$$\nabla^2 f(x,y) = \frac{\partial^2 f}{\partial x} + \frac{\partial^2 f}{\partial y} = f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) - 4f(x,y) \qquad (\mathbf{3}-57)$$

The Laplacian could easily be converted to a kernel (matrix) as follows:

$$\frac{\partial^2 f}{\partial x} = \begin{bmatrix} 0 & 0 & 0 \\ f(x-1,y) & -2f(x,y) & f(x+1,y) \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial^2 f}{\partial y} = \begin{bmatrix} 0 & f(x,y-1) & 0 \\ 0 & -2f(x,y) & 0 \\ 0 & f(x,y+1) & 0 \end{bmatrix}$$

$$\nabla^2 f(x,y) = \frac{\partial^2 f}{\partial x} + \frac{\partial^2 f}{\partial y} = \begin{bmatrix} 0 & f(x,y-1) & 0 \\ f(x-1,y) & -4f(x,y) & f(x+1,y) \\ 0 & f(x,y+1) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad (\mathbf{3}-58)$$

The Laplacian operator can be extended to produce multiple variations as shown in $(\mathbf{3}-59)$.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -8 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix} \qquad (\mathbf{3}-59)$$

Laplacian operator emphasizes the sharp changes (high frequencies) in the image and it de-emphasizes the slowly changing (low frequencies) regions. This will tend to produce images that have grayish edge lines and other discontinuities, all superimposed on a dark, featureless background. Background features can be “recovered” while still preserving the sharpening effect of the Laplacian by adding the Laplacian image to the original. If the kernel used has a negative center coefficient, then we subtract the Laplacian image from the original to obtain a sharpened result. Thus, the basic way in which we use the Laplacian for image sharpening is:

$$g(x,y) = f(x,y) + c[\nabla^2 f(x,y)] \qquad (\mathbf{3}-60)$$

In equation $(\mathbf{3}-60)$, c will be negative (-1) if center value of kernel is negative otherwise it will be positive (1).

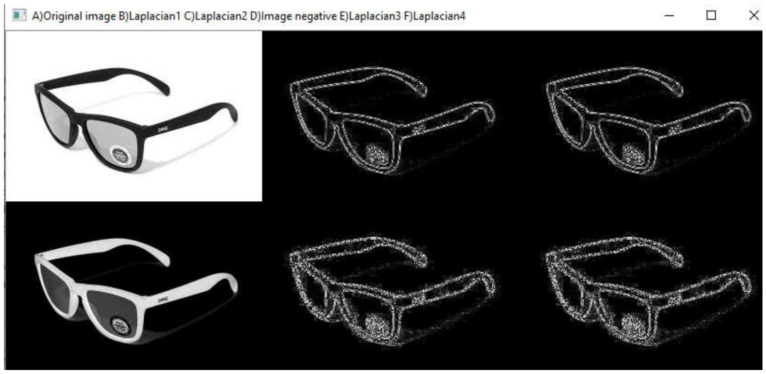


*Figure **Error! No text of specified style in document.**-12: Application of different Laplacian filters as given in equation 3-59*

3.5.6.4 Unsharp Masking and high boost filtering

Unsharp masking is a way to sharpen the image that consists of following three steps:

1. Blur the original image
2. Subtract the blurred image from the original to get the mask
3. Add the weighted mask to original image

Mathematically we can express this algorithm as follows:

$$g_{mask}(x,y) = f(x,y) - f_{blurred}(x,y) \qquad (\mathbf{3}-61)$$

$$g(x,y) = f(x,y) + k g_{mask}(x,y) \qquad (\mathbf{3}-62)$$

Different values of k affect the result differently

- $K = 1$, standard unsharp masking
- $k \geq 1$: high boost filtering
- $k \leq 1$: it reduces the effect of the sharpening

The gradient is used frequently in industrial inspection, either to aid humans in the detection of defects or, what is more common, as a preprocessing step in automated inspection.

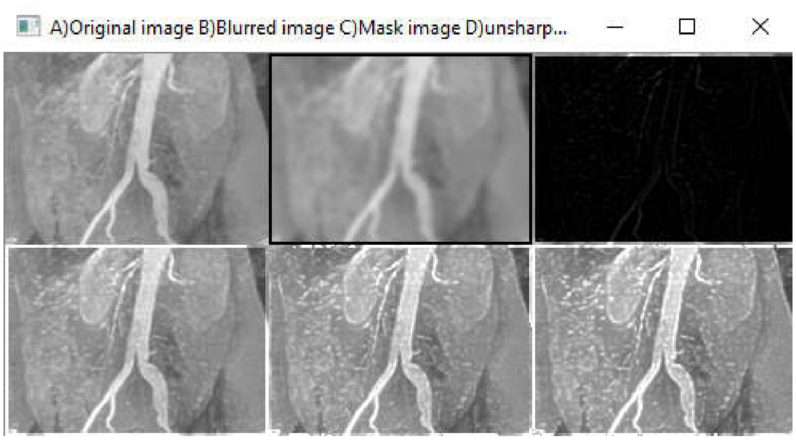


*Figure **Error! No text of specified style in document.**-13: Image enhancement using unsharp masking A) Original Image B)blurred image C)Image mask D) Unsharped using k=1 E) Unsharped using k=2 F)Unsharped using k=3*

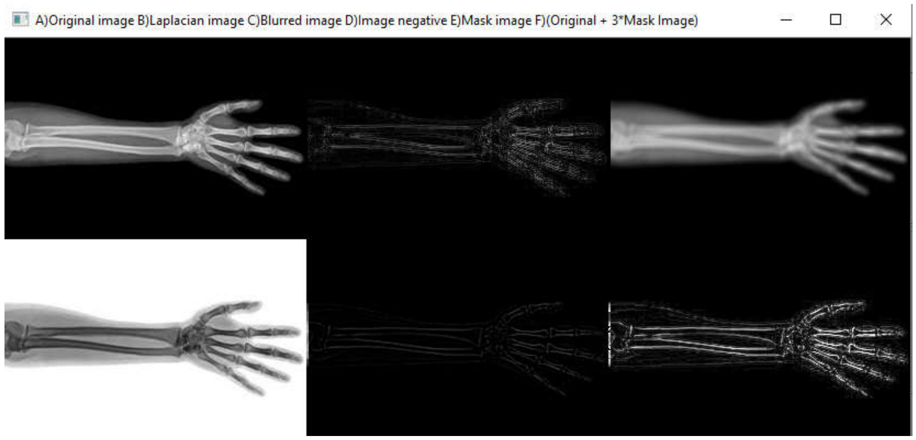


Figure ***Error! No text of specified style in document.****-14: Another example of image enhancement using unsharp masking*

We have shown image negative in order to fill the space of figure. It has no role in unsharp masking

3.5.6.5 Other Filters (High pass, Band pass and Band Reject filters)

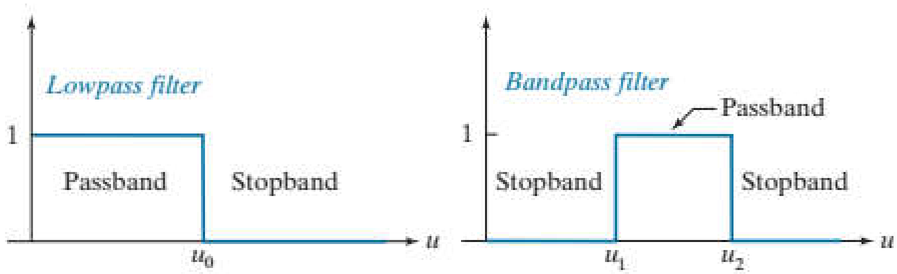
Filters are normally designed by considering signals and their frequencies. Every quantity that change by changing time or some other quantity can be considered as a signal. Generally speaking, any relationship of two quantities that can be written like $y = f(x)$ can be taken as signal. Every signal has a characteristic that is called the *frequency*. **Frequency** can be defined as repetition of signal behavior.

"*The term* ***filter*** *is taken from frequency of signal that is any mechanism that stops signal of some frequencies and allows others to pass*"

Frequencies are normally divided into two classes: low frequencies and high frequencies.

Low frequencies can find coarse features in an image whereas *high frequencies* can determine the fine details. There are four main categories of filters:

1. *Low pass filters*: These filters allow low frequencies to pass and stop high frequencies.
2. *Band pass filters*: These filters allow a range of frequencies to pass and stops frequencies that lie before and after the specified band.
3. *High pass filters*: These filters allow high frequencies to pass and stop all lower frequencies
4. *Band reject filters* (*notch filter*): These filters stop the specified band of frequencies and allow frequencies that lie before and after specified band to pass.



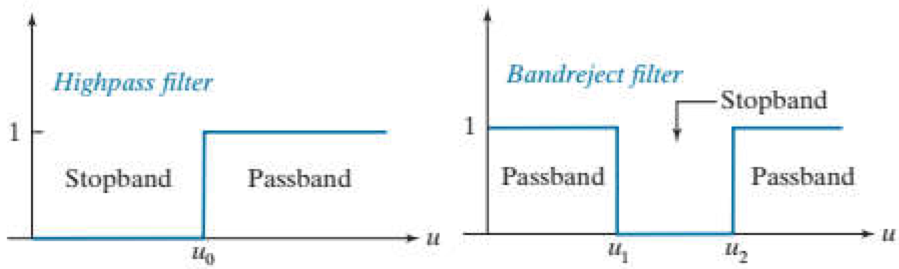


Figure Error! No text of specified style in document.**-15: Visual interpretation of filters**

Low pass filters can be used to construct rest of the three filters. In this section, we will see how can we construct other filters using low pass filters. Following table shows the construction of different filters

Filter Type	Mathematical Description
Low pass filter	$lp(x, y)$
High pass filter	$hp(x, y) = \delta(x, y) - lp(x, y)$
Band reject filter	$br(x, y) = lp_1(x, y) + hp_2(x, y)$
Band pass filter	$bp(x, y) = \delta(x, y) - br(x, y)$

3.5.6.6 Combining Spatial Enhancement Methods

In order to address a complex image enhancement task, we do combine several image processing methods. As we have studied that:

- Double derivative can be used to find (enhance) fine details in the image
- Single derivative is used to enhance edges i.e. coarser details
- Low pass filters are used to dump fine details

Reference Programs

***Program P3-28**: Box filter kernels*

***Program P3-29**: Gaussian Kernel in OpenCV*

***Program P3-30-32**: Smoothing based on median, min and max filter*

***Program P3-33**: First order image derivative*

***Program P3-34**: Second order image derivative*

***Program P3-35**: Application of Roberts and Sobel Operators*

***Program P3-36**: Use of Laplacian kernel in image sharpening*

***Program P3-37**: Unsharp Masking*

***Program P3-38**: Image edge enhancement*

1

Questions

1. write a program to apply log and power law transformation on given images. Also draw the graph between intensity levels and their log and power
2. write a program that applied multi-level thresholding on an input image
3. write a program that extracts each bit plane and displays it
4. Write a program to shift image histogram to different locations and show the shifted image
5. Change the histogram specification program to use descriptive names instead of mathematical symbols like “n_r” could be replaced with “levels”, “n_k→pixel_count”…
6. Write a program to match the histogram based on images
7. Write a program to find the distance between two histograms using correlation-based comparison.
8. There are other methods that can be used to find distance of two histograms. These include
    a. Chi-Square Distance
    $$d(H_1, H_2) = \sum_I \frac{(H_1(I) - H_2(I))^{\,2}}{H_1(I)}$$
    b. Intersection based distance
    $$d(H_1, H_2) = \sum_I \min\,(H_1(I) - H_2(I))$$
    c. Bhattacharyya distance
    $$d(H_1, H_2) = \sqrt{1 - (\frac{1}{\sqrt{\mathcal{H}_1 \mathcal{H}_2 N^2}}) \sum\nolimits_I \sqrt{H_1(I) \cdot H_2(I)}}$$

    You are required to write the programs to find these distances.
9. Write a program in python to perform the correlation and convolution of an image and filter.
10. By taking suitable filter and 1D image, show that Correlation can be high in image parts where the two (filter and image) match more or in the areas of high intensity where even both do not match well.
11. Write a program to find the similarity points between two images.
12. Consider following two matrices, using correlation, find similarity between the two matrices
    $$A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 4 & 1 \\ 0 & 5 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 2 & 1 \\ 3 & 4 & 1 \\ 5 & 3 & 6 \end{bmatrix}$$
13. Write python program to verify correlation and convolution properties visually i.e. using images.
14. Write a program that applies a separable kernel $G(x, y)$ on input image to produced its convolution image. Then apply sub-kernels ($G_1(x), G_2(y)$) on the same image and produce output image. Compare both outputs by using Euclidian distance formula to find the difference between two images.
15. Write a python program that counts the number of multiplication and addition operations when a 2D separable kernel is applied and also counts the operations when two sub-kernels ($G_1(x), G_2(y)$) are applied.
16.

References

1. Dumoulin, Vincent, and Francesco Visin. "A guide to convolution arithmetic for deep learning." *arXiv preprint arXiv:1603.07285* (2016).