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3 Chapter Three

Intensity Transformation and Spatial Filtering

This chapter will deal with different image processing techniques when an image is present in spatial domain. General representation of spatial domain image transformation is given as:

$$g(x,y) = T[f(x,y)] \quad (3-1)$$

Here are few basic concepts that are associated with spatial domain processing.

- $Intensity\ thresholding$: For the simplest case of binary thresholding, we decide that pixel values of lower than some specified threshold be changed to zero and higher values are changed to L-1 where L is the number of intensity levels.
- $Contrast\ stretching$: If k is intensity threshold, the process that darkens the intensity levels that are below k and brightens the pixels which are above threshold is called the contrast stretching
- $Image\ enhancement$: It is the process of manipulating the image pixel values such that the information in resulting image is clearer than the original.

3.1 Basic Image Transformations

There are three types of basic image transformation:

- Linear Transformation
- Non-Linear Transformation
  - Logarithmic Transformation
  - Power Law Transformation

3.1.1 Linear Transformation

In this transformation, the intensity value of a pixel is changed using some linear operation. This change may be increase or decrease in intensity value of an image. For example, to enhance the visibility of an image, intensity value of each pixel can be scaled up to some defined level. Linearly upscaled image shown in figure 3.1-1 have better visual information as compared to original image shown in figure 3.1-1 A. Similarly, linear intensity scaling can enhance the quality and color in color images.

$$s = r \pm i \quad (3-2)$$

Negative of an image is obtained by subtracting each pixel intensity value from maximum intensity value that can be in the image. In other words, each intensity value represents the intensity level. If there are L distinct intensity levels in an image and r is the input pixel intensity value then output pixel intensity value can be calculated as:

$$s = L - 1 - r \quad (3-3)$$

In normal gray level image, the intensity values range from $0-255$ $(0-L-1)$. If a pixel has intensity value 222 then its negative will be 255-222=33.

Python is very versatile and feature rich language. It supports both Single Instruction Single Data (SISD) and single instruction multiple data (SIMD) operations. SIMD operations are also called $vector\ operation$. For example, we can replace lines 8-12 in program P3-01 with single statement.

```
Img2=(L-1)-img1
```

This single subtraction operation will be applied on all pixels of `img1` and result will be assigned to `img2` on corresponding positions. There are some useful applications of image negative. In gray scale images, image negatives are used to enhance the gray level details which is present in darker areas. Specially if dark areas are dominant in image, information can't be seen directly. Taking image negative can enhance the contrast of image darker regions resulting in better visualization. For example, in image containing blood vessels, vessels can better be seen in negative image. Similarly, color image negative can provide information which is not clear in standard color space.

3.1.2 Log Transformation

In this transformation, log function is applied on intensity value of each pixel of input image. Equation 3-3 shows the process.

$$s = c * log(1 + r) \qquad (3-4)$$

Log of large values results in small values. For example, if $K = [1,2,4,8,16,32,64,128]$, taking its $\log_2 K$ produces values that lie in shorter range i.e. $[0, 1, 2, 3, 4, 5, 6, 7]$. It can be observed that log transformation has compressed the intensity values. The log function has the important characteristic that it compresses the dynamic range of pixel values. The term *dynamic range* refers to the ratio of max and min intensity values. There are the imaging modalities that can create images with pixel intensity values beyond the standard range of 0-255. In that case, log transformation can play its role to bring larger values in standard range. There are two major application areas of log transformation:

- Expanding the dark pixels and compressing the corresponding bright pixels
- Compressing the dynamic range

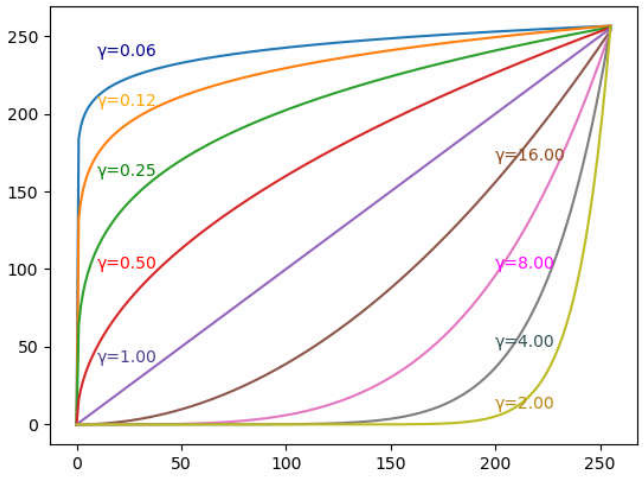
3.1.3 Power law (Gamma) Transformation

Power law transformation is given by

$$s = cr^{\gamma} \qquad where \qquad c, \gamma \geq 0 \qquad (3-5)$$

Where c and γ are some positive constants. If we consider c=1, we can easily describe the behavior of γ constant. It is a non-linear operation and like log transformation, gamma transformation also compresses and expands the intensity values. A gamma value $\gamma < 1$ is sometimes called an encoding gamma, and the process of encoding with this compressive power-law nonlinearity is called ***gamma compression***; conversely a gamma value $\gamma > 1$ is called a decoding gamma and the application of the expansive power-law nonlinearity is called ***gamma expansion***. Following graph represents the compression behavior for different values of gamma parameter.

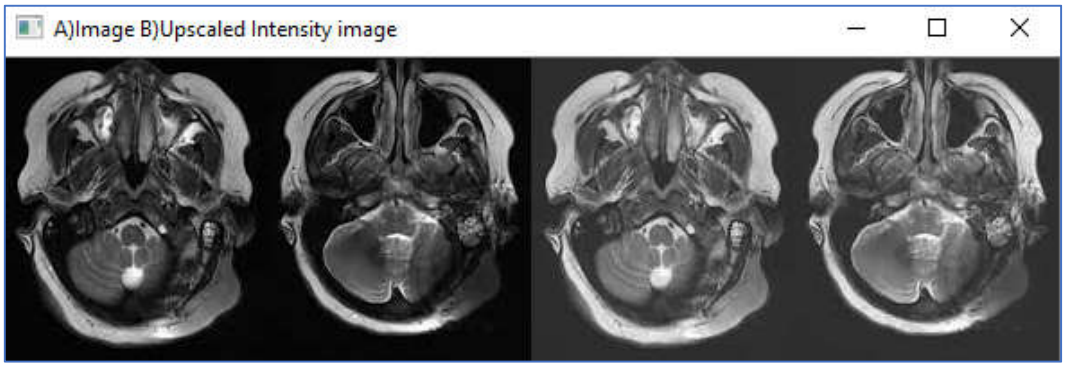
As the value of γ goes lower, low intnsity levels are compressed more and high intensity levels are stretched more. In case of γ value going higher than one, low intensity values are spread more and high intensity values are compressed more. When applying gamma transformation, one should be very careful about choosing the values of parameters. It would be good exercise, if you do some experimentation with the code used to generate this graph.



When working with transformations in OpenCV, you must be careful about image data types and their intensity ranges. Considering the intensity resolution, there are two modes to work with images 1) operate on raw intensity values and 2) work with normalized intensity values. Operations till now, we have seen, are based on raw intensity values where pixel intensity ranges from 0-255. However, some of the operations require to normalize the pixel intensity value in the range 0-1 like gamma transformation. This can be easily done by dividing image each pixel with the maximum intensity value found in that image i.e.

$$norm(img) = \frac{img(i)}{max(img)} \qquad \forall\, i \in L-1 \qquad (3-6)$$

A)



B)

C)


A)Image B)Negative

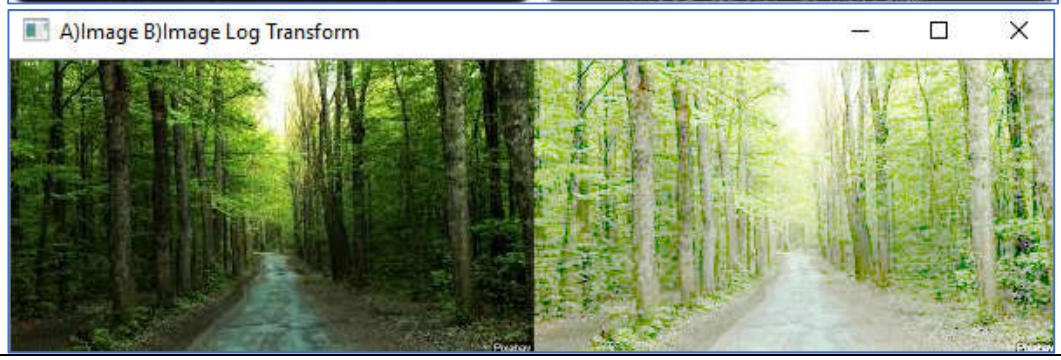
D)


A)Image B)Negative

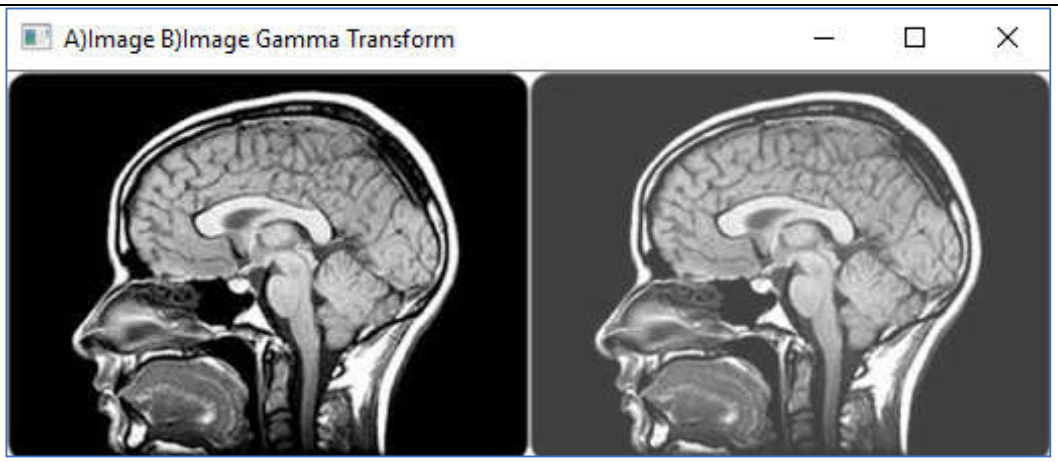
E)


A)Image B)Image Log Transform

F)


A)Image B)Image Log Transform

G)



H)

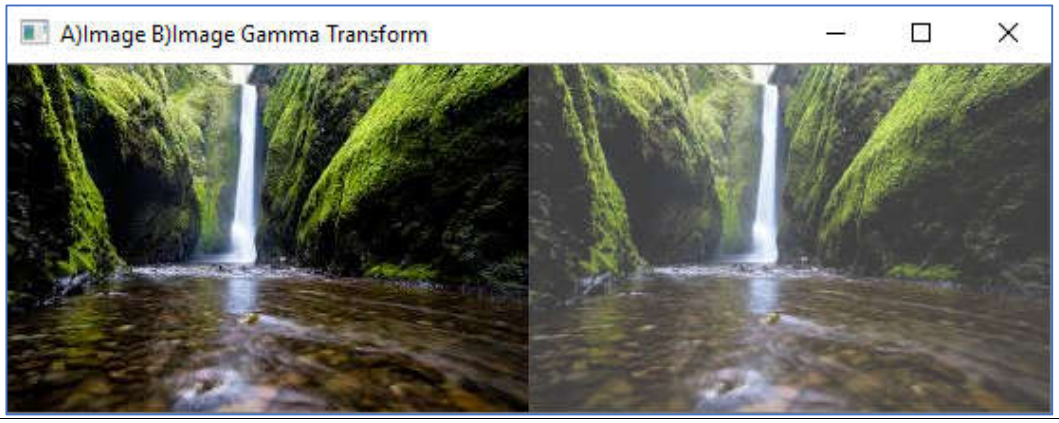


Figure 3-1: A-B) Linear Image Transformation in Grayscale and Color Images. C-D) Grayscale and Color Image Negatives E-F) Log Transformation on Grayscale and Color Images G-H) Gamma Transformation on Grayscale and color images.

Reference Programs
***Program P3-01**: Grayscale linear intensity transformation*
***Program P3-02**: Color linear intensity transformation*
***Program P3-03**: Grayscale image negative*
***Program P3-04**: Color image negative*
***Program P3-05**: Grayscale image log transformation*
***Program P3-06**: Color image log transformation*
***Program P3-07**: Drawing graph of gamma transformation*
***Program P3-08**: Grayscale image gamma transformation*
***Program P3-09**: Color image gamma transformation*

3.2 Piecewise Linear Transformation

Transformations discussed in last section operate on whole image. There may be the situations where we need to process different parts of an image differently by applying different type of transformations on difference pieces of image.

3.2.1 Piecewise Linear Thresholding (PLT)

An image contains pixel that have intensity range $0 \rightarrow L-1$. Simplest of such transform is to define a threshold or boundary T that divide all intensity levels to two parts i.e. $0 \leq p_1 \leq T \;\; and \; T+1 \leq p_2 \leq L-1$. For example, we may decide T=127 and change all intensity levels below this threshold to zero and all intensity levels above this threshold to 255.

$$s = T(r) = \begin{cases} r = 0 & if\ r \leq T \\ r = 255 & if\ r \geq T+1 \end{cases} \qquad (3-7)$$

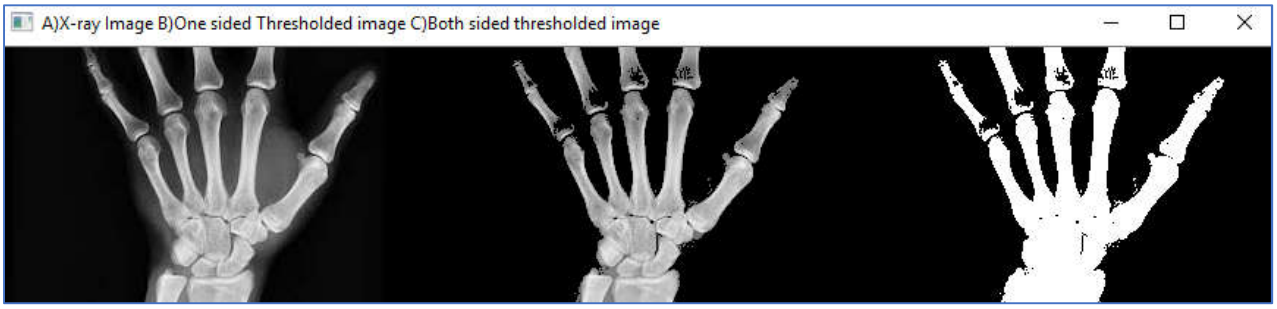


Figure Error! Use the Home tab to apply 0 to the text that you want to appear here.**-1: Application of multi-level intensity thresholding – T=105**

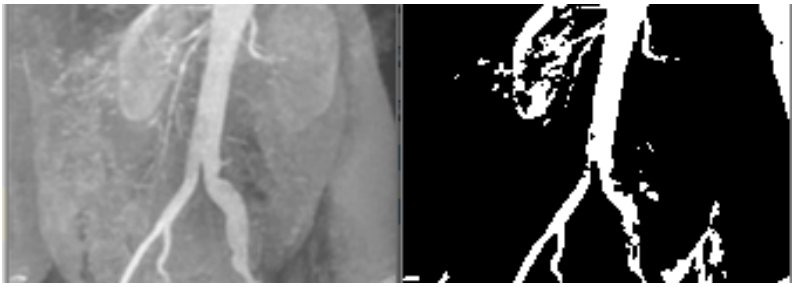
3.2.2 Intensity Level Slicing

There are applications in which it is of interest to highlight a specific range of intensities in an image. Some of these applications include enhancing features in satellite imagery, such as masses of water, and enhancing flaws in X-ray images. If we decide two thresholds T_1 and T_2 where

- Intensity values below T_1 are transformed using some function $T_1(r)$,

- Intensity values from $T_1 + 1$ to T_2 are transformed using function $T_2(r)$, and
- Intensity values above T_2, are transformed using function $T_3(r)$.

The values of T_1 and T_2 can be either found manually or using some sophisticated algorithm. This process is known as multi-thresholding.



3.2.3 Contrast Stretching

Contrast can be defined as the number of distinct colors in an image. In a grayscale image that contain intensity values from 0-255, it can be said that such image contains 256 colors ($L_{max} = 255$). Contrast can also be defined as difference between highest and lowest intensity values. If L is intensity value then

$$contrast = L_{highest} - L_{lowest} \quad (3-8)$$

A grayscale image may contain contrast value below L_{max}, this allows us to use all gray colors in image by expanding the intensity distribution known as **contrast stretching**. Here is simple formula for contrast stretching:

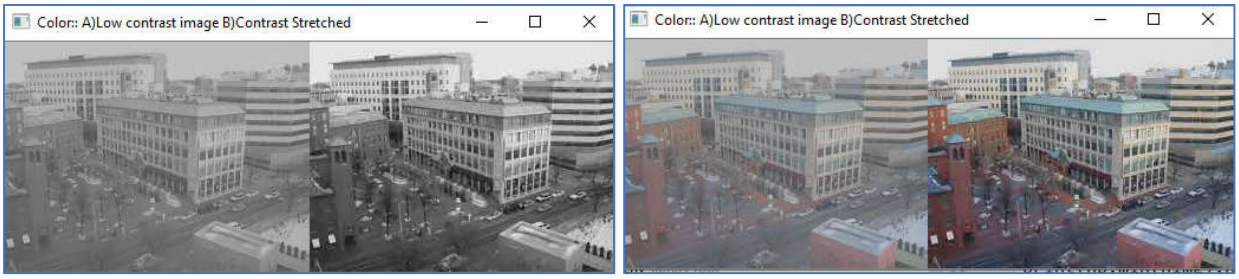
$$s_i = (r_i - r_{min}) * \frac{L_{max} - L_{min}}{r_{max} - r_{min}} + L_{min} \quad (3-9)$$

Where

$s_i, r_i = Intensity\ value\ of\ ith\ output\ and\ input\ pixel\ respectivelly$

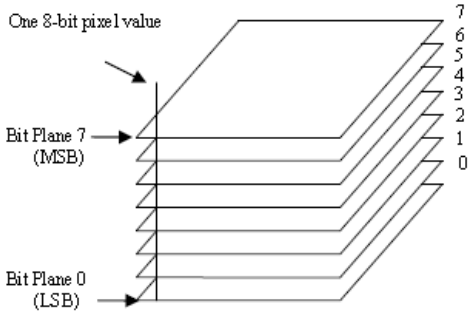
$r_{min}, r_{max} = Lowest\ and\ highest\ intensity\ value\ in\ input\ image$

$L_{min}, L_{max} = Lowest\ and\ highest\ intensity\ value\ of\ target\ contrast\ ranage$



3.2.4 Bit-Plane Slicing

Pixel values are represented by unsigned integers which are composed of bits. For example, a grayscale image is represented by 8-bit unsigned integer. Every bit in this representation of pixel intensity contains some information. Instead of slicing on intensity levels, we can slice the pixels on their bit levels. We can consider 8-bit grayscale image as a plane that consists of bits. For example, lowest plane contains 0^{th} bit of each pixel intensity, next higher plane contains 1^{st} bit and the topmost plane contains 8^{th} bit as shown in figure:



we can extract the 8^{th} bit of all pixels and form an image similarly we can extract all 7^{th} bits and form an image.

An image formed using nth bit from each pixel is called the nth bit plane. In case of images with 8-bit intensity values, there will be 8-bit planes.

Example: Bit-plane calculation

Nth bit plane is extracted by taking logical and with nth bit of each pixel. Let if binary value is 1101 1110 and we want to get 3^{rd} bit plane, we will compute as follows:

$$\begin{array}{l} 1101\ 1110 \\ \underline{0000\ 0100}\ \& \\ 0000\ 0100 \end{array}$$

Let there is an image consisting of 4 pixels and their values are as under

Decimal Intensity values	50	101	222	255
Binary values	0011 0010	0110 0101	1101 1110	1111 1111
1st Bit plane	0000 0000	0000 0001	0000 0000	0000 0001
2nd Bit plane	0000 0010	0000 0000	0000 0010	0000 0010
…	…	…	…	…
8th bit plane	0000 0000	0000 0000	1000 0000	1000 0000

We can see that most significant bit contains highest value of information and least significant bit contains least information in the image.

.

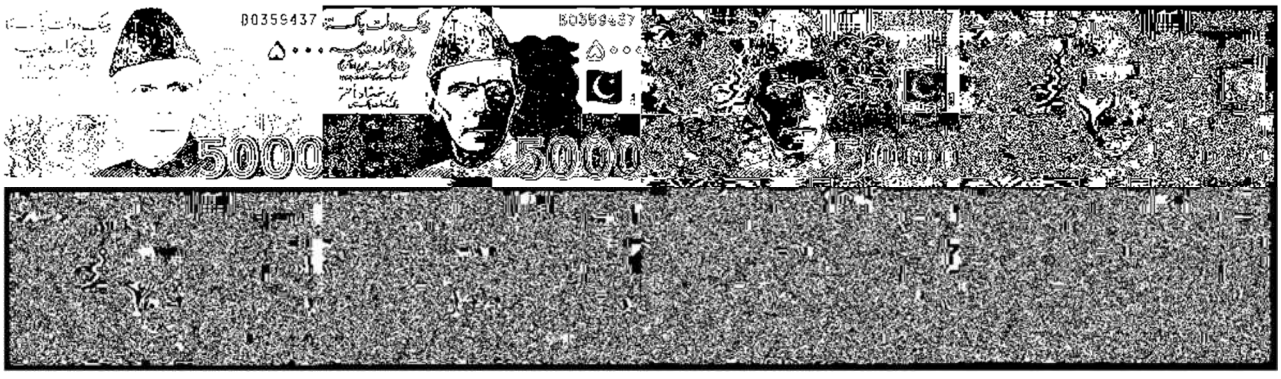


*Figure **Error! Use the Home tab to apply 0 to the text that you want to appear here.**-2: Bit-planes. Image intensities are enhanced for reader visualization. When you write your code, you may be able to visualize lower bit-planes as lower bits tends to be darker.*

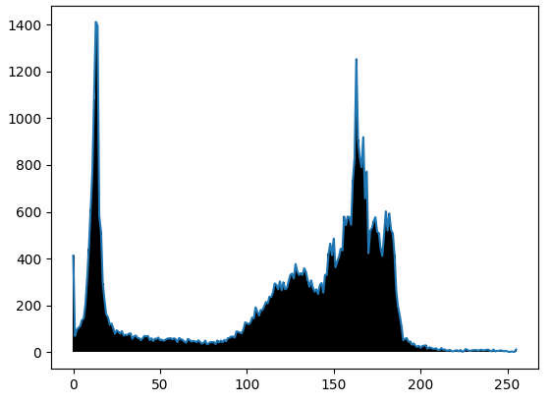
Reference Programs
***Program P3-10**: Single level image thresholding*
***Program P3-11**: Contrast stretching*
***Program P3-12**: Multi-level thresholding*
***Program P3-13**: Bit-plane slicing*

3.3 Histogram Processing

A histogram is a graphical representation that the number of pixels that lie on each intensity level in other words an image histogram shows the intensity distribution. Let for a given image

- L is the number of intensity levels ranging from $0 \rightarrow L-1$.
- M is the number of rows and N is the number of columns. Total number of pixels in the image are MN.
- There are n_k pixels for k intensity level. N_k is also known as k^{th} bin in image.
- For each intensity level there is one histogram bin and for $L-1$ levels there are $n_0, n_1, n_2, \ldots\ldots, n_{L-1}$ histogram bins.

A histogram made with such data is called the $un-normalized$ histogram. Here is program to calculate histogram of an input image.



A histogram can be normalized in order to keep its values in the range of 0-1. The normalization process produces the probability density function (pdf) of each intensity level. Histogram graph produced by un-normalized and normalized bins will be same with only difference that in un-normalized graph each vertical (black) line represents the density of corresponding intensity level whereas in normalized graph it is probability density.

$$p_r(k) = \frac{n_k}{MN} \qquad (3-10)$$

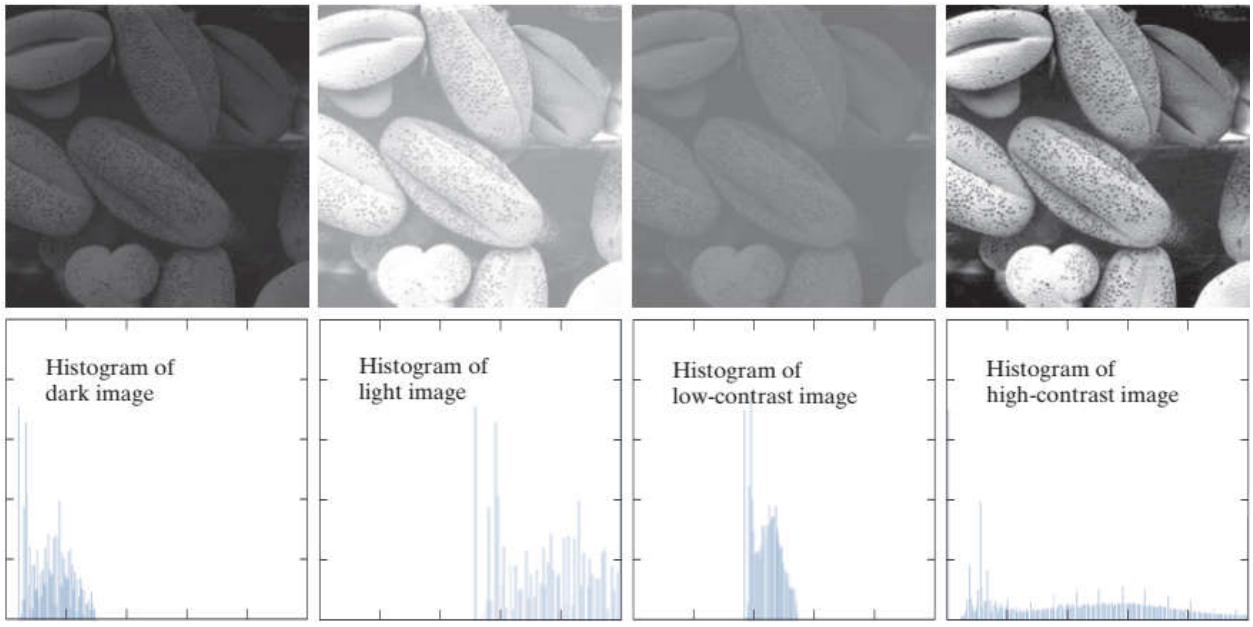
$$\sum_k p_r(k) = 1 \qquad (3-11)$$

Histogram shape is very much related to the appearance of image. Image quality and some characteristics can be described by looking at histogram. Here are some observations derived from histogram analysis:

- $Left\ shifted$: if a histogram is more on left and less on right side, such an image will be darker

- $Right\ shifted$: if a histogram is right shifted, most of the pixels will have brighter values and image will be more bright
- $Center\ located$: if intensity distribution is more located in center and less on both sides the image will be of low contrast
- $Equally\ distributed$: if histogram is equally distributed, image will have high contrast and better appearance.

Histogram spreading is a way to improve the quality of image. Following figures show different images and their corresponding histograms:



<Write a program to shift image histogram to different locations and show the shifted image>

An image histogram could be processed partially or as a whole. ***Global histogram process*** refers to spreading the histogram of complete image but histogram spreading of a part of image is known as ***local histogram processing***. In following sections, we discuss both Global and Local histogram processing techniques.

3.3.1 Global Histogram equalization (Contrast Stretching)

A special way of contrast stretching is called the histogram equalization where compressed histogram is spread over all available intensity range. Let r be intensity of input image pixel where $r = 0,1,2,3, \ldots, L-1$. Intensity transformation function is given by

$$s = T(r) \text{ where } 0 \leq r \leq L-1 \quad (3-12)$$

Before explaining the histogram equalization, two concepts are required to understand.

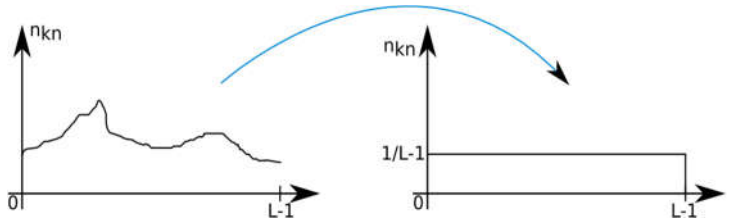
- $Probability\ density\ function\ (pdf)$: It is a function that tells the density (number of pixels) of particular intensity level i.e. probability.

$$pdf = p_k = \frac{n_k}{MN} = \frac{number\ of\ pixels\ with\ intensity\ k}{total\ number\ of\ pixels} \quad (3-13)$$

- $Cumulative\ distribution\ function\ (cdf)$: it is the function that tells total density till level k.

$$cdf = \sum_{i=0}^{k} p_i = \sum_{i=0}^{k} \frac{n_k}{MN} = \frac{number\ of\ pixels\ with\ intensity\ values\ from\ 0-k}{total\ number\ of\ pixels} \qquad (3-14)$$

If $0 \leq k \leq L-1$, pdf of all k levels is computed and summed, it is equal to 1. Idea behind histogram equalization is very simple, change the non-uniform histogram to uniform histogram where both are equal to 1 when summed.



Cdf of input and output images equal to 1 if computed all intensity levels i.e. $L-1$. This can be written mathematically as:

$$s_{L-1} = (L-1)\sum_{0}^{L-1} p_i = 1 \qquad (3-15)$$

Next, consider few cdf conditions

- The cdf of first level of output image must map to first value of pdf

$$s_0 = (L-1)\sum_{0}^{0} p_i \qquad (3-16)$$

- And the output should be uniformly distributed over all range of available intensity levels so that same formula could be used to compute cdf for all intensity levels.

$$s_k = (L-1)\sum_{0}^{k} p_i \qquad (3-17)$$

- The cdf computed for last intensity level must account for all intensities of input image as given in equation (3-15)

This is intuitive explanation of histogram equivalized distribution.

3.3.1.1 Mathematical Proof of Histogram Equalization

Now we provide mathematical basis for histogram equalization. We define two functions to understand our discussion.

- Monotonically increasing function is a function whose output is greater or equal to its previous output when input is given in ordered way i.e.

$$0 \leq T(r_k) \leq T(L-1) \qquad for\ \ 0 \leq k \leq L-1 \quad (3-18)$$

- Strictly monotonically increasing function is a function whose output is always greater than its previous output when input is given in ordered way i.e.

$$0 < T(r_k) < T(L-1) \qquad for\ \ 0 \leq k \leq L-1 \quad (3-19)$$

Probability density value for some intensity value k, in input image, is given by $p(k)$. If instead of writing pdf for particular level, we compute it for a range of intensity levels i.e. 2-4, can be written as $\Delta x = dx = k_2 - k_1$ in general form. Probability density for this range is given as:

$$p(x)d(x) \qquad (3-20)$$

Similarly, for output image we can write:

$$p(y)d(y) \qquad (3-21)$$

Equations (3-15), (3-16), (3-17), implies that

$$p(y)d(y) = p(x)d(x) \quad (3-22)$$

Equation (3-22) says that number of pixels mapped from input image to output image are same. To understand the concept, you can start by considering $d(x) = L - 1$ and then keep on decreasing $d(x)$. In other words, when level-0 pixels i.e. n_0, are transformed to output image, output image has total pixels equal to n_0. When k level pixels n_k are transformed, total of transformed pixels are $n_1 + n_2 + .. + n_k$ and output image will have same number of pixels. Hence the equation holds.

As we have assumed that output image has uniform distribution and according to equation (3-15), cdf of total image is equal to 1. If $\Delta y = L - 1$ then $p(y) = 1$ $i.e.$ it is cdf of complete output image. Equation (3-22) now can be written as:

$$p(y)d(y) = p(x)d(x) \rightarrow (1)d(y) = p(x)d(x) \rightarrow d(y) = p(x)d(x) \quad (3-23)$$

By rearranging the equation $(3-23)$, we can write

$$\frac{d(y)}{d(x)} = p(x) \qquad or \qquad p(x) = \frac{d(y)}{d(x)} \quad (3-24)$$

If we assume both x and y as continuous random variable, we can write the mapping function as:

$$y = f(x) = \int_0^x p(u)d(u) \qquad (3-25)$$

$$and$$

$$y = f(L-1) = \int_0^{L-1} p(u)d(u) = P(L-1) - P(0) = 1 \quad (3-26)$$

Equation 3-25 is cdf of input image that is used to produce output image. Equations 3-25 and 3-26 are used to show the relationship between differentiation and integration. As the gray levels in input image are discrete, we can convert the equations 3-25 and 3-26 as follows:

$$y' = f(x) = \sum_0^x p(u) \qquad (3-27)$$

And

$$y' = f(L-1) = \sum_{0}^{L-1} p(u) \qquad (3-28)$$

Now for input image, its $cdf = 1$ when intensity level ranges from $0 - L - 1$. We also know that cdf of output image is equal to 1 but we do not know the range of y. In order to keep same intensity range in input and output image, equations 3-26 and 3-28 suggest that it should be between $0 - L - 1$ instead of $0 - 1$. To fulfil this requirement, we multiply equation 3-26 and 3-28 with L-1 which scales the intensity range in output image to $0 - L - 1$ and equation 3-28 becomes as follows:

$$y' = f(L-1) = (L-1)\sum_{0}^{L-1} p(u) \qquad (3-29)$$

There is another condition which states that if equation 3-28 holds then

$$cdf(x) = T^{-1}(y)$$

It means that if there is some forward mapping from input image to output image with $L - 1$ levels, there should be a reverse mapping with same number of levels. Hence equation 3-29 is multiplied with $L - 1$.

Histogram Equalization Example

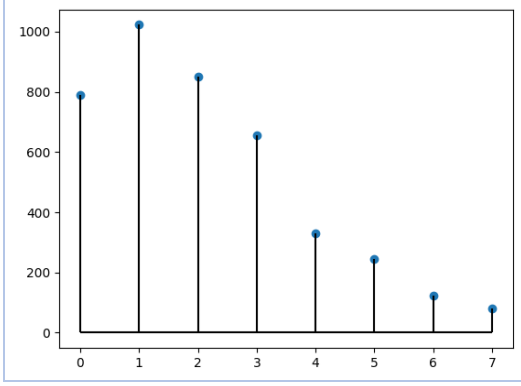
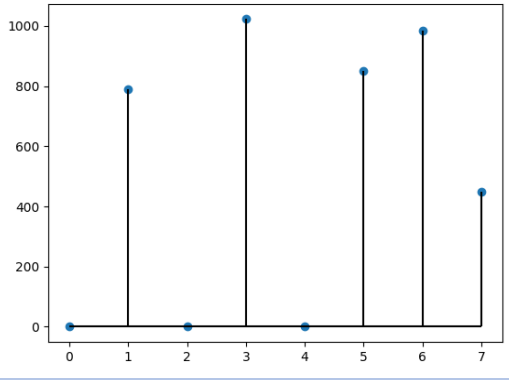
Let we have an image with following features:

- Intensity depth: $3 - bit$
- No. of intensity levels: $8\ (0 - 7)$
- Image spatial resolution: $64x64 \rightarrow 4096$
- Histogram equalization formula: $Equation\ (3-29)$

Intensity level r_k	Intensity density n_k	Probability density function $p_r(r_k) = \frac{n_k}{MN}$	$s_k = T(r_k)$ = $(L-1)\sum_0^k p_i$	$Round(s_k)$	n_{s_k}
$r_0 = 0$	790	$790/4096 \rightarrow 0.19$	$7 * (0.19) = 1.33$	1	790
$r_1 = 1$	1023	0.25	$7 * (0.19 + 0.25) = 3.08$	3	1023
$r_2 = 2$	850	0.21	**4.55**	**5**	**850**
$r_3 = 3$	656	0.16	**5.67**	**6**	**656+329**
$r_4 = 4$	329	0.08	**6.23**	**6**	**=985**
$r_5 = 5$	245	0.06	**6.65**	**7**	

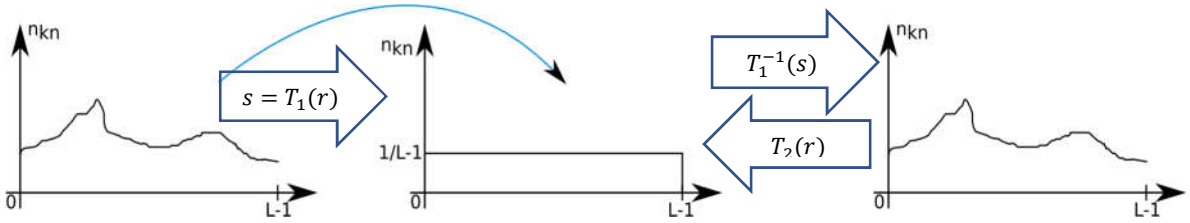
$r_6 = 6$	122	0.03	**6.86**	**7**	**245+122 +81 = 448**
$r_7 = 7$	81	0.02	**7.00**	**7**	

Now we create histograms of original and equalized images follows:

Before Equalization	After Equalization
	

3.3.1.2 Histogram Matching (Specification)

There are the situations where we need to transform the histogram of an input image that matches the histogram of some specified image. This process of matching histograms of two images is known as histogram matching or histogram specification.



Input Image Histogram **Equalized Histogram** **Specified Histogram**

Again, the idea behind histogram matching is simple. If you are given with two images A and B and you are asked to match the histogram of A with B. You will use the following algorithm:

1. Find and equalize the histogram of input image using some function T.
$$s = T(r) \qquad (3-30)$$
2. Find and equalize the histogram of specified image using some function G.
$$z = G(y) \qquad (3-31)$$
3. Now both histograms are equalized and are "equal" as both are uniform distributions. Applying G^{-1} on equalized histogram of specified image B will generate the specified image B.
$$T(r) = G(y) \qquad (3-32)$$
4. Instead of applying this inverse function on equalized histogram of B, apply it on equalized histogram of A. It will produce an image that will have histogram similar to histogram of specified image B.
$$y = G^{-1}(s) \qquad (3-33)$$

Now with this simple explanation of histogram matching, we discuss the mathematics behind this operation. Let s and r are the output and input intensities respectively then transformation is written as:

$$s_k = T(r_k) = (L-1)\sum_{i=0}^{k} p_i \qquad (3-34)$$

As before, we suppose that $p_r(r)$ is pdf of input image that we want to match with $p_z(z)$, the reference pdf. Then we can write

$$s = T(r) \qquad (3-35)$$

$$z = G^{-1}(s) = G^{-1}(T(r)) \qquad (3-36)$$

We can write these equations more specifically as (histogram equalization of specified image):

$$G(z_q) = (L-1)\sum_{i=0}^{q} p_z(z_i) \qquad (3-37)$$

For a value of q that

$$G(z_q) = s_k \qquad (3-38) \quad \text{and}$$

$$z_q = G^{-1}(s_k) \qquad (3-39)$$

Considering these two relationships, we can re-design the histogram matching algorithm as follows:

1. Compute the histogram $p_r(r)$ of the input image
2. Equalize this histogram using equation $s_k = T(r_k) = (L-1)\sum_{i=0}^{k} p_r(r_i)$ and round the output to integer values
3. Compute the values of $G(z_q)$ by using the relationship $G(z_q) = (L-1)\sum_{i=0}^{q} p_z(z_i)$ where $p_z(z_i)$ is the values of specified histogram. Round the value of G.
4. For every value of s_k, use the stored values of G from step 3 such that $G(z_q)$ is closest to s_k. When more than one values of z_q give the same match, choose the smallest value by convention. Form the output image by mapping every equalized pixel s_k to corresponding z_q using step 4
5. Compute the output image pdf

Example: Histogram Matching

Let we have an image with following features:

- Intensity depth: $3-bit$
- No. of intensity levels: $8\ (0-7)$
- Image spatial resolution: $64x64 \rightarrow 4096$

Following tables shows the details of input and reference histograms

1	2	3	4	5	6	7	8	9
r_k	n_k	$p_r k = \frac{n_k}{MN}$	$s_k = T(r_k)$	$Round(s_k)$	n_{s_k}	Specified $pdf = p_z(k)$	$G(z_q)$ $=(L-1)\sum_0^k p_i$	$G(z_q)$
$r_0 = 0$	790	0.19	1.33	1	790	0.00	0.00	0
$r_1 = 1$	1023	0.25	3.08	3	1023	0.00	0.00	0
$r_2 = 2$	850	0.21	4.55	5	850	0.00	0.00	0
$r_3 = 3$	656	0.16	5.67	6	985	0.15	7*(0.15) =1.05	1

$r_4 = 4$	329	0.08	6.23	6		0.20	2.45	2
$r_5 = 5$	245	0.06	6.65	7	448	0.30	4.55	5
$r_6 = 6$	122	0.03	6.86	7		0.20	5.95	6
$r_7 = 7$	81	0.02	7.00	7		0.15	7.00	7

1. Compute the pdf $p_r(r)$ for input image which is given in column 3 in above table
2. Equalize the histogram and find values that are given in column 5 in above table.
3. Compute the function $G(z_q)$ by considering the specified histogram as shown in column 8 and its rounded values are given in column 9
4. For every value of s_k, use the stored values of G from step 3 such that $G(z_q)$ is closest to s_k as shown in column 17
5. Output image pdf is shown in column 18.

10	11	12	13	14	15	16	17	18
r_k	n_k	s_k	n_{s_k}	Z_q	Specified pdf	$G(z_q)$	$n_z k$	$p_z(z_i)$
$r_0 = 0$	790	1	790	$Z_0 = 0$	0.00	0	0	0/4096=0
$r_1 = 1$	1024	3	1024	$Z_1 = 1$	0.00	0	0	0
$r_2 = 2$	850	5	850	$Z_2 = 2$	0.00	0	0	0
$r_3 = 3$	656	6	985	$Z_3 = 3$	0.15	1	790	0.19
$r_4 = 4$	329	6		$Z_4 = 4$	0.20	2	1024	0.25
$r_5 = 5$	245	7	448	$Z_5 = 5$	0.30	5	850	0.21
$r_6 = 6$	122	7		$Z_6 = 6$	0.20	6	985	0.24
$r_7 = 7$	81	7		$Z_7 = 7$	0.15	7	448	0.11

.

Histogram matching can only approximate the specified histogram. It is possible to use histogram matching to balance detector responses as a relative detector calibration technique. It can be used to normalize two images, when the images were acquired at the same local illumination (such as shadows) over the same location, but by different sensors, atmospheric conditions or global illumination. Exact histogram matching is the problem of finding a transformation for a discrete image so that its histogram exactly matches the specified histogram. Several techniques have been proposed for this. One simplistic approach converts the discrete-valued image into a continuous-valued image and adds small random values to each pixel so their values can be ranked without ties. However, this introduces noise to the output image. Because of this there may be holes or open spots in the output matched histogram. The concept of histogram matching can be extended to match multiple histograms.

3.3.1.3 Histogram Comparison

Histograms of two or more images can be compared in order to find the similarity between them. The process of comparing histogram finds distances between them and from these distances, a single score is obtained that is also named as distance i.e. $d(img1, img2)$. There are number of methods (algorithms) available that can be used to find the histogram-based distance of one image from another. Few of these algorithms are correlation, Chi-square and Intersection. Following figure shows different images and their similarity score. There are multiple uses of histogram comparison that may include image matching, image classification and searching similar images.

Reference Programs
***Program P3-14**: Calculating image histogram using python*
***Program P3-15**: Computing image histogram using OpenCV*
***Program P3-16**: Normalizing image histogram bins*
***Program P3-17**: Equalizing the histogram using python*
***Program P3-18**: Histogram equalization in OpenCV*
***Program P3-19**: Histogram Matching using book example*
***Program P3-20**: Histogram matching using images*
***Program P3-21**: Comparing image histograms*

3.3.1.4 Local Histogram Processing (Local Histogram Equalization-LHE)

The methods we have studied till now are based on whole image histogram processing. Complete image enhancement fails to improve the dark regions in an image where some hidden information is present. This is because the number of pixels in such small dark regions are either small in number or have little difference with surrounding pixels. This shortcoming of global histogram processing can be removed by using information present in intensity distribution of reference pixel neighborhood. This can be achieved using adaptive histogram processing.

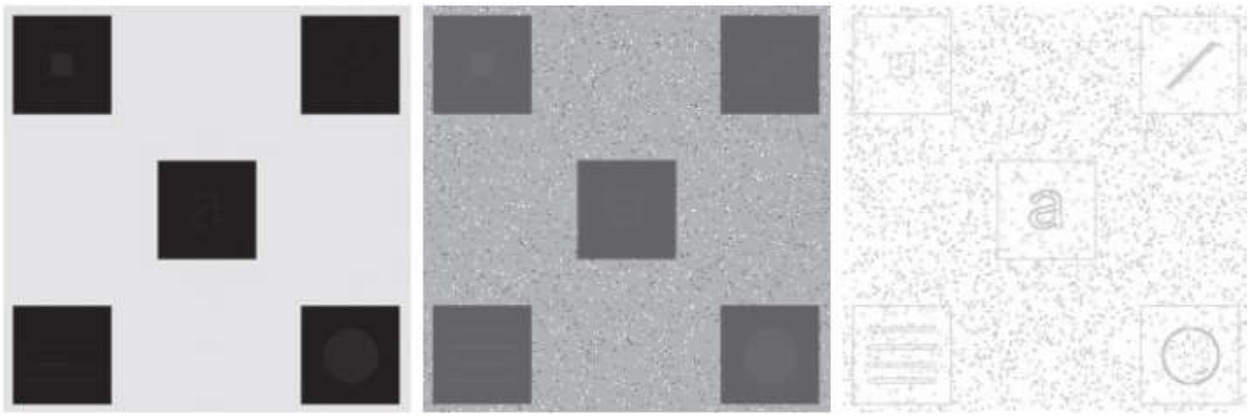


Figure Error! Use the Home tab to apply 0 to the text that you want to appear here.**-3: A) Original Image, B) Global Histogram Equalization C) Local Histogram Processing**

The local histogram processing is done by defining a window size and moving it pixel to pixel in horizontal and vertical directions. At each location, the histogram of the points in the window is computed, and either a $histogram\ equalization$ or $histogram\ specification$ transformation function is obtained. This function is used to map the intensity of the pixel centered in the neighborhood. The center of window is then moved one pixel to right and the transformation is computed again. When the column-wise moving is complete then window center is placed again on first column of next row. This process is also known as ***adaptive histogram processing (AHE)***. It differs from ordinary histogram equalization in the respect that the adaptive method computes several histograms, each corresponding to a distinct region (window) of the image, and uses them to redistribute the lightness values of the image. It is therefore suitable for improving the local contrast and enhancing the definitions of edges in each region of an image.

However, AHE has a tendency to overamplify noise in relatively homogeneous regions of an image. A variant of adaptive histogram equalization called contrast limited adaptive histogram equalization (CLAHE) which attempts to prevent this by limiting the amplification however small artifacts remains in processed image. This is due to fact that if noise is present in local window of image, that is also amplified. The method is named as contrast limited as If any histogram bin is above the specified contrast limit (by default 40 in OpenCV), those pixels are clipped and distributed uniformly to other bins before applying histogram equalization.

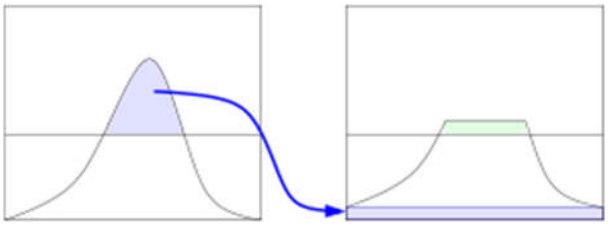


Figure Error! Use the Home tab to apply 0 to the text that you want to appear here.**-4: Contrast bin clipping**

Following figures show the application of CLAHE and enhancement of noise. As OpenCV uses 8x8 window, you can observe noise rectangles in processed image and it becomes more prominent when CLAHE is applied repeatedly.

3.3.1.5 Histogram statistics and image enhancement

Image enhancement can also be achieved by using statistics associated with an image. Most useful statistic measures are listed below:

Statistic	Formula
Mean	$m = \sum_{0}^{L-1} r_i p(r_i)$ $(3-40)$

First moment	$\mu_1 = \sum_{0}^{L-1}(r_i - m)p(r_i) \quad (3-41)$
Second Moment/ Variance σ^2	$\mu_2 = \sum_{0}^{L-1}(r_i - m)^2 p(r_i) \quad (3-42)$
Nth moment	$\mu_n = \sum_{0}^{L-1}(r_i - m)^n p(r_i) \quad (3-43)$

Among these statistics, mean and variance are most useful that can be employed for contrast enhancement either at global or local level. When used on global level, m and σ^2 are calculated for entire image and in case of local processing, these quantities are calculated for part of image (the window under consideration). If S_{xy} defines the neighborhood of window centered at pixel x, y. We can describe the m and σ^2 for this local image window as:

$$m_{s_{xy}} = \sum_{0}^{L-1} r_i p_{s_{xy}}(r_i) \quad (3-44)$$

And variance is given as:

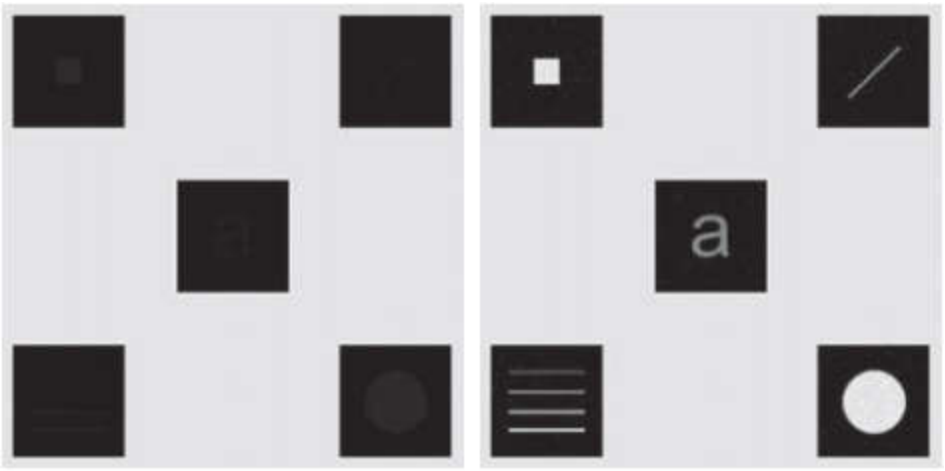
$$\mu_{s_{xy}} = \sum_{0}^{L-1} \left(r_i - m_{s_{xy}}\right)^2 p_{s_{xy}}(r_i) \quad (3-45)$$

Example: Image enhancement using image statistics

The problem at hand is to enhance the low contrast detail in the dark areas of the image, while leaving the light background unchanged. A method used to determine whether an area is relatively light or dark at a point (x, y) is to compare the average local intensity. Image enhancement algorithm using statistical measures works as follows:

- Firstly, find all four statistics $m, m_{s_{xy}}, \sigma^2, \sigma^2_{s_{xy}}$.
- A pixel $p(x, y)$ is candidate for processing if $k_0 m \leq m_{s_{xy}} \leq k_1 m$ for some non-negative value of k where $k_0 \leq k_1$. We choose k arbitrary. For example, if our focus is on areas that are darker than one-quarter of the mean intensity we would choose $k_0 = 0$ and $k_1 = 0.25$.
- A pixel is candidate for processing if $k_2\sigma \leq \sigma_{s_{xy}} \leq k_3\sigma$. For example, to enhance a dark area of low contrast, we might choose $k_2 = 0$ and $k_3 = 0.1$.
- A pixel that meets all the preceding conditions for local enhancement is processed by multiplying it by a specified constant, C, to increase (or decrease) the value of its intensity level relative to the rest of the image. Pixels that do not meet the enhancement conditions are not changed. Mathematically this procedure can be written as following formula

$$g(x,y) = f(x) = \begin{cases} cf(x,y), & if \quad k_0 m \leq m_{s_{xy}} \leq k_1 m \ \textbf{and} \ k_2\sigma \leq \sigma_{s_{xy}} \leq k_3\sigma \\ f(x,y), & otherwise \end{cases} \quad (3-46)$$



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The Spatial Filtering

An image is a matrix of values. The operations that are performed on images area also matrix operations. However new terms and names may be given to operations and operator matrices.

- $Filter$: A special type of matrix that is operated with image. A filter is also known as $kernel,\ mask,\ template\ and\ window$.
- $Linear\ Filter$: If the operation performed on image pixels is linear then operator matrix is called linear filter.
- $Non-Linear\ Filter$: If the operator filter (matrix) performs the operation on image pixels that is non-linear then the filter is called non-linear filter.

A linear filter performs the $sum\ of\ products$ operation and a filter kernel. Usually a kernel defines the neighborhood of reference pixel. Let we consider an image f of size 5x5 and kernel w of size 3x3 then general sum of products operation can be defined as:

Table 1: A) Filter Kernel w B) Image F

W(0,0)	W(0,1)	W(0,2)
W(1,0)	W(1,1)	W(1,2)
W(2,0)	W(2,1)	W(2,2)

F(0,0)	F(0,1)	F(0,2)	F(0,3)	F(0,4)
F(1,0)	F(1,1)	F(1,2)	F(1,3)	F(1,4)
F(2,0)	F(2,1)	F(2,2)	F(2,3)	F(2,4)
F(3,0)	F(3,1)	F(3,2)	F(3,3)	F(3,4)
F(4,0)	F(4,1)	F(4,2)	F(4,3)	F(4,4)

The filter maps over specific position on image which is shown by dark solid line in image. The operation is defined with reference to window center pixel, it is denoted as:

$$g(x,y) = f(0,0) * w(0,0) + f(0,1) * w(0,1) + f(0,2) * w(0,2) +$$
$$f(1,0) * w(1,0) + f(1,1) * w(1,1) + f(1,2) * w(1,2) +$$
$$f(2,0) * w(2,0) + f(2,1) * w(2,1) + f(2,2) * w(2,2)$$
$$= \sum_{x=0}^{2} \sum_{y=0}^{2} f(x,y) * w(x,y) \qquad (3-26)$$

If we consider the center pixel as reference pixel then it coordinates will be $(0,0)$. Then our matrices could be as follows:

Table 2: A) Filter Kernel w B) Image F

W(-1,-1)	W(0,-1)	W(1,-1)
W(-1,0)	W(0,0)	W(1,0)
W(-1,1)	W(0,1)	W(1,1)

F(-1,-1)	F(0,-1)	F(1,-1)	F(0,3)	F(0,4)
F(-1,0)	F(0,0)	F(1,0)	F(1,3)	F(1,4)
F(-1,1)	F(0,1)	F(1,1)	F(2,3)	F(2,4)
F(3,0)	F(3,1)	F(3,2)	F(3,3)	F(3,4)
F(4,0)	F(4,1)	F(4,2)	F(4,3)	F(4,4)

And resulting equation will be:

$$g(x,y) = f(-1,-1) * w(-1,-1) + f(-1,0) * w(-1,0) + f(-1,1) * w(-1,1) + f(-1,0) * w(-1,0) + f(0,0) * w(0,0) + f(1,0) * w(1,0) + f(-1,1) * w(-1,1) + f(0,1) * w(0,1) + f(1,1) * w(1,1)$$

$$= \sum_{x=-1}^{1} \sum_{y=-1}^{1} f(x,y) * w(x,y) \qquad (3-27)$$

As coordinates x and y are varied, the center of the kernel moves from pixel to pixel, generating the filtered image, g, in the process. As an example, for next processing the matrices will be like this:

W(-1,-1)	W(0,-1)	W(1,-1)
W(-1,0)	W(0,0)	W(1,0)
W(-1,1)	W(0,1)	W(1,1)

F(0,0)	F(-1,-1)	F(0,-1)	F(1,-1)	F(0,4)
F(1,0)	F(-1,0)	F(0,0)	F(1,0)	F(1,4)
F(2,0)	F(-1,1)	F(0,1)	F(1,1)	F(2,4)
F(3,0)	F(3,1)	F(3,2)	F(3,3)	F(3,4)
F(4,0)	F(4,1)	F(4,2)	F(4,3)	F(4,4)

Spatial Correlation and Convolution

Correlation and convolution are similar procedure except that in convolution, the filter is rotated by 180^0.

- $Correlation\ h \star f$: it is the measure of the similarity of two signals

$$g(x,y) = h \star f = \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s,t) * f(x+s, y+t) \qquad (3-28)$$

- $Convolution$: it measures the effect of one signal on another signal

$$g(x,y) = h * f = \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s,t) * f(x-s, y-t) \qquad (3-29)$$

We being by explaining 1-2 correlation. Let we have kernel of size 5 and signal (1D image) of size 8, the correlation and convolution are given as:

$$g(x) = w \star f = \sum_{-a}^{a} w(s) f(x+s) \qquad (3-30)$$

and

$$g(x) = w \star f = \sum_{-a}^{a} w(s) f(x-s) \qquad (3-31)$$

Example

Let we explain 1D correlation and convolution with the help of an example. Let $f = [2\ 1\ 3\ 1\ 0\ 1\ 2\ 1]$ and $w = [1\ 2\ 4\ 2\ 8]$

Correlation $(w \star f)$	Convolution $(w * f)$
[**2 1 3 1 0 1 2 1**] **[1 2 4 2 8]** **We position the center of filter kernel at first element of image**	[2 1 3 1 0 1 2 1] **[8** 2 4 2 1**]**
[0 0 2 1 3 1 0 1 2 1 0 0] **[1 2 4 2 8]** **Pad zeros on both sides of signal to make operation valid. If we do not pad then there will be no value to which 1,2 will be multiplied. Padded values are shown as blue zeros.**	[0 0 2 1 3 1 0 1 2 1 **0 0**] **[8** 2 4 2 1**]**
[0 0 2 1 3 1 0 1 2 1 0 0] **[1 2 4 2 8]** --- **0*1+0*2+2*4+1*2+3*8 = 0+0+8+2+24 = 34**	**[0 0** 2 1 3 1 0 1 2 1 0 0**]** **[8** 2 4 2 1**]** --- **0*8+0*2+2*4+1*2+3*1 = 0+0+8+2+3 = 13**
[0 0 2 1 3 1 0 1 2 1 0 0] **[1 2 4 2 8]** --- **0*1+2*2+1*4+3*2+1*8 = 0+4+4+6+8 = 22**	**[0 0** 2 1 3 1 0 1 2 1 0 0**]** **[8** 2 4 2 1**]** --- **0*8+2*2+1*4+3*2+1*1 = 0+4+4+6+1 = 15**
[0 0 2 1 3 1 0 1 2 1 0 0] **[1 2 4 2 8]** --- **2*1+1*2+3*4+1*2+0*8 = 2+2+12+2+0 = 18**	**[0 0** 2 1 3 1 0 1 2 1 0 0**]** **[8** 2 4 2 1**]** --- **2*8+1*2+3*4+1*2+0*1 = 16+2+12+2+0 = 32**
[0 0 2 1 3 1 0 1 2 1 0 0] **[1 2 4 2 8]** --- **1*1+3*2+1*4+0*2+1*8 = 1+6+4+0+8 = 19**	**[0 0** 2 1 3 1 0 1 2 1 0 0**]** **[8** 2 4 2 1**]** --- **1*8+3*2+1*4+0*2+1*1 = 8+6+4+0+1 = 19**
[0 0 2 1 3 1 0 1 2 1 0 0] **[1 2 4 2 8]** --- **3*1+1*2+0*4+1*2+2*8 = 3+2+0+2+16 = 23**	**[0 0** 2 1 3 1 0 1 2 1 0 0**]** **[8** 2 4 2 1**]** --- **3*8+1*2+0*4+1*2+2*1 = 24+2+0+2+2 = 30**
[0 0 2 1 3 1 0 1 2 1 0 0] **[1 2 4 2 8]** --- **1*1+0*2+1*4+2*2+1*8** **= 1+0+4+4+8 = 17**	**[0 0** 2 1 3 1 0 1 2 1 0 0**]** **[8** 2 4 2 1**]** --- **1*8+0*2+1*4+2*2+1*1 = 8+0+4+4+1 = 17**
[0 0 2 1 3 1 0 1 2 1 0 0] **[1 2 4 2 8]** ---	**[0 0** 2 1 3 1 0 1 2 1 0 0**]** **[8** 2 4 2 1**]** ---

0*1+1*2+2*4+1*2+0*8 **= 0+2+8+2+0 = 14**	**0*8+1*2+2*4+1*2+0*1 = 0+2+8+2+0 = 12**
[0 0 2 1 3 1 0 1 2 1 0 0] **[1 2 4 2 8]** **----------** **1*1+2*2+1*4+0*2+0*8** **= 1+4+4+0+0 = 9** **We end the procedure when the center of filter kernel reaches the end point in image.**	**[0 0** 2 1 3 1 0 1 2 1 **0 0]** **[8** 2 4 2 **1]** **----------** **1*8+2*2+1*4+0*2+0*1 = 8+4+4+0+0 = 16**
$w \star f$ **= [34 22 18 19 23 17 14 9]**	$w * f$ **= [13 15 32 19 30 17 12 16]**

Now we look at correlation and convolution in 2D. We write few steps of convolution only here, the reader can work out rest of the steps. Let our image f and filter w are:

$$W=\begin{bmatrix}1 & 1 & 2\\ 2 & 1 & 1\\ 1 & 1 & 1\end{bmatrix} \text{ and } f=\begin{bmatrix}1 & 2 & 0 & 1 & 2\\ 2 & 3 & 1 & 1 & 2\\ 1 & 4 & 2 & 2 & 0\\ 3 & 2 & 3 & 3 & 0\\ 1 & 0 & 0 & 2 & 1\end{bmatrix} \text{ then}$$

$$\begin{bmatrix}1 & 2 & 0 & 1 & 2\\ 2 & 3 & 1 & 1 & 2\\ 1 & 4 & 2 & 2 & 0\\ 3 & 2 & 3 & 3 & 0\\ 1 & 0 & 0 & 2 & 1\end{bmatrix} \rightarrow \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 2 & 0 & 1 & 2 & 0\\ 0 & 2 & 3 & 1 & 1 & 2 & 0\\ 0 & 1 & 4 & 2 & 2 & 0 & 0\\ 0 & 3 & 2 & 3 & 3 & 0 & 0\\ 0 & 1 & 0 & 0 & 2 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix} \rightarrow \begin{bmatrix}1 & 1 & 2\\ 2 & 1 & 1\\ 1 & 1 & 1\end{bmatrix} \star \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 2 & 0 & 1 & 2 & 0\\ 0 & 2 & 3 & 1 & 1 & 2 & 0\\ 0 & 1 & 4 & 2 & 2 & 0 & 0\\ 0 & 3 & 2 & 3 & 3 & 0 & 0\\ 0 & 1 & 0 & 0 & 2 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$= (0*1+0*1+0*2)+(0*2+1*1+2*1)+(0*1+2*1+3*1) = (0+3+5) = \mathbf{8}$$

$$\begin{bmatrix}1 & 1 & 2\\ 2 & 1 & 1\\ 1 & 1 & 1\end{bmatrix} \star \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 2 & 0 & 1 & 2 & 0\\ 0 & 2 & 3 & 1 & 1 & 2 & 0\\ 0 & 1 & 4 & 2 & 2 & 0 & 0\\ 0 & 3 & 2 & 3 & 3 & 0 & 0\\ 0 & 1 & 0 & 0 & 2 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$= (0*1+0*1+0*2)+(1*2+2*1+0*1)+(2*1+3*1+1*1) = 0+4+6$$

$$= \mathbf{10}$$

$$\begin{bmatrix}1 & 1 & 2\\ 2 & 1 & 1\\ 1 & 1 & 1\end{bmatrix} \star \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 2 & 0 & 1 & 2 & 0\\ 0 & 2 & 3 & 1 & 1 & 2 & 0\\ 0 & 1 & 4 & 2 & 2 & 0 & 0\\ 0 & 3 & 2 & 3 & 3 & 0 & 0\\ 0 & 1 & 0 & 0 & 2 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$= (0*1+0*1+0*2)+(2*2+0*1+1*1)+(3*1+1*1+1*1) = 0+5+5$$

$$= \mathbf{10}$$

$$\begin{bmatrix}1 & 1 & 2\\ 2 & 1 & 1\\ 1 & 1 & 1\end{bmatrix} \star \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 2 & 0 & 1 & 2 & 0\\ 0 & 2 & 3 & 1 & 1 & 2 & 0\\ 0 & 1 & 4 & 2 & 2 & 0 & 0\\ 0 & 3 & 2 & 3 & 3 & 0 & 0\\ 0 & 1 & 0 & 0 & 2 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$= (0*1+0*1+0*2)+(0*2+1*1+2*1)+(1*1+1*1+2*1) = 0+5+4$$

$$= \mathbf{9}$$

$$\begin{bmatrix}1 & 1 & 2\\ 2 & 1 & 1\\ 1 & 1 & 1\end{bmatrix} \star \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 2 & 0 & 1 & 2 & 0\\ 0 & 2 & 3 & 1 & 1 & 2 & 0\\ 0 & 1 & 4 & 2 & 2 & 0 & 0\\ 0 & 3 & 2 & 3 & 3 & 0 & 0\\ 0 & 1 & 0 & 0 & 2 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$= (0*1+0*1+0*2)+(1*2+2*1+0*1)+(1*1+2*1+0*1) = 0+4+3$$

$$= \mathbf{7}$$

$$\begin{bmatrix}1 & 1 & 2\\ 2 & 1 & 1\\ 1 & 1 & 1\end{bmatrix} \star \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 2 & 0 & 1 & 2 & 0\\ 0 & 2 & 3 & 1 & 1 & 2 & 0\\ 0 & 1 & 4 & 2 & 2 & 0 & 0\\ 0 & 3 & 2 & 3 & 3 & 0 & 0\\ 0 & 1 & 0 & 0 & 2 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$= (0*1+1*1+2*2)+(0*2+2*1+3*1)+(0*1+1*1+4*1) = 5+5+5$$

$$= \mathbf{15}$$

The resulting matrix becomes like

$$\begin{bmatrix}8 & 10 & 10 & 9 & 7\\ 15 & x & x & x & x\\ x & x & x & x & x\\ x & x & x & x & x\\ x & x & x & x & x\end{bmatrix}$$

<Do the rest of computations and fill the matrix>

<write a program in python to perform the correlation and convolution of an image and filter>

<you can get convolution idea from here

https://www.pyimagesearch.com/2016/07/25/convolutions-with-opencv-and-python/>

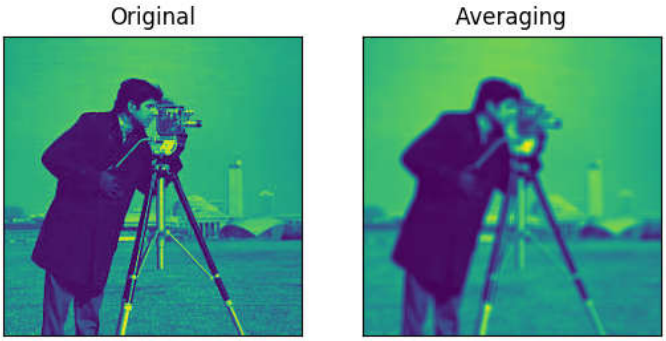
API	Description
cv.filter2D()	**`cv.filter2D(src, ddepth, kernel)`** This method convolves the kernel with input image. Its parameters are: • **`src`** – A Mat object representing the source (input image) for this operation. • **`ddepth`** – A variable of the type integer representing the depth of the output image • **`kernel`** – A Mat object representing the convolution kernel. **`Returns`**: • An image which is applied the filter

Here is an example program that uses this API.

P3-6: ConvolvingAnImage – HistogramMatching.py

```
1  import numpy as np
2  import cv2 as cv
3  import matplotlib.pyplot as plt
4
5  img=cv.imread("cameraman.jpg",0)
6  kernel = np.ones((5,5),np.float32)/25
7  dst = cv.filter2D(img,-1,kernel)
8  plt.subplot(121),plt.imshow(img),plt.title('Original')
9  plt.xticks([]), plt.yticks([])
10 plt.subplot(122),plt.imshow(dst),plt.title('Averaging')
11 plt.xticks([]), plt.yticks([])
12 plt.show()
```

Output ::



Program Description

***Line* 13: It computes the CLAHE on image**
***Line* 14: It applies the computed CLAHE on image**

To find correlation, you can easily rotate a matrix by writing few lines of python code. Finding a convolution with rotated matrix is same as finding the correlation with non-rotated matrix.

P3-6: RotatingMatrix – HistogramMatching.py

```
1  mat=np.random.randint(1,10,[5,5])
2  # reversing the matrix
3  rows,cols=mat.shape
4  rotated=np.zeros([5,5],np.uint8)
5  for i in range(rows):
6      rotated[i,:]=mat[-(i+1),:]
7  for j in range(cols):
8      rotated[:,j]=mat[:,-(j+1)]
9  print(mat)
10 print(rotated)
```

Output ::

```
Matrix:: [[7 4 2 9 3]          rotated:: [[3 9 2 4 7]
          [2 8 7 2 7]                     [7 2 7 8 2]
          [6 2 2 1 9]                     [9 1 2 2 6]
          [2 4 6 7 1]                     [1 7 6 4 2]
          [4 6 5 7 2]]                    [2 7 5 6 4]]
```

Correlation and convolution properties:

Property	Correlation	Convolution
Commutative	-	$f \star g = g \star f$
Associative	-	$f \star (g \star h) = f \star (g \star h)$
Distributive	$f \bigstar (g + h) = (f \bigstar g) + (f \bigstar h)$	$f \star (g + h) = (f \star g) + (f \star h)$

<Write python program to verify these properties visually>

Convolution filters are also known as $convolution\ masks$ or $convolution\ kernel$.

Convolving with multiple kernels

Sometime there may be the condition where we need to convolve an image with kernel-1 and then its result with 2nd kernel and so on. Such a situation for n-kernels can be written as:

$$w_n \star \dots w_3 \star w_2 \star (w_1 \star f) \quad (\mathbf{3}-31)$$

We can simplify the operation by some mathematical work on equation $(\mathbf{3}-31)$. We use the commutative property of convolution in this mathematical workaround.

$$\boldsymbol{w} = w_1 \star w_2 \star \dots \star w_n \qquad (\mathbf{3}-32).$$

and

$$convolution = \boldsymbol{w} \star f \qquad (\mathbf{3}-33).$$

Separable Kernel Filters

A 2D function is said to be separable if it can be written as product two 1D function. A kernel is also a function and if that can be written as product of two kernels, it is also called the separable kernel.

$$G(x,y) = G_1(x)G_2(y) \qquad (\mathbf{3}-34).$$

Example:

Let we have a kernel as follows:

$$w = G(x,y) = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} [1 \quad 2 \quad 1] \rightarrow G_1(x)G_2(y) \quad (3-35).$$

Separable kernels provide computational advantage in two ways:

- A product of two 1D kernels is equal to convolution of these vectors.
- Associative property of convolution. We can process the kernels and images in any order and by selecting suitable processing order, we can reduce the number of basic operations (multiplication and addition).

$$\boldsymbol{w} \star f = (w_1 \star w_2) \star f = w_1 \star (w_2 \star f) = (w_1 \star f) \star w_2 \quad (3-36).$$

Example:

If we have an image of size MN and kernel of size mn then convolving the kernel over image requires $MNmn$ multiplications and additions. For each pixel in an image, we need to compute mn multiplications and additions. If kernel has size $3x3$ then for each image pixel, we need to compute 9 multiplications and additions. If image has size MN then this count will be $MNmn$.

If kernels are separable then a 2D matrix of size mn is converted into two 1D matrices of size $1xm$ and $nx1$. Following the associative property of convolution, we can convolve the image in any order with kernel. By convolving with first kernel, we have to do MNm operations and performing convolution with second kernel requires MNn operations totaling to

$$MNm + MNn = MN(m+n) \ll MNmn \quad (3-37)$$

<Programming Exercise: Take a separable kernel and prove equation $3-36$ visually>

<Programming: Write a python program to show number of operations when convolved with combined filter ($G(x,y)$) and when it is convolved with partial filters ($G_1(x), G_2(y)$)

Process of Constructing the Spatial Domain Kernels

There are multiple ways to construct a filter

1. Kernel construction using mathematical properties.
   - $Integration$: A filter that works on neighborhood of a pixels and finds its value based on total effect of other pixels i.e. average filter, blurs the image.
   - $Differentiation$: A filter that finds the local difference of a pixel from its neighborhood, can be used to sharpen the image.
2. Considering the shape of 2D function
   - A Gaussian function can be used to weighted average filter
3. Using frequency response
   - We will discuss this type of filter in next chapter

The Smoothing (Low pass) filters

Smoothing or averaging filters are used to reduce the sharp changes in an image. The value of smoothing is determined by two things, size of kernel and value of its coefficients. For example, 5x5 kernel produces higher blur as compared to 3x3 filter. Similarly, a filter A will have more smoothing effect than filter B

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 4 & 8 & 4 \\ 2 & 4 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \qquad (3-37)$$

It can be applied in situations:

- $Noise\ reduction$: Noise can introduce sharp changes in image pixel intensities. In order to remove such sharp changes, an averaging filter can be used.
- $Reducing\ irrelevant\ details$: Smoothing can help in reducing unwanted details in given image. Hence it can be used in application where we want to prominent feature of given image. Usually details in a region that is smaller than kernel size is dumped.
- $False\ contour\ correction$: If insufficient intensity levels are used in an image, they introduce false lines, smoothing can be used to remove such false contours from image.
- $Image\ enhancement$: smoothing filter can be used in combination with other techniques to enhance the quality of an image.

Figure Error! Use the Home tab to apply 0 to the text that you want to appear here.**-5: A) Good quality image B) Image with false contours**

Box filter kernels

It is simple, separable, low pass filter kernel whose coefficients have same value, usually 1 as shown below:

$$box\ filter\ kernel = \frac{1}{C}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ where } 1 \leq C \qquad (3-38)$$

Here C is some normalizing constant. The constant C plays two roles:

- The average value of an area of constant intensity would equal to the intensity in the filtered image
- It prevents introducing a bias during filtering. The sum of the pixels in the original and filtered images will be the same

Box filters are good for quick experimentation as they are simple and efficient to apply. There are few limitations that are associated with box filter kernels i.e. box filters are poor approximations to the blurring characteristics of lenses and box filters favor blurring along perpendicular directions.

<write a program that applies box filter kernel on read image by using kernel of different sizes and using different constants C. Discuss the results and effect on output image>

Low pass Gaussian Filter Kernels

These filters are also known as $circularly\ symmetric$ or $isotropic$ because their response is independent of their orientation. These have usually shape of Gaussian distribution and given as follows:

$$w(s,t) = G(s,t) = Ke^{\frac{-(s^2+t^2)}{2\sigma^2}} = Ke^{\frac{-r}{2\sigma^2}} \qquad (3-39)$$

It is observed that small Gaussian kernels have little effect on output image. Useful size of Gaussian kernel should be of size $6\sigma \; x \; 6\sigma$.

Usually Gaussian filter produces more uniform smoothing than box filter kernels.

Non-Linear Filters

Non-linear spatial filters whose response is based on ordering (ranking) the pixels contained in the image region covered by the filter. Smoothing is achieved by replacing the value of the center pixel with the value determined by the ranking result.

Mean Filter

Median filters provide excellent noise reduction capabilities for certain types of random noise with less blurring as compared to box and gaussian filters. These filters are effective when salt-pepper noise (impulse noise) is to be reduced from image.

A median is the center value of a sequence of values when they are arranged. If there is no single center value then two center values are summed and average is used as median. For example, in following sequence A

$$A = \{10, 15, 20, 20, \mathbf{20}, 20, 20, 25, 100\}$$
$$B = \{10, 15, 20, 20, \mathbf{20}, \mathbf{22}, 25, 26, 50, 100\}$$

20 is the center value but in case of sequence B, median value is $(20 + 22)/2 = 21$. Median filter works as follows:

1. A window of size mxn is defined to scroll over the image
2. Image is padded with zeros by adding rows of zero values and columns of zero values in such a way that when window center element is placed on first pixel of picture, all elements of filter window match with either zero values or non-zero values.
3. Median filter is placed on first element of original image.
4. All image pixel values covered by median window are arranged in an order and center element of this arranged array is taken as output value for that pixel in output image.
5. Median window is slides to right by one column. And perform the step 4.
6. If the window has reached to last element in image row, slide it back to first column and also slide window down by one row and continue the process.

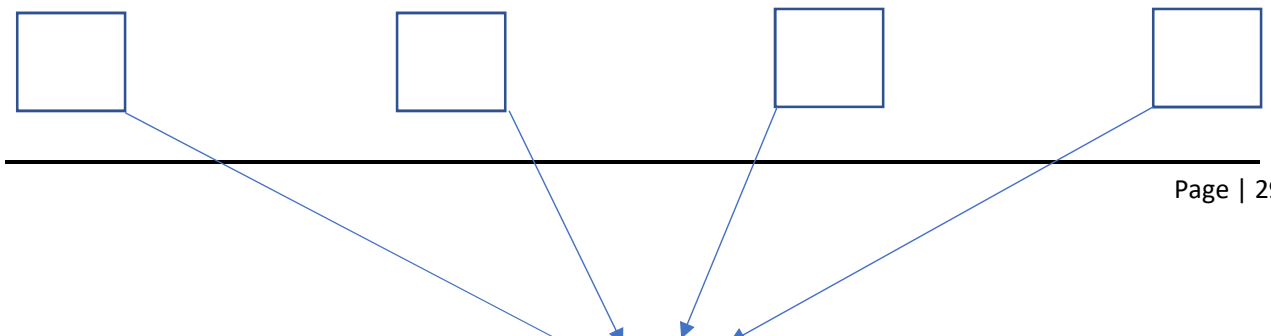
Example:

Let we consider the following image and 3x3 window for median filter:

$$f = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 2 & 3 & 1 & 1 & 2 \\ 1 & 4 & 2 & 2 & 0 \\ 3 & 2 & 3 & 3 & 0 \\ 1 & 0 & 0 & 2 & 1 \end{bmatrix} \qquad (3-40)$$

We list the picture elements covered by rectangle in ordered form $\{0,1,1,1,\mathbf{2},2,2,3,4\}$. Here 2 is our median. So, every pixel value in output image is replaced by median value of its corresponding window. Following figure shows few steps in this process:

Zero Padded Input Image $(3-41)$



$$\begin{bmatrix}0&0&0&0&0&0&0\\0&1&2&0&1&2&0\\0&2&3&1&1&2&0\\0&1&4&2&2&0&0\\0&3&2&3&3&0&0\\0&1&0&0&2&1&0\\0&0&0&0&0&0&0\end{bmatrix}\rightarrow\begin{bmatrix}0&0&0&0&0&0&0\\0&1&2&0&1&2&0\\0&2&3&1&1&2&0\\0&1&4&2&2&0&0\\0&3&2&3&3&0&0\\0&1&0&0&2&1&0\\0&0&0&0&0&0&0\end{bmatrix}\rightarrow\begin{bmatrix}0&0&0&0&0&0&0\\0&1&2&0&1&2&0\\0&2&3&1&1&2&0\\0&1&4&2&2&0&0\\0&3&2&3&3&0&0\\0&1&0&0&2&1&0\\0&0&0&0&0&0&0\end{bmatrix}\rightarrow\begin{bmatrix}0&0&0&0&0&0&0\\0&1&2&0&1&2&0\\0&2&3&1&1&2&0\\0&1&4&2&2&0&0\\0&3&2&3&3&0&0\\0&1&0&0&2&1&0\\0&0&0&0&0&0&0\end{bmatrix}$$

$$\begin{bmatrix}0&1&1&1&x\\x&x&x&x&x\\x&x&x&x&x\\x&x&x&x&x\\x&x&x&x&x\end{bmatrix}$$

$\underline{Median\ filtered\ output\ image}$

<fill the rest of the matrix>

<write a program to find the mean filter of an image>

Min and Max Filter

Just like median filter, where we select center value in ordered list, in min-filter, we select minimum value from that ordered list as value of output pixel. And for max-filter, maximum value is selected. As an example, in right most matrix in equation $(\mathbf{3}-40)$ value 0 will be output if min-filter is used and 2 will be outout if max-filter is used.

<find the min and max filtered output using input matrix of equation $(\mathbf{3}-40)$>

<write python program to apply min,max and median filter on input image and discuss your results>

The Sharpening (High pass) filters

This filtering process high lights sharp changes in intensity. It is useful in applications where user need to detect such changes in an image. For example, change of tissue intensity in brain MRI. There are numerous applications of sharpening in medical, industry, auto guided vehicles, agriculture, food quality measurement and military and this list is not exhaustive. Smoothing is an aggregate process that lowers the sharpness of an image. Sharpening is the inverse of aggregate and if smoothing is analogous to integration then differentiation is analogous to differentiation.

"$A\ differentiation\ is\ the\ difference\ in\ dependent\ variable\ with\ respect\ to\ independent\ variable$"

Differentiation and Its properties

Before discussing the sharpening techniques, let we have an overview of differentiation process and its properties. We define differentiation as:

$$if \qquad y = f(x) \qquad \text{then}$$

$$y' = \frac{dy}{dx} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad (\mathbf{3}-41)$$

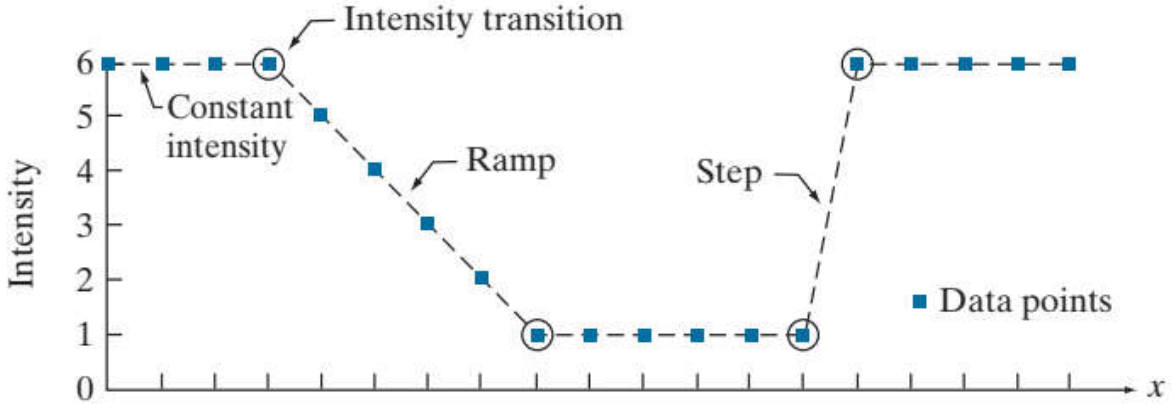
In case of digital image, the difference between two pixels is discrete. If we intend to find difference between two consecutive pixels, the distance (Δx) will be 1 and our equation can be simplified as:

$$\frac{dy}{dx} = f(x+1) - f(x) \quad (3-42)$$

This is called first order derivative as we find difference between two (consecutive) points.

First order derivatives have following properties:

1. Must be zero in areas of constant intensity i.e. there is zero difference between two pixels if they have same intensity value.
2. Must be non-zero on the onset of intensity step i.e. intensity value changes between two (consecutive) pixels.
3. Must be non-zero on intensity ramp i.e. if each pixel value changes along a line, there will be non-difference between two pixels



Second order derivatives have following properties:

1. Must be zero in areas of constant intensity
2. Must be nonzero at the onset and end of an intensity step or ramp
3. Must be zero along intensity ramps (where intensity changes at same rate)

The second order derivative (double difference) can be expressed mathematically as:

$$y'' = \frac{d^2y}{dx} = \frac{dy}{dx} - \frac{dy}{dx} = [f(x+1) - f(x)] - [f(x) - f(x-1)]$$

$$= f(x+1) - 2f(x) + f(x-1) \quad (3-43)$$

For double difference, there must be three points i.e. $(x+1), (x), (x-1)$.

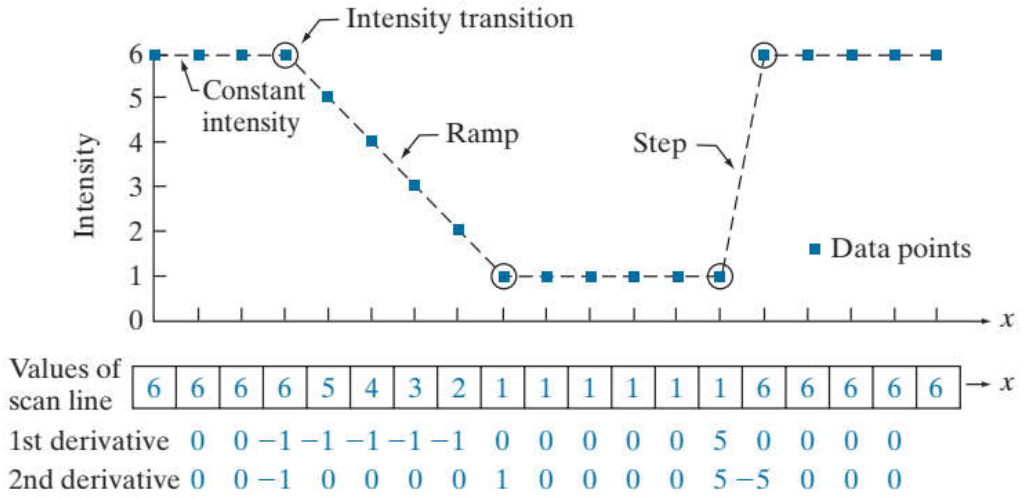
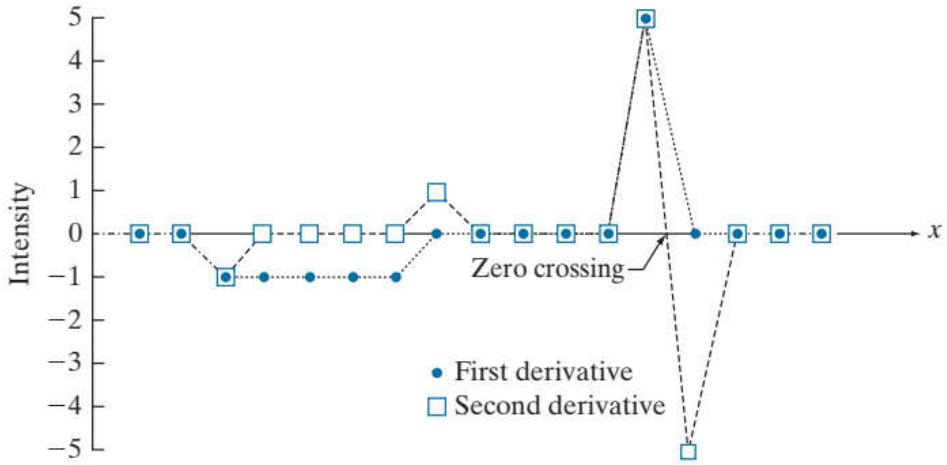


Figure Error! Use the Home tab to apply 0 to the text that you want to appear here.**-6: Derivatives in different image regions**



This *zero crossing* property is quite useful for locating edges.

In digital images, edges are often form the ramp profile in which when an intensity change starts, it remains constant change till it ends as shown in figure 3-5. In this type of intensity profile, first order derivative exists from start of ramp to its end. All the ramp area forms the image and a thicker image is obtained.

On the other hand, in case of second order derivative, 2nd order derivative exists on start of ramp and end of ramp. In between these two points, due to constant intensity change, 2nd order derivative is zero and black line is formed. As a result, double edge is formed. Both of these edges will be one pixel thick (thinner than that of first order derivative edge). Hence double order difference is useful for

- Enhancing fine details in an image
- Second order derivative requires fewer operations that first order derivatives i.e. efficient computationally.

If we find first order difference or 2nd order difference in an image, the result is again a matrix of values known as first order difference image and second order difference image.

<write a python program to compute first order difference image>

<Write a python program to compute second order difference image>

First derivative for image sharpening – The Gradient

First derivatives in image processing are implemented using the magnitude of the gradient. The ***gradient*** of an image f at coordinates (x, y) is defined as the two-dimensional column vector:

$$\nabla f = grad(f) = \begin{bmatrix} g_x \\ g_y \end{bmatrix} = \begin{bmatrix} \frac{\sigma f}{\sigma x} \\ \frac{\sigma f}{\sigma y} \end{bmatrix} \qquad (3-44)$$

Equation $(3-44)$ has two important properties:

- It points in the direction of the greatest rate of change of f at location (x, y)
- The magnitude of vector ∇f is the value of rate of change in the direction of gradient vector. As magnitude is matrix of differences in both x and y direction then it also forms an image called $gradient\ image$.

$$M(x, y) = \|\nabla f\| = mag(\nabla f) = \sqrt{g_x^2 + g_y^2} \qquad (3-45)$$

The magnitude can also be estimated using following formula:

$$M(x, y) = \|\nabla f\| = mag(\nabla f) = |g_x| + |g_y| \qquad (3-46)$$

- As the components of ∇f are derivatives (differences), they are linear operations however its magnitude is not linear as it incorporates square and square root.
- Partial derivatives are not rotation invariant however $M(x, y)$ is rotation invariant.

Using equation (3-46), we can define g_x and g_y as follows (as defined by Roberts [1965]):

$$g_x = f(x+1, y+1) - f(x, y) \qquad (3-47)$$

$$g_y = f(x, y+1) - f(x+1, y) \qquad (3-48)$$

We can now easily find the magnitude by plugging the values of equations $(3-47)\ and\ \ (3-48)$ in equation (3-45). Equations $3-47$ and $3-48$ are also called the ***Robert's cross gradient operators*** and in matrix form they are listed as follows:

$\underline{Robert's\ Operators}$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad (3-49)$$

we prefer to use kernels of odd sizes because they have a unique. Such smallest kernel is of size 3x3. For a pixel located at some arbitrary location (x, y), the 3x3 filter can be expressed as:

$$w = \begin{bmatrix} f(x-1, y-1) & f(x, y-1) & f(x+1, y-1) \\ f(x-1, y) & f(x, y) & f(x+1, y) \\ f(x-1, y+1) & f(x, y+1) & f(x+1, y+1) \end{bmatrix} \qquad (3-50)$$

Now the gradient vectors are:

$$g_x = \frac{\partial f}{\partial x} = [f(x-1, y+1) + 2f(x, y+1) + f(x+1, y+1)] - [f(x-1, y-1) + 2f(x, y-1) + f(x+1, y-1)] \quad (3-51)$$

And

$$y = \frac{\partial f}{\partial y} = [f(x+1, y-1) + 2f(x+1, y) + f(x+1, y+1)] - [f(x-1, y-1) + 2f(x-1, y) + f(x-1, y+1)] \quad (3-52)$$

These equations are called the ***Sobel's Gradient operators*** and can be implemented as follows:

Sobel's Operators

$$A = \begin{bmatrix} -1 & 2 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \quad (3-53)$$

It is to note that coefficients in all the kernels are equal to **zero**.

<write a program in python to apply Robert's operators, Sobel Filters and discuss their results in images>

Second derivative for image sharpening – The Laplacian

An isotropic kernel is one that is independent of rotation. Simplest isotropic kernel (derivative operator) is known as Laplacian kernel. For an image $f(x,y)$, the Laplacian kernel is given as:

$$\nabla^2 f(x,y) = \frac{\partial^2 f}{\partial x} + \frac{\partial^2 f}{\partial y} \qquad (3-54)$$

Equation $(3-54)$ states that Laplacian could be found by taking double difference in an image first along x-axis and the along y-axis. As defined earlier, for 2nd derivative (double difference), three points are needed. In case of 2D image, we can extend our equation $(3-43)$ as:

$$\frac{\partial^2 f}{\partial x} = f(x+1,y) - 2f(x,y) + f(x-1,y) \qquad (3-55)$$

And

$$\frac{\partial^2 f}{\partial y} = f(x,y+1) - 2f(x,y) + f(x,y-1) \qquad (3-56)$$

And finally

$$\nabla^2 f(x,y) = \frac{\partial^2 f}{\partial x} + \frac{\partial^2 f}{\partial y} = f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) - 4f(x,y) \quad (3-57)$$

The Laplacian could easily be converted to a kernel as follows

$$\frac{\partial^2 f}{\partial x} = \begin{bmatrix} 0 & 0 & 0 \\ f(x-1,y) & -2f(x,y) & f(x+1,y) \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial^2 f}{\partial y} = \begin{bmatrix} 0 & f(x,y-1) & 0 \\ 0 & -2f(x,y) & 0 \\ 0 & f(x,y+1) & 0 \end{bmatrix}$$

$$\nabla^2 f(x,y) = \frac{\partial^2 f}{\partial x} + \frac{\partial^2 f}{\partial y} = \begin{bmatrix} 0 & f(x,y-1) & 0 \\ f(x-1,y) & -4f(x,y) & f(x+1,y) \\ 0 & f(x,y+1) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad (3-58)$$

The Laplacian operator can be extended to produce multiple variations as shown in $(3-59)$.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -8 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix} \quad (3-59)$$

Laplacian operator emphasizes the sharp changes in the image and it de-emphasizes the slowly changing regions. This will tend to produce images that have grayish edge lines and other discontinuities, all

superimposed on a dark, featureless background. Background features can be "recovered" while still preserving the sharpening effect of the Laplacian by adding the Laplacian image to the original. If the definition used has a negative center coefficient, then we subtract the Laplacian image from the original to obtain a sharpened result. Thus, the basic way in which we use the Laplacian for image sharpening is:

$$g(x,y) = f(x,y) + c[\nabla^2 f(x,y)] \qquad (3-60)$$

In equation $(3-60)$, c will be negative (-1) if center value of kernel is negative otherwise it will be positive (1).

<Write a program in python to apply all four Laplacian kernels to sharpen images>

Unsharp Masking and high boost filtering

Unsharp masking operation is a way to sharpen the image that consists of following three steps:

1. Blur the original image
2. Subtract the blurred image from the original to get the mask
3. Add the weighted mask to original image

Mathematically we can express this algorithm as follows:

$$g_{mask}(x,y) = f(x,y) - f_{blurred}(x,y) \qquad (3-61)$$

$$g(x,y) = f(x,y) + kg_{mask}(x,y) \qquad (3-62)$$

Different values of k affect the result differently

- $K = 1$, standard unsharp masking
- $k \geq 1$: high boost filtering
- $k \leq 1$: it reduces the effect of the sharpening

<write a python program to see the effect of unsharp masking using different values of k>

Applications of Gradients

- The gradient is used frequently in industrial inspection, either to aid humans in the detection of defects or, what is more common, as a preprocessing step in automated inspection

<write a program to do the edge enhancement in given image>

Other Filters (High pass, Band pass and Band Reject filters)

Filters are normally designed by considering signals and their frequencies. Every quantity that change by changing time or some other quantity can be considered as a signal. Generally speaking, any relationship of two quantities that can be written like $y = f(x)$ can be taken as signal. Every signal has a characteristic that is called the $frequency$. **Frequency** can be defined as repetition of signal behavior.

"$The\ term\ \boldsymbol{filter}\ is\ taken\ from\ frequency\ of\ signal\ that\ is\ any\ mechanism\ that\ stops\ signal\ of\ some\ frequencies\ and\ allows\ others\ to\ pass$"

Frequencies are normally divided into two classes: low frequencies and high frequencies.

$Low\ frequencies$ can find coarse features in an image whereas $high\ frequencies$ can determine the fine details. There are four main categories of filters:

1. $Low\ pass\ filters$: These filters allow low frequencies to pass and stop high frequencies.
2. $Band\ pass\ filters$: These filters allow a range of frequencies to pass and stops frequencies that lie before and after the specified band.
3. $High\ pass\ filters$: These filters allow high frequencies to pass and stop all lower frequencies

4. $Band\ reject\ filters\ (notch\ filter)$: These filters stop the specified band of frequencies and allow frequencies that lie before and after specified band to pass.

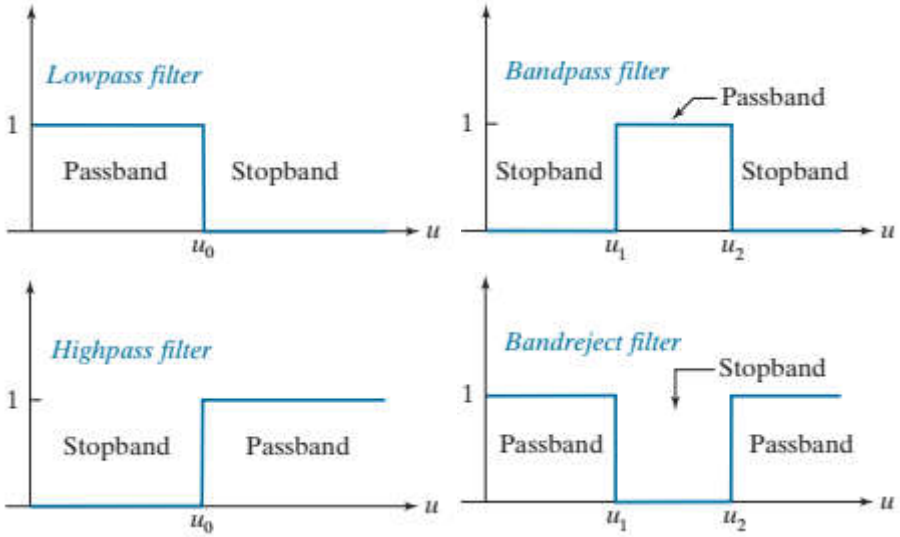


Figure Error! Use the Home tab to apply 0 to the text that you want to appear here.**-7: Visual interpretation of filters**

Low pass filters can be used to construct rest of the three filters. In this section, we will see how can we construct other filters using low pass filters. Following table shows the construction of different filters

Filter Type	Mathematical Description
Low pass filter	$lp(x,y)$
High pass filter	$hp(x,y) = \delta(x,y) - lp(x,y)$
Band reject filter	$br(x,y) = lp_1(x,y) + hp_2(x,y)$
Band pass filter	$bp(x,y) = \delta(x,y) - br(x,y)$

Combining Spatial Enhancement Methods

In order to address a complex image enhancement task, we do combine several image processing methods. As we have studied that:

- Double derivative can be used to find (enhance) fine details in the image
- Single derivative is used to enhance edges i.e. coarser details
- Low pass filters are used to dump fine details

Example

<write a program to apply different image enhancement methods to improve its quality>

Questions

1. write a program to apply log and power law transformation on given images. Also draw the graph between intensity levels and their log and power
2. write a program that applied multi-level thresholding on an input image
3. write a program that extracts each bit plane and displays it

4. Change the histogram specification program to use descriptive names instead of mathematical symbols like “n_r” could be replaced with “levels”, “n_k→pixel_count”…
5. Write a program to match the histogram based on images
6.