Answer

Step 1 > To determine

Calculate the output voltage across the capacitor v_o in the given circuit of Figure 7.112.

Step 2 > Answer

The output voltage across the capacitor v_o is $\left[20 - 15e^{-14.286t}\right]u(t)\mathrm{V}$.

Step 3 > Explanation

Given data:

Refer to Figure 7.112 in the textbook.

The value of capacitance (C) is $3\mu\mathrm{F}$.

The source voltage v_s is $30u(t)\mathrm{V}$.

The initial capacitor voltage $v_o(0)$ is 5 V.

Formula used:

Write the general expression to find the complete voltage response for an *RC* circuit.

$v_o(t) = v_o(\infty) + [v_o(0) - v_o(\infty)]e^{-\frac{t}{\tau}}$ (1)

Here,

τ is the time constant for the *RC* circuit,

$v_o(0)$ is the initial capacitor voltage, and

$v_o(\infty)$ is the final capacitor voltage.

Write the expression to find the time constant for an *RC* circuit.

$\tau = R_{\mathrm{Th}}C$ (2)

Here,

R_{Th} is the Thevenin resistance, and

C is the capacitance of the capacitor.

Write the general expression for the unit step function.

$$u(t) = \begin{cases} 0, t < 0 \\ 1, t > 0 \end{cases} \quad (3)$$

Calculation:

The initial capacitor voltage,

$$v_o(0) = 5\mathrm{V} \quad (4)$$

Apply the unit step function in equation (3) to equation (4).

$$\begin{aligned} v_o(0) &= (5\mathrm{V})u(t) \\ &= 5u(t)\mathrm{V} \end{aligned}$$

The given Figure 7.112 is redrawn as shown in Figure 1.

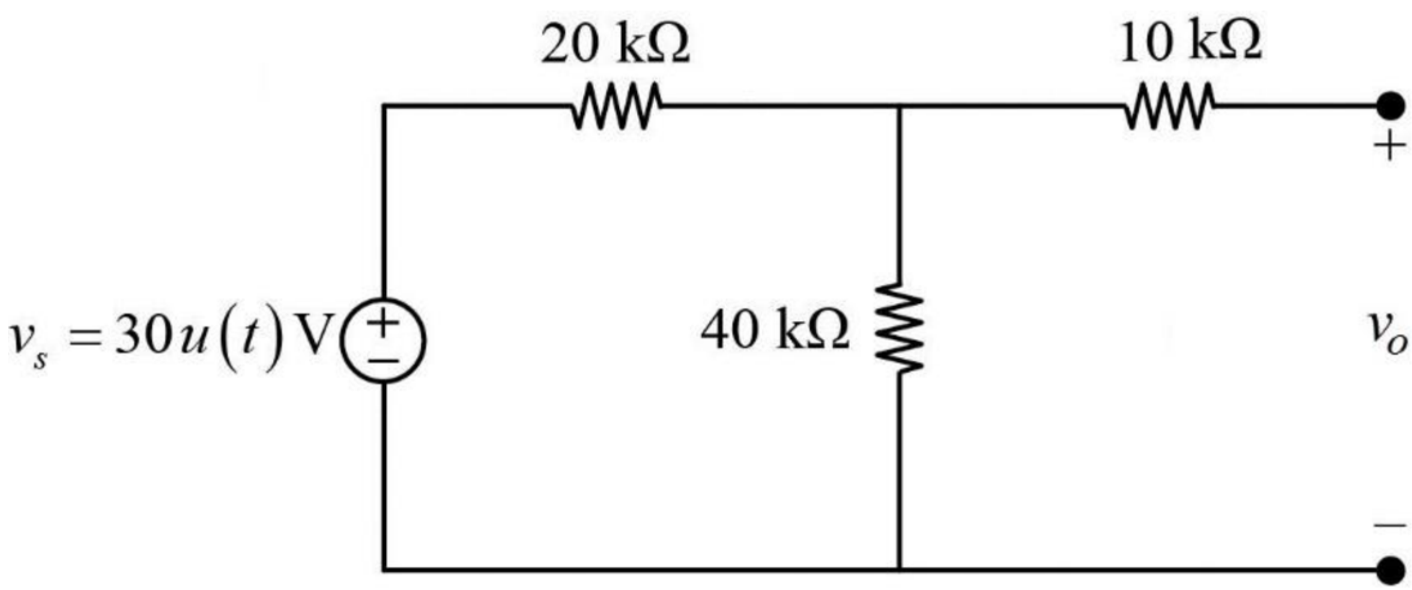

Figure 1

From Figure 1, the final capacitor voltage $v_o(\infty)$ calculated by using voltage division rule.

$$\begin{aligned} v_o(\infty) &= (30u(t)\mathrm{V})\left(\frac{40\mathrm{k}\Omega}{40\mathrm{k}\Omega+20\mathrm{k}\Omega}\right) \\ &= (30u(t)\mathrm{V})\left(\frac{4}{6}\right) \\ &= 20u(t)\mathrm{V} \end{aligned}$$

Figure 2 shows the Thevenin resistance R_{Th} at the capacitor terminal.

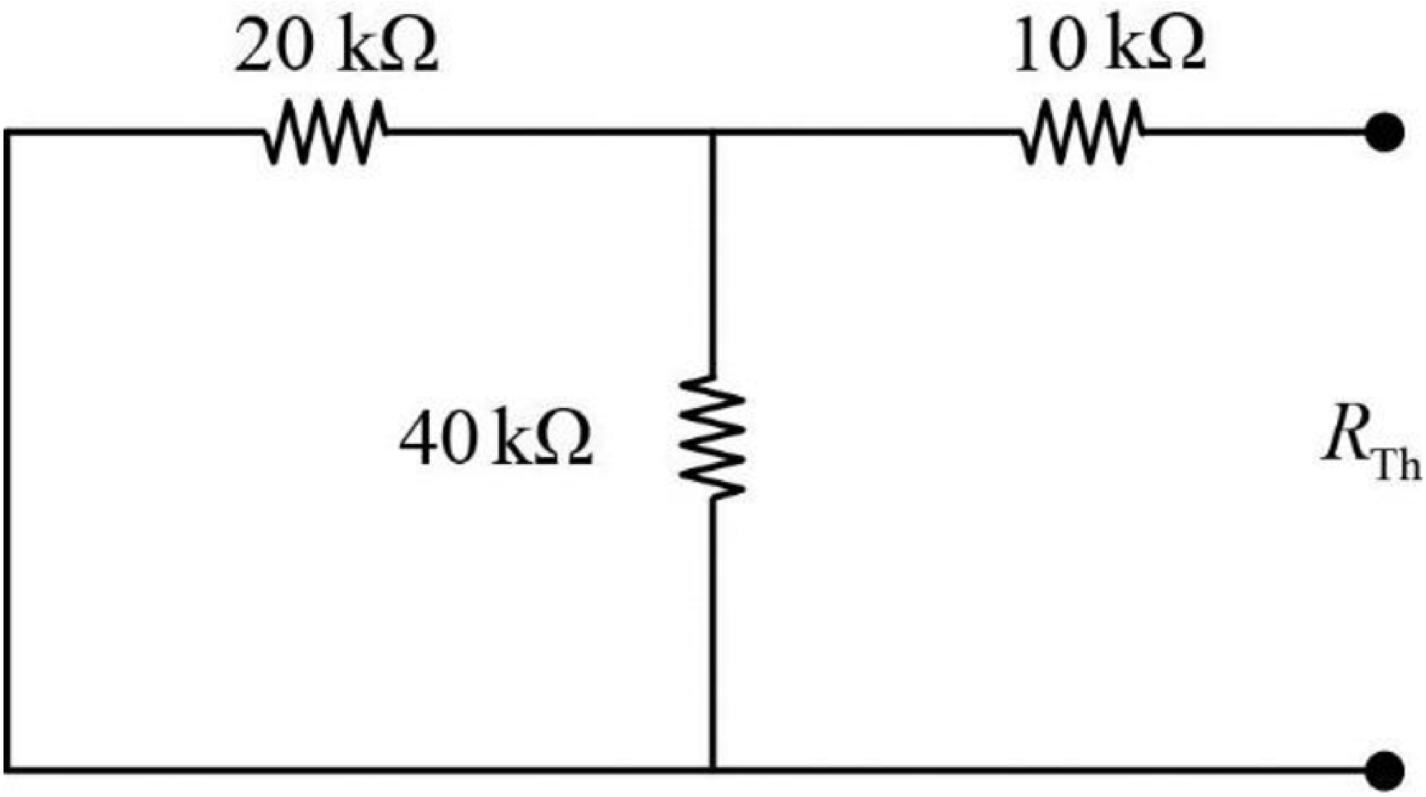

Figure 2

The Thevenin resistance R_{Th} at the capacitor terminal calculated as follows.

$$\begin{aligned} R_{\mathrm{Th}} &= (20\mathrm{k}\Omega \| 40\mathrm{k}\Omega) + 10\mathrm{k}\Omega \\ &= \left(\frac{20\mathrm{k}\Omega \times 40\mathrm{k}\Omega}{20\mathrm{k}\Omega + 40\mathrm{k}\Omega}\right) + 10\mathrm{k}\Omega \\ &= \left(\frac{20 \times 40}{60}\right)\mathrm{k}\Omega + 10\mathrm{k}\Omega \\ &= \left(\frac{40}{3}\right)\mathrm{k}\Omega + 10\mathrm{k}\Omega \end{aligned}$$

Simplify the equation as follows,

$$R_{\mathrm{Th}} = \frac{70}{3}\mathrm{k}\Omega$$

Substitute $\frac{70}{3}\mathrm{k}\Omega$ for R_{Th} and $3\mu\mathrm{F}$ for *C* in equation (2) to find the time constant τ.

$$\tau = \left(\frac{70}{3}\mathrm{k}\Omega\right)(3\mu\mathrm{F})$$

$$\tau = \left(\frac{70}{3} \times 10^3\Omega\right)\left(3 \times 10^{-6}\mathrm{F}\right)\left\{\because 1\mathrm{k} = 10^3, 1\mu = 10^{-6}\right\} \quad (5)$$

Substitute the units $\frac{\mathrm{V}}{\mathrm{A}}$ for Ω and $\frac{\mathrm{A \cdot s}}{\mathrm{V}}$ for F in equation (5) to find the time constant τ in seconds.

$$\tau = \left(\frac{70}{3} \times 10^3 \frac{\text{V}}{\text{A}}\right)\left(3 \times 10^{-6} \frac{\text{A}\cdot\text{s}}{\text{V}}\right)$$
$$= 0.07\text{s}$$

Substitute $5u(t)\text{V}$ for $v_o(0)$, $20u(t)\text{V}$ for $v_o(\infty)$, and 0.07s for τ in equation (1) to find the output voltage across the capacitor v_o in volts.

$$v_o = 20u(t)\text{V} + [5u(t)\text{V} - 20u(t)\text{V}]e^{-\frac{t}{0.07\text{s}}}\{\because v_o = v_o(t)\}$$
$$= 20u(t)\text{V} - 15u(t)\text{V}e^{-14.286t}$$

$$v_o = \left[20 - 15e^{-14.286t}\right]u(t)\text{V}$$

Conclusion:

Thus, the output voltage across the capacitor v_o is $\left[20 - 15e^{-14.286t}\right]u(t)\text{V}$.