Polynomial Invariants in Knot Theory

MTHM038 Project Dissertation
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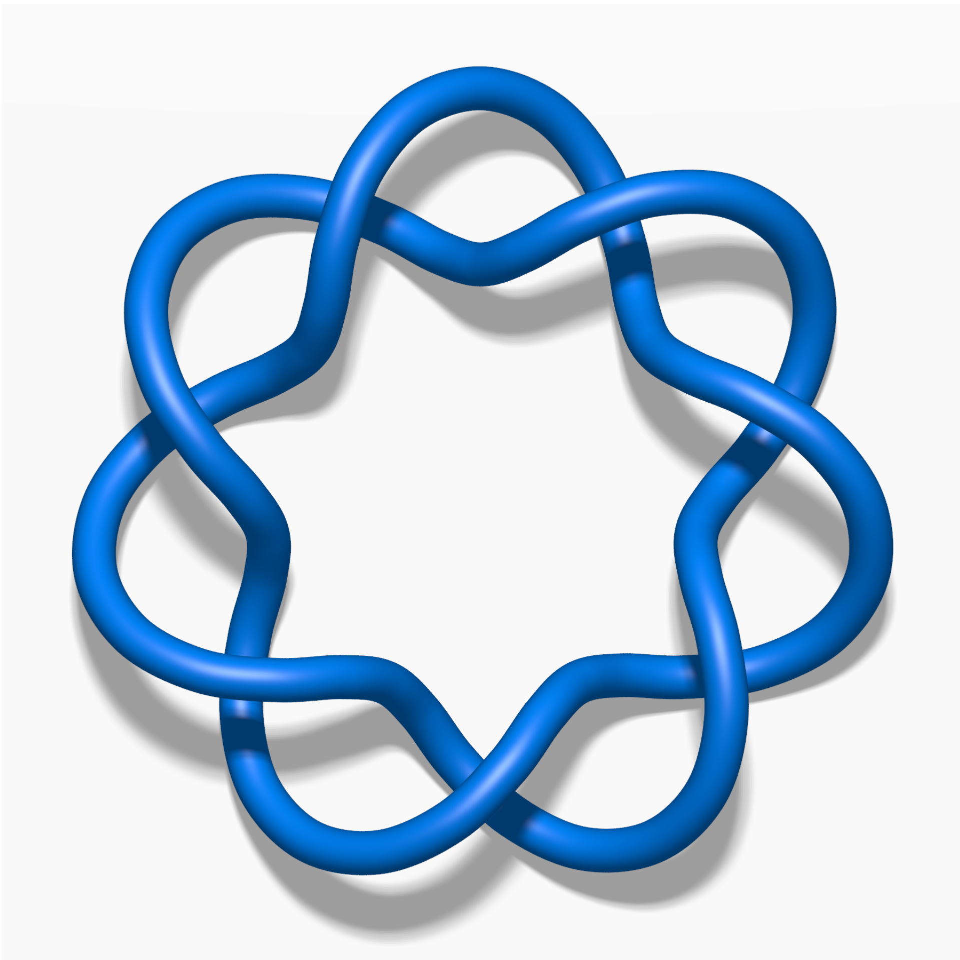

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Abstract

This paper reviews and describes in detail a set of polynomial invariants that are of fundamental importance in knot theory, namely, Alexander Polynomial, Conway Polynomial, Jones Polynomial, HOMFLY Polynomial, and Kauffman Polynomial. Each invariant is discussed in detail with a historical overview and many examples, including its relation to previous invariants. At last, we have drawn the picture to show how we finally arrive at invariants of finite type, namely Vassiliev Invariants.

Contents

1 Introduction

This project aims to provide a detailed introduction to the field of knot theory, outlining all the basics of the subject and all the necessary preliminaries required to study the polynomial invariants we wish to discuss in some depth. Apart from discussing each polynomial invariant in detail in their associated chapters, we are going to discuss the knot theory via three different approaches through combinatorics, geometry, and algebra. We also aim to explain at last, how knot theory ties together these areas and other areas of mathematics.

After giving some history and overview of the subject, we aim to assign a whole chapter to the basics of the subject. After which, a chapter is assigned to each polynomial invariant, namely five polynomial invariants are discussed in detail. These are Alexander Polynomial $\triangle_L(x)$, Conway Polynomial $\nabla_L(z)$, Jones Polynomial $V_L(t)$, Homfly Polynomial $P_L(v, z)$, Kauffman Polynomial $F_L(a, x)$.

As it is intended that this report be readable by anyone with a small background in mathematics, every here and there extra explanation is given in order to give further clarity on some topics such as topological spaces and so forth. As the space of this report is also limited, some of the in-depth or detailed proofs are dropped (it is attempted to give reference to all the proofs that are opted-out) or only a sketch of the proof is given.

The aim has been set to give proper examples throughout this report, and all the figures have been drawn in tikz environment or edited and imported as images from Paint. The proper reference is provided for the figures that are used from other articles or books. This is because some of the knot diagrams are really hard to draw, and can sometimes take a while to pin-point precisely. Hence, some of these diagrams are used from external sources.

In order that this report uses the most diversified sources, we have used some of the very well recognised materials and texts in the field such as [4], [8], and [9] which touches the subject at elementary, intermediate, and advanced level. It is also aimed, where possible, to give proper details of the discussion in this manner and order. I request the respected reader to bear any errors and mistakes which might have incurred in this report, they are entirely mine, and if possible to report it to me via contact page for future corrections.

Sajjad Abolverdi

2 Knot Theory

2.1 History

Please note that most of the key dates and reference points that is noted in this section has been borrowed from the source [7], as from the authors point of view, it was a rich reference point containing all the details required for the history of knot theory and its development. Knot theory began in the nineteenth century by the curiosity of three mathematicians, namely Karl Friedrich Gauss, James Clerk Maxwell and Peter Guthrie Tait, and a hefty but short-lived work of a physicist called Lord Kelvin (Sir William Thomson). The latter had a proposed physical theory which was based on the hypothesis that the atoms were three dimensional knotted vortices in the ever-expanding popular notion of the day, the ether of space. Thomson managed to enrol few other mathematicians, such as Tait, Little and Kirkman to help him produce the first ever tables of knots. He hoped that these tables can help him solve some of the mysteries of these chemical elements. His theory soon collapsed due the vast number of knotted forms, and the huge and impossible task of classifying and bringing all of these under one umbrella. The other reason of his failure was the end of viewing the space from etheric perspective.

But nevertheless, it was around this time that the theories of manifolds and topology came into existence through the work of Gauss, Riemann, and Poincaré. Their work managed to provide the right tools to analyse topological properties of knots and other characteristics within three dimensional manifolds. Poincaré managed to provide the first and most significant tool in knot theory, namely the fundamental group (of a topological space). In the early 1900's, Max Dehn used the properties of the fundamental group of the complement of the knot to prove the knottedness of the trefoil knot and show its mirror image is inequivalent to it [7]. After his work, one of the deep and fundamental questions in knot theory was raised, that is detecting the chirality of knots.

At this point of time in the late 1920's, James W. Alexander, from Princeton University managed to discover a polynomial invariant of knots and links [10] and made some huge computations possible. The Alexander polynomial was based entirely on the newly discovered Reidemeister moves (see the following section for further details) which expresses topological equivalence relation for isotopy of knots and links in terms of the so-called *knot diagrams*. Reidemeister exposition of moves can be found in his first book in knot theory written in 1932 [11]. The Alexander polynomial is expressed via determinant of a matrix which can directly be read from a knot diagram. One can prove the invariance of the polynomial by examining how this determinant is behaving under Reidemeister moves.

Topology began to gain a rapid expansion from the 1920's onward, with the outstanding work of Seifert [12], the algebraic topology was continuing its evolution. By mid 50's, Ralph Fox and his students showed [13] the significant role of the fundamental group in knot theory, by showing how one could obtain the equivalence of Alexander polynomial precisely from the presentations of the group using a non-commutative and discrete calculus, namely free differential calculus.

The ground-breaking paper of John Horton Conway [14] was published in the late 1960's and introduced a new method of computing the Alexander polynomial without the need of any matrices, free differential calculus, or the determinant. His technique was based upon a recursive formula which was expressing the Conway version of the Alexander polynomial in terms of simpler knots and links. The new Skein Theory found by Conway caused some puzzlement amongst topologists which lasted about a decade. After they found their heads with this new theory, they began thinking via Conway approach which confirmed the validity of his method and led to a normalised version of the Alexander polynomial.

In 1984, Vaughan Jones released an explosive paper [15], in which until today the mathematical world is still shaking and feeling its waves. He managed to come up with an entirely new Laurent polynomial invariant of knots and links by finding an equivalence relation between the structure of Artin Braid Group and some specific identities in a class of Neumann Algebras to find representations of the braid group which he used to come up with this new discovery. This was a blow to Alexander polynomial, as this new polynomial invariant could differentiate lots of knots and links from their mirror images as well as satisfying the same skein relation that was ruling the Conway polynomial, except the coefficients had to change here.

Soon after the Jones polynomial, a group of mathematicians working in separate groups and pairs, namely Hoste, Ocneanu, Millett, Freyd, Lickorish, Yetter, Przytycki and Traczyk generalised Jones invariant to a two-variable polynomial invariant of knots and links; and their initials was used to give the acronym for the name of the new polynomial invariant, namely HOMFLY polynomial (Originally this was named "Homflypt" but then as time passed it was simplified to HOMFLY). Few months after this discovery, Brandt, Millett and Ho came up with a new one-variable invariant of unoriented knots and links which had a different skein relation. Again, this invariant failed to distinguish knots and their mirror images. When Louis H. Kauffman received a copy this announcement, he realised that he can generalise this invariant to a two variable polynomial invariant

$L_K(z, a)$ of knots and links that could distinguish knots from their mirror images. This was also named after him, as Kauffman Polynomial. As we have covered the history of all five polynomials, we prefer to end this section here, and allow the reader to continue to follow and read the rest of history of the field with the details from [6] if interested.

2.2 Basics of Knot Theory

After having a comprehensive introduction into the history of knot theory, it is time to give a taste of the stuff we are dealing with in this beautiful subject. The first thing we need to settle is the rough idea of what a knot is, and then give a precise definition of a knot mathematically. I am sure that one way or the other we have all came across a knot throughout our lives. Whether it is the first task at pre-school age of learning how to tie a shoelace [5] on our own; or playing around with a piece of string made out of plastic or other materials. So this assumption can be assumed with confidence that we have all came across knots at some point.

The simplest knot that can be thought of is the *unknot*, which can be formed simply by connecting two ends of an open string together to form a loop. Now one of the key questions in knot theory and the invention of all these different invariants is to tell if a knot is simply the unknot. Even though this may seem to be a simple and easy question to answer, in reality and practical experience, it is not so and turn out to be one of the most difficult. Please look at figure 2.1 which demonstrated three different variants of the unknot.

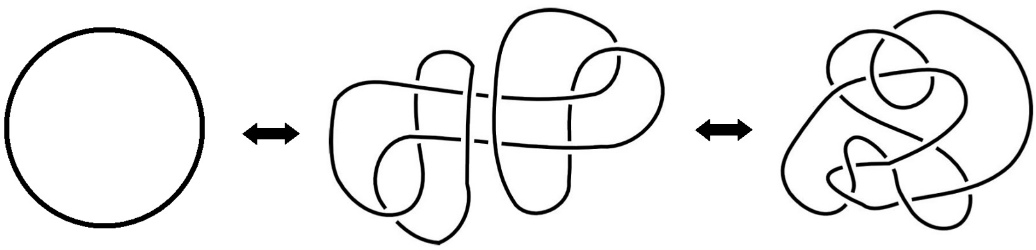


Figure 2.1: The unknot, and two non-trivial versions.[P]

The next kind of knot that we can consider is the first *prime* knot other than the unknot, and the only knot with *crossing number* 3, namely the *trefoil* knot with special notation 3_1. (We will see official definitions of prime and crossing number later on). The trefoil knot and two other versions of it is demonstrated in figure 2.2. The reason why we have given again another two non-trivial versions of the trefoil is because we want to show you how these simpler to imagine knots can also be difficult to distinguish with their original versions. Another important questions in knot theory is telling whether two knots are the same or not; and we will see how different invariants and in particular polynomial invariants of knots and links come into scene to help us overcome this problem.

Figure 2.2: The trefoil, and two non-trivial versions.[P]

Remark. The notation we used above, "3_1" for trefoil, or one such as 7_5 beside a diagram simply intends to show the fifth knot with crossing number 7 in a traditional ordering. This format of numbering for knots begun in the nineteenth century by P. G. Tait [15] and C. N. Little [16].

Now looking at the knots mathematically, we first need to look at it as a one-dimensional curve or the "string" embedded in three-dimensional space that has no self intersections. Lets denote $\mathbb{R}^n$ to be the n-dimensional Euclidean space and S^n be the n-dimensional sphere.

Definition 2.1. A **Knot** $K \subset \mathbb{R}^3$ is a locally flat subset of points homeomorphic to a circle.

The imposed condition on the points in the above definition to be "locally flat" is equivalent to having a piecewise linear condition that implies the knot K to be composed of a finite number of straight line segments placed end to end. By straight we mean that it is in the linear structure of $\mathbb{R}^3 \subset \mathbb{R}^3 \cup \infty = S^3$. As we can assume there to be so many straight line segments, we can always draw the curves in the diagrams of knots and links to look pretty well rounded. Once we require the number of straight line segments to be finite, we are preventing a knot from having an infinite number of kinks, that is going to get smaller as they converge to a point. We call these kind of knots, being "wild". In order to avoid such wildness, we need K to be a smooth 1-dimensional sub-manifold of the smooth 3-manifold

S^3. The only alternative class of knots which avoid the wildness, is the polygonal knots. Figure 2.3 is demonstrating this idea very clearly, and shows two possible ways the circle can be arranged in space based on this definition.

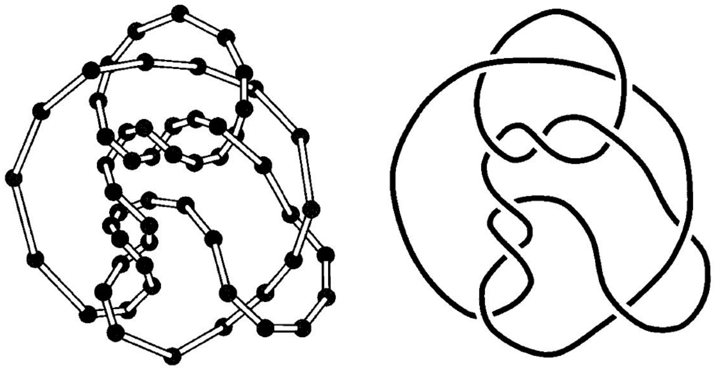


Figure 2.3: A polygonal and a smooth representation of the knot 10_{50}. [9]

Next we wish to formally define when two knots are equivalent mathematically. This can be done via a notion called *isotopy* from algebraic topology. Two knots are considered equivalent if they are *ambient isotopic.* Formally we have the following two definitions.

Definition 2.2. An **isotopy** is a continuous map $h : X \times [0,1] \to \mathbb{R}^3$ such that each $h_t = h \mid X \times \{t\}$ *is one-to-one.* By convention, h_0 is the identity map.

and now we define two isotopic knots.

Definition 2.3. Two knots K_1 and K_2 are **ambient isotopic** if there is an isotopy $h : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ such that $h(K_1, 0) = h_0(K_1) = K_1$ and $h(K_1, 1) = h_1(K_1) = K_2$.

The second definition is implying that the ambient isotopy is an equivalence relation on knots, or in other words two knots are equivalent if they can be deformed into one another. Hence, these two definitions opens our hand in deformation of any type of knot we wish, whether it is moving or bending the arcs through space or shrinking or enlarging the entire knot. The only forbidden thing we are not allowed to do, is Bachelors' unknotting, which is a continuous transformation (pulling the knots so tight) that makes knots vanish.

Now it is time to end this section with a precise meaning of a *knot diagram.* In order that we can all work with knots more easily and always refer to the exact knot in discussion, we need to be able to represent them on some diagram that is universal to us all. The easiest way we can do this is via projecting the whole knot into a plane, and enforce the image to contain all the "over and under" information at the crossings. In this way we can detect which arc is passing over or under the other arcs. This means there is an additional condition that all the singularities to be double points, with the approaching arcs having distinct tangents. We call such diagram to be a regular projection of the knot. Formally we can also give the following definition.

Definition 2.4. The image of K in $\mathbb{R}^2$ in addition with the "over and under" information at the crossings is called a ***knot diagram*** of K.

Remark. Note that a crossing is a point of intersection of the projections of two line segments of K and the "over and under" information is pointing to the relative heights above $\mathbb{R}^2$ of the two inverse images of a crossing. These details are always shown in pictures by breaks in the under-passing segments. In figure 2.4 and 2.5, we can see the moves that allow a projection to be regarded as regular.

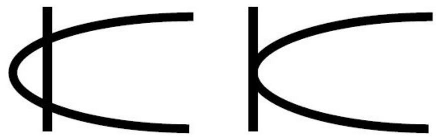


Figure 2.4: Regular projection vs double point with indistinct tangents.[P]

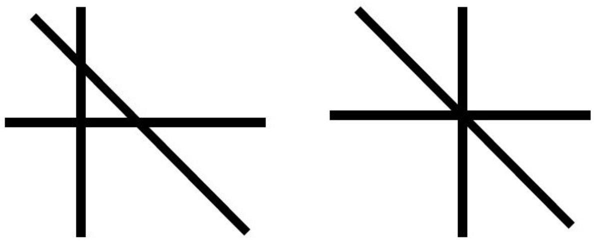


Figure 2.5: Regular projection vs Triple point singularity.[P]

Since the first publication of the Alexander's paper in 1928, a lot has changed, including the common notation of the crossings. He had a unique way of marking the crossing points in his paper, by placing two dots to the left hand side of the underpass as you follow the given orientation of the knot. The usual and common notation in today's literature is considering the crossing as a broken line, which is either a underpass or overpass. In this report, we will use the latter. This is demonstrated in figure 2.6.

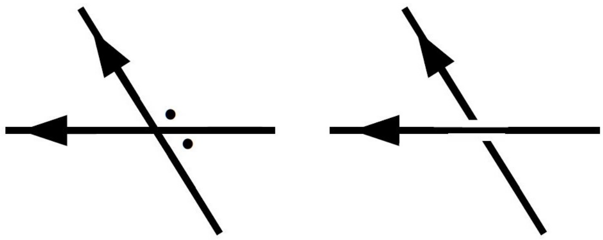


Figure 2.6: Alexander's notation vs The broken notation. [P]

As we may face or need to deal with some diagrams which have a specific orientation in this report, we would rather demonstrate how the left-hand crossing and the right-hand crossing looks like right at the end of this section. Supposing that we are travelling in car through some one-way highways, which have under-pass and over-pass, at a time which we are crossing the underpass, the cars that are travelling in the junctions above our head are either passing from left to right, or right to left. This is known as left-hand or right-hand crossing, which is demonstrated in figure 2.7.

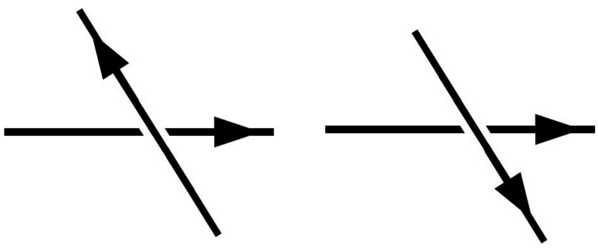


Figure 2.7: Left-hand crossing vs Right-handed crossing. [P]

Now with all of these rules applied to knot diagrams, it seems we are in a stable position in regards to precisely pin-pointing knot diagrams onto the plane; however, we face a new problem which is caused when we project a knot from a different angle, which as a result produce a different diagram for the same knot. If we allow continuous deformation of the knot, this will not result in a stable diagram, and hence we need to apply a new rule invented by Reidemeister so that it contains everything we have discussed so far.

2.3 Reidemeister Moves

In the 1920's Kurt Reidemeister managed to come up with a theorem and prove that the problem of distinguishing the topological type of a knot is in fact a problem in combinatorics. As we saw previously in section 2.2, he realised that via fixing the coordinates of a knot onto a plane, the sketch of the knot is equivalent to a rigorous and exact representation of the topological properties (type) of the knot. Then he successfully demonstrated that two diagrams of a knot are showing the same type of knot (with the same topological properties of knottedness) if and only if one diagram can be gained from the other diagram through homeomorphisms of the plane with some additional (and finite) sequence of Reidemeister moves. By applying the Reidmeister moves, the diagrams only changes locally but remain unchanged topologically. These three types of moves is demonstrated in figure 2.8.

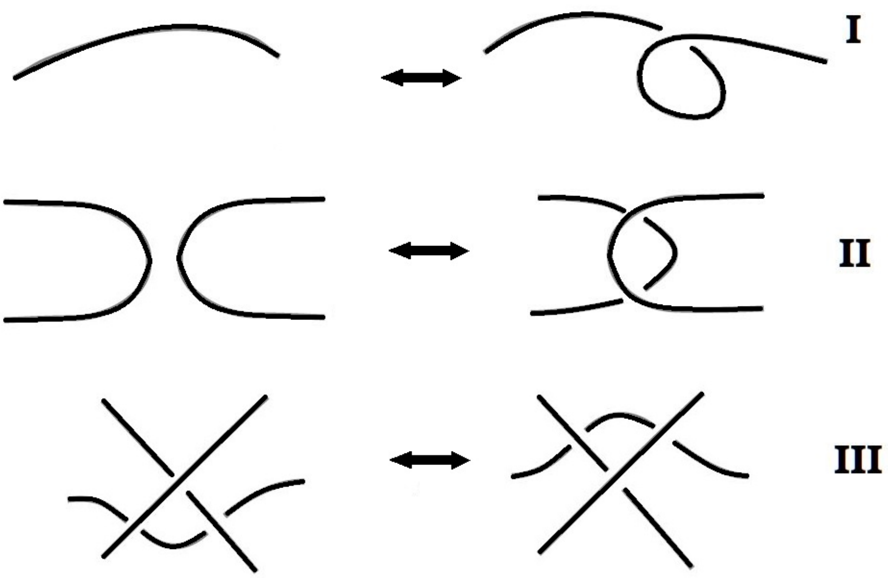


Figure 2.8: Reidemeister three types of moves. [P]

Now with these moves at hand, we may assume that we can easily solve most of equivalence problems in knot theory, but again we are wrong. These moves comes handy when we are dealing with simple knots of few crossings, and as the number of crossings increase, it is becoming increasingly difficult to distinguish knots, and Reidemeister moves loses their efficiency. Just as a reference to this fact, please refer to figure 2.9 which shows how the number of prime knots increase, as the crossing number increase. The data of this table was obtained by M. B. Thistlethwaite, and proof checked by two other mathematicians, named J. Hoste and J. Weeks [8].

TABLE 2.9.

Crossing number	3	4	5	6	7	8	9	10	11	12	13	14	15
Number of knots	1	1	2	3	7	21	49	165	552	2176	9988	46972	253293

Figure 2.9: Number of knots vs the crossing number. [8]

Hence we need to find an alternative and practical way to deal with this problem. The solution would be to find an algorithm that can detect if two knots are equivalent without touching the topological properties of the diagrams. In reality, such algorithm was found by W. Haken [1], which is a theoretical algorithm which always decides if two knots are of the same type or not. But practically, it is impossible to use it, however we can be rest assured to know that it exists out there. So our useful and practical solution would be to use *invariants* to distinguish between two knot types. Here is a definition of what we mean by an invariant.

Definition 2.5. An **invariant** of an object, with respect to some transformation of the object, is some quantity or characteristic that does not change under the transformation.

For knots, these are simply some well-defined mathematical entity such as a number, a polynomial, or a group that is associated to a knot or its diagram. The crossing number of a knot, which was discussed above, is one of the simplest knot invariants. As knots with different crossing number cannot be equivalent. But, as we need to do a lot of calculations to convert a knot to its prime representation, in order to find its crossing number, this invariant is one of the difficult ones to measure and use. The table 2.9 is another prove for this claim. In the next section we are going to discuss Seifert Surfaces and the idea of genus of a knot. These will come up in the chapters that will follow.

2.4 Seifert Surfaces and Genus of a Knot

There remain few more principles which we would need later that are essential in our discussion. We are going to give these here in this section. At the beginning we are also going to give few other definitions which seems to be also compulsory for the topics we are going to open up. The first thing we need make clear is the idea of a *link* and we only going to give a brief explanation and few small examples of links.

Definition 2.6. A **link** is a finite disjoint union of knots: $L = K_1 \cup \cdots \cup K_n$.

That means a link is made up of a finite number of components that are disjoint, simple closed curves in the desired space. A knot, is simply a link of one component. Similar to knots, links also have similar topological properties. They cannot also be broken, in the same manner that knots cannot be pulled apart. We have given three of the most famous links that are known in knot theory in figure 2.10.

Definition 2.7. An **oriented link** is a link for which each connected component has been given an orientation.

Remark. As oriented links are equivalent, if there is an orientation preserving homeomorphism of $\mathbb{R}^r$ which sends one link to the other, assignments of different orientations to the different components of a link will often result in inequivalent oriented links.

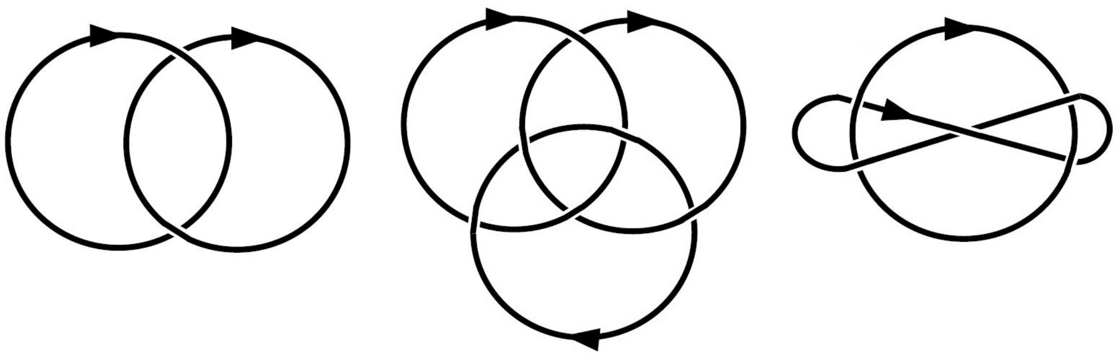


Figure 2.10: The Hopf, Borromean, and Whitehead links respectively.[P]

Definition 2.8. Let D be an oriented diagram of a 2-component link $K_1 \cup K_2$, and let D_i denote the component of D corresponding to K_i. The crossings of D are of three types: D_1 with itself, D_2 with itself, and $D1$ with D_2. We shall concentrate on the last group which we shall denote by $D_1 \cap D_2$. The **linking number** of D_1 with D_2 is defined to be

$$lk(D_1, D_2) = \frac{1}{2} \sum_{c \in D_1 \cap D_2} \varepsilon(c)$$

Remark. It is proved in Theorem 3.8.2 in [9] that the linking number is independent of the diagram D and therefore is an invariant of the link $K_1 \cup K_2$, and thus we can write $lk(K_1, K_2)$. The links in figure 2.10 have the following linking numbers: Hopf = 1, Whitehead = 0, Any two components of Borromean link has linking number 0, but through Milnor Invariants [5] we can prove that the Borromean rings are linked.

Next we need to define the primeness of a knot. Here is the definition of what we mean by a prime knot.

Definition 2.9. A knot K is a **prime** knot if it is not the unknot, and $K = K_1 + K_2$ implies that K_1 or K_2 is the unknot.

Lets also give the official definition of a crossing number

Definition 2.10. The **crossing number** of a knot is the minimal number of crossings needed for a diagram of the knot.

At the end of this section we have also provided a full table of knots (35 of them) which demonstrated the diagrams of these knots with crossing number at most 8. This table has been borrowed from the source [8]. Even though we could have gathered together each of these knots individually and put them all into a table, in source [8] this was done in a neat way, and we decided to give this in a form that uses less space. We will use this table as a reference point for all the polynomials we are going to discuss, and we will use the knots in this table as an example for calculating the values of these polynomials. The values of these examples will be given in their respective chapters and in separate tables.

We are now going to show you that any link in S^3 can be considered as the boundary of some surface embedded in S^3. We can use these surfaces, named Seifert surfaces, to study links from different ways. Later in this chapter we will also study Seifert Matrix. Here is the definition of these surfaces.

Definition 2.11. A **Seifert surface** for an oriented link L in S^3 is a connected compact oriented surface contained in S^3 that has L as its oriented boundary.

We can provide an example of a link with Seifert surface, simply by looking at any embedding that goes into S^3 of a compact, connected, and oriented surface with non-empty boundary. We also know that a surface is non-orientable if and only if it contains a Mobius band. Now we need to give a theorem which has an important proof. The proof of this theorem gives us a way of constructing a Seifert surface from a diagram of a link.

Theorem 2.1. *Any oriented link in S^3 has a Seifert surface.*

Proof. Let D be an oriented diagram for the oriented link L and let $\hat{D}$ be D modified as shown in Figure 2.11. $\hat{D}$ is the same as D except in a small neighbourhood of each crossing where the crossing has been removed in the only way compatible with the orientation. This $\hat{D}$ is just a disjoint union of oriented simple closed curves in S^2. Thus $\hat{D}$ is the boundary of the union of some disjoint discs all on one side of (above) S^2. Join these discs together with half-twisted strips at the crossings. This forms an oriented surface with L as boundary; each disc gets an orientation from the orientation of $\hat{D}$, and the strips faithfully relay this orientation. If this surface is not connected, connect components together by removing small discs and inserting long, thin tubes.[8] $\square$

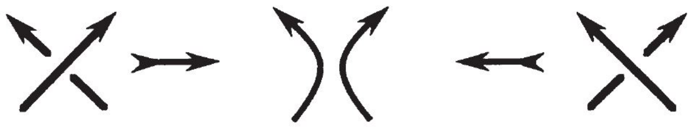


Figure 2.11: Opening all the oriented crossings, as shown [P]

In the proof of Theorem 2.1, $\hat{D}$ was consisted of disjoint, simple, and closed curves of D, we call these curves the *Seifert Circuits* of D. Lets look at an example which shows the Seifert circuit of the knot 8_{20}. This is shown in figure 2.12.

Figure 2.12: Seifert Circuit of the knot 8_{20}. [8]

Now in order to construct the Seifert surface for the knot 8_{20}, we need to add three discs on top of the page and 8 half-twisted strips near the crossings to connect the discs together. Lets have a look at another example which shows how a knot can have two very different Seifert surfaces. This can be seen as the two thin circles joined by a tube after chasing the narrow strip or after in-taking that part of the picture. This is shown in figure 2.13.

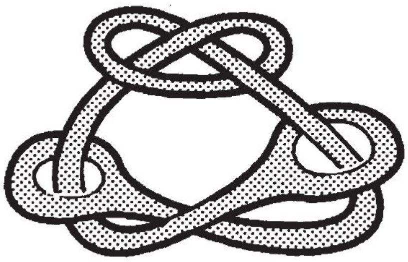


Figure 2.13: Two very different Seifert surfaces. [8]

The knot in the diagram of figure 2.13 has two genuses. I mention this before giving the definition of a genus, as it is intuitive, and you can imagine what we might mean by a genus. Now lets give the precise definition of the genus.

Definition 2.12. The **genus** $g(K)$ of a knot K is defined by

$$g(K) = min.\{genus(F) : \text{F is Seifert surface for K}\}$$

Looking at the boundary F in the above definition as a disk, adding a number of "handles" to this surface, the genus is this number of handles for each knot. The only sufficient result we would add before going into the Seifert Matrix, is the addition of the genus of two different knots.

Theorem 2.2. *For any two knots K_1 and K_2,*

$$g(K_1 + K_2) = g(K_1) + g(K_2)$$

Proof. The proof of this theorem is a long one and we will omit it, but it can be found in [8], page 17. □

There are few results and principles that needs to be cleared before arriving at the Seifert Matrix. We first need to consider some connected oriented surfaces in $\mathbb{R}^3$. Now we need to consider thickening this surface as follows: A result from [9] has shown that an oriented surface F has positive and negative sides, and we will take this for granted. Next let $b : F \times [-1, 1] \to \mathbb{R}^3$ be a homeomorphism such that $b(F \times \{0\}) = F$ and $b(F \times \{1\}$ lies on the positive side of F. Any subset $X \subset F$ can be lifted out of the surface on either side, hence we let $X^+ = b(X \times \{1\})$ and $X^- = b(X \times \{-1\})$. What we are interested in is the case in which X is a loop in the surface F. We can consider the intersection of two loops in the surface F, but if one of the loops is lifted out of the surface, then it is not possible to intersect them. Moreover, the two loops are going to have linking number. With this result in hand, we can arrive at the following definition.

Definition 2.13. The mapping

$$\begin{aligned} \Theta : H_1(F) \times H_1(F) &\to \mathbb{Z} \\ (a,b) &\mapsto lk(a, b^+) \end{aligned}$$

is called the **Seifert pairing** or the *linking form* of the embedded surface F.

In order to save time and space, we will simply state some partial results in order to obtain results rapidly, and therefore no requirement to go deeply into some concepts. The group defined above as $H_1(F)$ is called the first **homology group** of F, where F is a simplicial complex as shown in figure 2.14.

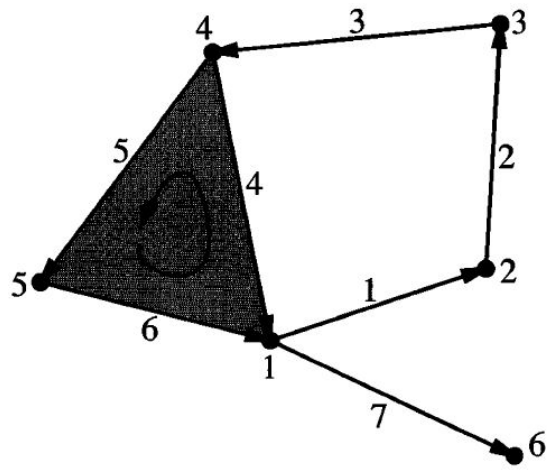


Figure 2.14: A simplicial complex. [9]

This group is in fact the quotient group composed of the following elements

$$H_1(F) = Z_1(F)/B_1(F) = ker(\partial_1)/Im(\partial_2)$$

where ∂_1 and ∂_2 are two boundary operators defined on individual simplices and extended to chains by linearity as follows

$$\begin{aligned} \partial_1 : C_1(F) &\longrightarrow C_0(F) \\ [v_1, v_2] &\longmapsto v_2 - v_1; \end{aligned}$$

$$\begin{aligned} \partial_2 : C_2(F) &\longrightarrow C_1(F) \\ [v_1, v_2, v_3] &\longmapsto [v_1, v_2] + [v_2, v_3] + [v_3, v_1]; \end{aligned}$$

The kernel of ∂_1 is made up of 1-chains that have zero boundary. These elements of $C_1(F)$ are called 1-cycles and we denote the set of these cycles as $Z_1(F)$. As some chains are boundaries of 2-chains, we denote the set of these boundaries as $B_1(F)$.

As an example we have computed the linking form for the projection surface F which has been constructed from the diagram of the knot 10_{165} with additional 6 loops added to the surface. These loops are the boundaries of the faces of the Seifert graph and they form a basis for $H_1(F)$. This new diagram has been shown in figure 2.15.

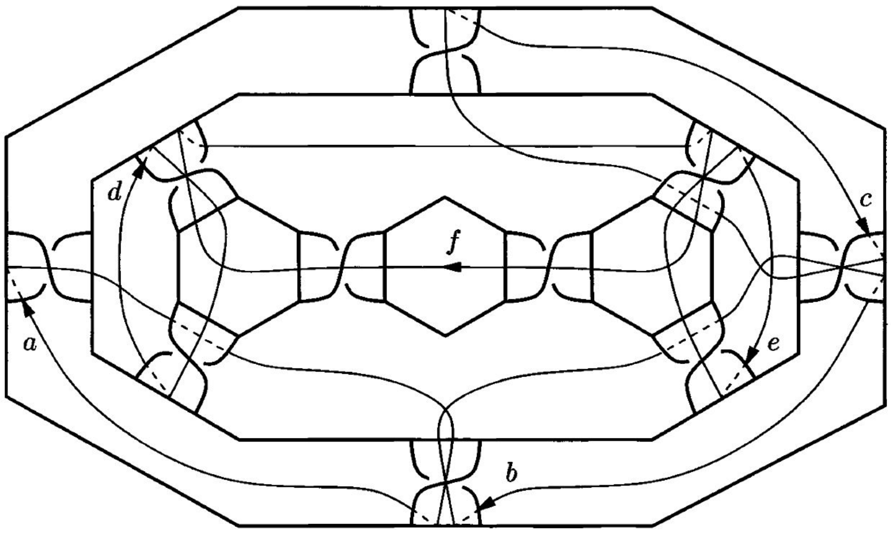


Figure 2.15: A projection surface for the knot 10_{165} [9]

Definition 2.14. A **Seifert diagram** of a knot is constructed by eliminating all crossings such that the edges of a crossing are disconnected, then each is reconnected to the other crossing edge with compatible orientation. The resulting **Seifert circles** are then connected with lines where the crossings used to be.

The cut vertex in the Seifert graph refers to the inner of the two octagonal Seifert circles. We can see that this type two Seifert circle is dividing the surface naturally into upper and lower parts. The loops d,e,f lie in the upper part, and the loops a,b,c lie in the lower part. We should now assume F is oriented such that the upper surface of the disk spanning the outermost Seifert circle is positive. Then the linking form or the Seifert pairing is rounded up in the following table, which we call **Seifert Matrix**.

	a^+	b^+	c^+	d^+	e^+	f^+
a	1	-1	0	0	0	0
b	0	0	0	0	0	0
c	0	1	0	0	0	0
d	1	0	0	-1	0	1
e	0	-1	1	0	1	-1
f	0	0	-1	0	0	-1

Figure 2.16: Seifert Matrix.

The proof on how the entries of the table 2.16 is calculated can be found in [9] on page 140. We now feel that we have all the required tools under our belt, and we shall continue our discussion into the polynomial invariants. As promised earlier, we have also given the table of 35 prime knots with up to eight crossings at the end of this chapter.

TABLE 2.17 The Knot Table to Eight Crossings

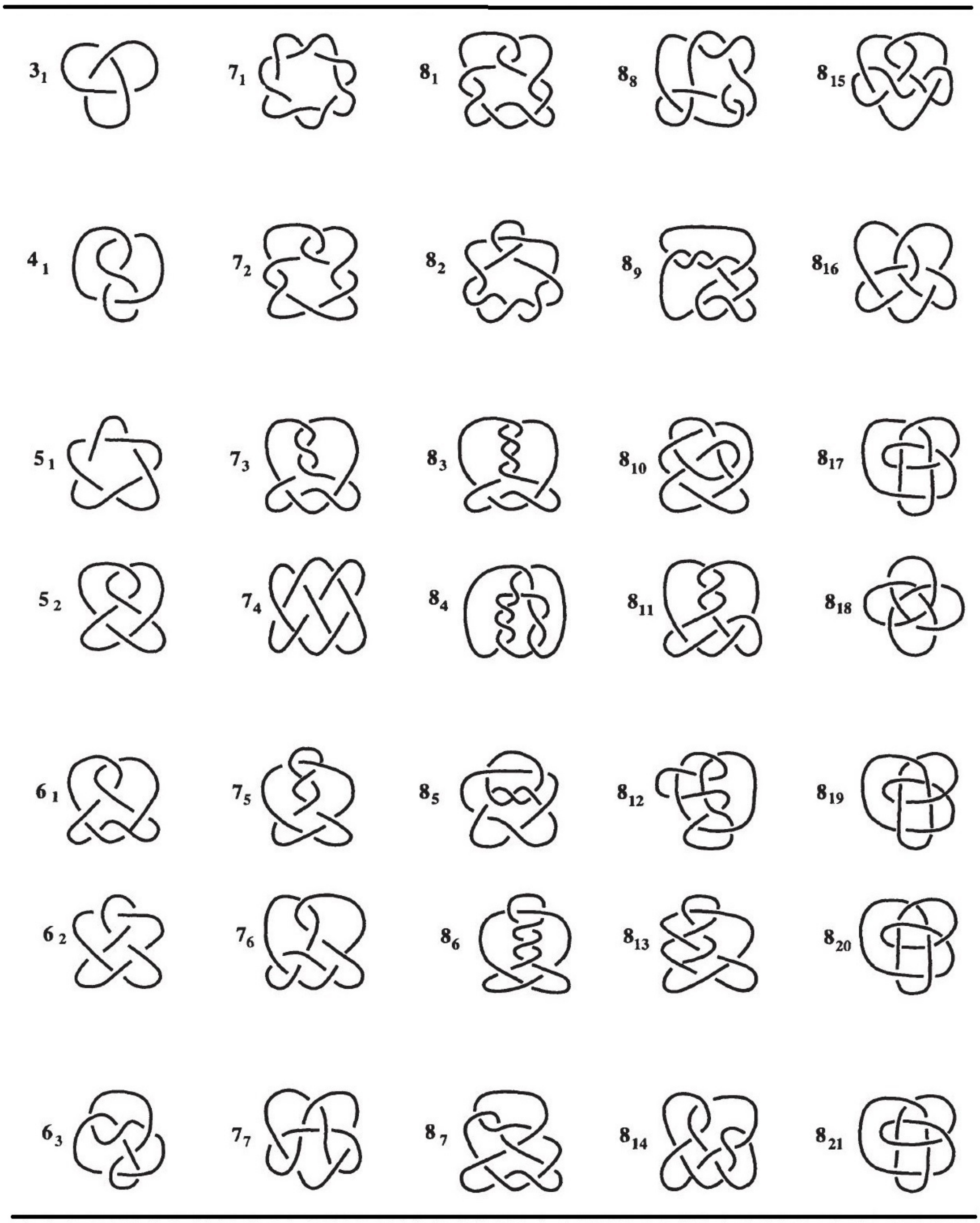


Figure 2.17: The first 35 prime knots with upto 8 crossings. [8]

3 Alexander Polynomial $\triangle_L(t)$

3.1 Historical Overview

Despite the fact that we should start our investigation into polynomial invariants with Jones polynomial, due to the ease of development, use, and understanding of this polynomial; we will stick to the historical development of these polynomials, and start with the first one, known as Alexander polynomial which was developed in 1928 [10]. The other arrangement, which follows as Jones, Alexander, Conway, and at last HOMFLY and Kauffman was deployed by Lickorish in [8]. Here, we are going to use the order on which they were invented, and hence going to use the arrangement of Alexander (1928), Conway (1967), Jones(1984), HOMFLY(1984), and Kauffman(1987).

The Mathematician who invented this break-through invariant, namely James Waddell Alexander, was born in United States of America, in New Jersey in 1888. He gained his PhD from Princeton University in 1915. He was a Mathematician that contributed his early mathematical career to military and in a way service to government. During WWI, and WWII, he worked at a weapons testing site and the office of scientific research and development which was operating under US Air Force. He managed to gain few positions at Princeton and later join newly established Institute for Advanced Study. At the end of his career, he got coupled with Politics during the McCarthy scandal in the 50's due to his left-views, and was forced to follow a recluse life. He last appeared in public in 1954, and died in 1971. Lets now look at his mathematical contribution.

We can say in a way that his main,and maybe the only, contribution to Mathematics, and consequently to knot theory is this polynomial invariant which earned his name to be put on it, namely Alexander Polynomial. This polynomial, which we are next going to investigate in detail is roughly an invariant which can be gained and calculated from a knot diagram. In short, we analyse each crossing point in a diagram and gain an equation (in variable r_i) for each crossing. Then we put all of these equations into a matrix and calculate its determinant. Then we gain a polynomial (in powers of t) and normalise it to yield the actual polynomial invariant known as Alexander Polynomial that can be used to distinguish equivalent knots from each other.

3.2 Arriving at Alexander Polynomial

As we will see later in the next chapter how the Conway polynomial is calculated, but let me give you a hint now, the Alexander polynomial and the Conway polynomial are essentially the same invariants, and only a change of variable can make this clear to us. By looking at it from Alexander perspective, we can gain a lower bound on the genus of a link, whereas the algebra of tangles by Conway shows us some of the symmetry properties of the polynomial.

After exploring the approach given by Lickorish in [8], which is a route via homology theory, we came to a conclusion that despite some simplicity of most of the results, still a major part of the results remain at a very advanced level of mathematics which require more time and space and hence would not be suitable and feasible to follow that approach here. Thus, we decided to follow the approach of Cromwell in [9] which can be considered as a more intermediate level mathematics and hence more matching to our discussion for this report.

Despite giving most of the required definitions and theorems in chapter 2, which we will use here, there remain a few essential results which we will give at the beginning of this chapter. The first thing we need to define mathematically, is the idea of determinant and signature of a link.

Definition 3.1. The **determinant** of a link L, written $det(L)$, is the absolute value of the determinant of $M + M^T$ where M is any Seifert matrix for L.

Next we need to define what we mean by signature of a link

Definition 3.2. The **signature** of a link L, written $\sigma(L)$, is the signature of $M + M^T$ where M is any Seifert matrix for L.

The following table shows the relationship of the determinant and the absolute value of the signature for prime knots up to 9 crossings. Knots of up to 6 crossings can easily be distinguished, but it fails when we move to 7 crossings. For example the knots of 7_2 and 6_2 are considered the same. In this table, at some instances, up to five knots are considered the same. In total we can distinguish 48 distinct non-trivial knots.

det	$\sigma = 0$	$\sigma = 2$	$\sigma = 4$	$\sigma = 6$	$\sigma = 8$
1	0_1				
3		3_1		8_{19}	
5	4_1		5_1		
7		5_2, 9_{42}		7_1	
9	6_1, 8_{20}, 9_{46}				9_1
11		6_2, 7_2			
13	6_3, 8_1		7_3, 9_{43}		
15		7_4, 8_{21}, 9_2			
17	8_3, 9_{44}		7_5, 8_2		
19		7_6, 8_4		9_3	
21	7_7		8_5, 9_4		
23		8_6, 8_7, 9_5, 9_{45}			
25	8_8, 8_9		9_{49}		
27		8_{10}, 8_{11}, 9_{35}, 9_{47}, 9_{48}		9_6	
29	8_{12}, 8_{13}		9_7		
31		8_{14}, 9_8		9_9	
33			8_{15}, 9_{10}, 9_{11}		
35		8_{16}, 9_{12}			
37	8_{17}. 9_{14}		9_{13}, 9_{36}		
39		9_{15}. 9_{17}		9_{16}	
41	9_{19}		9_{18}, 9_{20}		
43		9_{21}. 9_{22}			
45	8_{18}. 9_{24}. 9_{37}		9_{23}		
47		9_{25}. 9_{26}			
49	9_{27}, 9_{41}				
51		9_{28}. 9_{29}			
53	9_{30}				
55		9_{31}. 9_{39}			
57			9_{38}		
59		9_{32}			
61	9_{33}				
69	9_{34}				
75		9_{40}			

Figure 3.1: Determinant vs Signature for prime knots up to 9 crossings. [9]

We are now going to give an important lemma which we will use later in the proof of the invariance of the Alexander polynomial.

Lemma 3.1. *Let M be a Seifert matrix of a connected surface F, and let $\hat{F}$ be a surface derived from F by adding a tube. Then there is a basis for $H_1(\hat{F})$ which contains the basis elements of $H_1(F)$ as a subset, and such that the Seifert matrix of $\hat{F}$ has the form*

$$\begin{pmatrix} & & & * & 0 \\ & M & & \vdots & \vdots \\ & & & * & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} & & & 0 & 0 \\ & M & & \vdots & \vdots \\ & & & 0 & 0 \\ * & \cdots & * & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

where the asterisks indicate unknown integers.

Proof. Suppose we have added a tube to the surface as shown in figure 3.1. Let m be the meridian of the tube and let l be another curve that runs along the tube and back through F. Assume the outside of the tube is the positive side of the surface.

We can see that $lk(m, l^+) = 0 = lk(m, m^+)$ and $lk(l, m^+) = 1$. We can also choose l so that $lk(l, l^+) = 0$ for if $lk(l, l^+) = \lambda \neq 0$ we can replace l with $l - \lambda m$ as follows

$$\begin{aligned} & lk(l - \lambda m), (l - \lambda m)^+) \\ & = lk(l, l^+) - \lambda lk(m, l^+) - \lambda lk(l, m^+) + \lambda^2 lk(m, m^+) \\ & = lk(l, l^+) - \lambda \\ & = 0 \end{aligned}$$

Let $(a_1, \cdots, a_n)$ be a basis for $H_1(F)$. Clearly $lk(a_i, m^+) = lk(m, a_i^+) = 0$ for all i. The added tube may be knotted or linked around other parts of the surface so we have no control over the ways that a_i^+ links with l or a_i links with l^+. Let $\lambda_i = lk(l, a_i^+)$. Then adding a tube to F corresponds to an enlargement of the matrix M of the form

$$\begin{array}{c|ccccc} & a_1^+ & \cdots & a_n^+ & l^+ & m^+ \\ \hline a_1 & & & & * & 0 \\ \vdots & & M & & \vdots & \vdots \\ a_n & & & & * & 0 \\ l & \lambda_1 & \cdots & \lambda_n & 0 & 1 \\ m & 0 & \cdots & 0 & 0 & 0 \end{array}$$

Using the same trick as above, we shall change the basis of $H_1(\hat{F})$ so that the λ_i become zero. Replace each generator a_i with $b_i = a_i - \lambda_i m$. Then

$$lk(l, b_i^+) = lk(l, (a_i - \lambda_i m)^+) = lk(l, a_i^+) - \lambda_i lk(l, m^+) = 0$$

As before, $lk(b_i, m^+) = lk(m_i, b^+) = lk(b_i, l^+) = 0$. Also

$$\begin{aligned} lk(b_i, b_j^+) &= lk((a_i - \lambda_i m), (a_j - \lambda_j m)^+) \\ &= lk(a_i, a_j^+) - \lambda_i lk(m, a_j^+) - \lambda_j lk(a_i, m^+) + \lambda_i \lambda_j lk(m, m^+) \\ &= lk(a_i, a_j^+). \end{aligned}$$

This last result can be seen another way. If we slide the loop m along the tube until it comes off the end then we can think of it as a loop in F. Now m bounds a disc so it is trivial in $H_1(F)$. This means that a_i and b_i are homologous in F and so the change of basis does not affect M.

The enlarged matrix now looks like the one on the left in the statement of the lemma. If the outside of the tube is the negative side of the surface, we get the other form.[9] □

In the following theorem we are going to show that the determinant and signature of the link, as defined above, are independent of the Seifert matrix M, and therefore are well-defined. First lets look at some results in linear algebra known as *Sylvester's Theorem.*

Any symmetric matrix A which has entries in $\mathbb{R}$ is congruent to a diagonal matrix. This is evident from the existence of an invertible orthogonal matrix P with real entries and determinant ± 1 such that $P^T AP$ has all of its non-zero entries on the diagonal. The number of positive entries minus the negative entries is equivalent to the signature of a diagonal matrix. Now we have two diagonal matrices considered as congruent if and only if they have the same numbers of negative, positive, and zero entries. The result which is known as *Sylvester's Theorem* is that congruence is preserving signature.

Theorem 3.1. *Let M be a Seifert matrix constructed from a surface F spanning a link L. Then $|det(M + M^T)|$ and the signature of $M + M^T$ are link invariants which depend only on L.*

Proof. We need to show that the determinant and signature are preserved by the congruence transformations and the enlargement operations. The first part is straightforward. For signature we rely on Sylvester's Theorem given above, which is that signature is preserved by congruence. For the determinant, note that $det(P) = \pm 1$ so

$$\begin{aligned} det(P^T M P + (P^T M P)^T) &= det(P^T (M + M^T) P) \\ &= det(P^T) det(M + M^T) det(P) \\ &= det(M + M^T). \end{aligned}$$

Now we consider the enlargement operations, described in Lemma 3.1. An enlarged matrix has the following block structure:

$$\left(\begin{array}{ccc|cc} & & & * & 0 \\ & \boldsymbol{M} + \boldsymbol{M}^{\mathrm{T}} & & \vdots & \vdots \\ & & & * & 0 \\ \hline * & \cdots & * & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{array}\right)$$

Expanding the (matrix) determinant using the 1's in the lower-right block, we get -$det(M + M^T$ so enlargement does not change the (link) determinant. To compute the signature, first note that the values shown as asterisks can be set to zero by performing congruence transformations. It is then easy to see that the signature of the matrix is the sum of the signatures of the two diagonal blocks. The 2×2 block introduced by the enlargement has signature zero. □

Now that we have all ingredients in hand, we are ready to give the full definition of the Alexander polynomial, which is as follows

Definition 3.3. The **Alexander polynomial** of an oriented link L, written variously as $\triangle(L)$ or $\triangle_L(x)$ depending on the required emphasis, is the determinant $det(xM - x^{-1}M^T)$ where M is any Seifert matrix for L. It is a Laurent polynomial in $\mathbb{Z}[x^{\pm 1}]$, that is a polynomial in positive and negative powers of x with integer coefficients.

Remark. Similar to the way that it was not clear the determinant and the signature are independent of the Seifert matrix M, this is also the case for the Alexander polynomial. Hence, the proof of theorem 3.1 can also be used as a proof to show it is well-defined. This polynomial is more commonly defined as $\triangle(t) = det(t^{1/2}M - t^{1/2}M^T)$ in order that it matches the value from other form of determinations. This version is related to above definition by the equality $\triangle(x) = \triangle(t^2)$, and in our version, the square roots are cancelled out. In the above definition, we have also given a normalised form of the polynomial, whereas in other ways it is only constructed up to multiplication by $\pm t^{\pm n}$.

Now in the following theorem, we will prove that this polynomial invariant, which we defined above simply as the determinant of the Seifert matrix of the link L, is indeed a link invariant.

Theorem 3.2. *Let M be a Seifert matrix constructed from a surface F spanning an oriented link L. Then $det(xM - x^{-1}M^T)$ is a link invariant.*

Proof. To prove that $\triangle(L)$ is independent of the choice of F and the basis for $H_1(F)$ that gives M, we need to check invariance under congruence and enlargement operations. Congruence is straightforward:

$$\begin{aligned} det(xP^TMP - x^{-1}(P^TMP)^T) &= det(P^T(xM - x^{-1}M^T)P) \\ &= det(P^T)det(tM - x^{-1}M^T)det(P) \\ &= det(xM - x^{-1}M^T) \end{aligned}$$

Applying Lemma 3.1 we see that an enlarged matrix has the matrix $xM - x^{-1}M^T$ in the top-left corner, and the following matrix (or its transpose) in the bottom-right:

$$\begin{pmatrix} 0 & x \\ -x^{-1} & 0 \end{pmatrix}$$

Because the only non-zero entries in the last row and the last column are in the bottom-right block, expanding the determinant gives $det(xM - x^{-1}M^T)$ as desired. [9] □

In the next chapter (Conway Polynomial), we will prove a theorem about the existence of infinite number of knots with a given polynomial. But lets take this result for granted for now. Despite this fact, instead of knots, the Alexander polynomial comes handy when distinguishing links. From the table in figure 3.1, only 5 pairs has not been solved yet: this result is true by the time which the book [9] was published in 2004, and until today (2021) some or all of these results may have been solved fully.

$$\begin{aligned} &\triangle(6_1) = \triangle(9_{46}) = -2x^2 + 5 - 2x^{-2} \\ &\triangle(7_4) = \triangle(9_2) = 4x^2 - 7 + 4x^{-2} \\ &\triangle(8_{14}) = \triangle(9_8) = -2x^4 + 8x^2 - 11 + 8x^{-2} - 2x^{-4} \\ &\triangle(8_{18}) = \triangle(9_{24}) = -x^6 + 5x^4 - 10x^2 + 13 - 10x^{-2} + 5x^{-4} - x^{-6} \\ &\triangle(9_{28}) = \triangle(9_29) = x^6 - 5x^4 + 12x^2 - 15 + 12x^{-2} - 5x^{-4} + x^{-6} \end{aligned}$$

By looking at the above examples, we can spot some general properties of the Alexander polynomial such as the symmetry of the coefficients and the even powers of x that appear throughout. In the next section we will look at some extra properties of Alexander polynomial in more detail. But lets look at an example and see how we calculate the Alexander polynomial for the unknot.

Looking at the Seifert surface of the unknot, we can see it has a disk D^2. Then cutting the exterior of the unknot around this disk will yield $D^2 \times [-1, 1]$ which can be glued countably many times to produce $X_\infty = D^2 \times \mathbb{R}$. So now the zero module $H_1(X_\infty)$ is given by the 1×1 unit matrix. The determinant of this matrix will yield what we desire. That is $\triangle_{unknot}(t) \doteq 1$. This notation means "is equal to, up to multiplication by a unit" and used by Lickorish throughout [8].

Before we close this section, we have also given a table at the end of this section which includes the calculated Alexander polynomial (in normalised form) for all the knots previously shown in Table 2.17 (These are the prime knots with upto 8 crossings). We will produce and show a similar table for all the polynomials in the chapters that will follow. Despite being able to calculate each of these polynomials one by one (which is very time consuming, and may incur errors) and put them all in a table; we decided to borrow this table from [8], as we are certain that it has been proof-checked and is from a more reliable source. We need the following Lemma for the notation used in the table.

Lemma 3.2. *For any knot K,*

$$\triangle_K(t) \doteq a_0 + a_1(t^{-1} + t) + a_2(t^{-2} + t^2) + \cdots ,$$

where the a_i are integers and a_0 is odd.

Proof. Omitted for quick progress to other results. The proof is a short one and can be found in [8]-page 58. □

The table 3.2 is going to display the values of $a_0, a_1, a_2, \cdots$ as in Lemma 3.2. The signs have been chosen so that the polynomial is normalised in a way that $\triangle_K(1) = +1$. Lets look at the knot 8_7 for example

$$\triangle_{8_7}(t) = -5 + 5(t^{-1} + t) - 3(t^{-2} + t^2) + (t^{-3} + t^3)$$

TABLE 3.2 Alexander Polynomial Table

Knot	a_0	a_1	a_2	a_3
3_1	−1	1		
4_1	3	−1		
5_1	1	−1	1	
5_2	−3	2		
6_1	5	−2		
6_2	−3	3	−1	
6_3	5	−3	1	
7_1	−1	1	−1	1
7_2	−5	3		
7_3	3	−3	2	
7_4	−7	4		
7_5	5	−4	2	
7_6	−7	5	−1	
7_7	9	−5	1	
8_1	7	−3		
8_2	3	−3	3	−1
8_3	9	−4		
8_4	−5	5	−2	
8_5	5	−4	3	−1
8_6	−7	6	−2	
8_7	−5	5	−3	1
8_8	9	−6	2	
8_9	7	−5	3	−1
8_{10}	−7	6	−3	1
8_{11}	−9	7	−2	
8_{12}	13	−7	1	
8_{13}	11	−7	2	
8_{14}	−11	8	−2	
8_{15}	11	−8	3	
8_{16}	−9	8	−4	1
8_{17}	11	−8	4	−1
8_{18}	13	−10	5	−1
8_{19}	1	0	−1	1
8_{20}	3	−2	1	
8_{21}	−5	4	−1	

Figure 3.2: The Alexander Polynomial Table. [8]

3.3 Some Properties of $\triangle_L(t)$

Now that we finally met our first polynomial invariant, in this section we are going to look at some of other properties of the Alexander polynomial. As we now know our invariant is in fact consisted of the determinant of the Seifert matrix of a link L, most of these properties are essentially the properties and rules of the determinant and manipulation of matrices. Hence, we are basically exploring some results of linear algebra and its relation to the homology theory. The following theorem looks at few of these relations.

Theorem 3.3. *If L is a link with μ components then*

$$(a)\triangle_L(-x) = (-1)^{\mu-1}\triangle_L(x).$$
$$(b)\triangle_L(x^{-1}) = (-1)^{\mu-1}\triangle_L(x).$$

Proof. As predicted, these relations are consequences of the following properties of the matrix determinant: if A is a square matrix then $det(A) = det(A^T)$ and $det(-A) = (-1)^n det(A)$ where n is the number of rows in A.

If a surface F spanning L has genus g then the Seifert matrix M for F has $2g+\mu-1$ rows.

Let $A = xM - x^{-1}M^T$. Part (a) is the relation between $det(A)$ and $det(-A)$; and Part (b) is the relation between $det(A)$ and $det(-A^T)$. [9] □

Therefore by the above theorem, as we also saw previously, the Alexander polynomial of a knot K is an even function of x and hence only powers of x^2 emerges. The symmetry of the coefficients are also apparent from the equality $\triangle_K(x^{-1}) = \triangle_K(x)$. As we predicted earlier, there are a bunch of properties of the determinant, which we could have considered separately and in thorough manner. But we have gathered all of these properties in a condense Theorem which contain all the rules and relations connecting the determinant and the Alexander polynomial. This is given in Theorem 3.4.

Theorem 3.4. *The Alexander polynomial has the following properties*

(a) If L is a link the $|\triangle_L(\pm i)| = det(L)$ where $i = \sqrt{-1}$.

(b) If K is knot then $\triangle_K(1) = 1$.

(c) If L is a link with more than one component the $\triangle_L(1) = 0$.

(d) If L is a link then $\triangle(-L) = \triangle(L) = \triangle(L^*)$.

(e) If $L_1 \sqcup L_2$ is a split link then $\triangle(L_1 \sqcup L_2) = 0$.

Proof. As part (f) and (g) of the theorem is dropped due to its irrelevance to our discussion, we will skip this proof in order to save space for the rest of the chapters. The proof is not a difficult one and can be found in [9] - 159. $\square$

These were the general properties of the Alexander polynomial, and in the next section we will briefly touch the concepts of the genus and the skein relation of the Alexander polynomial.

3.4 The Alexander Skein Relation and Genus

Despite the title, in this section we are going to briefly look at the relation between the Alexander polynomial and the genus, and separately the Alexander skein relation on its own. We know this fact, that there must be a relation between the genus of a link L and its Alexander polynomial $\triangle_L(t)$, from the fact that the definition and calculation of $\triangle_L(t)$ is based upon the spanning surfaces. First lets look at the following definition which helps us prove a spanning surface contain a minimal genus.

Definition 3.4. The **breadth** of a polynomial is the difference between its highest and lowest degrees.

Next we are going to look at a theorem which has the capability to determine the genus of all knots with up to 10 crossings. We will explain why it fails for knots with crossings of more than this limit. Lets have a look at this result first.

Theorem 3.5. *The genus of a non-split link L is bounded below by the breadth of the Alexander polynomial, as follows*

$$2g(L) + \mu(L) - 1 \geqslant \frac{1}{2} \text{ breadth } \triangle_L(x)$$

Proof. Let F be a connected minimal-genus spanning surface for L and let $r = 2g + \mu - 1$. A basis for $H_1(F)$ has r generators, so the Seifert matrix for F is an $r \times r$ square matrix. Thus the largest possible degree for x is r and the smallest degree is $-r$. Hence $\mathit{breadth}\triangle_L(x) \leqslant 2r$.

Now we have

$$
\begin{aligned}
2g(F) &= 2 - \chi(F) - \mu(L) \\
&= 2r - \mu(L) + 1 \\
&= \frac{1}{2}\ breadth\ \triangle_L(x) - \mu(L) + 1\ .
\end{aligned}
$$
[9]

$\square$

The reason why the above theorem fails with knots of 11 crossing or more, is that if we consider a knot such as the one in figure 3.3, namely the Kinoshita-Terasaka knot, then we have the crossing number $c(K) = 11$, but the Alexander polynomial $\triangle(K) = 1$. As this knot is considered to be non-trivial, we can prove it must have genus at least 1. However, it has been shown in [17] that it has $g(K) = 2$. Lets look at another knot, which we have see already, that is knot 10_{165} in figure 2.15. The Alexander polynomial of this knot is as follows.

$$\triangle(10_{165}) = 3x^4 - 11x^2 + 17 - 11x^{-2} + 3x^{-4}$$

In figure 2.15, the projection surface has genus 3, but the deduced bound from the Alexander polynomial is equal to 2. Looking at the figure, we can show $g(10_{165}) = 2$ by observing loop b is bounding a compressing disk for the projection surface.

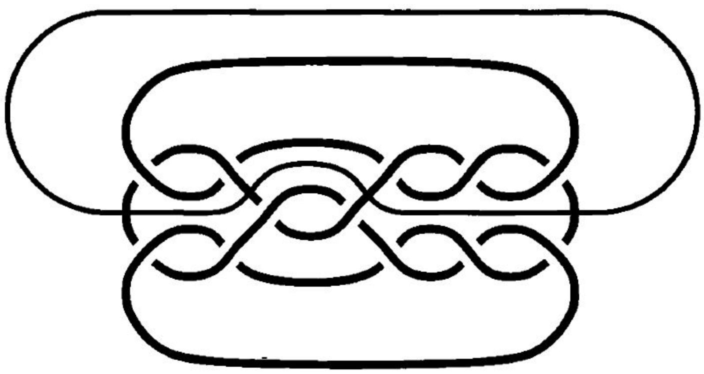


Figure 3.3: The Kinoshita-Terasaka Knot with 11-crossings [9]

Lets now go back to figure 2.11, we can also label this figure in a way that they are named $D+, D_0, D_-$ respectively. We call changing D_+ to D_- or the other way round, as **switching** a crossing. We also call what the figure is actually showing, namely changing D_+ or D_- to D_0, as **smoothing** a crossing. This notion will come into play later when we define the Conway polynomial. The moves of figure 2.11 which we just emphasised has the ability to change a link type, and the smoothing operation will always increase or decrease the number of components by one. There remain only one main result, before we close this chapter, which is the following theorem. The links in this theorem are labelled according to this new notion.

Theorem 3.6. *If three oriented links L_+, L_- and L_0 have diagrams D_+, D_-, and D_0 which differ only in a small neighbourhood as shown in* 2.11 *then*

$$\triangle(L_+) - \triangle(L_-) = (x^{-1} - x)\triangle(L_0)$$

Proof. A sketch, the proof is done by constructing projection surfaces from the three diagrams, and then verifying the connectivity of these. At last, we create Seifert matrix for each of these surfaces and expand the determinants to arrive at the desired result shown. Nevertheless, the complete proof can be found in [9] - page 162. □

The above theorem is an example of a **skein relation** which displays the connection between these three links and their diagrams. We will use this skein relation to define an axiomatic definition of the Conway polynomial in the next chapter.

4 Conway Polynomial $\nabla_L(z)$

4.1 Historical Overview

For the history section, we have used [5] extensively - specific page of John Conway - in order to gain some useful information about his personal and mathematical life. John Horton Conway was born on 26 December 1937 in Liverpool, he became interested in becoming a mathematician from a young age and managed to get admitted at Cambridge University, where he completed the Tripos and got his BA in 1959. He began research under Harold Davenport in Number Theory, and was awarded his doctorate in 1964. He stayed in Cambridge until 1986, and moved to Princeton University to hold John von Neumann professorship for the rest of his mathematical career.

Throughout his entire career, he left his name in a number of mathematical fields, from Algebra to Combinatorics, and from Geometric Topology to Theoretical Physics, and few other main areas. But for the purpose of this report, we wish to know more about what he did in Knot Theory. As we could have guessed from the title of this section, he invented a new variation of the Alexander Polynomial, which is named after him as Conway Polynomial [14]. He also developed "Tangle Theory", which is a system of notation for tabulating knots and links. Today, this is known as "Conway notation". He managed to correct a number of errors in the knot tables from 19th century and extend this to include all but four of the non-alternating prime knots with 11 crossings. In knot theory, we also have a specific knot with 11 crossings named after him.

In brief, the Conway polynomial for an oriented link is just the Alexander polynomial without puzzlement of multiplication by units of $\mathbb{Z}[t^{-1}, t]$. This seemingly little change, will enable us to add two such polynomials, otherwise it would be undefined if the signs were in doubt. In the next section, we will look at the Conway polynomial in detail and use the skein relation in the previous chapter to define it mathematically. If the space allows us, we may also look into Conway notation and show you how it can be used to tabulate knots, otherwise we will skip it. At last, we should admit the truth, despite being very bitter, that how the life on earth today has faced very harsh consequences. Unfortunately, John Conway passed away from Covid-19 symptoms on 11 April 2020 at the age of 82, may god bless his soul.

4.2 Formulating the Conway Polynomial

As we mentioned earlier, the Conway polynomial is simply just the Alexander polynomial without the ambiguity of multiplication by units of $\mathbb{Z}[t^{-1}, t]$. This in turn enables us to add two of these polynomials without worrying about the signs. The skein relation which we defined at the end of the previous section, namely from Theorem 3.6, is now going to be used as a basis of an axiomatic definition which is given below. Fortunately, in this section we do not need to give many of the required definitions and notions, as we have done this in Chapter 2, and explored it further in Chapter 3. Hence we can and will jump directly into the current discussion.

Definition 4.1. The **Conway polynomial** of an oriented link L, denoted variously by $\nabla(L)$ or $\nabla_L(z)$ depending on the required emphasis, is defined by the three following axioms.

1. Invariance: $\nabla_L(z)$ is invariant under ambient isotopy of L.
2. Normalisation: if K is the trivial knot (unknot), then $\nabla_K(z) = 1$.
3. Skein relation: $\nabla(L_+) - \nabla(L_-) = z\nabla(L_0)$ where L_+, L_- and L_0 have diagrams D_+, D_- and D_0 which differ as shown in Figure 2.11.

It is a polynomial in $\mathbb{Z}[z]$.

We can compute the Conway polynomial recursively by using the above three axioms. This means we can rewrite the skein relation in the following two form.

$$\nabla(L_+) = \nabla(L_-) + z\nabla(L_0)$$
$$\nabla(L_-) = \nabla(L_+) + z\nabla(L_0)$$

Switching a crossing enables us to convert any knot or link into a trivial knot or link. The other process, which is smoothing, allows us to reduce the number of crossings, and hence by an inductive process arrive at more simpler knots and links. This procedure will eventually leads us to produce the Conway polynomial of a knot or link, in terms of other trivial knots or links. We have looked into this procedure in the following example, which also has a very clear calculation given in figure 4.1. This example looks into the Conway polynomial of the knot 4_1, and shows how it can be simplified and written in terms of other knots and links.

We are going to apply the three axioms in the above definition to this example. The diagram in figure 4.1 is in fact showing a resolving **tree** which is simply showing different levels of the knot as it simplifies further and further at each stage, at the top of this tree we have the original diagram of the knot 4_1 which we wish to calculate its Conway polynomial. At the first stage, we apply Axiom 3 to the positive crossing at the top right of the diagram. The outcome of switching, and smoothing is then shown at the next level of the tree diagram. Axiom 1 implies that we can simplify these diagrams further, without altering their polynomials. By switching once more, the left hand side of the tree arrives at the isotopic unknot. Axiom 2 applied to this part, implies $\nabla(z) = 1$ and hence we stop simplying any further. By smoothing once more, the right hand side of the tree, arrives at the Hopf link and will be simplified further as shown in the figure. As the polynomial remains unknown, we apply Axiom 3 again. At the last stage, we use a negative crossing to produce the products of two component trivial link, and another unknot. The ending (or the terminal nodes) of the tree diagram are now displaying the known trivial links.

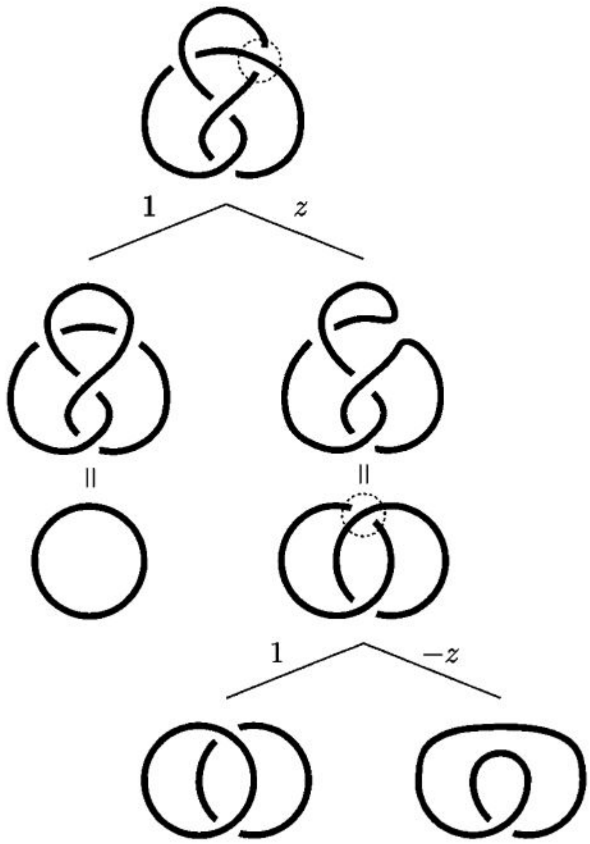


Figure 4.1: A resolving tree for the Conway polynomial of 4_1. [9]

We will label each of the right-hand branches of the tree diagram with $\pm z$ based on whether a positive or negative crossing was smooth or not. Then the Conway polynomial is constructed by combining the polynomials of the links which we obtained at the terminal nodes. For the example which we explored above, this will be as follows.

$$\nabla(4_1) = 1 + z(0 - z \cdot 1) = 1 - z^2$$

In the following theorem, we will show that the Conway polynomial of a split link is zero.

Theorem 4.1. *If L is a split link then $\nabla(L) = 0$.*

Proof. Let D_0 be a disconnected diagrams of the split link $L_1 \sqcup L_2$ arranged so that projections of L_1 and L_2 are disjoint, and so that the neighbourhood defining D_0 contains one arc from each L_i. Then D_+ will be arranged as shown in figure 4.2, and D_- is obtained by turning the right half of the diagram through a full twist. Therefore D_+ and D_- are isotopic and, by axiom 1, $\nabla(D_+) = \nabla(D_-)$. Applying axiom 3 completes the argument:

$$z\nabla(D_0) = \nabla(D+) - \nabla(D_-) = 0.$$ [9]

□

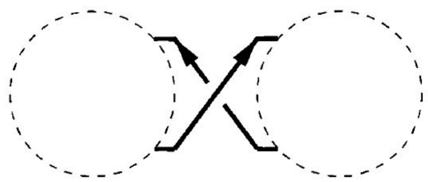


Figure 4.2: [9]

We require the Axioms I-III given in the definition of the Conway polynomial to be consistent. Otherwise, we may obtain different results from the computations made on a single link. The invariance of the Conway polynomial under Reidmeister moves, and few other conditions has been established through combinatorial ways in [18]. But if we let z to be $x^{-1} - x$, we get back to the Alexander polynomial in which case the axioms still hold. Once we allow our link L to be replaced by a knot K, we still get a genuine polynomial $\nabla_K(z)$ in z^2 with positive powers only and always 1 as the constant term. With these limitations, we can get hold of any possible polynomial in terms of some Conway polynomial of some knot K.

The following diagram shown in figure 4.3, does not have a specific name, and is not a wild knot, but it has been defined sequentially in a way that induction can be applied to n, and it can be kept repeated from a_1 to a_n where n is any natural number. We will use this digram and its proof in a number of results that will follow.

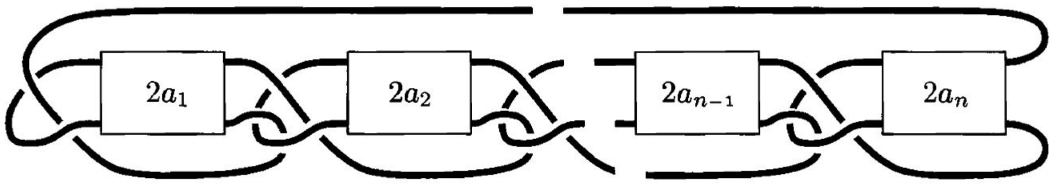


Figure 4.3: [9]

In the following Proposition, we have defined the Conway polynomial for the above knot.

Proposition 4.1. *The knot K shown in Figure 4.3 has Conway polynomial*

$$\nabla_K(z) = 1 + (-1)^{n+1} a_n z^{2n} + \sum_{i=1}^{n-1} (-1)^{i+1}(a_i - 1) z^{2i}$$

Proof. The proof is short and proceeds by induction on n. Concentrate on the left-hand end of the knot. First reduce the number of twists in box a_1: switching a crossing reduces the number of twists, smoothing a crossing produces a Hopf link. The proof needs to cope with the cases of a_i positive, negative, and zero. [9] □

Next we will use the above diagram in the proof of the following theorem which states that we have infinitely many knots with a given Conway polynomial.

Theorem 4.2. *There are infinitely many knots with a given Conway polynomial.*

Proof. Let K_A be the knot shown in Figure 4.3 with the given polynomial, and let K_B be the untwisted double of the figure-8 knot. Our example will be formed by taking the products

$$K_A, K_A \# K_B, K_A \# K_B \# K_B, K_A \# K_B \# K_B \# K_B, \cdots$$

These knots all have the same Conway polynomial since $\nabla(K_B) = 1$ and, by Theorem 3.4 (e), $\nabla(K_A \# K_B) = \nabla(K_A)\nabla(K_B)$.

To show that the knots are distinct, we show they have different genera. The figure-8 knot is shown to be non-trivial by its determinant. Theorem 4.4.1 of

[9] shows that the double of a non-trivial knot is non-trivial. A non-trivial knot has genus at least 1, and a double knot has a genus-1 spanning surface, hence $g(K_B) = 1$. Applying Theorem 5.6.1 of [9] we see that the genus of the product of K_A with n copies of K_B is $g(K_A) + n$. [9] □

The knots we made in the above theorem are not prime nor simple. It has been shown that we can construct infinitely many simple knots with a given Conway polynomial in [19]. However, in order to show the knots are distinct remain a more difficult task obviously.

We will have the following three Corollaries before we close this section and chapter.

Corollary 4.1. *There is a knot with unknotting number 1 and any given Conway polynomial.*

Proof. The Kinoshita-Terasaka knot shown in Figure 3.3 has Conway polynomial $\nabla(z) = 1$ and unknotting number 1 (switch the central crossing). For other polynomials, we use the knot in Figure 4.3: switching one of the two left-most crossings produces the trivial knot (unknot).[9] □

This final result touches the idea of a genus of the knot in figure 4.3.

Corollary 4.2. *There are infinitely many knots with a given genus.*

Proof. When $a_n \neq 0$, the knot shown in Figure 4.3 has genus n. [9] □

Corollary 4.3. *Knots with unknotting number 1 can have arbitrarily high genus.*

Proof. Easy exercise. □

The above results show that the unknotting number has no direct connection to the genus or the Conway polynomial. This is a usual way for verifying the independence of invariants in knot theory. At the end of this section, we have given a table similar to the one we have seen for Alexander polynomial, that is the Conway polynomials for knots of up to 9 crossings. It is using a different notation, however we will see the previous notation again for the Jones polynomial. This is provided in figure 4.4, and shows the actual calculated polynomials. As we wish to explore other polynomial invariants in this report, we will skip going into topics such as Resolving Trees (which is the method of simplifying the knots as seen in figure 4.1), and Homogeneous Links (which is relating the Conway polynomial to other diagram-based constructions like Seifert's algorithm). Hence, we will close this chapter here and will move to our next invariant, namely the Jones Polynomial.

Conway polynomial

3_1	$1+z^2$	8_6	$1-2z^2-2z^4$
4_1	$1-z^2$	8_7	$1+2z^2+3z^4+z^6$
5_1	$1+3z^2+z^4$	8_8	$1+2z^2+2z^4$
5_2	$1+2z^2$	8_9	$1-2z^2-3z^4-z^6$
6_1	$1-2z^2$	8_{10}	$1+3z^2+3z^4+z^6$
6_2	$1-z^2-z^4$	8_{11}	$1-z^2-2z^4$
6_3	$1+z^2+z^4$	8_{12}	$1-3z^2+z^4$
7_1	$1+6z^2+5z^4+z^6$	8_{13}	$1+z^2+2z^4$
7_2	$1+3z^2$	8_{14}	$1-2z^4$
7_3	$1+5z^2+2z^4$	8_{15}	$1+4z^2+3z^4$
7_4	$1+4z^2$	8_{16}	$1+z^2+2z^4+z^6$
7_5	$1+4z^2+2z^4$	8_{17}	$1-z^2-2z^4-z^6$
7_6	$1+z^2-z^4$	8_{18}	$1+z^2-z^4-z^6$
7_7	$1-z^2+z^4$	8_{19}	$1+5z^2+5z^4+z^6$
8_1	$1-3z^2$	8_{20}	$1+2z^2+z^4$
8_2	$1-3z^4-z^6$	8_{21}	$1-z^4$
8_3	$1-4z^2$	9_1	$1+10z^2+15z^4+7z^6+z^8$
8_4	$1-3z^2-2z^4$	9_2	$1+4z^2$
8_5	$1-z^2-3z^4-z^6$	9_3	$1+9z^2+9z^4+2z^6$
9_4	$1+7z^2+3z^4$	9_{27}	$1-z^4-z^6$
9_5	$1+6z^2$	9_{28}	$1+z^2+z^4+z^6$
9_6	$1+7z^2+8z^4+2z^6$	9_{29}	$1+z^2+z^4+z^6$
9_7	$1+5z^2+3z^4$	9_{30}	$1-z^2-z^4-z^6$
9_8	$1-2z^4$	9_{31}	$1+2z^2+z^4+z^6$
9_9	$1+8z^2+8z^4+2z^6$	9_{32}	$1-z^2+z^6$
9_{10}	$1+8z^2+4z^4$	9_{33}	$1+z^2-z^6$
9_{11}	$1+4z^2-z^4-z^6$	9_{34}	$1-z^2-z^6$
9_{12}	$1+z^2-2z^4$	9_{35}	$1+7z^2$
9_{13}	$1+7z^2+4z^4$	9_{36}	$1+3z^2-z^4-z^6$
9_{14}	$1-z^2+2z^4$	9_{37}	$1-3z^2+2z^4$
9_{15}	$1+2z^2-2z^4$	9_{38}	$1+6z^2+5z^4$
9_{16}	$1+6z^2+7z^4+2z^6$	9_{39}	$1+2z^2-3z^4$
9_{17}	$1-2z^2+z^4+z^6$	9_{40}	$1-z^2-z^4+z^6$
9_{18}	$1+6z^2+4z^4$	9_{41}	$1+3z^4$
9_{19}	$1-2z^2+2z^4$	9_{42}	$1-2z^2-z^4$
9_{20}	$1+2z^2-z^4-z^6$	9_{43}	$1+z^2-3z^4-z^6$
9_{21}	$1+3z^2-2z^4$	9_{44}	$1+z^4$
9_{22}	$1-z^2+z^4+z^6$	9_{45}	$1+2z^2-z^4$
9_{23}	$1+5z^2+4z^4$	9_{46}	$1-2z^2$
9_{24}	$1+z^2-z^4-z^6$	9_{47}	$1-z^2+2z^4+z^6$
9_{25}	$1-3z^4$	9_{48}	$1+3z^2-z^4$
9_{26}	$1+z^4+z^6$	9_{49}	$1+6z^2+3z^4$

Figure 4.4: The Conway Polynomial Table. [9]

5 Jones Polynomial $V_L(t)$

5.1 Historical Overview

Similar to the last chapter, again we have used source [5] extensively - specific page of Vaughan Jones - to gain an insight into the personal and mathematical life of this mathematician. Sir Vaughan Frederick Randal Jones was born on 31 December 1952 in Gisborne, New Zealand. He completed his undergraduate studies at University of Auckland, and gained his BSc and MSc in 1972 and 1973 respectively. He completed his postgraduate studies at the University of Geneva, and was awarded his PhD in 1979. He moved to US in the 80's and held various professorships at different universities until 2011, where he taught at Vanderbilt University until his death (on 6 September 2020).

He is best known for the polynomial invariant named after him, called the Jones Polynomial. His discovery came from an unexpected area in analysis, where he was working on von Neumann algebras. His polynomial led to the solution of a number of classical problems in knot theory, and increased interest in low-dimensional topology, as well as development of quantum topology. As mentioned earlier, it would have been easier for us to establish Jones polynomial first rather than Alexander polynomial. We will see if it is easier to work with or not later on.

It is a Laurent polynomial with integer coefficients that can be associated to every knot and link. The Jones polynomial has been developed into a theory and is more generalised, as it has connections with quantum theory through invariants for three-dimensional manifolds (we may open up this topic at the end of our discussion if time and space allows).The whole theory is defined upon the fact, that if a knot diagram is changed by some Reidemeister move, the Jones polynomial stays the same. Hence, if two links can be shown to have different polynomials (through their diagrams), then they are certainly distinct.

Despite the simplicity and its powerful theory, it is remarkable that it remained unknown until 1984 [20]. In May 1984, he travelled to New York, and along with Joan Birman showed that this polynomial is not a new variant of the Alexander polynomial, and is in fact a new invariant, proved by a simple test of distinguishing the left-handed and right-handed trefoils. In order to start our discussion (defining Jones Polynomial), we will need to introduce a different polynomial called the bracket polynomial introduced by L.H.Kauffman in [21].

5.2 Jones Polynomial, A New Approach

As we indicated at the end of the previous section, we will have to introduce the Kauffman bracket polynomial in order to define the Jones polynomial. It is defined as follows.

Definition 5.1. The **bracket polynomial** of an unoriented diagram D, denoted by $\langle D \rangle$, is a Laurent polynomial in a single variable A defined by the three following axioms.

1. Normalisation: $\langle \bigcirc \rangle = 1$ where $\bigcirc$ denotes the diagram of one component and no crossings.
2. Delta: $\langle D \sqcup \bigcirc \rangle = \delta \langle D \rangle$ where $\delta = -A^{-2} - A^2$ and $D \sqcup \bigcirc$ denotes the diagram D together with a single component that does not cross itself or D.
3. Skein relation: $\langle D_+ \rangle = A\langle D_0 \rangle + A^{-1}\langle D_\infty \rangle$ where diagrams D_+, D_0 and D_∞ differ as shown in figure 5.1.

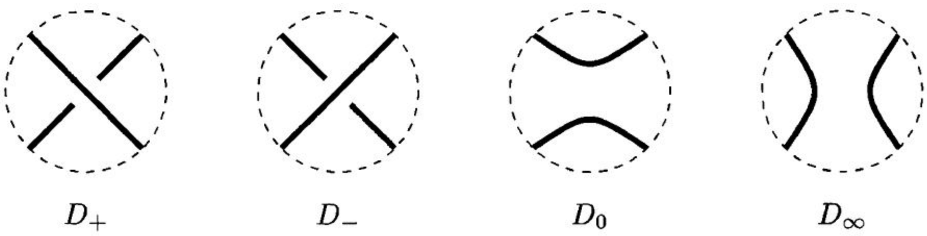


Figure 5.1: Four unoriented diagrams [9]

Remark. There are several important points we have to bear in mind about this definition. First of all, this definition of polynomial is based on the diagram of knot, and not on the knot or link itself. This is evident from the missing invariance axiom (part 1) of Definition 4.1 we gave for Conway polynomial. Secondly, the oriented diagrams we gave in figure 2.11 are different to the ones we gave in figure 5.1, as the latter one are only defined in relation to each other. Thirdly, adding this unoriented condition with the skein relation produces $\langle D_- \rangle = A\langle D_\infty \rangle + A^{-1}\langle D_0 \rangle$. Finally, the unknot in the second condition can be placed anywhere on the plane.

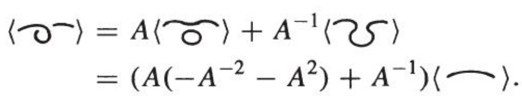
Lets now give the definition of what we mean by writhe of an oriented diagram.

Definition 5.2. The **writhe** $w(D)$ of a diagram D of an oriented link is the sum of the signs of the crossings of D, where each crossing has sign $+1$ or -1 as defined (by convention) in Figure 2.11.

We can compute $\langle D \rangle$ recursively by repeatedly applying the skein relation to transform a diagram into other diagrams with no crossings, and hence the value of each of these diagrams can be calculated by using axiom 1 and axiom 2. Now in order to complete the construction of the bracket polynomial, we need to show it is well -defined under all three Reidemeister moves via the following two lemmas.

Lemma 5.1. *If a diagram D is changed by a Type I Reidemeister move, its bracket polynomial changes in the following way:*

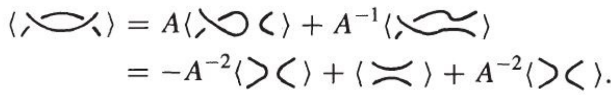
Proof. From the diagram, we have



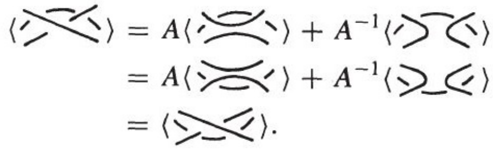
That produces the first equation; the second follows in the same way. [8] □

Lemma 5.2. *If a diagram D is changed by a Type II or Type III Reidemeister move, then $\langle D \rangle$ does not change. That is*

Proof. For the first part we have



and for the second part



Here the second line follows from the first by using (i) twice. [8]

□

We now have a theorem that gives us the main ingredient of the Jones polynomial, which is as follows.

Theorem 5.1. *Let D be a diagram of an oriented link L. Then the normalised bracket polynomial*

$$\widetilde{V}_D(A) = (-A)^{-3w(D)}\langle D\rangle$$

is an invariant of the oriented link L.

Proof. It follows from Lemma 5.2 that the given expression is unchanged by Reidemeister moves of Type II and III; Lemma 5.1 and definition of $w(D)$ show it is unchanged by a Type I move. As any two diagrams of two equivalent links are related by a sequence of such moves, the result follows at once. [8] $\square$

Now we can present the main definition of this chapter, as follows.

Definition 5.3. The **Jones polynomial** $V(L)$ of an oriented link L is the Laurent polynomial in $t^{1/2}$, with integer coefficients, defined by

$$V(L) = \left((-A)^{-3w(D)}\langle D\rangle\right)_{t^{1/2}=A^{-2}} \in \mathbb{Z}[t^{-1/2}, t^{1/2}]$$

where D is any oriented diagram for L.

From the proof of Theorem 5.1, we can also conclude the Jones polynomial in the above definition is an invariant under all three Reidemeister moves and hence well-defined. We also did not provide specific diagrams of the behaviour of the bracket polynomial under Type I,II, and III moves, as with some attention to details, these are provided in the proof of Lemma 5.1 and Lemma 5.2. Before we move on to look at some properties of the Jones polynomial, and its connection with the normalised bracket polynomial, lets look at the simplicity of this route in overall in compare to the Alexander polynomial.

As we saw in previous chapters, in order to build the Alexander polynomial, we had to go through obtaining a spanning surface, and then apply the homology theory to produce the Seifert matrices, and finally prove the invariance via surgery of surfaces. However, via the route of bracket polynomial, we proved the invariance of Jones polynomial in just few pages. In the next section, we will briefly mention why this polynomial is more powerful in distinguishing knots than the Alexander polynomial. Then we will show some extra properties of $V(L)$.

5.3 Some Properties of $V(L)$

Lets first unveil some of the features and power of the Jones polynomial which we did not touch in the previous section. As we mentioned earlier, the two table of knots we saw previously for the Alexander and Conway polynomials which was showing prime knots of upto order 8 and 9 respectively, had five pairs unresolved (Please refer to [9] - page 158). The normalised bracket polynomial can differentiate all of these five pairs and hence complete the proof of classification of these prime knots (up to order 9), and show they are in fact distinct knots. In addition to this, the Jones polynomial can also distinguish some difficult polygonal knots (such as 8_{20} and $3_1\#3_1^*$ - Look at [9] Figure 1.6, page 6 for details), and complete the classification of this class of knot type.

Despite the power and simplicity of the Jones polynomial, there remain some counter examples to disprove this claim. One example is the 8_9 and $4_1\#4_1$ which have the same Jones polynomial but different $\triangle_L(t)$ polynomials. Another example which produce the same polynomials (both $V(L)$ and $\triangle_L(t)$) for two knots, are 8_8^* and 10_{129}. Lets denote $\bar{D}$ as the reflection diagram of an oriented link with diagram D. Then the following holds for the writhe of the diagram D.

$$w(D) = -w(\bar{D}) \Rightarrow \langle \bar{D} \rangle = \overline{\langle D \rangle}$$

This means that by switching $t^{1/2}$ and $t^{-1/2}$ we can obtain $V(\bar{L})$ from the oriented link L, where $\bar{L}$ is the reflection of the link L. Lets now look at an example to clarify this result. By calculating the bracket polynomial of the right-handed, and its reflection, left-handed trefoil knot 3_1, we get two different Jones polynomials. These are

$$\begin{aligned} V(3_1^*) &= -t^4 + t^{-3} + t \\ V(3_1) &= -t^{-4} + t^{-3} + t^{-1} \end{aligned}$$

This difference in polynomials indicates that these two knots are distinct knots. In another words, we can say the 3_1 is not amphicheiral. Lets now look at two theorems that give some more detail about the bracket polynomial and its relation with the Jones polynomial, before we close this chapter with a table of knots, similar to the ones we saw before for the Alexander polynomial and the Conway polynomial. This table is using the same notation as the $\triangle_L(t)$.

Theorem 5.2. *The normalised bracket polynomial $\widetilde{V}_D(A)$ has the following properties.*

(a) If L is a link then $\widetilde{V}(-L) = \widetilde{V}(L)$.

(b) If L is a link then $\widetilde{V}(L^*)(A) = \widetilde{V}(L)(A^{-1})$.

(c) If $L_1 \sqcup L_2$ is a split link then $\widetilde{V}(L_1 \sqcup L_2) = \delta \widetilde{V}(L_1)\widetilde{V}(L_2)$.

(d) If a link can be factorised as $L_1 \# L_2$ then $\widetilde{V}(L_1 \# L_2) = \widetilde{V}(L_1)\widetilde{V}(L_2)$.

Proof. The proof is left as an exercise.[9] □

Now lets finally look at the equivalence of Kauffman's combinatorics and the Jones algebra through this final theorem of this section.

Theorem 5.3. *The normalised bracket polynomial is equivalent to the Jones polynomial under a simple change of variable:*

$$V_L(A^{-4}) = \widetilde{V}_L(A)$$

Proof. From the bracket skein relation we have

$$\begin{aligned} \langle D_+ \rangle &= A\langle D_0 \rangle + A^{-1}\langle D_\infty \rangle, \\ \langle D_- \rangle &= A^{-1}\langle D_0 \rangle + A\langle D_\infty \rangle. \end{aligned}$$

Multiplying by A and A^{-1} respectively and subtracting gives

$$A\langle D_+ \rangle - A^{-1}\langle D_- \rangle = (A^2 - A^{-2})\langle D_0 \rangle$$

These three diagrams can be oriented consistently with each other so we can compare their writhes: $w(D_+) - 1 = w(D_0) = w(D_-) + 1$. The definition of $\widetilde{V}$ can be rearranged to give $\langle D \rangle(A) = (-A^3)^{w(D)}\widetilde{V}_D(A)$. Combining this with the last displayed equation gives

$$(-A^3)A\widetilde{V}(D_+) - (-A^{-3})A^{-1}\widetilde{V}(D_-) = (A^2 - A^{-2})\widetilde{V}(D_0)$$

and hence

$$A^4\widetilde{V}(D_+) - A^{-4}\widetilde{V}(D_-) = (A^2 - A^{-2})\widetilde{V}(D_0)$$

.

This is the skein relation of the Jones polynomial with the substitution given in the statement of the theorem. Both V and $\widetilde{V}$ are normalised to be 1 on the trivial knot (unknot). [9] □

TABLE 5.2. Jones Polynomial Table

3_1	-1	1	0	1	**0**						
4_1	1	-1	**1**	-1	1						
5_1	-1	1	-1	1	0	1	0	**0**			
5_2	-1	1	-1	2	-1	1	**0**				
6_1	1	-1	1	-2	**2**	-1	1				
6_2	1	-2	2	-2	2	**-1**	1				
6_3	-1	2	-2	**3**	-2	2	-1				
7_1	-1	1	-1	1	-1	1	0	1	0	0	**0**
7_2	-1	1	-1	2	-2	2	-1	1	**0**		
7_3	**0**	0	1	-1	2	-2	3	-2	1	-1	
7_4	**0**	1	-2	3	-2	3	-2	1	-1		
7_5	-1	2	-3	3	-3	3	-1	1	0	**0**	
7_6	-1	2	-3	4	-3	3	**-2**	1			
7_7	-1	3	-3	**4**	-4	3	-2	1			
8_1	1	-1	1	-2	2	-2	**2**	-1	1		
8_2	1	-2	2	-3	3	-2	2	-1	**1**		
8_3	1	-1	2	-3	**3**	-3	2	-1	1		
8_4	1	-2	3	-3	3	**-3**	2	-1	1		
8_5	**1**	-1	3	-3	3	-4	3	-2	1		
8_6	1	-2	3	-4	4	-4	3	**-1**	1		
8_7	-1	2	**-2**	4	-4	4	-3	2	-1		
8_8	-1	2	-3	**5**	-4	4	-3	2	-1		
8_9	1	-2	3	-4	**5**	-4	3	-2	1		
8_{10}	-1	2	**-3**	5	-4	5	-4	2	-1		
8_{11}	1	-2	3	-5	5	-4	4	**-2**	1		
8_{12}	1	-2	4	-5	**5**	-5	4	-2	1		
8_{13}	-1	2	-3	5	-5	**5**	-4	3	-1		
8_{14}	1	-3	4	-5	6	-5	4	**-2**	1		
8_{15}	1	-3	4	-6	6	-5	5	-2	1	0	**0**
8_{16}	-1	3	-5	6	-6	6	**-4**	3	-1		
8_{17}	1	-3	5	-6	**7**	-6	5	-3	1		
8_{18}	1	-4	6	-7	**9**	-7	6	-4	1		
8_{19}	**0**	0	0	1	0	1	0	0	-1		
8_{20}	-1	1	-1	2	-1	**2**	-1				
8_{21}	1	-2	2	-3	3	-2	2	**0**			

Figure 5.2: The Jones Polynomial Table. Bold entries are coefficients of t^0. [8]

6 HOMFLY and Kauffman Polynomials

6.1 Historical Overview

We have decided to combine this last two chapters into one chapter, as we will only penetrate into the Kauffman polynomial and would not go very deep into the subject, and hence it would be more feasible to attach this last polynomial at the end of HOMFLY polynomial chapter. But we will give a combined and comprehensive historical overview of both polynomials here. The first polynomial, namely HOMFLY, has a very interesting story and history where we see the fairness and justice of mathematics towards recognizing and appreciating the work of independent mathematicians working and achieving the same results.

Once Jones managed to demonstrate clear connection between the skein relation of his new polynomial and the Alexander polynomial, several independent mathematicians (and some in teams) started to look for a generalisation of the Jones polynomial as a linear skein invariant for oriented knots and links. These were: Jim **H**oste [22],Adrian **O**cneanu [23],Ken **M**illet [18], Peter **F**reyd [24], Raymond **L**ickorish [18], David **Y**etter [24], and Jozef Przytycki and Pawel Traczyk [25], respectively. Once the American Mathematical Society (AMS) received these works within some time of each other, it requested them to prepare a joint statement [26] and the name of this new polynomial, as you could guess from the bold emphasis above, formed an acronym from the initials of these mathematicians. It is a two variable polynomial $P_L(v, z)$, and has several form of skein relation, but all of these are equivalent via change of variable. In the following section, we will mostly refer to HOMFLY polynomial az $P(L)$. One common and general form of these skein relations for $P(L)$ is the following.

$$vP(L_+) - v^{-1}P(L_-) = zP(L_0)$$

We already saw the bracket polynomial $\widetilde{V}_D(A)$ discovered by Louis Kauffman in the last section. Just for clarity, we have given its skein relation again, which is as follows.

$$\langle D_+ \rangle = A\langle D_0 \rangle + A^{-1}\langle D_\infty \rangle$$

The Kauffman polynomial of our interest in this section is a different polynomial, and before we get to it, we need to introduce a different polynomial named as Absolute polynomial $Q_L(x)$. Briefly and roughly, this polynomial was discovered by Robert Brandt, Lickorish and Millet [27], and separately by Chi Fai Ho [28,29].This link invariant has the following skein relation.

$$Q(D_+) + Q(D_-) = x[Q(D_0) + Q(D_\infty)]$$

The Kauffman polynomial was discovered by Louis Kauffman in 1987 [30]. $F_L(a, x)$ or as we will call it in the following section as $F(L)$, is a 2-variable generalisation of the Absolute polynomial $Q_L(x)$ we mentioned above. It has the following skein relation.

$$F(D_+) + F(D_-) = z(D_0) + z(D_\infty)$$

The Kauffman polynomial is obtained by an additional axiom for the Type I Reidemeister move, and has used the writhe of the diagram for the normalisation of the final result. All the skein relations we gave above, are using notation of diagrams we gave in figure 5.1. In the next section we will look at the relation of this polynomial with the HOMFLY, and the other previous ones. We will also look at some of the properties of these two polynomials.

6.2 Exploring $P(L)$ and $F(L)$

Before we start looking at some of common properties of the HOMFLY and Kauffman polynomials, we will have to define them precisely and rigorously. First we will give the axiomatic definition of the HOMFLY polynomial, and the Kauffman is then defined through a function.

Definition 6.1. The **Homfly polynomial** of an oriented link L, denoted variously by $P(L)$ or $P_L(v, z)$ depending on the required emphasis, is defined by the three following axioms.

1. Invariance: $P_L(v, z)$ is invariant under ambient isotopy of L.
2. Normalisation: of K is the trivial knot (unknot) then $P_K(v, z) = 1$.
3. Skein relation: $v^{-1}P(L_+) - vP(L_-) = zP(L_0)$ where L_+,L_- and L_0 have diagrams D_+, D_- and D_0 which differ as shown in Figure 2.11.

   We can calculate this polynomial using similar method we used in Figure 4.1 for the Conway polynomial, which is by resolving trees and simplifying the knot diagrams until they are showing trivial knots at the terminal nodes.

   We can then apply the skein relation in the following ways.

$$P(L_+) = v^2P(L_) + vzP(L_0),$$
$$P(L_-) = v^{-2}P(L_+) - v^{-1}zP(L_0).$$

   The following relationship is required to complete the calculation.

4. Delta: $P(L_1 \sqcup L_2) = \delta P(L_1)P(L_2)$ where $\delta = \frac{v^{-1}-v}{z}$.

   Thus the polynomial of an n-component trivial link is δ^{n-1}.

We can either prove the invariance of $P(L)$ through algebraic methods or via combinatorial routes, one can be found in [26], but the proof is long, and we will skip it, but we will provide the claim in form of a theorem.

Theorem 6.1. *The HOMFLY polynomial is a well-defined invariant of oriented links that takes values in $\mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ or $\mathbb{Z}[v^{\pm 1}, z, \delta]$, depending on whether or not we choose to expand the factors of δ.*

Proof. Refer to [26] for details and method of this proof. $\square$

By substitution, the Alexander, Conway, and Jones polynomials can be recovered, as follows.

$$\begin{aligned} \triangle_L(x) &= P_L(1, x^{-1} - x), \\ \nabla_L(z) &= P_L(1, z), \\ V_L(t) &= P_L(t, t^{-1/2} - t^{1/2}) \end{aligned}$$

Next we need to define the Kauffman polynomial, and for this we first need the result of the following theorem.

Theorem 6.2. *There exists a function*

$$\Lambda \ : \ \{\textit{Unoriented link diagrams in } S^2\} \to \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$$

that is defined uniquely by the following:

(i) $\Lambda(U) = 1$, *where* U *is the zero-crossing diagram of the unkot;*

(ii) $\Lambda(D)$ *is unchanged by Type II, and III Reidemeister moves on the diagram* D*;*

(iii) $\Lambda($  $) = a\Lambda($  $);$

(iv) If D_+, D_-, D_0 *and* D_∞ *are four diagrams exactly the same except with switched crossings, and near a point where they are as shown in Figure 5.1, then*

$$\Lambda(D_+) + \Lambda(D_-) = z(\Lambda(D_0) + \Lambda(D_\infty))$$

Proof. This theorem has a very long proof, and we have omitted it due to lack of space. But it can be found in [8] - p.174. □

Now we are ready to give the full definition of the Kauffman polynomial, which is given below

Definition 6.2. The **Kauffman polynomial** is the function

$$F \ : \ \{\textit{Oriented links in } S^3\} \to \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$$

defined by $F(L) = a^{-w(D)}\Lambda(D)$, where D is a diagram with writhe $w(D)$ of the oriented link L and Λ is the function defined in Theorem 6.2.

In the previous section, we looked at some of the properties of the Jones polynomial. Now in the following two propositions, we have given some results which are still valid for the HOMFLY and Kauffman polynomials. We have omitted the proofs, as they are almost identical to the ones we gave in chapter 5, and left it as an exercise. Lets now see some of these properties which holds for both of these polynomials.

Proposition 6.1. *If L is an oriented link and $\overline{L}$ is its reflection, then the following holds*

(i) *changing the signs of both variables leaves $P(L)$ and $F(L)$ unchanged;*

(ii) $\overline{P(L)} = P(\overline{L})$ *where* $\bar{l} = l^{-1}$ *and* $\overline{m} = m$*;*

(iii) $\overline{F(L)} = F(\overline{L})$ *where* $\overline{a} = a^{-1}$ *and* $\overline{z} = z$

Proof. Left as an exercise. □

The next proposition looks at the addition of two oriented links (and the split links) in these polynomials.

Proposition 6.2. *If L_1 and L_2 are oriented links, then the following holds*

(i) $P(L_1 + L_2) = P(L_1)P(L_2)$*;*

(ii) $F(L_1 + L_2) = F(L_1)F(L_2)$*;*

(iii) $P(L_1 \sqcup L_2) = -(l + l^{-1})m^{-1}P(L_1)P(L_2)$*;*

(iv) $F(L_1 \sqcup L_2) = ((a + a^{-1})z^{-1} - 1)F(L_1)F(L_2)$*.*

Proof. Left as an exercise. □

Lets now look at two real examples to see the difference of these two polynomials in practice. These two polynomials, namely the HOMFLY and the Kauffman, are independent link invariants. This means that they differentiate different pairs of knots. Hence, we cannot arrive from either polynomial to the other one, by a change of variable. The two pair of knots shown in figure 6.1, is showing the knots 8_8 and 10_{129} which have the same HOMFLY, but different Kauffman polynomials. The other two knots are showing 11_{255} and 11_{257} that have the same Kauffman, but different HOMFLY polynomials.

Figure 6.1: Two pairs of knots with inequivalent polynomials.

We already saw the relation of the HOMFLY polynomial with the other previous polynomials. The Kauffman polynomial has no connection with the Alexander polynomial, but the Jones polynomial is hidden in the Kauffman polynomial in the following two ways.

$$V(L) = F(L) \ \textit{when} \ (a, z) = (-t^{-3/4}, (t^{-1/4} + t^{1/4})),$$
$$(V(L))^2 = (-1)^{\#L-1} F(L) \ \textit{when} \ t = -q^{-2}, (a, z) = (q^3, q^{-1} + q)$$

The above claim has been proven as a proposition in [8] - p.181. But we skipped present it as a formal result due to its long proof. Now that we have seen all the required polynomials in this report, we need to give the two table of knots, similar to the ones we provided for the previous three polynomials, at the end of this chapter.

In the next and last section of this last chapter, we will have a look at the journey we came through and see what we managed to demonstrate in this report. We will also mention invariants of finite type which is the next topic to these polynomial invariants, and is currently one of the main research areas in mathematics.

TABLE **6.2** HOMFLY Polynomial Table

$\mathbf{3_1}$	(**0** -2 -1)	(**0** 1)		
$\mathbf{4_1}$	(-1 **-1** -1)	(**1**)		
$\mathbf{5_1}$	(**0** 0 3 2)	(**0** 0 -4 -1)	(**0** 0 1)	
$\mathbf{5_2}$	(**0** -1 1 1)	(**0** 1 -1)		
$\mathbf{6_1}$	(-1 **0** 1 1)	(**1** -1)		
$\mathbf{6_2}$	(**2** 2 1)	(**-1** -3 -1)	(**0** 1)	
$\mathbf{6_3}$	(1 **3** 1)	(-1 **-3** -1)	(**1**)	
$\mathbf{7_1}$	(**0** 0 0 -4 -3)	(**0** 0 0 10 4)	(**0** 0 0 -6 -1)	(**0** 0 0 1)
$\mathbf{7_2}$	(**0** -1 0 -1 -1)	(**0** 1 -1 1)		
$\mathbf{7_3}$	(-2 -2 1 0 **0**)	(1 3 -3 0 **0**)	(-1 1 0 **0**)	
$\mathbf{7_4}$	(-1 0 2 0 **0**)	(1 -2 1 **0**)		
$\mathbf{7_5}$	(**0** 0 2 0 -1)	(**0** 0 -3 2 1)	(**0** 0 1 -1)	
$\mathbf{7_6}$	(**1** 1 2 1)	(**-1** -2 -2)	(**0** 1)	
$\mathbf{7_7}$	(1 2 **2**)	(-2 **-2** -1)	(**1**)	
$\mathbf{8_1}$	(-1 **0** 0 -1 -1)	(**1** -1 1)		
$\mathbf{8_2}$	(**0** -3 -3 -1)	(**0** 4 7 3)	(**0** –1 -5 -1)	(**0** 01)
$\mathbf{8_3}$	(1 0 **-1** 0 1)	(-1 **2** -1)		
$\mathbf{8_4}$	(-2 **-2** 0 1)	(1 **3** -2 -1)	(**-1** 1)	
$\mathbf{8_5}$	(-2 -5 -4 **0**)	(3 8 4 **0**)	(-1 -5 -1 **0**)	(1 0 **0**)
$\mathbf{8_6}$	(**2** 1 -1 -1)	(**-1** -2 2 1)	(**0** 1 -1)	
$\mathbf{8_7}$	(-2 -4 **-1**)	(3 8 **3**)	(-1 -5 **-1**)	(1 **0**)
$\mathbf{8_8}$	(-1 -1 **2** 1)	(1 2 **-2** -1)	(-1 **1**)	
$\mathbf{8_9}$	(-2 **-3** -2)	(3 **8** 3)	(-1 **-5** -1)	(**1**)
$\mathbf{8_{10}}$	(-3 -6 **-2**)	(3 9 **3**)	(-1 -5 **-1**)	(1 **0**)
$\mathbf{8_{11}}$	(**1** -1 -2 -1)	(**-1** -1 2 1)	(**0** 1 -1)	
$\mathbf{8_{12}}$	(1 1 **1** 1 1)	(-2 **-1** -2)	(**1**)	
$\mathbf{8_{13}}$	(**0** -2 -1)	(-1 **-1** 2 1)	(**1** -1)	
$\mathbf{8_{14}}$	(**1**)	(**-1** -1 1 1)	(**0** 1 -1)	
$\mathbf{8_{15}}$	(**0** 0 1 -3 -4 -1)	(**0** 0 -2 5 3)	(**0** 0 1 -2)	
$\mathbf{8_{16}}$	(**0** -2 -1)	(**2** 5 2)	(**-1** -4 -1)	(**0** 1)
$\mathbf{8_{17}}$	(-1 **-1** -1)	(2 **5** 2)	(-1 **-4** -1)	(**1**)
$\mathbf{8_{18}}$	(1 **3** 1)	(1 **1** 1)	(-1 **-3** -1)	(**1**)
$\mathbf{8_{19}}$	(-1 -5 -5 0 0 **0**)	(5 10 0 0 **0**)	(-1 -6 0 0 **0**)	(1 0 0 **0**)
$\mathbf{8_{20}}$	(**-1** -4 -2)	(**1** 4 1)	(**0** -1)	
$\mathbf{8_{21}}$	(**0** -3 -3 -1)	(**0** 2 3 1)	(**0** 0 -1)	

Figure 6.2: The HOMFLY Polynomial Table. [8]

TABLE **6.3** Kauffman Polynomial Table

3_1	(**0** -2 -1)	(⋆ 0 1 1)	(**0** 1 1)	
4_1	(-1 **-1** -1)	(-1 ⋆ -1)	(1 **2** 1)	(1 ⋆ 1)
5_1	(**0** 0 3 2)	(⋆ 0 0 -2 -1 1)	(**0** 0 -4 -3 1)	(⋆ 0 0 1 1)
	(**0** 0 1 1)			
5_2	(**0** -1 1 1)	(⋆ 0 0 -2 -2)	(**0** 1 -1 -2)	(⋆ 0 1 2 1)
	(**0** 0 1 1)			
6_1	(-1 **0** 1 1)	(⋆ 2 2)	(1 **0** -4 -3)	(1⋆ -2 -3)
	(**1** 2 1)	(⋆ 1 1)		
6_2	(**2** 2 1)	(⋆ 0 -1 -1)	(**-3** -6 -2 1)	(⋆ -2 0 2)
	(**1** 3 2)	(⋆ 1 1)		
6_3	(1 **3** 1)	(-1 -2 ⋆ -2 -1)	(-3 **-6** -3)	(1 1 ⋆ 1 1)
	(2 **4** 2)	(1 ⋆ 1)		
7_1	(**0** 0 0 -4 -3)	(⋆ 0 0 0 3 1 -1 1)	(**0** 0 0 10 7 -2 1)	(⋆ 0 0 0 -4 -3 1)
	(**0** 0 0 -6 -5 1)	(⋆ 0 0 0 1 1)	(**0** 0 0 1 1)	
7_2	(**0** -1 0 -1 -1)	(⋆ 0 0 0 3 3)	(**0** 1 0 3 4)	(⋆ 0 1 -1 -6-4)
	(**0** 0 1 -3 -4)	(⋆ 0 0 1 2 1)	(**0** 0 0 1 1)	
7_3	(-2 -2 1 0 **0**)	(-2 1 3 0 0 0 ⋆)	(-1 6 4 -3 0 **0**)	(1 -1 -4 -2 0 0 ⋆)
	(1 -3 -3 1 0 **0**)	(1 2 1 0 0 ⋆)	(1 1 0 0 **0**)	
7_4	(-1 0 2 0 **0**)	(4 4 0 0 0 ⋆)	(2 -3 -4 1 **0**)	(-4 -8 -2 2 0 ⋆)
	(-3 0 3 0 **0**)	(1 3 2 0 0 ⋆)	(1 1 0 0 **0**)	
7_5	(**0** 0 2 0 -1)	(⋆ 0 0 -1 1 1 -1)	(**0** 0 -3 0 1 -2)	(⋆ 0 0 -1 -4 -2 1)
	(**0** 0 1 -1 0 2)	(⋆ 0 0 1 3 2)	(**0** 0 0 1 1)	
7_6	(**1** 1 2 1)	(⋆ 1 2 0 -1)	(**-2** -4 -4 -2)	(⋆ -4 -6 -1 1)
	(**1** 1 2 2)	(⋆ 2 4 2)	(**0** 1 1)	
7_7	(1 2 **2**)	(2 3 ⋆ **1**)	(-2 -6 **-7** -3)	(-4 -8 ⋆ -3 1)
	(1 2 **4** 3)	(2 5 ⋆ 3)	(1 **1**)	
8_1	(-1 **0** 0 -1 -1)	(⋆ 0 -3 -3)	(1 **0** 0 7 6)	(1 ⋆ -1 5 7)
	(**1** -2 -8 -5)	(⋆ 1 -4 -5)	(**0** 1 2 1)	(⋆ 0 1 1)
8_2	(**0** -3 -3 -1)	(⋆ 0 1 1 -1 -1)	(**0** 7 12 3 -1 1)	(⋆ 0 3 -1 -2 2)
	(**0** -5 -12 -5 2)	(⋆ 0 -4 -2 2)	(**0** 1 3 2)	(⋆ 0 1 1)
8_3	(1 0 **-1** 0 1)	(-4 ⋆ -4)	(-3 1 **8** 1 -3)	(-2 8 ⋆ 8 -2)
	(1 -2 **-6** -2 1)	(1 -4 ⋆ -4 1)	(1 **2** 1)	(1 ⋆ 1)
8_4	(-2 **-2** 0 1)	(-1 ⋆ 1 2)	(7 **10** -1 -3 1)	(4 ⋆ -3 -5 2)
	(-5 **-11** -3 3)	(-4 ⋆ -1 3)	(1 **3** 2)	(1 ⋆ 1)
8_5	(-2 -5 -4 **0**)	(4 7 3 0 ⋆)	(1 -2 4 15 8 **0**)	(2 -8 -10 0 0 ⋆)
	(3 -7 -15 -5 **0**)	(4 1 -3 0 ⋆)	(3 4 1 **0**)	(1 1 0 ⋆)
8_6	(**2** 1 -1 -1)	(⋆ -1 -3 -1 1)	(**-3** -2 6 3 -2)	(⋆ -1 5 2 -4)
	(**1** 0 -6 -4 1)	(⋆ 1 -2 -1 2)	(**0** 1 3 2)	(⋆ 0 1 1)
8_7	(-2 -4 **-1**)	(-1 0 2 2 ⋆ 1)	(-2 4 12 **6**)	(1 -1 -2 -3 ⋆ -3)
	(2 -3 -12 **-7**)	(2 0 -1 ⋆ 1)	(2 4 **2**)	(1 1 ⋆)

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Figure 6.3: The Kauffman Polynomial Table. [8]

6.3 Conclusion

We believe we managed to successfully arrive at the destination we aimed for, that is, constructing the knot theory from scratch by giving a detailed history of the subject for all the polynomials we investigated and also by filling all the gaps and using previously constructed knowledge as a building block to base the upcoming topics that would follow. Throughout this journey, we used very useful and rich resources to help us deliver the message we wished to transfer to the reader. All of these are listed in the References section. Now apart from recapping what we achieved from the first chapter to the last, we will also round up our conversation with a conclusion to look at the connection between all these polynomial invariants.

Throughout this report, we gave lots of examples to show the independence of these polynomial invariants, but we also gave examples where some pairs of knots have the same polynomial of one type, but a different polynomial of another type. Despite giving some indication and hints in these examples, the following diagram shows the relationship between all the invariants we discussed in this report.

$$
\begin{array}{ccccccccc}
 & & P(L) & & & & F(L) & & \\
 & \swarrow & \downarrow & \searrow & & \swarrow & & \searrow & \\
\Delta(L) & & \nabla(L) & & V(L) & & & & Q(L)
\end{array}
$$

Even though, these invariants are so powerful at differentiating different knots and links, it has been demonstrated in [31] that we can still construct infinite families of knots and links which have the same polynomial invariant. Now its time to look at the next topic of interest, which we did not find space and time to discuss in detail, but we will give it a very short analysis and the way it is defined.

The last two polynomials we discussed in this chapter, namely HOMFLY and Kauffman, are best described and their significance are demonstrated in the language of invariants of finite type, or more commonly referred to as *Vassiliev invariants*. This type of invariants provides a radically new way of looking at knots. We will just mention how it is defined, and we will omit giving the details of the notions we have not covered previously.

Definition 6.3. If we suppose V is any invariant of oriented links with input values from some abelian group, then this invariant can be extended to be an invariant of singular links. By *singular link* we mean an immersion of simple closed curves in S^3 with finitely many transverse double points. These self-intersections are required to remain to remain transverse in any isotopy demonstrating the equivalence of such singular links. If the definition of V has been extended over singular links with $n-1$ double points, define it on a singular link $L_\times$ with n singularities by

$$V(L_\times) = V(L_+) - V(L_-)$$

where $V(L_\times)$, $V(L_+)$ and $V(L_-)$ are identical except near a point. Note that $V(L_+)$ and $V(L_-)$ each has $n-1$ double points. Then V is called a **Vassiliev invariant** of order n, or an invariant of finite type n, if $V(L) = 0$ for every L with $n+1$ or more singularities.

The above definition was borrowed from [8], but if one is interested to read about this invariant in more detail, please refer to [32] and [33]. Vassiliev invariants have been the centre of attention mainly because they provide a structured view of the polynomials we discussed in this report, as well as associating themselves to a mixture of linear algebras, visual combinatorics, and also Lie algebras. More details about these connections have been discussed and can be found in [32].

7 References

[1] G. Hemion. The Classification of Knots and 3-Dimensional Spaces, Oxford University Press (1992).

[2] C. M. Gordon and J. Luecke. Knots are determined by their complements, J. American Mathematical Society 2 (1989) 371-415.

[3] G. Burde and H. Zieschang. Knots, de Gruyter (1986).

[4] C. C. Adams, The Knot Book, American Mathematical Society, 2004.

[5] https://en.wikipedia.org/wiki/Knot_(mathematics)

[6] A. Sossinsky, Knots: Mathematics with a twist, Harvard University Press, 2002.

[7] Review of the book [6], written by Louis H. Kauffman, 2003. Accessed via https://arxiv.org/pdf/math/0312168.pdf

[8] W. B. R. Lickorish, An Introduction to Knot Theory, Graduate Texts in Mathematics, Springer, 1997.

[9] P. Cromwell, Knots and Links, Cambridge University Press, 2004.

[10] J. W. Alexander, Topological invariants of knots and links, Transactions of the American Mathematical Society, Vol. 30, (1928), 275-306.

[11] K. Reidemeister, “Knotentheorie”, Ergebn. Math. Grenzgeb., Bd. 1; Berlin: Springer-Verlag (1932).

[12] H. Seifert and W. Threllfall, “Lehrbuch der Topologie”, Leipzig:Teubner (1934).

[13] R. H. Fox, A quick trip through knot theory, in “Topology of ThreeManifolds - Proceedings of 1961 Topology Institute University of Georgia (ed. by M. K. Fort), 120-167, Englewood Cliffs, N.J.: Prentice-Hall (1962).

[14] J. H. Conway, An enumeration of knots and links and some of their related properties, in “Computational Problems in Abstract Algebra Proceedings of Conference in Oxford 1967” (ed. by J. Leech), 329-358, New York: Pergamon Press (1970).

[15] P. G. Tait. On knots I, II, III., Scientific papers I. Cambridge University Press London (1898) 273-437.

[16] C. N. Little. Non-alternate ± knots, Trans. Roy. Soc. Edinburgh 39 (1900) 771-778.

[17] D.GABAI, 'Genera of the arborescent links', Memoirs of the American Mathematical Society. **59** no.339,1986.

[18] W.B.R. Lickorish and K.C. Millett, 'A polynomial invariant of oriented links', Topology **26** (1987) 107-141.

[19] P.R.Cromwell, 'Some infinite families of satellite knots with given Alexander Polynomial', Mathematika **38** (1991) 156-169.

[20] V. F. R. Jones. Hecke algebra representations of braid groups and link polynomials, Ann. of Math. **126** (1987) 335-388.

[21] L. H. Kauffman. States models and the Jones polynomial, Topology **26** (1987) 395-407.

[22] J.Hoste, 'A polynomial invariant of knots and links', Pacific J.Math. **124**(1986) 295-320.

[23] A.Ocneanu. A Polynomial Invariant for Knots: A combinatorial and Algebraic Approach. Preprint (unpublished), MSRI, Univ.California, 1985.

[24] P.Freyd and D.Yetter, A new Invariant for knots and links, Preprint (unpublished), Univ. Pennsylvania, 1984.

[25] J.H.Przytycki and P.Traczyk. 'Invariants of links of Conway type', Kobe J.Math. **4** (1987) 115-139.

[26] P.Freyd, D.Yetter, J.Hoste, W.B.R. Lickorish, K.C. Millet and A.Ocneanu, 'A new polynomial invariant of knots and links', Bull. American Math.Soc. **12** (1985) 239-246.

[27] R.D.Brandt, W.B.R.Lickorish and K.C.Millet, 'A polynomial invariant for unoriented knots and links', Inventions Math. **84**(1986) 563-573.

[28] C.F.Ho, 'A new polynomial invariant for knots and links - preliminary report', American Math Soc. Abstract **6**(1985) p.300.

[29] C.F.Ho, On Polynomial Invariants for Knots and Links, Ph.D. Thesis, California Institute of Technology, 1986.

[30] L.H. Kauffman, 'An invariant of regular isotopy', Trans.American Math. Soc. **318** (1990) 417-471.

[31] T.Kanenobu, 'Infinitely many knots with the same polynomial invariants', Proc.American Math.Soc.**97**(1986) 158-162.

[32] D. Bar-Natan. On the Vassiliev knot invariants, Topology **34** (1995) 423–472.

[33] J. S. Birman and X.-S. Lin. Knot polynomials and Vassiliev invariants, Invent. Math. **111**(1993)225-270.

[P] The diagrams that have a [P] as their reference point indicate that they are produced in the software "Paint" and then exported to latex as an external image (JPG).