Discrete Mathematics	6-AUG-2014

Lecture 1: Catalan Numbers and Recurrence Relations

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1 Catalan Numbers[2]

1.1 Introduction

In combinatorial mathematics, the Catalan numbers form a sequence of natural numbers that occur in various counting problems, often involving recursively-defined objects. They are named after the Belgian mathematician Eugne Charles Catalan.
The nth Catalan number is given directly in terms of binomial coefficients by:

$$C_n = \frac{1}{n+1}.\binom{2n}{n} = \frac{2n!}{(n+1)!n!} \forall n \geq 0 \tag{1}$$

The first Catalan numbers for n = 0, 1, 2, 3, are 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420, 24466267020, 91482563640, 343059613650, 1289904147324,.....

1.2 Proofs for Catalan Numbers :

A large number of formal and informal proofs have been developed for Catalan Numbers.Some of them are quite involved. Two simple ones among them have been discussed below :

1.2.1 First Proof by Andre's Reflection Method

This method is based on counting the total number of monotonically increasing paths from bottom left corner to the top right corner in an nXn square.We need to count the number of paths possible without crossing the diagonal.Suppose we are given a monotonic path in an n n grid that does cross the diagonal. Find the first edge in the path that lies above the diagonal, and flip the portion of the path occurring after that edge, along a line parallel to the diagonal. Observe that now we have taken into account k+1 vertical edges and k horizontal edges for some k between 1 and n-1. This leaves l-1 vertical edges and l horizontal edges, where l + k = n. By reflecting the rest of the vertical and horizontal edges, we will have k + 1 + l = n +1 vertical edges and k + l - 1 = n - 1 horizontal edges. The resulting path is a monotonic path in an $(n-1)$ (n + 1) grid. Figure 1 illustrates this procedure; the green portion of the path is the portion being flipped.This method can be mapped to the Dyke words problem where we start with a sequence of sequence of n X's and n Y's which is not a Dyck word, and exchanging all X's with Y's after the first Y that violates the Dyck condition. At that first Y, there are k + 1 Y's and k X's for some k between 1 and $n-1$.

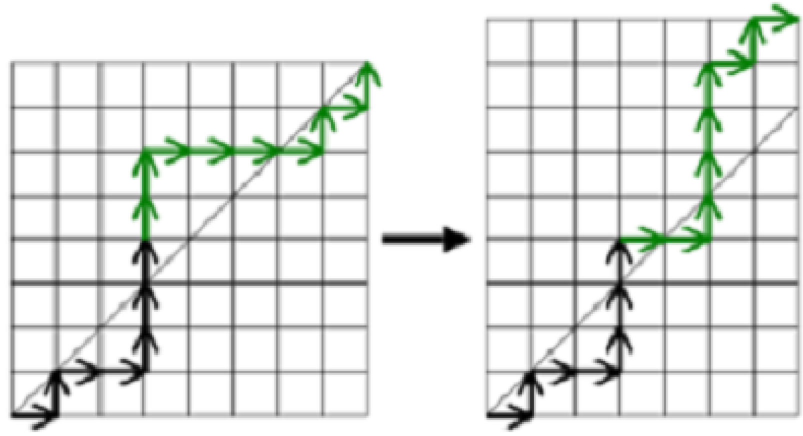

Figure 1: The green portion of the path is flipped. The old grid is n x n. The new grid is (n - 1) x (n + 1).

Since every monotonic path in the $(n-1)$ (n + 1) grid is of length 2n and every such path involves n + 1 vertical edges in any order, the number of these paths is equal to:

$$\binom{2n}{n+1} \tag{2}$$

Therefore, to calculate the number of monotonic n n paths which do not cross the diagonal, we need to subtract this from the total number of monotonic n n paths, so we finally obtain

$$\binom{2n}{n} - \binom{2n}{n+1} \tag{3}$$

which is the nth Catalan number C_n.

1.3 Second Proof of Catalan Numbers Rukavicka Josef[1]

In order to understand this proof, we need to understand the concept of exceedance number, defined as follows :
Exceedance number, for any path in any square matrix, is defined as the number of vertical edges above the diagonal.
For example, in Figure 1, the edges lying above the diagonal are marked in red, so the exceedance of the path is 5.

This is a bijective proof, while being more involved than the previous one, provides a more natural explanation for the term n + 1 appearing in the denominator of the formula for Cn. A generalized version of this proof can be found in a paper of Rukavicka Josef.

1.3.1 Method to reduce the exceedance number of the path

We need to reduce the exceedance number of the path till we get path with exceedance number zero implying that all the vertical edges are below the diagonal. For a given a

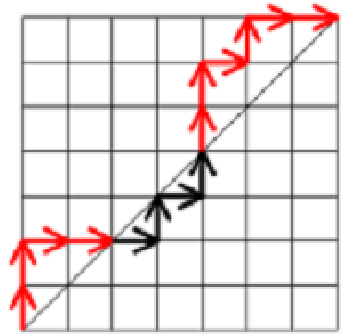

Figure 2: A path with exceedance 5..

monotonic path whose exceedance is not zero, then we may apply the following method to construct a new path whose exceedance is one less than the one we started with.

1. Starting from the bottom left, follow the path until it first travels above the diagonal.
2. Continue to follow the path until it touches the diagonal again. Denote by X the first such edge that is reached.
3. Swap the portion of the path occurring before X with the portion occurring after X.

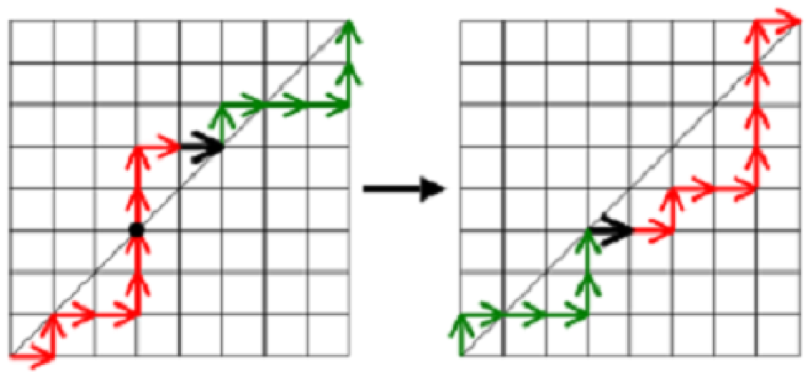

Figure 3: The green and red portions are being exchanged.

The following diagrams depict an example of reduction of exceedance number following the steps as metioned above.

Notice that the exceedance has dropped from three to two. In fact, the algorithm will cause the exceedance to decrease by one, for any path that we feed it, because the first vertical step starting on the diagonal (at the point marked with a black dot) is the unique vertical edge that under the operation passes from above the diagonal to below it; all other vertical edges stay on the same side of the diagonal.It is also not difficult to see that this process is reversible: given any path P whose exceedance is less than n, there is exactly one path which yields P when the algorithm is applied to it. Indeed, the (black) edge X, which originally was the first horizontal step ending on the diagonal, has become the last horizontal step starting on the diagonal. This implies that the number of paths of exceedance n is equal to the number of paths of exceedance n 1, which is equal to the number of paths of exceedance n 2, and so on, down to zero. In other words, we have split up the set of all monotonic paths into n + 1 equally sized classes, corresponding to the possible exceedances between 0 and n. Since there are:

$$\binom{2n}{n} \tag{4}$$

Thus we get the desired result as below

$$C_n = \frac{1}{n+1} . \binom{2n}{n} \tag{5}$$

2 Linear Recurrence Solutions with constant coefficients

[3] Let us suppose a_n, a_{n-1}, a_{n-1}..........a_{n-k} are the values of the relation at n, n-1 upto n - k, A linear recurrence relation of degree with coefficients is a recurrence relation of the form:

$$c_0 a^n + c_1 a^{n-1} ......c_k a^{(n-k)} = f_n \tag{6}$$

If $f_n = 0$ then the equation above becomes homogeneous. Total solution for a recurrence relation = homoegeneous solution + particular.In this particular section, however , we shall discuss about the homoegneous equation only.

A homogeneous solution of linear equation with constant coefficient is of the form $A\alpha^r$ which is called the characteristic root and A is a constant which are determined by boundary conditions, since $A\alpha^r$ is a root,it will satisfy the homogeneous equation.

$$c_0 A\alpha^n + c_1 A\alpha^{n-1} ......c_k A\alpha^{(n-k)} = 0 \tag{7}$$

$\implies$

$$c_0 A\alpha^n + c_1 A\alpha^{n-1} ......c_k A\alpha^{(n-k)} = 0 \tag{8}$$

so, α is a characteristic root for this equation.

e .g.

$$a_n - a_{n-1} - a_{n-1} \tag{9}$$

$$\alpha^n - \alpha^{n-1} - \alpha^{n-1} \tag{10}$$

Solving,we get

$$\alpha_1 = \frac{1+\sqrt{5}}{2} \quad \alpha_1 = \frac{1-\sqrt{5}}{2} \tag{11}$$

$$ar^n = A_1 (\frac{1+\sqrt{5}}{2})^r + A_2 (\frac{1-\sqrt{5}}{2})^r \tag{12}$$

Using Boundary Conditions, $a_0 = 0 a_1 = 1$, we get

$$A_1 = \frac{1}{2}(1 + \frac{1}{\sqrt{5}}) \tag{13}$$

and

$$A_2 = \frac{1}{2}(1 - \frac{1}{\sqrt{5}}) \tag{14}$$

If the roots of the solution are not all distinct,$\alpha_1 = \alpha$ is a root with multiplicity m. The corresponding homogeneous equation

$$c_0 A\alpha^n + c_1 A\alpha^{n-1} ......c_k A\alpha^{(n-k)} = 0 \tag{15}$$

becomes the following :

$$c_0(n)A\alpha^{n-1} + c_1(n-2)A\alpha^{n-2}......c_k(n-k)A\alpha^{(n-k-1)} = 0 \tag{16}$$

as the derivative of the original root becomes the another solution.Now multiplying all the terms of the equation by α :

$$c_0(n)A\alpha^{n} + c_1(n-2)A\alpha^{n-1}......c_k(n-k)A\alpha^{(n-k)} = 0 \tag{17}$$

Now, $A_{(n-1)}$ n $\alpha^{(n-1)}$ is a solution for this homogeneous equation. Similarly we will have solutions for other derivatives, as follows :
$A_{(n-2)}n^2\alpha^n A_{(n-3)}n^3\alpha^n$

References

[1] Rukavika Joseph. On generalized Dyck paths. http://www.emis.de/journals/EJC/Volume_18/PDF/v18i1p40.pdf, [Online, accessed 28 August, 2014].

[2] Wikipedia. Catalan Numbers. http://en.wikipedia.org/wiki/Catalan_number, [Online, accessed 28 August, 2014].

[3] Wikipedia. Recurrence relation http://en.wikipedia.org/wiki/Recurrence_relation, [Online, accessed 28 August, 2014].