Other important formulae

$$\sum_{k=1}^{n} log\, k \approx nlogn$$

$$\sum_{k=1}^{n} k^p = 1^p + 2^p + \cdots + n^p \approx \frac{1}{p+1} n^{p+1}$$

1.21 Master Theorem for Divide and Conquer

All divide and conquer algorithms (In detail, we will discuss them in *Divide and Conquer* chapter) divides the problem into subproblems, each of which is part of the original problem, and then perform some additional work to compute the final answer. As an example, merge sort algorithm [for details, refer *Sorting* chapter] operates on two subproblems, each of which is half the size of the original and then performs O(n) additional work for merging. This gives the running time equation:

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$

The following theorem can be used to determine the running time of divide and conquer algorithms. For a given program (algorithm), first we try to find the recurrence relation for the problem. If the recurrence is of the below form then we can directly give the answer without fully solving it.

If the recurrence is of the form $T(n) = aT(\frac{n}{b}) + \Theta(n^k log^p n)$, where $a \geq 1$, $b > 1, k \geq 0$ and p is a real number, then:

1) If $a > b^k$, then $T(n) = \Theta\left(n^{log_b^a}\right)$
2) If $a = b^k$
   a. If $p > -1$, then $T(n) = \Theta\left(n^{log_b^a} log^{p+1} n\right)$
   b. If $p = -1$, then $T(n) = \Theta\left(n^{log_b^a} loglogn\right)$
   c. If $p < -1$, then $T(n) = \Theta\left(n^{log_b^a}\right)$
3) If $a < b^k$
   a. If $p \geq 0$, then $T(n) = \Theta(n^k log^p n)$
   b. If $p < 0$, then $T(n) = O(n^k)$

1.22 Problems on Divide and Conquer Master Theorem

For each of the following recurrences, give an expression for the runtime $T(n)$ if the recurrence can be solved with the Master Theorem. Otherwise, indicate that the Master Theorem does not apply.

Problem-1 $T(n) = 3T(n/2) + n^2$

Solution: $T(n) = 3T(n/2) + n^2 \Rightarrow T(n) = \Theta(n^2)$ (Master Theorem Case 3.a)

Problem-2 $T(n) = 4T(n/2) + n^2$

Solution: $T(n) = 4T(n/2) + n^2 \Rightarrow T(n) = \Theta(n^2 logn)$ (Master Theorem Case 2.a)

Problem-3 $T(n) = T(n/2) + n^2$

Solution: $T(n) = T(n/2) + n^2 \Rightarrow \Theta(n^2)$ (Master Theorem Case 3.a)

Problem-4 $T(n) = 2^n T(n/2) + n^n$

Solution: $T(n) = 2^n T(n/2) + n^n \Rightarrow$ Does not apply (a is not constant)

Problem-5 $T(n) = 16T(n/4) + n$

Solution: $T(n) = 16T(n/4) + n \Rightarrow T(n) = \Theta(n^2)$ (Master Theorem Case 1)

Problem-6 $T(n) = 2T(n/2) + nlogn$

Solution: $T(n) = 2T(n/2) + nlogn \Rightarrow T(n) = \Theta(nlog^2n)$ (Master Theorem Case 2.a)

Problem-7 $T(n) = 2T(n/2) + n/logn$

Solution: $T(n) = 2T(n/2) + n/logn \Rightarrow T(n) = \Theta(nloglogn)$ (Master Theorem Case 2.b)

Problem-8 $T(n) = 2T(n/4) + n^{0.51}$

Solution: $T(n) = 2T(n/4) + n^{0.51} \Rightarrow T(n) = O(n^{0.51})$ (Master Theorem Case 3.b)

Problem-9 $T(n) = 0.5T(n/2) + 1/n$

Solution: $T(n) = 0.5T(n/2) + 1/n \Rightarrow$ Does not apply ($a < 1$)

Problem-10 $T(n) = 6T(n/3) + n^2 logn$

Solution: $T(n) = 6T(n/3) + n^2logn \Rightarrow T(n) = \Theta(n^2logn)$ (Master Theorem Case 3.a)

Problem-11 $T(n) = 64T(n/8) - n^2logn$

Solution: $T(n) = 64T(n/8) - n^2logn \Rightarrow$ Does not apply (function is not positive)

Problem-12 $T(n) = 7T(n/3) + n^2$

Solution: $T(n) = 7T(n/3) + n^2 \Rightarrow T(n) = \Theta(n^2)$ (Master Theorem Case 3.as)

Problem-13 $T(n) = 4T(n/2) + logn$

Solution: $T(n) = 4T(n/2) + logn \Rightarrow T(n) = \Theta(n^2)$ (Master Theorem Case 1)

Problem-14 $T(n) = 16T(n/4) + n!$

Solution: $T(n) = 16T(n/4) + n! \Rightarrow T(n) = \Theta(n!)$ (Master Theorem Case 3.a)

Problem-15 $T(n) = \sqrt{2}T(n/2) + logn$

Solution: $T(n) = \sqrt{2}T(n/2) + logn \Rightarrow T(n) = \Theta(\sqrt{n})$ (Master Theorem Case 1)

Problem-16 $T(n) = 3T(n/2) + n$

Solution: $T(n) = 3T(n/2) + n \Rightarrow T(n) = \Theta(n^{log3})$ (Master Theorem Case 1)

Problem-17 $T(n) = 3T(n/3) + \sqrt{n}$

Solution: $T(n) = 3T(n/3) + \sqrt{n} \Rightarrow T(n) = \Theta(n)$ (Master Theorem Case 1)

Problem-18 $T(n) = 4T(n/2) + cn$

Solution: $T(n) = 4T(n/2) + cn \Rightarrow T(n) = \Theta(n^2)$ (Master Theorem Case 1)

Problem-19 $T(n) = 3T(n/4) + nlogn$

Solution: $T(n) = 3T(n/4) + nlogn \Rightarrow T(n) = \Theta(nlogn)$ (Master Theorem Case 3.a)

Problem-20 $T(n) = 3T(n/3) + n/2$

Solution: $T(n) = 3T(n/3) + n/2 \Rightarrow T(n) = \Theta(nlogn)$ (Master Theorem Case 2.a)

1.23 Master Theorem for Subtract and Conquer Recurrences

Let $T(n)$ be a function defined on positive n, and having the property

$$T(n) = \begin{cases} c, & \text{if } n \leq 1 \\ aT(n-b) + f(n), & \text{if } n > 1 \end{cases}$$

for some constants $c, a > 0, b > 0, k \geq 0$, and function $f(n)$. If $f(n)$ is in $O(n^k)$, then

$$T(n) = \begin{cases} O(n^k), & \text{if a} < 1 \\ O(n^{k+1}), & \text{if a} = 1 \\ O\left(n^k a^{\frac{n}{b}}\right), & \text{if a} > 1 \end{cases}$$

1.24 Variant of subtraction and conquer master theorem

The solution to the equation $T(n) = T(\alpha n) + T((1-\alpha)n) + \beta n$, where $0 < \alpha < 1$ and $\beta > 0$ are constants, is $O(nlogn)$.

1.25 Amortized Analysis

Amortized analysis refers to determining the time-averaged running time for a sequence of operations. It is different from average case analysis, because amortized analysis does not make any assumption about the distribution of the data values, whereas average case analysis assumes the data are not "bad" (e.g., some sorting algorithms do well on "average" over all input orderings but very badly on certain input orderings). That is, amortized analysis is a worst case analysis, but for a sequence of operations, rather than for individual operations.

The motivation for amortized analysis is to better understand the running time of certain techniques, where standard worst case analysis provides an overly pessimistic bound. Amortized analysis generally applies to a method that consists of a sequence of operations, where the vast majority of the operations are cheap, but some of the operations are expensive. If we can show that the expensive operations are particularly rare we can "charge them" to the cheap operations, and only bound the cheap operations.

The general approach is to assign an artificial cost to each operation in the sequence, such that the total of the artificial costs for the sequence of operations bounds total of the real costs for the sequence. This artificial cost is called the amortized cost of an operation. In order to analyze the running time, the amortized cost thus is a correct way of understanding the overall running time — but note that particular operations can still take longer so it is not a way of bounding the running time of any individual operation in the sequence.

When one event in a sequence affects the cost of later events:

- One particular task may be expensive.
- But it may leave data structure in a state that next few operations becomes easier.

Example: Let us consider an array of elements from which we want to find k^{th} smallest element. We can solve this problem using sorting. After sorting the given array, we just need to return the k^{th} element from it. Cost of performing sort (assuming comparison based sorting algorithm) is $O(nlogn)$. If we perform n such selections then the average cost of each selection is $O(nlogn/n) = O(logn)$. This clearly indicates that sorting once is reducing the complexity of subsequent operations.

1.26 Problems on Algorithms Analysis

Note: From the following problems, try to understand the cases which give different complexities ($O(n), O(logn), O(loglogn)$ etc...).

Problem-21 Find the complexity of the below recurrence:

$$T(n) = \begin{cases} 3T(n-1), if\ n > 0, \\ 1, \quad otherwise \end{cases}$$

Solution: Let us try solving this function with substitution.

$$T(n) = 3T(n-1)$$

$$T(n) = 3(3T(n-2)) = 3^2T(n-2)$$

$$T(n) = 3^2(3T(n-3))$$

.

.

$$T(n) = 3^nT(n-n) = 3^nT(0) = 3^n$$

This clearly shows that the complexity of this function is $O(3^n)$.
Note: We can use the *Subtraction and Conquer* master theorem for this problem.

Problem-22 Find the complexity of the below recurrence:

$$T(n) = \begin{cases} 2T(n-1) - 1, if\ n > 0, \\ 1, \qquad\qquad otherwise \end{cases}$$

Solution: Let us try solving this function with substitution.

$$T(n) = 2T(n-1) - 1$$

$$T(n) = 2(2T(n-2) - 1) - 1 = 2^2T(n-2) - 2 - 1$$

$$T(n) = 2^2(2T(n-3) - 2 - 1) - 1 = 2^3T(n-4) - 2^2 - 2^1 - 2^0$$

$$T(n) = 2^nT(n-n) - 2^{n-1} - 2^{n-2} - 2^{n-3} \ldots . 2^2 - 2^1 - 2^0$$

$$T(n) = 2^n - 2^{n-1} - 2^{n-2} - 2^{n-3} \ldots . 2^2 - 2^1 - 2^0$$

$$T(n) = 2^n - (2^n - 1)\ [note: 2^{n-1} + 2^{n-2} + \cdots + 2^0 = 2^n]$$

$$T(n) = 1$$

∴Complexity is O(1). Note that while the recurrence relation looks exponential the solution to the recurrence relation here gives a different result.

Problem-23 What is the running time of the following function?

```
void Function(int n) {
        int i=1, s=1;
        while( s <= n) {
                i++;
                s= s+i;
                printf("*");
        }
}
```

Solution: Consider the comments in below function:

```
void Function (int n) {
        int i=1, s=1;
        // s is increasing not at rate 1 but i
        while( s <= n) {
                i++;
                s= s+i;
                printf("*");
        }
}
```

We can define the terms '*s*' according to the relation $s_i = s_{i-1} + i$. The value of '*i*' increases by one for each iteration. The value contained in '*s*' at the i^{th} iteration is the sum of the first '*i*' positive integers. If k is the total number of iterations taken by the program, then *while* loop terminates if:

$$1 + 2 + \ldots + k = \frac{k(k+1)}{2} > n \Rightarrow k = O(\sqrt{n}).$$

Problem-24 Find the complexity of the function given below.

```
void Function(int n) {
        int i, count =0;
        for(i=1; i*i<=n; i++)
                count++;
}
```

Solution:

```
void Function(int n) {
        int i, count =0;
        for(i=1; i*i<=n; i++)
                count++;
}
```

In the above function the loop will end, if $i^2 \leq n \Rightarrow T(n) = O(\sqrt{n})$. The reasoning is same as that of Problem-23.

Problem-25 What is the complexity of the below program:

```
void function(int n) {
        int i, j, k , count =0;
        for(i=n/2; i<=n; i++)
                for(j=1; j + n/2<=n; j= j++)
                        for(k=1; k<=n; k= k * 2)
                                count++;
}
```

Solution: Consider the comments in the following function.

```
void function(int n) {
        int i, j, k , count =0;
        //outer loop execute n/2 times
        for(i=n/2; i<=n; i++)
                //Middle loop executes n/2 times
                for(j=1; j + n/2<=n; j= j++)
                        //outer loop execute logn times
                        for(k=1; k<=n; k= k * 2)
                                count++;
}
```

The complexity of the above function is $O(n^2 logn)$.

Problem-26 What is the complexity of the below program:

```
void function(int n) {
        int i, j, k , count =0;
        for(i=n/2; i<=n; i++)
                for(j=1; j<=n; j= 2 * j)
                        for(k=1; k<=n; k= k * 2)
                                count++;
}
```

Solution: Consider the comments in the following function.

```
void function(int n) {
        int i, j, k , count =0;
        //outer loop execute n/2 times
        for(i=n/2; i<=n; i++)
```

```
        //Middle loop executes logn times
        for(j=1; j<=n; j= 2 * j)
                //outer loop execute logn times
                for(k=1; k<=n; k= k*2)
                        count++;
}
```

The complexity of the above function is $O(nlog^2n)$.

Problem-27 Find the complexity of the below program.

```
function( int n ) {
        if(n == 1) return;
        for(int i = 1 ; i <= n ; i + + ) {
                for(int j= 1 ; j <= n ; j + + ) {
                        printf("*" );
                        break;
                }
        }
}
```

Solution: Consider the comments in the following function.

```
function( int n ) {
        //constant time
        if( n == 1 ) return;
        //outer loop execute n times
        for(int i = 1 ; i <= n ; i + + ) {
                // inner loop executes only time due to break statement.
                for(int j= 1 ; j <= n ; j + + ) {
                        printf("*" );
                        break;
                }
        }
}
```

The complexity of the above function is $O(n)$. Even though the inner loop is bounded by n, but due to the break statement it is executing only once.

Problem-28 Write a recursive function for the running time $T(n)$ of the function given below. Prove using the iterative method that $T(n) = \Theta(n^3)$.

```
        function( int n ) {
                if( n == 1 ) return;
                for(int i = 1 ; i <= n ; i + + )
                        for(int j = 1 ; j <= n ; j + + )
                                printf("*" ) ;
                function( n-3 );
        }
```

Solution: Consider the comments in below function:

```
function (int n) {
        //constant time
        if( n == 1 ) return;
        //outer loop execute n times
        for(int i = 1 ; i <= n ; i + + )
                //inner loop executes n times
```

```
            for(int j = 1 ; j <= n ; j + + )
                //constant time
                printf("*" ) ;
        function( n-3 );
}
```

The recurrence for this code is clearly $T(n) = T(n-3) + cn^2$ for some constant $c > 0$ since each call prints out n^2 asterisks and calls itself recursively on n - 3. Using the iterative method we get: $T(n) = T(n-3) + cn^2$. Using the *Subtraction and Conquer* master theorem, we get $T(n) = \Theta(n^3)$.

Problem-29 Determine Θ bounds for the recurrence relation: $T(n) = 2T\left(\frac{n}{2}\right) + nlogn$.

Solution: Using Divide and Conquer master theorem, we get $O(nlog^2n)$.

Problem-30 Determine Θ bounds for the recurrence: $T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right) + T\left(\frac{n}{8}\right) + n$.

Solution: Substituting in the recurrence equation, we get: $T(n) \leq c1 * \frac{n}{2} + c2 * \frac{n}{4} + c3 * \frac{n}{8} + cn \leq k * n$, where k is a constant. This clearly says $\Theta(n)$.

Problem-31 Determine Θ bounds for the recurrence relation: $T(n) = T(\lceil n/2 \rceil) + 7$.

Solution: Using Master Theorem we get $\Theta(logn)$.

Problem-32 Prove that the running time of the code below is $\Omega(logn)$.

```
void Read(int n) {
        int k = 1;
        while( k < n )
                k = 3*k;
}
```

Solution: The *while* loop will terminate once the value of *'k'* is greater than or equal to the value of *'n'*. In each iteration the value of *'k'* is multiplied by 3. If i is the number of iterations, then *'k'* has the value of $3i$ after i iterations. The loop is terminated upon reaching i iterations when $3^i \geq n \leftrightarrow i \geq \log_3 n$, which shows that $i = \Omega(logn)$.

Problem-33 Solve the following recurrence.

$$T(n) = \begin{cases} 1, & if\ n = 1 \\ T(n-1) + n(n-1), & if\ n \geq 2 \end{cases}$$

Solution: By iteration:

$$T(n) = T(n-2) + (n-1)(n-2) + n(n-1)$$

...

$$T(n) = T(1) + \sum_{i=1}^{n} i(i-1)$$

$$T(n) = T(1) + \sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} i$$

$$T(n) = 1 + \frac{n((n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$$

$$T(n) = \Theta(n^3)$$

Note: We can use the *Subtraction and Conquer* master theorem for this problem.

Problem-34 Consider the following program:

```
Fib[n]
if(n==0) then return 0
else if(n==1) then return 1
```

else return Fib[n-1]+Fib[n-2]

Solution: The recurrence relation for running time of this program is: $T(n) = T(n-1) + T(n-2) + c$. Notice T($n$) has two recurrence calls indicating a binary tree. Each step recursively calls the program for n reduced by 1 and 2, so the depth of the recurrence tree is O(n). The number of leaves at depth n is 2^n since this is a full binary tree, and each leaf takes at least O(1) computation for the constant factor. Running time is clearly exponential in n and it is O(2^n).

Problem-35 Running time of following program?

```
function(n) {
        for(int i = 1; i <= n ; i + + )
                for(int j = 1 ; j <= n ; j+ = i )
                        printf(" * ") ;
}
```

Solution: Consider the comments in below function:

```
function (n) {
        //this loop executes n times
        for(int i = 1;  i <= n ; i + + )
                //this loop executes j times with j increase by the rate of i
                for(int j = 1 ; j <= n ; j+ = i )
                        printf( " * " ) ;
}
```

In the above code, inner loop executes n/i times for each value of i. Its running time is $n \times (\sum_{i=1}^{n} n/i) = O(nlogn)$.

Problem-36 What is the complexity of $\sum_{i=1}^{n} log\, i$?

Solution: Using the logarithmic property, $logxy = logx + logy$, we can see that this problem is equivalent to

$$\sum_{i=1}^{n} logi = log\,1 + log\,2 + \cdots + log\,n = log(1 \times 2 \times \ldots \times n) = log(n!) \leq log(n^n) \leq nlogn$$

This shows that that the time complexity = O($nlogn$).

Problem-37 What is the running time of the following recursive function (specified as a function of the input value n)? First write the recurrence formula and then find its complexity.

```
function(int n) {
        if(n <= 1) return;
        for (int i=1 ; i <= 3; i++ )
                f(⌈n/3⌉);
}
```

Solution: Consider the comments in below function:

```
function (int n) {
        //constant time
        if(n <= 1) return;
        //this loop executes with recursive loop of n/3 value
        for (int i=1 ; i <= 3; i++ )
                f(⌈n/3⌉);
}
```

We can assume that for asymptotical analysis $k = \lceil k \rceil$ for every integer $k \geq 1$. The recurrence for this code is $T(n) = 3T(\frac{n}{3}) + \Theta(1)$. Using master theorem, we get $T(n) = \Theta(n)$.

Problem-38 What is the running time of the following recursive function (specified as a function of the input value n)? First write a recurrence formula, and show its solution using induction.

```
function(int n) {
```

```
        if(n <= 1) return;
        for (int i=1 ; i <= 3 ; i++ )
                function (n – 1).
    }
```

Solution: Consider the comments in below function:

```
    function (int n) {
        //constant time
        if(n <= 1) return;
        //this loop executes 3 times with recursive call of n-1 value
        for (int i=1 ; i <= 3 ; i++ )
                function (n – 1).
    }
```

The *if* statement requires constant time [O(1)]. With the *for* loop, we neglect the loop overhead and only count three times that the function is called recursively. This implies a time complexity recurrence:

$$T(n) = c, if\ n \leq 1;$$
$$= c + 3T(n-1), if\ n > 1.$$

Using the *Subtraction and Conquer* master theorem, we get $T(n) = \Theta(3^n)$.

Problem-39 Write a recursion formula for the running time $T(n)$ of the function whose code is below.

```
    function (int n) {
        if(n <= 1)  return;
        for(int i = 1; i < n; i + +)
                printf(" * ");
        function ( 0.8n ) ;
    }
```

Solution: Consider the comments in below function:

```
    function (int n) {
        if(n <= 1)  return;          //constant time
        // this loop executes n times with constant time loop
        for(int i = 1; i < n; i + +)
                printf(" * ");
        //recursive call with 0.8n
        function ( 0.8n ) ;
    }
```

The recurrence for this piece of code is $T(n) = T(.8n) + O(n) = T(4/5n) + O(n) = 4/5\ T(n) + O(n)$. Applying master theorem, we get $T(n) = O(n)$.

Problem-40 Find the complexity of the recurrence: $T(n) = 2T(\sqrt{n}) + logn$

Solution: The given recurrence is not in the master theorem form. Let us try to convert this to master theorem format by assuming $n = 2^m$. Applying logarithm on both sides gives, $logn = mlog2 \Rightarrow m = logn$. Now, the given function becomes,

$$T(n) = T(2^m) = 2T\left(\sqrt{2^m}\right) + m = 2T\left(2^{\frac{m}{2}}\right) + m.$$

To make it simple we assume $S(m) = T(2^m) \Rightarrow S(\frac{m}{2}) = T(2^{\frac{m}{2}}) \Rightarrow S(m) = 2S\left(\frac{m}{2}\right) + m$. Applying the master theorem would result $S(m) = O(mlogm)$. If we substitute $m = logn$ back, $T(n) = S(logn) = O((logn)\ loglogn)$.

Problem-41 Find the complexity of the recurrence: $T(n) = T(\sqrt{n}) + 1$

Solution: Applying the logic of Problem-40, gives $S(m) = S\left(\frac{m}{2}\right) + 1$. Applying the master theorem would result $S(m) = O(logm)$. Substituting $m = logn$, gives $T(n) = S(logn) = O(loglogn)$.

Problem-42 Find the complexity of the recurrence: $T(n) = 2T(\sqrt{n}) + 1$

Solution: Applying the logic of Problem-40, gives: $S(m) = 2S\left(\frac{m}{2}\right) + 1$. Using the master theorem results $S(m) = O(m^{log_2^2}) = O(m)$. Substituting $m = logn$ gives $T(n) = O(logn)$.

Problem-43 Find the complexity of the below function.

```
int Function (int n) {
        if(n <= 2) return 1;
        else return (Function (floor(sqrt(n))) + 1);
}
```

Solution: Consider the comments in below function:

```
int Function (int n) {
        //constant time
        if(n <= 2) return 1;
        else    // executes √n + 1 times
                return (Function (floor(sqrt(n))) + 1);
}
```

For the above code, the recurrence function can be given as: $T(n) = T(\sqrt{n}) + 1$. This is same as that of Problem-41.

Problem-44 Analyze the running time of the following recursive psuedocode as a function of n.

```
void function(int n) {
        if( n < 2 ) return;
        else counter = 0;
        for i = 1 to 8 do
                function (n/2);
        for i =1 to n^3 do
                counter = counter + 1;
}
```

Solution: Consider the comments in below psuedocode and call running time of function(n) as $T(n)$.

```
void function(int n) {
        if( n < 2 ) return;   //constant time
        else    counter = 0;
        // this loop executes 8 times with n value half in every call
        for i = 1 to 8 do
                function (n/2);
        // this loop executes n^3 times with constant time loop
        for i =1 to n^3 do
                counter = counter + 1;
}
```

$T(n)$ can be defined as follows:

$$T(n) = 1 \; if \; n < 2,$$
$$= 8T(\frac{n}{2}) + n^3 + 1 \; otherwise.$$

Using the master theorem gives, $T(n) = \Theta(n^{log_2^8} logn) = \Theta(n^3 logn)$.

Problem-45 Find the complexity of the below psuedocode.

```
temp = 1
repeat
        for i = 1 to n
                temp = temp + 1;
```

```
            n = n/2;
        until n <= 1
```

Solution: Consider the comments in below psuedocode:

```
        temp = 1  //const time
        repeat  // this loops executes n times
                for i = 1 to n
                        temp = temp + 1;
                //recursive call with n/2 value
                n = n/2;
        until n <= 1
```

The recurrence for this function is $T(n) = T(n/2) + n$. Using master theorem we get, $T(n) = O(n)$.

Problem-46 Running time of following program?

```
function(int n) {
        for(int i = 1 ; i <= n ; i + + )
                for(int j = 1 ; j <= n ; j * = 2 )
                        printf( " * " );
}
```

Solution: Consider the comments in below function:

```
function(int n) {
        for(int i = 1 ; i <= n ; i + + ) // this loops executes n times
                // this loops executes logn times from our logarithms guideline
                for(int j = 1 ; j <= n ; j * = 2 )
                        printf( " * " );
}
```

Complexity of above program is : $O(nlogn)$.

Problem-47 Running time of following program?

```
function(int n) {
        for(int i = 1 ; i <= n/3 ; i + + )
                for(int j = 1 ; j <= n ; j += 4 )
                        printf( " * " );
}
```

Solution: Consider the comments in below function:

```
function(int n) { // this loops executes n/3 times
        for(int i = 1 ; i <= n/3 ; i + + )
                // this loops executes n/4 times
                for(int j = 1 ; j <= n ; j += 4)
                        printf( " * " );
}
```

The time complexity of this program is : $O(n^2)$.

Problem-48 Find the complexity of the below function.

```
void function(int n) {
    if(n <= 1) return;
    if(n > 1) {
            printf (" * ");
            function( n/2 );
            function( n/2 );
      }
```

```
}
```

Solution: Consider the comments in below function:

```
void function(int n) {
        //constant time
        if(n <= 1) return;
        if(n > 1) {
                //constant time
                printf (" * ");
                //recursion with n/2 value
                function( n/2 );
                //recursion with n/2 value
                function( n/2 );
        }
}
```

The recurrence for this function is: $T(n) = 2T\left(\frac{n}{2}\right) + 1$. Using master theorem, we get $T(n) = O(n)$.

Problem-49 Find the complexity of the below function.

```
function(int n) {
        int i=1;
        while (i < n) {
                int j=n;
                while(j > 0)
                        j = j/2;
                i=2*i;
        } // i
}
```

Solution:

```
function(int n) {
        int i=1;
        while (i < n) {
                int j=n;
                while(j > 0)
                        j = j/2;   //logn code
                i=2*i; //logn times
        } // i
}
```

Time Complexity: $O(logn * logn) = O(log^2 n)$.

Problem-50 $\sum_{1 \le k \le n} O(n)$, where $O(n)$ stands for order n is:

(a) $O(n)$ (b) $O(n^2)$ (c) $O(n^3)$ (d) $O(3n^2)$ (e) $O(1.5n^2)$

Solution: (b). $\sum_{1 \le k \le n} O(n) = O(n) \sum_{1 \le k \le n} 1 = O(n^2)$.

Problem-51 Which of the following three claims are correct

I $(n + k)^m = \Theta(n^m)$, where k and m are constants II $2^{n+1} = O(2^n)$ III $2^{2n+1} = O(2^n)$

(a) I and II (b) I and III (c) II and III (d) I, II and III

Solution: (a). (I) $(n + k)^m = n^k + c1*n^{k-1} + \ldots k^m = \Theta(n^k)$ and (II) $2^{n+1} = 2*2^n = O(2^n)$

Problem-52 Consider the following functions:

$f(n) = 2^n$ $g(n) = n!$ $h(n) = n^{logn}$

Which of the following statements about the asymptotic behaviour of $f(n)$, $g(n)$, and $h(n)$ is true?

(A) f(n) = O(g(n)); g(n) = O(h(n)) (B) f(n) = Ω (g(n)); g(n) = O(h(n))
(C) g(n) = O(f(n)); h(n) = O(f(n)) (D) h(n) = O(f(n)); g(n) = Ω (f(n))

Solution: (D). According to rate of growths: h(n) < f(n) < g(n) (g(n) is asymptotically greater than f(n) and f(n) is asymptotically greater than h(n)). We can easily see above order by taking logarithms of the given 3 functions: $lognlogn < n < log(n!)$. Note that, $log(n!) = O(nlogn)$.

Problem-53 Consider the following segment of C-code:

```
int j=1, n;
while (j <=n)
        j = j*2;
```

The number of comparisons made in the execution of the loop for any $n > 0$ is:
(A) ceil(log_2^n)+ 1 (B) n (C) ceil(log_2^n) (D) floor(log_2^n) + 1

Solution: (a). Let us assume that the loop executes k times. After k^{th} step the value of j is 2^k. Taking logarithms on both sides gives $k = log_2^n$. Since we are doing one more comparison for exiting from loop, the answer is ceil(log_2^n)+ 1.

Problem-54 Consider the following C code segment. Let T(n) denotes the number of times the for loop is executed by the program on input n. Which of the following is TRUE?

```
int IsPrime(int n){
 for(int i=2;i<=sqrt(n);i++)
  if(n%i == 0)
   {       printf("Not Prime\n"); return 0;    }
 return 1;
}
```

(A) T(n) = O($\sqrt{n}$) and T(n) = Ω($\sqrt{n}$) (B) T(n) = O($\sqrt{n}$) and T(n) = Ω(1)
(C) T(n) = O(n) and T(n) = Ω($\sqrt{n}$) (D) None of the above

Solution: (B). Big O notation describes the tight upper bound and Big Omega notation describes the tight lower bound for an algorithm. The for loop in the question is run maximum $\sqrt{n}$ times and minimum 1 time. Therefore, T(n) = O($\sqrt{n}$) and T(n) = Ω(1).

Problem-55 In the following C function, let $n \geq m$. How many recursive calls are made by this function?

```
int gcd(n,m){
        if (n%m ==0) return m;
        n = n%m;
        return gcd(m,n);
}
```

(A) Θ(log_2^n) (B) Ω(n) (C) Θ($log_2 log_2^n$) (D) Θ(n)

Solution: No option is correct. Big O notation describes the tight upper bound and Big Omega notation describes the tight lower bound for an algorithm. For $m = 2$ and for all $n = 2^i$, running time is O(1) which contradicts every option.

Problem-56 Suppose $T(n) = 2T(n/2) + n$, T(0)=T(1)=1. Which one of the following is FALSE?

(A) $T(n) = O(n^2)$ (B) $T(n) = \Theta(nlogn)$ (C) $T(n) = \Omega(n^2)$ (D) $T(n) = O(nlogn)$

Solution: (C). Big O notation describes the tight upper bound and Big Omega notation describes the tight lower bound for an algorithm. Based on master theorem, we get $T(n) = \Theta(nlogn)$. This indicates that tight lower bound and tight upper bound are same. That means, O($nlogn$) and Ω($nlogn$) are correct for given recurrence. So option (C) is wrong.

Chapter 2 ANALYSIS OF ALGORITHMS

Introduction

The objective of this chapter is to explain the importance of analysis of algorithms, their notations, relationships and solving as many problems as possible. We first concentrate on understanding the importance of analysis and then slowly move towards analyzing the algorithms with different notations and finally, the problems. After completion of this chapter you should be able to find the complexity of any given algorithm (especially recursive functions).

What is an Algorithm?

Just to understand better, let us consider the problem of preparing an omelet. For preparing omelet, general steps which we follow are:

1) Get the frying pan.
2) Get the oil.
    a. Do we have oil?
        i. If yes, put it in the pan.
        ii. If no, do we want to buy oil?
            1. If yes, then go out and buy.
            2. If no, we can terminate.
3) Turn on the stove, etc..

What we are doing is, for a given problem (preparing an omelet), giving step by step procedure for solving it. Formal definition of an algorithm can be given as:

An algorithm is the step-by-step instructions to a given problem.

One important note to remember while writing the algorithms is: we do not have to prove each step of the algorithm.

Why Analysis of Algorithms?

If we want to go from city "*A*" to city "*B*". There can be many ways of doing this: by flight, by bus, by train and also by cycle. Depending on the availability and convenience we choose the one which suits us. Similarly, in computer science there can be multiple algorithms exist for solving the same problem (for

example, sorting problem has lot of algorithms like insertion sort, selection sort, quick sort and many more). Algorithm analysis helps us determining which of them is efficient in terms of time and space consumed.

Goal of Analysis of Algorithms?

The goal of *analysis of algorithms* is to compare algorithms (or solutions) mainly in terms of running time but also in terms of other factors (e.g., memory, developer's effort etc.)

What is Running Time Analysis?

It is the process of determining how processing time increases as the size of the problem (input size) increases. Input size is number of elements in the input and depending on the problem type the input may be of different types. In general, we encounter the following types of inputs.

- Size of an array
- Polynomial degree
- Number of elements in a matrix
- Number of bits in binary representation of the input
- Vertices and edges in a graph

How to Compare Algorithms?

To compare algorithms, let us define some *objective measures*.

Execution times? *Not a good measure* as execution times are specific to a particular computer.

Number of statements executed? *Not a good measure* since the number of statements varies with the programming language as well as the style of the individual programmer.

Ideal Solution?

Let us assume that we expressed running time of given algorithm as a function of the input size n (i.e., $f(n)$). We can compare these different functions corresponding to running times and this kind of comparison is independent of machine time, programming style, etc..

What is Rate of Growth?

The rate at which the running time increases as a function of input is called *rate of growth*. Let us assume that you went to a shop for buying a car and a cycle. If your friend sees you there and asks what you are buying then in general we say *buying a car*. This is because cost of car is too big compared to cost of cycle (approximating the cost of cycle to cost of car).

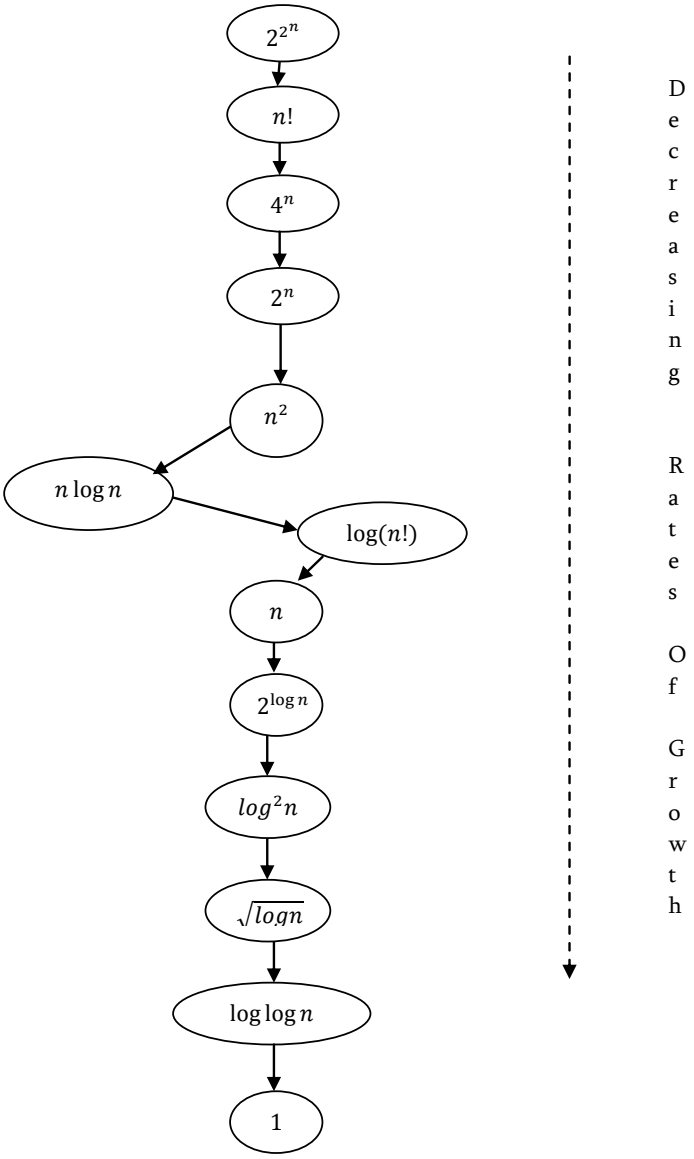
Total Cost = cost_of_car + cost_of_cycle
Total Cost ≈ cost_of_car (approximation)

For the above example, we can represent the cost of car and cost of cycle in terms of function and for a given function we ignore the low order terms that are relatively insignificant (for large value of input size, n). As an example in the below case, n^4, $2n^2$, $100n$ and 500 are the individual costs of some function and we approximate it to n^4. Since, n^4 is the highest rate of growth.

$$n^4 + 2n^2 + 100n + 500 \approx n^4$$

Commonly used Rate of Growths

Below diagram shows the relationship between different rates of growth.



Below is the list of rate of growths which come across in remaining chapters.

Time complexity	Name	Example
1	Constant	Adding an element to the front of a linked list
$logn$	Logarithmic	Finding an element in a sorted array
n	Linear	Finding an element in an unsorted array
$nlogn$	Linear Logarithmic	Sorting n items by 'divide-and-conquer'-Mergesort
n^2	Quadratic	Shortest path between two nodes in a graph
n^3	Cubic	Matrix Multiplication
2^n	Exponential	The Towers of Hanoi problem

Types of Analysis

If we have an algorithm for a problem and want to know on what inputs the algorithm is taking less time (performing well) and on what inputs the algorithm is taking huge time.

We have already seen that an algorithm can be represented in the form of an expression. That means we represent the algorithm with multiple expressions: one for case where it is taking the less time and other for case where it is taking the more time. In general the first case is called the best case and second case is called the worst case for the algorithm.

To analyze an algorithm we need some kind of syntax and that forms the base for asymptotic analysis/notation. There are three types of analysis:

- **Worst case**
  - Defines the input for which the algorithm takes huge time.
  - Input is the one for which the algorithm runs the slower.
- **Best case**
  - Defines the input for which the algorithm takes lowest time.
  - Input is the one for which the algorithm runs the fastest.
- **Average case**
  - Provides a prediction about the running time of the algorithm
  - Assumes that the input is random

$$\text{Lower Bound} <= \text{Average Time} <= \text{Upper Bound}$$

For a given algorithm, we can represent best case, worst case, and average case analysis in the form of expressions. As an example, let $f(n)$ be the function which represents the given algorithm.

$f(n) = n^2 + 500$, for worst case
$f(n) = n + 100n + 500$, for best case

Similarly, for average case too. The expression defines the inputs with which the algorithm takes the average running time (or memory).

Asymptotic Notation?

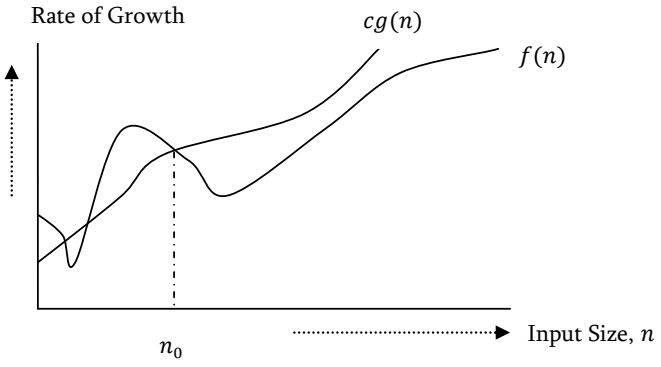
Having the expressions for best case, average case and worst case, for all the three cases we need to identify the upper bound, lower bounds. In order to represent these upper bound and lower bounds we need some syntax and that is the subject of following discussion. Let us assume that the given algorithm is represented in the form of function $f(n)$.

Big-O Notation

This notation gives the $tight$ upper bound of the given function. Generally we represent it as $f(n) = O(g(n))$. That means, at larger values of n, the upper bound of $f(n)$ is $g(n)$.

For example, if $f(n) = n^4 + 100n^2 + 10n + 50$ is the given algorithm, then n^4 is $g(n)$. That means $g(n)$ gives the maximum rate of growth for $f(n)$ at larger values of n.

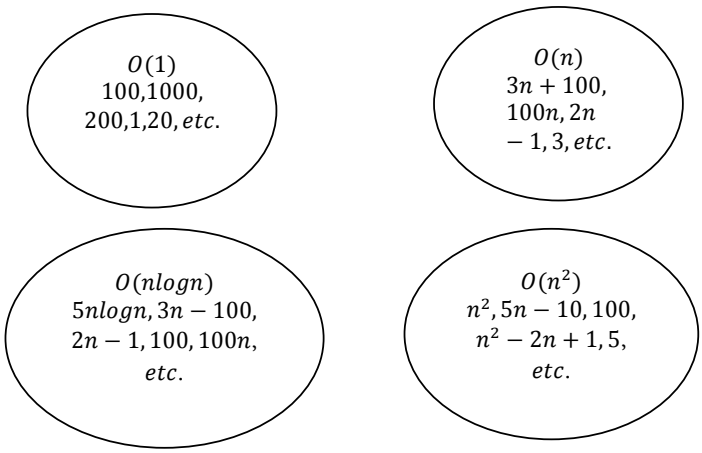
Let us see the O −notation with little more detail. O −notation defined as $O(g(n)) = \{f(n)$: there exist positive constants c and n_0 such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0\}$. $g(n)$ is an asymptotic tight upper bound for $f(n)$. Our objective is to give smallest rate of growth $g(n)$ which is greater than or equal to given algorithms rate of growth $f(n)$.
In general, we discard lower values of n. That means the rate of growth at lower values of n is not important. In the below figure, n_0 is the point from which we need to consider the rate of growths for a given algorithm. Below n_0 the rate of growths could be different.



Big-O Visualization

$O(g(n))$ is the set of functions with smaller or same order of growth as $g(n)$. For example, $O(n^2)$ includes $O(1), O(n), O(nlogn)$ etc..

Note: Analyze the algorithms at larger values of n only. What this means is, below n_0 we do not care for rate of growth.



Big-O Examples

Example-1 Find upper bound for $f(n) = 3n + 8$

Solution: $3n + 8 \leq 4n$, for all $n \geq 1$
$\therefore 3n + 8 = O(n)$ with c = 4 and $n_0 = 8$

Example-2 Find upper bound for $f(n) = n^2 + 1$

Solution: $n^2 + 1 \leq 2n^2$, for all $n \geq 1$
$\therefore n^2 + 1 = O(n^2)$ with $c = 2$ and $n_0 = 1$

Example-3 Find upper bound for $f(n) = n^4 + 100n^2 + 50$

Solution: $n^4 + 100n^2 + 50 \leq 2n^4$, for all $n \geq 1$
$\therefore n^4 + 100n^2 + 50 = O(n^4)$ with $c = 2$ and $n_0 = 100$

Example-4 Find upper bound for $f(n) = 2n^3 - 2n^2$

Solution: $2n^3 - 2n^2 \leq 2n^3$, for all $n \geq 1$
$\therefore 2n^3 - 2n^2 = O(2n^3)$ with $c = 2$ and $n_0 = 1$

Example-5 Find upper bound for $f(n) = n$

Solution: $n \leq n^2$, for all $n \geq 1$
$\therefore n = O(n^2)$ with $c = 1$ and $n_0 = 1$

Example-6 Find upper bound for $f(n) = 410$

Solution: $410 \leq 410$, for all $n \geq 1$
$\therefore 100 = O(1)$ with $c = 1$ and $n_0 = 1$

No Uniqueness?

There is no unique set of values for n_0 and c in proving the asymptotic bounds. Let us consider, $100n + 5 = O(n^2)$. For this function there are multiple n_0 and c values possible.

Solution1: $100n + 5 \leq 100n + n = 101n \leq 101n^2$ for all $n \geq 5$, $n_0 = 5$ and $c = 101$ is a solution.

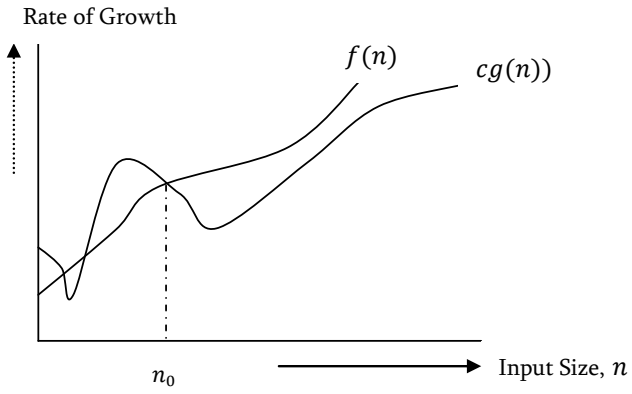
Solution2: $100n + 5 \leq 100n + 5n = 105n \leq 105n^2$ for all $n \geq 1, n_0 = 1$ and $c = 105$ is also a solution.

Omega-Ω Notation

Similar to O discussion, this notation gives the tighter lower bound of the given algorithm and we represent it as $f(n) = \Omega(g(n))$. That means, at larger values of n, the tighter lower bound of $f(n)$ is $g(n)$.

For example, if $f(n) = 100n^2 + 10n + 50$, $g(n)$ is $\Omega(n^2)$.

The Ω notation can be defined as $\Omega(g(n)) = \{f(n)$: there exist positive constants c and n_0 such that $0 \leq cg(n) \leq f(n)$ for all n $\geq n_0\}$. $g(n)$ is an asymptotic tight lower bound for $f(n)$. Our objective is to give largest rate of growth $g(n)$ which is less than or equal to given algorithms rate of growth $f(n)$.



Ω Examples

Example-1 Find lower bound for $f(n) = 5n^2$

Solution: $\exists c$, n_0 Such that: $0 \leq cn \leq 5n^2 \Rightarrow cn \leq 5n^2 \Rightarrow c = 1$ and $n_0 = 1$
$\therefore 5n^2 = \Omega(n)$ with $c = 1$ and $n_0 = 1$

Example-2 Prove $f(n) = 100n + 5 \neq \Omega(n^2)$

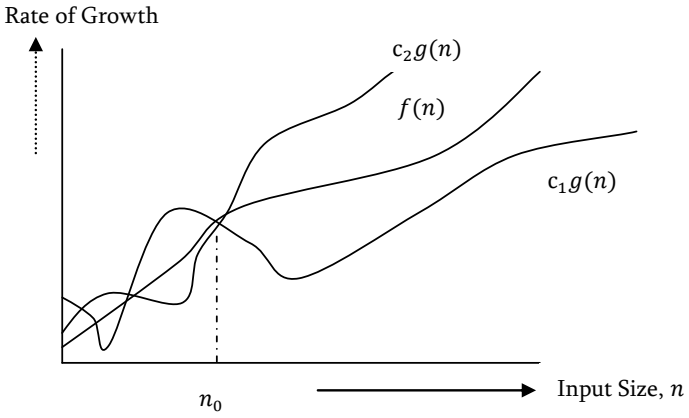
Solution: $\exists$ c, n_0 Such that: $0 \leq cn^2 \leq 100n + 5$
$100n + 5 \leq 100n + 5n\ (\forall n \geq 1) = 105n$

$cn^2 \leq 105n \Rightarrow n(cn - 105) \leq 0$
Since n is positive $\Rightarrow cn - 105 \leq 0 \Rightarrow n \leq 105/c$
$\Rightarrow$ Contradiction: n cannot be smaller than a constant

Example-3 $n = \Omega(2n),\ n^3 = \Omega(n^2),\ n = \Omega(logn)$

Theta-θ Notation

This notation decides whether the upper and lower bounds of a given function (algorithm) are same or not. The average running time of algorithm is always between lower bound and upper bound. If the upper bound (O) and lower bound (Ω) gives the same result then θ notation will also have the same rate of growth. As an example, let us assume that $f(n) = 10n + n$ is the expression. Then, its tight upper bound $g(n)$ is $O(n)$. The rate of growth in best case is $g(n) = O(n)$. In this case, rate of growths in best case and worst are same. As a result, the average case will also be same. For a given function (algorithm), if the rate of growths (bounds) for O and Ω are not same then the rate of growth θ case may not be same.



Now consider the definition of θ notation. It is defined as $\Theta(g(n)) = \{f(n)$: there exist positive constants c_1, c_2 and n_0 such that $0 \leq c_1g(n) \leq f(n) \leq c_2g(n)$ for all $n \geq n_0\}$. $g(n)$ is an asymptotic tight bound for $f(n)$. $\theta(g(n))$ is the set of functions with the same order of growth as $g(n)$.

Θ Examples

Example-1 Find θ bound for $f(n) = \frac{n^2}{2} - \frac{n}{2}$

Solution: $\frac{n^2}{5} \leq \frac{n^2}{2} - \frac{n}{2} \leq n^2$, for all, $n \geq 1$
$\therefore \frac{n^2}{2} - \frac{n}{2} = \theta(n^2)$ with $c_1 = 1/5, c_1 = 1$ and $n_0 = 1$

Example-2 Prove $n \neq \theta(n^2)$

Solution: $c_1 n^2 \leq n \leq c_2n^2 \Rightarrow$ only holds for: $n \leq 1/c_1$
$\therefore n \neq \Theta(n^2)$

Example-3 Prove $6n^3 \neq \theta(n^2)$

Solution: $c_1 n^2 \leq 6n^3 \leq c_2 n^2 \Rightarrow$ only holds for: $n \leq c_2/6$
$\therefore 6n^3 \neq \Theta(n^2)$

Example-4 Prove $n \neq \theta(logn)$

Solution: $c_1 logn \leq n \leq c_2 log\, n \Rightarrow c_2 \geq \frac{n}{\log n}, \forall n \geq n_0$ – Impossible

Important Notes

For analysis (best case, worst case and average) we try to give upper bound (O) and lower bound (Ω) and average running time (θ). From the above examples, it should also be clear that, for a given function (algorithm) getting upper bound (O) and lower bound (Ω) and average running time (θ) may not be possible always. For example, if we are discussing the best case of an algorithm, then we try to give upper bound (O) and lower bound (Ω) and average running time (θ).

In the remaining chapters we generally concentrate on upper bound (O) because knowing lower bound (Ω) of an algorithm is of no practical importance and we use θ notation if upper bound (O) and lower bound (Ω) are same.

Why is it called Asymptotic Analysis?

From the above discussion (for all the three notations: worst case, best case and average case), we can easily understand that, in every case for a given function $f(n)$ we are trying to find other function $g(n)$ which approximates $f(n)$ at higher values of n. That means, $g(n)$ is also a curve which approximates $f(n)$ at higher values of n. In mathematics we call such curve as *asymptotic curve*. In other terms, $g(n)$ is the asymptotic curve for $f(n)$. For this reason, we call algorithm analysis as *asymptotic analysis*.

Guidelines for Asymptotic Analysis?

There are some general rules to help us in determining the running time of an algorithm. Below are few of them.

1) **Loops**: The running time of a loop is, at most, the running time of the statements inside the loop (including tests) multiplied by the number of iterations.

```
// executes n times
for (i=1; i<=n; i++)
{
    m = m + 2; // constant time, c
}
```

Total time = a constant $c \times n = cn = O(n)$.

2) **Nested loops:** Analyze from inside out. Total running time is the product of the sizes of all the loops.

```
//outer loop executed n times
for (i=1; i<=n; i++)
{
    // inner loop executed n times
    for (j=1; j<=n; j++)
    {
            k = k+1; //constant time
    }
}
```

Total time = $c \times n \times n = cn^2 = O(n^2)$.

3) **Consecutive statements:** Add the time complexities of each statement.

```
x = x +1; //constant time

// executed n times
for (i=1; i<=n; i++)
{
    m = m + 2; //constant time
}

//outer loop executed n times
for (i=1; i<=n; i++)
{
    //inner loop executed n times
    for (j=1; j<=n; j++)
    {
            k = k+1; //constant time
    }
}
```

Total time = $c_0 + c_1 n + c_2 n^2 = O(n^2)$.

4) **If-then-else statements:** Worst-case running time: the test, plus *either* the *then* part *or* the *else* part (whichever is the larger).

```
//test: constant
if (length ( ) != otherStack. length ( ) )
{
    return false; //then part: constant
```

```
}
else
{
    // else part: (constant + constant) * n
    for (int n = 0; n < length( ); n++)
    {
            // another if : constant + constant (no else part)
            if (!list[n].equals(otherStack.list[n]))
            //constant
            return false;
    }
}
```

Total time = $c_0 + c_1 + (c_2 + c_3) * n = O(n)$.

5) **Logarithmic complexity:** An algorithm is $O(logn)$ if it takes a constant time to cut the problem size by a fraction (usually by ½).

As an example let us consider the following program:

```
for (i=1; i<=n;)
{
    i = i*2;
}
```

If we observe carefully, the value of i is doubling every time. Initially $i = 1$, in next step $i = 2$, and in subsequent steps $i = 4, 8$ and so on.

Let us assume that the loop is executing some k times. That means at k^{th} step $2^i = n$ and we come out of loop. So, if we take logarithm on both sides,

$log(2^i) = logn$
$ilog2 = logn$
$i = logn$ //if we assume base-2

So the total time = $O(logn)$.

Note: Similarly, for the below case also, worst case rate of growth is $O(logn)$. That means, the same discussion holds good for decreasing sequence also.

```
for (i=n; i<=1;)
{
    i = i/2;
}
```

Another example algorithm is binary search: finding a word in a dictionary of n pages

- Look at the centre point in the dictionary
- Is word towards left or right of centre?
- Repeat process with left or right part of dictionary until the word is found

Properties of Notations

- Transitivity: $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$. Valid for O and Ω also.
- Reflexivity: $f(n) = \Theta(f(n))$. Valid for O and Ω also.
- Symmetry: $f(n) = \Theta(g(n))$ if and only if g(n) $= \Theta$(f(n)).
- Transpose symmetry: $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$.

Commonly used Logarithms and Summations

Logarithms

$$\log x^y = y \log x$$
$$\log n = \log_e^n$$
$$\log xy = \log x + \log y$$
$$\log^k n = (\log n)^k$$
$$\log \log n = \log(\log n)$$
$$\log \frac{x}{y} = \log x - \log y$$
$$a^{\log_b^x} = x^{\log_b^a} \log_b^x = \frac{\log_a^x}{\log_a^b}$$

Arithmetic series

$$\sum_{K=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Geometric series

$$\sum_{k=1}^{n} x^k = 1 + x + x^2 \ldots + x^n = \frac{x^{n+1} - 1}{x - 1} (x \neq 1)$$

Harmonic series

$$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \ldots + \frac{1}{n} \approx \log n$$

Other important formulae

$$\sum_{k=1}^{n} \log k \approx n \log n$$
$$\sum_{k=1}^{n} k^p = 1^p + 2^p + \cdots + n^p \approx \frac{1}{p+1} n^{p+1}$$

Master Theorem for Divide and Conquer

In all divide and conquer algorithms we divide the problem into subproblems, each of which is some part of the original problem, and then perform some additional work to compute the final answer. As an example, if we consider merge sort [for details, refer *Sorting* chapter], it operates on two subproblems, each of which is half the size of the original, and then uses O(n) additional work for merging. This gives the running time equation:

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$

The following theorem can be used to determine the running time of divide and conquer algorithms. For a given program (algorithm), first we try to find the recurrence relation for the problem. If the recurrence is of the below form then we directly give the answer without fully solving it.

If the recurrence is of the form $T(n) = aT(\frac{n}{b}) + \Theta(n^k log^p n)$, where $a \geq 1, b > 1, k \geq 0$ and p is a real number, then:

1) If $a > b^k$, then $T(n) = \Theta\left(n^{log_b^a}\right)$
2) If $a = b^k$
   a. If $p > -1$, then $T(n) = \Theta\left(n^{log_b^a} log^{p+1} n\right)$
   b. If $p = -1$, then $T(n) = \Theta\left(n^{log_b^a} loglogn\right)$
   c. If $p < -1$, then $T(n) = \Theta\left(n^{log_b^a}\right)$
3) If $a < b^k$
   a. If $p \geq 0$, then $T(n) = \Theta\left(n^k log^p n\right)$
   b. If $p < 0$, then $T(n) = O\left(n^k\right)$

Problems Divide and Conquer Master Theorem

For each of the following recurrences, give an expression for the runtime $T(n)$ if the recurrence can be solved with the Master Theorem. Otherwise, indicate that the Master Theorem does not apply.

Problem-1 $T(n) = 3T\ (n/2) + n^2$

Solution: $T(n) = 3T\ (n/2) + n^2 \Rightarrow T\ (n) = \Theta(n^2)$ (Master Theorem Case 3.a)

Problem-2 $T(n) = 4T\ (n/2) + n^2$

Solution: $T(n) = 4T\ (n/2) + n^2 \Rightarrow T\ (n) = \Theta(n^2 logn)$ (Master Theorem Case 2.a)

Problem-3 $T(n) = T(n/2) + 2^n$

Solution: $T(n) = T(n/2) + 2^n \Rightarrow \Theta(2^n)$ (Master Theorem Case 3.a)

Problem-4 $T(n) = 2^n T(n/2) + n^n$

Solution: $T(n) = 2^n T(n/2) + n^n$ => Does not apply (a is not constant)

Problem-5 $T(n) = 16T(n/4) + n$

Solution: $T(n) = 16T(n/4) + n \Rightarrow T(n) = \Theta(n^2)$ (Master Theorem Case 1)

Problem-6 $T(n) = 2T(n/2) + n \log n$

Solution: $T(n) = 2T(n/2) + n \log n \Rightarrow T(n) = \Theta(n\log^2 n)$ (Master Theorem Case 2.a)

Problem-7 $T(n) = 2T(n/2) + n/\log n$

Solution: $T(n) = 2T(n/2) + n/\log n \Rightarrow T(n) = \Theta(n\text{loglogn})$ (Master Theorem Case 2.b)

Problem-8 $T(n) = 2T(n/4) + n^{0.51}$

Solution: $T(n) = 2T(n/4) + n^{0.51} \Rightarrow T(n) = O(n^{0.51})$ (Master Theorem Case 3.b)

Problem-9 $T(n) = 0.5T(n/2) + 1/n$

Solution: $T(n) = 0.5T(n/2) + 1/n \Rightarrow$ Does not apply ($a < 1$)

Problem-10 $T(n) = 6T(n/3) + n^2 \log n$

Solution: $T(n) = 6T(n/3) + n^2\log n \Rightarrow T(n) = \Theta(n^2\log n)$ (Master Theorem Case 3.a)

Problem-11 $T(n) = 64T(n/8) - n^2\log n$

Solution: $T(n) = 64T(n/8) - n^2\log n \Rightarrow$ Does not apply (function is not positive)

Problem-12 $T(n) = 7T(n/3) + n^2$

Solution: $T(n) = 7T(n/3) + n^2 \Rightarrow T(n) = \Theta(n^2)$ (Master Theorem Case 3.as)

Problem-13 $T(n) = 4T(n/2) + \log n$

Solution: $T(n) = 4T(n/2) + \log n \Rightarrow T(n) = \Theta(n^2)$ (Master Theorem Case 1)

Problem-14 $T(n) = 16T(n/4) + n!$

Solution: $T(n) = 16T(n/4) + n! \Rightarrow T(n) = \Theta(n!)$ (Master Theorem Case 3.a)

Problem-15 $T(n) = \sqrt{2}T(n/2) + \log n$

Solution: $T(n) = \sqrt{2}T(n/2) + \log n \Rightarrow T(n) = \Theta(\sqrt{n})$ (Master Theorem Case 1)

Problem-16 $T(n) = 3T(n/2) + n$

Solution: $T(n) = 3T(n/2) + n \Rightarrow T(n) = \Theta(n^{\log 3})$ (Master Theorem Case 1)

Problem-17 $T(n) = 3T(n/3) + \sqrt{n}$

Solution: $T(n) = 3T(n/3) + \sqrt{n} \Rightarrow T(n) = \Theta(n)$ (Master Theorem Case 1)

Problem-18 $T(n) = 4T(n/2) + cn$

Solution: $T(n) = 4T(n/2) + cn \Rightarrow T(n) = \Theta(n^2)$ (Master Theorem Case 1)

Problem-19 $T(n) = 3T(n/4) + nlogn$

Solution: $T(n) = 3T(n/4) + nlogn \Rightarrow T(n) = \Theta(nlogn)$ (Master Theorem Case 3.a)

Problem-20 $T(n) = 3T(n/3) + n/2$

Solution: $T(n) = 3T(n/3) + n/2 \Rightarrow T(n) = \Theta(nlogn)$ (Master Theorem Case 2.a)

Master Theorem for Subtract and Conquer Recurrences

Let $T(n)$ be a function defined on positive n, and having the property

$$T(n) = \begin{cases} c, & \text{if } n \leq 1 \\ aT(n-b) + f(n), & \text{if } n > 1 \end{cases}$$

for some constants $c, a > 0, b > 0, k \geq 0$, and function $f(n)$. If $f(n)$ is in $O(n^k)$, then

$$T(n) = \begin{cases} O(n^k), & \text{if a} < 1 \\ O(n^{k+1}), & \text{if a} = 1 \\ O\left(n^k a^{\frac{n}{b}}\right), & \text{if a} > 1 \end{cases}$$

Variant of subtraction and conquer master theorem

The solution to the equation $T(n) = T(\alpha n) + T((1-\alpha)n) + \beta n$, where $0 < \alpha < 1$ and $\beta > 0$ are constants, is $O(nlogn)$.

Problems on Algorithms Analysis

Note: From the following problems, try to understand in what cases we get different complexities ($O(n), O(logn), O(loglogn)$ etc..).

Problem-21 Find the complexity of the below recurrence:

$$T(n) = \begin{cases} 3T(n-1), if\ n > 0, \\ 1, \quad otherwise \end{cases}$$

Solution: Let us try to solve this function with substitution.

$T(n) = 3T(n-1)$

$T(n) = 3(3T(n-2)) = 3^2T(n-2)$

$T(n) = 3^2(3T(n-3))$

.

.

$T(n) = 3^nT(n-n) = 3^nT(0) = 3^n$

This clearly shows that the complexity of this function is $O(3^n)$.

Note: We can use the *Subtraction and Conquer* master theorem for this problem.

Problem-22 Find the complexity of the below recurrence:

$$T(n) = \begin{cases} 2T(n-1) - 1, if\ n > 0, \\ 1, \qquad\qquad otherwise \end{cases}$$

Solution: Let us try to solve this function with substitution.

$T(n) = 2T(n-1) - 1$

$T(n) = 2(2T(n-2) - 1) - 1 = 2^2T(n-2) - 2 - 1$

$T(n) = 2^2(2T(n-3) - 2 - 1) - 1 = 2^3T(n-4) - 2^2 - 2^1 - 2^0$

$T(n) = 2^nT(n-n) - 2^{n-1} - 2^{n-2} - 2^{n-3} \ldots . 2^2 - 2^1 - 2^0$

$T(n) = 2^n - 2^{n-1} - 2^{n-2} - 2^{n-3} \ldots . 2^2 - 2^1 - 2^0$

$T(n) = 2^n - (2^n - 1)$ $[note: 2^{n-1} + 2^{n-2} + \cdots + 2^0 = 2^n]$

$T(n) = 1$

∴ Complexity is $O(1)$. Note that while the recurrence relation looks exponential the solution to the recurrence relation here gives a different result.

Problem-23 What is the running time of the following function (specified as a function of the input value n)

```
void Function(int n)
{
        int i=1 ;
        int s=1 ;
        while( s <= n)
        {
                i++ ;
                s= s+i ;
                print(\*");
        }
}
```

Solution: Consider the comments in below function:

```
void Function (int n)
{
        int i=1 ;
        int s=1 ;
        // s is increasing not at rate 1 but i
        while( s <= n)
        {
                i++ ;
```

```
                s= s+i ;
                print("*");
        }
}
```

We can define the terms 's' according to the relation $s_i = s_{i-1} + i$. The value of 'i' increases by one for each iteration. So the value contained in 's' at the i^{th} iteration is the sum of the first 'i' positive integers. If k is the total number of iterations taken by the program, then the while loop terminates once.

$$1 + 2 + \ldots + k = \frac{k(k+1)}{2} > n \Longrightarrow k = O(\sqrt{n}).$$

Problem-24 Find the complexity of the function given below.

```
void Function(int n)
{
        int i, count =0;;
        for(i=1; i*i<=n; i++)
                count++;
}
```

Solution: Consider the comments in below function:

```
void Function(int n)
{
        int i, count =0;;
        for(i=1; i*i<=n; i++)
                count++;
}
```

In the above function the loop will end, if $i^2 \leq n \Longrightarrow T(n) = O(\sqrt{n})$. The reasoning is same as that of Problem-23.

Problem-25 What is the complexity of the below program:

```
void function(int n)
{
        int i, j, k , count =0;
        for(i=n/2; i<=n; i++)
                for(j=1; j + n/2<=n; j= j++)
                        for(k=1; k<=n; k= k * 2)
                                count++;
}
```

Solution: Consider the comments in the following function.

```
void function(int n)
{
        int i, j, k , count =0;
```

```
        //outer loop execute n/2 times
        for(i=n/2; i<=n; i++)
                //Middle loop executes n/2 times
                for(j=1; j + n/2<=n; j= j++)
                        //outer loop execute logn times
                        for(k=1; k<=n; k= k * 2)
                                count++;
}
```

The complexity of the above function is $O(n^2 \log n)$.

Problem-26 What is the complexity of the below program:

```
void function(int n)
{
        int i, j, k , count =0;
        for(i=n/2; i<=n; i++)
                for(j=1; j<=n; j= 2 * j)
                        for(k=1; k<=n; k= k * 2)
                                count++;
}
```

Solution: Consider the comments in the following function.

```
void function(int n)
{
        int i, j, k , count =0;
        //outer loop execute n/2 times
        for(i=n/2; i<=n; i++)
                //Middle loop executes logn times
                for(j=1; j<=n; j= 2 * j)
                        //outer loop execute logn times
                        for(k=1; k<=n; k= k*2)
                                count++;
}
```

The complexity of the above function is $O(n \log^2 n)$.

Problem-27 Find the complexity of the below program.

```
    function( int n )
    {
        if (n == 1) return;
        for( i = 1 ; i <= n ; i + + )
        {
                for( j= 1 ; j <= n ; j + + )
                {
                        print("*" ) ;
```

```
                    break;
                }
            }
        }
```

Solution: Consider the comments in the following function.

```
function( int n )
{
        //constant time
        if ( n == 1 ) return;
        //outer loop execute n times
        for( i = 1 ; i <= n ; i + + )
        {
                // inner loop executes only time due to break statement.
                for( j= 1 ; j <= n ; j + + )
                {
                        print("*" ) ;
                        break;
                }
        }
}
```

The complexity of the above function is $O(n)$. Even though the inner loop is bounded by n, but due to the break statement it is executing only once.

Problem-28 Write a recursive function for the running time $T(n)$ of the function function, whose code is below. Prove using the iterative method that $T(n) = \Theta(n^2)$.

```
function( int n )
{
        if ( n == 1 ) return ;
        for( i = 1 ; i <= n ; i + + )
                for( j = 1 ; j <= n ; j + + )
                        print("*" ) ;
        function( n-3 );
}
```

Solution: Consider the comments in below function:

```
function (int n)
{
        //constant time
        if ( n == 1 ) return ;

        //outer loop execute n times
        for( i = 1 ; i <= n ; i + + )
```

```
                    //inner loop executes n times
                    for( j = 1 ; j <= n ; j + + )
                            //constant time
                            print("*" ) ;
            function( n-3 );
}
```

The recurrence for this code is clearly $T(n) = T(n-3) + cn^2$ for some constant $c > 0$ since each call prints out n^2 asterisks and calls itself recursively on n - 3. Using the iterative method we get:

$$T(n) = T(n-3) + cn^2$$

Using the *Subtraction and Conquer* master theorem, we get $T(n) = \Theta(n^3)$.

Problem-29 Determine Θ bounds for the recurrence relation: $T(n) = 2T\left(\frac{n}{2}\right) + nlogn$

Solution: Using Divide and Conquer master theorem, we get $O(n\ log^2 n)$.

Problem-30 Determine Θ bounds for the recurrence: $T(n) = T(n/2) + T(n/4) + T(n/8) + n$

Solution: Substituting in the recurrence equation, we get:

$$T(n) \leq c1 * \frac{n}{2} + c2 * \frac{n}{4} + c3 * \frac{n}{8} + cn$$
$$\leq k * n, \text{ where } k \text{ is a constant.}$$

Problem-31 Determine Θ bounds for the recurrence relation: $T(n) = T(\lceil n/2 \rceil) + 7$

Solution: Using Master Theorem we get $\Theta(logn)$.

Problem-32 Prove that the running time of the code below is $\Omega(logn)$.

```
Read(int n);
{
        int k = 1 ;
        while( k < n )
                k = 3k;
}
```

Solution: The while loop will terminate once the value of *'k'* is greater than or equal to the value of *'n'*. Since each loop the value of *'k'* is being multiplied by 3, if i is the number of iterations, then *'k'* has the value of 3i after i iterations. That is the loop is terminated upon reaching i iterations when $3^i \geq n \leftrightarrow i \geq \log_3 n$, which shows that $i = \Omega(logn)$.

Problem-33 Solve the following recurrence.

$$T(n) = \begin{cases} 1, & if\ n = 1 \\ T(n-1) + n(n-1), & if\ n \geq 2 \end{cases}$$

Solution: By iteration:

$$T(n) = T(n-2) + (n-1)(n-2) + n(n-1)$$
$$\dots$$
$$T(n) = T(1) + \sum_{i=1}^{n} i(i-1)$$
$$T(n) = T(1) + \sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} i$$
$$T(n) = 1 + \frac{n((n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$$
$$T(n) = \Theta(n^3)$$

Note: We can use the *Subtraction and Conquer* master theorem for this problem.

Problem-34 Consider the following program:

```
Fib[n]
if (n==0) then return 0
else if (n==1) then return 1
else return Fib[n-1]+Fib[n-2]
```

Solution: The recurrence relation for running time of this program is

$$T(n) = T(n-1) + T(n-2) + c.$$

Notice T(n) has two recurrence calls indicating a binary tree. Each step recursively calls the program for n reduced by 1 and 2, so the depth of the recurrence tree is $O(n)$. The number of leaves at depth n is 2^n since this is a full binary tree, and each leaf takes at least $O(1)$ computation for the constant factor. Running time is clearly exponential in n.

Problem-35 Running time of following program?

```
function(n)
{
        for( i = 1 ; i <= n ; i + + )
                for( j = 1 ; j <= n ; j+ = i )
                        print( "*" ) ;
}
```

Solution: Consider the comments in below function:

```
function (n)
{
        //this loop executes n times
        for( i = 1 ; i <= n ; i + + )
```

```
        //this loop executes j times with j increase by the rate of i
        for( j = 1 ; j <= n ; j+ = i )
                print( "*" ) ;
}
```

Its running time is $n \times (\sqrt{n}) = O(n^{\frac{3}{2}})$[since the inner loop is same as that of Problem-23].

Problem-36 What is the complexity of $\sum_{i=1}^{n} log\ i$?

Solution: Using the logarithmic property, $logxy = logx + logy$, we can see that this problem is equivalent to

$$\sum_{i=1}^{n} \log i = \log 1 + \log 2 + \cdots + \log n$$

$$= \log(1 \times 2 \times \ldots \times n)$$

$$= \log(n!)$$

$$\leq \log(n^n)$$

$$\leq n\log n$$

This shows that that the time complexity $= O(nlogn)$.

Problem-37 What is the running time of the following recursive function (specified as a function of the input value n)? First write the recurrence formula and then find its complexity.

```
function(int n)
{
        if (n <= 1)
                return ;

        for (int i=1 ; i <= 3; i++ )
                f(⌈n/3⌉);
}
```

Solution: Consider the comments in below function:

```
function (int n)
{
        //constant time
        if (n <= 1)
                return ;

        //this loop executes with recursive loop of n/3 value
        for (int i=1 ; i <= 3; i++ )
                f(⌈n/3⌉);
}
```

We can assume that for asymptotical analysis $k = \lceil k \rceil$ for every integer $k \geq 1$. The recurrence for this code is $T(n) = 3T(\frac{n}{3}) + \Theta(1)$.
Using master theorem, we get $T(n) = \Theta(n)$.

Problem-38 What is the running time of the following recursive function (specified as a function of the input value n)? First write a recurrence formula, and show its solution using induction.

```
function(int n)
{
        if (n <= 1)
                return;
        for (i=1 ; i <= 3 ; i++ )
                function (n – 1).
}
```

Solution: Consider the comments in below function:

```
function (int n)
{
        //constant time
        if (n <= 1)
                return;
        //this loop executes 3 times with recursive call of n-1 value
        for (i=1 ; i <= 3 ; i++ )
                function (n – 1).
}
```

The $if\ statement$ requires constant time ($O(1)$). With the $for\ loop$, we neglect the loop overhead and only count the three times that the function is called recursively. This implies a time complexity recurrence:

$T(n) = c, if\ n \leq 1;$
$= c + 3T(n-1), if\ n > 1.$

Now we use repeated substitution to guess at the solution when we substitute k times:

$$T(n) = c + 3T(n-1)$$

Using the $Subtraction\ and\ Conquer$ master theorem, we get $T(n) = \Theta(3^n)$.

Problem-39 Write a recursion formula for the running time $T(n)$ of the function f, whose code is below. What is the running time of $function$, as a function of n?

```
function (int n)
{
        if (n <= 1)
                return;
```

```
        int i = 1 ;
        for(i = 1; i < n; i + +)
                print("*");
        function ( 0.8n ) ;
}
```

Solution: Consider the comments in below function:

```
function (int n)
{
        //constant time
        if (n <= 1)
                return;
        //constant time
        int i = 1 ;
        // this loop executes n times with constant time loop
        for(i = 1; i < n; i + +)
                print("*");

        //recursive call with 0.8n
        function ( 0.8n ) ;
}
```

The recurrence for this piece of code is $T(n) = T(.8n) + O(n)$

$$T(n) = T(\frac{4}{5}n) + \Theta(n)$$

$$T(n) = \frac{4}{5}T(n) + \Theta(n)$$

Applying master theorem, we get $T(n) = O(n)$.

Problem-40 Find the complexity of the recurrence: $T(n) = 2T(\sqrt{n}) + logn$

Solution: The given recurrence is not in the master theorem form. Let try to convert this master theorem format. For that let use assume that $n = 2^m$.

Applying logarithm on both sides gives, $logn = mlog2 \Rightarrow m = logn$
Now, the given function becomes,

$$T(n) = T(2^m) = 2T\left(\sqrt{2^m}\right) + m = 2T\left(2^{\frac{m}{2}}\right) + m.$$

To make it simple we assume $S(m) = T(2^m) \Rightarrow S(\frac{m}{2}) = T(2^{\frac{m}{2}}) \Rightarrow S(m) = 2S\left(\frac{m}{2}\right) + m$
Applying the master theorem would result $S(m) = O(mlogm)$
If we substitute $m = logn$ back, $T(n) = S(logn) = O((logn)\, loglogn)$.

Problem-41 Find the complexity of the recurrence: $T(n) = T(\sqrt{n}) + 1$

Solution: We apply the same logic as that of Problem-40 and we get

$$S(m) = S\left(\frac{m}{2}\right) + 1$$

Applying the master theorem would result $S(m) = O(logm)$.
Substituting $m = logn$, gives $T(n) = S(log\ n) = O(loglogn)$.

Problem-42 Find the complexity of the recurrence: $T(n) = 2T(\sqrt{n}) + 1$

Solution: Applying the logic of Problem-40, gives:

$$S(m) = 2S\left(\frac{m}{2}\right) + 1$$

Using the master theorem results $S(m) = O\left(m^{log_2^2}\right) = O(m)$.
Substituting $m = logn$ gives $T(n) = O(logn)$.

Problem-43 Find the complexity of the below function.

```
int Function (int n)
{
        if (n <= 2)
                return 1;
        else
                return (Function (floor(sqrt(n))) + 1);
}
```

Solution: Consider the comments in below function:

```
int Function (int n)
{
        //constant time
        if (n <= 2)
                return 1;
        else
                // executes √n + 1 times
                return (Function (floor(sqrt(n))) + 1);
}
```

For the above function, the recurrence function can be given as: $T(n) = T(\sqrt{n}) + 1$. And, this is same as that of Problem-41.

Problem-44 Analyze the running time of the following recursive procedure as a function of n.

```
void function(int n)
        {
                if ( n < 2 )
                        return;
                else
```

```
                counter = 0;
        for i = 1 to 8 do
                function (n/2);
        for I =1 to n^3 do
                counter = counter + 1;
}
```

Solution: Consider the comments in below function and let us refer to the running time of function (n) as $T(n)$.

```
void function(int n)
{
        //constant time
        if ( n < 2 )
                return;
        else
                counter = 0;
        // this loop executes 8 times with n value half in every call
        for i = 1 to 8 do
                function (n/2);
        // this loop executes n^3 times with constant time loop
        for I =1 to n^3 do
                counter = counter + 1;
}
```

$T(n)$ can be defined as follows:

$$T(n) = 1 \; if \; n < 2,$$
$$= 8T(\frac{n}{2}) + n^3 + 1 \; otherwise.$$

Using the master theorem gives, $T(n) = \Theta(n^{log_2^8} \log n) = \Theta(n^3 \log n)$.

Problem-45 Find the complexity of the below function.

```
temp = 1
repeat
        for i = 1 to n
                temp = temp + 1;
        n = n/2;
until n <= 1
```

Solution: Consider the comments in below function:

```
//const time
temp = 1
repeat
```

```
            // this loops executes n times
            for i = 1 to n
                  temp =  temp + 1;
            //recursive call with n/2 value
            n  = n/2;
      until n <= 1
```

The recurrence for this function is $T(n) = T(\frac{n}{2}) + n$.
Using master theorem, we get, $T(n) = O(n)$.

Problem-46 Running time of following program?

```
function(int n)
{
        for( i = 1 ; i <= n ; i + + )
              for( j = 1 ; j <= n ; j * = 2 )
                    print( "*" ) ;
}
```

Solution: Consider the comments in below function:

```
function(int n)
{
      // this loops executes n times
      for( i = 1 ; i <= n ; i + + )
            // this loops executes logn times from our logarithms
            //guideline
            for( j = 1 ; j <= n ; j * = 2 )
                  print( "*" ) ;
}
```

Complexity of above program is : $O(nlogn)$.

Problem-47 Running time of following program?

```
function(int n)
{
        for( i = 1 ; i <= n/3 ; i + + )
              for( j = 1 ; j <= n ; j += 4 )
                    print( "*" ) ;
}
```

Solution: Consider the comments in below function:

```
function(int n)
{
```