

Optimal Control Design of a Repairable Multistate system

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Abstract:

Keywords:

1. PROBLEM DESCRIPTION

Carefully provide a mathematical description of the problem discussed in this report.

$$\frac{dp_0}{dt} = -\lambda_0 p_0(t) + \int_0^1 \mu_1(x) p_1(x, t) dx + \int_0^1 u^*(x, t) dx \quad (1)$$

$$\frac{\partial p_1(x, t)}{\partial t} + \frac{\partial p_1(x, t)}{\partial x} = -\mu_1(x) p_1(x, t) - u^*(x, t) \quad (2)$$

Given Initial Conditions:

$$p_1(x, 0) = 0 \quad (3)$$

$$p_0(0) = 1 \quad (4)$$

Given Boundary Conditions:

$$p_1(0, t) = \lambda_0 p_0(t) \quad (5)$$

$$p_1(1, t) = 0 \quad (6)$$

The function u^* is given by

$$u^*(x, t) = (0.3 + 0.1 \sin(x)) b(t) \quad (7)$$

The function $b(t)$ represents the input. It is related to the cost constraint function $c(t)$ as given below.

$$b(t) + \int_0^1 \mu_1(x) f(x, t) dx - 0.3 p_0^*(t) = c(t) \quad (8)$$

$$f(x, t) = 0.1 \cos(\pi t) \sin^2(1 - x) \quad (9)$$

Our objective is to find the input $b(t)$ such that the resulting distribution $p_0(t)$ is closest (as measured by the 2-norm) to the optimal distribution $p_0^*(t)$ given below.

$$p_0^*(t) = 0.85 + 0.05 \cos(2\pi t) \quad (10)$$

2. METHODOLOGY

We make couple of substitutions, following the notation that z_i^j refers to the value of z evaluated at time point

i and at position j . The repair time is divided into m subintervals, while the system running time is divided into n subintervals. For the purposes of numerical implementation, we chose m and n to be 20 and 400 respectively.

So

$$p_0(t_j) = v_j \quad 0 \leq j \leq n \quad (11)$$

$$p_1(x_i, t_j) = w_j^i \quad 0 \leq i \leq m, 0 \leq j \leq n \quad (12)$$

$$\mu_1(x_i) = \mu^i \quad 0 \leq i \leq m \quad (13)$$

$$\lambda = \lambda_0 \quad (14)$$

$$(15)$$

Using the new notation, the boundary conditions and initial conditions may be written as follows.

Initial conditions:

$$w_0^i = 0 \quad \forall 0 \leq i \leq m \quad (16)$$

$$v_0 = 1 \quad (17)$$

Boundary Conditions:

$$w_j^{20} = 0 \quad \forall 0 \leq j \leq n \quad (18)$$

$$w_j^0 = \lambda v_j \quad (19)$$

Also condensing,

$$I_j^* = u^*(x_i, t_j) = g^i b_j \quad (20)$$

$$\int_0^1 u^*(x, t_j) dx = \alpha b_j \quad (21)$$

$$\text{where } g^i = (0.3 + 0.1 \sin(x^i))$$

And,

$$b(t) + \int_0^1 \mu_1(x) f(x, t) dx - 0.3 p_0^*(t) = c(t) \quad (22)$$

$$b_j = c_j - f_j$$

$$\text{where } f_j = \int_0^1 \mu_1(x) f(x, t_j) dx - 0.3 p_0^*(t_j) \quad (23)$$

Discretizing (1)

$$\frac{v_{j+1} - v_j}{\tau} = -\lambda v_j + I_j + I_j^* \quad (24)$$

$$I_j = h \left[\frac{\mu^0 w_j^0}{2} + \sum_{k=1}^{19} \mu^k w_j^k + \frac{\mu^{20} w_j^{20}}{2} \right] \quad (25)$$

$$= h \left[\frac{\mu^0 w_j^0}{2} + \sum_{k=1}^{19} \mu^k w_j^k \right]$$

$$I_j^* = \alpha b_j \quad (26)$$

Discretizing (2)

$$\frac{w_{j+1}^i - w_j^i}{\tau} + \frac{w_j^{i+1} - w_j^{i-1}}{2h} = -\mu^i w_j^i - g^i b_j$$

$$w_{j+1}^i = w_j^i - \frac{\tau}{2h} (w_j^{i+1} - w_j^{i-1}) - \tau \mu^i w_j^i - \tau g^i b_j \quad (27)$$

Applying LAX scheme $w_j^i = \frac{w_j^{i-1} + w_j^{i+1}}{2}$ we get,

$$w_{j+1}^i = \left(\frac{w_j^{i+1} + w_j^{i-1}}{2} \right) - \frac{\tau}{2h} (w_j^{i+1} - w_j^{i-1})$$

$$- \tau \mu^i \left(\frac{w_j^{i+1} + w_j^{i-1}}{2} \right)$$

$$- \tau g^i b_j$$

$$w_{j+1}^i = \frac{1}{2} \left(1 - \mu^i \tau + \frac{\tau}{h} \right) w_j^{i-1} +$$

$$\frac{1}{2} \left(1 - \mu^i \tau - \frac{\tau}{h} \right) w_j^{i+1} -$$

$$\tau g^i b_j$$

Under an appropriately defined matrix A , we can re-write the above equation to read

$$\mathbf{w}_{j+1} = A \mathbf{w}_j - b_j \tau \mathbf{g} + \mathbf{e}_1 v_{j+1} \quad (28)$$

$$= (A)^{j+1} \mathbf{w}_0 - \left[\sum_{k=0}^j b_k (A)^{j-k} \right] \mathbf{g} \tau$$

$$+ \left[\sum_{k=0}^j v_{k+1} (A)^{j-k} \right] \mathbf{e}_1 \quad (29)$$

where \mathbf{e}_1 is an $m \times 1$ matrix given by

$$\mathbf{e}_1 = [\lambda, 0, \dots, 0]^T \quad (30)$$

Matrix A has the form:

$$\begin{pmatrix} w_{j+1}^0 \\ w_{j+1}^1 \\ \vdots \\ w_{j+1}^{n-2} \\ w_{j+1}^{n-1} \\ w_{j+1}^n \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & 0 & a_3 & \cdots & 0 & 0 \\ 0 & a_2 & 0 & a_4 & \cdots & 0 \\ 0 & 0 & a_3 & 0 & a_5 & 0 \\ \vdots & & & & & \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_j^0 \\ w_j^1 \\ \vdots \\ w_j^{n-2} \\ w_j^{n-1} \\ w_j^n \end{pmatrix} \quad (31)$$

$$+ b \tau \begin{pmatrix} g^0 \\ g^1 \\ \vdots \\ g^{n-2} \\ g^{n-1} \\ g^n \end{pmatrix} + v_{j+1} \begin{pmatrix} \lambda \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad (32)$$

From (24)

$$v_{j+1} = (1 - \lambda \tau) v_j + \tau I_j + \tau I_j^* \quad (33)$$

$$= (1 - \lambda \tau + \frac{h\tau}{2}) v_j + h \tau \boldsymbol{\mu}^T \mathbf{w}_j + \alpha b_j \tau \quad (34)$$

Substitute the expression for the time evolution for \mathbf{w} in the above to obtain,

$$v_{j+1} = (1 - \lambda \tau + \frac{h\tau}{2}) v_j$$

$$+ h \tau \boldsymbol{\mu}^T (A)^j \mathbf{w}_0$$

$$- \boldsymbol{\mu}^T \left[\sum_{k=0}^{j-1} b_k (A)^{j-1-k} \right] \mathbf{g} \tau$$

$$+ \alpha b_j \tau$$

Let's define

$$\beta_{j,k} = \boldsymbol{\mu}^T (A)^{j-1-k} \mathbf{g} \quad (35)$$

$$\omega_j = \boldsymbol{\mu}^T (A)^j \mathbf{w}_0 \quad (36)$$

$$\gamma = (1 - \lambda \tau + \frac{h\tau}{2}) \quad (37)$$

$$\boldsymbol{\beta}_0 = \begin{pmatrix} \alpha \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (38)$$

$$(39)$$

$$v_{j+1} = \gamma v_j + h \tau \omega_j - \tau \sum_{k=0}^{j-1} \beta_{j,k} b_k + \alpha b_j \tau \quad (40)$$

$$= \gamma v_j + h \tau \omega_j - \tau \boldsymbol{\beta}_j^T \mathbf{b} \quad (41)$$

$$= \gamma^{j+1} v_0 + h \tau \sum_{k=0}^j \gamma^{j-k} \omega_k - \left(\sum_{k=0}^j \gamma^{j-k} \boldsymbol{\beta}_k \right)^T \mathbf{b} \quad (42)$$

Under appropriately defined strictly lower triangular matrices G and B ,

$$\mathbf{v} = -B \mathbf{b} + \mathbf{d} + h \tau G \boldsymbol{\omega} \quad (43)$$

$$\mathbf{d} = \begin{pmatrix} 1 \\ \gamma \\ \gamma^2 \\ \gamma^3 \\ \vdots \\ \gamma^n \end{pmatrix} v_0 \quad (44)$$

where row $j+1$ of matrix B is given by

$$\left(\sum_{k=0}^j \gamma^{j-k} \boldsymbol{\beta}_k \right)^T$$

From the initial conditions, we have $\mathbf{w}_0 = \mathbf{0}$ Thus $\boldsymbol{\omega} = \mathbf{0}$. Consequently, we have,

$$\mathbf{v} = -B \mathbf{b} + \mathbf{d} \quad (45)$$

Note that \mathbf{d} is a known quantity. Hence our optimization problem reduces to finding \mathbf{b} that best fits the equation

$$B \mathbf{b} = \mathbf{d} - \mathbf{v}^* \quad (46)$$

The matrix B is invertible in this case and hence our estimate of \mathbf{v} is given by

$$\hat{\mathbf{b}} = B^{-1}(\mathbf{d} - \mathbf{v}^*) \quad (47)$$

$$\mathbf{v} = -B\hat{\mathbf{b}} + \mathbf{d} \quad (48)$$

$$= \mathbf{v}^* \quad (49)$$

2.1 Computing the cost constraint function

We approximate the integral using:

$$\int_{-1}^1 f(x, t) dx = \sum_{i=1}^n \rho_i f(x_i, t) \quad (50)$$

We used Gauss-Legendre quadrature with $n = 4$. Thus,

$$\int_0^1 f(x, t) dx = \frac{1}{2} \sum_{i=1}^n \rho_i f\left(\frac{x_i}{2} + \frac{1}{2}, t\right) \quad (51)$$

$$\therefore c_j = b_j + f_j \quad (52)$$

3. OBSERVATION AND CONCLUSIONS

The inversion of matrix B does present some difficulties. It is easily seen from the definition of B that it is lower triangular. In addition,

$$(B)_{ii} = \alpha\tau \quad \forall 0 \leq i \leq n$$

As B is lower triangular, its determinant is the product of the main diagonal terms.

$$|B| = (\alpha\tau)^{n+1}$$

We have $\alpha \approx 0.354$, $\tau = 0.025$ and $n = 401$. Thus,

$$|B| \approx 10^{-802}$$

which makes B dangerously close to being singular. In fact, taking a direct inverse in Julia results in multiple entries going to `NaN`. We resolve this issue by computing the Moore-Penrose pseudoinverse B^+ of B . For an invertible matrix B , $B^- = B^+$

4. RESULTS



