# Optimal Control Design of a Reparable Multistate system

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**Abstract:**

*Keywords:*

## 1. PROBLEM DESCRIPTION

Model of a two state reparable system:

$$\frac{dp_0}{dt} = -\lambda_0 p_0(t) + \int_0^1 \mu_1(x) p_1(x,t)dx + \int_0^1 u^*(x,t)dx \quad (1)$$

$$\frac{\partial p_1(x,t)}{\partial t} + \frac{\partial p_1(x,t)}{\partial x} = -\mu_1(x)p_1(x,t) - u^*(x,t) \quad (2)$$

where:

$p_0(t)$: Probability that the device is in good mode 0 at time $t$.

$p_1(x,t)$: Probability density (with respect to repair time $x$) that the failed device is in failure mode 1 at time $t$ and has an elapsed repair time of $x$

$\mu_1(x)$: Time-dependent nonnegative repair rate when the device is in failure state and has an elapsed repair time of $x$.

**Given Initial Conditions:**

$$p_1(x,0) = 0 \quad (3)$$

$$p_0(0) = 1 \quad (4)$$

**Given Boundary Conditions:**

$$p_1(0,t) = \lambda_0 p_0(t) \quad (5)$$

$$p_1(1,t) = 0 \quad (6)$$

The function $u^*$ is given by

$$u^*(x,t) = (0.3 + 0.1 sin(x))b(t) \quad (7)$$

The function $b(t)$ represents the input. It is related to the cost constraint function $c(t)$ as given below.

$$b(t) + \int_0^1 \mu_1(x)f(x,t)dx - 0.3p_0^*(t) = c(t) \quad (8)$$

$$f(x,t) = 0.1\cos(\pi t)\sin^2(1-x) \quad (9)$$

Our objective is to find the input $b(t)$ such that the resulting distribution $p_0(t)$ is closest (as measured by the 2-norm) to the optimal distribution $p_0^*(t)$ given below.

$$p_0^*(t) = 0.85 + 0.05\cos(2\pi t) \quad (10)$$

Optimize:

$$J(b) = \|p_0(t;b) - p_0^*(t)\|_2 \quad (11)$$

## 2. METHODOLOGY

We make couple of substitutions, following the notation that $z_i^j$ refers to the value of $z$ evaluated at time point $i$ and at position $j$. The repair time is divided into $m$ subintervals, while the system running time is divided into $n$ subintervals. For the purposes of numerical implementation, we chose $m$ and $n$ to be 20 and 400 respectively.

So

$$p_0(t_j) = v_j \quad 0 \le j \le n \quad (12)$$

$$p_1(x_i,t_j) = w_j^i \quad 0 \le i \le m, 0 \le j \le n \quad (13)$$

$$\mu_1(x_i) = \mu^i \quad 0 \le i \le m \quad (14)$$

$$\lambda = \lambda_0 \quad (15)$$

$$(16)$$

Using the new notation, the boundary conditions and initial conditions may be written as follows.

Initial conditions:

$$w_0^i = 0 \quad \forall 0 \le i \le m \quad (17)$$

$$v_0 = 1 \quad (18)$$

Boundary Conditions:

$$w_j^{20} = 0 \quad \forall 0 \le j \le n \quad (19)$$

$$w_j^0 = \lambda v_j \quad (20)$$

Also condensing,

$$I_j^* = u^*(x_i,t_j) = g^i b_j \quad (21)$$

$$\int_0^1 u^*(x,t_j)dx = \alpha b_j \quad (22)$$

$$where\ g^i = (0.3 + 0.1 sin(x^i))$$

And,

$$b(t) + \int_0^1 \mu_1(x) f(x,t) dx - 0.3 p_0^*(t) = c(t)$$

$$b_j = c_j - f_j \tag{23}$$

$$where\ f_j = \int_0^1 \mu_1(x) f(x,t_j) dx - 0.3 p_0^*(t_j) \tag{24}$$

Discretizing (1)

$$\frac{v_{j+1} - v_j}{\tau} = -\lambda v_j + I_j + I_j^* \tag{25}$$

$$I_j = h[\frac{\mu^0 w_j^0}{2} + \sum_{k=1}^{19} \mu^k w_j^k + \frac{\mu^{20} w_j^{20}}{2}] \tag{26}$$

$$= h[\frac{\mu^0 w_j^0}{2} + \sum_{k=1}^{19} \mu^k w_j^k]$$

$$I_j^* = \alpha b_j \tag{27}$$

Discretizing (2)

$$\frac{w_{j+1}^i - w_j^i}{\tau} + \frac{w_j^{i+1} - w_j^{i-1}}{2h} = -\mu^i w_j^i - g^i b_j$$

$$w_{j+1}^i = w_j^i - \frac{\tau}{2h}(w_j^{i+1} - w_j^{i-1}) - \tau \mu^i w_j^i - \tau g^i b_j \tag{28}$$

Applying LAX scheme $w_j^i = \frac{w_j^{i-1} + w_j^{i+1}}{2}$ we get,

$$\begin{aligned} w_{j+1}^i = & \left(\frac{w_j^{i+1} + w_j^{i-1}}{2}\right) - \frac{\tau}{2h}(w_j^{i+1} - w_j^{i-1}) \\ & - \tau \mu^i \left(\frac{w_j^{i+1} + w_j^{i-1}}{2}\right) \\ & - \tau g^i b_j \\ w_{j+1}^i = & \frac{1}{2}\left(1 - \mu^i \tau + \frac{\tau}{h}\right) w_j^{i-1} + \\ & \frac{1}{2}\left(1 - \mu^i \tau - \frac{\tau}{h}\right) w_j^{i+1} - \\ & \tau g^i b_j \end{aligned}$$

Under an appropriately defined matrix $A$, we can re-write the above equation to read

$$\boldsymbol{w}_{j+1} = A\boldsymbol{w}_j - b_j \tau \boldsymbol{g} + \boldsymbol{e_1} v_{j+1} \tag{29}$$

$$= (A)^{j+1} \boldsymbol{w}_0 - \left[\sum_{k=0}^{j} b_k (A)^{j-k}\right] \boldsymbol{g}\tau \tag{30}$$

$$+ \left[\sum_{k=0}^{j} v_{k+1} (A)^{j-k}\right] \boldsymbol{e_1}$$

where $\boldsymbol{e_1}$ is an $m \times 1$ matrix given by

$$\boldsymbol{e_1} = [\lambda, 0, \ldots, 0]^T \tag{31}$$

Matrix A has the form:

$$\begin{pmatrix} w_{j+1}^0 \\ w_{j+1}^1 \\ \cdots \\ w_{j+1}^{n-2} \\ w_{j+1}^{n-1} \\ w_{j+1}^n \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & 0 & a_3 & \cdots & 0 & 0 \\ 0 & a_2 & 0 & a_4 & \cdots & 0 \\ 0 & 0 & a_3 & 0 & a_5 & 0 \\ \vdots & & & & & \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_j^0 \\ w_j^1 \\ \cdots \\ w_j^{n-2} \\ w_j^{n-1} \\ w_j^n \end{pmatrix} \tag{32}$$

$$+ b\tau \begin{pmatrix} g^0 \\ g^1 \\ \vdots \\ g^{n-2} \\ g^{n-1} \\ g^n \end{pmatrix} + v_{j+1} \begin{pmatrix} \lambda \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{33}$$

From (25)

$$v_{j+1} = (1 - \lambda\tau) v_j + \tau I_j + \tau I_j^* \tag{34}$$

$$= (1 - \lambda\tau + \frac{h\tau}{2}) v_j + h\tau \boldsymbol{\mu}^T \boldsymbol{w}_j + \alpha b_j \tau \tag{35}$$

Substitute the expression for the time evolution for $\boldsymbol{w}$ in the above to obtain,

$$\begin{aligned} v_{j+1} = & (1 - \lambda\tau + \frac{h\tau}{2}) v_j \\ & + h\tau \boldsymbol{\mu}^T (A)^j \boldsymbol{w}_0 \\ & - \boldsymbol{\mu}^T \left[\sum_{k=0}^{j-1} b_k (A)^{j-1-k}\right] \boldsymbol{g}\tau \\ & + \alpha b_j \tau \end{aligned}$$

Let's define

$$\beta_{j,k} = \boldsymbol{\mu}^T (A)^{j-1-k} \boldsymbol{g} \tag{36}$$

$$\omega_j = \boldsymbol{\mu}^T (A)^j \boldsymbol{w}_0 \tag{37}$$

$$\gamma = (1 - \lambda\tau + \frac{h\tau}{2}) \tag{38}$$

$$\boldsymbol{\beta_0} = \begin{pmatrix} \alpha \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{39}$$

$$\tag{40}$$

$$v_{j+1} = \gamma v_j + h\tau\omega_j - \tau \sum_{k=0}^{j-1} \beta_{j,k} b_k + \alpha b_j \tau \tag{41}$$

$$= \gamma v_j + h\tau\omega_j - \tau \boldsymbol{\beta}_j^T \boldsymbol{b} \tag{42}$$

$$= \gamma^{j+1} v_0 + h\tau \sum_{k=0}^{j} \gamma^{j-k} \omega_k - \left(\sum_{k=0}^{j} \gamma^{j-k} \boldsymbol{\beta}_k\right)^T \boldsymbol{b} \tag{43}$$

Under appropriately defined strictly lower triangular matrices $G$ and $B$,

$$\boldsymbol{v} = -B\boldsymbol{b} + \boldsymbol{d} + h\tau G\boldsymbol{\omega} \tag{44}$$

$$\boldsymbol{d} = \begin{pmatrix} 1 \\ \gamma \\ \gamma^2 \\ \gamma^3 \\ \vdots \\ \gamma^n \end{pmatrix} v_0 \tag{45}$$

where row $j+1$ of matrix $B$ is given by

$$\left( \sum_{k=0}^{j} \gamma^{j-k} \boldsymbol{\beta}_k \right)^T$$

From the initial conditions, we have $\boldsymbol{w_0} = \mathbf{0}$ Thus $\boldsymbol{\omega} = \mathbf{0}$. Consequently, we have,

$$\boldsymbol{v} = -B\boldsymbol{b} + \boldsymbol{d} \tag{46}$$

Note that $\boldsymbol{d}$ is a known quantity. Hence our optimization problem reduces to finding $\boldsymbol{b}$ that best fits the equation

$$B\boldsymbol{b} = \boldsymbol{d} - \boldsymbol{v}^* \tag{47}$$

The matrix $B$ is invertible in this case and hence our estimate of $\boldsymbol{v}$ is given by

$$\hat{\boldsymbol{b}} = B^{-1} (\boldsymbol{d} - \boldsymbol{v}^*) \tag{48}$$

$$\boldsymbol{v} = -B\hat{\boldsymbol{b}} + \boldsymbol{d} \tag{49}$$

$$= \boldsymbol{v}^* \tag{50}$$

### *2.1 Computing the cost constraint function*

We approximate the integral using:

$$\int_{-1}^{1} f(x,t)dx = \sum_{i=1}^{n} \rho_i f(x_i, t) \tag{51}$$

We used Gauss-Legendre quadrature with $n = 4$. Thus,

$$\int_{0}^{1} f(x,t)dx = \frac{1}{2} \sum_{i=1}^{n} \rho_i f\left( \frac{x_i}{2} + \frac{1}{2}, t \right) \tag{52}$$

$$\therefore c_j = b_j + f_j \tag{53}$$

## 3. OBSERVATION AND CONCLUSIONS

The inversion of matrix $B$ does present some difficulties. It is easily seen from the definition of $B$ that it is lower triangular. In addition,

$$(B)_{ii} = \alpha\tau \quad \forall 0 \leq i \leq n$$

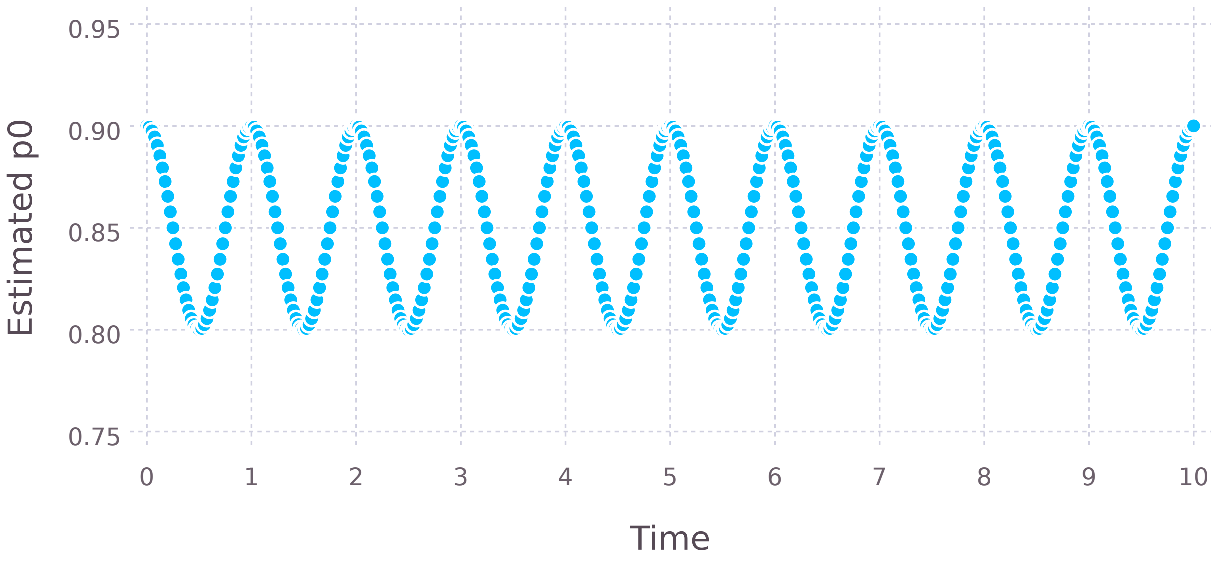
As $B$ is lower triangular, its determinant is the product of the main diagonal terms.

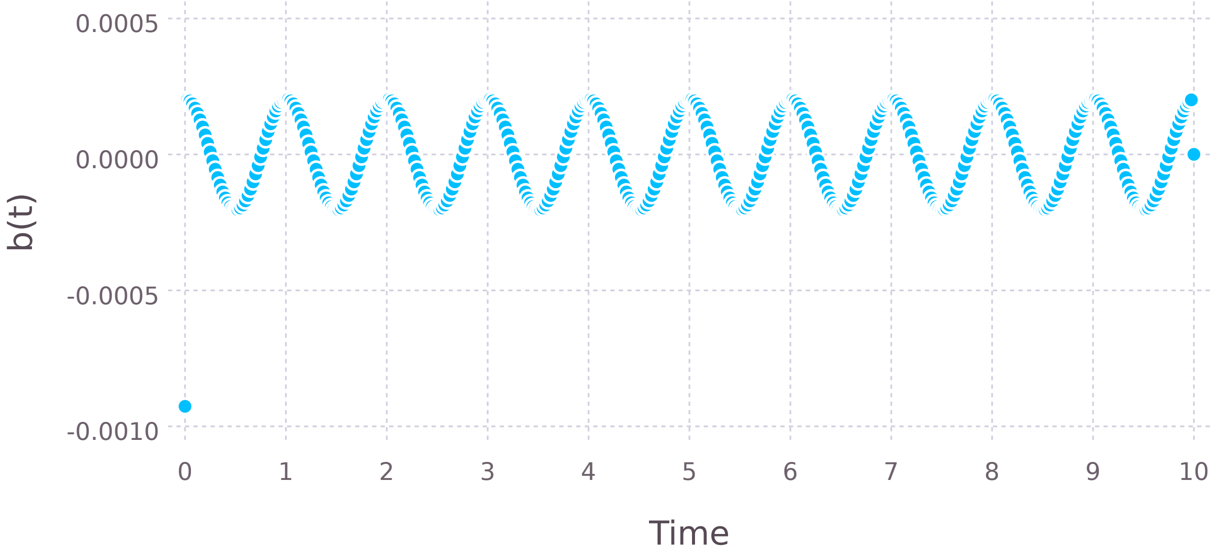
$$|B| = (\alpha\tau)^{n+1}$$

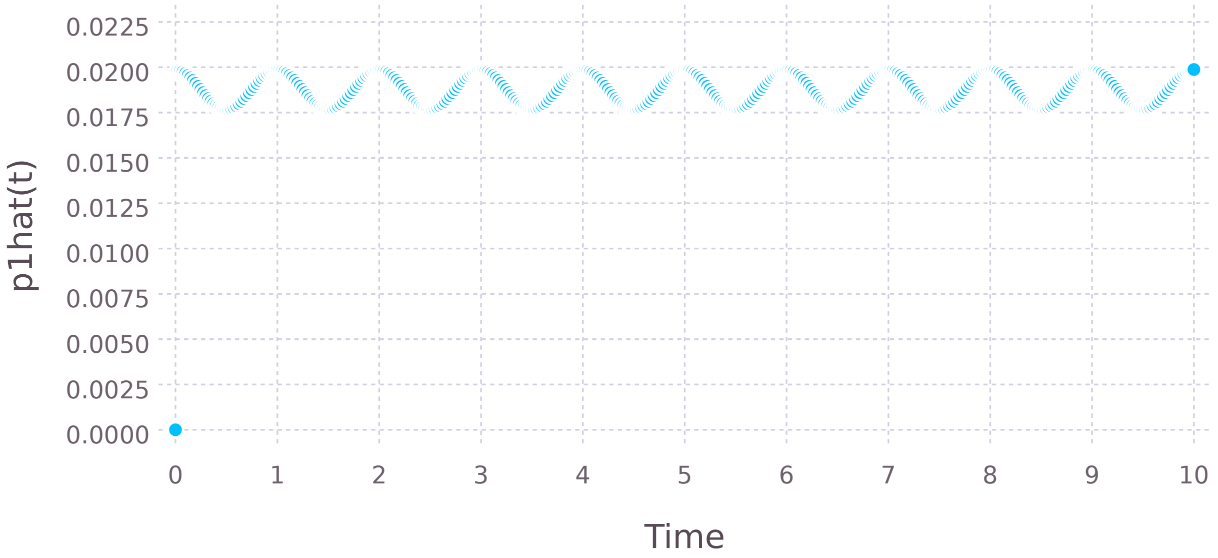
We have $\alpha \approx 0.354$, $\tau = 0.025$ and $n = 401$. Thus,

$$|B| \approx 10^{-802}$$

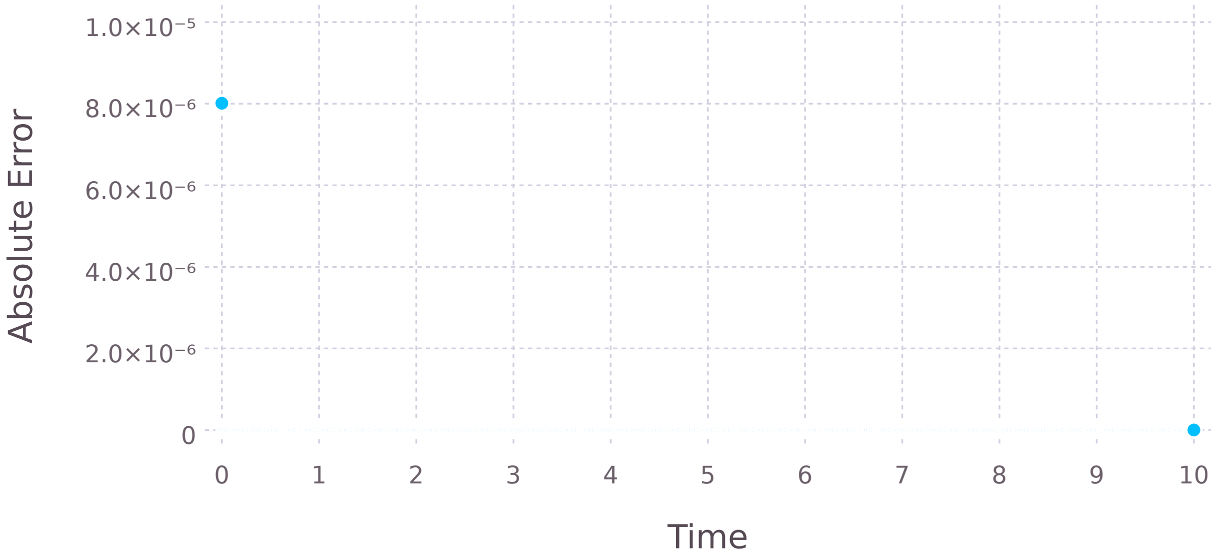
which makes $B$ dangerously close to being singular. In fact, taking a direct inverse in Julia results in multiple entries going to `NaN`. We resolve this issue by computing the Moore-Penrose pseudoinverse $B^+$ of $B$. For an invertible matrix $B$, $B^- = B^+$


Estimated p0
0.95
0.90
0.85
0.80
0.75
0
1
2
3
4
5
6
7
8
9
10
Time


b(t)
0.0005
0.0000
-0.0005
-0.0010
0
1
2
3
4
5
6
7
8
9
10
Time



p1hat(t)
0.0225
0.0200
0.0175
0.0150
0.0125
0.0100
0.0075
0.0050
0.0025
0.0000
0
1
2
3
4
5
6
7
8
9
10
Time



Absolute Error
1.0×10⁻⁵
8.0×10⁻⁶
6.0×10⁻⁶
4.0×10⁻⁶
2.0×10⁻⁶
0
0
1
2
3
4
5
6
7
8
9
10
Time