

# Optimal Control Design of a Repairable Multistate system

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## Abstract:

Keywords:

## 1. PROBLEM DESCRIPTION

Carefully provide a mathematical description of the problem discussed in this report.

$$p_0(t_j) = v_j$$

$$p_1(x_i, t_j) = w_j^i \quad (11)$$

$$\mu_1(x_i) = \mu^i \quad (12)$$

$$\lambda = \lambda_0 \quad (13)$$

$$0 \leq i \leq 19 \quad (14)$$

$$0 \leq j \leq 399 \quad (15)$$

## 2. METHODOLOGY

$$\frac{dp_0}{dt} = -\lambda_0 p_0(t) + \int_0^1 \mu_1(x) p_1(x, t) dx + \int_0^1 u^*(x, t) dx \quad (1)$$

Using the new notations boundary conditions and initial values:

Initial conditions:

$$\frac{\partial p_1(x, t)}{\partial t} + \frac{\partial p_1(x, t)}{\partial x} = -\mu_1(x) p_1(x, t) - u^*(x, t) \quad (2)$$

$$w_j^{20} = 0 \quad (16)$$

$$w_0^i = 0 \quad (17)$$

$$v_0 = 1 \quad (18)$$

Given Initial Conditions:

$$p_1(1, t) = 0 \quad (3)$$

$$p_1(x, 0) = 0 \quad (4)$$

$$p_0(0) = 1 \quad (5)$$

Boundary Conditions:

$$w_j^0 = \lambda v_j \quad (19)$$

Also condensing,

Given Boundary Conditions:

$$p_1(0, t) = \lambda_0 p_0(t) \quad (6)$$

$$I_j^* = u^*(x_i, t_j) = g^i b_j \quad (20)$$

$$u^*(x, t) = (0.3 + 0.1 \sin(x)) b(t) \quad (7)$$

$$\int_0^1 u^*(x, t_j) dx = \alpha b_j \quad (21)$$

$$\text{where } g^i = (0.3 + 0.1 \sin(x_i))$$

And,

And,

$$b(t) + \int_0^1 \mu_1(x) f(x, t) dx - 0.3 p_0^*(t) = c(t) \quad (8)$$

$$f(x, t) = 0.1 \cos(\pi t) \sin^2(1 - x) \quad (9)$$

$$p_0^*(t) = 0.85 + 0.05 \cos(2\pi t) \quad (10)$$

$$b(t) + \int_0^1 \mu_1(x) f(x, t) dx - 0.3 p_0^*(t) = c(t)$$

$$b_j = c_j - f_j \quad (22)$$

$$\text{where } f_j = \int_0^1 \mu_1(x) f(x, t_j) dx - 0.3 p_0^*(t_j) \quad (23)$$

We make couple of substitutions, following the notation that  $z_i^j$  refers to the value of  $z$  evaluated at time point  $i$  and at position  $j$

Discretizing (1)

So

$$(24)$$

$$\frac{v_{j+1} - v_j}{\tau} = -\lambda v_j + I_j + I_j^* \quad (25)$$

$$I_j = h \left[ \frac{\mu^0 w_j^0}{2} + \sum_{k=1}^{19} \mu^k w_j^k + \frac{\mu^{20} w_j^{20}}{2} \right] \quad (26)$$

$$= h \left[ \frac{\mu^0 w_j^0}{2} + \sum_{k=1}^{19} \mu^k w_j^k \right] \quad (27)$$

Discretizing (2)

$$\frac{w_{j+1}^i - w_j^i}{\tau} + \frac{w_j^{i+1} - w_j^{i-1}}{2h} = -\mu^i w_j^i - g^i b_j \quad (28)$$

Applying LAX scheme  $w_j^i = \frac{w_j^{i-1} + w_j^{i+1}}{2}$  we get,

$$\begin{aligned} w_{j+1}^i &= \left( \frac{w_j^{i+1} + w_j^{i-1}}{2} \right) - \frac{\tau}{2h} (w_j^{i+1} - w_j^{i-1}) \\ &\quad - \tau \mu^i \left( \frac{w_j^{i+1} + w_j^{i-1}}{2} \right) \\ &\quad - \tau g^i b_j \\ w_{j+1}^i &= \frac{1}{2} \left( 1 - \mu^i \tau + \frac{\tau}{h} \right) w_j^{i-1} + \\ &\quad \frac{1}{2} \left( 1 - \mu^i \tau - \frac{\tau}{h} \right) w_j^{i+1} - \\ &\quad \tau g^i b_j \end{aligned}$$

Under an appropriately defined matrix  $A$ , we can re-write the above equation to read

$$\mathbf{w}_{j+1} = A \mathbf{w}_j - b_j \tau \mathbf{g} + \mathbf{e}_1 v_{j+1} \quad (29)$$

$$\begin{aligned} &= (A)^{j+1} \mathbf{w}_0 - \left[ \sum_{k=0}^j b_k (A)^{j-k} \right] \mathbf{g} \tau \\ &\quad + \left[ \sum_{k=0}^j v_{k+1} (A)^{j-k} \right] \mathbf{e}_1 \end{aligned} \quad (30)$$

where  $\mathbf{e}_1$  is an  $m \times 1$  matrix given by

$$\mathbf{e}_1 = [\lambda, 0, \dots, 0]^T \quad (31)$$

Matrix  $A$  has the form:

$$\begin{pmatrix} w_{j+1}^0 \\ w_{j+1}^1 \\ \vdots \\ w_{j+1}^{n-2} \\ w_{j+1}^{n-1} \\ w_{j+1}^n \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & 0 & a_3 & \cdots & 0 & 0 \\ 0 & a_2 & 0 & a_4 & \cdots & 0 \\ 0 & 0 & a_3 & 0 & a_5 & 0 \\ \vdots & & & & & \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_j^0 \\ w_j^1 \\ \vdots \\ w_j^{n-2} \\ w_j^{n-1} \\ w_j^n \end{pmatrix} \quad (32)$$

$$+ b \tau \begin{pmatrix} g^0 \\ g^1 \\ \vdots \\ g^{n-2} \\ g^{n-1} \\ g^n \end{pmatrix} + v_{j+1} \begin{pmatrix} \lambda \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad (33)$$

From (25)

$$v_{j+1} = (1 - \lambda \tau) v_j + \tau I_j + \tau I_j^* \quad (34)$$

$$= (1 - \lambda \tau + \frac{h\tau}{2}) v_j + h \tau \boldsymbol{\mu}^T \mathbf{w}_j + \alpha b_j \tau \quad (35)$$

Substitute the expression for the time evolution for  $\mathbf{w}$  in the above to obtain,

$$\begin{aligned} v_{j+1} &= (1 - \lambda \tau + \frac{h\tau}{2}) v_j \\ &\quad + h \tau \boldsymbol{\mu}^T (A)^j \mathbf{w}_0 \\ &\quad - \boldsymbol{\mu}^T \left[ \sum_{k=0}^{j-1} b_k (A)^{j-1-k} \right] \mathbf{g} \tau \\ &\quad + \alpha b_j \tau \end{aligned}$$

Let's define

$$\beta_{j,k} = \boldsymbol{\mu}^T (A)^{j-1-k} \mathbf{g} \quad (36)$$

$$\omega_j = \boldsymbol{\mu}^T (A)^j \mathbf{w}_0 \quad (37)$$

$$\gamma = (1 - \lambda \tau + \frac{h\tau}{2}) \quad (38)$$

$$\boldsymbol{\beta}_0 = \begin{pmatrix} \alpha \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (39)$$

$$(40)$$

$$v_{j+1} = \gamma v_j + h \tau \omega_j - \tau \sum_{k=0}^{j-1} \beta_{j,k} b_k + \alpha b_j \tau \quad (41)$$

$$= \gamma v_j + h \tau \omega_j - \tau \boldsymbol{\beta}_j^T \mathbf{b} \quad (42)$$

$$= \gamma^{j+1} v_0 + h \tau \sum_{k=0}^j \gamma^{j-k} \omega_k - \left( \sum_{k=0}^j \gamma^{j-k} \boldsymbol{\beta}_k \right)^T \mathbf{b} \quad (43)$$

Under appropriately defined strictly lower triangular matrices  $G$  and  $B$ ,

$$\mathbf{v} = \mathbf{c} + h \tau G \boldsymbol{\omega} - B \mathbf{b} \quad (44)$$

$$\mathbf{c} = \begin{pmatrix} 1 \\ \gamma \\ \gamma^2 \\ \gamma^3 \\ \vdots \\ \gamma^n \end{pmatrix} v_0 \quad (45)$$

where row  $j+1$  of matrix  $B$  is given by

$$\left( \sum_{k=0}^j \gamma^{j-k} \boldsymbol{\beta}_k \right)^T$$

Consequently, we have,

$$(\gamma G - I) \mathbf{v} = B \mathbf{b} - h \tau G \boldsymbol{\omega} \quad (46)$$

$$\mathbf{v} = (\gamma G - I)^{-1} B \mathbf{b} - h \tau (\gamma G - I)^{-1} G \boldsymbol{\omega} \quad (47)$$

$$= B' \mathbf{b} - \mathbf{c} \quad (48)$$

for appropriate matrix  $B'$  and vector  $\mathbf{c}$ . Note that  $\mathbf{c}$  is a known quantity. Hence our optimization problem reduces to finding  $\mathbf{b}$  that best fits the equation

$$B'\mathbf{b} = \mathbf{v}^* + \mathbf{c} \quad (49)$$

### 3. RESULTS

Illustrate the results using the methodology proposed in Section. 2

### 4. OBSERVATION AND CONCLUSIONS

State your observation from the results and make conclusions.

### REFERENCE

All materials (books, papers, and websites) mentioned in your reports.

### APPENDIX

Attach the Matlab code here.