Optimal Control Design of a Reparable Multistate system

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Abstract:

Keywords:

1. PROBLEM DESCRIPTION

Carefully provide a mathematical description of the problem discussed in this report.

2. METHODOLOGY

$$\frac{dp_0}{dt} = -\lambda_0 p_0(t) + \int_0^1 \mu_1(x) p_1(x,t)dx + \int_0^1 u^*(x,t)dx \quad (1)$$

$$\frac{\partial p_1(x,t)}{\partial t} + \frac{\partial p_1(x,t)}{\partial x} = -\mu_1(x)p_1(x,t) - u^*(x,t) \quad (2)$$

Given Initial Conditions:

$$p_1(1,t) = 0 \quad (3)$$
$$p_0(0) = 1 \quad (4)$$

Given Boundary Conditions:

$$p_1(0,t) = \lambda_0 p_0(t) \quad (5)$$

$$u^*(x,t) = (0.3 + 0.1sin(x))b(t) \quad (6)$$

And,

$$b(t) + \int_0^1 \mu_1(x) f(x,t)dx - 0.3p_0^*(t) = c(t) \quad (7)$$
$$f(x,t) = 0.1\cos(\pi t)\sin^2(1-x) \quad (8)$$
$$p_0^*(t) = 0.85 + 0.05\cos(2\pi t) \quad (9)$$

We make couple of substitutions, following the notation that z_i^j refers to the value of z evaluated at time point i and at position j

So

$$p_0(t_j) = v_j$$
$$p_1(x_i, t_j) = w_j^i \quad (10)$$
$$\mu_1(x_i) = \mu^i \quad (11)$$
$$\lambda_= \lambda_0 \quad (12)$$

Using the new notations boundary conditions and initial values:

Initial conditions:

$$w_j^{20} = 0 \quad (13)$$
$$v_0 = 1 \quad (14)$$

Boundary Conditions:

$$w_j^0 = \lambda v_j \quad (15)$$

Also condensing,

$$I_j^* = u^*(x_i, t_j) = g^i b_j \quad (16)$$
$$\int_0^1 u^*(x, t_j)dx = Gb_j \quad (17)$$
$$where\ g^i = (0.3 + 0.1sin(x_i))$$

And,

$$b(t) + \int_0^1 \mu_1(x) f(x,t)dx - 0.3p_0^*(t) = c(t)$$
$$b_j = c_j - F_j \quad (18)$$
$$where\ F_j = \int_0^1 \mu_1(x) f(x,t_j)dx - 0.3p_0^*(t_j) \quad (19)$$
$$(20)$$

Discretizing (1)

$$\frac{v_{j+1} - v_j}{\tau} = -\lambda v_j + I_j + I_j^* \quad (21)$$
$$I_j = h[\frac{\mu^0 w_j^0}{2} + \sum_{k=1}^{19} \mu^k w_j^k + \frac{\mu^{20} w_j^{20}}{2}] \quad (22)$$
$$= h[\frac{\mu^0 w_j^0}{2} + \sum_{k=1}^{19} \mu^k w_j^k]$$
$$I_j^* = Gb_j \quad (23)$$

Discretizing (2)

$$\frac{w_{j+1}^i - w_j^i}{\tau} + \frac{w_j^{i+1} - w_j^{i-1}}{2h} = -\mu^i w_j^i - g^i b_j$$

$$w_{j+1}^i = w_j^i - \frac{\tau}{2h}(w_j^{i+1} - w_j^{i-1}) - \tau\mu^i w_j^i - \tau g^i b_j \qquad (24)$$

Applying LAX scheme $w_j^i = \frac{w_j^{i-1}+w_j^{i+1}}{2}$ we get,

$$w_{j+1}^i = \frac{1}{2}\left(1 - \mu^i\tau + \frac{\tau}{h}\right) w_j^{i-1} + \frac{1}{2}\left(1 - \mu^i\tau - \frac{\tau}{h}\right) w_j^{i+1} - \tau g^i b_j$$

Under an appropriately defined matrix A, we can re-write the above equation to read

$$\boldsymbol{w}_{j+1} = A\boldsymbol{w}_j - b_j\tau\boldsymbol{g} + \boldsymbol{e_1}v_{j+1} \qquad (25)$$

$$= (A)^{j+1}\boldsymbol{w}_0 - \left[\sum_{k=0}^{j} b_k (A)^{j-k}\right]\boldsymbol{g}\tau \qquad (26)$$

$$+ \left[\sum_{k=0}^{j} v_{k+1} (A)^{j-k}\right]\boldsymbol{e_1}$$

where $\boldsymbol{e_1}$ is an $m \times 1$ matrix given by

$$\boldsymbol{e_1} = [\lambda, 0, \ldots, 0]^T \qquad (27)$$

From (21)

$$v_{j+1} = (1 - \lambda\tau)v_j + \tau I_j + \tau I_j^* \qquad (28)$$

$$= (1 - \lambda\tau + \frac{h\tau}{2})v_j + h\tau\boldsymbol{\mu}^T\boldsymbol{w}_j + \alpha b_j\tau \qquad (29)$$

Substitute the expression for the time evolution for $\boldsymbol{w}$ in the above to obtain,

$$v_{j+1} = (1 - \lambda\tau + \frac{h\tau}{2})v_j + h\tau\boldsymbol{\mu}^T(A)^j\boldsymbol{w}_0 - \boldsymbol{\mu}^T\left[\sum_{k=0}^{j-1} b_k (A)^{j-1-k}\right]\boldsymbol{g}\tau + \alpha b_j\tau$$

Let's define

$$\beta_k = \boldsymbol{\mu}^T(A)^{j-1-k}\boldsymbol{g} \qquad (30)$$

$$\omega_j = \boldsymbol{\mu}^T(A)^j\boldsymbol{w}_0 \qquad (31)$$

$$\gamma = (1 - \lambda\tau + \frac{h\tau}{2}) \qquad (32)$$

$$v_{j+1} = \gamma v_j + h\tau\omega_j - \tau\sum_{k=0}^{j-1}\beta_k b_k + \alpha b_j\tau \qquad (33)$$

$$= \gamma v_j + h\tau\omega_j - \tau\boldsymbol{\beta}_j^T\boldsymbol{b} \qquad (34)$$

$$= \gamma\sum_{k=0}^{j}\gamma^{j-k}v_k + h\tau\sum_{k=0}^{j}\gamma^{j-k}\omega_k - \left(\sum_{k=0}^{j}\gamma^{j-k}\boldsymbol{\beta}_k\right)^T\boldsymbol{b} \qquad (35)$$

Under appropriately defined strictly lower triangular matrices G and B,

$$\boldsymbol{v} = \gamma G\boldsymbol{v} + h\tau G\boldsymbol{\omega} - B\boldsymbol{b} \qquad (36)$$

where row $j+1$ of matrix B is given by

$$\left(\sum_{k=0}^{j}\gamma^{j-k}\boldsymbol{\beta}_k\right)^T$$

Consequently, we have,

$$(\gamma G - I)\boldsymbol{v} = B\boldsymbol{b} - h\tau G\boldsymbol{\omega} \qquad (37)$$

$$\boldsymbol{v} = (\gamma G - I)^{-1}B\boldsymbol{b} - h\tau(\gamma G - I)^{-1}G\boldsymbol{\omega} \qquad (38)$$

$$= B'\boldsymbol{b} - \boldsymbol{c} \qquad (39)$$

for appropriate matrix B' and vector $\boldsymbol{c}$. Note that $\boldsymbol{c}$ is a known quantity. Hence our optimization problem reduces to finding $\boldsymbol{b}$ that best fits the equation

$$B'\boldsymbol{b} = \boldsymbol{v}^* + \boldsymbol{c} \qquad (40)$$

3. RESULTS

Illustrate the results using the methodology proposed in Section. 2

4. OBSERVATION AND CONCLUSIONS

State your observation from the results and make conclusions.

REFERENCE

All materials (books, papers, and websites) mentioned in your reports.

APPENDIX

Attach the Matlab code here.