Optimal Control Design of a Reparable Multistate system

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Abstract:

Keywords:

1. PROBLEM DESCRIPTION

Carefully provide a mathematical description of the problem discussed in this report.

2. METHODOLOGY

$$\frac{dp_0}{dt} = -\lambda_0 p_0(t) + \int_0^1 \mu_1(x) p_1(x,t)dx + \int_0^1 u^*(x,t)dx \quad (1)$$

$$\frac{\partial p_1(x,t)}{\partial t} + \frac{\partial p_1(x,t)}{\partial x} = -\mu_1(x)p_1(x,t) - u^*(x,t) \quad (2)$$

Given Initial Conditions:

$$p_1(1,t) = 0 \quad (3)$$
$$p_1(x,0) = 0 \quad (4)$$
$$p_0(0) = 1 \quad (5)$$

Given Boundary Conditions:

$$p_1(0,t) = \lambda_0 p_0(t) \quad (6)$$

$$u^*(x,t) = (0.3 + 0.1sin(x))b(t) \quad (7)$$

And,

$$b(t) + \int_0^1 \mu_1(x)f(x,t)dx - 0.3p_0^*(t) = c(t) \quad (8)$$
$$f(x,t) = 0.1\cos(\pi t)\sin^2(1-x) \quad (9)$$
$$p_0^*(t) = 0.85 + 0.05\cos(2\pi t) \quad (10)$$

We make couple of substitutions, following the notation that z_i^j refers to the value of z evaluated at time point i and at position j

So

$$p_0(t_j) = v_j$$
$$p_1(x_i,t_j) = w_j^i \quad (11)$$
$$\mu_1(x_i) = \mu^i \quad (12)$$
$$\lambda = \lambda_0 \quad (13)$$
$$0 \leq i \leq 19 \quad (14)$$
$$0 \leq j \leq 399 \quad (15)$$

Using the new notations boundary conditions and initial values:

Initial conditions:

$$w_j^{20} = 0 \quad (16)$$
$$w_0^i = 0 \quad (17)$$
$$v_0 = 1 \quad (18)$$

Boundary Conditions:

$$w_j^0 = \lambda v_j \quad (19)$$

Also condensing,

$$I_j^* = u^*(x_i,t_j) = g^i b_j \quad (20)$$
$$\int_0^1 u^*(x,t_j)dx = \alpha b_j \quad (21)$$
$$where\ g^i = (0.3 + 0.1sin(x_i))$$

And,

$$b(t) + \int_0^1 \mu_1(x)f(x,t)dx - 0.3p_0^*(t) = c(t)$$
$$b_j = c_j - f_j \quad (22)$$
$$where\ f_j = \int_0^1 \mu_1(x)f(x,t_j)dx - 0.3p_0^*(t_j) \quad (23)$$
$$(24)$$

Discretizing (1)

$$\frac{v_{j+1}-v_j}{\tau} = -\lambda v_j + I_j + I_j^* \tag{25}$$

$$I_j = h[\frac{\mu^0 w_j^0}{2} + \sum_{k=1}^{19} \mu^k w_j^k + \frac{\mu^{20} w_j^{20}}{2}] \tag{26}$$

$$= h[\frac{\mu^0 w_j^0}{2} + \sum_{k=1}^{19} \mu^k w_j^k]$$

$$I_j^* = \alpha b_j \tag{27}$$

Discretizing (2)

$$\frac{w_{j+1}^i - w_j^i}{\tau} + \frac{w_j^{i+1} - w_j^{i-1}}{2h} = -\mu^i w_j^i - g^i b_j$$

$$w_{j+1}^i = w_j^i - \frac{\tau}{2h}(w_j^{i+1} - w_j^{i-1}) - \tau \mu^i w_j^i - \tau g^i b_j \tag{28}$$

Applying LAX scheme $w_j^i = \frac{w_j^{i-1}+w_j^{i+1}}{2}$ we get,

$$\begin{aligned} w_{j+1}^i = & \left(\frac{w_j^{i+1} + w_j^{i-1}}{2}\right) - \frac{\tau}{2h}(w_j^{i+1} - w_j^{i-1}) \\ & - \tau\mu^i \left(\frac{w_j^{i+1} + w_j^{i-1}}{2}\right) \\ & - \tau g^i b_j \\ w_{j+1}^i = & \frac{1}{2}\left(1 - \mu^i\tau + \frac{\tau}{h}\right) w_j^{i-1} + \\ & \frac{1}{2}\left(1 - \mu^i\tau - \frac{\tau}{h}\right) w_j^{i+1} - \\ & \tau g^i b_j \end{aligned}$$

Under an appropriately defined matrix A, we can re-write the above equation to read

$$\boldsymbol{w}_{j+1} = A\boldsymbol{w}_j - b_j\tau\boldsymbol{g} + \boldsymbol{e_1} v_{j+1} \tag{29}$$

$$= (A)^{j+1}\boldsymbol{w}_0 - \left[\sum_{k=0}^{j} b_k (A)^{j-k}\right] \boldsymbol{g}\tau \tag{30}$$

$$+ \left[\sum_{k=0}^{j} v_{k+1} (A)^{j-k}\right] \boldsymbol{e_1}$$

where $\boldsymbol{e_1}$ is an $m \times 1$ matrix given by

$$\boldsymbol{e_1} = [\lambda, 0, \ldots, 0]^T \tag{31}$$

Matrix A has the form:

$$\begin{pmatrix} w_{j+1}^0 \\ w_{j+1}^1 \\ \cdots \\ w_{j+1}^{n-2} \\ w_{j+1}^{n-1} \\ w_{j+1}^n \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & 0 & a_3 & \cdots & 0 & 0 \\ 0 & a_2 & 0 & a_4 & \cdots & 0 \\ 0 & 0 & a_3 & 0 & a_5 & 0 \\ \vdots & & & & & \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_j^0 \\ w_j^1 \\ \cdots \\ w_j^{n-2} \\ w_j^{n-1} \\ w_j^n \end{pmatrix} \tag{32}$$

$$+ b\tau \begin{pmatrix} g^0 \\ g^1 \\ \vdots \\ g^{n-2} \\ g^{n-1} \\ g^n \end{pmatrix} + v_{j+1} \begin{pmatrix} \lambda \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{33}$$

From (25)

$$v_{j+1} = (1 - \lambda\tau)v_j + \tau I_j + \tau I_j^* \tag{34}$$

$$= (1 - \lambda\tau + \frac{h\tau}{2})v_j + h\tau\boldsymbol{\mu}^T\boldsymbol{w}_j + \alpha b_j\tau \tag{35}$$

Substitute the expression for the time evolution for $\boldsymbol{w}$ in the above to obtain,

$$\begin{aligned} v_{j+1} = & (1 - \lambda\tau + \frac{h\tau}{2})v_j \\ & + h\tau\boldsymbol{\mu}^T(A)^j\boldsymbol{w}_0 \\ & - \boldsymbol{\mu}^T \left[\sum_{k=0}^{j-1} b_k (A)^{j-1-k}\right] \boldsymbol{g}\tau \\ & + \alpha b_j \tau \end{aligned}$$

Let's define

$$\beta_{j,k} = \boldsymbol{\mu}^T (A)^{j-1-k}\boldsymbol{g} \tag{36}$$

$$\omega_j = \boldsymbol{\mu}^T (A)^j \boldsymbol{w}_0 \tag{37}$$

$$\gamma = (1 - \lambda\tau + \frac{h\tau}{2}) \tag{38}$$

$$\boldsymbol{\beta_0} = \begin{pmatrix} \alpha \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{39}$$

$$\tag{40}$$

$$v_{j+1} = \gamma v_j + h\tau\omega_j - \tau\sum_{k=0}^{j-1}\beta_{j,k}b_k + \alpha b_j\tau \tag{41}$$

$$= \gamma v_j + h\tau\omega_j - \tau\boldsymbol{\beta}_j^T\boldsymbol{b} \tag{42}$$

$$= \gamma^{j+1}v_0 + h\tau\sum_{k=0}^{j}\gamma^{j-k}\omega_k - \left(\sum_{k=0}^{j}\gamma^{j-k}\boldsymbol{\beta}_k\right)^T \boldsymbol{b} \tag{43}$$

Under appropriately defined strictly lower triangular matrices G and B,

$$\boldsymbol{v} = -B\boldsymbol{b} + \boldsymbol{c} + h\tau G\boldsymbol{\omega} \tag{44}$$

$$c = \begin{pmatrix} 1 \\ \gamma \\ \gamma^2 \\ \gamma^3 \\ \vdots \\ \gamma^n \end{pmatrix} v_0 \tag{45}$$

where row $j+1$ of matrix B is given by

$$\left(\sum_{k=0}^{j}\gamma^{j-k}\boldsymbol{\beta}_k\right)^T$$

From the initial conditions, we have $\boldsymbol{w_0} = \boldsymbol{0}$ Thus $\boldsymbol{\omega}_0 = \boldsymbol{0}$. Consequently, we have,

$$\boldsymbol{v} = -B\boldsymbol{b} + \boldsymbol{c} \tag{46}$$

Note that $\boldsymbol{c}$ is a known quantity. Hence our optimization problem reduces to finding $\boldsymbol{b}$ that best fits the equation

$$B\boldsymbol{b} = \boldsymbol{c} - \boldsymbol{v}^* \tag{47}$$

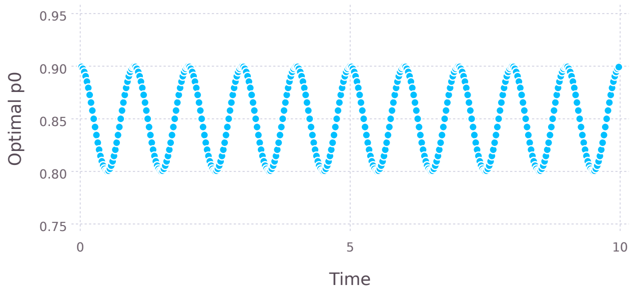

Fig. 1. $v(t)$ v/s t

3. RESULTS

Illustrate the results using the methodology proposed in Section. 2

4. OBSERVATION AND CONCLUSIONS

State your observation from the results and make conclusions.

REFERENCE

All materials (books, papers, and websites) mentioned in your reports.

APPENDIX

Attach the Matlab code here.