Can You Bake The Optimal Cake? – The Riddler

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September 2, 2016

We consider a simple case of maximizing the volume of a cylinder radius r_3, height h_3 inside a cone of radius R and height H

Similar triangles: $\frac{H-h_3}{r_3} = \frac{H}{R} \implies h_3 = H\left(1-\frac{r_3}{R}\right)$

Volume of cylinder $= V_3 = \pi r_3^2 h_3 = \pi r_3^2 H\left(1-\frac{r_3}{R}\right)$

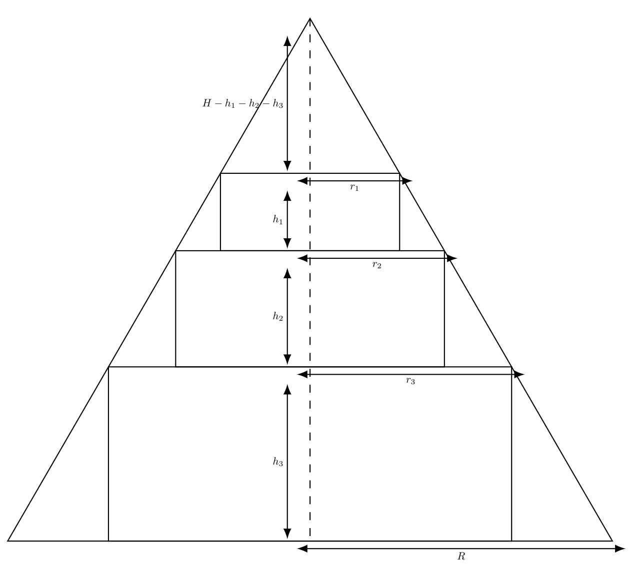
Maximizing:

$$\begin{aligned}
\frac{dV_3}{dr_3} &= 0 \\
2r_3R - 3r_3^2 &= 0 \\
r_3 &= \frac{2R}{3} \\
h_3 &= \frac{H}{3}
\end{aligned}$$

This result now trickles down to the cone with radius r_3 and height H_3:

$$\begin{aligned}
r_2 &= \frac{2r_3}{3} \\
&= \frac{4R}{9} \\
h_2 &= (H-h_3)\left(1-\frac{r_2}{R}\right) \\
&= \frac{2H}{3}(1-\frac{4}{9}) \\
&= \frac{10H}{27}
\end{aligned}$$

Similarly,



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\begin{aligned}
r_1 &= \frac{2r_2}{3} \\
&= \frac{8R}{27} \\
h_1 &= (H - h_2 - h_3)(1 - \frac{r_1}{R}) \\
&= \frac{8H}{27}\frac{19}{27} \\
&= \frac{162H}{729}
\end{aligned}
$$

Now,

$$
\begin{aligned}
V_3 &= \pi \left(r_1^2 h_1 + r_2^2 h_2 + r_3^2 h_3 \right) \\
&= \pi R^2 H \left((\frac{2}{3})^2 \frac{1}{3} + (\frac{2}{3})^4 (1 - \frac{1}{3})(1 - (\frac{2}{3})^2) + (\frac{2}{3})^9 (1 - \frac{1}{3})(1 - (\frac{2}{3})^2) \right)
\end{aligned}
$$