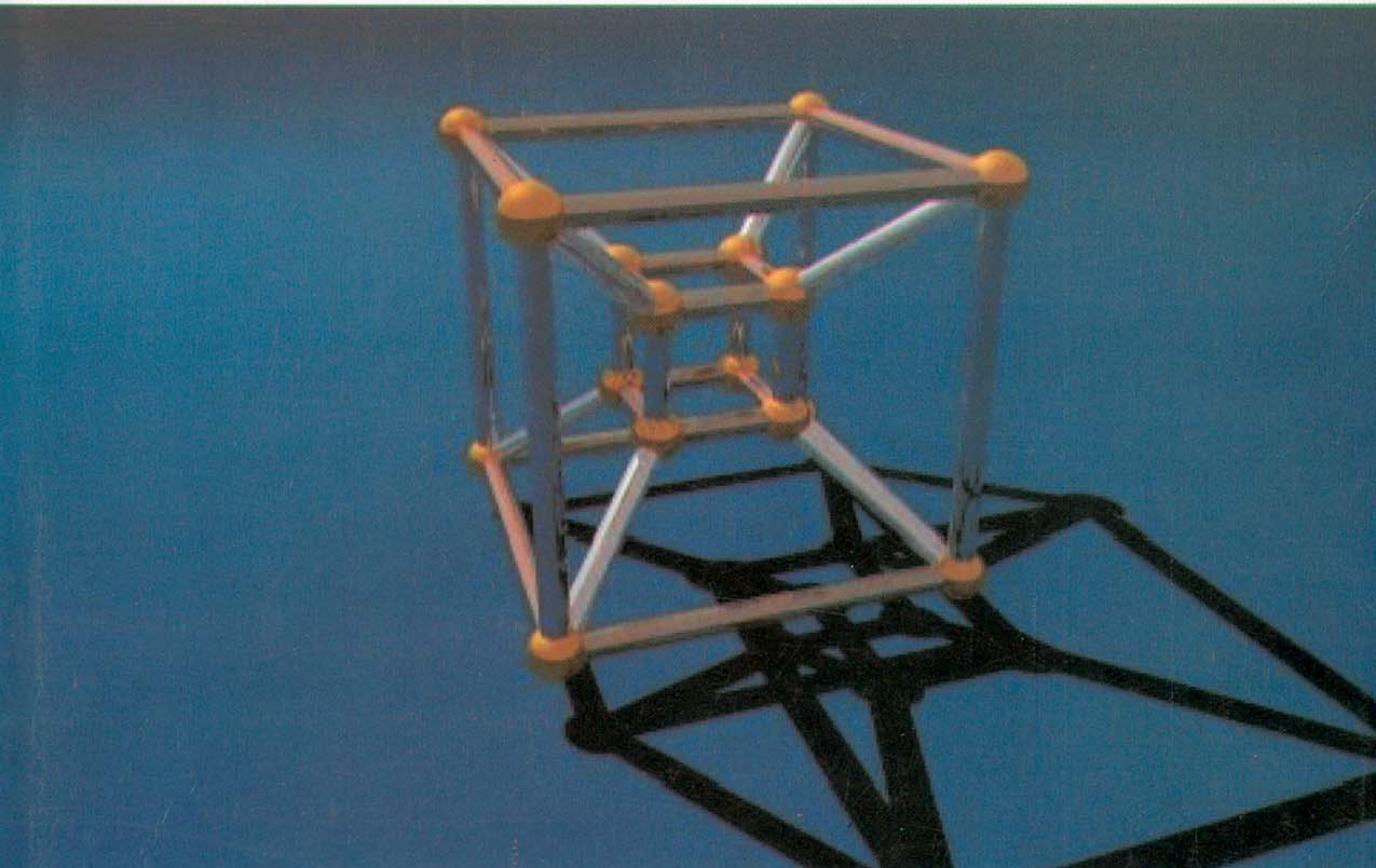
NEW AGE

SECOND EDITION

FUNDAMENTAL APPROACH TO DISCRETE MATHEMATICS



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Set Theory

■ 2.0 INTRODUCTION

An ordinary understanding of a set is a collection of objects. In our day-to-day life we use phrases like a set of utensils, a bunch of flowers, a set of books, a herd of cattle, a set of birds and etc., which are all examples of sets.

In the 19th century the German Mathematician George Cantor developed the theory of sets to define numbers and to base mathematics on a solid logical foundation. In late 19th century, Frege developed these ideas further, but his work did not attract much attention. In 20th century Bertrand Russell rediscovered his analysis independently. His works in 1903 led to the monumental work with North Whitehead the principia Mathematica a landmark in the foundations of mathematics. It was observed in 1940s that all mathematics could develop from the idea of sets and mathematics was systematized.

In this chapter we try to impart fundamental concepts and approach to the problem, that is how to proceed for the expected solution as for as set theory is concerned. By the way, we will study and learn about the basic concepts of sets, some of the operations on sets, Venn diagrams, Cartesian product of sets and its applications.

■ 2.1 SETS

Collection of well defined objects is called a set. Well defined means distinct and distinguishable. The objects are called as elements of the set. The ordering of elements in a set does not change the set, *i.e.*, the ordering of elements can not play a vital role in the set theory. For example

$$A = \{a, b, c, d\} \text{ and } B = \{b, a, d, c\} \text{ are equal sets.}$$

The symbol $\in$ stands for 'belongs to'. $x \in A$ means x is an element of the set A. It is observed that if A be a set and x is any object, then either $x \in A$ or $x \notin A$ but not both. Generally sets are denoted by capital letters A, B, C and etc.

Consider the examples of set:

$$A = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}$$

$$B = \{x, y, z, u, v, w\}$$
$$N = \{1, 2, 3, \ldots.\}$$
$$I = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots.\}$$

In general the set can be expressed in two ways, Tabular method (Roster method) and Set-builder method (Specification method).

2.1.1 Tabular Method

Expressing the elements of a set within a parenthesis where the elements are separated by commas is known as tabular method, roster method or method of extension.

Consider the example

$$A = \{1, 3, 5, 7, 9, 11, 13, 15\}$$

2.1.2 Set Builder Method

Expressing the elements of a set by a rule or formula is known as set-builder method, specification method or method of intension. Mathematically

$$S = \{x \mid P(x)\}$$

where $P(x)$ is the property that describes the elements of the set. The symbol $\mid$ stands for 'such that'. It is not possible to write every set in tabular form. Consider an example

$$S = \{x \mid x \text{ is an Italian}\}$$

The above set S can not be expressed in tabular form as it is impossible to list all Italians. Consider the examples

$$A = \{x \mid x = 2n + 1; 0 \leq n \leq 7; n \in I\}$$
$$= \{1, 3, 5, 7, 9, 11, 13, 15\}$$

and

$$B = \{x \mid x = 1, x = a, x = \text{Book}, x = \text{Pen}\}$$
$$= \{1, a, \text{Book}, \text{Pen}\}$$

From the second example given above it is clear that the elements of a set do not have any common property also.

■ 2.2 TYPES OF SETS

On considering real life problems, it is observed that the sets are of different types. Keeping in view to these problems, we discuss different types of sets in this section.

2.2.1 Finite Set

A set which contains finite number of elements is known as finite set. Consider the example of finite set as

$$A = \{a, b, c, d, e\}$$

2.2.2 Infinite Set

A set which contains infinite number of elements is known as infinite set. Consider the example of infinite set as

$$N = \{1, 2, 3, 4, \ldots\}$$
$$I = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots.\}$$

2.2.3 Singleton Set

A set which contains only one element is known as a singleton set. Consider the example

$$S = \{9\}$$

2.2.4 Pair Set

A set which contains only two elements is known as a pair set. Consider the examples

$$S = \{e, f\}$$
$$S = \{\{a\}, \{1, 3, 5\}\}$$

2.2.5 Empty Set

A set which contains no element is known as empty set. The empty set is also known as void set or null set. Generally denoted by ϕ. Consider the examples

(*i*) $\phi = \{x : x \neq x\}$
(*ii*) $\phi = \{x : x$ is a month of the year containing 368 days$\}$

2.2.6 Set of Sets

A set which contains sets is known as set of sets. Consider the example

$$A = \{\{a, b\}, \{1\}, \{1, 2, 3, 4\}, \{u, v\}, \{\text{Book, Pen}\}\}$$

2.2.7 Universal Set

A set which is superset of all the sets under consideration or particular discussion is known as universal set. Generally denoted by U or E or Ω. Generally, the universal set can be chosen arbitrarily for discussion, but once chosen it is fixed for discussion. Consider the example:

Let $A = \{a, b, c\}$
$B = \{a, e, i, o, u\}$
$C = \{p, q, r, s\}$

So, we can take the universal set U as $\{a, b, c, \ldots, z\}$

i.e., $U = \{a, b, c, d, e, \ldots, z\}$

■ 2.3 CARDINALITY OF A SET

If S be a set, then the number of elements present in the set S is known as cardinality of S and is denoted by $|S|$. Mathematically if $S = \{s_1, s_2, s_3, \ldots, s_k\}$, then $|S| = k$; $k \in N$.

Consider the example

Let $A = \{2, 4, 8, 16, 32, 64, 128, 256\}$

So, $|A| = 8$

2.3.1 Equivalent Sets

Two sets A and B are said to be equivalent if they contains equal number of elements. In other words A and B are said to be equivalent if they have same cardinality, *i.e.* $|A| = |B|$. The equivalent sets are also known as similar sets and is denoted as $A \approx B$.

Consider the example of two sets.

$$A = \{a, e, i, o, u\}$$
$$B = \{7, 9, 11, 13, 15\}$$

Here, $|A| = 5 = |B|$. Thus A and B are similar.

■ 2.4 SUBSET AND SUPERSET

Set A is said to be a subset of B or set B is said to be the superset of A if each element of A is also an element of the set B. We write $A \subseteq B$.

i.e., $A \subseteq B \leftrightarrow \{x \in A \to x \in B; \forall x \in A\}$

Consider the examples

(*i*) Let $A = \{1, 2, 3, 4, 5, 6\}$
$B = \{1, 2, 3, 4, 5, 6, 7, 8\}$
So $A \subseteq B$.

(*ii*) Let $A = \{a, b, c\}$
$B = \{b, c, a\}$
so, $A \subseteq B$ and $B \subseteq A$.

(*iii*) Let $A = \{\ \}$ and $B = \{1, 2, 3\}$
So, $A \subseteq B$.

2.4.1 Equal Sets

Two sets A and B are said to be equal if and only if every element of A is in B and every element of B is in A, *i.e.* $A \subseteq B$ and $B \subseteq A$. Mathematically

$$A = B \leftrightarrow \{A \subseteq B \text{ and } B \subseteq A\}$$

i.e.,

$$A = B \leftrightarrow \{x \in A \leftrightarrow x \in B\}$$

Consider the example: Let $A = \{x, y, z, p, q, r\}$

$$B = \{p, q, r, x, y, z\}$$

So, $B \subseteq A$ and $A \subseteq B$. Thus $A = B$.

2.4.2 Proper Subset

Set A is said to be a proper subset of B if each element of A is also an element of B and set B has at least one element which is not an element of set A. We write $A \subset B$.

Mathematically

$$A \subset B \leftrightarrow \{x \in A \to x \in B \text{ and for at least one } y \in B \to y \notin A\}.$$

Consider an example

Let $A = \{a, b, c, d\}$

$$B = \{a, b, c, d, e, f, g\}$$

Here for $x \in A$ we have $x \in B$ and $y = e \in B$ such that $y = e \notin A$. Thus $A \subset B$.

Note: 1. Every set is a subset of itself, *i.e.* $A \subseteq A$.

2. Empty set is a subset of every set, *i.e.* $\phi \subseteq A$.

■ 2.5 COMPARABILITY OF SETS

Two sets A and B are said to be comparable if any one of the following relation holds.

i.e., (*i*) $A \subset B$ or (*ii*) $B \subset A$ or (*iii*) $A = B$.

Consider the following sets

$$A = \{a, b, c, d, e\};\ B = \{2, 3, 5\} \text{ and } C = \{c, d, e\}.$$

It is clear that $A \not\subset B$, $B \not\subset A$ and $A \neq B$. So, A and B are not comparable. Similarly $B \not\subset C$, $C \not\subset B$ or $C \neq B$. So, B and C are also not comparable. At the same time it is clear that, $C \subset A$, thus A and C are comparable.

■ 2.6 POWER SET

Power set is of great importance while studying finite state systems such as non-deterministic finite automation. Here, we present the concept of power set that will be useful while studying finite state systems.

If A be a set, then the set of all subsets of A is known as power set of A and is denoted as P(A).

Mathematically, $P(A) = \{X : X \subseteq A\}$

Consider the example:

Let $A = \{a\}$

$\Rightarrow$ $P(A) = \{\Phi, \{a\}\}$

Let $A = \{a, b\}$

$\Rightarrow$ $P(A) = \{\{a\}, \{b\}, \{a, b\}, \Phi\}$

Let $A = \{a, b, c\}$

$\Rightarrow$ $P(A) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}, \Phi\}$

From the above examples it is clear that if a set A contains n elements, then the power set of A, *i.e.* P(A) contains 2^n elements.

i.e., $|A| = n \Rightarrow |P(A)| = 2^n$.

■ 2.7 OPERATIONS ON SETS

It is observed that set theory is a tool to solve many real life problems. In order to solve these problems, it is essential to study different set operations. Here we discuss certain operations such as union, intersection and difference in order to develop an algebra of sets.

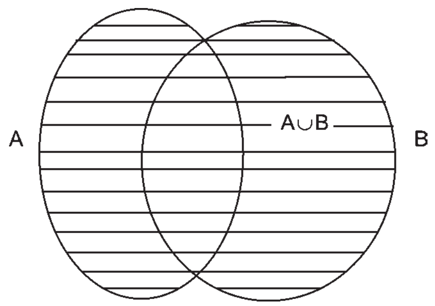
2.7.1 Union

If A and B be two sets, then the union $(A \cup B)$ is defined as a set of all those elements which are either in A or in B or in both.

Symbolically,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

Venn diagram



Consider the example:

Let $A = \{a, b, c, d, e\}$

$B = \{a, e, i, o, u\}$

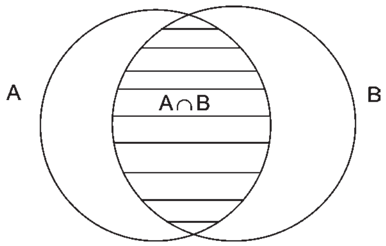
Therefore, $(A \cup B) = \{a, b, c, d, e, i, o, u\}$

2.7.2 Intersection

If A and B be two sets, then the intersection $(A \cap B)$ is defined as a set of all those elements which are common to both the sets. Symbolically

$$(A \cap B) = \{x : x \in A \text{ and } x \in B\}$$

Venn diagram



Consider the example:

Let $A = \{a, b, c, d, e\}$

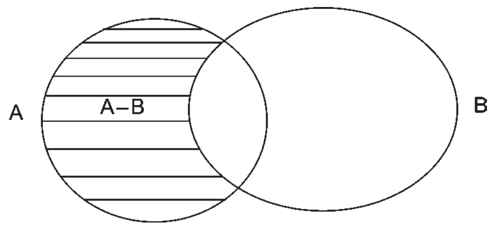
$B = \{a, e, i, o, u\}$

Therefore, $(A \cap B) = \{a, e\}$

2.7.3 Difference

If A and B be two sets, then the difference $(A - B)$ is defined as a set of all those elements of A which are not in B. Symbolically, $(A - B) = \{x \mid x \in A \text{ and } x \notin B\}$

Venn diagram



Consider the example:

Let $A = \{a, b, c, d, e, f\}$

$B = \{a, c, i, o, u, k\}$

Therefore, $(A - B) = \{b, d, e, f\}$

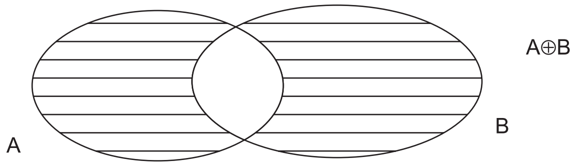
2.7.4 Symmetric Difference

If A and B be two sets, then the symmetric difference $(A \Delta B)$ or $(A \oplus B)$ is defined as a set of all those elements which are either in A or in B but not in both.

Symbolically,

$$(A \oplus B) = (A - B) \cup (B - A)$$

Venn diagram



Consider the example:

Let $A = \{a, b, c, k, p, q, r, s\}$

$B = \{b, k, q, m, n, o, t\}$

So, $(A - B) = \{a, c, p, r, s\}$

and $(B - A) = \{m, n, o, t\}$

Therefore, $(A \oplus B) = (A - B) \cup (B - A)$

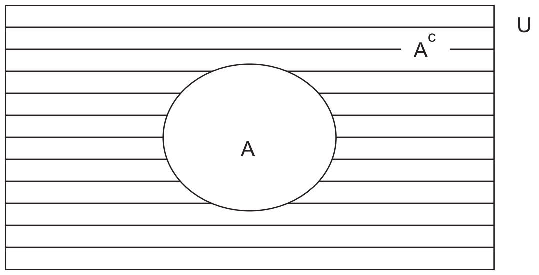
$= \{a, c, p, r, s, m, n, o, t\}$

2.7.5 Complement of a Set

If A be a set, then the complement of A is given as A^c, A' or $\overline{A}$ and is defined as a set of all those elements of the universal set U which are not in A. Symbolically,

$$A^c = \{x \mid x \in U \text{ and } x \notin A\}$$

Venn diagram



Consider the example:

Let $A = \{b, c, k, d, i, p, q, r, s, t\}$

So, we can take the universal set $U = \{a, b, c, \ldots, x, y, z\}$.

Therefore, $A^c = U - A$

$= \{a, e, f, g, h, j, l, m, n, o, u, v, w, x, y, z\}$

2.7.6 Theorem

Let A, B and C be subsets of the universal set U. Then the following important laws hold.

(*a*) Commutative laws:
$(A \cup B) = (B \cup A)$; $(A \cap B) = (B \cap A)$

(*b*) Associative laws:
$A \cup (B \cup C) = (A \cup B) \cup C$; $A \cap (B \cap C) = (A \cap B) \cap C$

(*c*) Idempotent laws:
$(A \cup A) = A$; $(A \cap A) = A$

(*d*) Identity laws:
$(A \cup \phi) = A$; $(A \cap U) = A$

(*e*) Bound laws:
$(A \cup U) = U$; $(A \cap \phi) = \phi$

(*f*) Absorption laws:
$A \cup (A \cap B) = A$; $A \cap (A \cup B) = A$

(*g*) Complement laws:
$(A \cup A^c) = U$; $(A \cap A^c) = \phi$

(*h*) Involution law:
$(A^c)^c = A$

(*i*) Distributive laws :
(*i*) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(*ii*) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof: Proofs of (*a*), (*b*), (*c*), (*d*), (*e*), (*f*), (*g*) and (*h*) are immediate consequences of the definitions. We prove only the distributive laws.

(*i*) $x \in A \cup (B \cap C)$
$\Leftrightarrow$ $x \in A$ or $x \in (B \cap C)$
$\Leftrightarrow$ $x \in A$ or ($x \in B$ and $x \in C$)
$\Leftrightarrow$ ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$)
$\Leftrightarrow$ $x \in (A \cup B)$ and $x \in (A \cup C)$
$\Leftrightarrow$ $x \in (A \cup B) \cap (A \cup C)$
So, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(*ii*) $x \in A \cap (B \cup C)$
$\Leftrightarrow$ $x \in A$ and $x \in (B \cup C)$
$\Leftrightarrow$ $x \in A$ and ($x \in B$ or $x \in C$)
$\Leftrightarrow$ ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$)
$\Leftrightarrow$ $x \in (A \cap B)$ or $x \in (A \cap C)$
$\Leftrightarrow$ $x \in (A \cap B) \cup (A \cap C)$
So, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

2.7.7 Theorem

Let A, B and C be subsets of the universal set U. Then the following properties hold.

(*a*) $(A \Delta A) = \phi$
(*b*) $(A \Delta B) = (B \Delta A)$
(*c*) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$
(*d*) $(A \Delta B) = (A \cup B) - (A \cap B)$

Proof: Proofs of (*a*) and (*b*) are immediate consequences of definitions. Here, we prove (*c*) and (*d*).

(*c*) $x \in A \cap (B \Delta C)$

$\Leftrightarrow$ $x \in A$ and $x \in (B \Delta C)$

$\Leftrightarrow$ $x \in A$ and $x \in ((B - C) \cup (C - B))$

$\Leftrightarrow$ $x \in A$ and ($x \in (B - C)$ or $x \in (C - B)$)

$\Leftrightarrow$ ($x \in A$ and $x \in (B - C)$) or ($x \in A$ and $x \in (C - B)$)

$\Leftrightarrow$ ($x \in A$ and ($x \in B$ and $x \notin C$))

or ($x \in A$ and ($x \in C$ and $x \notin B$))

$\Leftrightarrow$ (($x \in A$ and $x \in B$) and ($x \in A$ and $x \notin C$))

or (($x \in A$ and $x \in C$) and ($x \in A$ and $x \notin B$))

$\Leftrightarrow$ ($x \in (A \cap B)$ and $x \notin (A \cap C)$) or

($x \in (A \cap C)$ and $x \notin (A \cap B)$)

$\Leftrightarrow$ $x \in ((A \cap B) - (A \cap C))$ or $x \in ((A \cap C) - (A \cap B))$

$\Leftrightarrow$ $x \in ((A \cap B) - (A \cap C)) \cup ((A \cap C) - (A \cap B))$

$\Leftrightarrow$ $x \in (A \cap B) \Delta (A \cap C)$

So, $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$.

(*d*) $x \in (A \cup B) - (A \cap B)$

$\Leftrightarrow$ $x \in (A \cup B)$ and $x \notin (A \cap B)$

$\Leftrightarrow$ $x \in (A \cup B)$ and ($x \notin A$ or $x \notin B$)

$\Leftrightarrow$ ($x \in (A \cup B)$ and $x \notin A$) or ($x \in (A \cup B)$ and $x \notin B$)

$\Leftrightarrow$ (($x \in A$ or $x \in B$) and $x \notin A$)

or (($x \in A$ or $x \in B$) and $x \notin B$)

$\Leftrightarrow$ (($x \in A$ and $x \notin A$) or ($x \in B$ and $x \notin A$))

or (($x \in A$ and $x \notin B$) or ($x \in B$ and $x \notin B$))

$\Leftrightarrow$ ($x \in \phi$ or $x \in (B - A)$) or ($x \in (A - B)$ or $x \in \phi$)

$\Leftrightarrow$ $x \in (\phi \cup (B - A))$ or $x \in ((A - B) \cup \phi)$

$\Leftrightarrow$ $x \in (B - A) \cup (A - B)$ [By identity law]

$\Leftrightarrow$ $x \in (B \Delta A)$

$\Leftrightarrow$ $x \in (A \Delta B)$ [By commutative law]

So, $(A \Delta B) = (A \cup B) - (A \cap B)$

2.7.8 de Morgan's Law

Let A and B be subsets of the universal set U. Then

(*a*) $(A \cup B)^c = (A^c \cap B^c)$

(*b*) $(A \cap B)^c = (A^c \cup B^c)$

Proof: (*a*) $x \in (A \cup B)^c$

$\Leftrightarrow$ $x \notin (A \cup B)$

$\Leftrightarrow$ $x \notin A$ and $x \notin B$

$\Leftrightarrow$ $x \in A^c$ and $x \in B^c$

$\Leftrightarrow$ $x \in A^c \cap B^c$

So, $(A \cup B)^c = (A^c \cap B^c)$

(*b*) $x \in (A \cap B)^c$

$\Leftrightarrow$ $x \notin (A \cap B)$

$\Leftrightarrow$ $x \notin A$ or $x \notin B$

$\Leftrightarrow$ $x \in A^c$ or $x \in B^c$

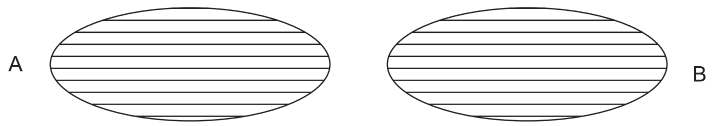
$\Leftrightarrow$ $x \in A^c \cup B^c$

So, $(A \cap B)^c = (A^c \cup B^c)$

■ 2.8 DISJOINT SETS

Two sets A and B are called disjoint or non-overlapping if both sets have no common element. Mathematically, $(A \cap B) = \phi$.

Venn diagram



■ 2.9 APPLICATION OF SET THEORY

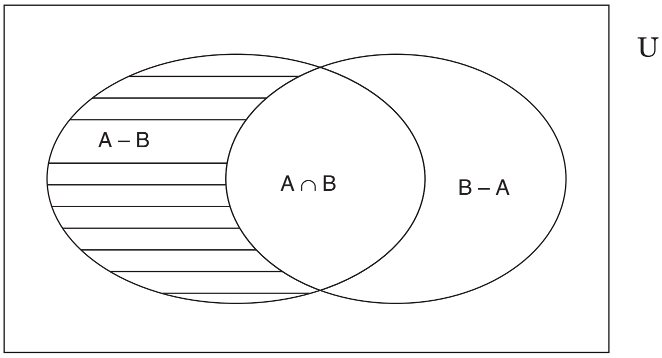
Let A and B be finite sets. Let $n(A)$ be the number of distinct elements of the set A. Then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

Further if A and B are disjoint, then

$$n(A \cup B) = n(A) + n(B)$$

Proof: A and B be finite sets and $n(A)$ represent the number of distinct elements of the set A.



From the above Venn diagram it is clear that

$$n(A) = n(A - B) + n(A \cap B)$$

and $$n(B) = n(B - A) + n(A \cap B)$$

and $$\begin{aligned} n(A \cup B) &= n(A - B) + n(A \cap B) + n(B - A) \\ &= n(A) - n(A \cap B) + n(A \cap B) + n(B) - n(A \cap B) \\ &= n(A) + n(B) - n(A \cap B) \end{aligned}$$

i.e., $$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

If A and B are disjoint, then $(A \cap B) = \phi$ *i.e.*, $n(A \cap B) = 0$

Therefore, $n(A \cup B) = n(A) + n(B)$.

■ 2.10 PRODUCT OF SETS

The product of sets is defined with the help of an order pair. An order pair is usually denoted by (x, y) such that $(x, y) \neq (y, x)$ whenever $x \neq y$. The product of two sets A and B is the set of all those order pairs whose first coordinate is an element of A and the second coordinate is an element of B. The set is denoted by $(A \times B)$. Mathematically,

$$(A \times B) = \{(x, y) \mid x \in A \text{ and } x \in B\}$$

Consider the example:

Let $A = \{1, 2, 3, 5, 7\}$

$B = \{4, 9, 25\}$

So, $(A \times B) = \{(1,4), (1, 9), (1, 25), (2, 4), (2, 9), (2, 25), (3, 4), (3, 9), (3, 25), (5, 4), (5, 9), (5, 25), (7, 4), (7, 9), (7, 25)\}$

Note: The product of sets can be extendable for n sets $A_1, A_2, A_3, ..., A_n$.

Thus, $A_1 \times A_2 \times A_3 \times \ldots \times A_n$ can be defined as

$A_1 \times A_2 \times A_3 \times \ldots \times A_n = \{(x_1, x_2, x_3, \ldots, x_n) \mid x_1 \in A_1$ and $x_2 \in A_2$ and $x_3 \in A_3$ and … and $x_n \in A_n\}$where $(x_1, x_2, x_3, \ldots, x_n)$ is called as n-tuple of $x_1, x_2, x_3, \ldots, x_n$. To explain this consider the example in which $A = \{a, b, c\}$; $B = \{1, 2\}$ and $C = \{\alpha, \beta\}$.

Therefore, $A \times B \times C = \{(a, 1, \alpha), (a, 1, \beta), (a, 2, \alpha), (a, 2, \beta), (b, 1, \alpha), (b, 1, \beta), (b, 2, \alpha), (b, 2, \beta), (c, 1, \alpha), (c, 1, \beta), (c, 2, \alpha), (c, 2, \beta)\}$.

From the above example, it is very clear that $|A \times B \times C| = |A| \times |B| \times |C|$.

In general, $|A_1 \times A_2 \times A_3 \times \ldots \times A_n| = |A_1| \times |A_2| \times |A_3| \times \ldots \times |A_n|$.

2.10.1 Theorem

Let A, B and C be three subsets of the universal set U. Then

(*a*) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(*b*) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Proof: (*a*) $(x, y) \in A \times (B \cup C)$

$\Leftrightarrow$ $x \in A$ and $y \in (B \cup C)$

$\Leftrightarrow$ $x \in A$ and $(y \in B$ or $y \in C)$

$\Leftrightarrow$ $(x \in A$ and $y \in B)$ or $(x \in A$ and $y \in C)$

$\Leftrightarrow (x, y) \in (A \times B)$ or $(x, y) \in (A \times C)$

$\Leftrightarrow (x, y) \in (A \times B) \cup (A \times C)$

Therefore, $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

(*b*) $(x, y) \in A \times (B \cap C)$

$\Leftrightarrow$ $x \in A$ and $y \in (B \cap C)$

$\Leftrightarrow$ $x \in A$ and $(y \in B$ and $y \in C)$

$\Leftrightarrow$ $(x \in A$ and $y \in B)$ and $(x \in A$ and $y \in C)$

$\Leftrightarrow (x, y) \in (A \times B)$ and $(x, y) \in (A \times C)$

$\Leftrightarrow (x, y) \in (A \times B) \cap (A \times C)$

Therefore, $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

2.11 FUNDAMENTAL PRODUCTS

Let $A_1, A_2, A_3, \ldots, A_n$ be n sets. A fundamental product of these n sets is an expression of the form $(B_1 \cap B_2 \cap B_3 \cap \ldots \cap B_n)$ where B_i is either A_i or A_i^c.

Consider an example with three sets A, B and C. The fundamental products of these three sets are as follows, which are 2^3 in number.

i.e., $A \cap B \cap C$; $A^c \cap B \cap C$; $A \cap B^c \cap C$; $A \cap B \cap C^c$;
$A^c \cap B^c \cap C$; $A \cap B^c \cap C^c$; $A^c \cap B \cap C^c$; $A^c \cap B^c \cap C^c$.

SOLVED EXAMPLES

Example 1 *Let A, B and C be any three subsets of the universal set U. Then prove that*

(a) $A - (B \cup C) = (A - B) \cap (A - C)$
(b) $A - (B \cup C) = (A - B) - C$
(c) $(A \cap B) - C = A \cap (B - C)$

Solution: (*a*) $x \in A - (B \cup C)$

$\Leftrightarrow \quad x \in A$ and $x \notin (B \cup C)$
$\Leftrightarrow \quad x \in A$ and ($x \notin B$ and $x \notin C$)
$\Leftrightarrow \quad$ ($x \in A$ and $x \notin B$) and ($x \in A$ and $x \notin C$)
$\Leftrightarrow \quad x \in (A - B)$ and $x \in (A - C)$
$\Leftrightarrow \quad x \in (A - B) \cap (A - C)$

Therefore, $A - (B \cup C) = (A - B) \cap (A - C)$

(*b*) $x \in A - (B \cup C)$

$\Leftrightarrow \quad x \in A$ and $x \notin (B \cup C)$
$\Leftrightarrow \quad x \in A$ and ($x \notin B$ and $x \notin C$)
$\Leftrightarrow \quad$ ($x \in A$ and $x \notin B$) and $x \notin C$
$\Leftrightarrow \quad x \in (A - B)$ and $x \notin C$
$\Leftrightarrow \quad x \in (A - B) - C$

Therefore, $A - (B \cup C) = (A - B) - C$

(*c*) $x \in (A \cap B) - C$

$\Leftrightarrow \quad$ ($x \in A$ and $x \in B$) and $x \notin C$
$\Leftrightarrow \quad x \in A$ and ($x \in B$ and $x \notin C$)
$\Leftrightarrow \quad x \in A$ and $x \in (B - C)$
$\Leftrightarrow \quad x \in A \cap (B - C)$

Therefore, $(A \cap B) - C = A \cap (B - C)$

Example 2 *Show that* $A - \bigcup_{i=1}^{n} B_i = \bigcap_{i=1}^{n} (A - B_i)$

Solution: $x \in A - \bigcup_{i=1}^{n} B_i$

$\Leftrightarrow \quad x \in A$ and $x \notin \bigcup_{i=1}^{n} B_i$
$\Leftrightarrow \quad x \in A$ and $x \notin (B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_n)$
$\Leftrightarrow \quad x \in A$ and ($x \notin B_1$ and $x \notin B_2$ and $x \notin B_3$ and … and $x \notin B_n$)

$\Leftrightarrow \quad (x \in A \text{ and } x \notin B_1) \text{ and } (x \in A \text{ and } x \notin B_2) \text{ and } \ldots. \text{ and } (x \in A \text{ and } x \notin B_n)$

$\Leftrightarrow \quad x \in (A - B_1) \text{ and } x \in (A - B_2) \text{ and } \ldots \text{ and } x \in (A - B_n)$

$\Leftrightarrow \quad x \in (A - B_1) \cap (A - B_2) \cap \ldots \cap x \in (A - B_n)$

$$\Leftrightarrow \quad x \in \bigcap_{i=1}^{n} (A - B_i)$$

Therefore, $A - \bigcup_{i=1}^{n} B_i = \bigcap_{i=1}^{n} (A - B_i)$

Example 3 *If A and B subsets of the universal set U, then show that*

(a) $(A^c)^c = A$

(b) $A - B = A \cap B^c$

(c) $(A - B) \cap B = \phi$

Solution: (*a*) $x \in (A^c)^c$

$\Leftrightarrow \quad x \notin A^c$

$\Leftrightarrow \quad x \in A$

So, $(A^c)^c = A$

(*b*) $x \in (A - B)$

$\Leftrightarrow \quad x \in A \text{ and } x \notin B$

$\Leftrightarrow \quad x \in A \text{ and } x \in B^c$

$\Leftrightarrow \quad x \in (A \cap B^c)$

So, $(A - B) = (A \cap B^c)$

(*c*) $x \in (A - B) \cap B$

$\Leftrightarrow \quad x \in (A - B) \text{ and } x \in B$

$\Leftrightarrow \quad (x \in A \text{ and } x \notin B) \text{ and } x \in B$

$\Leftrightarrow \quad x \in A \text{ and } (x \notin B \text{ and } x \in B)$

$\Leftrightarrow \quad x \in A \text{ and } x \in \phi$

$\Leftrightarrow \quad x \in (A \cap \phi)$

$\Leftrightarrow \quad x \in \phi$

So, $(A - B) \cap B = \phi$

Example 4 *Let A, B be the subsets of the universal set U, then prove that*

(a) $A - (A \cap B) = A \cap B^c$

(b) $(A \cap B^c)^c = A^c \cup B$

Solution: (*a*) $x \in A - (A \cap B)$

$\Leftrightarrow \quad x \in A \text{ and } x \notin (A \cap B)$

$\Leftrightarrow \quad x \in A \text{ and } (x \notin A \text{ or } x \notin B)$

$\Leftrightarrow \quad (x \in A \text{ and } x \notin A) \text{ or } (x \in A \text{ and } x \notin B)$

$\Leftrightarrow \quad x \in \phi \text{ or } (x \in A \text{ and } x \in B^c)$

$\Leftrightarrow \quad x \in \phi \text{ or } x \in (A \cap B^c)$

$\Leftrightarrow \quad x \in \phi \cup (A \cap B^c)$

$\Leftrightarrow \quad x \in (A \cap B^c)$

So, $A - (A \cap B) = A \cap B^c$

(*b*) $x \in (A \cap B^c)^c$

$\Leftrightarrow$ $x \notin (A \cap B^c)$

$\Leftrightarrow$ $x \notin A$ or $x \notin B^c$

$\Leftrightarrow$ $x \in A^c$ or $x \in B$

$\Leftrightarrow$ $x \in (A^c \cup B)$

So, $(A \cap B^c)^c = (A^c \cup B)$

Example 5 *Let A, B and C be three subsets of the universal set U. Then show that*

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

Solution: Let $(B \cup C) = D$. So, we have

$$\begin{aligned} n(A \cup B \cup C) &= n(A \cup D) \\ &= n(A) + n(D) - n(A \cap D) \\ &= n(A) + n(B \cup C) - n(A \cap (B \cup C)) \\ &= n(A) + n(B) + n(C) - n(B \cap C) - n((A \cap B) \cup (A \cap C)) \\ &= n(A) + n(B) + n(C) - n(B \cap C) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C) \end{aligned}$$

Therefore, $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$.

Example 6 *In the IEEE conference held at New York, 500 delegates attended. 200 of them could take tea, 350 could take coffee and 10 did not take either coffee or tea. Then answer the following questions.*

(a) How many can take both tea and coffee.
(b) How many can take tea only and
(c) How many can take coffee only.

Solution: Let T: Set of persons who take tea.
C: Set of persons who take coffee.
U: Total number of delegates.

Hence we have $n(U) = 500$; $n(T) = 200$; $n(C) = 350$

Number of delegates did not take either coffee or tea = 10

Therefore number of delegates who take either coffee or tea = 500 –10 = 490

i.e., $n(T \cup C) = 490$

i.e., $n(T) + n(C) - n(T \cap C) = 490$

i.e., $n(T \cap C) = n(T) + n(C) - 490 = 200 + 350 - 490 = 60$

So, the number of persons who take both coffee and tea = $n(T \cap C) = 60$

Number of persons take tea only = $n(T) - n(T \cap C) = 140$

Number of persons take coffee only = $n(C) - n(T \cap C) = 290$.

Example 7 *If 65% of students like apples where 75% like grapes, then what percentage of students likes both apples and grapes?*

Solution: Let $n(S)$: Total number of students = 100
$n(A)$: Total number of students who like apples = 65
$n(B)$: Total number of students who like grapes = 75

Therefore, $n(S) = n(A \cup B) = n(A) + n(B) - n(A \cap B)$

i.e., $100 = 65 + 75 - n(A \cap B)$

i.e., $n(A \cap B) = 40$

So, 40% of students like both apples and grapes.

Example 8 *If $A = \{2, 3, 4, 5, 6\}$, $B = \{3, 4, 5, 6, 7\}$ and $C = \{4, 5, 6, 7, 8\}$ then find the followings.*

(i) $(A \cup B) \cap (A \cup C)$ *(ii)* $(A \cap B) \cup (A \cap C)$

(iii) $A - (B - C)$ *(iv)* $(A \Delta B)$.

Solution: Given $A = \{2, 3, 4, 5, 6\}$, $B = \{3, 4, 5, 6, 7\}$ and $C = \{4, 5, 6, 7, 8\}$

(*i*) $(A \cup B) = \{2, 3, 4, 5, 6, 7\}$

$(A \cup C) = \{2, 3, 4, 5, 6, 7, 8\}$

Therefore, $(A \cup B) \cap (A \cup C) = \{2, 3, 4, 5, 6, 7\}$

(*ii*) $(A \cap B) = \{3, 4, 5, 6\}$

$(A \cap C) = \{4, 5, 6\}$

Therefore, $(A \cap B) \cup (A \cap C) = \{3, 4, 5, 6\}$

(*iii*) $(B - C) = \{3\}$

Therefore, $A - (B - C) = \{2, 4, 5, 6\}$

(*iv*) $(A \cup B) = \{2, 3, 4, 5, 6, 7\}$

$(A \cap B) = \{3, 4, 5, 6\}$

Therefore, $(A \Delta B) = (A \cup B) - (A \cap B) = \{2, 7\}$

Example 9 *Find the power sets of the following sets.*

(i) $\{0\}$

(ii) $\{1, \{1, 2\}\}$ *and*

(iii) $\{4, 1, 8\}$.

Solution: (*i*) Let $A = \{0\}$

Therefore, $P(A) = \{\{0\}, \phi\}$

(*ii*) Let $A = \{1, \{1, 2\}\}$

So, $P(A) = \{\{1\}, \{\{1, 2\}\}, A, \phi\}$

(*iii*) Let $A = \{4, 1, 8\}$

So, $P(A) = \{\{4\}, \{1\}, \{8\}, \{4, 1\}, \{4, 8\}, \{1, 8\}, A, \phi\}$.

Example 10 *If $A = \{4, 5\}$, $B = \{7, 8\}$ and $C = \{9, 10\}$, then find the followings.*

(a) $(A \times B) \cup (B \times C)$ *and (b)* $A \times (B \cup C)$.

Solution: Given $A = \{4, 5\}$, $B = \{7, 8\}$ and $C = \{9, 10\}$

(*a*) $(A \times B) = \{(4, 7), (4, 8), (5, 7), (5,8)\}$

$(B \times C) = \{(7, 9), (7, 10), (8, 9), (8, 10)\}$

So, $(A \times B) \cup (B \times C) = \{(4, 7), (4, 8), (5, 7), (5,8), (7, 9), (7, 10), (8, 9), (8, 10)\}$

(*b*) $(B \cup C) = \{7, 8, 9, 10\}$

So, $A \times (B \cup C) = \{(4, 7), (4, 8), (4, 9), (4,10), (5,7), (5, 8), (5, 9), (5, 10)\}$.

Example 11 *If $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ and $C = \{3, 4, 5\}$, then verify the product laws.*

Solution: Given $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ and $C = \{3, 4, 5\}$

Therefore, $(B \cup C) = \{2, 3, 4, 5\}$ and

$A \times (B \cup C) = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 2), (2, 3), (2, 4), (2, 5), (3, 2), (3, 3), (3, 4), (3, 5)\}$

$(A \times B) = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$

$(A \times C) = \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 3), (3, 4), (3, 5)\}$

Thus, $(A \times B) \cup (A \times C) = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 2), (2, 3), (2, 4), (2, 5), (3, 2), (3, 3), (3, 4), (3, 5)\}$

$= A \times (B \cup C)$

Similarly, the second product law $A \times (B \cap C) = (A \times B) \cap (A \times C)$ can be verified.

Example 12 *If P = {a, c, e}, Q ={100, 101, 102} and R = {m, c, e, 101}. Compute ((Q ∪ P) – (P ∩ Q)) × R, where ∩, ∪, – and × are well known set theoretic binary operations.*

Solution: Given P = {a, c, e}; Q = {100, 101, 102} and R = {m, c, e, 101}.

So, $(Q \cup P) = \{100, 101, 102, a, c, e\}$ and $(P \cap Q) = \phi$

Therefore, $((Q \cup P) - (P \cap Q)) = \{100, 101, 102, a, c, e\}$

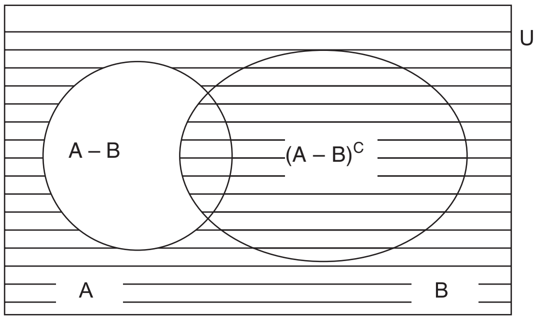
Thus, $((Q \cup P) - (P \cap Q)) \times R = \{(100, m), (100, c), (100, e), (100, 101), (101, m), (101, c), (101, e), (101, 101), (102, m), (102, c), (102, e), (102, 101), (a, m), (a, c), (a, e), (a, 101), (c, m), (c, c), (c, e), (c, 101), (e, m), (e, c), (e, e), (e, 101)\}$.

Example 13 *Show the following sets by Venn diagram.*

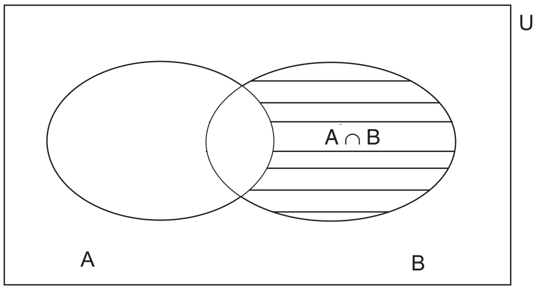
(a) $(A - B)^c$ *(b)* $A^c \cap B$ *(c)* $A \cap B \cap C$

Solution:

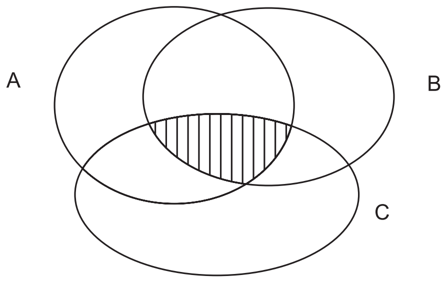
(a) $(A - B)^c$



(b) $(A^c \cap B)$



(c) $A \cap B \cap C$



Example 14 *In a group of 64 students 26 can speak French only, 14 can speak English only. How many can speak both French and English?*

Solution: Let F: Set of students who can speak French.

E: Set of students who can speak English.

Let $n(S)$: Total number of students = 64

i.e., $n(S) = n(F \cup E) = 64$

Given: $n(F - E)$: Number of students speak French only = 26

and $n(E - F)$: Number of students speak English only = 14

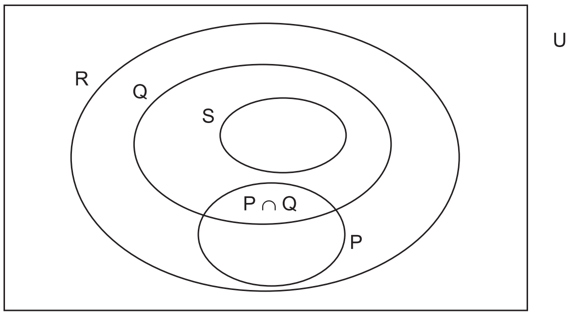
Therefore, $n(F \cup E) = n(F - E) + n(E - F) + n(F \cap E)$

i.e., $n(F \cap E) = 64 - 26 - 14 = 24$

So, 24 students can speak both French and English.

Example 15 *Draw a Venn diagram to represent the following facts for the sets P, Q, R and S. $(P \cap Q) \neq \phi$, $S \subseteq Q \subseteq R$ and $(P \cap S) = \phi$.*

Solution: Given conditions are $(P \cap Q) \neq \phi$, $S \subseteq Q \subseteq R$ and $(P \cap S) = \phi$. The Venn diagram for the above facts is given below.



Example 16 *If in a city 60% of the residents can speak German and 50% can speak French. What percentage of residents can speak both the languages, if 20% residents can not speak any of these two languages?*

Solution: Let $n(S)$: Total number of residents = 100

$n(G)$: Total number of residents who speak German = 60

$n(F)$: Total number of residents who speak French = 50

$n(G \cup F)^c$: Total number of residents who cannot speak any of these two languages = 20

So, $n(G \cup F) = n(S) - n(G \cup F)^c = 100 - 20 = 80$

i.e., $n(G) + n(F) - n(G \cap F) = 80$

i.e., $n(G \cap F) = 60 + 50 - 80 = 30$

Therefore, 30% of the residents can speak both the languages German and French.

Example 17 *In a survey about liking for colours, it was found that everyone who was surveyed had a liking for at least one of the three colours namely Red, Green and Blue. Further 30% liked Red; 40% liked Green and 50% liked Blue. Further 10% people liked both Red and Green, 5% liked both Green and Blue and 10% liked both Red and Blue. Find the percentage of the surveyed people who like all the colours.*

Solution: Let R: Set of people who like Red colour

G: Set of people who like Green colour

B: Set of people who like Blue colour

and S: Set of all people who was surveyed.

Therefore, $n(S) = 100$; $n(R) = 30$; $n(G) = 40$; $n(B) = 50$; $n(R \cap G) = 10$; $n(G \cap B) = 5$; $n(R \cap B) = 10$.

Thus $n(S) = n(R \cup G \cup B) = 100$

i.e., $n(R) + n(G) + n(B) - n(R \cap G) - n(G \cap B) - n(R \cap B) + n(R \cap G \cap B) = 100$

i.e., $n(R \cap G \cap B) = 100 - 30 - 40 - 50 + 10 + 5 + 10 = 5$

So, 5% of the surveyed people like all the colours, i.e. Red, Green and Blue.

Example 18 *If $A \subset B$ and $B \subset C$, then show that $A \subset C$.*

Solution: Given $B \subset C$, *i.e.* $x \in B \Rightarrow x \in C$

Again $A \subset B$, *i.e.* $x \in A \Rightarrow x \in B$ $\quad \forall x \in A$

i.e., $x \in A \Rightarrow x \in B \Rightarrow x \in C$

i.e., $x \in A \Rightarrow x \in C$

Therefore, $A \subset C$.

Example 19 *For all sets A and B prove that $\overline{A \times B} = \overline{A} \times \overline{B}$.*

Solution: $(x, y) \in \overline{A \times B}$

$\Leftrightarrow$ $(x, y) \notin A \times B$

$\Leftrightarrow$ $x \notin A$ and $y \notin B$

$\Leftrightarrow$ $x \in \overline{A}$ and $y \in \overline{B}$

$\Leftrightarrow$ $(x, y) \in \overline{A} \times \overline{B}$

So, $(x, y) \in \overline{A \times B} \Leftrightarrow (x, y) \in \overline{A} \times \overline{B}$

Therefore, $\overline{A \times B} = \overline{A} \times \overline{B}$

Example 20 *For all sets A, B and C prove that $A \times (B - C) = (A \times B) - (A \times C)$.*

Solution: $(x, y) \in A \times (B - C)$

$\Leftrightarrow$ $x \in A$ and $y \in (B - C)$

$\Leftrightarrow$ $x \in A$ and ($y \in B$ and $y \notin C$)

$\Leftrightarrow$ ($x \in A$ and $y \in B$) and ($x \in A$ and $y \notin C$)

$\Leftrightarrow$ $(x, y) \in (A \times B)$ and $(x, y) \notin (A \times C)$

$\Leftrightarrow$ $(x, y) \in (A \times B) - (A \times C)$

Therefore, $A \times (B - C) = (A \times B) - (A \times C)$.

Example 21 *In a group of 191 students, 10 are taking English, Computer Science and Music; 36 are taking English and Computer Science; 20 are taking English and Music; 18 are taking Computer Science and Music; 65 are taking English; 76 are taking Computer Science and 63 are taking Music. Then answer the followings*

(a) How many are taking English and Music but not Computer Science.
(b) How many are taking Computer Science and Music but not English.
(c) How many are taking Computer Science and neither English nor Music.
(d) How many are taking none of the three subjects.

Solution: Let S: Set of students
E: Set of students taking English
C: Set of students taking Computer Science
M: Set of students taking Music.

Given that $n(S) = 191$; $n(E) = 65$; $n(C) = 76$; $n(M) = 63$; $n(E \cap C \cap M) = 10$; $n(E \cap C) = 36$; $n(E \cap M) = 20$; $n(C \cap M) = 18$

(*a*) Number of students taking English and Music but not Computer Science
$$= n(E \cap M) - n(E \cap C \cap M) = 20 - 10 = 10$$

(*b*) Number of students taking Computer Science and Music but not English
$$= n(C \cap M) - n(E \cap C \cap M) = 18 - 10 = 8$$

(*c*) Number of students taking Computer Science and neither English nor Music
$$= n(C) - n(E \cap C) - n(C \cap M) + n(E \cap C \cap M) = 76 - 36 - 18 + 10 = 32$$

(*d*) Number of students taking none of the three subjects
$$= n(E \cup C \cup M)^c$$
$$= n(S) - n(E \cup C \cup M)$$
$$= n(S) - \{n(E) + n(C) + n(M) - n(E \cap C) - n(C \cap M) - n(E \cap M) + n(E \cap C \cap M)\}$$
$$= 191 - (65 + 63 + 76 - 20 - 36 - 18 + 10)$$
$$= 51.$$

Example 22 *Examine whether the following sets are equivalent or not.*

(*a*) $A = \{ x \mid x^2 - 7x + 12 = 0; x \in N\}$ (*b*) $B = \{x \mid x = a \text{ and } x = b\}$

(*c*) $C = \{a, b, c, d, e\}$ (*d*) $D = \{x \mid x^2 - 4 = 0; x \in I\}$

Solution: Given that $A = \{x \mid x^2 - 7x + 12 = 0; x \in N\}$

Therefore $A = \{3, 4\}$

i.e., $|A| = 2$

Similarly $B = \{x \mid x = a \text{ and } x = b\}$
$= \{a, b\}$

i.e., $|B| = 2$

Also $C = \{a, b, c, d, e\}$

i.e., $|C| = 5$

Again $D = \{x \mid x^2 - 4 = 0; x \in I\} = \{2, -2\}$

i.e., $|D| = 2$

Therefore, $|A| = |B| = |D| = 2 \neq |C| = 5$; So A, B and D are equivalent.

Example 23 *For all Sets A and B prove that* $(A \cap B) \cup (B - A) = B$.

Solution:
$$\begin{aligned}
(A \cap B) \cup (B - A) &= (A \cap B) \cup (B \cap A^c) \\
&= ((A \cap B) \cup B) \cap ((A \cap B) \cup A^c) && \text{[Distributive law]} \\
&= B \cap ((A \cap B) \cup A^c) && \text{[Absorption law]} \\
&= B \cap ((A \cup A^c) \cap (B \cup A^c)) && \text{[Distributive law]} \\
&= B \cap (U \cap (B \cup A^c)) && \text{[Complement law]} \\
&= B \cap (B \cup A^c) \\
&= B && \text{[Absorption law]}
\end{aligned}$$

Example 24 *By applying properties of sets prove that* $(A - B) \cap (B - A) = \phi$ *for all sets A and B.*

Solution:
$$\begin{aligned}
(A - B) \cap (B - A) &= (A \cap B^c) \cap (B \cap A^c) \\
&= A \cap (B^c \cap (B \cap A^c)) && \text{[Associative law]} \\
&= A \cap ((B^c \cap B) \cap A^c) && \text{[Associative law]} \\
&= A \cap (\phi \cap A^c) && \text{[Complement law]} \\
&= (A \cap \phi) && \text{[Bound law]} \\
&= \phi && \text{[Bound law]}
\end{aligned}$$

Example 25 *For all sets X, Y and Z prove that* $X \cap (Y - Z) = (X \cap Y) - (X \cap Z)$.

Solution:

$x \in X \cap (Y - Z)$

$\Leftrightarrow$ $x \in X$ and $x \in (Y - Z)$

$\Leftrightarrow$ $x \in X$ and ($x \in Y$ and $x \notin Z$)

$\Leftrightarrow$ ($x \in X$ and $x \in Y$) and ($x \in X$ and $x \notin Z$)

$\Leftrightarrow$ $x \in (X \cap Y)$ and $x \notin (X \cap Z)$

$\Leftrightarrow$ $x \in (X \cap Y) - (X \cap Z)$

Therefore, $X \cap (Y - Z) = (X \cap Y) - (X \cap Z)$

Example 26 *For all sets X, Y and Z prove that* $X - (Y \cup Z) = (X - Y) \cap Z^c$.

Solution:

$x \in X - (Y \cup Z)$

$\Leftrightarrow$ $x \in X$ and $x \notin (Y \cup Z)$

$\Leftrightarrow$ $x \in X$ and ($x \notin Y$ and $x \notin Z$)

$\Leftrightarrow$ ($x \in X$ and $x \notin Y$) and $x \notin Z$

$\Leftrightarrow$ $x \in (X - Y)$ and $x \in Z^c$

$\Leftrightarrow$ $x \in (X - Y) \cap Z^c$

Therefore, $X - (Y \cup Z) = (X - Y) \cap Z^c$.

Example 27 *Determine the equality for the following pair of sets.*

$A = \{1, 2, 3\}$ and $B = \{x \mid x \in N;\ x^3 - 6x^2 + 11x - 6 = 0\}$

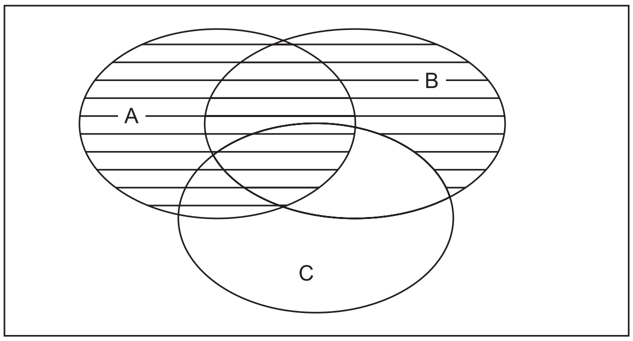
Solution: Given $A = \{1, 2, 3\}$ and

$$B = \{x \mid x \in N;\ x^3 - 6x^2 + 11x - 6 = 0\}$$
$$= \{x \mid x \in N;\ (x - 1)(x - 2)(x - 3) = 0\}$$
$$= \{1, 2, 3\}$$

Therefore, sets A and B are equal as $A \subseteq B$ and $B \subseteq A$.

Example 28 *Express* $A \cup (B - C)$ *as the union of fundamental products.*

Solution: The figure given below represents the Venn diagram for $A \cup (B - C)$. From this it is clear that $A \cup (B - C)$ consists of the five areas of the Venn diagram corresponding to the fundamental products $(A \cap B \cap C)$, $(A \cap B \cap C^c)$, $(A \cap B^c \cap C)$, $(A^c \cap B \cap C^c)$ and $(A \cap B^c \cap C^c)$.



$A \cup (B - C)$ is shaded

Thus, $A \cup (B - C) = (A \cap B \cap C) \cup (A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C^c) \cup (A \cap B^c \cap C^c)$.

EXERCISES

1. Express the following sets in tabular form.

(*a*) A = {x | x is a letter in the word MATHEMATICS}

(*b*) B = {x | $x = 2n + 1$; $1 \le n < 5$; $n \in$ N}

(*c*) C = {x | x = Book and $x = 1$ and $x = a$ and x = Pen}

(*d*) D = {x | x is an even integer and $1 \le x \le 15$}

(*e*) E = {x | $x \in$ I and $x^2 + x - 20 = 0$}

2. Express the following sets in set builder form.

(*a*) A = {1, 8, 27, 64, 125}

(*b*) B = {a, e, i, o, u}

(*c*) C = {2, 9, 28, 65, 126}

(*d*) D = {a, b, 2, 4, 6, Book}

(*e*) E = {1, 2, 3, 4, 5, 6, 7, ...}

(*f*) F = {1, 3}

3. Find the power sets of the following sets.

(*a*) {ϕ}

(*b*) {k, l, m, n}

(*c*) {x | $x \in$ N and $x^2 - 4x + 3 = 0$}

(*d*) {1, {1, 2}, {1, 2, 3}}

(*e*) {x | x is a letter of the word wolf}

4. Let the universal set U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, Let A = {1, 2, 3, 4, 5}, B = {2, 4, 6, 8} and C = {1, 4, 7, 10}, then find the followings.

(*i*) $(A \cup B)$

(*ii*) $(A \cup B) \cap C$

(*iii*) $(A \cup B) \cap (A \cup C)$

(*iv*) $(A \cap B) \cup (A \cap C)$

(*v*) $A - (B \cup C)$

(*vi*) $(A \cap B) - C$

(*vii*) $B^c - (C - A)$

(*viii*) $A^c \cap B^c$

(*ix*) AB

(*x*) A^c

(*xi*) $C - B$

(*xii*) $(A \cup B) - (C - B)$

5. Draw the Venn diagram and indicate the region for the given sets.

(*a*) $A \cup (B \cap C)$

(*b*) $A \cap (B \cup C)$

(*c*) $A^c - B$

(*d*) $(A \cup B) - B$

(*e*) $(A^c \cup B) \cap (C^c - A)$

(*f*) $B \cap (C \cup A)^c$

(*g*) $(B \cup C) - A$

(*h*) $(A \cup B \cup C)^c$

6. In a group of 1000 people, there are 800 people who can speak English and 500 people who can speak German. Except 100 people in the group, each person speaks at least one of English and German. Find how many people can speak both English and German.

7. If P = {a, c, e}, Q = {100, 101, 102} and R = {m, c, e, 101}, then compute $((P \cup R) - (P \cap R)) \times Q$.

8. If G = {p, q, r}, H = {20, 70, 90} and K = {r, 70, s}, then compute $(G - K) \times (K - H)$.

9. Let X = {a, b, c} and Y = {1, 2}, then compute the followings.

(*a*) $X \times Y$

(*b*) $Y \times X$

(*c*) $Y \times Y$

(*d*) $X \times X$

(*e*) $(X \,\Delta\, Y) \times Y$

10. If $B_1, B_2, \ldots, B_n$ and A are sets, then prove that $A - \bigcap_{i=1}^{n} B_i = \bigcup_{i=1}^{n} (A - B_i$

11. If $B_1, B_2, \ldots, B_n$ are sets, then prove the following de Morgan's laws.

(*a*) $\left(\bigcup_{i=1}^{n}\right)' = \bigcap_{i=1}^{n} B_i'$

(*b*) $\left(\bigcap_{i=1}^{n} B_i\right)' = \bigcup_{i=1}^{n} B_i{}'$

12. Let X, Y and Z be three sets. Show that $X - (Y \cap Z) = (X - Y) \cup (X - Z)$.

13. In a class of 120 students, 80 students study Mathematics, 45 study History and 20 students neither study History nor study Mathematics. What is the number of students who study both Mathematics and History?

14. Let $A = \{1, 2\}$, $B = \{\alpha\}$ and $C = \{\alpha, \beta\}$, then compute the followings.

(*a*) $A \times B \times C$ (*b*) $A \times B \times B$

(*c*) $B \times A \times C$ (*d*) $A \times A \times A$

(*e*) $(A - B) \times C$.

15. Examine the comparability with the following sets.

(*i*) $A = \{a, b, c\}$ (*ii*) $B = \{a, e, i, o, u\}$

(*iii*) $C = \{b, c, o, u\}$ (*iv*) $D = \{b, c, i, o, u, k\}$.

16. In a class containing 100 students, 30 play tennis; 40 play cricket; 40 do athletics; 6 play tennis and cricket; 12 play cricket and do athletics; and 10 play tennis and do athletics; while 14 play no game or do athletics at all. How many play cricket, tennis and do athletics?

17. If in a city 70% of the residents can speak French and 50% can speak English, what percentage of residents can speak both the languages, if 10% residents cannot speak any of these two languages?

18. Let X, Y, Z and T be four sets. Then prove that $(X \cap Z) \times (Y \cap T) = (X \times Y) \cap (Z \times T)$.

19. Write the following sets as the union of fundamental products.

(*a*) $A \cap (B \cup C)$ (*b*) $A^c \cap (B \cup C)$

(*c*) $A \cup (B \cap C)$ (*d*) $A \cup (B - C)$.

20. Identify the smallest set X containing the sets.
{Book, Pen}; {Pen, Pencil, Box}; {Book, Box, Ball}.

21. One hundred students were asked whether they had taken courses in any of the three subjects, Mathematics, Computer Science and Information Technology. The results were given below. 45 had taken Mathematics; 18 had taken Mathematics and Computer Science; 38 had taken Computer Science; 21 had taken Information Technology; 9 had taken Mathematics and Information Technology; 4 had taken Computer Science and Information Technology and 23 had taken no courses in any of the subjects. Draw a Venn diagram that will show the results of the survey.