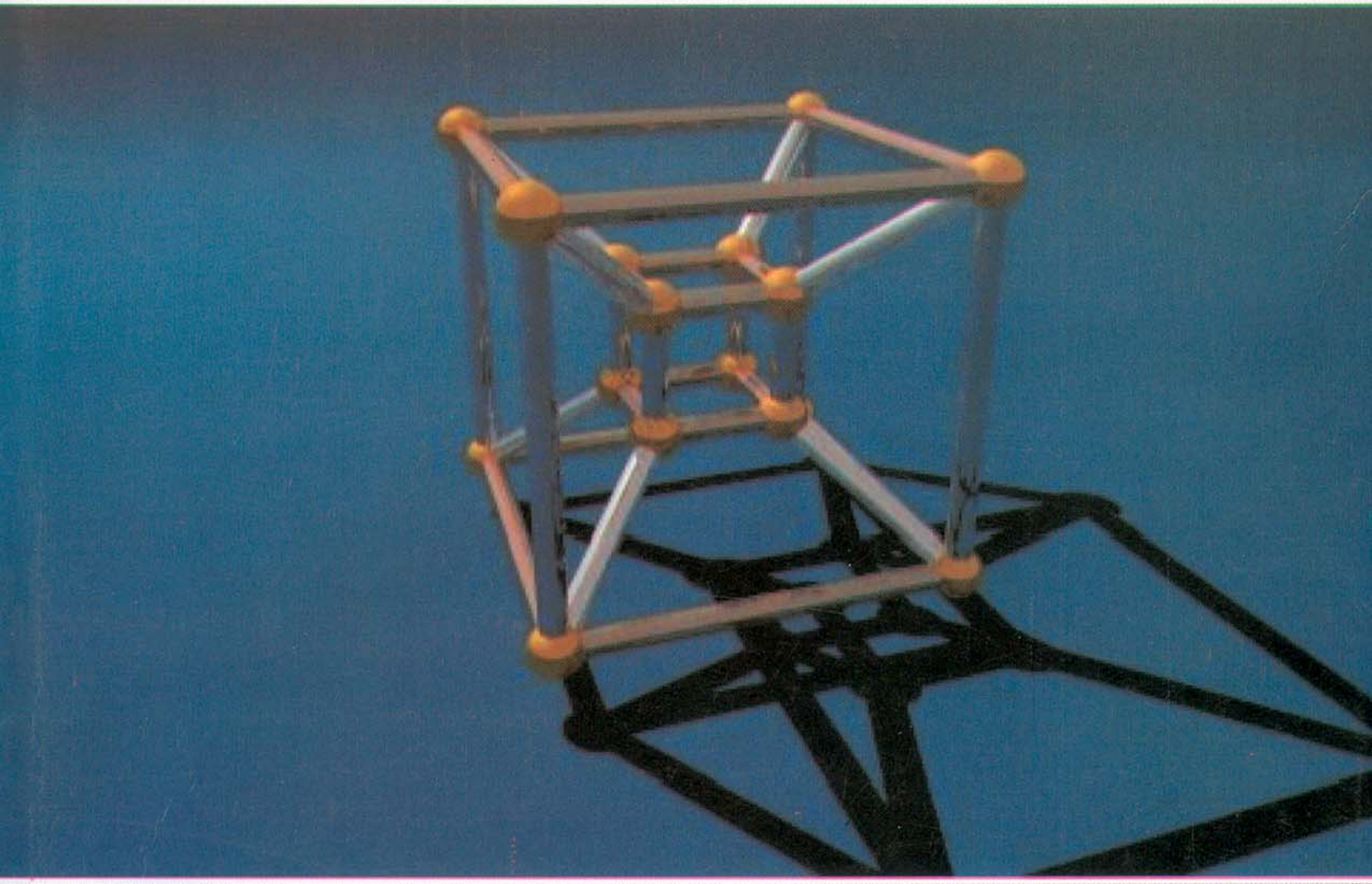
NEW AGE

SECOND EDITION

FUNDAMENTAL APPROACH TO DISCRETE MATHEMATICS



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Mathematical Logic

■ 1.0 INTRODUCTION

Mathematics is considered to be a deductive science. We infer things from certain premises through logical reasoning. Consider an example.

Three caps problem. A certain father had three sons: Smith, Clark and Jones. The father brought three caps of different colours; say Red, Blue and Black. He showed them the caps. After which they are folded blind. The father put three caps on the heads of three sons. Then the sons were taken away from his father to another room. Few minutes after father called Jones and removed the blindfold of Jones and asked him to tell the colour of his cap. Jones said he could not infer about the colour of his own cap. Then he called Clark and removed the blindfold of Clark and asked him to tell the colour of his cap by looking at the colour of the cap of Jones. He too could not infer. Then he called Smith and asked him to tell the colour of his cap without removing the blindfold of Smith. Smith replied he could tell the colour of the cap on his own head.

How Smith come to that conclusion? Let us see. Smith asked two questions one to Jones and another to Clark. He asked to Jones about the colour of Clark's cap and asked to Clark about the colour of Jones's cap. By the way he got two colours of the cap. As a result Smith got the colour of his own cap.

In the above reasoning we have certain premises and we conclude from them by a pure deductive reasoning. In the following passages we shall formalize the process of deduction.

■ 1.1 STATEMENT (PROPOSITION)

A statement is a declarative sentence which is either true or false but not both. The statement is also known as proposition. The truth value True and False are denoted by the symbols **T** and **F** respectively. Sometimes it is also denoted by **1** and **0**, where **1** stands for true and **0** stands for false. As it depends on only two possible truth values, we call it as two-valued logic or bi-valued logic.

Consider the following examples

(*a*) Man is mortal.

(*b*) Sun rises in the east.
(*c*) Two is less than five.
(*d*) May God bless you!
(*e*) x is a Dog.
(*f*) Lopez is a nice Cat.
(*g*) It is too cold today.
(*h*) 6 is a composite number.

From the above example, it is very clear that (*a*), (*b*), (*c*) and (*h*) are statements as they declare a definite truth value T or F. The other example (*d*), (*e*), (*f*), and (*g*) are not statements as they do not declare any truth value T or F.

Consider the sentence 111011 + 11 = 111110

The above sentence is a statement but its truth value depends on the context. If we consider the binary number system, the statement is True (T) but in decimal number system the statement is False (F).

1.2 LOGICAL CONNECTIVES

Another important aspect is that logical connectives. We use some logical connectives to connect several statements into a single statement. The most basic and fundamental connectives are Negation, Conjunction and Disjunction.

1.2.1 Negation

It is observed that the negation of a statement is also a statement. We use the connective **Not** for negation. Usually the statements are denoted by single letters P, Q, R etc. If P be a statement, then the negation of P is denoted as $\neg$ P.

Consider the example of a statement.

P: New York is the capital of France.

$\neg$ P: New York is not the capital of France.

As we all know that Paris is the capital of France, the truth value for the statements P is false (F) and $\neg$ P is true (T). From the above it is clear that P and $\neg$ P has opposite truth values. $\neg$ P can also be written as

$\neg$ P: It is not true that New York is the capital of France.

Rule: If P is true, then $\neg$ P is false and if P is false, then $\neg$ P is true.

Truth Table (Negation)

P	$\neg$ P
T	F
F	T

1.2.2 Conjunction

The conjunction of two statements P and Q is also a statement denoted by (P $\wedge$ Q). We use the connective **And** for conjunction.

Consider the example where P and Q are two statements.

P : 2 + 3 = 5

Q : 5 is a composite number.

So, $(P \wedge Q)$: 2 + 3 = 5 and 5 is a composite number.

As another example if P: Smith went to the college and Q: Mary went to the college, then $(P \wedge Q)$: Smith and Mary went to the college.

It is clear that $(P \wedge Q)$ stand for P and Q. In order to make $(P \wedge Q)$ true, P and Q have to be simultaneously true.

Rule: $(P \wedge Q)$ is true if both P and Q are true, otherwise false.

Truth Table (Conjunction)

P	**Q**	**$(P \wedge Q)$**
T	T	T
T	F	F
F	T	F
F	F	F

1.2.3 Disjunction

The disjunction of two statements P and Q is also a statement denoted by $(P \vee Q)$. We use the connective **Or** for disjunction. Consider the example where P and Q are two statements

P : 2 + 3 is not equal to 5

Q : 5 is a prime number

So, $(P \vee Q)$: 2 + 3 is not equal to 5 or 5 is a prime number.

It is observed that $(P \vee Q)$ is true when P may be true or Q may be true and this also includes the case when both are true, that is the truth value of one statement is not assumed in exclusion of the truth value of the other statement. We call it as also inclusive or.

Rule: $(P \vee Q)$ is true, if either P or Q is true and it is false when both P and Q are false.

Truth Table (Disjunction)

P	**Q**	**$(P \vee Q)$**
T	T	T
T	F	T
F	T	T
F	F	F

■ 1.3 CONDITIONAL

Let P and Q be any two statements. Then the statement $P \rightarrow Q$ is called a conditional statement. This can be put in any one of the following forms.

(*a*) If P, then Q

(*b*) P only if Q

(*c*) P implies Q

(*d*) Q if P

In an implication $P \rightarrow Q$, P is called the antecedent (hypothesis) and Q is called the consequent (conclusion). To explain the conditional statement, consider the example

A boy promises a girl, "I will take you boating on Sunday, if it is not raining".

Now if it is raining, then the boy would not be deemed to have broken his promise. The boy would be deemed to have broken his promise only when it is not raining and the boy did not take the girl for boating on Sunday.

Let us break the above conditional statement to symbolic from.

P: It is not raining.

Q: I will take you boating on Sunday.

So, the above statement reduces to $P \to Q$.

From the above discussion it is clear that if P is false, then $P \to Q$ is true, whatever be the truth value of Q. The conditional $P \to Q$ is false, if P is true and Q is false.

Rule: An implication (conditional) $P \to Q$ is false only when the hypothesis (P) is true and conclusion (Q) is false, otherwise true.

Truth Table (Conditional)

P	**Q**	**$(P \to Q)$**
T	T	T
T	F	F
F	T	T
F	F	T

1.4 BI-CONDITIONAL

Let P and Q be any two statements. Then the statement $P \leftrightarrow Q$ is called a bi-conditional statement. This $P \leftrightarrow Q$ can be put in any one of the following forms.

(*a*) P if and only if Q

(*b*) P is necessary and sufficient of Q

(*c*) P is necessary and sufficient for Q

(*d*) P is implies and implied by Q

The bi-conditional (double implication) $P \leftrightarrow Q$ is defined as

$$(\mathbf{P} \leftrightarrow \mathbf{Q}) : (\mathbf{P} \to \mathbf{Q}) \wedge (\mathbf{Q} \to \mathbf{P})$$

From the truth table discussed below, it is clear that $P \leftrightarrow Q$ has the truth value T whenever both P and Q have identical truth values.

Truth Table (Bi-Conditional)

P	**Q**	**$P \to Q$**	**$Q \to P$**	**$(P \leftrightarrow Q)$**
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Rule: $(P \leftrightarrow Q)$ is true only when both P and Q have identical truth values, otherwise false.

1.5 CONVERSE

Let P and Q be any two statements. The converse statement of the conditional $P \rightarrow Q$ is given as $Q \rightarrow P$.

Consider the example "all concurrent triangles are similar". The above statement can also be written as "if triangles are concurrent, then they are similar".

Let P: Triangles are concurrent
Q: Triangles are similar

So, the statement becomes $P \rightarrow Q$. The converse statement is given as "if triangles are similar, then they are concurrent" or all similar triangles are concurrent.

1.6 INVERSE

Let P and Q be any two statements. The inverse statement of the conditional $(P \rightarrow Q)$ is given as $(\neg P \rightarrow \neg Q)$.

Consider the example "all concurrent triangles are similar". The above statement can also be written as "if triangles are concurrent, then they are similar".

Let P: Triangles are concurrent
Q: Triangles are similar

So, the statement becomes $P \rightarrow Q$. The inverse statement is given as "if triangles are not concurrent, then they are not similar".

1.7 CONTRA POSITIVE

Let P and Q be any two statements. The contra positive statement of the conditional $(P \rightarrow Q)$ is given as $(\neg Q \rightarrow \neg P)$. Consider the Example "all concurrent triangles are similar". The above statement can also be written as "if triangles are concurrent, then they are similar".

Let P: Triangles are concurrent and
Q: Triangles are similar

So, the statement becomes $P \rightarrow Q$. The contra positive statement is given as "if triangles are not similar, then they are not concurrent".

Truth Table (Contra Positive)

P	**Q**	**$P \rightarrow Q$**	**$\neg Q$**	**$\neg P$**	**$(\neg Q \rightarrow \neg P)$**
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

From the truth table it is observed that both conditional $(P \rightarrow Q)$ and contra positive $(\neg Q \rightarrow \neg P)$ have same truth values.

1.8 EXCLUSIVE OR

Let P and Q be any two statements. The exclusive OR of two statements P and Q is denoted by $(P \barwedge Q)$. We use the connective XOR for exclusive OR. The exclusive OR $(P \barwedge Q)$ is true if either P or Q is true but not both. The exclusive OR is also termed as exclusive disjunction.

Consider the example where P and Q be two statements such that $P \equiv 2 + 3 = 5$ and $Q \equiv 5 - 3 = 2$. Here both the statements are true. Therefore $(P \barwedge Q)$ is false.

Rule: $(P \barwedge Q)$ is true if either P or Q is true but not both, otherwise false.

Truth Table (Exclusive OR)

P	Q	$(P \barwedge Q)$
T	T	F
T	F	T
F	T	T
F	F	F

1.9 NAND

The word NAND stands for NOT and AND. The connective NAND is denoted by the symbol $\uparrow$. If P and Q be two statements, then NAND of P and Q is given as $(P \uparrow Q)$ defined by

$$(P \uparrow Q) \equiv \neg (P \wedge Q).$$

Rule: $(P \uparrow Q)$ is true if either P or Q is false, otherwise false.

Truth Table (NAND)

P	Q	$(P \uparrow Q)$
T	T	F
T	F	T
F	T	T
F	F	T

1.10 NOR

The word NOR stands for NOT and OR. The connective NOR is denoted by the symbol $\downarrow$. If P and Q be two statements, then NOR of P and Q is given as $(P \downarrow Q)$ defined by

$$(P \downarrow Q) \equiv \neg (P \vee Q)$$

Rule: $(P \downarrow Q)$ is true only when both P and Q are false, otherwise false.

Truth Table (NOR)

P	Q	$(P \downarrow Q)$
T	T	F
T	F	F
F	T	F
F	F	T

■ 1.11 TAUTOLOGY

If the truth values of a composite statement are always true irrespective of the truth values of the atomic (individual) statements, then it is called a tautology.

For example the composite statement $(P \wedge (P \to Q)) \to Q$ is a tautology. To verify this draw the truth table with composite statement as $(P \wedge (P \to Q)) \to Q$

Truth Table

P	Q	$(P \to Q)$	$P \wedge (P \to Q)$	$(P \wedge (P \to Q)) \to Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

So, $(P \wedge (P \to Q)) \to Q$ is a tautology.

■ 1.12 CONTRADICTION

If the truth values of a composite statement are always false irrespective of the truth values of the atomic statements, then it is called a contradiction or unsatisfiable.

For example the composite statement $\neg (P \to (Q \to (P \wedge Q)))$ is a contradiction.

To verify this draw the truth table of $\neg (P \to (Q \to (P \wedge Q)))$. Let $R \equiv P \to (Q \to (P \wedge Q))$

Truth Table

P	Q	$(P \wedge Q)$	$Q \to (P \wedge Q)$	$(P \to (Q \to (P \wedge Q))$	$\neg R$
T	T	T	T	T	F
T	F	F	T	T	F
F	T	F	F	T	F
F	F	F	T	T	F

So, $\neg R \equiv \neg (P \to (Q \to (P \wedge Q)))$ is a contradiction.

■ 1.13 SATISFIABLE

If the truth values of a composite statement are some times true and some times false irrespective of the truth values of the atomic statements, then it is called a satisfiable.

Consider the composite statement $(P \to Q) \to (Q \to P)$. To verify this draw the truth table of $(P \to Q) \to (Q \to P)$.

Truth Table

P	Q	$P \to Q$	$Q \to P$	$(P \to Q) \to (Q \to P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

So, the composite statement $(P \to Q) \to (Q \to P)$ is satisfiable.

■ 1.14 DUALITY LAW

Two formulae P and P* are said to be duals of each other if either one can be obtained from the other by interchanging $\wedge$ by $\vee$ and $\vee$ by $\wedge$. The two connectives $\wedge$ and $\vee$ are called dual to each other.

Consider the formulae $P \equiv (P \vee Q) \wedge R$ and $P^* \equiv (P \wedge Q) \vee R$ which are dual to each other.

■ 1.15 ALGEBRA OF PROPOSITIONS

If P, Q and R be three statements, then the following laws hold good.

(*a*) Commutative Laws: $P \wedge Q \equiv Q \wedge P$ and
$P \vee Q \equiv Q \vee P$

(*b*) Associative Laws: $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$ and
$P \vee (Q \vee R) \equiv (P \vee Q) \vee R$

(*c*) Distributive Laws: $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ and
$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

(*d*) Idempotent Laws: $P \wedge P \equiv P$ and
$P \vee P \equiv P$

(*e*) Absorption Laws: $P \vee (P \wedge Q) \equiv P$ and
$P \wedge (P \vee Q) \equiv P$

1.15.1 de Morgan's Laws

If P and Q be two statements, then

(*i*) $\neg (P \wedge Q) \Leftrightarrow (\neg P) \vee (\neg Q)$ and
(*ii*) $\neg (P \vee Q) \Leftrightarrow (\neg P) \wedge (\neg Q)$

■ 1.16 MATHEMATICAL INDUCTION

Generally direct methods are adopted for proving theorems and propositions. Sometimes it is too difficult and tedious. As a result the other methods are developed for proving theorems and propositions. These are (*i*) method of contra positive, (*ii*) method of contradiction and (*iii*) method of induction. Here, we will discuss method of induction. The method of induction is otherwise known as mathematical induction.

Suppose that n be a natural number. Our aim is to show that some statement P(n) involving n is true for any n. The following steps are used in mathematical induction.

1. Suppose that P(n) be a statement.
2. Show that P(1) and P(2) are true. *i.e.*, P(n) is true for $n = 1$ and $n = 2$.
3. Assume that P(k) is true. *i.e.*, P(n) is true for $n = k$.
4. Show that P(k + 1) follows from P(k).

Consider an example $1+ 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$

Suppose that $P(n) \equiv 1+ 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$

So, $$P(1) \equiv 1 = \frac{1(1+1)}{2}$$

and $$P(2) \equiv 1 + 2 = 3 = \frac{2(2+1)}{2}$$

So, P(1) and P(2) are true.

Assume that P(k) is true. So,

$$1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$$

So,
$$\begin{aligned} P(k+1) &\equiv 1 + 2 + 3 + \ldots + k + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \qquad [\because \ P(k) \text{ is true.}] \\ &= \left(\frac{k+1}{2}\right)(k+2) = \frac{(k+1)(k+2)}{2} \end{aligned}$$

which shows that P(k + 1) is also true. Hence, P(n) is true for all n.

SOLVED EXAMPLES

Example 1 *Find the negation of* $P \to Q$.

Solution: $P \to Q$ is equivalently written as $(\neg P \vee Q)$

So, negation of
$$\begin{aligned} P \to Q &\equiv \neg(\neg P \vee Q) \\ &\equiv \neg(\neg P) \wedge (\neg Q), \text{ (By de Morgan's Law)} \\ &\equiv P \wedge (\neg Q) \end{aligned}$$

Hence the negation of $P \to Q$ is $P \wedge (\neg Q)$.

Example 2 *Construct the truth table for* $(P \to Q) \leftrightarrow (\neg P \vee Q)$.

Solution: The given compound statement is $(P \to Q) \leftrightarrow (\neg P \vee Q)$ where P and Q are two atomic statements.

P	Q	$\neg$ P	P$\to$ Q	$\neg$ P $\vee$ Q	(P $\to$ Q) $\leftrightarrow$ ($\neg$ P $\vee$ Q)
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Example 3 *Construct the truth table for* $P \to (Q \leftrightarrow P \wedge Q)$.

Solution: The given compound statement is $P \to (Q \leftrightarrow P \wedge Q)$, where P and Q are two atomic statements.

P	Q	P $\wedge$ Q	Q $\leftrightarrow$ P $\wedge$ Q	P $\to$ (Q $\leftrightarrow$ P $\wedge$ Q)
T	T	T	T	T
T	F	F	T	T
F	T	F	F	T
F	F	F	T	T

Example 4 *Find the negation of the following statement. "If Cows are Crows then Crows are four legged".*

Solution: Let P: Cows are Crows

Q: Crows are four legged

Given statement : If Cows are Crows then Crows are four legged.

$$\equiv P \to Q$$

So, the negation is given as $P \wedge (\neg Q)$ *i.e.* Cows are Crows and Crows are not four legged.

Example 5 *Find the negation of the following statement.*

He is rich and unhappy.

Solution: Let $P \equiv$ He is rich

$Q \equiv$ He is unhappy

Given statement: He is rich and unhappy

$$\equiv P \wedge Q$$

By de Morgan's law $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

$\equiv$ He is neither rich nor unhappy.

Example 6 *Prove by constructing truth table*

$$P \to (Q \vee R) \equiv (P \to Q) \vee (P \to R)$$

Solution: Our aim to prove $P \to (Q \vee R) \equiv (P \to Q) \vee (P \to R)$

Let P, Q and R be three atomic statements.

P	**Q**	**R**	$Q \vee R$	$P \to (Q \vee R)$	$P \to Q$	$P \to R$	$(P \to Q) \vee (P \to R)$
T	T	T	T	T	T	T	T
T	F	F	F	F	F	F	F
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	T	T	T	T	T	T	T
T	F	T	T	T	F	T	T
T	T	F	T	T	T	F	T
F	F	F	F	T	T	T	T

From the truth table it is clear that $P\to (Q \vee R) \equiv (P \to Q) \vee (P \to R)$.

Example 7 *Find the negation of $P \leftrightarrow Q$.*

Solution: $P \leftrightarrow Q$ is equivalently written as $(P \to Q) \wedge (Q \to P)$

So, $\neg(P \leftrightarrow Q) \equiv \neg((P \to Q) \wedge (Q \to P))$

$\equiv \neg(P \to Q) \vee \neg(Q \to P)$; (de Morgan's Law)

$\equiv \neg(\neg P \vee Q) \vee \neg(\neg Q \vee P)$

$\equiv (P \wedge \neg Q) \vee (Q \wedge \neg P)$; (de Morgan's Law)

Hence, $\neg(P \leftrightarrow Q) \equiv (P \wedge \neg Q) \vee (Q \wedge \neg P)$.

Example 8 *With the help of truth table prove that $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$.*

Solution: Our claim is $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$.

Let P and Q be two atomic statements.

P	**Q**	**P ∧ Q**	**¬ (P∧ Q)**	**¬ P**	**¬ Q**	**¬ P ∨ ¬ Q**
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

From the truth table it is clear that $\neg (P \wedge Q) \equiv \neg P \vee \neg Q$.

Example 9 *Show that $(P \to Q) \leftrightarrow (\neg P \vee Q)$ is a tautology.*

Solution: Let P and Q be two atomic statements. Our aim is to show $(P \to Q) \leftrightarrow (\neg P \vee Q)$ is a tautology.

P	**Q**	**P → Q**	**¬ P**	**¬ P ∨ Q**	**(P → Q) ↔ (¬ P ∨ Q)**
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Hence $(P \to Q) \leftrightarrow (\neg P \vee Q)$ is a tautology.

Example 10 *Show that the following statements are equivalent.*

Statement 1: Good food is not cheap

Statement 2: Cheap food is not good.

Solution: Let P≡ Food is good and Q ≡ Food is cheap

Statement 1: Good food is not cheap

i.e., $P \to \neg Q$

Statement 2: Cheap food is not good

i.e., $Q \to \neg P$

Truth Table

P	**Q**	**¬ P**	**¬ Q**	**P → ¬ Q**	**Q → ¬ P**
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

From truth table it is clear that both statements are equivalent.

Example 11 *Express $P \to Q$ using $\downarrow$ and $\uparrow$ only.*

Solution:

$$\begin{aligned} P \to Q &\equiv \neg P \vee Q \\ &\equiv \neg P \vee \neg\ (\neg Q) \\ &\equiv \neg (P \wedge \neg Q) \equiv P \uparrow \neg Q \\ &\equiv P \uparrow (\neg Q \vee \neg Q) \\ &\equiv P \uparrow \neg (Q \wedge Q) \equiv P \uparrow (Q \uparrow Q) \end{aligned}$$

i.e., $P \to Q \equiv P \uparrow (Q \uparrow Q)$

Example 12 *Prove that $(P \wedge Q) \wedge \neg (P \vee Q)$ is a contradiction.*

Solution: Truth table for $(P \wedge Q) \wedge \neg (P \vee Q)$

P	Q	$(P \wedge Q)$	$(P \vee Q)$	$\neg (P \vee Q)$	$(P \wedge Q) \wedge \neg (P \vee Q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Hence, $(P \wedge Q) \wedge \neg (P \vee Q)$ is a contradiction.

Example 13 *Express $P \leftrightarrow Q$ using $\downarrow$ and $\uparrow$ only.*

Solution:

$$\begin{aligned}
P \leftrightarrow Q &\equiv (P \rightarrow Q) \wedge (Q \rightarrow P)\\
&\equiv (\neg P \vee Q) \wedge (P \vee \neg Q)\\
&\equiv ((\neg P \vee Q) \wedge P) \vee ((\neg P \vee Q) \wedge \neg Q)\\
&\equiv \neg (\neg ((\neg P \vee Q) \wedge P) \wedge \neg ((\neg P \vee Q) \wedge \neg Q))\\
&\equiv \neg ((\neg P \vee Q) \wedge P) \uparrow \neg ((\neg P \vee Q) \wedge \neg Q)\\
&\equiv ((\neg P \vee Q) \uparrow P) \uparrow ((\neg P \vee Q) \uparrow \neg Q)\\
&\equiv (\neg (P \wedge \neg Q) \uparrow P) \uparrow (\neg (P \wedge \neg Q) \uparrow \neg Q)\\
&\equiv ((P \uparrow \neg Q) \uparrow P) \uparrow ((P \uparrow \neg Q) \uparrow \neg Q)\\
&\equiv ((P \uparrow (\neg Q \vee \neg Q) \uparrow P) \uparrow ((P \uparrow (\neg Q \vee \neg Q)) \uparrow (\neg Q \vee \neg Q))\\
&\equiv ((P \uparrow \neg (Q \wedge Q) \uparrow P) \uparrow ((P \uparrow \neg (Q \wedge Q)) \uparrow \neg (Q \wedge Q))\\
&\equiv ((P \uparrow (Q \uparrow Q) \uparrow P) \uparrow ((P \uparrow (Q \uparrow Q)) \uparrow (Q \uparrow Q))
\end{aligned}$$

Note: These expressions are not unique.

Alternative Solution:

$$\begin{aligned}
P \leftrightarrow Q &\equiv (P \rightarrow Q) \wedge (Q \rightarrow P)\\
&\equiv (\neg P \vee Q) \wedge (P \vee \neg Q)\\
&\equiv ((\neg P \vee Q) \wedge P) \vee ((\neg P \vee Q) \wedge \neg Q)\\
&\equiv ((\neg P \wedge P) \vee (Q \wedge P)) \vee ((\neg P \wedge \neg Q) \vee (Q \wedge \neg Q))\\
&\equiv (Q \wedge P) \vee (\neg P \wedge \neg Q)\\
&\equiv \neg (\neg (Q \wedge P)) \vee \neg (P \vee Q)\\
&\equiv \neg (\neg (Q \wedge P) \wedge (P \vee Q))\\
&\equiv \neg (Q \wedge P) \uparrow (P \vee Q)\\
&\equiv (Q \uparrow P) \uparrow \neg (\neg (P \vee Q))\\
&\equiv (Q \uparrow P) \uparrow \neg (\neg P \wedge \neg Q))\\
&\equiv (Q \uparrow P) \uparrow (\neg P \uparrow \neg Q)\\
&\equiv (Q \uparrow P) \uparrow ((\neg P \vee \neg P) \uparrow (\neg Q \vee \neg Q))\\
&\equiv (Q \uparrow P) \uparrow (\neg (P \wedge P) \uparrow \neg (Q \wedge Q))\\
&\equiv (Q \uparrow P) \uparrow ((P \uparrow P) \uparrow (Q \uparrow Q))
\end{aligned}$$

Example 14 *Prove that $n (n + 1)$ is an even natural number.*

Solution: Suppose that $P(n) \equiv n (n + 1)$ is even.

So, $P(1) \equiv 1(1 + 1) = 2$, which is even and

$P(2) \equiv 2 (2 + 1) = 6$, which is also even.

Hence, P(1) and P(2) are true.

Assume that $P(k) \equiv k\,(k + 1)$ is even

i.e., $k\,(k + 1) = 2m;\ m \in N$

So,
$$\begin{aligned} P(k + 1) &\equiv (k + 1)\,(k + 2) = k\,(k + 1) + 2\,(k + 1) \\ &= 2m + 2\,(k + 1) \qquad [\because\ P(k) \text{ is true.}] \\ &= 2(m + k + 1), \text{ which is even.} \end{aligned}$$

Which shows that $P(k + 1)$ is also true.

So, $P(n)$ is true for all n.

Example 15 *Show by truth table the following statements are equivalent.*

Statement 1: Rich men are unhappy.

Statement 2: Men are unhappy or poor.

Solution: Let P ≡ Men are Rich and Q ≡ Men are unhappy.

Statement 1: Rich men are unhappy.

i.e., If men are rich then they are unhappy.

i.e., $P \rightarrow Q$.

Statement 2: Men are unhappy or poor.

i.e., $Q \vee \neg P$; (Here poor indicates not rich)

Truth Table

P	Q	P → Q	¬ P	Q ∨ ¬ P
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

So, it is clear that both statements are equivalent.

Example 16 *A boy promises a girl "I will take you park on Monday if it is not raining". When the boy would be deemed to have broken his promise. Explain with the help of truth table.*

Solution: Let P: I will take you park on Monday

Q: It is raining.

Given statement: I will take you park on Monday if it is not raining

i.e., P if $\neg$ Q

i.e., $\neg Q \rightarrow P$

Truth Table

P	Q	¬ Q	¬ Q → P
T	T	F	T
T	F	T	T
F	T	F	T
F	F	T	F

It indicates that if ¬ Q is true and P is false, then the boy is deemed to have broken his promise. *i.e.* when it is not raining and the boy does not take her park on Monday, then the boy is deemed to have broken his promise.

Example 17 *Prove by method of induction*

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Solution: Suppose that $P(n) \equiv 1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

So, $$P(1) = 1^3 = 1 = \left(\frac{1(1+1)}{2}\right)^2$$

and $$P(2) = 1^3 + 2^3 = 9 = \left(\frac{2(2+1)}{2}\right)^2$$

Hence, P(1) and P(2) are true.

Assume that $P(k)$ is true, so

$$P(k) \equiv 1^3 + 2^3 + 3^3 + \dots + k^3 = \left(\frac{k(k+1)}{2}\right)^2$$

$$P(k+1) \equiv 1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3$$

$$= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \qquad [\because \ P(k) \text{ is true.}]$$

$$= (k+1)^2 (k^2 + 4(k+1)) / 4$$

$$= \left(\frac{(k+1)(k+2)}{2}\right)^2$$

Which shows that $P(k + 1)$ is also true.

So, $P(n)$ is true for all n .

Example 18 *Show by method of induction*

$$\frac{1}{1*2} + \frac{1}{2*3} + \frac{1}{3*4} + \dots + \frac{1}{n*(n+1)} = \frac{n}{n+1}$$

Solution: Suppose that

$$P(n) \equiv \frac{1}{1*2} + \frac{1}{2*3} + \frac{1}{3*4} + \dots + \frac{1}{n*(n+1)} = \frac{n}{n+1}$$

So, $$P(1) \equiv \frac{1}{1*2} = \frac{1}{2} = \frac{1}{1+1}$$

and $$P(2) \equiv \frac{1}{1*2} + \frac{1}{2*3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3} = \frac{2}{2+1}$$

Assume that P(k) is true. So,

$$P(k) \equiv \frac{1}{1*2} + \frac{1}{2*3} + \frac{1}{3*4} + \ldots + \frac{1}{k*(k+1)} = \frac{k}{k+1}$$

$$\therefore \quad P(k+1) \equiv \frac{1}{1*2} + \frac{1}{2*3} + \frac{1}{3*4} + \ldots + \frac{1}{k*(k+1)} + \frac{1}{(k+1)*(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)*(k+2)}; \qquad [\because \quad P(k) \text{ is true}]$$

$$= \frac{1}{(k+1)}\left(k + \frac{1}{(k+2)}\right)$$

$$= \frac{1}{(k+1)}\left(\frac{k^2 + 2*k + 1}{(k+2)}\right)$$

$$= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

Which shows that P(k + 1) is also true.

So, P(n) is true for all n.

EXERCISES

1. Find the negation of the following statements.

(*a*) Today is Sunday or Monday.
(*b*) If I am tired and busy, then I cannot study.
(*c*) Either it is raining or some one left the shower on.
(*d*) The moon rises in the west.
(*e*) The triangles are equilateral is necessary and sufficient for three equal sides.
(*f*) $2 + 3 \neq 18$.

2. Prove the following by using truth table.

(*a*) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
(*b*) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
(*c*) $\neg (P \vee Q) \equiv \neg P \wedge \neg Q$
(*d*) $P \rightarrow (Q \wedge R) \equiv (P \rightarrow Q) \wedge (P \rightarrow R)$
(*e*) $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$
(*f*) $P \vee Q \equiv \neg (\neg P \wedge \neg Q)$
(*g*) $P \,\overline{\vee}\, Q \equiv (P \vee Q) \wedge \neg (P \wedge Q)$
(*h*) $(P \downarrow Q) \downarrow (P \downarrow Q) \equiv P \vee Q$
(*i*) $P \wedge Q \equiv (P \downarrow P) \downarrow (Q \downarrow Q)$
(*j*) $\neg (P \vee Q) \vee (\neg P \wedge Q) \equiv \neg P$

3. For each of the following formulas tell whether it is (*i*) tautology, (*ii*) satisfiable, or (*iii*) contradiction.

(*a*) $(P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R))$
(*b*) $(P \rightarrow (Q \rightarrow R)) \leftrightarrow ((P \wedge Q) \rightarrow R)$
(*c*) $P \wedge \neg Q$
(*d*) $(P \vee Q) \rightarrow P$
(*e*) $\neg (P \rightarrow Q) \rightarrow (P \wedge \neg Q)$
(*f*) $(P \rightarrow Q) \rightarrow (Q \rightarrow P)$
(*g*) $((P \rightarrow Q) \leftrightarrow Q) \rightarrow P$
(*h*) $\neg P \wedge (P \vee Q) \rightarrow P$
(*i*) $P \rightarrow (P \wedge Q)$
(*j*) $P \rightarrow (Q \rightarrow (P \wedge Q))$
(*k*) $(P \vee Q) \leftrightarrow (Q \wedge P)$
(*l*) $(P \rightarrow (Q \rightarrow (P \wedge Q))) \leftrightarrow P$

4. Prove by using different laws.

(*a*) $\neg (P \vee Q) \vee (\neg P \wedge Q) \equiv \neg P$ (*b*) $P \vee (P \wedge Q) \equiv P$

(*c*) $(P \vee Q) \wedge \neg P \equiv \neg P \wedge Q$

5. Write each of the following in symbolic form by indicating statements.

(*a*) Brown is rich and unhappy.

(*b*) Jackson speaks English or French.

(*c*) I am hungry and I can study.

(*d*) I am tired if and only if I work hard.

(*e*) If New York is a city, then it is the capital of US.

(*f*) $5 + 2 = 7$ if $7 - 2 = 5$.

6. Write the truth value of each of the following statements.

(*a*) Sun rises in the south.

(*b*) Man is mortal.

(*c*) London is the capital of UK.

(*d*) If three sides of a triangle are equal, then it is an equilateral triangle.

(*e*) $(11101)_2 + (1)_2 = (11110)_2$

(*f*) $(11101)_{10} + (1)_{10} = (11110)_{10}$

(*g*) $(11111)_2 + (1)_2 = (100000)_2$ and $(111)_2 = (7)_{10}$

(*h*) $(270)_8 + (5)_8 = (184)_{10}$ or $(11101)_2 + (111)_2 = (100101)_2$

(*i*) $2^2 = 9$ if and only if $2 \neq 3$

(*j*) $(111)_2 + (010)_2 = (1001)_2$ if and only if $(1001)_2 - (010)_2 = (111)_2$.

7. Write the converse, inverse and contra positive of the following statement by indicating the conditional statement.

(*a*) In binary number system $1 + 1 = 10$.

(*b*) Good food are not cheap.

(*c*) If $9x + 36 = 9$, then $x \neq 17$.

(*d*) If $\cos(x) = 1$, then $x = 0$.

(*e*) Two sets are similar, if they contains equal number of elements.

8. Prove by using method of induction.

(*a*) $1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(*b*) $1 + r + r^2 + \ldots + r^{n-1} = \dfrac{1-r^n}{1-r}; r \neq 1$

(*c*) $1 + r + r^2 + \ldots + r^n = \dfrac{1-r^{n+1}}{1-r}; r \neq 1$

(*d*) $a + ar + ar^2 + \ldots + ar^n = \dfrac{a\left(1-r^{n+1}\right)}{1-r}; r \neq 1$

(*e*) $a + (a+d) + (a+2d) + \ldots + (a+(n-1)d) = \dfrac{n\left(2a+(n-1)d\right)}{2}$

(*f*) $3 + 7 + 11 + \ldots + (4n - 1) = n\,(2n + 1)$

(*g*) $2 + 4 + 6 + \ldots + 2n = n\,(n + 1)$

(*h*) $1^2 + 4^2 + 7^2 + \ldots + (3n - 2)^2 = \dfrac{n\left(6n^2 - 3n - 1\right)}{2}$

(*i*) $3 * 6 + 6 * 9 + \ldots + 3n\,(3n + 3) = 3n\,(n + 1)\,(n + 2)$

(*j*) $1 * 2 + 2 * 3 + 3 * 4 + \ldots + n(n + 1) = \dfrac{n\,(n+1)\,(n+2)}{3}$

(*k*) $1 * 2 * 3 + 2 * 3 * 4 + \ldots + n(n + 1)\,(n + 2) = \dfrac{n\,(n+1)\,(n+2)\,(n+3)}{4}$

(*l*) $1 + 2 * 3 + 3 * 5 + \ldots + n(2n - 1) = \dfrac{n\,(n+1)\,(4n-1)}{6}$

(*m*) $1 * 3 * 5 + 3 * 5 * 7 + \ldots + (2n - 1)\,(2n + 1)(2n + 3) = n\,(n + 2)(2n^2 + 4n - 1).$

(*n*) $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \ldots + (1^2 + 2^2 + \ldots + n^2) = \dfrac{n\,(n+1)^2\,(n+2)}{12}$

(*o*) $1 * 2^2 + 2 * 3^2 + \ldots + n\,(n + 1)^2 = \dfrac{n\,(n+1)\,(n+2)\,(3n+5)}{12}$

(*p*) $3 * 8 + 6 * 11 + \ldots + 3n\,(3n + 5) = 3n\,(n + 1)\,(n + 3)$

(*q*) $1 + (1 + 4) + (1 + 4 + 7) + \ldots + (1 + 4 + 7 + \ldots + (3n - 2)) = \dfrac{n^2\,(n+1)}{2}$

(*r*) $2 + 6 + 12 + 20 + \ldots + \dfrac{n\,(2n+2)}{2} = \dfrac{n\,(n+1)\,(n+2)}{3}$

(*s*) $1 + \dfrac{1}{2} + \dfrac{1}{4} + \ldots + \dfrac{1}{2^{n-1}} = \dfrac{2^n - 1}{2^{n-1}}$

(*t*) $1 * 4 + 2 * 7 + 3 * 10 + \ldots + n\,(3n + 1) = n\,(n + 1)^2.$