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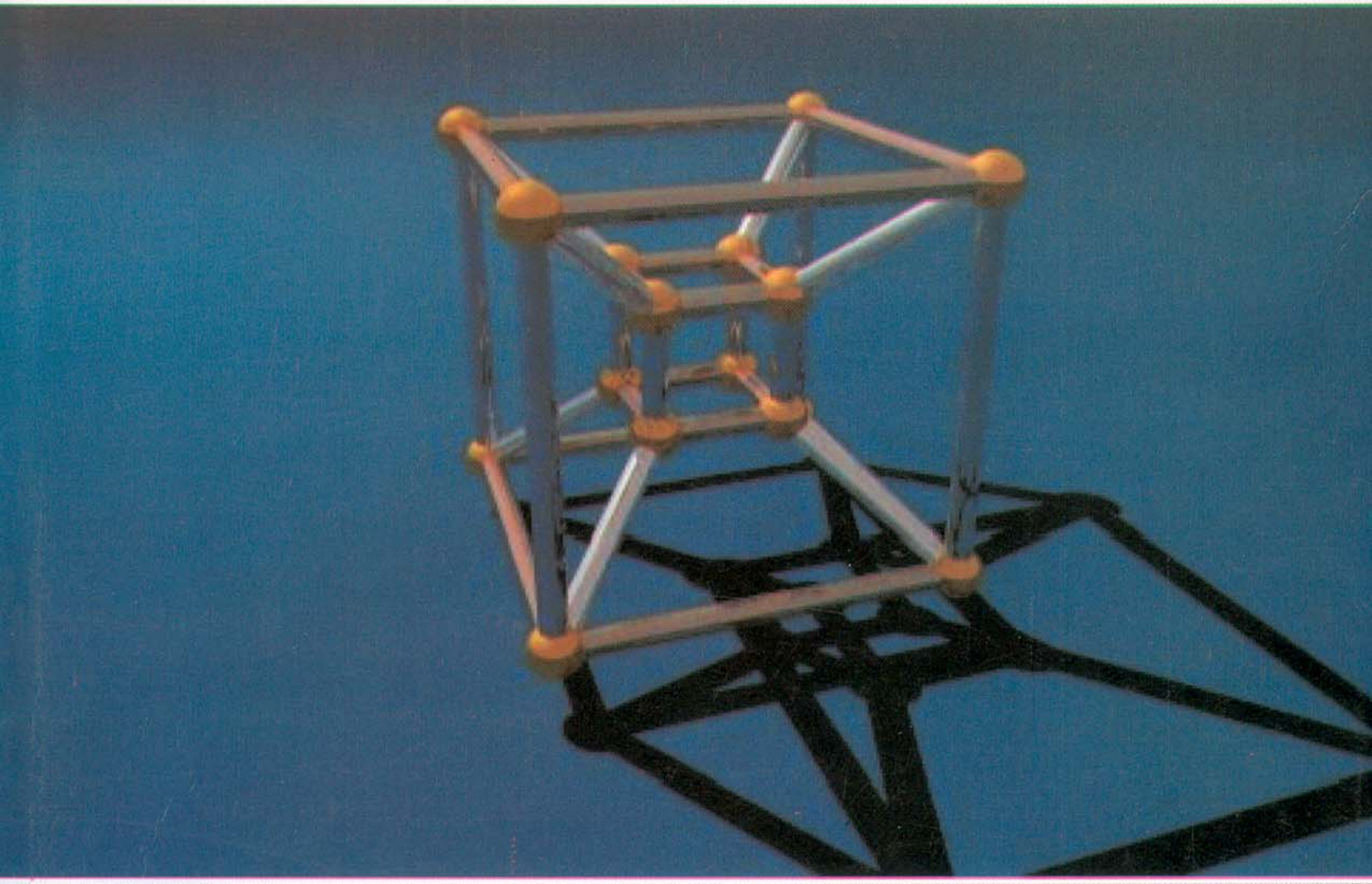
FUNDAMENTAL APPROACH TO DISCRETE MATHEMATICS



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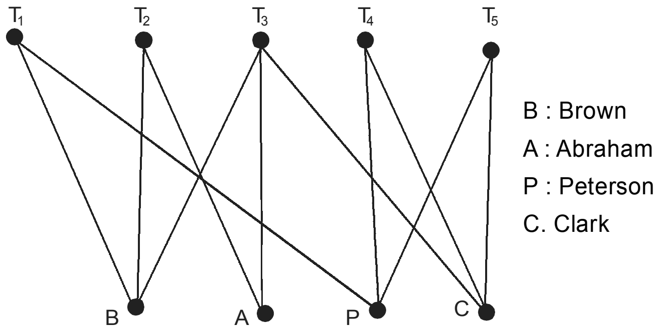
12

Graph Theory

■ 12.0 INTRODUCTION

Graph theory has applications in many areas like Mathematics, Computer Science, Engineering, Communication Science etc. Oystein Ore, the prominent graph theorist and author of the first graph theory book said in that "the theory of graphs is one of the few fields of mathematics with a definite birth date". Graph theory is considered to have begun in 1736 with the publication of Euler's solution of the Konigsberg Bridge problem. In 1936, Denes Konig wrote the first book on graph theory. The major developments of graph theory occurred by the ever growing importance of Computer Science and its connection with graph theory.

Now the question arises "what is a graph"? Consider the example. Suppose there are four sales persons Brown, Abraham, Peterson, Clark and five territories T_1, T_2, T_3, T_4, T_5. Brown is interested to work in the territories T_1, T_2, T_3. Abraham is interested to work in the territories T_2, T_3. Peterson is interested to work in the territories T_1, T_4, T_5 whereas Clark is interested for the territories T_3, T_4, T_5. This is explained in the following figure. This is nothing but a graph, a concept which we are about to study extensively.



In this chapter, we will study the basic components of graph theory.

■ 12.1 GRAPH

A graph G consists of a finite set of vertices V and a finite set of edges E. Mathematically,

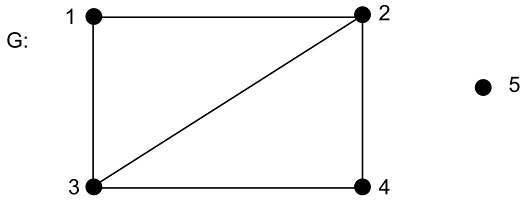
$$G = (V, E)$$

Where, $E = \{(v_i, v_j) \mid v_i, v_j \in V\}$

Let us consider $V = \{1, 2, 3, 4, 5\}$

and $E = \{(1, 2), (1, 3), (2, 3), (2, 4), (3, 4)\}$.

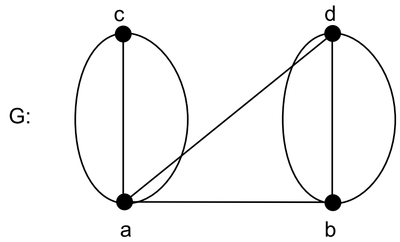
Hence the graph G = (V, E) becomes



12.1.1 Order and Size

The number of vertices in a graph G(V, E) is called its order, and the number of edges is its size. That is the order of G is $|V|$ and its size $|E|$

Consider the following graph G



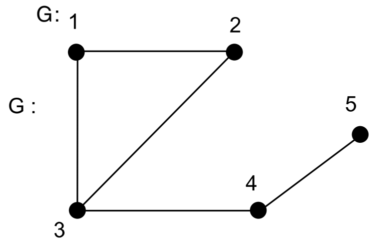
The order of G *i.e.*, $|V| = 4$

The size of G *i.e.*, $|E| = 8$

12.1.2 Adjacent Vertices

Two vertices v_i and v_j are said to be adjacent if there exists an edge (v_i, v_j) in the graph G(V, E).

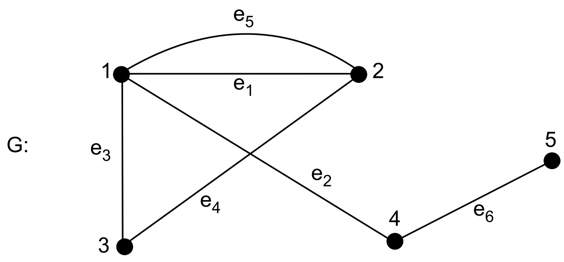
Consider the graph G as



Here the vertices 1 and 2 are adjacent. Similarly, the vertices 1 and 3 are also adjacent.

12.1.3 Parallel Edges

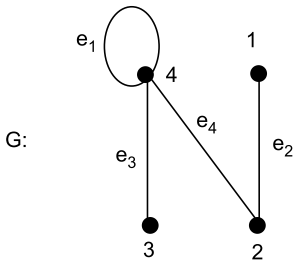
If there is more than one edge between the same pair of vertices, then the edges are termed as parallel edges. Consider the graph G as



Here the edges e_1 and e_5 are parallel edges.

12.1.4 Loop

An edge whose starting and ending vertex are same is known as a loop. Mathematically $e = (v_i , v_i)$. Consider the graph G as



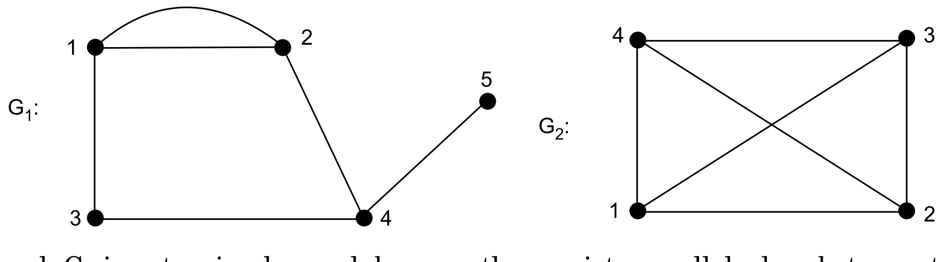
From the graph, it is clear that the edge e_1 is a loop.

■ 12.2 KINDS OF GRAPH

In this section, we will discuss different kinds of graph.

12.2.1 Simple Graph

A graph G(V, E) that has no self-loop or parallel edges is called a simple graph. Consider the graphs G_1 and G_2 as

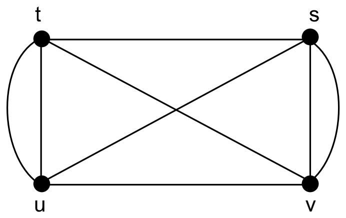


The graph G_1 is not a simple graph because there exists parallel edges between the vertices 1 and 2 whereas the graph G_2 is a simple graph.

12.2.2 Multi Graph

A graph G(V, E) is known as a multi graph if it contains parallel edges, *i.e.*, two or more edges between a pair of vertices. It is to be noted that every simple graph is a multi graph but the converse is not true. Consider the graph G as

G:



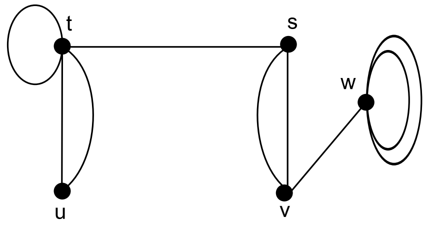
The above graph is a multi graph because there are parallel edges between the vertices u, t and v, s.

12.2.3 Pseudo Graph

A graph G(V,E) is known as a pseudo graph if we allow both parallel edges and loops. It is to be noted that every simple graph and multi graph are pseudo graph but the converse is not true.

Consider the graph G as

G:



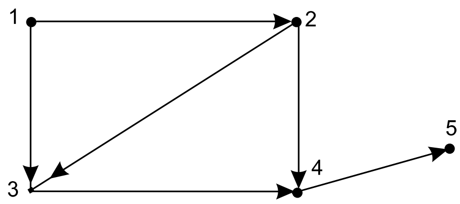
■ 12.3 DIGRAPH

A graph G(V, E) where V is the set of nodes or vertices and E is the set of edges having direction. If (v_i, v_j) is an edge, then there is an edge from the vertex v_i to the vertex v_j. A digraph is also called a directed graph. Let us consider

$$V = \{1, 2, 3, 4, 5\} \text{ and } E = \{(1, 2), (1, 3), (2, 3), (2, 4), (3, 4), (4, 5)\}$$

Hence, the digraph G becomes

G:



■ 12.4 WEIGHTED GRAPH

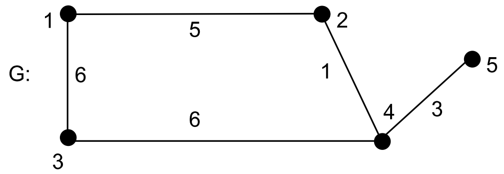
A graph (or digraph) is known as a weighted graph (or digraph) if each edge of the graph has some weights. Let us consider

$$V = \{1, 2, 3, 4, 5\} \text{ and } E = \{e_1, e_2, e_3, e_4, e_5\}$$

Where $\quad e_1 = (1, 2), e_2 = (1, 3), e_3 = (2, 4), e_4 = (3, 4), e_5 = (4, 5)$

and $\quad w(e_1) = 5, w(e_2) = 6, w(e_3) = 1, w(e_4) = 6, w(e_5) = 3$

Hence, the weighted graph G becomes



■ 12.5 DEGREE OF A VERTEX

The number of edges connected to the vertex 'v' is known as degree of vertex 'v', generally denoted by degree (v). In case of a digraph, there are two degrees *i.e.*, indegree and outdegree.

The number of edges coming to the vertex 'v' is known as indegree of 'v' whereas the number of edges emanating from the vertex 'v' is known as outdegree of 'v'. Generally, the indegree is denoted by indegree (v) and the outdegree is denoted by outdegree (v).

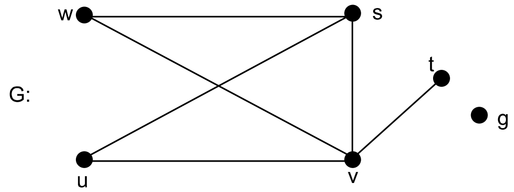
Note: In case of a loop, it contributes 2 to the degree of a vertex.

12.5.1 Isolated Vertex

A vertex is said to be an isolated vertex if there is no edge connected from any other vertex to the vertex. In other words a vertex is said to be an isolated vertex if the degree of that vertex is zero.

i.e., If degree (v) = 0, then v is isolated.

Consider the graph G as



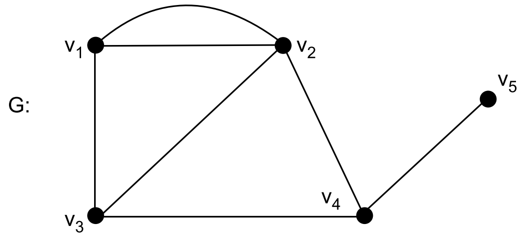
Now, degree (u) = 2; degree (v) = 4; degree (t) = 1
degree (g) = 0; degree (s) = 3; degree (w) = 2

Therefore, it is clear that 'g' is an isolated vertex.

■ 12.6 PATH

A path in a graph is a sequence $v_1, v_2, \ldots, v_k$ of vertices each adjacent to the next, and a choice of an edge between each 'v_i' to 'v_{i+1}' so that no edge is chosen more than once.

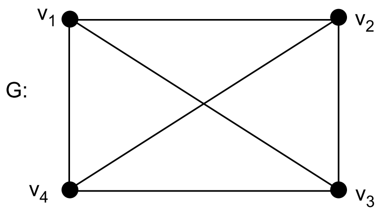
Consider the graph G as



Here one path is $v_1\, v_2\, v_1\, v_3\, v_4\, v_5$.

■ 12.7 COMPLETE GRAPH

A graph (digraph) G is said to be complete if each vertex 'u' is adjacent to every other vertex 'v' in G. In other words, there are edges from any vertex to all other vertices. Consider the graph G as



The above graph G is a complete graph.

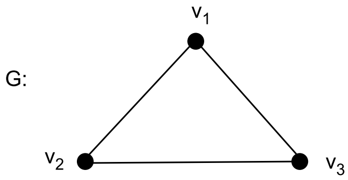
■ 12.8 REGULAR GRAPH

A graph G (V, E) is said to be regular if the degree of every vertex are equal. Mathematically, G is denoted as regular if

$$\text{degree } (v_i) = \text{degree } (v_j) \ \forall \ i, j.$$

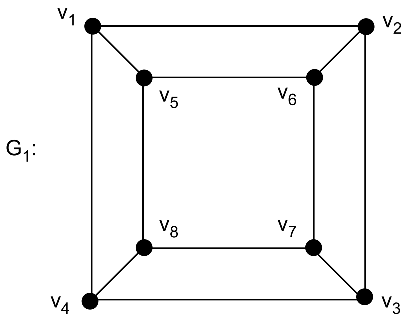
Where, $v_i, v_j \in$ G (V, E).

Consider the graph G as



In the above graph, degree (v_1) = degree (v_2) = degree (v_3) = 2. Therefore, the graph G is regular (2 regular). The above graph is also complete.

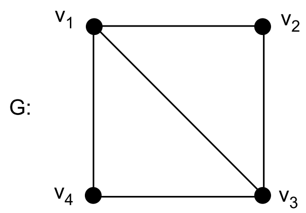
Consider another example G_1 as



Here the degree of every vertex is 3. So, the above graph is 3-regular but not complete.

■ 12.9 CYCLE

If there is a path containing one or more edges which starts from a vertex 'v' and terminates into the same vertex, then the path is known as a cycle. Consider the graph G as



In the above graph G, one cycle is $v_1 v_2\ v_3\ v_1$. Similarly, another cycle is $v_1\ v_2\ v_3\ v_4\ v_1$.

■ 12.10 PENDANT VERTEX

A vertex 'v' in a graph G is said to be a pendant vertex if the degree (v) =1. In case of a digraph, a vertex 'v' is said to be a pendant vertex if the indegree (v) = 1 and outdegree (v) = 0. In the graph 'G(figure 1)' given below, indegree of the vertices v_4 , v_5 , v_6 and v_7 is equal to 1 and the outdegree is equal to 0. Therefore, these vertices are pendant vertices. Similarly, in the graph 'G(figure 2)' given below the vertices v_1, v_5 and v_6 are pendent vertices.

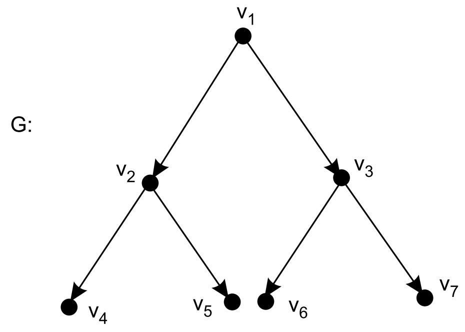


Figure 1

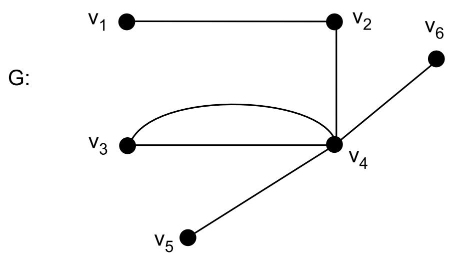
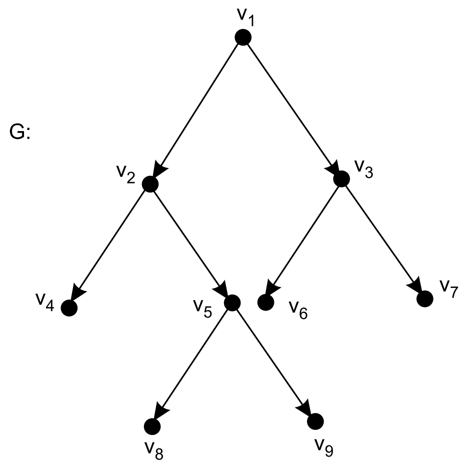


Figure 2

■ 12.11 ACYCLIC GRAPH

A graph (digraph) which does not have any cycle is known as an acyclic graph (digraph). Consider the graph G as



Here, G is an acyclic graph.

■ 12.12 MATRIX REPRESENTATION OF GRAPHS

There are many ways to represent a graph in computer. Generally, graphs are represented diagrammatically, but this is possible only when the number of vertices and edges are reasonably small. So, the concept of matrix representation of graphs is developed. The major advantage of this representation is that the calculation of paths and cycles in graph theoretical problems such as communication networks, power distribution, transportation etc. However, the disadvantage is that this representation takes away from the visual aspect of graphs.

12.12.1 Adjacency Matrix

The most useful way of representing any graph is the matrix representation. It is a square matrix of order $(n \times n)$ where n is the number of vertices in the graph G. Generally denoted by A $[a_{ij}]$ where a_{ij} is the ith row and jth column element. The general form of adjacency matrix is given as below:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & . & . & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & . & . & . & a_{2n} \\ a_{31} & a_{32} & a_{33} & . & . & . & a_{3n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{n1} & a_{n2} & a_{n3} & . & . & . & a_{nn} \end{bmatrix}$$

where, $$a_{ij} = \begin{cases} 1; & \text{if there is an edge from '}v_i\text{' to '}v_j\text{'} \\ 0; & \text{Otherwise} \end{cases}$$

This matrix is termed as adjacency matrix, because an entry stores the information whether two vertices are adjacent or not. This is also known as bit matrix or Boolean matrix as each entry is either 1 or 0.

Notes: 1. In the adjacency matrix if the main diagonal elements are zero, then the graph is said to be a simple graph.

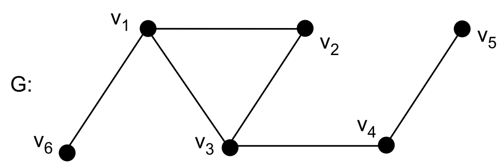
2. In case of a multi graph the adjacency matrix can be found out with the relation.

$$a_{ij} = \begin{cases} n; & n \text{ be the number of edges from '}v_i\text{' to '}v_j\text{'} \\ 0; & \text{Otherwise} \end{cases}$$

3. In case of a weighted graph the adjacency matrix can be found out with the relation

$$a_{ij} = \begin{cases} w; & w \text{ is the weight of the edges from '}v_i\text{' to '}v_j\text{'} \\ 0; & \text{Otherwise} \end{cases}$$

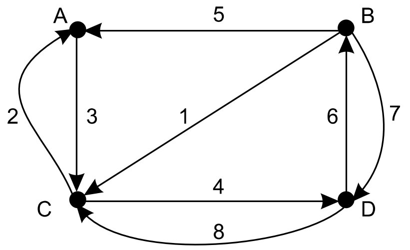
Consider the graph G as



Hence, the adjacency matrix is given as

$$A = \begin{array}{c c} & \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

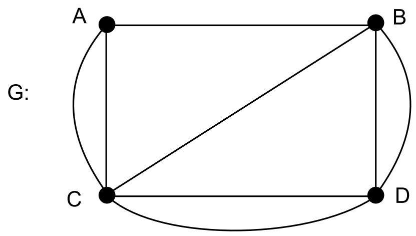
Consider the graph G as



The adjacency matrix of the above graph with respect to the ordering A, B, C and D is given below:

$$A = \begin{bmatrix} 0 & 0 & 3 & 0 \\ 5 & 0 & 1 & 7 \\ 2 & 0 & 0 & 4 \\ 0 & 6 & 8 & 0 \end{bmatrix}$$

Consider the graph G as



The adjacency matrix of the above graph with respect to the ordering A, B, C and D is given below:

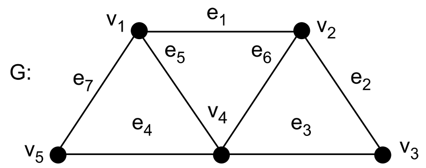
$$A = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 2 \\ 0 & 2 & 2 & 0 \end{bmatrix}$$

12.12.2 Incidence Matrix

Suppose that G be a simple undirected graph with m vertices and n edges, then the incidence matrix $I[a_{ij}]$ is a matrix of order $(m \times n)$ where the element a_{ij} is defined as

$$a_{ij} = \begin{cases} 1; & \text{If vertex } i \text{ belongs to edges } j. \\ 0; & \text{Otherwise} \end{cases}$$

Consider the graph G as



Hence, the incidence matrix of the graph G is of order (5×7). The incidence matrix relative to the ordering v_1, v_2, v_3, v_4, v_5 and $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ is given below:

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

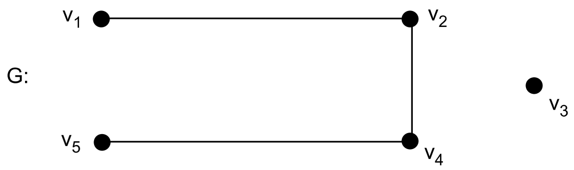
12.12.3 Path Matrix

Suppose that G be simple graph with n-vertices. Then the $(n \times n)$ matrix $P = [p_{ij}]_{(n \times n)}$ defined by

$$p_{ij} = \begin{cases} 1; & \text{if there is a path from } v_i \text{ to } v_j \\ 0; & \text{Otherwise} \end{cases}$$

is known as the path matrix or reachability matrix of the graph G.

Consider the graph G as



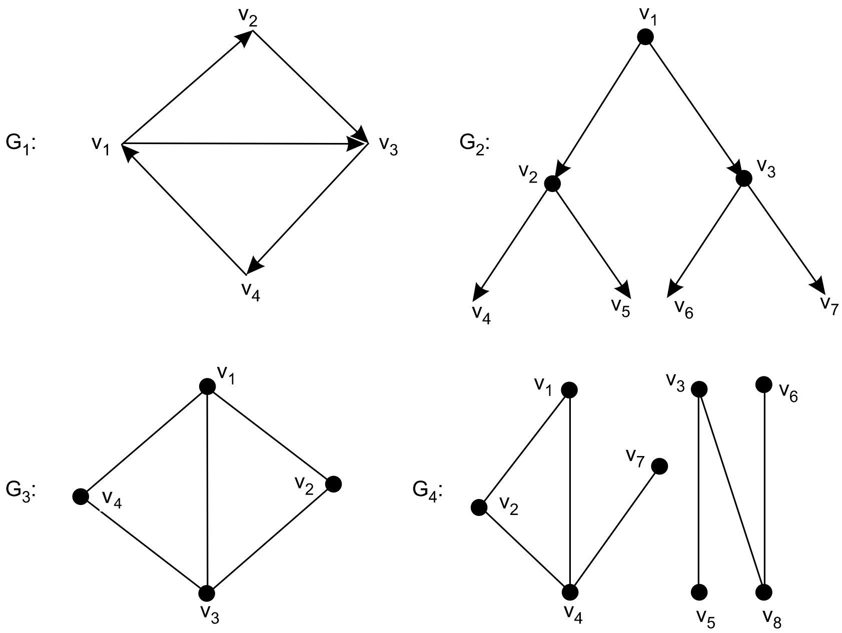
Therefore, the path matrix of the above graph relative to the ordering v_1, v_2, v_3, v_4, v_5 is given as

$$P = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

■ 12.13 CONNECTED GRAPH

A graph (not digraph) G (V, E) is said to be connected if for every pair of distinct vertices 'u' and 'v' in G, there is a path. A directed graph is said to be strongly connected if for every pair of distinct vertices 'u' and 'v' in G, there is a directed path from 'u' to 'v' and also from 'v' to 'u'. A directed graph is said to be weakly connected if for every pair of distinct vertices, there is a path without taking the direction.

Consider the following graphs



From the above graphs, it is clear that

G_1 : Strongly Connected; G_2: Weakly Connected

G_3 : Connected; G_4: Disconnected

12.13.1 Theorem

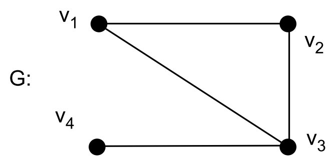
Suppose that G be a graph with n-vertices $v_1, v_2, ..., v_n$ and let A be the adjacency matrix of G. Let us define $B = [b_{ij}]$ such that

$$B = A + A^2 + A^3 + ... + A^{n-1}.$$

If for every pair of distinct indices i and j, $b_{ij} \neq 0$, then the graph is said to be connected.

The proof of the above theorem is beyond the scope of this book.

Consider the graph G as



Hence, the adjacency matrix A is given as

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Here, number of vertices $(n) = 4$. Therefore, $B = A + A^2 + A^3$

Now, $$A^2 = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}; \; A^3 = A^2\, A = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 3 & 2 & 4 & 1 \\ 4 & 4 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{pmatrix}$$

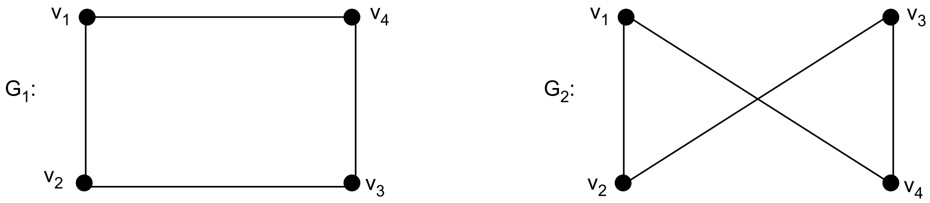
Therefore, $$B = A + A^2 + A^3 = \begin{pmatrix} 4 & 5 & 6 & 2 \\ 5 & 4 & 6 & 2 \\ 6 & 6 & 5 & 4 \\ 2 & 2 & 4 & 1 \end{pmatrix}$$

Since, all $b_{ij} \neq 0$ for $i \neq j$; the graph G is connected. All elements except the diagonal elements must not be zero for connected graph.

■ 12.14 GRAPH ISOMORPHISM

Suppose $G_1 : (V_1, E_1)$ and $G_2 : (V_2, E_2)$ be two graphs. Then the two graphs G_1 and G_2 are said to be isomorphic if there is one to one correspondence between the edges E_1 of G_1 and E_2 of G_2 which indicates that if $(u_1, v_1) \in G_1$, then $(u_1, v_1) \in G_2$.

Such a pair of correspondence is known as graph isomorphism. The different way of representing the same graph is known as graph isomorphism. Consider graph G_1 and G_2 as



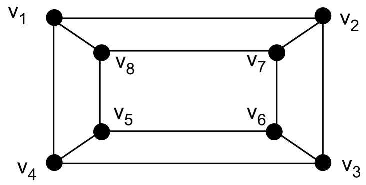
Therefore, the graphs G_1 and G_2 are isomorphic to each other.

■ 12.15 BIPARTITE GRAPH

Suppose that G: (V, E) be the graph. If the vertex set V can be partitioned into two non-empty disjoint sets V_1 and V_2 such that each edge of the graph G has one end in V_1 and other end in V_2, then the graph is said to be bipartite graph.

Consider the graph G as

G:



Let $V_1 = \{v_1, v_3, v_5, v_7\}$ and $V_2 = \{v_4, v_2, v_6, v_8\}$

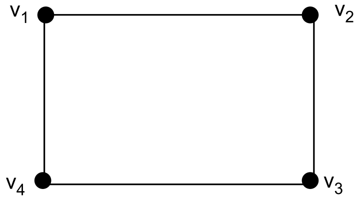
Now, $(V_1 \cap V_2) = \phi$ and each edge of G has one vertex in V_1 and other vertex at V_2 . So, G is said to be a bipartite graph.

12.15.1 Complete Bipartite Graph

Suppose that G: (V, E) be the graph. If the vertex set $V = (V_1 \cup V_2)$ and $V_1, V_2 \neq \phi$, $(V_1 \cap V_2) = \phi$, such that each edge of the graph G has one end in V_1 and other end in V_2 , then the graph G is termed as bipartite.

If every vertex of V_1 is joined to every vertex of V_2, then the graph G is termed as complete bipartite graph. Consider the graph G as

G:



Let $V_1 = \{v_1, v_3\}$ and $V_2 = \{v_2, v_4\}$.

Therefore, $V = (V_1 \cup V_2)$; $V_1, V_2 \neq \phi$, and $(V_1 \cap V_2) = \phi$.

Also, every vertex of V_1 is joined to every vertex of V_2. So, G is a complete bipartite graph.

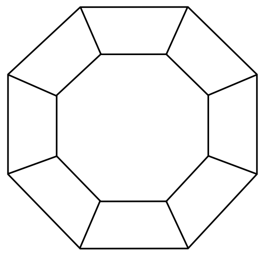
■ 12.16 SUBGRAPH

Suppose that G and H be two graphs with vertex sets V(G) and V(H). Let the edge sets be E(G) and E(H). Now H is said to be subgraph of G if

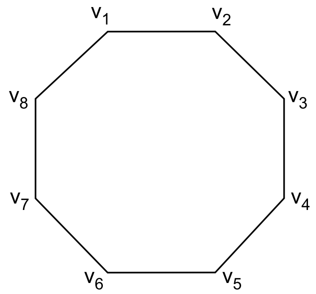
$$V(H) \subseteq V(G) \text{ and } E(H) \subseteq E(G)$$

Consider two graphs G and H as

G:



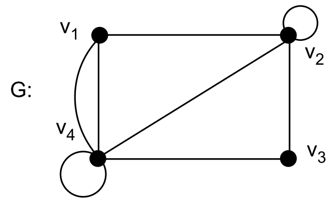
H:



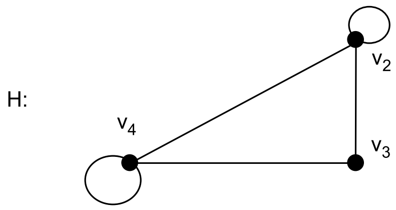
Therefore, it is clear that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. So, H is a subgraph of G.

12.16.1 Vertex Deleted Subgraph

Suppose that G(V, E) be a graph. If we delete a subset U of the set V and all the edges, which have a vertex in U as an end, then the resultant graph is termed as vertex deleted subgraph of G. Consider the graph G as

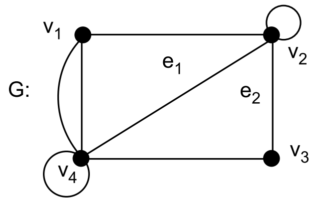


On deleting the vertex v_1, the vertex deleted subgraph H is given as

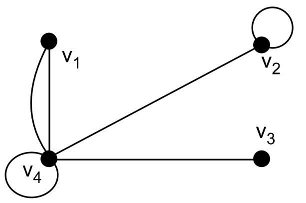


12.16.2 Edge Deleted Subgraph

Suppose that G: (V, E) be a graph. If a subset F from the set of edges E is deleted from the graph G, then the resultant graph is edge deleted subgraph of G. Consider the graph G as



On deleting the edges e_1 and e_2, the edge deleted subgraph is given as



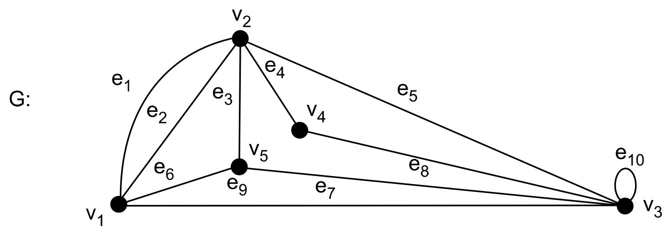
■ 12.17 WALKS

Let G be a graph, then a walk W in a graph G is a finite sequence W = $v_0\, e_1\, v_1 e_2\, v_2\, e_3$ .... $v_{i-1}\, e_i\, v_i \ldots v_{k-1}\, e_k\, v_k$. Whose terms are alternately vertices and edges such that for $1 \le i \le k$, the edge e_i has ends v_{i-1} and v_1 . The starting vertex v_0 is the origin and the end vertex v_k is the terminus of the walk W. The vertices v_1, v_2, ... v_{k-1} are known as internal vertices. The walk is termed as $v_0 - v_k$ walk.

The number of edges present in the walk W is known as the length of walk W. Note that in a walk W there may be repetition of vertices and edges. In a simple graph, a walk W = $v_0e_1\, v_1e_2 \ldots e_kv_k$ is determined by a sequence of vertices $v_0\, v_1\, v_2 \ldots v_{k-1}v_k$ because each pair of vertices $v_{i-1}\, v_i$ has one edge only. Even if a graph is not simple, a walk is often simply denoted by a sequence of vertices $v_0\, v_1\, v_2 \ldots v_{k-1}v_k$ where the consecutive vertices are adjacent.

Notes : 1. A walk containing no edges is known as a trivial walk.
2. A walk containing no repeated edges is termed as a trail.
3. A walk containing no repeated vertices is termed as a path. Which indicates that if the sequence of vertices $v_0\, v_1\, v_2 \ldots\ldots v_{k-1}v_k$ of the walk W = $v_0\, e_1 v_1\, e_2\, v_2 \ldots\ldots\ldots v_{k-1}\, e_k\, v_k$ are distinct, then the walk is a path.
4. Every path is a trail but the converse is not true always.

Consider the graph G as



Consider the following walks

$$W_1 = v_1e_1\, v_2e_2\, v_1e_6\, v_5e_7\, v_3e_{10}\, v_3e_8\, v_4$$
$$W_2 = v_1e_1\, v_2e_1\, v_1e_1\, v_2e_2\, v_1e_1\, v_2$$
$$W_3 = v_3e_{10}\, v_3e_9\, v_1e_1\, v_2e_2\, v_1$$
$$W_4 = v_1e_2\, v_2e_5\, v_3e_7\, v_5$$

The length of W_1 is 6. Similarly, the length of other walks can be found out. Here W_1 and W_2 are walks; W_3 is trail and W_4 is a path.

12.17.1 Open and Closed Walk

Suppose that u and v be two vertices of a graph. An $u - v$ walk is said to be open or closed according to $u \neq v$ or $u = v$ respectively. In other words a walk is closed if the starting vertex (u) and the terminus (v) are same otherwise it is open.

■ 12.18 OPERATIONS ON GRAPHS

There are many operations that gives new graphs from old ones. They are mainly separated into three categories such as elementary operation, unary operation and binary operation. In elementary operation, a new graph may be produced from the original graph by a simple local change such as addition or deletion of a vertex or an edge. In unary operation we create a significantly new graph from the old one whereas in binary operation we create a new graph from two initial graphs.

12.18.1 Union

If G_1 and G_2 be two graphs, then their union $(G_1 \cup G_2)$ is a graph with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

12.18.2 Intersection

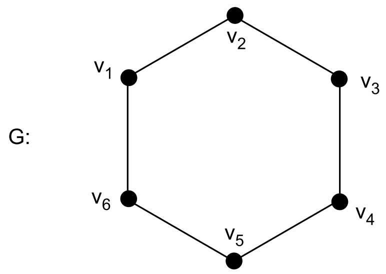
If G_1 and G_2 be two graphs with at least one vertex in common, then their intersection $(G_1 \cap G_2)$ is a graph with

$$V(G_1 \cap G_2) = V(G_1) \cap V(G_2)$$

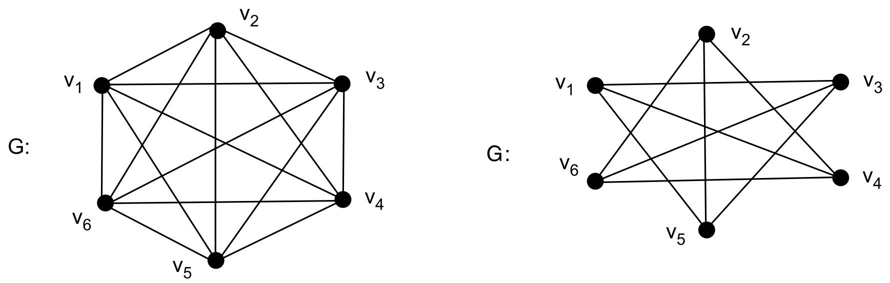
and $$E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$$

12.18.3 Complement

Suppose that G be a simple graph with n-vertices. Then the complement of G is given by $\overline{G}$ and is defined to be the simple graph with the same vertices of G and where two vertices (u, v) are adjacent in $\overline{G}$, if u and v are not adjacent in G. In other words the complement of G can be obtained from the complete graph K_n by deleting all the edges of G. Consider the graph G as



To obtain the complement of G construct the complete graph with the same vertices and then delete the edges of the graph G. The complement graph of G *i.e.*, $\overline{G}$ is given below.



12.18.4 Product of Graphs

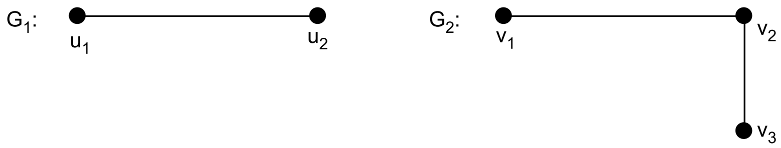
Suppose that G_1: (V_1, E_1) and G_2: (V_2, E_2) be two graphs. Then the product of graphs G_1 and G_2 is given as $(G_1 \times G_2)$ and is defined as $(G_1 \times G_2)$: (V, E). Where $V = (V_1 \times V_2)$ and the edge set E can be found out from the following relation.

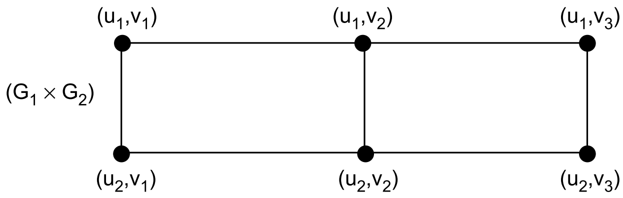
If (u_1, u_2) and (v_1, v_2) be two vertices of $(G_1 \times G_2)$. Then there is an edge between them if

(*i*) ($u_1 = v_1$ and u_2 is adjacent to v_2) or

(*ii*) (u_1 is adjacent to v_1 and $u_2 = v_2$).

Consider the graphs G_1 and G_2 as





12.18.5 Composition

Suppose that G_1: (V_1, E_1) and G_2: (V_2, E_2) be two graphs. Then the composition of $G_1[G_2]$ and is defined as

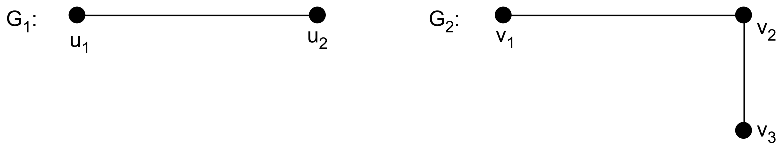
$$G_1[G_2]: (V, E)$$

Where, $V = (V_1 \times V_2)$ and the edge set E can be found out from the following relation. If (u_1, u_2) and (v_1, v_2) be two vertices of $G_1[G_2]$, then there is an edge between them if

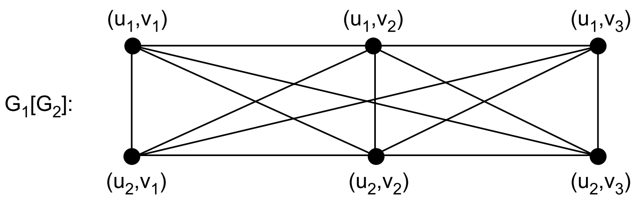
(*i*) u_1 is adjacent to v_1 or

(*ii*) ($u_1 = v_1$ and u_2 is adjacent to v_2)

Consider the graphs G_1 and G_2 as



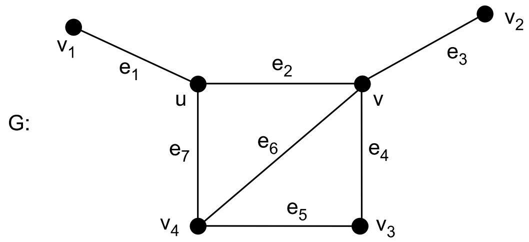
The composition $G_1[G_2]$ is defined as



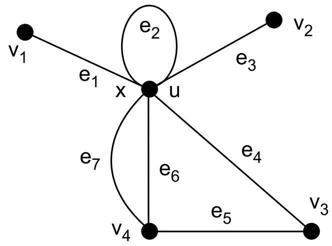
■ 12.19 FUSION OF GRAPHS

Let u and v be distinct vertices of a graph G, we can construct a new graph G_1 by fusing the two vertices. This means by replacing them by a single new vertex 'x' such that every edge that was incident with either 'u' or 'v' is now incident with x.

Consider the graph G as



On fusing the vertices 'u' and 'v' the graph becomes



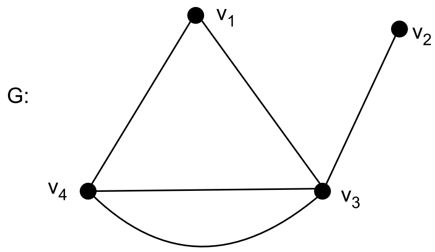
12.19.1 Adjacency Matrix (After fusion of two adjacent vertices)

The following steps are used to find the new adjacency matrix after fusion of two adjacent vertices 'u' and 'v':

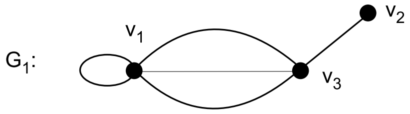
Step 1. Replace the uth row by the sum of uth row and vth row. Similarly, replace the uth column by the sum of uth column and vth column.

Step 2. Delete the row and column corresponding to the vertex v. The resulting matrix is the new adjacency matrix.

Consider the graph G as



After fusing v_1 and v_4 we have the new graph G_1 as



Relative to the ordering v_1, v_2, v_3 and v_4 we have $A(G) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{pmatrix}$

Now on replacing Row $(v_1) \leftarrow$ Row (v_1) + Row (v_4) and Col $(v_1) \leftarrow$ Col (v_1) + Col (v_4), we get

$$A(G) = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{pmatrix}$$

On deleting the row and column corresponding to v_4 the adjacency matrix of G_1 is given as

$$A(G_1) = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

12.19.2 Fusion Algorithm (Connectedness)

The following steps are used to check the connectedness of a graph G:

Step 1. Replace the graph G by its underlying simple graph. The adjacency matrix can be obtained by replacing all non-zero entries off the diagonal by 1 and all entries on the diagonal by 0.

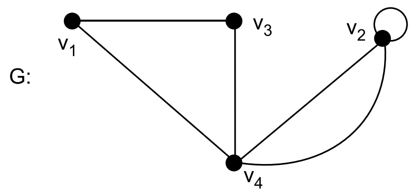
Step 2. Fuse vertex v_1 to the first of the vertices v_2, v_3 ..., v_n with which it is adjacent to give a new graph. Denote it by G in which the new vertex is also denoted by v_1.

Step 3. Carry out step1 on the new graph G.

Step 4. Carry out step 2 to step 3 repeatedly with v_1 until v_1 is not adjacent to any of the other vertices.

Step 5. Carry out steps 2 to 4 on the vertex v_2 of the latest graph and than on all the remaining vertices of the resulting graphs in turn. The final graph is empty and the number of its isolated vertices is the number of connected components of the initial graph G.

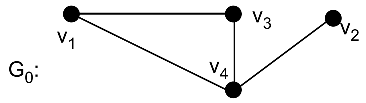
Consider the following graph G.



The adjacency matrix A (G) relative to the ordering v_1, v_2, v_3 and v_4 becomes

$$A(G) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

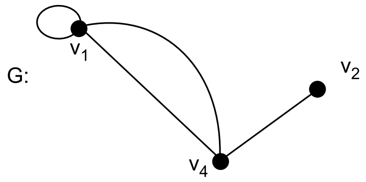
The underlying simple graph of G is given as



The adjacency matrix becomes

$$A(G_0) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

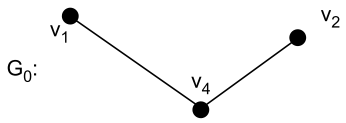
On fusing v_1 with v_3 we have the graph G as



Therefore, on replacing Row $(v_1) \leftarrow$ Row (v_1) + Row (v_3); Col $(v_1) \leftarrow$ Col. (v_1) + Col. (v_3) and on removing the row and column corresponding to v_3 the adjacency matrix relative to the ordering v_1, v_2 and v_4 becomes

$$A(G) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

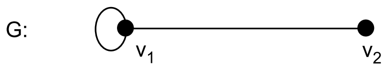
The underlying simple graph of G is given as



The adjacency matrix becomes

$$A(G_0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

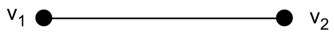
On fusing v_1 with v_4 we have the graph G as



Therefore, on replacing Row $(v_1) \leftarrow$ Row (v_1) + Row (v_4); Col $(v_1) \leftarrow$ Col (v_1) + Col. (v_4) and on removing the row and column corresponding to v_4, the adjacency matrix relative to the ordering v_1 and v_2 becomes

$$A(G) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

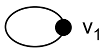
The underlying spanning graph of G is given as

G_0: 

The adjacency matrix becomes

$$A(G_0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

On fusing v_1 with v_2 we have the graph G as

G: 

Therefore, on replacing Row $(v_1) \leftarrow$ Row (v_1) + Row (v_2); Col $(v_1) \leftarrow$ Col (v_1) + Col (v_2) and on removing the row and column corresponding to v_2, the adjacency matrix relative to v_1 becomes

$$A(G) = (1)$$

The underlying spanning graph of G is given as

G_0: 

The adjacency matrix becomes

$$A(G_0) = (0)$$

As the final graph is empty, the process terminates. Here the number of isolated point is one. So, the graph is said to be connected.

SOLVED EXAMPLES

Example 1 *Draw the graph having the following matrix as its adjacency matrix*

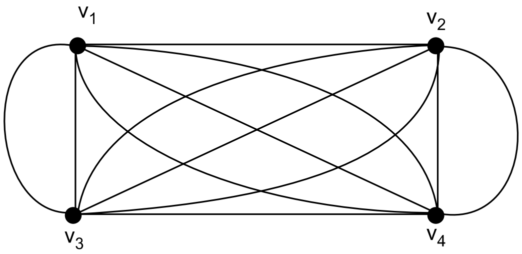
$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

Solution: Given that the adjacency matrix is

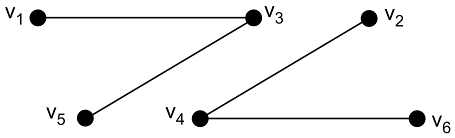
$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

The order of the adjacency matrix is (4×4). So, the graph G has four vertices, say v_1, v_2, v_3 and v_4 . Relative to the ordering v_1, v_2, v_3 and v_4 the graph G is given below.

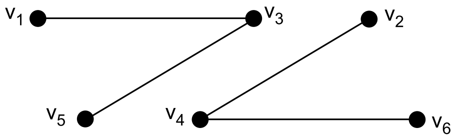
G:



Example 2 *Write down the path matrix of the following graph.*



Solution: Given that the graph is

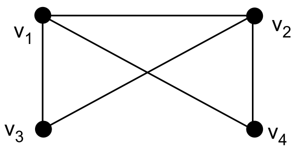


The path matrix relative to the ordering v_1, v_2, v_3, v_4, v_5 and v_6 is given as

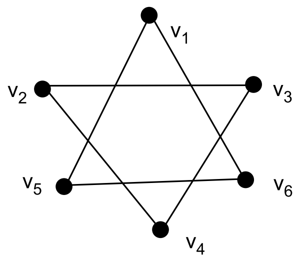
$$P(G) = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Example 3 *Write the adjacency matrix of the following graphs*

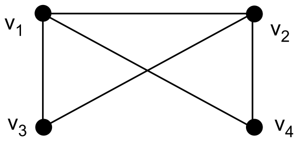
(a)



(b)



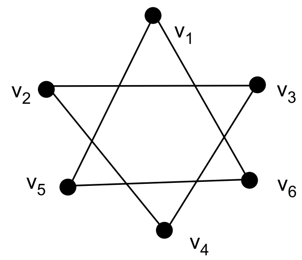
Solution: (a) Given graph is



The adjacency matrix relative to the ordering v_1, v_2, v_3 and v_4 is given as

$$A(G) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

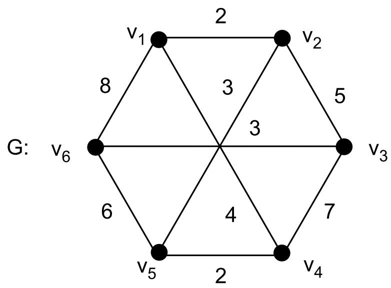
(*b*) Given graph is



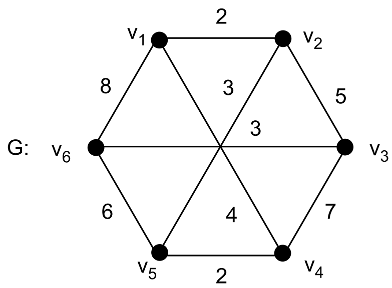
The adjacency matrix relative to the ordering v_1, v_2, v_3, v_4, v_5 and v_6 is given as

$$A(G) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Example 4 *Write the adjacency matrix of the following weighted graph.*



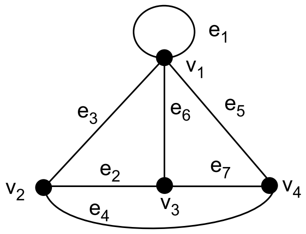
Solution: The weighted graph is



The adjacency matrix relative to the ordering v_1, v_2, v_3, v_4, v_5 and v_6 is given as

$$A(G) = \begin{pmatrix} 0 & 2 & 0 & 4 & 0 & 8 \\ 2 & 0 & 5 & 0 & 3 & 0 \\ 0 & 5 & 0 & 7 & 0 & 3 \\ 4 & 0 & 7 & 0 & 2 & 0 \\ 0 & 3 & 0 & 2 & 0 & 6 \\ 8 & 0 & 3 & 0 & 6 & 0 \end{pmatrix}$$

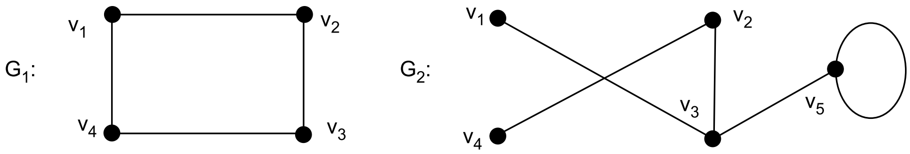
Example 5 *Write down the incidence matrix of the following graph G.*



Solution: In the above graph G, V = $\{v_1, v_2, v_3, v_4\}$ and E = $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$. Therefore the order of incidence matrix is (4×7). Relative to the ordering of V and E, the incidence matrix is given as

$$I(G) = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

Example 6 *Find the union of the following graphs.*



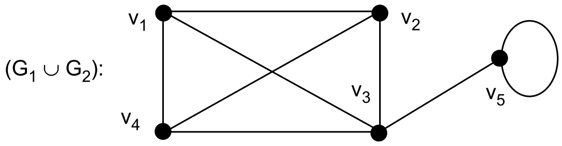
Solution: Here, $V(G_1) = \{v_1, v_2, v_3, v_4\}$

and $V(G_2) = \{v_1, v_2, v_3, v_4, v_5\}$.

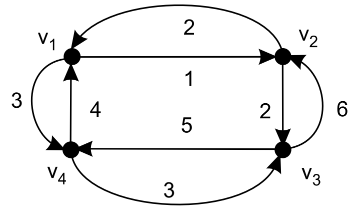
Therefore, $V(G_1 \cup G_2) = \{v_1, v_2, v_3, v_4, v_5\}$.

Similarly, $E(G_1 \cup G_2) = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1), (v_1, v_3), (v_2, v_4), (v_3, v_5), (v_5, v_5)\}$.

Therefore, the graph $(G_1 \cup G_2)$ becomes



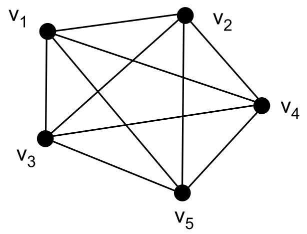
Example 7 *Write the adjacency matrix of the following directed weighted graph*

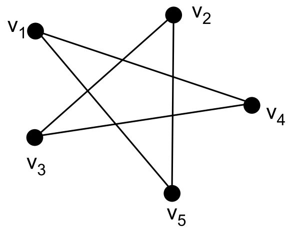


Solution: In the above directed weighted graph the total number of vertices are 4. So, the adjacency matrix is of order (4 × 4). The adjacency matrix relative to the ordering v_1, v_2, v_3 and v_4 is given as below.

$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 3 \\ 2 & 0 & 2 & 0 \\ 0 & 6 & 0 & 5 \\ 4 & 0 & 3 & 0 \end{pmatrix}$$

Example 8 *Find the intersection of the following graphs.*

G_1: 

G_2: 

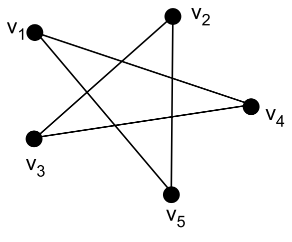
Solution: Here, $V(G_1) = \{v_1, v_2, v_3, v_4, v_5\}$

and $V(G_2) = \{v_1, v_2, v_3, v_4, v_5\}$.

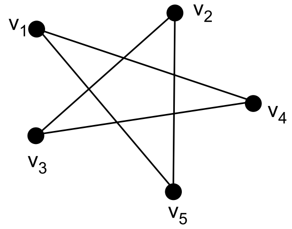
Therefore, $V(G_1 \cap G_2) = \{v_1, v_2, v_3, v_4, v_5\}$.

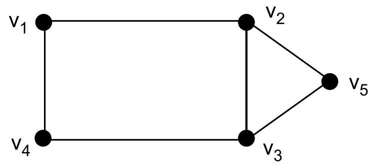
Similarly, $E(G_1 \cap G_2) = \{(v_1, v_4), (v_4, v_3), (v_3, v_2), (v_2, v_5), (v_5, v_1)\}$.

Therefore, the graph $(G_1 \cap G_2)$ becomes

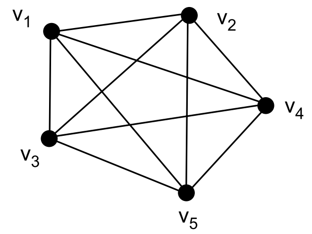
$(G_1 \cap G_2)$: 

Example 9 *Find the complement of the following graphs.*

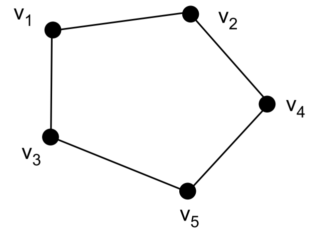
(a) 

(b) 

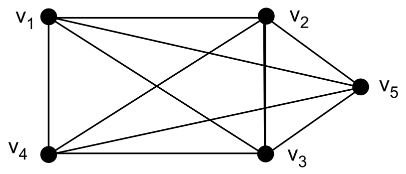
Solution: (*a*) To obtain the complement of G, find the complete graph with the same vertices. This is given below.



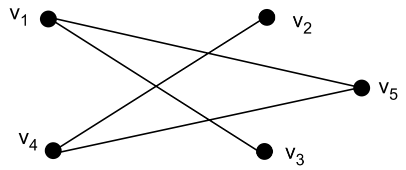
On deleting the edges of G, the complement $\overline{G}$ of G is given below.



(*b*) To obtain the complement of G, find the complete graph with the same vertices. This is given below.

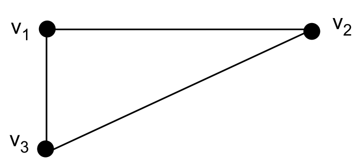


On deleting the edges of G, the complement $\overline{G}$ of G is given below.

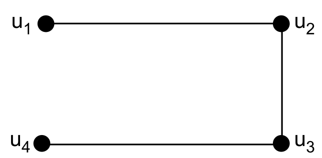


Example 10 *If G_1 and G_2 be two graphs given below, then find the product of graphs $(G_1 \times G_2)$. Where*

G_1:



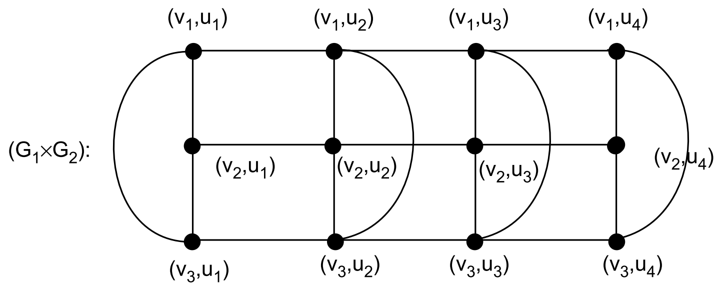
G_2:



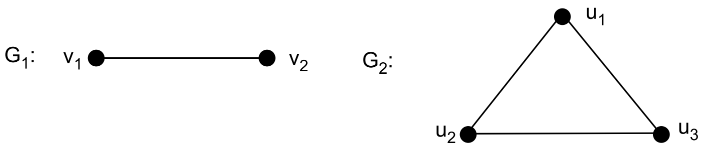
Solution: Here $V(G_1) = \{v_1, v_2, v_3\}$

and $V(G_2) = \{u_1, u_2, u_3, u_4\}$.

Therefore, $V(G_1 \times G_2) = \{(v_1, u_1), (v_1, u_2), (v_1, u_3), (v_1, u_4), (v_2, u_1), (v_2, u_2), (v_2, u_3), (v_2, u_4), (v_3, u_1), (v_3, u_2), (v_3, u_3), (v_3, u_4)\}$



Example 11 *Given G_1 and G_2 be two graphs. Find the composition $G_1[G_2]$ where*

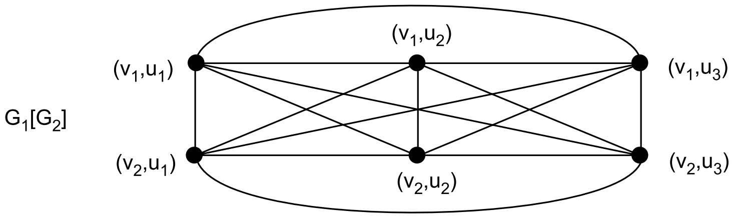


Solution: In the above graph

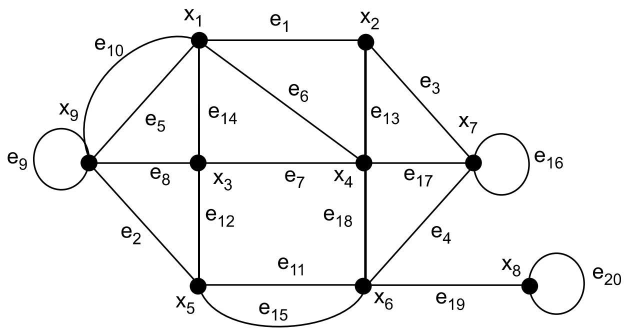
$$V(G_1) = \{v_1, v_2\}$$

and $$V(G_2) = \{u_1, u_2, u_3\}.$$

Therefore, the vertex set of $G_1[G_2]$ is $\{(v_1, u_1), (v_1, u_2), (v_1, u_3), (v_2, u_1), (v_2, u_2), (v_2, u_3)\}$. Thus the composition graph $G_1[G_2]$ is given below.



Example 12 *Let G be the graph given below.*

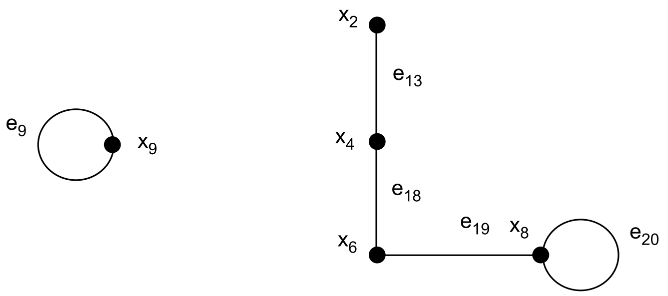


(*a*) *Find G – U; where* $U = \{x_1, x_3, x_5, x_7\}$

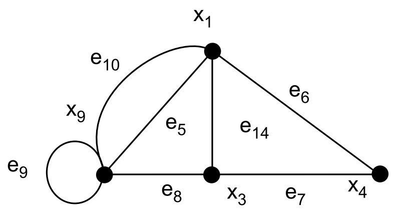
(b) Find G(U); where $U = \{x_1, x_3, x_4, x_9\}$
(c) Find G – V; where $V = \{e_2, e_5, e_8, e_{12}, e_{14}, e_1, e_6, e_{18}, e_4, e_{19}, e_{20}\}$
(d) Find G(V); where $V = \{e_1, e_6, e_7, e_{11}, e_{15}\}$

Solution: (*a*) Given U = $\{x_1, x_3, x_5, x_7\}$.

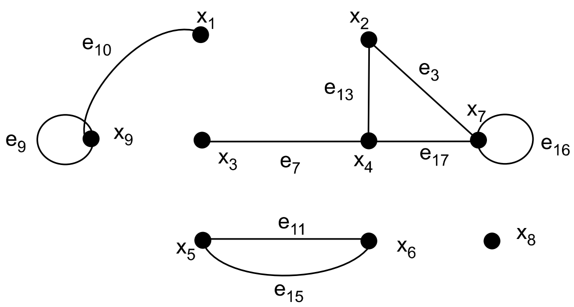
Therefore, G – U becomes



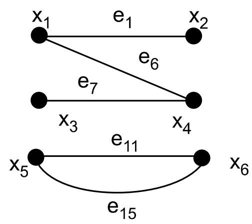
(*b*) Given: U = $\{x_1, x_3, x_4, x_9\}$. Therefore, G(U) becomes



(*c*) Given: V = $\{e_2, e_5, e_8, e_{12}, e_{14}, e_1, e_6, e_{18}, e_4, e_{19}, e_{20}\}$. Therefore, G – V becomes



(*d*) Given: V = $\{e_1, e_6, e_7, e_{11}, e_{15}\}$. Therefore, G(V) becomes



Example 13 *Determine whether the graph given below by its adjacency matrix is connected or not.*

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Solution: The adjacency matrix A is given as

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Here, the number of vertices $(n) = 4$. Let $B = A + A^2 + A^3$

Now, $$A^2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

Again, $$A^3 = A^2A = \begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 5 & 5 \\ 5 & 2 & 5 & 2 \\ 5 & 5 & 4 & 5 \\ 5 & 2 & 5 & 2 \end{pmatrix}$$

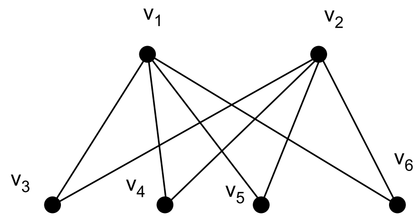
Therefore, $$B = A + A^2 + A^3 = \begin{pmatrix} 7 & 7 & 8 & 7 \\ 7 & 4 & 7 & 4 \\ 8 & 7 & 7 & 7 \\ 7 & 4 & 7 & 4 \end{pmatrix}$$

As all $b_{ij} \neq 0$ for $i \neq j$, the graph G is connected.

Example 14 *Draw a complete bipartite graph on two and four vertices.*

Solution: Complete bipartite graph on m and n is the simple graph whose vertex set is partitioned into sets V_1 and V_2 with m and n vertices respectively. Generally, denoted by K_{mn}.

The complete bipartite graph on two and four vertices is shown in the following figure.



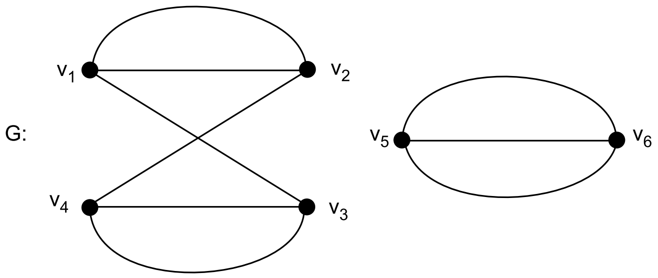
Example 15 *Use the fusion algorithm to determine whether the graph given below by its adjacency matrix is connected or not.*

$$\begin{pmatrix} 0 & 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 3 & 0 \end{pmatrix}$$

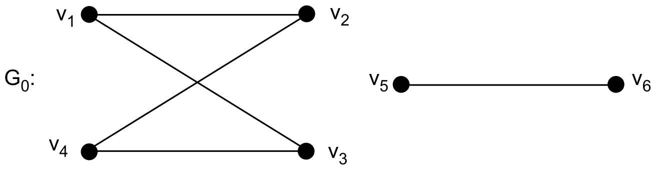
Solution: The adjacency matrix of the graph G is given as

$$A(G) = \begin{pmatrix} 0 & 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 3 & 0 \end{pmatrix}$$

Therefore, the graph G becomes



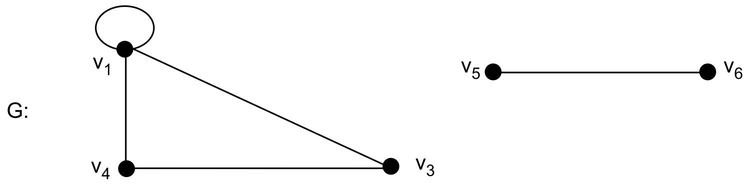
The underlying simple graph of G is given as



The adjacency matrix is given as

$$A(G_0) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

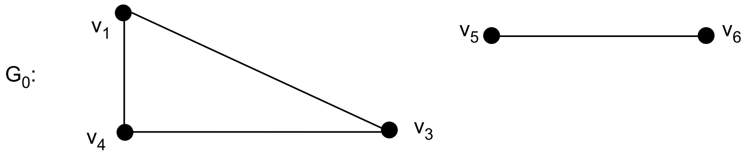
On fusing vertex v_1 with v_2 we have the graph G as



So, on replacing Row $(v_1) \leftarrow$ Row (v_1) + Row (v_2); Col $(v_1) \leftarrow$ Col (v_1) + Col (v_2) and on removing the row and column corresponding to v_2 the adjacency matrix becomes

$$A(G) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

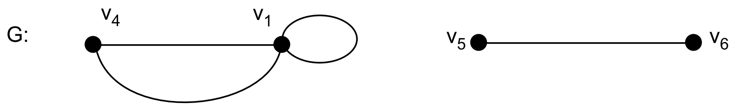
The underlying simple graph of G is given as



The adjacency matrix becomes

$$A(G_0) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

On fusing v_1 with v_3 we have the graph G as



So, on replacing Row $(v_1) \leftarrow$ Row (v_1) + Row (v_3); Col $(v_1) \leftarrow$ Col (v_1) + Col (v_3) and on removing the row and column corresponding to v_3 the adjacency matrix becomes

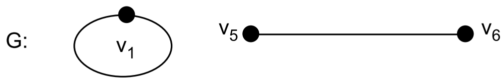
$$A(G) = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The underlying simple graph of G is given as

The adjacency matrix becomes

$$A(G_0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

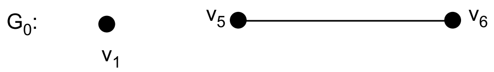
On fusing v_1 with v_4 we have the graph G as



So, on replacing Row (v_1) ← Row (v_1) + Row (v_4); Col (v_1) ← Col (v_1) + Col (v_4) and on removing the row and column corresponding to v_4 the adjacency matrix becomes

$$A(G) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

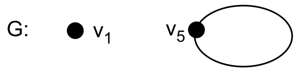
The underlying simple graph of G is given as



The adjacency matrix becomes

$$A(G_0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

On fusing v_5 with v_6 we have the graph G as



So, on replacing Row (v_5) ← Row (v_5) + Row (v_6); Col (v_5) ← Col (v_5) + Col (v_6) and on removing the row and column corresponding to v_6 the adjacency matrix becomes

$$A(G) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The underlying simple graph of G is given as

The adjacency matrix becomes

$$A(G_0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

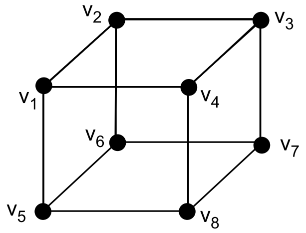
As the final graph is empty, the process terminates. The number of isolated points is the order of the matrix *i.e.*, two. So, the graph is not connected.

Example 16 *Draw the following graphs.*

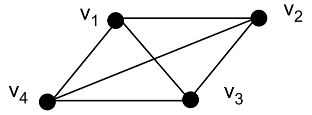
(i) *3 regular but not complete*
(ii) *3 regular and complete*
(iii) *4 regular but not complete*
(iv) *2 regular but not complete.*

Solution:

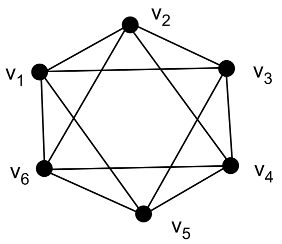
(*i*) In the graph given below, the degree of every vertex is 3 but for vertices v_1 and v_6 there is no edge. Hence, the graph is 3 regular but not complete.



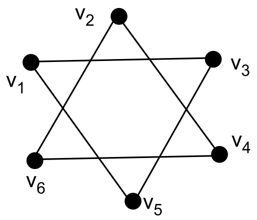
(*ii*) In the graph given below, the degree of every vertex is 3 and for any two vertices v_i and v_j there is an edge. Hence the graph is 3 regular and complete.



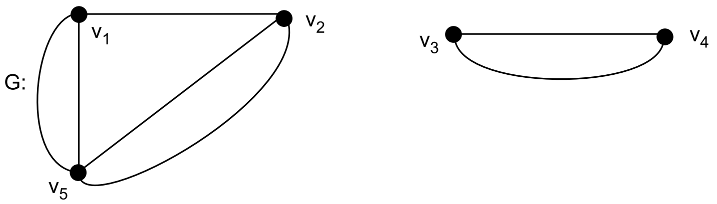
(*iii*) In the graph given below, the degree of every vertex is 4 and for the vertices v_2 and v_5 there exists no edge. Hence, the graph is 4 regular but not complete.



(*iv*) In the graph given below, the degree of every vertex is 2 and for the vertices v_2 and v_5 there exists no edge. Hence, the graph is 2 regular but not complete.



Example 17 *Find whether the graph given below is connected or not.*



Solution: The adjacency matrix A(G) of the above graph relative to the ordering v_1, v_2, v_3, v_4 and v_5 is given as

$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 0 & 0 \end{pmatrix}$$

Here, the number of vertices $(n) = 5$. Let $B = A + A^2 + A^3 + A^4$. Therefore, we get

$$A^2 = AA = \begin{pmatrix} 5 & 4 & 0 & 0 & 2 \\ 4 & 5 & 0 & 0 & 2 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 2 & 5 & 0 \\ 2 & 2 & 0 & 0 & 8 \end{pmatrix};\ A^3 = A^2 A = \begin{pmatrix} 8 & 9 & 0 & 0 & 18 \\ 9 & 8 & 0 & 0 & 18 \\ 0 & 0 & 4 & 10 & 0 \\ 0 & 0 & 10 & 9 & 0 \\ 18 & 18 & 0 & 0 & 8 \end{pmatrix}$$

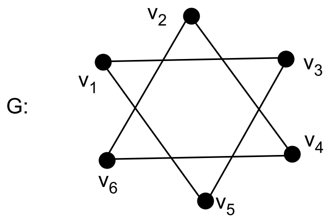
$$A^4 = A^3A = \begin{pmatrix} 45 & 44 & 0 & 0 & 34 \\ 44 & 45 & 0 & 0 & 34 \\ 0 & 0 & 20 & 18 & 0 \\ 0 & 0 & 18 & 29 & 0 \\ 34 & 34 & 0 & 0 & 72 \end{pmatrix}$$

Therefore, $$B = A + A^2 + A^3 + A^4$$

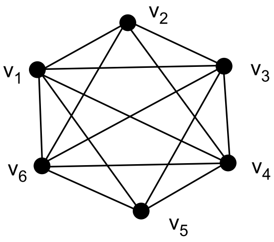
$$= \begin{pmatrix} 58 & 58 & 0 & 0 & 56 \\ 58 & 58 & 0 & 0 & 56 \\ 0 & 0 & 28 & 32 & 0 \\ 0 & 0 & 32 & 44 & 0 \\ 56 & 56 & 0 & 0 & 88 \end{pmatrix}$$

As some $b_{ij} = 0$ for $i \neq j$, the graph G is not connected.

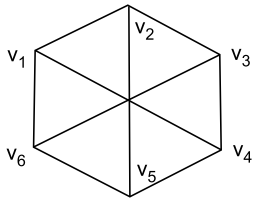
Example 18 *Find the complement graph of the following graph G, where*



Solution: On considering the above graph G, construct the complete graph with the same vertices as G. The graph is given below.

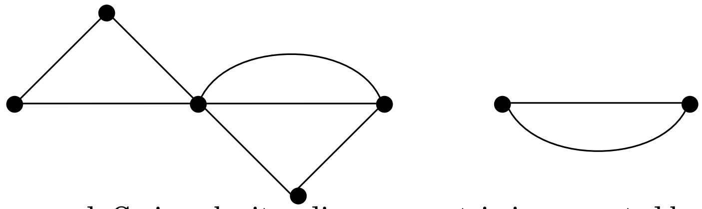


On deleting the edges of the graph G, the complement $\overline{G}$ of G is given below.



EXERCISES

1. Using fusion algorithm determine whether the graph given below is connected or not.

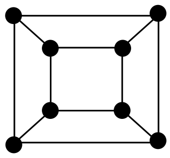


2. Show that the graph G given by its adjacency matrix is connected by using fusion algorithm.

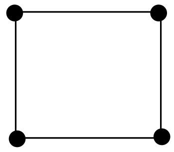
$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

3. Find the complement of the following graphs.

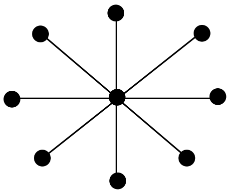
(a)



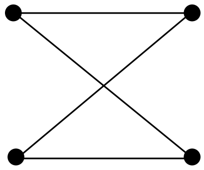
(b)



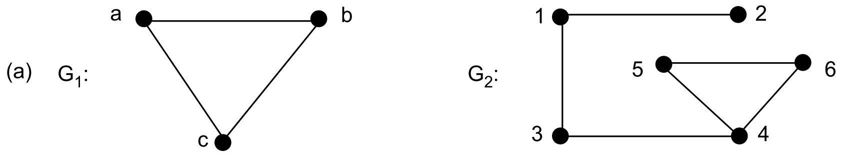
(c)

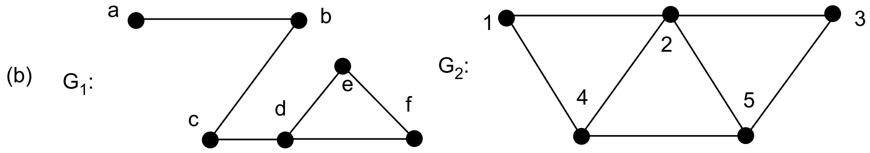


(d)



4. Find the product graph where G_1 and G_2 are given below.

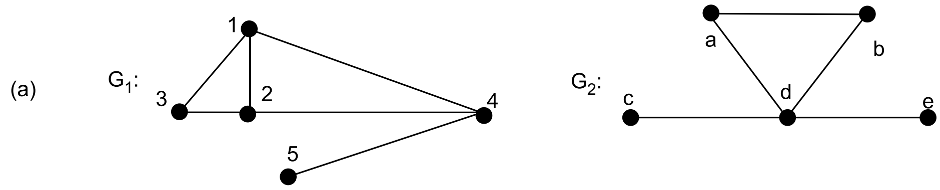


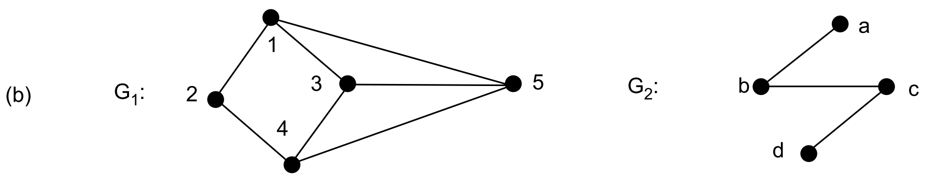


5. Construct a graph of order 5, whose vertices have degrees 1, 2, 2, 3 and 4. What is the size of this graph?

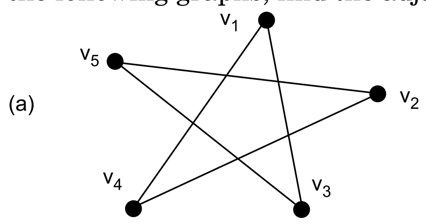
6. Construct a 3-regular graph G. Determine the complement of G. Show that $\overline{G}$ is also regular.

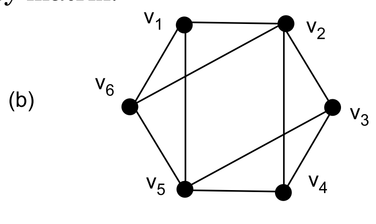
7. Write the graph which is the composition of the following graphs G_1 and G_2.

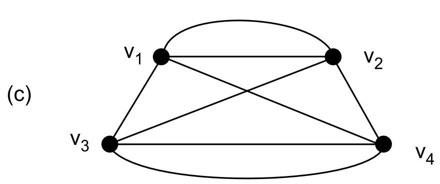


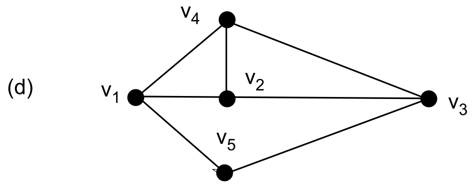


8. For the following graphs, find the adjacency matrix.

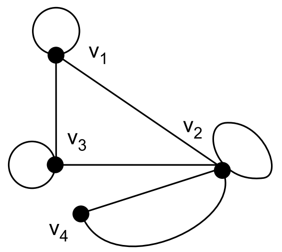




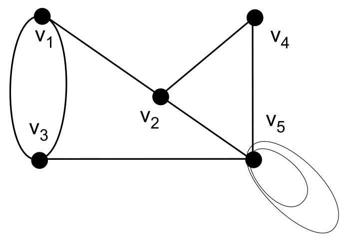




(e)

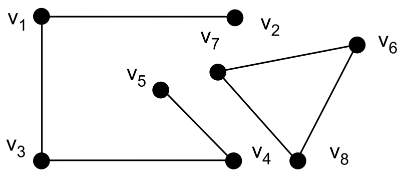


(f)

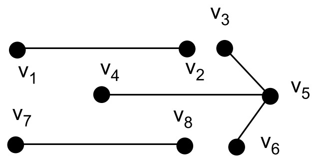


9. Find the path matrix of the following graphs.

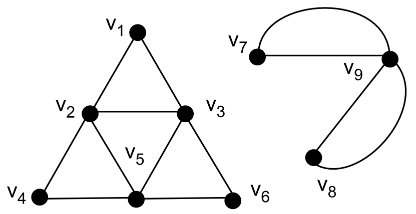
(a)



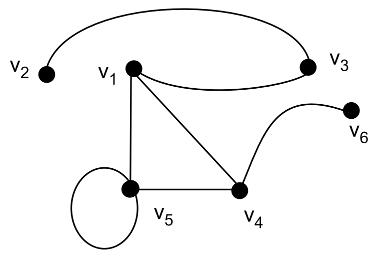
(b)



(c)

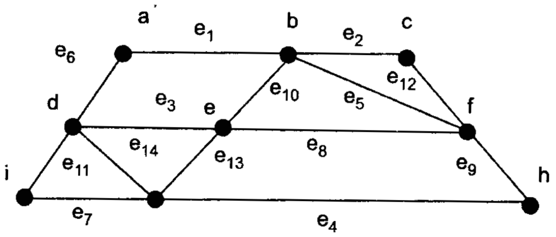


(d)

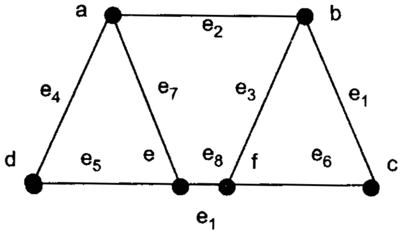


10. Write the incidence matrix of the following graphs.

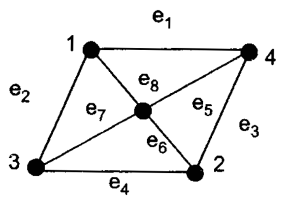
(a)



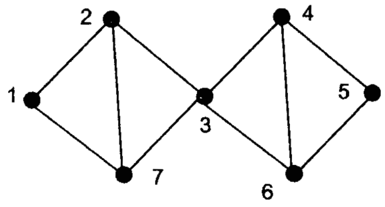
(b)



(c)



(d)



11. Let the graph G is given below. Find the followings.

(*a*) $G - V_1$; where $V_1 = \{1, 3, 5, 6, 7, 8\}$

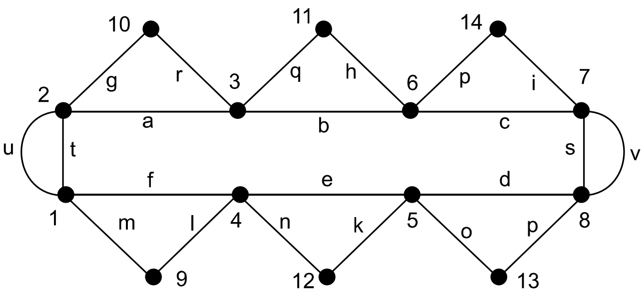
(*b*) $G - E_1$; where $E_1 = \{a, c, d, f, g, i, j, m, n, q, r, t\}$

(*c*) $G - V_2$; where $V_2 = \{1, 3, 5, 7, 9, 11, 13\}$

(*d*) $G - E_2$; where $E_2 = \{m, l, n, k, o, j, f, e, d\}$

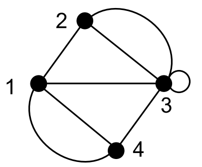
(*e*) G(U); where $U = \{1, 2, 3, 4\}$

(*f*) G(V); where $V = \{a, b, c, d, e, f\}$

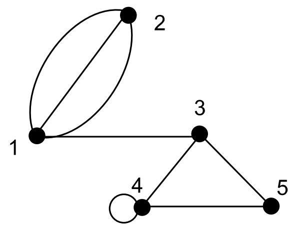


12. Write the union and intersection of the following graphs.

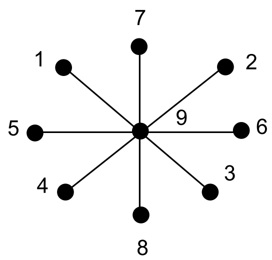
(a) G_1:



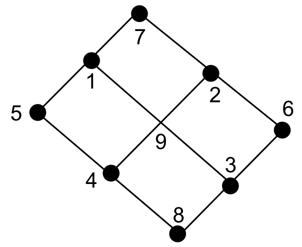
G_2:



(b) G_1:



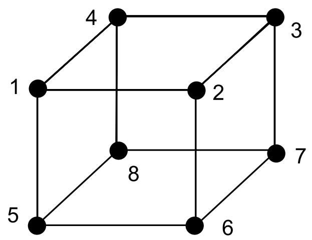
G_2:



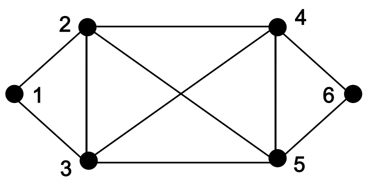
13. Let G be the set of all graphs. Show that the relation "is isomorphic" is an equivalence relation on the set G.

14. Find the degree of every vertex for the following graphs.

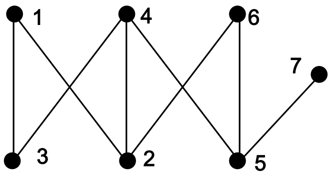
(a)



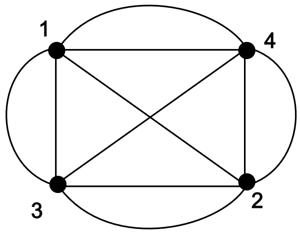
(b)



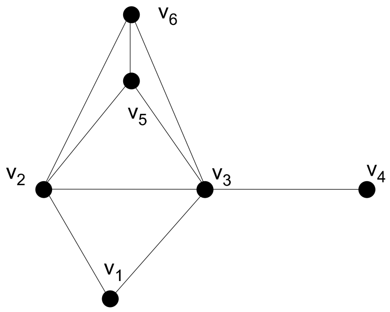
(c)



(d)



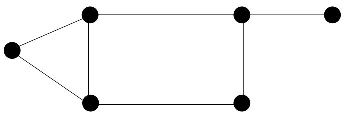
15. Determine the degrees of the vertices v_i ; I = 1, 2, 3, 4, 5 and 6 of the graph G shown below. Compute $\sum_i deg\,(v_i)$. Use this to determine the size of G.



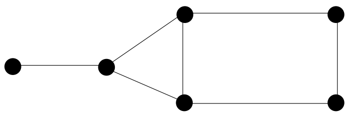
16. Determine which pairs of the graphs G_1, G_2 and G_3 are isomorphic.

G_1:

G_2:



G_3:



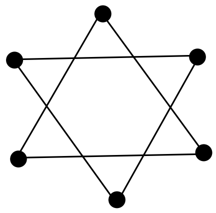
17. From the graphs given below identify

(*i*) Regular graphs

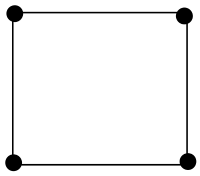
(*ii*) Complete graphs and

(*iii*) Neither regular nor complete graphs.

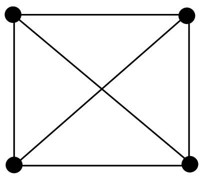
(a)



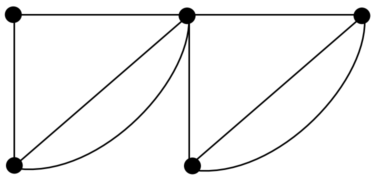
(b)



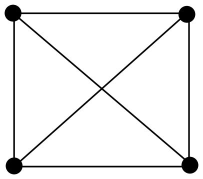
(c)



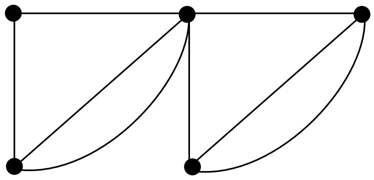
(d)



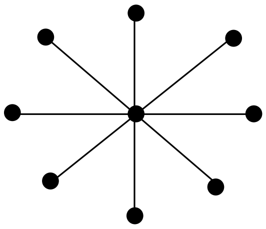
(c)



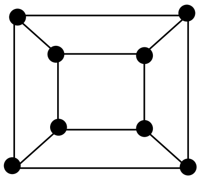
(d)



(e)

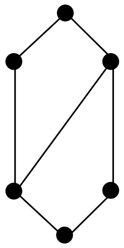


(f)

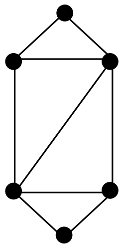


18. Determine whether the graphs G_1 and G_2 shown below are isomorphic.

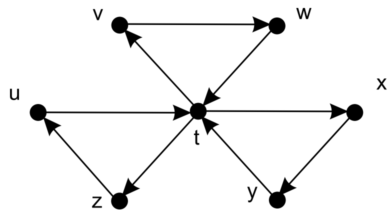
G_1:



G_2:



19. Determine whether the graph G shown below is strongly connected or weakly connected.



20. In the digraph G shown below, find the indegree and outdegree of every vertex.

G: