PROPOSITIONAL AND PREDICATE CALCULUS

October 21, 2020

Proposition Calculus

A declarative sentence that is either true or false is called "**PROPOSITION**"

Example

It rained yesterday.

"True" or "False" are called the truth vales of the proposition and are denoted by T and F respectively. values

A proposition that is true under all circumstances is called "**Tautology**"

Example

15 is divisible by 3.

A proposition that is false under all circumstances is called "**Contradiction**"

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Example

-3 is a natural number.

Definition

Two or more propositions can be combined using words like "and, "or", "iff", "if, then" etc. These are called **Logical Connectivities**.

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A proposition having one or more logical connectivities is called a **Compound Proposition**. Otherwise is called **Simple**/ **Atom**

Definition

Two propositions p and q are said to be **Equivalent** if when p is T, q is also T and when p is F, q is also F and conversely.

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Example

p: He was born in 1934

q: He'll be 60 years old in 1994

p and q are equivalent

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Example

p: He was born in 1934
q: He'll be 60 years old in 1994
p and q are equivalent

Example

p: x is a prime number
q: x is not divisible by 2
p and q are not equivalent, as x not divisible by 2 doesn't mean its prime

Definition

Let p be a proposition, we define **Negation** of p denoted by ¬**p** to be a proposition which is true when p is false and is false when p is true

$\neg$	p	$\neg p$
	T	F
	F	T

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$\neg$	p	$\neg p$
	T	F
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Example

If p is " monthly volume of sales is less than 20K", then negation p is " monthly volume of sales exceeds or equal to 20K"

Definition

Let p and q be two propositions. The **Disjunction** of two propositions is denoted by **p** $\vee$ **q(read as p or q)**

$\vee$	p	q	$p \vee q$
	T	T	T
	T	F	T
	F	T	T
	F	F	F

Definition

Let p and q be two propositions. The **Conjunction** of two propositions is denoted by **p $\wedge$ q(read as p and q)**

$\wedge$	p	q	$p \wedge q$
	T	T	T
	T	F	F
	F	T	F
	F	F	F

Definition

The **Conditional** statement is denoted by $\mathbf{p \rightarrow q}$**(read as if p then q)**

$\rightarrow$	p	q	$p \rightarrow q$
	T	T	T
	T	F	F
	F	T	T
	F	F	T

Note: 1) p is called the " first component" or "ANTECEDENT" and q is called the "second component" or "CONSEQUENT"

Note: 2) For the conditional $p \to q$,

(i) $q \to p$ is called "converse"

p	q	$p \to q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$	$\neg p \vee q$
T	T	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	F	T	T	T	T	T

Table: 1

Note: 2) For the conditional $p \to q$,

(i) $q \to p$ is called "converse"

(ii) $\neg p \to \neg q$ is called " inverse"

p	q	$p \to q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$	$\neg p \vee q$
T	T	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	F	T	T	T	T	T

Table: 1

Note: 2) For the conditional $p \to q$,

(i) $q \to p$ is called "converse"

(ii) $\neg p \to \neg q$ is called " inverse"

(iii) $\neg q \to \neg p$ is called "contrapositive"

p	q	$p \to q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$	$\neg p \vee q$
T	T	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	F	T	T	T	T	T

Table: 1

Observations

Note: 3) From Table 1 we make the following observations:

(i) $p \to q$ and $\neg q \to \neg p$ are logically equivalent

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(ii) Inverse and Converse are logically equivalent. i.e., $q \to p$ and $\neg p \to \neg q$ are logically equivalent

Observations

Note: 3) From Table 1 we make the following observations:

(i) $p \to q$ and $\neg q \to \neg p$ are logically equivalent

(ii) Inverse and Converse are logically equivalent. i.e., $q \to p$ and $\neg p \to \neg q$ are logically equivalent

(iii) $p \to q$ and $\neg p \vee q$ are logically equivalent

Problem 1

Question: There are two restaurants next to each other. One has a sign that says " Good food is not cheap". The other has a sign that says " Cheap food is not good". Are the signs saying the same thing?

Solution : Let A: Food is good

B: Food is cheap

We have to examine $A \rightarrow \neg B$ and $B \rightarrow \neg A$

A	B	$A \rightarrow \neg B$	$B \rightarrow \neg A$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

Inference: Both are saying the same thing.

Exercise 1

Question: John made two statements:

- **I love Lucy**

Given that John either told the truth or lied in both the cases, determine whether John really loves Lucy?

Exercise 1

Question: John made two statements:

- **I love Lucy**
- **If I love lucy, then I also love Vivian.**

Given that John either told the truth or lied in both the cases, determine whether John really loves Lucy?

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p : John loves lucy

q : Joh loves vivian

Definition

Let p and q be two propositions. The **Biconditional** is denoted by $p \leftrightarrow q$ read as "**p iff q**

$\leftrightarrow$	p	q	$p \leftrightarrow q$
	T	T	T
	T	F	~~T~~ F
	F	T	F
	F	F	T

Problem 2

Question: An island has 2 tribes of natives. Any native from the first tribe always tells the truth, while any native from the other tribe always lies. You arrive at island and ask a native if there is gold at the island. He answers "there is gold on the island iff I always tell the truth". Which tribe is he from? Is there gold on the island?

Solution : Let p: There is gold on the island

q: I always tell the truth

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Consider the following cases:

Case 1: If the person belongs to first tribe. Then q is true and the statement $p \leftrightarrow q$ is true. From the truth table above, p is also true. Therefore, "there is gold"

Thus there is gold on the island and the native could have been from either tribe.

Consider the following cases:

Case 1: If the person belongs to first tribe. Then q is true and the statement $p \leftrightarrow q$ is true. From the truth table above, p is also true. Therefore, "there is gold"

Case 2: If the person belongs to second tribe. Then q is false and the statement $p \leftrightarrow q$ must be false. From the truth table above, p is true. Therefore, "there is gold"

Thus there is gold on the island and the native could have been from either tribe.

Well formed formulas

A WFF is a formula generated using the following groups:

i) A statement variable is a WFF

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i) A statement variable is a WFF

ii) If A is WFF, then $\neg A$ is also a WFF

iii) If A and B are WFF's, $A \vee B$, $A \wedge B$, $A \to B$, $A \leftrightarrow B$ are also WFFs.

Well formed formulas

A WFF is a formula generated using the following groups:

i) A statement variable is a WFF

ii) If A is WFF, then $\neg A$ is also a WFF

iii) If A and B are WFF's, $A \vee B$, $A \wedge B$, $A \rightarrow B$, $A \leftrightarrow B$ are also WFFs.

iv) A string of symbols consisting of statement variables, connectivities and parenthesis is a WFF iff it can be obtained by finite many applications of the rules (i), (ii) and (iii)

Well formed formulas

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i) A statement variable is a WFF
ii) If A is WFF, then $\neg A$ is also a WFF
iii) If A and B are WFF's, $A \vee B$, $A \wedge B$, $A \rightarrow B$, $A \leftrightarrow B$ are also WFFs.
iv) A string of symbols consisting of statement variables, connectivities and parenthesis is a WFF iff it can be obtained by finite many applications of the rules (i), (ii) and (iii)

Example

(1) $p \wedge q$, $\neg(p \wedge q)$, $(\neg(p \rightarrow q)) \vee r$, $((p \rightarrow q) \rightarrow r$ are WFFs.
(2) $p \wedge q \rightarrow r$ is not a WFF as it can be $(p \wedge q) \rightarrow r$ or $p \wedge (q \rightarrow r)$

Equivalence of formulas

Definition

Let A and B be two statement formulas and $P_1, P_2, \cdots P_n$ denote all the variables occurring in A and B. If the truth value of A is same as that of B for each of 2^n possible set of assignments to the variables $P_1, P_2, \cdots P_n$, then A and B are said to be equivalent. We write as $A \Leftrightarrow B$.

Two statement formulas A and B are equivalent iff $A \leftrightarrow B$ is a Tautology.

Table of equivalence

(1) $\neg\neg p \Leftrightarrow q$

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(2) Commutative: (a) $p \vee q \Leftrightarrow q \vee p$
(b) $p \wedge q \Leftrightarrow q \wedge p$

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(2) Commutative: (a) $p \vee q \Leftrightarrow q \vee p$
(b) $p \wedge q \Leftrightarrow q \wedge p$

(3) Associative: (a) $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
(b) $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

Table of equivalence

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(2) Commutative: (a) $p \vee q \Leftrightarrow q \vee p$
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(b) $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

(4) Distributive: (a) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
(b) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

Table of equivalence

(1) $\neg\neg p \Leftrightarrow q$

(2) Commutative: (a) $p \vee q \Leftrightarrow q \vee p$
(b) $p \wedge q \Leftrightarrow q \wedge p$

(3) Associative: (a) $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
(b) $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

(4) Distributive: (a) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
(b) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

(5) Absorption: (a) $p \vee (p \wedge q) \Leftrightarrow p$
(b) $p \wedge (p \vee q) \Leftrightarrow p$

Table of equivalence

(1) $\neg\neg p \Leftrightarrow q$

(2) Commutative: (a) $p \vee q \Leftrightarrow q \vee p$
(b) $p \wedge q \Leftrightarrow q \wedge p$

(3) Associative: (a) $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
(b) $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

(4) Distributive: (a) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
(b) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

(5) Absorption: (a) $p \vee (p \wedge q) \Leftrightarrow p$
(b) $p \wedge (p \vee q) \Leftrightarrow p$

(6) Idempotent: (a) $(p \wedge p) \Leftrightarrow p$
(b) $(p \vee p) \Leftrightarrow p$

Table of equivalence

(7) (a) $p \wedge (\neg p) \Leftrightarrow F$
(b) $p \vee (\neg p) \Leftrightarrow T$

Table of equivalence

(7) (a) $p \wedge (\neg p) \Leftrightarrow F$
(b) $p \vee (\neg p) \Leftrightarrow T$

(8) (a) $p \vee F \Leftrightarrow p$
(b) $p \wedge F \Leftrightarrow F$

Table of equivalence

(7) (a) $p \wedge (\neg p) \Leftrightarrow F$
    (b) $p \vee (\neg p) \Leftrightarrow T$

(8) (a) $p \vee F \Leftrightarrow p$
    (b) $p \wedge F \Leftrightarrow F$

(9) (a) $p \vee T \Leftrightarrow T$
    (b) $p \wedge T \Leftrightarrow p$

Table of equivalence

(7) (a) $p \wedge (\neg p) \Leftrightarrow F$
(b) $p \vee (\neg p) \Leftrightarrow T$

(8) (a) $p \vee F \Leftrightarrow p$
(b) $p \wedge F \Leftrightarrow F$

(9) (a) $p \vee T \Leftrightarrow T$
(b) $p \wedge T \Leftrightarrow p$

(10) (a) $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$
(b) $\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$

Table of equivalence

(7) (a) $p \wedge (\neg p) \Leftrightarrow F$
(b) $p \vee (\neg p) \Leftrightarrow T$

(8) (a) $p \vee F \Leftrightarrow p$
(b) $p \wedge F \Leftrightarrow F$

(9) (a) $p \vee T \Leftrightarrow T$
(b) $p \wedge T \Leftrightarrow p$

(10) (a) $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$
(b) $\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$

(11) (a) $p \rightarrow q \Leftrightarrow \neg p \vee q$
(b) $\neg(p \rightarrow q) \Leftrightarrow p \wedge \neg q$

Table of equivalence

(7) (a) $p \wedge (\neg p) \Leftrightarrow F$
(b) $p \vee (\neg p) \Leftrightarrow T$

(8) (a) $p \vee F \Leftrightarrow p$
(b) $p \wedge F \Leftrightarrow F$

(9) (a) $p \vee T \Leftrightarrow T$
(b) $p \wedge T \Leftrightarrow p$

(10) (a) $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$
(b) $\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$

(11) (a) $p \rightarrow q \Leftrightarrow \neg p \vee q$
(b) $\neg(p \rightarrow q) \Leftrightarrow p \wedge \neg q$

(12) (a) $p \rightarrow q \Leftrightarrow (\neg q \rightarrow \neg p)$
(b) $q \rightarrow p \Leftrightarrow (\neg p \rightarrow \neg q)$

Problem 4

Question: Show that $(\neg p \wedge (\neg q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) \Leftrightarrow r$

$$\begin{aligned}
(\neg p \wedge (\neg q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) &\Leftrightarrow (\neg p \wedge (\neg q \wedge r)) \vee (r \wedge (q \vee p)) \\
&\Leftrightarrow ((\neg p \wedge \neg q) \wedge r) \vee (r \wedge (p \vee q)) \\
&\Leftrightarrow (\neg(p \vee q) \wedge r) \vee (r \wedge (p \vee q)) \\
&\Leftrightarrow r \wedge [(p \vee q) \vee \neg(p \vee q)] \\
&\Leftrightarrow r \wedge T \\
&\Leftrightarrow r
\end{aligned}$$

Excerise 5

Question: Show that $p \rightarrow (q \rightarrow r) \Leftrightarrow p \rightarrow (\neg q \vee r) \Leftrightarrow (p \wedge q) \rightarrow r$

$$\begin{aligned} p \rightarrow (q \rightarrow r) &\Leftrightarrow p \rightarrow (\neg q \vee r) \\ &\Leftrightarrow \neg p \vee (\neg q \vee r) \\ &\Leftrightarrow (\neg p \vee \neg q) \vee r \\ &\Leftrightarrow \neg (p \wedge q) \vee r \\ &\Leftrightarrow (p \wedge q) \rightarrow r \end{aligned}$$

Problem 4

Question: Show that
$((p \vee q) \wedge \neg[(\neg p) \wedge (\neg q \vee \neg r)]) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$ **is a tautology**.
Solution: $[(p \vee q) \wedge \neg((\neg p) \wedge (\neg q \vee \neg r))] \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$
$= [(p \vee q) \wedge \neg(\neg p \wedge (\neg q \vee \neg r))] \vee (\neg p \wedge (\neg q \vee \neg r))$
$= [(p \vee q) \wedge \neg\neg p \vee (q \wedge r)] \vee (\neg p \wedge \neg(q \wedge r))$
$= [(p \vee q) \wedge (p \vee (q \wedge r))] \vee \neg(p \vee (q \wedge r))$
$= [(p \vee q) \wedge (p \vee q) \wedge (p \vee r)] \vee \neg(p \vee (q \wedge r))$
$= [(p \vee q) \wedge (p \vee r)] \vee \neg(p \vee (q \wedge r))$
$= [(p \vee (q \wedge r)] \vee \neg(p \vee (q \wedge r))$
$= T$

Excerise 6

Question: Show that $q \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$ **is a tautology**

Tautological Implications

Definition

A is said to tautologically imply to statement B if $A \to B$ is a tautology. In this case, we write $A \Rightarrow B$ (read as A implies B)

Table of Tautological Implications

(1) $p \wedge q \implies p$

$p \wedge q \implies q$

Table of Tautological Implications

(1) $p \wedge q \implies p$

$p \wedge q \implies q$

(2) $p \implies p \vee q$

$q \implies p \vee q$

Table of Tautological Implications

(1) $p \wedge q \implies p$

$p \wedge q \implies q$

(2) $p \implies p \vee q$

$q \implies p \vee q$

(3) $\neg p \implies p \to q$

Table of Tautological Implications

(1) $p \wedge q \implies p$

$p \wedge q \implies q$

(2) $p \implies p \vee q$

$q \implies p \vee q$

(3) $\neg p \implies p \to q$

(4) $q \implies p \to q$

Table of Tautological Implications

(1) $p \wedge q \implies p$

$p \wedge q \implies q$

(2) $p \implies p \vee q$

$q \implies p \vee q$

(3) $\neg p \implies p \to q$

(4) $q \implies p \to q$

(5) $\neg(p \to q) \implies p$

Table of Tautological Implications

(1) $p \wedge q \implies p$
$p \wedge q \implies q$

(2) $p \implies p \vee q$
$q \implies p \vee q$

(3) $\neg p \implies p \to q$

(4) $q \implies p \to q$

(5) $\neg(p \to q) \implies p$

(6) $\neg(p \to q) \implies \neg q$

Table of Tautological Implications

(1) $p \wedge q \implies p$

$p \wedge q \implies q$

(2) $p \implies p \vee q$

$q \implies p \vee q$

(3) $\neg p \implies p \to q$

(4) $q \implies p \to q$

(5) $\neg(p \to q) \implies p$

(6) $\neg(p \to q) \implies \neg q$

(7) $p \wedge (p \to q) \implies q$

Table of Tautological Implications

(8) $\neg q \wedge (p \to q) \implies \neg p$

Table of Tautological Implications

(8) $\neg q \wedge (p \rightarrow q) \implies \neg p$

(9) $\neg p \wedge (p \vee q) \implies q$

Table of Tautological Implications

(8) $\neg q \wedge (p \to q) \implies \neg p$

(9) $\neg p \wedge (p \vee q) \implies q$

(10) $(p \to q) \wedge (q \to r) \implies p \to r$

Table of Tautological Implications

(8) $\neg q \wedge (p \to q) \implies \neg p$

(9) $\neg p \wedge (p \vee q) \implies q$

(10) $(p \to q) \wedge (q \to r) \implies p \to r$

(11) $(p \vee q) \wedge (p \to r) \wedge (q \to r) \implies r$

Problem 7

Question: Show that $\neg q \wedge (p \to q) \implies \neg p$

Solution: Suppose $\neg q \wedge (p \to q)$ is true.

$\neg q$ is true and $p \to q$ is true

q is false and $p \to q$ is true

$\implies p$ is false

$\implies \neg p$ is true

$\therefore \neg q \wedge (p \to q) \implies \neg p$

Remark

To show that $A \implies B$, we can assume B is false and show that A is false. So the above problem can also be analyzed as follows: Consider again $\neg p$ is false, $\implies p$ is true. If q is true, $\neg q$ is false and its understood that $\neg q \wedge (p \to q)$ is false. If q is false, $\neg q$ is true and $p \to q$ is false. Again $\neg q \wedge (p \to q)$ is false.

Problem 8

Question: Show that $\neg(p \rightarrow q) \implies \neg q$

Solution: We say that $A \implies b$ if $A \rightarrow B$ is true in all conditions

p	q	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg q$	$\neg(p \rightarrow q) \rightarrow \neg q$
T	T	T	F	F	T
T	F	F	T	T	T
F	T	T	F	F	T
F	F	T	F	T	T

Exercise 9

Question: Show that $(p \vee q) \wedge (p \to r) \wedge (q \to r) \implies r$

Exercise 10

Question: Prove that

(i) $\neg p \implies p \to q$

Exercise 10

Question: Prove that

(i) $\neg p \implies p \to q$

(ii) $p \wedge (p \to q) \implies q$

Exercise 10

Question: Prove that

(i) $\neg p \implies p \to q$

(ii) $p \wedge (p \to q) \implies q$

(iii) $p \wedge q \implies p$