1. At an art exhibition there are 12 paintings of which 10 are original. A visitor selects a painting at random & before he decides to buy, he asks the opinion of an expert about the authenticity of the painting. The expert is right 9 out of 10 cases on an average. (i) Given that the expert decides that the painting is authentic, what is the probability that this is really the case. (ii) If the expert decides that the painting is a copy, then the visitor returns it & chooses another one. What is the P that his second choice is actually the original?

Sol: A: Expert decides painting is authentic
B_1: Painting is authentic
B_2: Painting is Fake

$P(A|B_1) = \frac{9}{10}$

$P(A|B_2) = \frac{1}{10}$

~~$P(A) = P(A$~~

$P(B_1) = \frac{10}{12}$

$P(B_2) = \frac{2}{12}$

(i) $P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2)$

$= \frac{9}{10} \times \frac{10}{12} + \frac{1}{10} \times \frac{2}{12}$

$= \frac{90+2}{120} = \frac{92}{120} = \frac{23}{30}$

$$P(B_1|A) = \frac{P(A|B_1)P(B_1)}{P(A)}$$

$$= \frac{9/10 \times 10/12}{23/30}$$

$$= \frac{\cancel{9}^{3}}{\cancel{10}} \times \frac{\cancel{10}}{\cancel{12}_{\cancel{4}_2}} \times \frac{\cancel{30}^{15}}{23}$$

$$= \frac{45}{46}$$

$$\boxed{P(B_1|A) = \frac{45}{46}}$$

(ii)

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2. A, B, C play a game and the chances of their winning it in an attempt are 2/3, 1/2 & 1/4 resp. A has the first chance, followed by B & then C. This cycle is repeated till one of them wins the game. Find their respective chances of winning the game.

Sol: A: A wins in an attempt
B: B wins in an attempt
C: C wins in an attempt

$$P(A) = \frac{2}{3}$$

$$P(B) = \frac{1}{2}$$

$$P(C) = \frac{1}{4}$$

AW: A wins
BW: B wins
CW: C wins

$$P(AW) = \frac{2}{3} + \frac{1}{3}\times\frac{1}{2}\times\frac{3}{4}\times\frac{2}{3} + \frac{1}{3}\times\frac{1}{2}\times\frac{3}{4}\times\frac{1}{3}\times\frac{1}{2}\times\frac{3}{4}\times\frac{2}{3} + \dots$$

$$= \frac{2}{3}\left[1 + \frac{1}{3}\times\frac{1}{2}\times\frac{3}{4} + \frac{1}{3}\times\frac{1}{2}\times\frac{3}{4}\times\frac{1}{3}\times\frac{1}{2}\times\frac{3}{4} + \dots\right]$$

$$= \frac{2}{3}\left(\frac{1}{1 - \frac{1}{\cancel{3}}\times\frac{1}{2}\times\frac{\cancel{3}}{4}}\right)$$

$$= \frac{2}{3}\left(\frac{1}{1 - 1/8}\right) = \cancel{\frac{2}{3}\left(\frac{1}{5/6}\right)} = \frac{2}{3} \times \frac{1}{7/8}$$

$$\cancel{= \frac{2}{3} \times \frac{\cancel{6}^{2}}{5} = \frac{4}{5}}$$

$$= \frac{2 \times 8}{3 \times 7} = \frac{16}{21}$$

$$\boxed{P(AW) = \frac{16}{21}}$$

$$P(BW) = \frac{1}{3}\times\frac{1}{2} + \frac{1}{3}\times\frac{1}{2}\times\frac{3}{4}\times\frac{1}{3}\times\frac{1}{2} + \dots$$

$$= \frac{1}{3}\times\frac{1}{2}\left[1 + \frac{1}{3}\times\frac{1}{2}\times\frac{3}{4} + \frac{1}{3}\times\frac{1}{2}\times\frac{3}{4}\times\frac{1}{3}\times\frac{1}{2}\times\frac{3}{4} + \dots\right]$$

$$= \frac{1}{3}\times\frac{1}{2}\left[\frac{1}{1-\frac{1}{\cancel{3}}\times\frac{1}{2}\times\frac{\cancel{3}}{4}}\right]$$

$$= \frac{1}{3}\times\frac{1}{2}\left(\frac{1}{1-\frac{1}{8}}\right)$$

$$= \frac{1}{3}\times\frac{1}{2}\left(\frac{1}{7/8}\right)$$

$$= \frac{1}{3}\times\frac{1}{\cancel{2}}\times\frac{\cancel{8}\,4}{7}$$

$$\boxed{P(BW) = \frac{4}{21}}$$

$$P(CW) = \frac{1}{3}\times\frac{1}{2}\times\frac{1}{4} + \frac{1}{3}\times\frac{1}{2}\times\frac{3}{4}\times\frac{1}{3}\times\frac{1}{2}\times\frac{1}{4} + \dots$$

$$= \frac{1}{3}\times\frac{1}{2}\times\frac{1}{4}\left[1 + \frac{1}{3}\times\frac{1}{2}\times\frac{3}{4} + \dots\right]$$

$$= \frac{1}{3}\times\frac{1}{2}\times\frac{1}{4}\left[\frac{1}{1-\frac{1}{\cancel{3}}\times\frac{1}{2}\times\frac{\cancel{3}}{4}}\right]$$

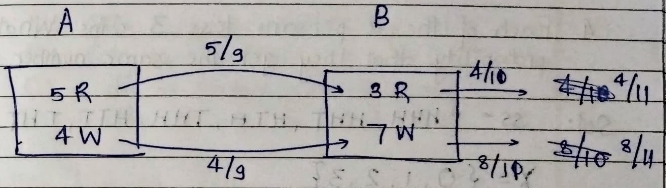
$$= \frac{1}{3}\times\frac{1}{2}\times\frac{1}{4}\left[\frac{1}{7/8}\right]$$

$$= \frac{1}{3}\times\frac{1}{\cancel{2}}\times\frac{1}{\cancel{4}}\times\frac{\cancel{8}}{7}$$

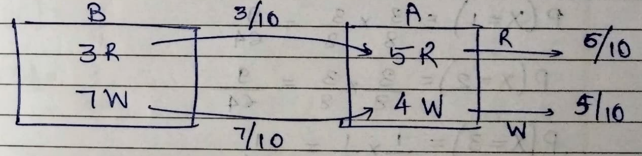
$$\boxed{P(CW) = \frac{1}{21}}$$

3. Box A contains 5 red and 4 white balls. Box B contains 3 red and 7 white balls. A box is selected at random. A ball is drawn & put into the other. Then a ball is drawn from that box. Find the P that both are of same colour.

Sol:



~~$\text{Req}^d = \frac{1}{2}\left[\frac{5}{9}\times\frac{4}{10}\right.$~~



$$\text{Req}^d = \frac{1}{2}\left[\frac{5}{9}\times\frac{4}{11} + \frac{3}{10}\times\frac{6}{10} + \frac{4}{9}\times\frac{8}{11} + \frac{7}{10}\times\frac{5}{10}\right]$$

~~$= \frac{1}{2}\left[\frac{20+32}{99} + \frac{15+28}{100}\right]$~~ $\rightarrow$ ~~$\frac{1}{2}\left[\frac{20+18}{99}\right.$~~ $\rightarrow \frac{1}{2}\left[\frac{20+32}{99} + \frac{18+35}{100}\right]$

$$= \frac{1}{2}\left[\frac{52}{99} + \frac{53}{100}\right] = \frac{1}{2}\left[\frac{5200+5247}{9900}\right]$$

~~$= \frac{1}{2}\left[\frac{9457}{9900}\right]$~~ $= \frac{1}{2}\left(\frac{10447}{9900}\right)$

~~5. In a coin tossing experiment, if the coin shows head, one die is thrown and the result is recorded. But if the coins show tail, two dice are thrown and the sum of the numbers is recorded. What is the probability that recorded number is 2?~~

4. Each of the 2 persons toss 3 coins. What is the probability that they get the same number of heads.

Sol: SS = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

$X = \{0, 1, 2, 3\}$

$P(X=0) = \frac{1}{8} \times \frac{1}{8} = \frac{1}{64}$

$P(X=1) = \frac{3}{8} \times \frac{3}{8} = \frac{9}{64}$

$P(X=2) = \frac{3}{8} \times \frac{3}{8} = \frac{9}{64}$

$P(X=3) = \frac{1}{8} \times \frac{1}{8} = \frac{1}{64}$

$P(H) = \frac{1}{64} + \frac{9}{64} + \frac{9}{64} + \frac{1}{64}$

$= \frac{20}{64}$

$\boxed{P(H) = \frac{5}{16}}$

5.

Sol: A: Coin shows heads
B: Coin shows tails
C: 1 die is thrown and 2 shows
D: 2 dies thrown & 2 shows

$P(A) = \frac{1}{2}$

$P(B) = \frac{1}{2}$

$P(C) = \frac{1}{6}$

$P(D) = \frac{1}{36}$

Case 1: ~~P(A~~ Coin shows head and die shows 2

$P(A) \times P(C)$

$\Rightarrow \frac{1}{2} \times \frac{1}{6}$

$\Rightarrow \frac{1}{12}$

Case 2: Coin shows tails & 2 dice sum shows 2

$\Rightarrow P(B) \times P(D)$

$\Rightarrow \frac{1}{2} \times \frac{1}{36} = \frac{1}{72}$

$$Req^d = Case1 + Case\ 2$$

$$= \frac{1}{12} + \frac{1}{72}$$

$$= \frac{6+1}{72}$$

$$\boxed{= \frac{7}{72}}$$

6.

Sol: X : X becomes manager
Y : Y becomes manager
Z : Z becomes manager

A : Bonus scheme is introduced

$P(X) = \frac{4}{9}$

$P(Y) = \frac{2}{9}$

$P(Z) = \frac{3}{9}$

$P(A|X) = \frac{3}{10}$

$P(A|Y) = \frac{5}{10}$

$P(A|Z) = \frac{8}{10}$

$$P(X|A) = \frac{P(A|X)P(X)}{P(A)}$$

$$P(A) = P(A|X)P(X) + P(A|Y)P(Y) + P(A|Z)P(Z)$$

$$= \frac{3}{10} \times \frac{4}{9} + \frac{5}{10} \times \frac{2}{9} + \frac{8}{10} \times \frac{3}{9}$$

$$= \frac{12}{90} + \frac{10}{90} + \frac{24}{90}$$

$$= \frac{46}{90}$$

$$= \frac{23}{45}$$

$$P(X|A) = \frac{3/10 \times 4/9}{23/\overset{}{45}_{2}}$$

$$= \frac{3}{\cancel{10}_{2}} \times \frac{\cancel{4}}{\cancel{9}} \times \frac{\cancel{45}^{5}}{23}$$

$$\boxed{P(X|A) = \frac{6}{23}}$$

7.

Sol: $f(x) \begin{cases} 4x-4x^3, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$

$$E(X) = \int_0^1 x(4x-4x^3)\,dx$$

$$= \int_0^1 (4x^2-4x^4)\,dx$$

$$= 4\int_0^1 \cancel{4}x^2 dx - 4\int_0^1 x^4\,dx$$

$$= 4\left[\left[\frac{x^3}{3}\right]_0^1 - \cancel{4}\left[\frac{x^5}{5}\right]_0^1\right]$$

$$= 4\left[\left(\frac{1}{3}-\frac{0}{3}\right) - \left(\frac{1}{5}-\frac{0}{5}\right)\right]$$

$$= 4\left[\frac{1}{3}-\frac{1}{5}\right]$$

$$= 4\left[\frac{5-3}{15}\right]$$

$$= 4\left[\frac{2}{15}\right]$$

$$\boxed{E(x) = \frac{8}{15}}$$

$$\boxed{E(X) = 0.5333}$$

$$\sigma^2 = \int_0^1 (x-\mu)^2 f(x)\,dx$$

~~$= \int_0^1 (x-0.5333)^2(4x-4x^3)\,dx$~~

~~$= \int_0^1 (x^2-1.0666x+0.2844)(4x-4x^3)\,dx$~~

~~$= \int_0^1 4x^3-4.2664x^2+1.1376x-4x^5+4.2664x^4 -1.1376x^3\,dx$~~

~~$= \int_0^1 -4x^5+4.2$~~

$$= \int_0^1 (x-0.5333)^2(4x-4x^3)\,dx$$

$$= 4\int_0^1 (x-0.5333)^2(x-x^3)\,dx$$

$$= 4\int_0^1 (x^2-1.0666x+0.2844)(x-x^3)\,dx$$

$$= 4\int_0^1 (x^3-1.0666x^2+0.2844x-x^5+1.0666x^4 - 0.2844x^3)\,dx$$

$$= 4\int_0^1 (-x^5+1.0666x^4+0.7156x^3-1.0666x^2 + 0.2844x)\,dx$$

$$= 4\left[\frac{-x^6}{6}+\frac{1.0666x^5}{5}+\frac{0.7156x^4}{4}-\frac{1.05666x^3}{3}+\frac{0.2844x^2}{2}\right]_0^1$$

$$= 4\left[\left(\left(\frac{-1}{6}+\frac{0}{6}\right)+\left(\frac{1.0666\cancel{x}}{5}-\frac{0}{5}\right)+\left(\frac{0.7156}{4}-\frac{0}{4}\right)\right.\right.$$

$$\left.-\cancel{8}\left(\frac{1.0666}{3}-0\right)+\left(\frac{0.2844}{2}-0\right)\right]$$

$$= 4\left(\frac{-1}{6}+\frac{1.0666}{5}+\frac{0.7156}{4}-\frac{1.0666}{3}+\frac{0.2844}{2}\right]$$

$$= \cancel{4}\left(\frac{-10+12.7992+10.734-21.332+8.532}{\cancel{60}\ 15}\right]$$

$$= \cancel{4}\ \frac{0.7332}{15}$$

$$\boxed{\sigma^2 = 0.04888}$$

8 (a) If the probability that A solves a problem is $\frac{1}{2}$ and that of B is $\frac{3}{4}$ and if they aim at solving a problem independently. What is the probability that the problem is solved?

Sol: A : Problem is solved by A

B : Problem is solved by B

$P(A) = \frac{1}{2}$

$P(\bar{A}) = \frac{1}{2}$

$P(B) = \frac{3}{4}$

$P(\bar{B}) = \frac{1}{4}$

The probability of problem being not solved by both A & B

$= \frac{1}{2}\times\frac{1}{4}$

$P(\bar{S}) = \frac{1}{8}$

$\therefore$ The probability of problem being solved $= 1 - P(\bar{S})$

$P(S) = 1 - \frac{1}{8}$

$$\boxed{P(S) = \frac{7}{8}}$$

(b) Let G be a simple graph with 6 vertices. The degree of 6 vertices are (2,3,3,3,4,5). Then the number of edges of graph G

Ans: We know that sum of degrees of vertices of graph G is equal to twice the no. of edges

Let E be the no of edges

$\therefore$ Sum of degrees $= 2\times E$

$2+3+3+3+4+5 = 2\times E$

$20 = 2\times E$

$$\boxed{E = 10}$$

(c) The graph in which, there is a walk which includes every vertex of the graph exactly once is known as

Ans: Hemiltonian Graph

(d) Generators of the group $(G,\cdot)$, where $G=\{1,-1,i,-i\}$ is

Ans: The generators are i and $-i$.

(e) Write all the subgroups of $(G,\cdot)$, where $G=\{1,-1,i,-i\}$

Sol: (i) Trivial Subgroup: $\{1\} = H_1$

~~H_2~~

(ii) Subgroups generated by -1:

$H_2 = \{1,-1\}$

(iii) Subgroups generated by i:

$H_3 = \{1,i,-1,-i\}$

9. For $a,b \in Z$ define the binary operation $*$ as $a|b$ to mean that a divides b. Is $(Z,*)$ a group? Also verify if $(R,*)$ is a group under same operation.

Sol: To check if $(Z,*)$ is a group:

(i) Closure law: a/b where $a,b \in Z$

$2/3 = 0.66 \notin Z$

∴ Closure law does not hold good

∴ $(Z,*)$ is not a group

Now, to check if $(R,*)$ is a group

(i) Closure law: a/b where $a,b \in Z$

$2/3 = 0.66 \in R$

$5/4 = 1.25 \in R$

$5/2 = 2.5 \in R$

∴ Closure law holds

(ii) Associative law: Let $a,b,c \in R$

$$\text{LHS} = (a*b)*c$$
$$= (a/b)/c$$
$$= \frac{a}{bc}$$

$$RHS = a*(b*c)$$
$$= a/(b/c)$$
$$= \frac{ac}{b}$$

$LHS \neq RHS$

∴ Associative law doesn't hold good.

∴ $(R,*)$ is not a group.

10. Check which of the following Binary operations is a group under Z, Q, N and R.

(1) $a*b = (a-b)$

① For Z

Sol: (i) Closure law: Let $a, b \in Z$

$$a*b = a-b$$
$$= 7-3$$
$$= 4 \in Z$$

$$= 4-6$$
$$= -2 \in Z$$

∴ Closure law holds

(ii) Associative law: Let $a, b, c \in Z$

LHS : $(a*b)*c$

$$\Rightarrow (a-b)-c$$
$$\Rightarrow a-b-c$$

RHS : $a*(b*c)$

$$\Rightarrow a-(b-c)$$
$$\Rightarrow a-b+c$$

$LHS \neq RHS$

∴ Associative law doesn't hold good

∴ $a*b = (a-b)$ under Z is not a group

② For Q

(i) Closure law : Let $a, b \in Z$

$$a*b = (a-b)$$
$$= 7-10$$
$$= -3 \in Q$$

$$= 9-2$$
$$= 7 \in Q$$

∴ Closure law holds good

(ii) Associative law: Let $a,b,c \in Q$

LHS : $(a*b)*c$

$\Rightarrow (a-b)-c$

$\Rightarrow a-b-c$

RHS : $a*(b*c)$

$\Rightarrow a-(b-c)$

$\Rightarrow a-b+c$

LHS $\neq$ RHS

Associative law doesn't hold

$\therefore$ $a*b = (a-b)$ is not a group under Q.

Similarly $a*b = a-b$ won't be a groupo under N & R.

(ii) $a*b = (a^2+b)$

Sol: ① Under Z

(i) Closure law : Let $a,b \in Z$

$(5)^2+2 = 27 \in Z$

$(2)^2+3 = 7 \in Z$

$\therefore$ Closure law holds

(ii) Associative law : Let $a,b,c \in Z$

LHS : $(a*b)*c$

$\Rightarrow (a^2+b)*c$

$\Rightarrow (a^2+b)^2+c$

$\Rightarrow ((a^2)^2+2a^2b+b^2)+c$

$\Rightarrow a^4+2a^2b+b^2+c$

RHS : $a*(b*c)$

$\Rightarrow a*(b^2+c)$

$\Rightarrow a^2*(b^2+c)$

$\Rightarrow a^2+b^2+c$

$\therefore$ Associative doesn't hold

$\therefore$ a^2+b is not a group under Z

② Under Q

(i) Closure law : Let $a,b \in Q$

$\left(\frac{1}{2}\right)^2+3 = \frac{7}{2} \in Q$

$(2)^2+\frac{1}{2} = \frac{5}{2} \in Q$

$\therefore$ Closure law holds good

Associative law will not hold again ∴

∴ a^2+b won't be a group under Z, Q, N or R

(iii) $a*b = (ab)/2$

Sol: ① Under Z

(i) Closure law : $a, b \in Z$

$\cancel{4}$ $\frac{\cancel{4}^{2}\times 5}{\cancel{2}} = 10 \in Z$

$\frac{5\times \cancel{10}^{5}}{\cancel{2}} = 25 \in Z$

~~∴ Closure law holds~~ $\frac{5\times 3}{2} = 7.5 \notin Z$

~~(ii) Associative law : Let a, b, c ∈ Z~~

~~LHS : (a * b) * c~~

~~⇒ (ab/2) * c~~

~~⇒ ((ab/2) * c)/2~~

~~⇒ abc~~

∴ Closure law doesn't hold good

∴ ab/2 isn't a group under Z

② Under Q

(i) Closure law : $a, b \in Q$

$\frac{4\times 9}{2} = 18 \in Q$

$\frac{1\times 7}{2} = \frac{7}{2} \in Q$

∴ Closure law holds good

(ii) Associative law : Let $a, b, c \in Q$

LHS : $(a*b)*c$

$\Rightarrow \frac{ab}{2} * c$

$\Rightarrow \frac{ab/\cancel{2}\times c}{\cancel{2}}$

$\Rightarrow abc$

RHS : ~~(a*b)*~~ $a*(b*c)$

$\Rightarrow a * \frac{bc}{2}$

$\Rightarrow \frac{abc/\cancel{2}}{\cancel{2}}$

$\Rightarrow abc$

LHS = RHS

∴ ~~(ab)/2 :~~ Associative law holds good

(iii) Identity law : $a \in Z$

$\Rightarrow a * e = a$

$\Rightarrow \frac{ae}{2} = a$

$\Rightarrow \cancel{a}e = 2\cancel{a}$

$\Rightarrow e = 2 \in Q$

∴ Identity law holds good

(iv) Inverse law : $a * a^{-1} = e$

$\Rightarrow \frac{aa^{-1}}{2} = e$

$\Rightarrow aa^{-1} = 2e$

$\Rightarrow a^{-1} = \frac{2 \times 2}{a}$

$\therefore a^{-1} = \frac{4}{a} \in Q$

∴ Inverse law holds good

∴ (ab)/2 is a group under Q

③ Under N

(i) Closure law : $a, b \in N$

$\frac{4 \times \cancel{2}}{\cancel{2}} = 4 \in N$

$\frac{3 \times 5}{2} = \frac{15}{2} \notin N$

∴ Closure law doesn't hold good

⇒ (ab)/2 isn't a group under N

④ Under R

(i) Closure law : $a, b \in R$

$\frac{3 \times 7}{2} = \frac{21}{2} \in R$

$\frac{3 \times 11}{2} = \frac{33}{2} \in R$

∴ Closure law holds good

(ii) Associative law : Let $a, b, c \in R$

LHS : $(a * b) * c$

$\Rightarrow \left(\frac{ab}{2}\right) * c$

$\Rightarrow \frac{ab/\cancel{2}\ c}{\cancel{2}}$

$\Rightarrow abc$

RHS : $a*(b*c)$

$\Rightarrow a*\frac{bc}{2}$

$\Rightarrow \frac{abc/\cancel{2}}{\cancel{2}}$

$\Rightarrow abc$

LHS = RHS

∴ Associative law holds good

(iii) Identity law : $a*e=a$

$\Rightarrow \frac{\cancel{a}e}{2} = \cancel{a}$

$\Rightarrow e=2 \in \mathbb{R}$

∴ Identity law holds good

(iv) Inverse law : $a*a^{-1}=e$

$\Rightarrow \cancel{\Rightarrow} \frac{aa^{-1}}{2} = e$

$\Rightarrow aa^{-1}=4$

$\Rightarrow a^{-1}=\frac{4}{a} \in \mathbb{R}$

∴ Inverse law holds good

∴ $(a*b) = (ab)/2$ is a group under $\mathbb{R}$.