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$$\underbrace{Y = X\beta + \varepsilon}$$ $\qquad E[\varepsilon | X] = 0$

$$E[\hat{\beta}] = \beta$$

$$E[\hat{\beta}] = (X'X)^{-1}X'Y$$

$$= (X'X)^{-1}X'(X\beta + \varepsilon)$$

$$= \underbrace{(X'X)^{-1}(X'X)\beta}_{\beta} + E[\underbrace{(X'X)^{-1}X'\varepsilon}]$$

$$E[\varepsilon | X] = 0$$

$$X = \begin{bmatrix} 1 & r_1 \\ \vdots & \vdots \\ 1 & r_{T-1} \end{bmatrix} \qquad \varepsilon = \begin{bmatrix} \varepsilon_2 \\ \varepsilon_3 \\ \vdots \\ \varepsilon_T \end{bmatrix}$$

$$E[\varepsilon_2 | X] \overset{?}{=} 0$$

$r_1, r_2 \cdots r_{T-1}$

$$r_{t+1} = a + b\,x_t + \varepsilon_{t+1}$$

$$E[\varepsilon_{t+1} | X]$$

x_t, x_{t+1}

$$r_{t+1} = \phi_0 + \phi_1 r_t + \varepsilon_{t+1}$$

$$E[r_t] = ?$$

$$E[r_{t+1}] = \phi_0 + \phi_1 E[r_t] + \underbrace{E[\varepsilon_{t+1}]}_{} = 0$$

If $|\phi_1| < 1$ $\qquad E[r_t] = \mu$

$$\mu = \phi_0 + \phi_1 \mu \Rightarrow \mu = \frac{\phi_0}{1-\phi_1}$$

$$\underline{Var(r_{t+1})} = \phi_1^2 \underline{Var(r_t)} + \sigma^2 + \cancel{2\phi Cov(r_t, \varepsilon_{t+1})}$$

$$\gamma_0 = \phi_1^2 \gamma_0 + \sigma^2 \Rightarrow \gamma_0 = \frac{\sigma^2}{1-\phi_1^2}$$

$$\mu = \frac{\phi_0}{1-\phi_1} \Rightarrow \phi_0 = \mu(1-\phi_1)$$

$$r_{t+1} = \underline{\phi_0} + \phi_1 r_t + \varepsilon_{t+1}$$

$$= \mu - \underline{\phi_1 \mu} + \underline{\phi_1 r_t} + \varepsilon_{t+1}$$

$$r_{t+1} - \mu = \phi_1 (r_t - \mu) + \varepsilon_{t+1}$$

$$Cov(r_{t+1}, r_t) = E[(\underline{r_{t+1}-\mu})(r_t-\mu)]$$

$$= E[(\underline{\phi_1(r_t-\mu)} + \varepsilon_{t+1})(\underline{r_t-\mu})]$$

$$= E[\phi_1 (r_t-\mu)^2 + \varepsilon_{t+1}(r_t-\mu)]$$

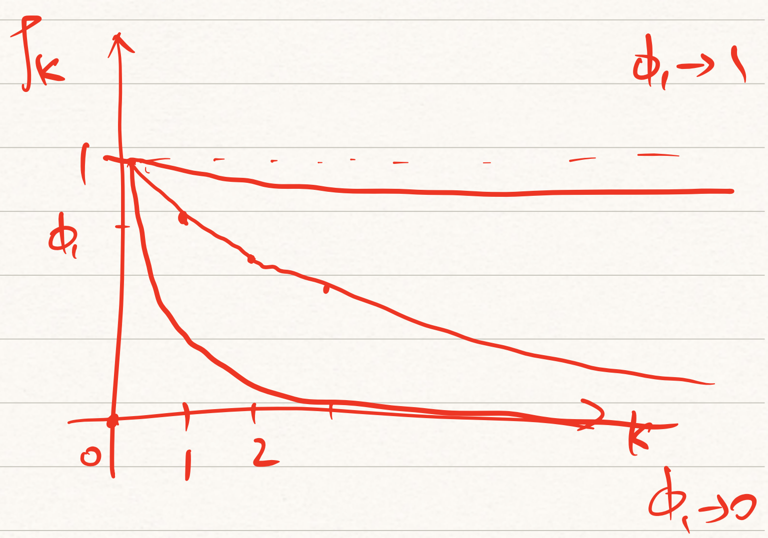
$$= \phi_1 \cdot \gamma_0 + E[\cancel{E_t[\varepsilon_{t+1}(r_t-\mu)]}]$$

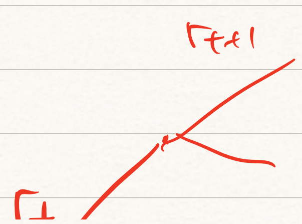
$$=0$$

$$E_t[\varepsilon_{t+1}] = 0$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\phi_1 \gamma_0}{\gamma_0} = \phi_1$$

$$\rho_k = \phi_1^k$$





$\underline{E_t[r_{t+1}]}$ $\qquad$ $\underline{E[r_{t+1}]} = \mu$

$$E_t[r_{t+1}] = E_t[\phi_0 + \phi_1 \underline{r_t} + \varepsilon_{t+1}]$$
$$= \underline{\phi_0 + \phi_1 r_t} + \cancel{E_t[\varepsilon_{t+1}]}$$

$$\underline{E_t[r_{t+1} - \mu]} = E_t[\phi_1 (r_t - \mu) + \varepsilon_{t+1}]$$
$$= \phi_1 (r_t - \mu)$$

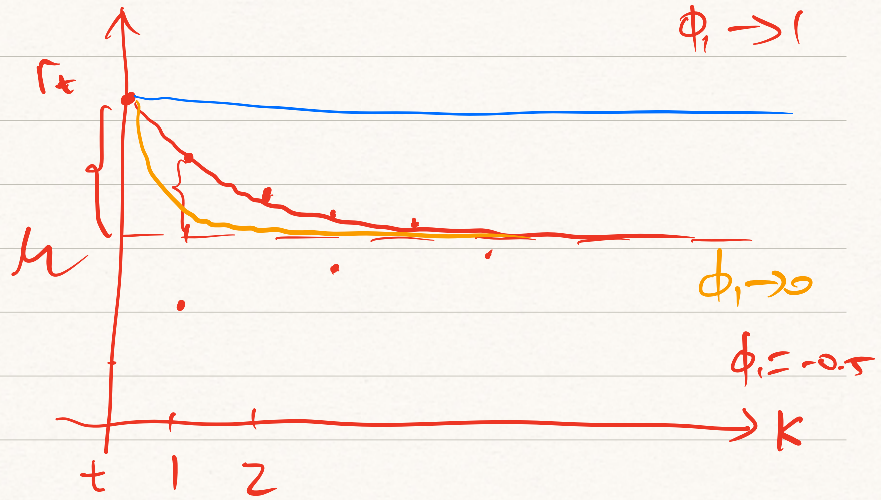
$$E_t[r_{t+1}] = \mu + \phi_1 (r_t - \mu)$$

$$E_t[r_{t+2} - \mu] = E_t[\phi_1 \underline{(r_{t+1} - \mu)} + \varepsilon_{t+2}]$$
$$= E_t[\phi_1 (\phi_1 (r_t - \mu) + \underline{\varepsilon_{t+1}}) + \underline{\varepsilon_{t+2}}]$$
$$= \phi_1^2 (r_t - \mu)$$

$$E_t[r_{t+2}] = \mu + \phi_1^2 (r_t - \mu)$$

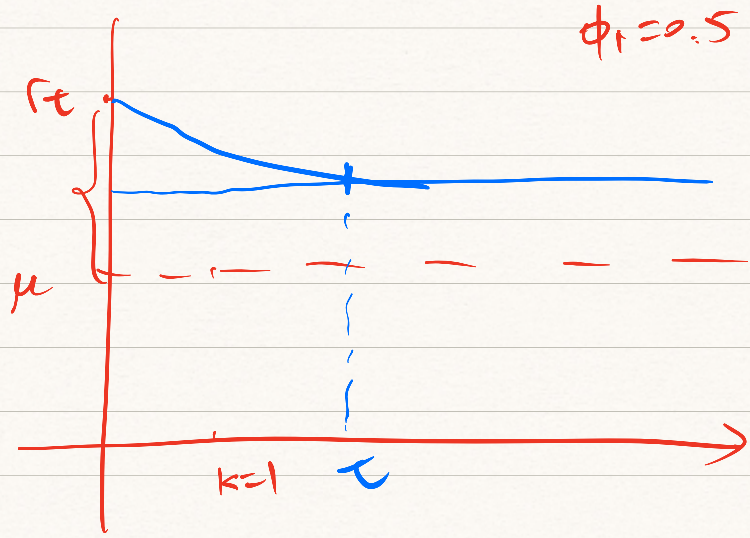
$$E_t\{r_{t+k}\} = \mu + \phi_1^k (r_t - \mu)$$

$\phi_1 = 0.5$



Mean-reversion vs. negative autocorrelation

$\phi_1 = 0.5$



$$E_t\{r_{t+\tau} - \mu\}$$

$$= \phi_1^\tau \cdot (r_t - \mu) = \frac{1}{2}(r_t - \mu)$$

$$\phi_1^\tau = \frac{1}{-}$$

$$\tau = -\frac{\ln(2)}{\underbrace{\ln(|\phi_1|)}_{-(1-\phi_1)}}$$