MAT223H5S - Linear Algebra I - Winter 2025
Term Test 2 - Version B

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Duration - 100 Minutes Aids: None

Other Instructions:

- Make sure your name and student number are written *neatly, in ink* above.
- **Do not open the test until instructed to do so.**
- If you wish to have rough work graded, make sure to indicate which page it is on. (For example, "This question continued on Page 2".) Otherwise the graders may not see your work.
- **Do not remove any pages, or write on the QR Codes.**

- There are **5 questions** worth a total of **50 points** (10 points each). Each question has several parts.
- Throughout the exam you should **show all of your work** unless specified otherwise.
- This test has **12 pages** including this page.

1

1.1 **(2 points)** Let $\mathbf{d} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{a} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$. Determine whether or not $\mathbf{d}$ and $\mathbf{a}$ are orthogonal.

1.2 **(3 points)** Compute $||\text{proj}_{\mathbf{a}}(\mathbf{d})||$. *Use* $\mathbf{a}$ *and* $\mathbf{d}$ *as above. You do not need to simplify fractions, roots, etc.*

1.3 **(1 point)** Let T be a linear transformation with matrix $A_T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. What is the domain of T? (Is it $\mathbb{R}^2$, $\mathbb{R}^3$, or something else?) *You do not need to justify your answer.*

1.4 **(4 points)** Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 2 & 2 & 4 \end{bmatrix}$. Is $\mathbf{d}$ in $\operatorname{im}(A)$? *(Use* $\mathbf{d}$ *as on the previous page.)*

2

2.1 **(5 points)** Find the equation of the plane through the points

$$P = (1, 2, 3), \quad Q = (-1, 9, -2), \quad R = (2, 0, 5).$$

Give your answer in the form $ax + by + cz = d$.

You should only use the methods taught to you in this course to solve this problem. In particular, do not apply any formulas or shortcuts from outside the course.

2.2 **(5 points)** Show that the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 + y^2 \\ x - y \end{bmatrix}$ is *not* a linear transformation.

3 3.1 **(4 points)**

Show that the set U defined below is a subspace. As part of your answer, find a spanning set for U.

$$U = \left\{ \begin{bmatrix} 2x - 4y \\ z \\ x - y \\ y \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; x, y, z \in \mathbb{R} \right\}$$

3.2 **(6 points)**

Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; x + y < 0 \right\}$.

For each of the three subspace axioms below, determine whether the set V satisfies it or not.

In particular, if you think that V satisfies a given axiom, explain why. If you think that V does not satisfy a given axiom, give an example showing that.

(1) $\mathbf{0} \in V$.

(2) For all $\mathbf{u}, \mathbf{v} \in V$, we have $\mathbf{u} + \mathbf{v} \in V$.

(3) For all $\mathbf{v} \in V$ and $t \in \mathbb{R}$, we have $t\mathbf{v} \in V$.

4 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which reflects across the line $y = 2x$ and then stretches in the x and y directions, each by a factor of 5.

You may assume that $A_T = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$.

4.1 **(5 points)** Sketch a single copy of $\mathbb{R}^2$ which contains the following:

- $\mathbf{e}_1$, $\mathbf{e}_2$, $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$,
- The **fundamental parallelogram** for T (i.e. the image of the unit square under T),
- The lines $y = 2x$ and $y = -\frac{1}{2}x$.

Note: *For the questions on this page, you must justify your answers using only* ***geometric*** *explanations explicitly relying on and referring to your drawing from the previous part, or to a new drawing. Algebraic work (e.g. computations involving A_T) can be used to check your work, but will not be marked.*

4.2 **(2.5 points)** Explain **geometrically** why your drawing from the previous part implies that neither $\mathbf{e}_1$ nor $\mathbf{e}_2$ is an eigenvector for A_T.

4.3 **(2.5 points)** Determine two basic eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ for A_T **geometrically** using your drawing from the first part of this question

(You do not need to determine the eigenvalue(s) associated to those basic eigenvectors.)

5 **(2.5 points each = 10 points)**

Determine if the statements below are **true or false**.

Make sure to justify your answers! You will receive no credit for simply selecting “true" or “false", or providing little explanation.

5.1 **True or False:** Suppose that U is a subset of $\mathbb{R}^3$ such that $\begin{bmatrix}1\\2\\3\end{bmatrix}$ is in U, but $\mathbf{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\mathbf{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$ are not in U. Then U is not a subspace of $\mathbb{R}^3$.

5.2 **True or False:** If $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ and $\operatorname{span}\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is a plane, then $\operatorname{span}\{\mathbf{a}, \mathbf{b}\}$ is a line.

5.3 **True or False:** Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, and let $\mathbf{b}$ be a fixed element of $\mathbb{R}^n$. Then the set $\{\mathbf{x} \in \mathbb{R}^n \mid T(\mathbf{x}) = \mathbf{b}\}$ is a subspace.

5.4 **True or False:** The system of equations below represents three planes in $\mathbb{R}^3$ that intersect in a point:

$$\left\{ \begin{array}{c} x + y + z = 1 \\ x + y + z = 0 \\ x + y + z = 2 \end{array} \right\}$$

Additional space for rough work. If you wish to have work on this page marked, make sure to indicate this clearly.

End of Test