MAT223H5S - Linear Algebra I - Winter 2025
Term Test 2 SOLUTIONS (Both Versions)

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Duration - 100 Minutes   Aids: None

1

1.1 **(2 points)** Let $\mathbf{d} = \begin{bmatrix}1\\1\\2\end{bmatrix}$ and $\mathbf{a} = \begin{bmatrix}1\\-1\\-1\end{bmatrix}$. Determine whether or not $\mathbf{d}$ and $\mathbf{a}$ are orthogonal.

VerB: Instead, $\mathbf{d} = \begin{bmatrix}1\\2\\0\end{bmatrix}$ and $\mathbf{a} = \begin{bmatrix}1\\-1\\-2\end{bmatrix}$.

1.2 **(3 points)** Compute $||\text{proj}_{\mathbf{d}}(\mathbf{a})||$. *Use* $\mathbf{d}$ *and* $\mathbf{a}$ *as above. You do not need to simplify fractions, roots, etc.*

VerB: This question asks for the projection of $\mathbf{d}$ onto $\mathbf{a}$, and $\mathbf{a}$ and $\mathbf{d}$ are different (in the previous part.)

1.3 **(1 point)** Let T be a linear transformation with matrix $A_T = \begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}$. What is the domain of T? (Is it $\mathbb{R}^2$, $\mathbb{R}^3$, or something else?) *You do not need to justify your answer.*

VerB: Here A is 2×3.

1.4 **(4 points)** Let $A = \begin{bmatrix}1 & 1 & 2\\0 & 0 & 0\\2 & 2 & 4\end{bmatrix}$. Is $\mathbf{d}$ in $\text{im}(A)$? *(Use* $\mathbf{d}$ *as on the previous page.)*

VerB: same (again, $\mathbf{d}$ in part 1.1 is different though).

Solution

1.1 $\mathbf{d} \cdot \mathbf{a} = -2 \neq 0$, so they are not orthogonal. VerB: Same, but the dot product is equal to -1.

1.2 $\frac{-1}{3}\begin{bmatrix}1\\1\\2\end{bmatrix}$. VerB: $\frac{-1}{6}\begin{bmatrix}1\\-1\\-2\end{bmatrix}$.

1.3 A is a 3×2 matrix, so the domain is $\mathbb{R}^2$. VerB: $\mathbb{R}^3$.

1.4 No for both versions. You can show this by considering the augmented system $[A|\mathbf{d}]$ and noticing that there is a row $[000|*]$ where the last entry is non-zero. This means that the vector $\mathbf{d}$ is not in the span of the columns of A, or equivalently that it is not a linear combination of the columns of A. Alternately, you could just say that $im(A)$ is the set of $\mathbf{b}$ so that $A\mathbf{x} = \mathbf{b}$ is consistent, and we just saw

2

2.1 **(5 points)** Find the equation of the plane through the points

$$P = (1, 2, 3), \quad Q = (-1, 9, -2), \quad R = (1, 0, 4).$$

Give your answer in the form $ax + by + cz = d$.

VerB: $R = (2, 0, 5)$.

You should only use the methods taught to you in this course to solve this problem. In particular, do not apply any formulas or shortcuts from outside the course.

2.2 **(5 points)** Show that the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2y^2 \\ x-y \end{bmatrix}$ is *not* a linear transformation.

VerB: The top coordinate of the output is $x^2 + y^2$, not x^2y^2.

SOLUTION

2.1 Computing $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and taking their cross product (there are other reasonable, equivalent choices) we get $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. (In VerB, with the same vectors, but R changed, we get $\begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix}$.)

Plugging in a point on the plane to solve for d in the plane equation, we get 13 (or -7 in VerB). So the final answer is (up to multiplying both sides by a non-zero number $-3x + 2y + 4z = 13$ (or $4x - y - 3z = -7$ in VerB).

2.2 There are lots of ways to do this, using either axiom of a linear transformation. Here is one example (essentially the same example works for VerB): (I will write the vectors horizontally just to make it easier to type up)

$$-F(1, 2) = -(5, -1) = (-5, 1), \text{ while } F(-1, -2) = (5, 1),$$

which is different, so F fails the axiom that $kF(\mathbf{x}) = F(k\mathbf{x})$ for any k.

3

3.1 **(4 points)**

Show that the set U defined below is a subspace. As part of your answer, find a spanning set for U.

$$U = \left\{ \begin{bmatrix} 2x - 4y \\ y \\ x + y \\ z \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; x, y, z \in \mathbb{R} \right\}$$

VerB: The second and fourth coordinates ('y' and 'z') are swapped, and the third coordinate is '$x - y$' not '$x + y$'.

3.2 **(6 points)**

Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; x + y > 0 \right\}$.

For each of the three subspace axioms below, determine whether the set V satisfies it or not.

In particular, if you think that V satisfies a given axiom, explain why. If you think that V does not satisfy a given axiom, give an example showing that.

(1) $\mathbf{0} \in V$.

(2) For all $\mathbf{u}, \mathbf{v} \in V$, we have $\mathbf{u} + \mathbf{v} \in V$.

(3) For all $\mathbf{v} \in V$ and $t \in \mathbb{R}$, we have $t\mathbf{v} \in V$.

VerB: The set V is the same except the $>$ is replaced with $<$.

SOLUTION

3.1 In both versions, this set can be written as $im(A)$ where A is the matrix $\begin{bmatrix} 2 & -4 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, or in VerB, $\begin{bmatrix} 2 & -4 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Since images of matrices are always subspaces (from class), we have shown that U is a subspace.

A spanning set is given by the columns of the matrix above.

3.2 (1) No, because $0 + 0 \not> 0$. (For VerB, just replace $\not>$ with $\not<$.

(2) Yes. We are asked to verify whether $\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$ is in V, assuming $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ are in V. That is, assuming that $x_1 + y_1 > 0$ and $x_2 + y_2 > 0$. Butn $(x_1 + x_2) + (y_1 + y_2)$ can be rearranged into $(x_1 + y_1) + (x_2 + y_2)$, and by assumption, both halves of this are > 0, so the sum is > 0. *It is important for this argument to make sense that you start with '$(x_1 + x_2) + (y_1 + y_2)$' not '$(x_1 + y_1) + (x_2 + y_2)$'.*

VerB: The argument is the same, but with $<$ instead of $>$.

(3) No. There are lots of counterexamples, but consider that $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is in V since 2 + 1 > 0 (or swap the coordinates for VerB). But $-1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is not in V, since $-2 - 1 \not> 0$.

4 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which reflects across the line $y = -2x$ and then stretches in the x and y directions, each by a factor of 5.

You may assume that $A_T = \begin{bmatrix} 3 & -4 \\ -4 & 3 \end{bmatrix}$.

4.1 **(5 points)** Sketch a single copy of $\mathbb{R}^2$ which contains the following:

- $\mathbf{e}_1$, $\mathbf{e}_2$, $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$,
- The **fundamental parallelogram** for T (i.e. the image of the unit square under T),
- The lines $y = -2x$ and $y = \frac{1}{2}x$.

Note: *For the questions on this page, you must justify your answers using only* ***geometric*** *explanations explicitly relying on and referring to your drawing from the previous part, or to a new drawing. Algebraic work (e.g. computations involving A_T) can be used to check your work, but will not be marked.*

4.2 **(2.5 points)** Explain **geometrically** why your drawing from the previous part implies that neither $\mathbf{e}_1$ nor $\mathbf{e}_2$ is an eigenvector for A_T.

4.3 **(2.5 points)** Determine two basic eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ for A_T **geometrically** using your drawing from the first part of this question

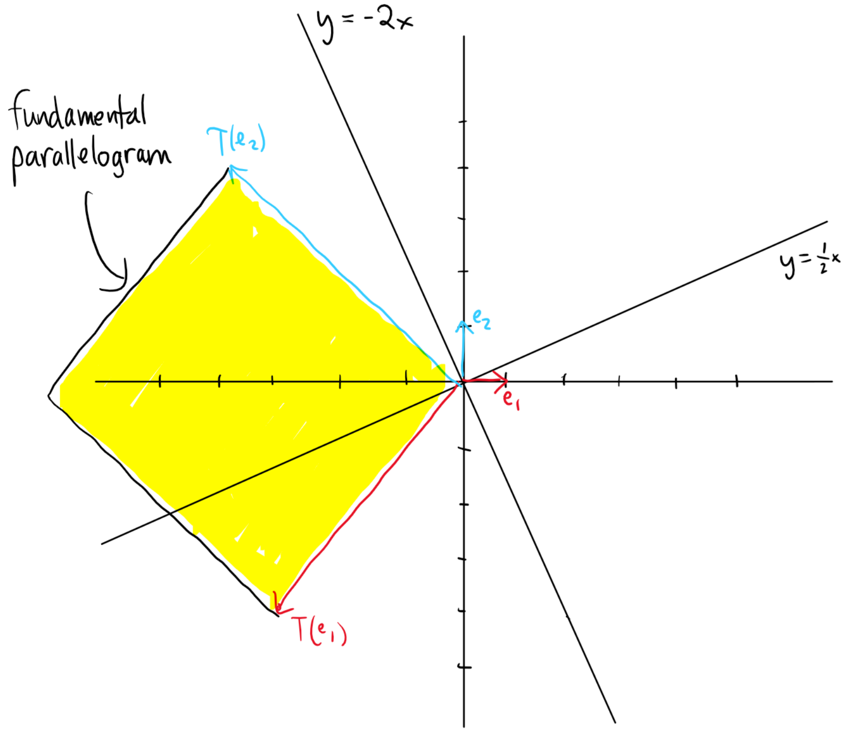
(You do not need to determine the eigenvalue(s) associated to those basic eigenvectors.)

VerB was the same, except the transformation reflected across $y = 2x$ instead, changing A_T to $\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$, and it asks you to graph $y = 2x$ and $y = -1/2x$ instead of $y = -2x$ and $y = 1/2x$.

SOLUTION

For VerB, while the question is different, the reasoning is the same, and the picture is just rotated compared to the picture for VerA. The eigenvectors you find (for 4.3) are the same, except with the coordinates swapped compared to VerA.

4.1 This is an example of a drawing you could have given.



4.2 As you can see from the picture, $\mathbf{e}_1$ and $T(\mathbf{e}_1)$ do not lie on the same line through the origin, so $\mathbf{e}_1$ is not an eigenvector. The argument is the same for $\mathbf{e}_2$.

4.3 From inspecting the drawing, we can see that any (non-zero) vector on the line of reflection $y = -2x$, like $\mathbf{u} = (1, -2)$ say, will not move upon reflecting, and then after scaling by 5, will still be on the line $y = -2x$ ($\mathbf{u}$ will be sent to $5\mathbf{u}$ in particular). That is, $\mathbf{u}$ and $T(\mathbf{u})$ are on the same line through the origin (which we can see from the picture!), so $\mathbf{u}$ is an eigenvector.

Similarly, if we consider a vector on the perpendicular line $y = \frac{1}{2}x$, like $\mathbf{v} = (1, 1/2)$ say, it will be reflected to $-\mathbf{v}$ by the transformation and then scaled to $-5\mathbf{v}$, remaining on the line. So again, $\mathbf{v}$ and $T(\mathbf{v})$ are on the same line through the origin, which we can see from the picture as well, and so $\mathbf{v}$ is an eigenvector.

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5 **(2.5 points each = 10 points)**

Determine if the statements below are **true or false**.

Make sure to justify your answers! You will receive no credit for simply selecting “true" or “false", or providing little explanation.

5.1 **True or False:** Let A be an $n \times n$ matrix, and $\mathbf{b} \in \mathbb{R}^n$ be arbitrary. Then the set of solutions to the system $A\mathbf{x} = \mathbf{b}$ is a subspace.

5.2 **True or False:** If $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ and span$\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is a plane, then span$\{\mathbf{a}, \mathbf{b}\}$ is also plane.

5.3 **True or False:** The system of equations below represents three planes in $\mathbb{R}^3$ that intersect in a line:

$$\left\{\begin{array}{r} x+y+z=1 \\ x+y+z=0 \\ x+y+z=-1 \end{array}\right\}$$

5.4 **True or False:** Suppose that U is a subset of $\mathbb{R}^3$ such that $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is in U, but $\mathbf{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\mathbf{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$ are not in U. Then U is not a subspace of $\mathbb{R}^3$.

VerB had similar questions, but in a different order. Ver B 5.1, 5.2, 5.3, 5.4 correspond to 5.4, 5.2, 5.1, 5.3 in VerA. (There were some slight changes of notation and numbers as well.)

Solution

5.1 False. If $\mathbf{b} \neq \mathbf{0}$, then this won't work. So there are lots of counterexamples. For instance, $A = I_2, \mathbf{b} = \mathbf{e}_1$, then the solutions to $A\mathbf{x} = \mathbf{b}$ will just contain one vector, $\mathbf{e}_1$, and a single vector isn't a subspace, unless it is $\{\mathbf{0}\}$.

5.2 False. For example, $\{\mathbf{e}_1, 2\mathbf{e}_1, \mathbf{e}_2\}$ spans a plane (say in $\mathbb{R}^3$) but the span of $\{\mathbf{e}_1, 2\mathbf{e}_1\}$ is just a line.

5.3 False. If we convert this to an augmented system of equations and row reduce (though that's not strictly necessary), we will get a row $[0\ 0\ 0\ |1]$, showing the system to be inconsistent. Geometrically this means that the planes do not have any common intersection (i.e. there are no points that lie in all three planes).

5.4 False. Careful with the logic here: a counterexample would be a subset U which contained the first vector, but none of $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ *and* which is (nonetheless) a subspace. So just let U be the line spanned by the first vector - it is a subspace (it's just $span\{\mathbf{u}\}$ for a single vector $\mathbf{u}$!) and avoids the other three vectors. Equivalently, you could think of it this way: for this to be true, it would have to be that whenever a subspace contained $\mathbf{u}$ it would *have to* contain $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, which is false. (This is similar to Exercise 5, Week 8 Polls, by the way.)