MAT223H5S - Linear Algebra I - Winter 2025
Make-Up Term Test SOLUTIONS
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Duration - 100 Minutes   Aids: None

1

1.1 **(2 points)** Suppose that A is a matrix with $A^{-1} = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 1 & 5 \\ -1 & 0 & 3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Determine the unique solution to the equation $A\mathbf{x} = \mathbf{b}$.

1.2 **(3 points)** Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ first rotates the plane counterclockwise by $\pi/2$, then reflects across the line $y = x$. Determine the matrix A_T.

1.3 **(5 points)** For which values (if any) of $c \in \mathbb{R}$ is the following set of vectors in $\mathbb{R}^3$ linearly independent?

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} c \\ 0 \\ c^2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Solution

1.1 Multiplying both sides of $A\mathbf{x} = \mathbf{b}$ by A^{-1} we get $\mathbf{x} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$.

1.2 If you consider where $\mathbf{e}_1$ and $\mathbf{e}_2$ are sent by this transformation, you can see that $\mathbf{e}_1 \mapsto \mathbf{e}_2 \mapsto \mathbf{e}_1$ and $\mathbf{e}_2 \mapsto -\mathbf{e}_1 \mapsto -\mathbf{e}_2$, so that the matrix of the transformation is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Alternately, you could use a similar process to determine the matrices of each part of the transformation, and multiply them (in the correct order, of course).

1.3 Carefully row-reducing the matrix with these columns, we can see that as long as $c \neq 0, 1$, the matrix will have full rank, and therefore the set of vectors will be linearly independent. Alternately, we could compute the determinant of said matrix, and get $2c(1-c)$, which is non-zero for $c \neq 0, 1$.

2

Reminder: show your work and justify your steps using only techniques taught in this course.

2.1 **(1 point)** The equation $x - y - 6z = 6$ represents a plane in $\mathbb{R}^3$. Determine whether the point $P = (3, -3, 0)$ is on the plane or not.

2.2 **(1.5 points)** Determine the angle between vectors $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

2.3 **(2.5 points)** Provide an example, with justification, of a 2×2 matrix which is skew-symmetric and invertible, or explain why no such matrix exists.

2.4 **(5 points)** Find the shortest distance between the lines $\mathbf{L_1}$ and $\mathbf{L_2}$ with the following equations:

$$\mathbf{L_1}: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \quad \mathbf{L_2}: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + t \begin{bmatrix} -4 \\ 6 \\ -2 \end{bmatrix}$$

You should include a rough drawing illustrating the situation (e.g. it doesn't have to plot the lines accurately); the drawing should show any vectors and points that you compute as part of your solution.

SOLUTION

2.1 Just plugging in the point into the equation results in 3+3 - 0 = 6, so yes, the point is on the plane.

2.2 The dot product of these vectors is 0, so they are orthogonal, i.e. the angle is $\pi/2$.

2.3 For a 2×2 matrix to be skew-symmetric, it must be of this form $\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$. So, any matrix of that form with $a \neq 0$ would work, since it is clearly invertible, as required.

2.4 These lines are parallel, so notice that you cannot use the technique you learned to compute the distance between skew lines (it involves taking the cross product of the two lines' direction vectors, which would be 0 here.)

There are several ways to do this though. For example, letting P and Q be the points on these lines, compute $\overrightarrow{PQ}$, project it onto the (common) direction vector, subtract it from $\overrightarrow{PQ}$ and take the length.

Alternately, you can use the observation from one of the in-class activities that $||\mathbf{u}||||\mathbf{v}|| \sin\theta = ||\mathbf{u} \times \mathbf{v}||$ is the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$, but includes the fact that the height of the parallelogram (i.e. the vector we want the length of) is just $||\mathbf{u} \times \mathbf{v}||/||\mathbf{v}||$. In this case, we would be using $\mathbf{u} = \overrightarrow{PQ}$ and $\mathbf{v}$ is the direction vector of either line.

In any case, the final answer is $\sqrt{299}/\sqrt{14}$, or unsimplified, $\frac{\sqrt{11^2+3^2+13^2}}{\sqrt{14}}$.

3

Let $U = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{matrix} x - 2y + z = w \\ y + z + w = x \end{matrix} \right\}$.

3.1 **(5 points)** Show that U is a subspace.

3.2 **(5 points)** Determine a basis for U. Show your steps.

SOLUTION

3.1 You can simply note that this is the nullspace of a matrix, namely $U = \text{null}\left(\begin{bmatrix} 1 & -2 & 1 & -1 \\ -1 & 1 & 1 & 1 \end{bmatrix} \right)$, and null spaces (from class) are always subspaces.

3.2 By row reducing the above matrix, and pulling out basic solutions, we get two:

$$\begin{bmatrix}3\\2\\1\\0\end{bmatrix}, \quad \begin{bmatrix}5\\2\\0\\1\end{bmatrix}$$

4

4.1 **(5 points)** Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation given by $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}x-y\\y\\0\end{bmatrix}$.

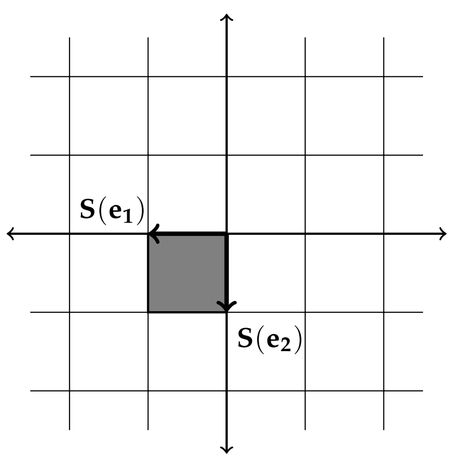
Show that T is linear *by verifying the two properties below*. Do **not** simply say that T is a matrix transformation, or use any other technique, or you will receive 0 points.

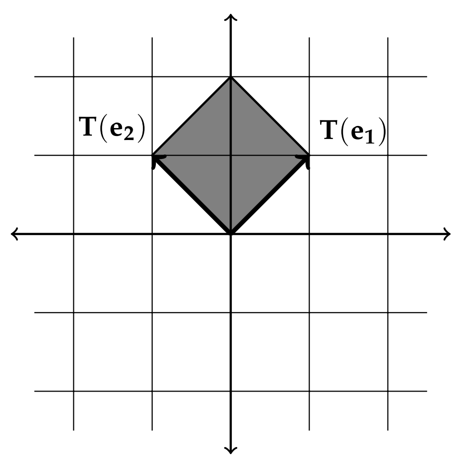
(1) For all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, we have $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.

(2) For all $\mathbf{u} \in \mathbb{R}^3$ and $r \in \mathbb{R}$, we have $T(r\mathbf{u}) = rT(\mathbf{u})$.

4.2 **(5 points)** In the pictures below, the fundamental parallelograms of two linear transformations, $S, T : \mathbb{R}^2 \to \mathbb{R}^2$ are shown. Determine all eigenvalues (if any) for each of S and T and for each eigenvalue determine a set of basic eigenvectors for that eigenvalue.

Make sure to justify your answers using ***geometric*** *arguments* ***only*** *(i.e. ones which reference the pictures of the transformations, not any algebra involving the matrices of the transformations.) Assume the grid lines are spaced 1 unit apart.*





SOLUTION

4.1 Letting $\mathbf{u} = (x_1, y_1, z_1)^T$ and $\mathbf{v} = (x_2, y_2, z_2)^T$, we compute

$$T(\mathbf{u}+\mathbf{v}) = \begin{bmatrix}(x_1+x_2)-(y_1+y_2)\\y_1+y_2\\0\end{bmatrix} = \begin{bmatrix}x_1-y_1\\y_1\\0\end{bmatrix} + \begin{bmatrix}x_2-y_2\\y_2\\0\end{bmatrix} = T(\mathbf{u}) + T(\mathbf{v})$$

The other axiom is similar to verify.

4.2 S has eigenvalue -1, with basic eigenvectors $\mathbf{e}_1, \mathbf{e}_2$. We can see this because $S(\mathbf{e}_1) = -1 \cdot \mathbf{e}_1$ and $S(\mathbf{e}_2) = -1 \cdot \mathbf{e}_2$ from the first picture (and in general we know that they are eigenvectors for S because they are sent to the same line through $\mathbf{0}$ that they began on.)

T has no eigenvalues, since we can see from where $\mathbf{e}_1$ and $\mathbf{e}_2$ are sent that T is a rotation by some angle (in this case $\pi/4$) which is neither 0 nor π (which result in eigenvalue 1 or -1, respectively). Since any vector $\mathbf{v} \neq \mathbf{0}$ will be sent to $T(\mathbf{v})$ which is at angle $\pi/4$ from $\mathbf{v}$, $T(\mathbf{v})$ will not lie on the same line through the origin as $\mathbf{v}$ and so $\mathbf{v}$ cannot be an eigenvector.

5 Determine if the statements below are **true or false**.

Make sure to justify your answers! You will receive no credit for simply selecting “true" or “false", or providing little explanation.

5.1 **True or False:** If $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$ and $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly independent, then $\{\mathbf{x}, \mathbf{x}+\mathbf{y}+\mathbf{z}, \mathbf{z}\}$ is also linearly independent.

5.2 **True or False:** If A is a 3×6 matrix, then $\dim(null(A)) > 2$.

5.3 **True or False:** If A is a 4×4 matrix and the systems $(A - I)\mathbf{x} = \mathbf{0}$ and $(A + I)\mathbf{x} = \mathbf{0}$ each have two basic solutions, then A is diagonalizable.

5.4 **True or False:** If two planes in $\mathbb{R}^3$ pass through the origin, and intersect in a line with direction vector $\mathbf{d}$, then $\mathbf{d}$ is orthogonal to the normal vectors both of those planes.

SOLUTION

5.1 **True.** Suppose that $a\mathbf{x} + b(\mathbf{x}+\mathbf{y}+\mathbf{z}) + c\mathbf{z} = \mathbf{0}$. Then we can factor and see that $(a+b)\mathbf{x} + b\mathbf{y} + (b+c)\mathbf{z} = \mathbf{0}$. Since $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly independent, the coefficients here must be 0; that is, $a + b = 0; b = 0; b + c = 0$. The only solution to this system is if $a = b = c = 0$, so the original set must be linearly independent. Notice that we strictly adhered to the definition of linear independence in this argument.

5.2 **True.** We know from class that $dim(null(A)) = n - r = 6 - r \geq 6 - 3 = 3$, which is the same as saying that it is > 2.

5.3 **True.** The assumptions tell us that A has two basic eigenvectors (each) for eigenvalues 1 and -1, which gives us four total, as needed.

5.4 **True.** Since the line is contained in both planes, and everything passes through the origin, the direction vector $\mathbf{d}$ is contained in each plane. But such a vector must be perpendicular to the normal of such a plane, i.e. $\mathbf{n}_1 \cdot \mathbf{d} = 0$ and $\mathbf{n}_2 \cdot \mathbf{d} = 0$, as desired.