MAT223H5S - Linear Algebra I - Winter 2025
Make-Up Term Test

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Duration - 100 Minutes Aids: None

Other Instructions:

- Make sure your name and student number are written *neatly, in ink* above.
- **Do not open the test until instructed to do so.**
- If you wish to have rough work graded, make sure to indicate which page it is on. (For example, "This question continued on Page 2".) Otherwise the graders may not see your work.
- **Do not remove any pages, or write on the QR Codes.**

- There are **5 questions** worth a total of **50 points** (10 points each). Each question has several parts.
- Throughout the exam you should **show all of your work** unless specified otherwise.
- This test has **12 pages** including this page.

1

1.1 **(2 points)** Suppose that A is a matrix with $A^{-1} = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 1 & 5 \\ -1 & 0 & 3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Determine the unique solution to the equation $A\mathbf{x} = \mathbf{b}$.

1.2 **(3 points)** Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ first rotates the plane counterclockwise by $\pi/2$, then reflects across the line $y = x$. Determine the matrix A_T.

1.3 **(6 points)** For which values (if any) of $c \in \mathbb{R}$ is the following set of vectors in $\mathbb{R}^3$ linearly independent?

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} c \\ 0 \\ c^2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

2 **Reminder:** show your work and justify your steps using only techniques taught in this course.

2.1 **(1 point)** The equation $x - y - 6z = 6$ represents a plane in $\mathbb{R}^3$. Determine whether the point $P = (3, -3, 0)$ is on the plane or not.

2.2 **(1.5 points)** Determine the angle between vectors $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

2.3 **(2.5 points)** Provide an example, with justification, of a 2×2 matrix which is skew-symmetric and invertible, or explain why no such matrix exists.

2.4 **(5 points)** Find the shortest distance between the lines $\mathbf{L_1}$ and $\mathbf{L_2}$ with the following equations:

$$\mathbf{L_1}: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \quad \mathbf{L_2}: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + t \begin{bmatrix} -4 \\ 6 \\ -2 \end{bmatrix}$$

You should include a rough drawing illustrating the situation (e.g. it doesn't have to plot the lines accurately); the drawing should show any vectors and points that you compute as part of your solution.

3

$$\text{Let } U = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{array}{c} x - 2y + z = w \\ y + z + w = x \end{array} \right\}.$$

3.1 **(5 points)** Show that U is a subspace.

3.2 **(5 points)** Determine a basis for U. Show your steps.

4

4.1 **(5 points)** Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - y \\ y \\ 0 \end{bmatrix}$.

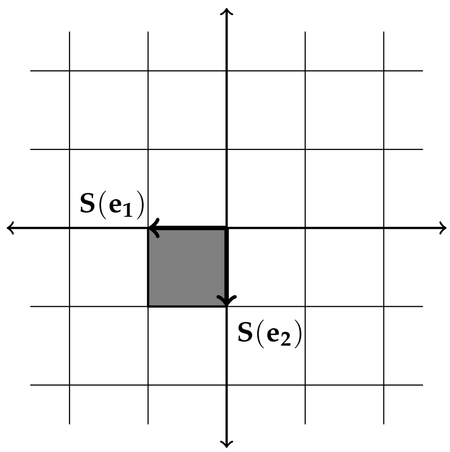
Show that T is linear *by verifying the two properties below*. Do **not** simply say that T is a matrix transformation, or use any other technique, or you will receive 0 points.

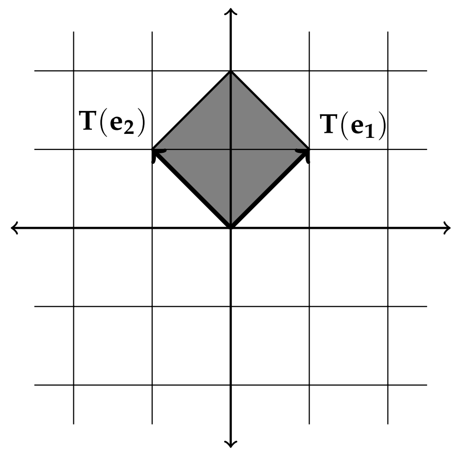
(1) For all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, we have $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.

(2) For all $\mathbf{u} \in \mathbb{R}^3$ and $r \in \mathbb{R}$, we have $T(r\mathbf{u}) = rT(\mathbf{u})$.

4.2 **(5 points)** In the pictures below, the fundamental parallelograms of two linear transformations, $S, T : \mathbb{R}^2 \to \mathbb{R}^2$ are shown. Determine all eigenvalues (if any) for each of S and T and for each eigenvalue determine a set of basic eigenvectors for that eigenvalue.

*Make sure to justify your answers using **geometric** arguments **only** (i.e. ones which reference the pictures of the transformations, not any algebra involving the matrices of the transformations.) Assume the grid lines are spaced 1 unit apart.*





5 Determine if the statements below are **true or false**.

Make sure to justify your answers! You will receive no credit for simply selecting “true" or “false", or providing little explanation.

5.1 **True or False:** If $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$ and $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly independent, then $\{\mathbf{x}, \mathbf{x}+\mathbf{y}+\mathbf{z}, \mathbf{z}\}$ is also linearly independent.

5.2 **True or False:** If A is a 3×6 matrix, then $\dim(null(A)) > 2$.

5.3 **True or False:** If A is a 4×4 matrix and the systems $(A - I)\mathbf{x} = \mathbf{0}$ and $(A + I)\mathbf{x} = \mathbf{0}$ each have two basic solutions, then A is diagonalizable.

5.4 **True or False:** If two planes in $\mathbb{R}^3$ pass through the origin, and intersect in a line with direction vector $\mathbf{d}$, then $\mathbf{d}$ is orthogonal to the normal vectors both of those planes.

Additional space for rough work. If you wish to have work on this page marked, make sure to indicate this clearly.

End of Test