Resilience in Transportation Networks

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Problem Definition

Given a graph $G(V, E)$ and a set of paths $P = \{(s_1, t_1), (s_2, t_2), \dots, (s_k, t_k)\}$, where each $p_i \equiv (s_i, t_i) \in P$ represents the source and destination node for a path p_i, find a set $E' \subseteq E$ consisting of all edges which when removed from the graph, maximum number of paths are affected. Let a path be represented by an ordered list of edges $p_i = (e_1, e_2, \dots, e_j)$. It is assumed that a path connecting the nodes (s_i, t_i) is affected if atleast one of the edge $e \in p_i$ is attacked. Also p_i is the only path connecting the nodes (s_i, t_i), and there is no other path connecting (s_i, t_i). Further, $\sum_{e_i \in E'} b(e_i) \leq B$,where B is the budget of the adversary and $b(e_i)$ is the cost of removing edge e_i.

Proposed Solution

Let us assume the graph has n nodes and m edges.

We construct a $m \times 1$ matrix $\mathbf{C}$ such that ith element in $\mathbf{C}$ represents the number of paths in set P passing through the i^{th} edge.This can be found by iterating over all the paths in P.

Let $\mathbf{b}$ be a $m \times 1$ matrix where b_i represents the cost of removing the i^{th} edge.

Let $\mathbf{X}$ be a $m \times 1$ matrix such that $x_i = 1$ if i^{th} edge belongs to E',otherwise it is 0.

The solution to the above problem is equivalent to solving a linear program

$$\arg\max_{X} \mathbf{C^T} \cdot \mathbf{X} \tag{1}$$

$$\text{such that } \mathbf{b^T} \cdot \mathbf{X} \leq B \text{ and } x_i \in \{0, 1\}$$

which is equivalent to

$$\arg\min_{X} -\mathbf{C^T} \cdot \mathbf{X} \tag{2}$$

$$\text{such that } \mathbf{b^T} \cdot \mathbf{X} \leq B \text{ and } x_i \in \{0, 1\}$$

Proposed solution 2

Let us assume the graph has n nodes, m edges and k paths. Let each path $p_i \in P$ have a weight $w_i \in W$, where the weight represents the importance of the path.Thus if $w_i < w_j; w_i, w_j \in W$ implies that it is beneficial to destroy path p_j over p_i.

We create a $k \times m$ matrix $\mathbf{A}$ whose entry $a_{ij} = 1$ if $e_j \in p_i$, else it is zero.

Let $\mathbf{X}'$ represent the $m \times 1$ uint vector $\vec{1}$.

Let $\mathbf{X}$ be a $m \times 1$ matrix such that $x_i = 1$ if $e_i \in \mathbf{E}'$, otherwise it is 0.

Let $\mathbf{W}$ be a vector such that each element $w_i \in \mathbf{W}$ represents weight of path p_i.

Then the above problem is equivalent to solving a linear program

$$\arg\max_{X} f(\mathbf{A} \cdot (\mathbf{X}' - \mathbf{X}))^{\mathbf{T}} \cdot \mathbf{W} \tag{3}$$

$$\text{such that } \mathbf{b}^{\mathbf{T}} \cdot \mathbf{X} \leq B \text{ and } x_i \in \{0, 1\}$$

where we define function f for every element $z_i \in \mathbf{Z}$ in any vector $\mathbf{Z}$ as $f(z_i) = \begin{cases} 0, & \text{if } z_i = 0 \\ 1, & \text{otherwise} \end{cases}$

Basically,$f(\mathbf{A} \cdot (\mathbf{X}' - \mathbf{X}))$ is k dimensional vector whose i^{th} entry is 1 if the path $p_i \in P$ is affected.