ISLAMIC UNIVERSITY OF TECHNOLOGY (IUT)

ORGANISATION OF ISLAMIC COOPERATION (OIC)

Department of Computer Science and Engineering (CSE)

SEMESTER FINAL EXAMINATION **WINTER SEMESTER, 2022-2023**

DURATION: 3 HOURS **FULL MARKS: 150**

Math 4341: Linear Algebra

Programmable calculators are not allowed. Do not write anything on the question paper.

Answer **all 6 (six)** questions. Figures in the right margin indicate full marks of questions whereas corresponding CO and PO are written within parentheses.

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1. Consider the matrices A and b to answer the following questions:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & 0 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

a) Is A invertible? Justify. 5 (CO1) (PO1)

b) Compute the LDU decomposition of A. 10 (CO2) (PO1)

c) Solve $Ax = b$. 10 (CO2) (PO1)

2. Let $a, b \in R$ and let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 & a \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & b \end{bmatrix}$$

a) Reduce A to U and prove that the determinant of A is equal to the product of the pivots. 10 (CO1) (PO1)

b) What are the dimensions of the four subspaces associated with the matrix B? These values will depend on the values of a and b, and you should distinguish all cases. 10 (CO1) (PO1)

c) For $a = b = 1$, give a basis for the columnspace of B. Is this also a basis for $\mathbb{R}^3$? Justify your answer. 5 (CO1) (PO1)

3. a) Consider a sequence of numbers defined using the following terms: 5 + 10+5 (CO3) (PO1)

$$G_0 = 0$$
$$G_1 = 1/2$$
$$G_{k+2} = (G_{k+1} + G_k)/2$$

i. Set up a 2×2 matrix A to get from $\begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$ to $\begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix}$

ii. Find an explicit formula for G_k.

iii. What is G_{1000}? What is the limit of G_k as $k \to \infty$?

b) Suppose a 3×3 matrix A has $row_1 + row_2 = row_3$. Explain why $Ax = [1\ 0\ 0]^T$ can not have a solution. 5 (CO1) (PO1)

4. a) Imagine a plane $z \equiv Cx + Dy$ that is closest (in the least squares sense) to these four measurements: 6 + 9 (CO2) (PO1)

$$\begin{aligned}&\text{At } x = 1; y = 0 \text{ measurement gives } z = 1\\ &\text{At } x = 1; y = 2 \text{ measurement gives } z = 3\\ &\text{At } x = 0; y = 1 \text{ measurement gives } z = 5\\ &\text{At } x = 0; y = 2 \text{ measurement gives } z = 0\end{aligned}$$

i. Show that this system $Ax = b$ has no solution.

ii. Find the best least squares solution, $\widehat{z} = (\widehat{C}, \widehat{D})$.

b) Assume q_1, q_2, and q_3 are the orthonormal basis of the independent columns of the matrix, A. 5 + 5 (CO2) (PO1)

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$$

i. Find q_1, q_2, and q_3.

ii. Express A as the product of an orthogonal matrix Q and an upper triangular matrix R.

5. If you convert matrix A (shown below) to row-reduced echelon form by the usual row elimination steps, you will achieve $R = rref(A)$ as: 5 × 5 (CO3) (PO1)

$$A = \begin{bmatrix} 1 & 2 & 1 & -7 \\ 2 & 4 & 1 & -5 \\ 1 & 2 & 2 & -16 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

a) What is the rank and dimension of A?

b) What are the minimum and the maximum number of columns of A that form a dependent and independent set of vectors, respectively?

c) Give an orthonormal basis for the rowspace of A. Note that it is not $C(A)$.

d) Given the vector $b = (2\ 5\ -9\ 3)^T$, compute a vector p that is closest to b in the rowspace $C(A^T)$.

e) In terms of your calculated p in Question 5.d), what is the closest vector to b in the nullspace $N(A)$?

6. a) Compute the Singular Value Decomposition $A = U\Sigma V^T$ for $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \\ 1 & 0 \end{bmatrix}$. 15 (CO3) (PO1)

b) The matrix A has a varying $(1 - x)$ in its $(1, 2)$ position: 10 (CO1) (PO1)

$$\begin{bmatrix} 2 & 1-x & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 3 & 9 \end{bmatrix}$$

Considering the properties of the determinant, show that $\det(A)$ is a linear function of x. For any x, compute $\det(A)$. For which value of x is the matrix singular?

3x3 3x2 2x2