**Project 11**

# TEMPERATURE DISTRIBUTION IN AN OVEN FOR CALIBRATING THERMOMETERS

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**Abstract**

In this paper we compute the temperature distribution of a small oven which is used for calibrating thermometers. The aim is to reach a uniform heat distribution inside the oven which depends on the placement and number of heating devices. We present numerical as well as analytical methods to solve the heat equation and as a conclusion we try to make some statements about the optimal placement and number of heating devices inside the oven.

# 1 Introduction

Correct calibration of measurement devices is an important issue for the industry. Quick and accurate measurements are an essential part of industrial manufacturing. In our project work we have been presented with a small oven which is used for calibrating thermometers.

The oven is made up by a block of metal, typically copper or aluminium connected to an accurate temperature measuring device. Inside the oven there is a cavity where you can insert a thermometer for calibration. The particular oven we have been studying consists of a block of length 92 mm and a square cross-section of 35 mm x 35 mm. In the longitudinal direction of this block is drilled a hole of diameter 18 mm and length 85 mm. In addition two smaller holes are drilled for inserting heating devices.

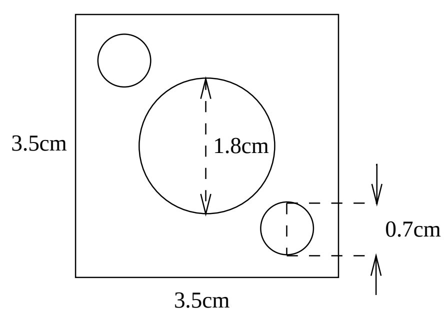


Figure 1: Cross section of the oven

To ensure a good calibration the temperature distribution in the cavity should be as uniform as possible. Clearly, this distribution is influenced by the size and geometry of the oven. The placement of heating devices and the boundary conditions at the surface will also be of some importance.

The main concern of the company is therefore, how should the oven be constructed to minimize the temperature fluctuations in the cavity. Also of interest are temperature development of the oven, and the power needed to sustain a certain temperature. Could normal batteries give enough power to heat the oven to a specific temperature within a reasonable amount of time?

# 2 Basic Equations

## 2.1 The Heat Equation

The basis of our model is the heat equation:

$$\frac{\partial T}{\partial t} = \frac{K}{\rho c}\Delta T + \frac{1}{\rho c} q, \tag{1}$$

or

$$\frac{\partial T}{\partial t} = \kappa \Delta T + \frac{\kappa}{K} q, \quad \kappa = \frac{K}{\rho c},$$

where $q(x, y, z, t)$ is the heat source at position $(x, y, z)$ at time $t$, $\rho$ the density, $K$ the thermal conductivity and $c$ the specific heat constant of the medium under examination. $T$ is the temperature. The thermal conductivity $K$, density $\rho_{air}$ and specific heat $c_{air}$ for air are temperature dependent, whereas $\rho_{Al,Cu}$ and $c_{Al,Cu}$ for copper and aluminium are constants (see Tables 1, 2, 3). In order to simplify the problem we consider all values as constants.

| $T[K]$ | $\rho[\frac{g}{cm^3}]$ | $c[\frac{J}{gK}]$ | $K[\frac{W}{cmK}]$ | $\kappa[\frac{cm^2}{s}]$ |
|---|---|---|---|---|
| 300 | $11.7 \cdot 10^{-4}$ | 1.00 | $2.64 \cdot 10^{-4}$ | 0.226 |
| 400 | $8.80 \cdot 10^{-4}$ | 1.01 | $3.26 \cdot 10^{-4}$ | 0.367 |
| 500 | $7.04 \cdot 10^{-4}$ | 1.03 | $3.85 \cdot 10^{-4}$ | 0.531 |
| 600 | $5.87 \cdot 10^{-4}$ | 1.05 | $4.44 \cdot 10^{-4}$ | 0.720 |
| 700 | $5.03 \cdot 10^{-4}$ | 1.07 | $5.06 \cdot 10^{-4}$ | 0.940 |
| 800 | $4.40 \cdot 10^{-4}$ | 1.10 | $5.65 \cdot 10^{-4}$ | 1.170 |

Table 1: Temperature dependence of $\rho, c$ and $K$ for air.

| $T[^0K]$ | $K_{Cu}[\frac{W}{cmK}]$ | $K_{Al}[\frac{W}{cmK}]$ |
|---|---|---|
| 300 | 4.01 | 2.37 |
| 400 | 3.96 | 2.40 |
| 500 | 3.93 | 2.40 |
| 600 | 3.86 | 2.36 |
| 700 | 3.79 | 2.31 |
| 800 | 3.66 | 2.18 |

Table 2: Temperature dependence of $K$ for Cu and Al.

| Material | $\rho[\frac{g}{cm^3}]$ | $c[\frac{J}{gK}]$ | $K[\frac{W}{cmK}]$ | $\kappa[\frac{cm^2}{s}]$ |
|---|---|---|---|---|
| Cu | 8.96 | 0.385 | 4.01 | 1.160 |
| Al | 2.70 | 0.897 | 2.37 | 0.979 |

Table 3: Constants for Cu and Al, $c$ at fixed pressure p and $K$ at $T = 300°K$.

## 2.2 Initial and Boundary Conditions

### 2.2.1 Initial Conditions

The initial condition gives information about the temperature $T_0$ of the medium at time $t_0 = 0$, i.e. the beginning of the heat transfer process:

$$T_0 = f(x, y, z) \tag{2}$$

For the computations of the next chapters we used the initial condition:

$$T_0 = 20^o C, \tag{3}$$

i.e. a constant initial temperature throughout the whole oven.

### 2.2.2 Boundary and Surface Conditions

There mainly exist four types of boundary conditions:

1. Prescribed surface temperature:

$$T(x) = \phi(x), x \in ?, \tag{4}$$

   where ? is the boundary. This condition is also called the Dirichlet-Condition.

2. Prescribed flux across the surface:

$$\frac{dT}{dn}(x) = \phi(x), x \in ?, \tag{5}$$

   where $\frac{dT}{dn}$ is the normal derivative. This condition is called the Neumann-Condition.

3. Linear heat transfer at the boundary:

$$\frac{dT}{dn} + h(T - T_s) = 0, \quad h = \frac{H}{K}, \tag{6}$$

   where $T$ is the temperature of the medium at the surface, $T_s$ is the temperature of the surrounding and $H$ is called the coefficient of surface heat transfer which depends on several facts such as the shape of the surface or the kind of medium which surrounds the surface (air, water, etc.) and its velocity. If for example air flows turbulently with velocity $u$ inside a circular pipe of diameter $d$, then forced convection takes place and $H$ becomes:

$$H = 5.5 \cdot 10^{-6} u^{0.8} d^{-0.2} \frac{cal}{cm^2 \cdot sec \cdot {}^oC}$$

Equation 6 includes the proportionality of the heat flux through the boundary to the temperature difference between the medium and the surrounding of the medium.

4. Nonlinear heat transfer at the boundary:
   An important example for this boundary condition is the natural convection. It takes place when e.g. a hot body is surrounded by water or air. In contrast to the forced convection the water or the air around the body are not in motion. The heat flux is then proportional to

$$H(T - T_s)^{\frac{5}{4}}. \tag{7}$$

   For small temperature differences this boundary condition can be linearized around $T_s$ and therefore approximated by the condition 6.

As the oven in our problem is just surrounded by air natural convection takes place which can be approximated by equation 6. As the initial temperature of the oven is $20^oC$ which is a value near room temperature (temperature of the surrounding air) this approximation is justified. We use $H = 5 \cdot 10^{-5}$ for a body surrounded by air.

# 3 Initial studies

Before the modelling of the temperature-distribution in the oven, we did some simple considerations of the processes involved in the heating of the oven.

We neglect the cavity and assume the oven to be totally out of copper or aluminium. If we assume a uniform temperature throughout the metal body, we can transform the heat equation into an ordinary differential equation. Let

$$\bar{T} = \frac{1}{V} \int_V T dV, \tag{8}$$

be the average temperature of the oven. We integrate both sides of (1) over the volume of the body and divide by the total volume to get.

$$\frac{d\bar{T}}{dt} = \frac{K}{\rho c V} \int_V \Delta T dV + \frac{1}{\rho c V} \int_V q dV. \tag{9}$$

Now using the Green Formula

$$\int_V \Delta u dV = \int_{\partial V} \frac{\partial u}{\partial n} d\sigma, \tag{10}$$

and inserting the proper boundary conditions, we will get the ODE we want. Using a linear relationship along the boundary, as is the case for forced convection, we get the following

$$\begin{aligned}\frac{d\bar{T}}{dt} &= \frac{K}{mc}\int_{\partial V}\frac{\partial T}{\partial n}d\sigma + \frac{P}{mc} \\ &= \frac{K}{mc}\left[-\frac{H}{K}A(\bar{T}-T_0)\right] + \frac{P}{mc} \\ &= -\frac{HA}{mc}(\bar{T}-T_0) + \frac{P}{mc} \qquad (11)\end{aligned}$$

where $A$ is the surface area, $P$ the power supplied and $m = \rho V$ for the mass.

We introduce $\theta = \bar{T} - T_0$ to write the equation as

$$\frac{d\theta}{dt} = -\frac{HA}{mc}\theta + \frac{P}{mc}. \qquad (12)$$

Putting $t = k_t\tau$ and $\theta = k_\theta y$ where $\tau$ and $y$ are dimensionless variables, we get by the chain rule

$$\begin{aligned}\frac{d\theta}{dt} &= k_\theta\frac{dy}{dt} \qquad (13) \\ &= \frac{k_\theta}{k_t}\frac{dy}{d\tau}. \qquad (14)\end{aligned}$$

Inserting this in (12), we get

$$mc\frac{k_\theta}{k_t}\frac{dy}{d\tau} = -HAk_\theta y + P, \qquad (15)$$

which can be simplified to

$$\frac{dy}{d\tau} = -\frac{AHk_t}{mc}y + \frac{k_tP}{mck_\theta}. \qquad (16)$$

Choosing both coefficients to be one, we can solve for $k_t$ and $k_\theta$ to get

$$\begin{aligned}k_t &= \frac{mc}{AH} \qquad (17) \\ k_\theta &= \frac{P}{AH}. \qquad (18)\end{aligned}$$

In dimensionless form we get the very simple equation

$$\frac{dy}{d\tau} = -y + 1, \qquad (19)$$

which together with the initial condition $y(0) = 0$ gives us the solution

$$y(\tau) = 1 - e^{-\tau}. \qquad (20)$$

That is, we will move towards a stable equilibrium at $y = 1$. This will correspond to $\theta = k_\theta = \frac{P}{AH}$. The solution is shown graphically in Figure 2.

It is also interesting to know some of the real values to get a feeling with the temperatures and times involved. In Table 4 the equilibrium temperature is calculated for typical values of $P$ and $H$. As these temperatures are obtained by assuming a uniform temperature throughout the oven, the temperature at the boundary will be too high and thereby the heat loss too big. The values therefore represent a lower bound for the real temperature in the oven.

Another interesting question is the time needed to reach a certain temperature. Would it be possible to heat the oven to high temperatures using only batteries in a reasonable amount of time? Table 5 shows the time needed to reach different temperatures using different amount of power. These times will represent an upper bound.

We can also easily calculate a lower bound by assuming that no heat is lost during the heating process. The time is then given by

$$t = \frac{mc(T - T_0)}{P}. \tag{21}$$

The results obtained with this method are shown in Table 6.

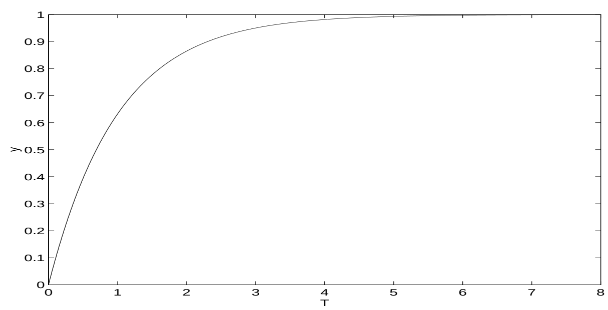


Figure 2: Temperature development in the oven.

| $P$ (W) | $H$ (J/cm$^2$ s °C ) | $T$=$\theta + T_0$ (°C) |
|---|---|---|
| 50 | $10^{-3}$ | 53.3 |
| 50 | $10^{-4}$ | 353.3 |
| 100 | $10^{-3}$ | 86.7 |
| 100 | $10^{-4}$ | 686.7 |

Table 4: Equilibrium temperature, $A$=150 cm$^2$ ,$T_0$=20 °C.

| T (°C) | $P$ (W) | $t$ (s) |
|---|---|---|
| 100 | 50 | 554.4 |
| | 100 | 258.2 |
| 200 | 50 | 1568.6 |
| | 100 | 635.7 |
| 300 | 50 | 3701.8 |
| | 100 | 1100.3 |

Table 5: Time to reach temperature, $T$ (upper bound). $A$=150 cm$^2$, $m$=787g, $c$=0.385 J/g °C and $H$=$10^{-4}$J/cm$^2$ s °C.

| T (°C) | $P$ (W) | $t$ (s) |
|---|---|---|
| 100 | 50 | 484.8 |
| | 100 | 242.4 |
| 200 | 50 | 1090.8 |
| | 100 | 545.4 |
| 300 | 50 | 1696.8 |
| | 100 | 848.4 |

Table 6: Time to reach temperature, $T$ (lower bound). $m$=100g and $c$=0.385 J/g °C .

# 4 Numerical Approximation of the Temperature Distribution

In this chapter we discuss the numerical method we use for approximating the temperature distribution inside the oven. We assume the oven to have reached the equilibrium state, i.e. $\frac{\partial T}{\partial t} = 0$. We use finite differences to discretize the heat equation. Furthermore we use the Gauss-Seidel method to solve the discretized heat equation.

## 4.1 Finite differences

We choose the geometry of the cross section such that we can use the finite difference method to approximate the derivatives in the heat equation and boundary conditions. For the numerical approximation we use an equidistant grid, with $N$ points in each direction. Let $h$ be the step size, then the coordinates of the grid points are $(x_i, y_j) = (h \cdot i, h \cdot j)$ for $i, j = 0, ..., N$. The points in the cavity are the points $(x_i, y_j)$ for $i, j = L, ..., M$, see Figure 3. Furthermore we denote the temperature at the point $(x_i, y_j)$ as $T_{ij}$, that is $T_{ij} = T(x_i, y_j)$, and the heat source as $q_{ij} = q(x_i, y_j)$.

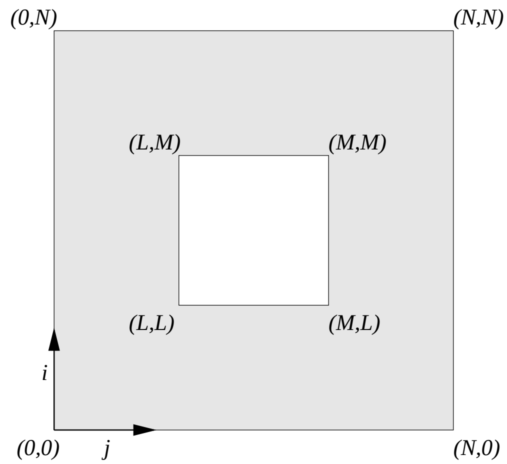


Figure 3: Discretized square oven.

The derivatives of the temperature with respect to $x$ and $y$ are approximated by a second order central scheme at the point $(x_i, y_j)$, that is

$$\frac{\partial^2 T_{ij}}{\partial x^2} \approx \frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{h^2} \tag{22}$$

$$\frac{\partial^2 T_{ij}}{\partial y^2} \approx \frac{T_{i,j-1} - 2T_{i,j} + T_{i,j+1}}{h^2} \tag{23}$$

The Laplacian then becomes, by summation of the latter two equations

$$\Delta T_{ij} \approx \frac{T_{i-1,j} + T_{i,j-1} + T_{i,j+1} + T_{i+1,j} - 4T_{i,j}}{h^2} \tag{24}$$

The normal derivative is equal (up to a plus or minus sign) to the first derivative of $x$ or $y$. Because we have chosen a second order scheme for the approximation of the Laplacian, we use the second order scheme to approximate the first derivative. For the outer boundary we use the following scheme:

$$\frac{\partial T_{ij}}{\partial x} \approx \frac{T_{i-1,1} - T_{i+1,j}}{2h} \text{ and } \frac{\partial T_{ij}}{\partial y} \approx \frac{T_{i,j-1} - T_{i,j+1}}{2h}. \tag{25}$$

On the outer boundary we approximate the normal derivative by using a ghostpoint. In Figure 4 the ghostpoint is denoted as a cross.

The boundary condition $-K_c \frac{\partial T}{\partial n} = H(T - T_0)$ is used to give an expression for the ghostpoints

$$T_{-1,j} = T_{1,j} + \tfrac{2hH}{K_c}(T_{i,j} + T_0) \qquad j = 0, 1 \dots N \tag{26}$$

$$T_{i,-1} = T_{i,1} + \tfrac{2hH}{K_c}(T_{i,j} + T_0) \qquad i = 0, 1 \dots N \tag{27}$$

$$T_{N+1,j} = T_{N-1,j} + \tfrac{2hH}{K_c}(T_{i,j} + T_0) \quad j = 0, 1 \dots N \tag{28}$$

$$T_{i,N+1} = T_{i,N+1} + \tfrac{2hH}{K_c}(T_{i,j} + T_0) \quad i = 0, 1 \dots N \tag{29}$$

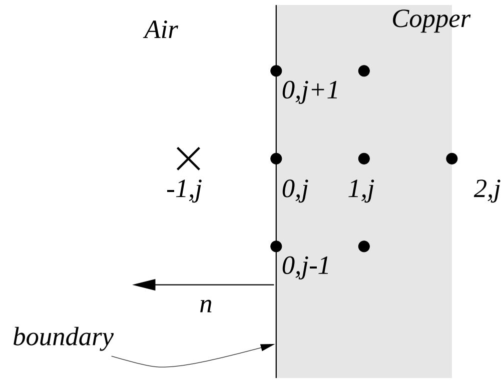


Figure 4: Discretizing the outer boundary.

Let the set $B_{in}$ be the inner boundary, that is

$$B_{in} = \{(L,j) \cup (i,L) \cup (M,j) \cup (i,M) | \text{ where } L \leq i,j \leq M\}. \tag{30}$$

The heat equation now becomes, for points in the copper which are not on the inner boundary $B_{in}$:

$$\frac{T_{i-1,j} + T_{i,j-1} + T_{i,j+1} + T_{i+1,j} - 4T_{i,j}}{h^2} + \frac{1}{K_c} q_{ij} = 0, \tag{31}$$

and for a point in the oven hole, also not on the inner boundary $B_{in}$:

$$\frac{T_{i-1,j} + T_{i,j-1} + T_{i,j+1} + T_{i+1,j} - 4T_{i,j}}{h^2} = 0. \tag{32}$$

On the inner boundary we have to be cautious. The normal derivative of the temperature of the metal can only be approximated using temperature values which are in the metal. The same yields for the temperature in the air. If we for instance evaluate the boundary conditions at the boundary shown in Figure 5 we use the points which are denoted by a cross in order to approximate the normal derivative of the metal on the boundary. For the temperature in the air we use the points that are denoted by a circle.

The approximation of the normal derivative for the temperature in the metal and in the air is:

$$\frac{\partial T_{ij}^{Cu}}{\partial n} = \frac{\partial T_{ij}^{Cu}}{\partial x} \approx \frac{-3T_{ij}^{Cu} + 4T_{i-1,j}^{Cu} + T_{i-2}^{Cu}}{2h}, \tag{33}$$

$$\frac{\partial T_{ij}^{a}}{\partial n} = \frac{\partial T_{ij}^{a}}{\partial x} \approx \frac{3T_{ij}^{a} + 4T_{i+1,j}^{a} - T_{i+2}^{a}}{2h}. \tag{34}$$

Furthermore the second inner boundary condition is

$$T_{ij}^{a} = T_{ij}^{Cu} \tag{35}$$

For the points on the inner boundary we find a different scheme

$$K_{Cu}(4T^{Cu}_{L-1,j} - T^{Cu}_{L-2,j}) - 3(K_a + K_{Cu})T_{L,j} + K_a(4T^a_{L+1,j} - T^a_{L+2,j}) = 0 \quad (36)$$
$$K_{Cu}(4T^{Cu}_{i,L-1} - T^{Cu}_{i,L-2}) - 3(K_a + K_{Cu})T_{i,L} + K_a(4T^a_{i,L+1} - T^a_{i,L+2}) = 0 \quad (37)$$
$$K_a(4T^a_{M-1,j} - T^a_{M-2,j}) - 3(K_{Cu} + K_a)T_{M,j} + K_{Cu}(4T^{Cu}_{M+1,j} - T^{Cu}_{M+2,j}) = 0 \quad (38)$$
$$K_a(4T^a_{i,M-1} - T^a_{i,M-2}) - 3(K_{Cu} + K_a)T_{i,M} + K_{Cu}(4T^{Cu}_{i,M+1} - T^{Cu}_{i,M+2}) = 0, \quad (39)$$

with $L < i, j < M$. Note that the discrete heat equation is not satisfied on the inner boundary points. In the corner points of the inner boundary we use the discrete heat equation. The temperature at a point $(i, j)$ can now be calculated, using the discrete heat equations (31) or (32) (if necessary with ghostpoints) when $(i, j) \notin B_{in}$ or the scheme from equations (36-39) when a $(i, j) \in B_{in}$.

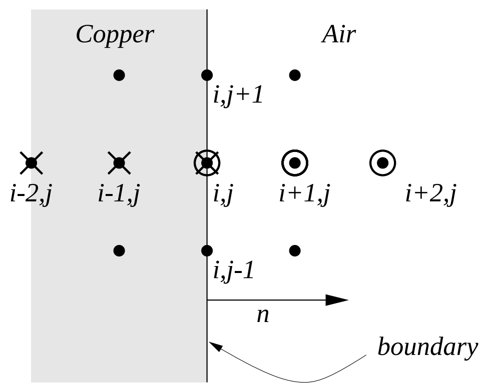


Figure 5: Discretizing the inner boundary.

## 4.2 Numerical solution: Gauss-Seidel Method

Equations (31), (32) and (36-39) are implicit schemes. In order to solve these equations we use the Gauss-Seidel method which can be described by the following iteration process: Given the temperature $T^k_{i,j}$ for $0 \le i, j \le N$ at a certain step $k$ we calculate a the next approximation of the temperature, $T^{k+1}_{i,j}$ using the numerical scheme's (31), (32) and (36-39). For example the implementation of the discretized heat equation is the following:

$$(31) \;\Rightarrow\; T^{k+1}_{i,j} = \frac{1}{4}\left(T^k_{i-1,j} + T^k_{i,j-1} + T^k_{i,j+1} + T^k_{i+1,j} + \frac{h^2}{K_c} q_{ij}\right). \quad (40)$$

When $T^k_{i,j}$ and $T^{k+1}_{i,j}$ are close to each other we say that $T_{i,j} = T^k_{i,j}$ is the solution of the problem. For values of $H$ not too small, say of larger than 0.1 this method is converging with in reasonable time. There are other methods which have better asymptotic behaviour, for example the Successive Over-Relaxation method.

# 5 A Cylindrical Oven

In this chapter we discuss an analytical solution of the problem. The analytical solution is obtained by using a cylindrical geometry and a Fourier representation of the temperature. The heat sources are approximated by a Dirac-delta function.

## 5.1 Computation of an analytic solution

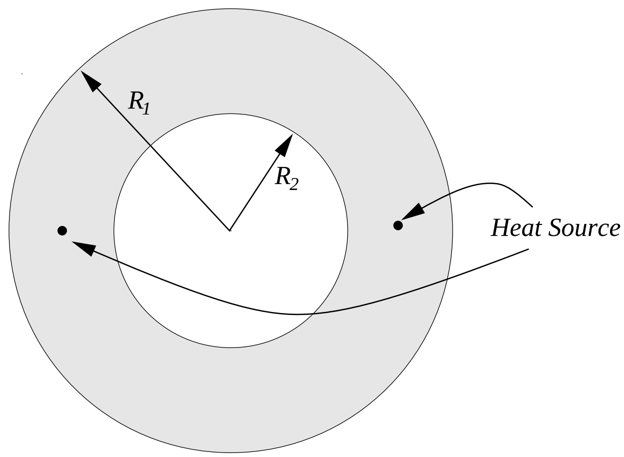


Figure 6: The two dimensional cylindrical model.

In this section we consider a cylindrical oven, see Figure 6. The oven is a circular disk with a circular cavity. The radius of the disk is $R_1$, the radius of the cavity is $R_2$. The heat equation in polar coordinates is the following:

$$\Delta T^c + \frac{q}{K_c} = 0 \quad \text{for } R_2 < r < R_1, \tag{41}$$

$$\Delta T^a = 0 \quad \text{for } 0 \leq r < R_2, \tag{42}$$

with

$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial f}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}, \tag{43}$$

and with the boundary conditions

$$\frac{\partial T^c}{\partial r} = \alpha(T^c - T_0) \quad \text{for } r = R_1, \tag{44}$$

$$T^c = T^a \quad \text{for } r = R_2, \tag{45}$$

$$\frac{\partial T^c}{\partial r} = \beta\frac{\partial T^a}{\partial r} \quad \text{for } r = R_2, \tag{46}$$

with $\beta = K_a/K_c$ and $\alpha = -H/K_c$.

The heat sources are approximated using Dirac-delta functions, that is

$$q(x) = Q_{tot}\delta(x - \xi), \tag{47}$$

where $\xi$ is the coordinate of the heat source and $Q_{tot}$ is the total heat delivered by the source to the oven. The Dirac-delta function has the property that $\delta(x) = 0$ almost everywhere except for an arbitrary small neighbourhood around $x = 0$, and that the area integral of $\delta(x)$ over the whole space $R^2$ is equal to one. We know that a solution of equation (41) can be written as a sum of the homogeneous solution, called $T_h$, and a fundamental solution, called $T_\delta$, that is

$$T(r,\theta) = T_h(r,\theta) + T_\delta(r,\theta). \tag{48}$$

From literature we know that the fundamental solution is equal to

$$T_\delta = \frac{-Q_{tot}}{2K_a\pi}\ln(|x-\xi|) \tag{49}$$

We write the homogeneous solution $T_h$ as a Fourier series in the following way:

$$T_h(r,\theta) = u_0(r) + \sum_{n=1}^{\infty} u_n(r)\cos(n\theta) + v_n(r)\sin(n\theta), \tag{50}$$

then the homogeneous equation results in the following equations for the functions $u_n$ and $v_n$

$$ru_0' + u_0'' = 0 \qquad \text{and} \tag{51}$$

$$\left.\begin{array}{r} ru_n' + r^2u_n'' - n^2u_n = 0 \\ rv_n' + r^2v_n'' - n^2v_n = 0 \end{array}\right\} \quad \text{for } n > 0. \tag{52}$$

The general solution of equation (51) is

$$u_0(r) = A_0 + B_0\ln(r) \tag{53}$$

and of the solution of equation (52) is

$$u_n(r) = A_nr^n + B_nr^{-n} \tag{54}$$

$$v_n(r) = C_nr^n + D_nr^{-n}. \tag{55}$$

The constants $A_n, B_n, C_n$ and $D_n$ are obtained by demanding that the solution is regular and satisfies the boundary conditions. For the temperature in the cavity $T^a$ we can construct a similar Fourier series representation. Because the origin lies inside the cavity, the constants $B_n$ and $D_n$ should be equal to zero to have a regular solution. Summarising we have that the equations (41) and (42) have solutions of the following form:

$$T^a(r,\theta) = A_0^a + \sum_{n=1}^{\infty} A_n^a r^n\cos(n\theta) + C_n^a r^n\sin(n\theta) \tag{56}$$

$$\begin{aligned} T^c(r,\theta) &= T_\delta(r,\theta) + A_0^c + B_0^c\ln(r) \\ &+ \sum_{n=1}^{\infty}(A_n^c r^n + B_n^c r^{-n})\cos(n\theta) + (C_n^c r^n + D_n^c r^{-n})\sin(n\theta), \end{aligned} \tag{57}$$

where

$$T_\delta(r,\theta) = \frac{-Q_{tot}}{2K_a\pi}\frac{1}{N}\sum_{i=1}^{N}\ln(|x-\xi_i|)\ . \tag{58}$$

$N$ is the number of heat sources which are the points $x = \xi_i$. When we are able to write the functions $T_\delta$ and $\frac{\partial T_\delta}{\partial n}$ for $r = R_1$ and $r = R_2$ as a Fourier series, we can derive a set of equations for the unknown constants using the boundary conditions at $r = R_1$ and $r = R_2$.

## 5.2 Solution for an oven with one and four heat sources

In this section we present a special solution for the problem. The values of $R_1$, $R_2$ and the position of the heat sources $\xi_i$ are chosen such that we can calculate the Fourier coefficients of $T$ and $\frac{\partial T}{\partial r}$ at the inner and outer boundary analytically. The problem is also solvable for other values of $R_1$, $R_2$ and $\xi_i$, however the Fourier coefficients of $T$ and $\frac{\partial T}{\partial r}$ cannot always be calculated analytically and therefore should be calculated numerically.

When we choose $R_1 = 1$, $R_2 = 1/4$ and $N = 1$ with $\xi_1 = (1/2, 0)$ we have that the function $T_\delta$ and $\frac{\partial T_\delta}{\partial r}$ are

$$T_\delta(r,\theta) = \frac{-Q_{tot}}{4K_a\pi}\ln(r^2+\frac{1}{4}-r\cos(\theta)) \tag{59}$$

$$\frac{\partial T_\delta}{\partial r}(r,\theta) = -\frac{Q_{tot}}{4K_a\pi}\frac{2r-\cos(\theta)}{r^2+\frac{1}{4}-r\cos(\theta)}, \tag{60}$$

and the Fourier series of these functions at the boundaries are

$$T_\delta(1,\theta) = \frac{Q_{tot}}{2K_a\pi}\sum_{n=1}^{\infty}\frac{2^{-n}}{n}\cos(n\theta) \tag{61}$$

$$\frac{\partial T_\delta}{\partial r}(1,\theta) = \frac{-Q_{tot}}{2K_a\pi}\left(1+\sum_{n=1}^{\infty}2^{-n}\cos(n\theta)\right) \tag{62}$$

$$T_\delta(1/4,\theta) = \frac{Q_{tot}}{4K_a\pi}\left(\ln(2)+\sum_{n=1}^{\infty}\frac{2^{-n}}{n}\cos(n\theta)\right) \tag{63}$$

$$\frac{\partial T_\delta}{\partial r}(1/4,\theta) = \frac{-2Q_{tot}}{K_a\pi}\sum_{n=1}^{\infty}2^{-n}\cos(n\theta) \tag{64}$$

Using these series we can calculate the temperature in the entire oven. The functions constants are the following:

$$A_0^c = \frac{Q_{tot}}{2\pi K_c}\left(\ln(2)-\frac{1}{\alpha}\right) \tag{65}$$

$$A_n^c = \frac{Q_{tot}}{\pi n K_c}\frac{(32^n+8^n)n-(32^n-8^n)\alpha}{16^n(\beta+1)(n-\alpha)-(1-\beta)(n+\alpha)} \tag{66}$$

$$A_0^m = -\frac{Q_{tot}}{2\pi K_c\alpha} \tag{67}$$

$$A_n^m = \frac{Q_{tot}}{2\pi K_c n} \frac{(2^n + 8^n + \beta 8^n - 2^n\beta)(n+\alpha)}{16^n(\beta+1)(n-\alpha) - (1-\beta)(n+\alpha)} \tag{68}$$

$$B_n^m = \frac{Q_{tot}}{2\pi K_c n} \frac{(1-\beta)(2^n(n-\alpha) + 2^{-n}(n+\alpha))}{16^n(\beta+1)(n-\alpha) - (1-\beta)(n+\alpha)} \tag{69}$$

$$B_0^m = C_n^c = C_n^m = D_n^m = 0, \tag{70}$$

with $\beta = K_a/K_c$ and $\alpha = -H/K_c$.

We choose the following values $K_a = 1.0 \cdot 10^{-3}$, $K_c = 1$, $H = 1.0 \cdot 10^{-2}$ and $Q_{tot} = 1$. A contour plot of the temperature is shown in Figure 7. The bright area represents a high temperature; the dark area a low temperature.

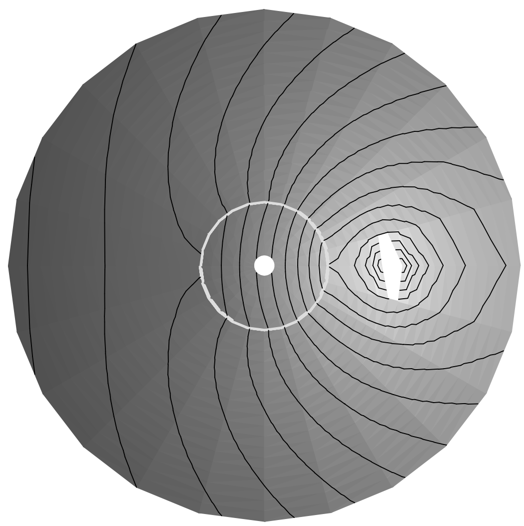

Figure 7: Contour plot of the temperature with one source.

Recall that the mean temperature difference of the steady state is given equal to the total heat divided by the area of the boundary times $H$. In our case we therefore have a mean temperature which is equal to $\bar{T} = 20 + 1/(2\pi H) = 35.915^o$C. The temperature distribution on the inner boundary is shown in Figure 8. The temperature difference over the inner boundary is of the order $0.45^o$C.

The temperature at the inner boundary attains its highest value near the source at $\theta = 0$ and attains its smallest value at $\theta = \pi$.

The solution for the oven with four heat sources at the points $\xi_1 = (1/2, 0)$, $\xi_2 = (0, 1/2)$, $\xi_1 = (-1/2, 0)$ and $\xi_1 = (0, -1/2)$ is shown in Figure 9 and 10. The temperature is a little higher than $36^o$C. This is due to the fact that the sources are close to the inner boundary. Furthermore we clearly see the influences of the heat sources. The temperature difference at the inner boundary is for this case of the order $0.01^o$C.

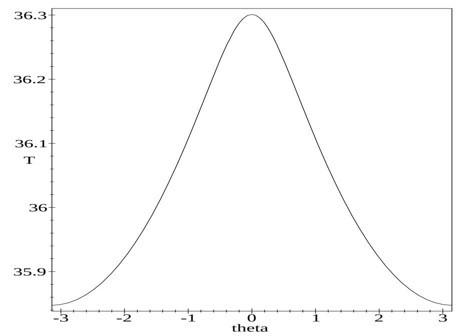


Figure 8: The temperature distribution on the inner boundary with one source.

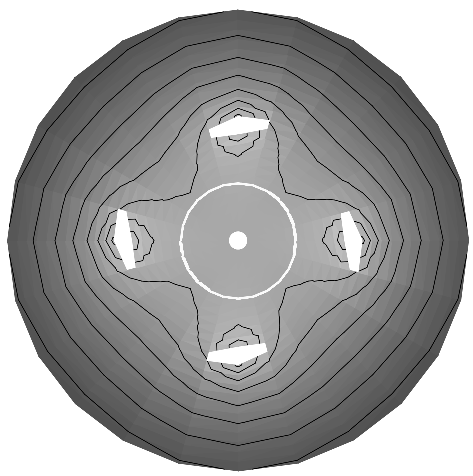

Figure 9: Contour plot of the temperature with four sources.

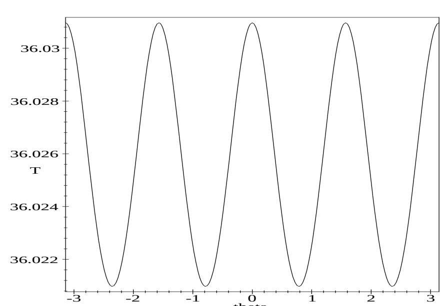


Figure 10: The temperature distribution on the inner boundary with four sources.

# 6 Conclusions

At the beginning of this modelling week our aim was to calculate the time evolution of heat within the oven. But implementing a scheme for the complete heat equation (1) we were confronted with problems of stability and convergence which could not be solved within this one week. Using the Finite Element Methods also seemed to be too time consuming.

Therefore we thought about upper and lower bounds for the time necessary in order to reach the steady state. Calculations in Chapter 3 show that these equilibrium temperatures can be reached within reasonable time.

That is why we could restrict our considerations in analysing the steady state. In this case, using a second order finite difference scheme (Chapter 4), problems of stability and convergence don't arise, although this method is a very slow and inaccurate one, but using SOR-methods the asymptotic behaviour can be improved. For special geometries also analytical results can be given (Chapter 5).

The analytical results for the cylindrical oven as well as the numerical results for an oven with quadratic square show us, that a symmetrical arrangement of two or four heat sources causes a very even temperature distribution in the cavity of the oven. Naturally, four heat sources would be an optimal solution of our problem as they cause a highly uniform heat distribution but also two heat sources (Figure 1) are a good compromise between cost and effect.

# References

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[2] D. R. Lide (editor), *Handbook of Chemistry and Physics*, $74^{th}$ edition, 1993–1994.