Convex Functions

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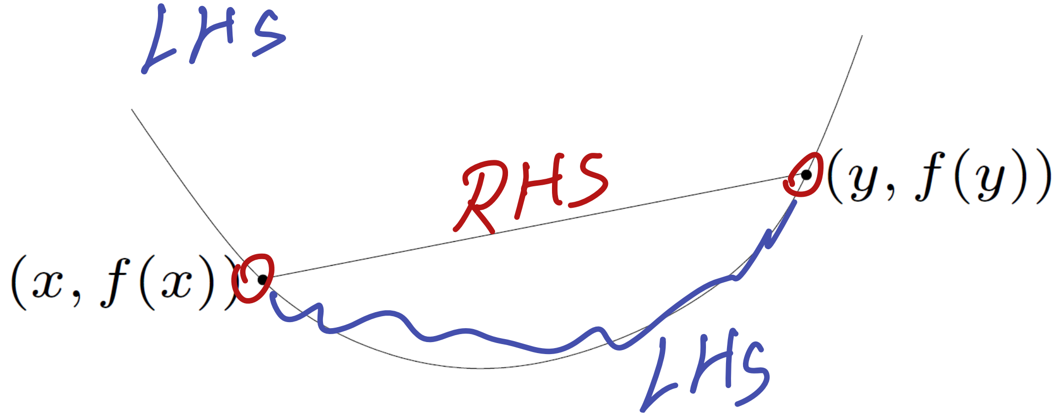
Outline

1. Definition of Convex Function
2. Restriction of a Convex Function to a Line
3. First and Second Order Conditions
4. Operations that Preserve Convexity
5. Quasi-Convexity, Log-Convexity, and Convexity w.r.t. Generalized Inequalities

Definition of Convex Function

- A function $f : \mathbb{R}^n \Rightarrow \mathbb{R}$ is said to be **convex** if the domain, $\mathsf{dom} f$, is convex and for any $\boldsymbol{x}, \boldsymbol{y} \in \mathsf{dom} f$ and $0 \le \theta \le 1$,

$$\underbrace{f(\theta \boldsymbol{x} + (1-\theta)\boldsymbol{y})}_{LHS} \le \underbrace{\theta f(\boldsymbol{x}) + (1-\theta) f(\boldsymbol{y})}_{RHS}$$



- f is **strictly convex** if the inequality is strict for $0 < \theta < 1$
- f is **concave** if $-f$ is convex

strongly convex: if $\exists\ \alpha > 0$, such that

$$g(x) = f(x) - \alpha \|x\|^2 \text{ is convex}$$

Lemma:

strong convexity $\underset{①}{\Rightarrow}$ strict convexity $\underset{②}{\Rightarrow}$ convexity

Proof: ① $\Rightarrow$ ②, strong convexity of f implies

$$f(\lambda x + (1-\lambda) y) - \alpha \|\lambda x + (1-\lambda) y\|^2 \leq$$

$$\lambda [f(x) - \alpha \|x\|^2] + (1-\lambda) [f(y) - \alpha \|y\|^2]$$

but

$$\lambda \alpha \|x\|^2 + (1-\lambda) \alpha \|y\|^2 - \alpha \|\lambda x + (1-\lambda) y\|^2 > 0$$

$$\forall x, y,\quad x \neq y,\quad \forall \lambda \in (0,1)$$

$$\Downarrow$$

$$f(\lambda x + (1-\lambda) y) < \lambda f(x) + (1-\lambda) f(y)$$

(strict convexity)

⚠ The converse statements are not true, e.g.,

$f(x) = x$ is convex, not strictly convex

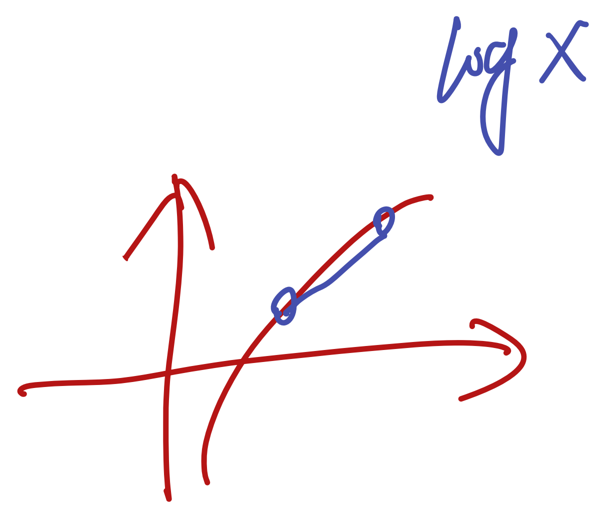
Examples on $\mathbb{R}$

Convex functions:

- affine: $ax + b$ on $\mathbb{R}$
- powers of absolute value: $|x|^p$ on $\mathbb{R}$, for $p \geq 1$ (e.g., $|x|$)
- powers: x^p on $\mathbb{R}_{++}$, for $p \geq 1$ or $p \leq 0$ (e.g., x^2)
- exponential: e^{ax} on $\mathbb{R}$
- negative entropy: $x \log x$ on $\mathbb{R}_{++}$

Concave functions:

- affine: $ax + b$ on $\mathbb{R}$
- powers: x^p on $\mathbb{R}_{++}$, for $0 \leq p \leq 1$
- logarithm: $\log x$ on $\mathbb{R}_{++}$



Examples on $\mathbb{R}^n$

- **Affine functions** $f(\boldsymbol{x}) = \boldsymbol{a}^T\boldsymbol{x} + b$ are convex and concave on $\mathbb{R}^n$
- **Norms** $\|\boldsymbol{x}\|$ are convex on $\mathbb{R}^n$ (e.g., $\|\boldsymbol{x}\|_\infty, \|\boldsymbol{x}\|_1, \|\boldsymbol{x}\|_2$)
- **Quadratic functions** $f(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x} + 2\boldsymbol{q}^T\boldsymbol{x} + r$ are convex $\mathbb{R}^n$ if and only if $\boldsymbol{P} \succeq \boldsymbol{0}$
- The **geometric mean** $f(\boldsymbol{x}) = \left(\prod_{i=1}^n x_i\right)^{1/n}$ is concave on $\mathbb{R}^n_{++}$
- The **log-sum-exp** $f(\boldsymbol{x}) = \log \sum_i e^{x_i}$ is convex on $\mathbb{R}^n$ (it can be used to approximate $\max_{i=1,\cdots,n} x_i$)

  $\max_i x_i \le f(x) \le \max_i x_i + \log n$
- **Quadratic over linear:** $f(\boldsymbol{x}, y) = \boldsymbol{x}^T\boldsymbol{x}/y$ is convex on $\mathbb{R}^n \times \mathbb{R}_{++}$

Examples on $\mathbb{R}^{n\times n}$

- **Affine functions:** (prove it!)

$$f(\boldsymbol{X}) = \mathrm{Tr}(\boldsymbol{A}\boldsymbol{X}) + b$$

are convex and concave on $\mathbb{R}^{n\times n}$

$C(Q) =$

- **Logarithmic determinant function:** (prove it!)

$\log\det(I + H^H Q H)$

$$f(\boldsymbol{X}) = \mathrm{logdet}(\boldsymbol{X})$$

channel capacity

is concave on $\mathbb{S}^n = \{\boldsymbol{X} \in \mathbb{R}^{n\times n} \mid \boldsymbol{X} \succeq \boldsymbol{0}\})$

- **Maximum eigenvalue function**: (prove it!)

$$f(\boldsymbol{x}) = \lambda_{\max}(\boldsymbol{X}) = \sup_{\boldsymbol{y}\neq \boldsymbol{0}} \frac{\boldsymbol{y}^T\boldsymbol{X}\boldsymbol{y}}{\boldsymbol{y}^T\boldsymbol{y}}$$

is convex on $\mathbb{S}^n$

$Xy = \lambda_{\max}(y)$

Epigraph

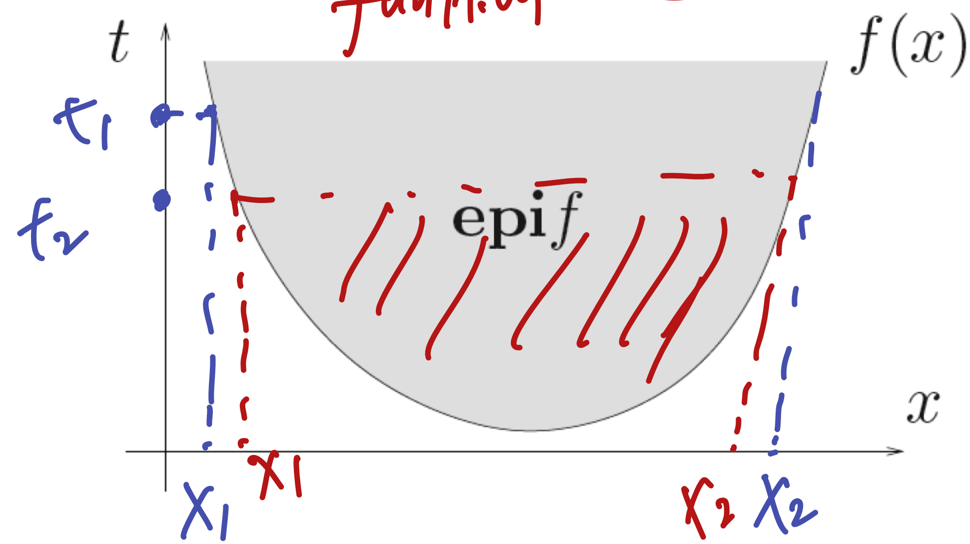
- The **epigraph** of f if the set

$$\operatorname{epi} f = \{\underbrace{(\boldsymbol{x}, t) \in \mathbb{R}^{n+1}} \mid \boldsymbol{x} \in \operatorname{dom} f,\ \underbrace{f(\boldsymbol{x}) \le t}\}$$

- Relation between convexity in sets and convexity in functions:

$$f \text{ is convex} \iff \operatorname{epi} f \text{ is convex}$$

function set



Outline

Restriction of a Convex Function to a Line

Proof is straight from the definition

- $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ is convex if and only if the function $g\colon \mathbb{R} \longrightarrow \mathbb{R}$

$$g(t) = f(\boldsymbol{x} + t\boldsymbol{v}), \quad \operatorname{dom} g = \{t \mid \boldsymbol{x} + t\boldsymbol{v} \in \operatorname{dom} f\}$$

  is convex for any $\boldsymbol{x} \in \operatorname{dom} f$, $\boldsymbol{v} \in \mathbb{R}^n$

- In words: a function is convex if and only if it is convex when restricted to an arbitrary line.

- Implication: we can check convexity of f by checking convexity of functions of one variable!

- Example: concavity of $\operatorname{logdet}(\boldsymbol{X})$ follows from concavity of $\log(x)$

Example

Example: concavity of $\operatorname{logdet}(\boldsymbol{X})$:

$$
\begin{aligned}
g(t) = \operatorname{logdet}(\boldsymbol{X} + t\boldsymbol{V}) = \quad & \operatorname{logdet}(\boldsymbol{X}) + \operatorname{logdet}(\boldsymbol{I} + t\boldsymbol{X}^{-1/2}\boldsymbol{V}\boldsymbol{X}^{-1/2}) \\
&= \operatorname{logdet}(\boldsymbol{X}) + \sum_{i=1}^{n} \log(1 + t\lambda_i)
\end{aligned}
$$

where λ_i's are the eigenvalues of $\boldsymbol{X}^{-1/2}\boldsymbol{V}\boldsymbol{X}^{-1/2}$.

The function g is concave in t for any choice of $\boldsymbol{X} \succ \boldsymbol{0}$ and $\boldsymbol{V}$; therefore, f is concave.

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First and Second Order Conditions I

- **Gradient** (for differentiable f):

$$\nabla f(\boldsymbol{x}) = \left[\frac{\partial f(\boldsymbol{x})}{\partial x_1} \quad \cdots \quad \frac{\partial f(\boldsymbol{x})}{\partial x_n} \right]^T \in \mathbb{R}^n$$

- **Hessian** (for twice differentiable f):

$$\nabla^2 f(\boldsymbol{x}) = \left(\frac{\partial^2 f(\boldsymbol{x})}{\partial x_i \partial x_j} \right)_{ij} \in \mathbb{R}^{n \times n}$$

- **Taylor series**:

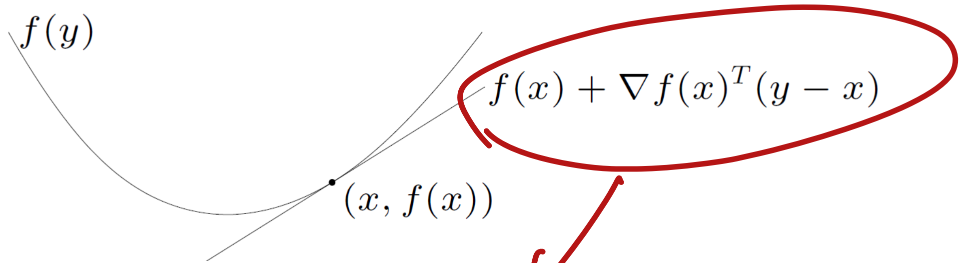
$$f(\boldsymbol{x} + \boldsymbol{\delta}) = f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^T \boldsymbol{\delta} + \frac{1}{2} \boldsymbol{\delta}^T \nabla^2 f(\boldsymbol{x}) \boldsymbol{\delta} + o\left(\|\boldsymbol{\delta}\|^2 \right)$$

$$\lim_{\delta \to 0} o(\|\delta\|^2) = 0$$

First and Second Order Conditions II

- **First-order condition:** a differentiable f with convex domain is convex if and only if

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^T(\boldsymbol{y}-\boldsymbol{x}) \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \operatorname{dom} f$$



- Interpretation: first-order approximation is a global under estimator

- **Second-order condition:** a twice differentiable f with convex domain is convex if and only if

$$\nabla^2 f(\boldsymbol{x}) \succeq \mathbf{0} \quad \forall \boldsymbol{x} \in \operatorname{dom} f$$

S_+^n

" the matrix cook book "
by Kaave Brandt

Examples

- **Quadratic function:** $f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x} + \boldsymbol{q}^T\boldsymbol{x} + r$(with $\boldsymbol{P} \in \mathbb{S}^n$)

$$\nabla f(\boldsymbol{x}) = \boldsymbol{P}\boldsymbol{x} + \boldsymbol{q}, \qquad \nabla^2 f(\boldsymbol{x}) = \boldsymbol{P}$$

A

  is convex if $\boldsymbol{P} \succeq \boldsymbol{0}$.

- **Least-squares objective:** $f(\boldsymbol{x}) = \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_2^2$

$$\nabla f(\boldsymbol{x}) = 2\boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}), \qquad \nabla^2 f(\boldsymbol{x}) = 2\boldsymbol{A}^T\boldsymbol{A}$$

A

  is convex.

- **Quadratic-over-linear:** $f(x, y) = x^2/y$

$$\nabla^2 f(x, y) = \frac{2}{y^3}\begin{bmatrix} y \\ -x \end{bmatrix}\begin{bmatrix} y & -x \end{bmatrix} \succeq \boldsymbol{0}$$

  is convex for $y > 0$.

A

Outline

Operations that Preserve Convexity I

How to establish the convexity of a given function?

- Applying the definition
- With first- or second-order conditions
- By restricting to a line
- Showing that the functions can be obtained from simple functions by operations that preserve convexity:
  - nonnegative weighted sum
  - composition with affine function (and other compositions)
  - pointwise maximum and supremum, minimization
  - perspective

Operations that Preserve Convexity II

- **Nonnegative weighted sum:** if f_1, f_2 are convex, then $\alpha_1 f_1 + \alpha_2 f_2$ is convex, with $\alpha_1, \alpha_2 \geq 0$.

- **Composition with affine functions:** if f is convex, then $f(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b})$ is convex (e.g., $\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\|$ is convex, $\operatorname{logdet}(\boldsymbol{I} + \boldsymbol{H}\boldsymbol{X}\boldsymbol{H}^T)$ is concave).

- **Pointwise maximum:** $f := \max\{f_1, \cdots, f_m\}$ is convex, if $f_1, \cdots, f_m$ are convex

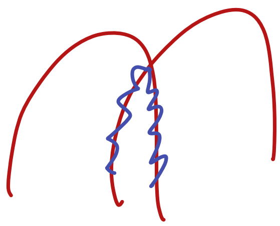
Example: sum of r largest components of $\boldsymbol{x} \in \mathbb{R}^n$:

$$f(\boldsymbol{x}) = x_{[1]} + x_{[2]} + \cdots + x_{[r]}$$

where $x_{[i]}$ is the ith largest component of $\boldsymbol{x}$.

Proof: $f(\boldsymbol{x}) = \max\{x_{i_1} + x_{i_2} + \cdots + x_{i_r} | 1 \leq i_1 < i_2 < \cdots < i_r \leq n\}$.

② pointwise minimum of two concave functions is concave



pointwise maximum of convex functions is convex

$$f = \max_{i\in\{1,\dots,m\}} f_i$$

proof: $\forall x, y \in \mathrm{dom}(f)$, $\lambda \in (0,1)$. Then

$$\begin{aligned}
f(\lambda x+(1-\lambda)y) &= f_j(\lambda x+(1-\lambda)y), \text{ for some } j\in\{1,\dots,m\} \\
&\le \lambda f_j(x)+(1-\lambda)f_j(y) \\
&\le \lambda \underbrace{\max\{f_1(x),\dots,f_m(x)\}}_{:f(x)} \\
&\quad + (1-\lambda)\underbrace{\max\{f_1(y),\dots,f_m(y)\}}_{:f(y)}
\end{aligned}$$

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proof via epigraphs

recall: f is convex $\iff$ $\mathrm{epi}(f)$ is a convex set

$$\mathrm{epi}(f) = \bigcap_{i=1}^{m} \underbrace{\mathrm{epi}(f_i)}_{\text{convex set}} \Rightarrow \text{convex set}$$

$f = \max_i f_i$

fact: the intersection of convex set is convex

Operations that Preserve Convexity III

❧ **Pointwise supremum:** if $f(\boldsymbol{x}, \boldsymbol{y})$ is convex in $\boldsymbol{x}$ for each $\boldsymbol{y} \in \mathcal{A}$, then

$$g(\boldsymbol{x}) = \sup_{\boldsymbol{y} \in \mathcal{A}} f(\boldsymbol{x}, \boldsymbol{y})$$

$\text{epi}(g) = \bigcap_{y \in A} \text{epi}(f(x,y))$

is convex.

Example: distance to farthest point in a set C:

$$f(\boldsymbol{x}) = \sup_{\boldsymbol{y} \in C} \|\boldsymbol{x} - \boldsymbol{y}\|$$

Example: maximum eigenvalue of symmetric matrix: for $\boldsymbol{X} \in \mathbb{S}^n$,

$$\lambda_{\max}(\boldsymbol{X}) = \sup_{\boldsymbol{y} \neq \boldsymbol{0}} \frac{\boldsymbol{y}^T \boldsymbol{X} \boldsymbol{y}}{\boldsymbol{y}^T \boldsymbol{y}}$$

$f(X, y) = \frac{y^T X y}{y^T y}$ is convex in X given y

Operations that Preserve Convexity IV

- **Composition with scalar functions:** let $g : \mathbb{R}^n \longrightarrow \mathbb{R}$, $h : \mathbb{R} \longrightarrow \mathbb{R}$, then the function $f(\boldsymbol{x}) = h(g(\boldsymbol{x}))$ satisfies:

$$f(\boldsymbol{x}) \text{ is convex if } \begin{array}{l} g \text{ convex, } h \text{ convex nondecreasing} \\ g \text{ concave, } h \text{ convex nonincreasing} \end{array}$$

- **Minimization:** if $f(\boldsymbol{x}, \boldsymbol{y})$ is convex in $(\boldsymbol{x}, \boldsymbol{y})$ and C is a convex set, then

$$g(\boldsymbol{x}) = \inf_{\boldsymbol{y} \in C} f(\boldsymbol{x}, \boldsymbol{y})$$

is convex (e.g., distance to a convex set).

Example: distance to a set C:

$$f(\boldsymbol{x}) = \inf_{\boldsymbol{y} \in C} \|\boldsymbol{x} - \boldsymbol{y}\|$$

is convex if C is convex.

Assume the infimum over $y \in C$ is attained for each x, we have

$$\operatorname{epi}(g) = \{(x, t) \mid (x, \underline{y}, t) \in \operatorname{epi}(f)\}$$

for some $y \in C$

the projection of a convex set on some of its $\underline{\text{components}}$

(y)

Operations that Preserve Convexity V

- **Perspective:** if $f(\boldsymbol{x})$ is convex, then its perspective

$$g(\boldsymbol{x}, t) = t f(\boldsymbol{x}/t), \quad \mathrm{dom}\, g = \{(\boldsymbol{x}, t) \in \mathbb{R}^{n+1} | \boldsymbol{x}/t \in \mathrm{dom}\, f, t > 0\}$$

  is convex.

Example: $f(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{x}$ is convex; hence $g(\boldsymbol{x}, t) = \boldsymbol{x}^T \boldsymbol{x}/t$ is convex for $t > 0$.

Example: the negative logarithm $f(\boldsymbol{x}) = -\log \boldsymbol{x}$ is convex; hence the relative entropy function $g(\boldsymbol{x}, t) = t \log t - t \log \boldsymbol{x}$ is convex on $\mathbb{R}^2_{++}$.

$$(x, t, s) \in \text{epi}(g) \iff g(x,t) = t f\left(\frac{x}{t}\right) \le s$$

$$\iff f\left(\frac{x}{t}\right) \le \frac{s}{t}$$

$$\iff \left(\frac{x}{t}, \frac{s}{t}\right) \in \text{epi}(f)$$

$$\Rightarrow (u, v, w) \text{ to } (u, w)/v$$

Outline

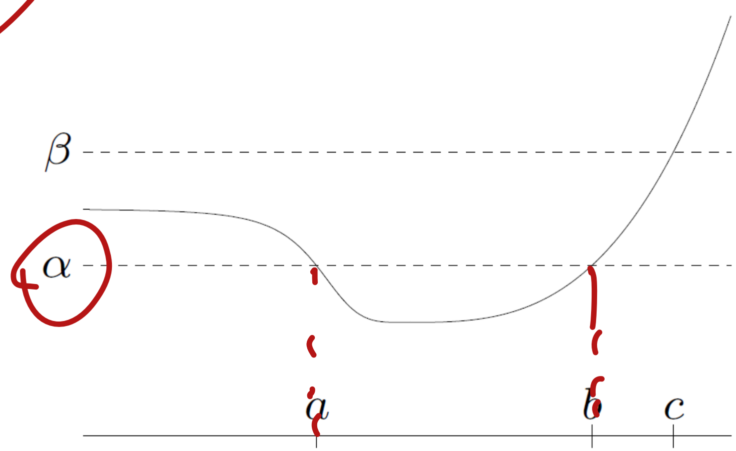
Quasi-Convexity Functions

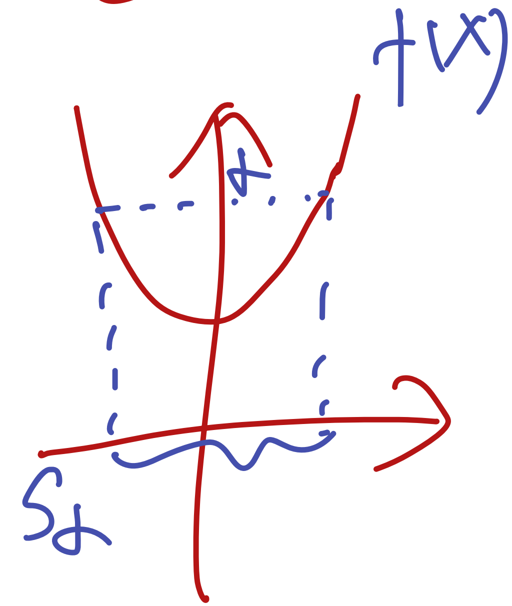
- A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is quasi-convex if $\operatorname{dom} f$ is convex and the sublevel sets

$$S_\alpha = \{\boldsymbol{x} \in \operatorname{dom} f \mid f(\boldsymbol{x}) \leq \alpha\}$$

are convex for all α.

fixed





- f is quasiconcave if $-f$ is quasiconvex.

Proof: $x, y \in S_\alpha$, $\lambda \in (0,1)$

$x \in S_\alpha \Rightarrow f(x) \leq \alpha$ — ①

$y \in S_\alpha \Rightarrow f(y) \leq \alpha$ — ②

f convex $\Rightarrow \underbrace{f(\lambda x + (1-\lambda) y)}$

$\leq \lambda f(x) + (1-\lambda) f(y)$

$\leq \lambda \alpha + (1-\lambda)\alpha$

$= \alpha$

$\lambda x + (1-\lambda) y \in S_\alpha$

convexity $\Rightarrow$ quasi-convexity

$\Leftarrow$ ⊗

Examples

- $\sqrt{|x|}$ is quasiconvex on $\mathbb{R}$
- $\text{ceil}(x) = \inf\{z \in \mathbb{Z} \mid z \geq x\}$ is quasilinear
- $\log x$ is quasilinear on $\mathbb{R}_{++}$

set $\{x \in \mathbb{R} \mid x_1 x_2 \geq \alpha\}$ is convex, for any α

- $f(x_1, x_2) = x_1 x_2$ is quasiconcave on $\mathbb{R}^2_{++}$
- the linear-fractional function

$\to [l, u] \to$ bi-section

$l \quad \frac{l+u}{2} \quad u$

$$f(\boldsymbol{x}) = \frac{\boldsymbol{a}^T \boldsymbol{x} + b}{\boldsymbol{c}^T \boldsymbol{x} + d}, \qquad \text{dom}\, f = \{\boldsymbol{x} \mid \boldsymbol{c}^T \boldsymbol{x} + d > 0\}$$

feasible infeasible

is quasilinear

$S_\alpha = \{x \mid f(x) \leq \alpha\} = \{x \mid c^T x + d > 0,\ a^T x + b \leq \alpha(c^T x + d)\}$

Log-Convexity

$\log(f(\theta x + (1-\theta)y)) \geq \theta \log f(x) + (1-\theta) \log f(y)$

- A positive function f is log-concave if $\log f$ is concave:

$$f(\theta \boldsymbol{x} + (1-\theta)\boldsymbol{y}) \geq f(\boldsymbol{x})^{\theta} f(\boldsymbol{y})^{1-\theta} \quad \text{for} \quad 0 \leq \theta \leq 1$$

- f is log-convex if $\log f$ is convex.

- Example: x^a on $\mathbb{R}_{++}$ is log-convex for $a \leq 0$ and log-concave for $a \geq 0$

$\log x$ is convex

- Example: many common probability densities are log-concave

$f(x) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \exp\left(-\frac{1}{2}(x-\bar{x})\Sigma^{-1}(x-\bar{x})\right)$

multivariate normal distribution

Convexity w.r.t. Generalized Inequalities

- $\boldsymbol{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is K-convex if $\operatorname{dom} \boldsymbol{f}$ is convex and for any $\boldsymbol{x}, \boldsymbol{y} \in \operatorname{dom} \boldsymbol{f}$ and $0 \leq \theta \leq 1$,

b a $\Rightarrow a \cdot b \in X$

$$\underbrace{\boldsymbol{f}(\theta \boldsymbol{x} + (1-\theta)\boldsymbol{y})} \preceq_K \underbrace{\theta \boldsymbol{f}(\boldsymbol{x}) + (1-\theta)\boldsymbol{f}(\boldsymbol{y})}$$

- Example: $\boldsymbol{f} : \mathbb{S}^m \longrightarrow \mathbb{S}^m$, $\boldsymbol{f}(\boldsymbol{X}) = \underbrace{\boldsymbol{X}^2}_{XX^T}$ is $\mathbb{S}^m_+$-convex

Reference

Chapter 3 of:

- Stephen Boyd and Lieven Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge University Press, 2004.

Book:

- Petersen, Kaare Brandt, and Michael Syskind Pedersen. “The matrix cookbook." Technical University of Denmark 7 (2008): 15.