ℓ_1-norm Methods for Convex-Cardinality Problems

- problems involving cardinality
- the ℓ_1-norm heuristic
- convex relaxation and convex envelope interpretations
- examples
- recent results

ℓ_1-norm heuristics for cardinality problems

- cardinality problems arise often, but are hard to solve exactly
- a simple heuristic, that relies on ℓ_1-norm, seems to work well
- used for many years, in many fields
  - sparse design
  - LASSO, robust estimation in statistics regression
  - support vector machine (SVM) in machine learning classifier $\{-1, 1\}$
  - total variation reconstruction in signal processing, geophysics
  - compressed sensing $y = \underset{n\times m}{A} x$ $n \ll m$
- new theoretical results guarantee the method works, at least for a few problems

$\mathrm{support}(x) = \{i \mid x_i \neq 0\}$

Cardinality

$\mathrm{card}(x) = |\mathrm{support}(x)|$

- the **cardinality** of $x \in \mathbf{R}^n$, denoted $\mathbf{card}(x)$, is the number of nonzero components of x
- $\mathbf{card}$ is separable; for scalar x, $\mathbf{card}(x) = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}$
- $\mathbf{card}$ is quasiconcave on $\mathbf{R}^n_+$ (but not $\mathbf{R}^n$) since

$$\mathbf{card}(x + y) \geq \min\{\mathbf{card}(x), \mathbf{card}(y)\}$$

  holds for $x, y \succeq 0$

- but otherwise has no convexity properties
- arises in many problems

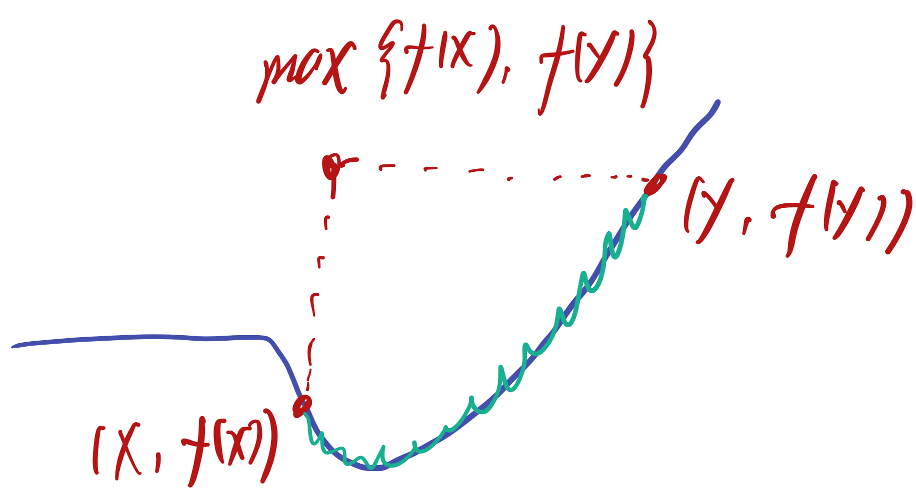
$f: R^n \to R$ is quasi-convex, then all the sub-level sets $\underline{S_\alpha} = \{x \in dom(f) \mid f(x) \underline{\le \alpha}\}$

for $\alpha \in R$, are convex

Basic properties:

A function f is quasi-convex if and only if $dom(f)$ is convex and any $x, y \in dom(f)$, and $0 \le \theta \le 1$

$$f(\theta x + (1-\theta) y) \le \max\{f(x), f(y)\}$$



General convex-cardinality problems

a **convex-cardinality problem** is one that would be convex, except for appearance of $\mathbf{card}$ in objective or constraints

examples (with $\mathcal{C}$, f convex):

- convex minimum cardinality problem:

$$\begin{array}{ll} \text{minimize} & \mathbf{card}(x) \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

convex constraint.

- convex problem with cardinality constraint:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C}, \quad \mathbf{card}(x) \leq k \end{array}$$

A

$\min f(x)$
s.t. $x \in C$, $x_i = 0, i \in B$, $|B| \le k$ | find x
s.t. $x \in C$, $x_i = 0, i \in B$
fixed

Solving convex-cardinality problems

convex-cardinality problem with $x \in \mathbf{R}^n$

- if we fix the sparsity pattern of x (*i.e.*, which entries are zero/nonzero) we get a convex problem

$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k} \approx 2^n$

- by solving 2^n convex problems associated with all possible sparsity patterns, we can solve convex-cardinality problem (possibly practical for $n \le 10$; not practical for $n > 15$ or so . . .)

exponential

- general convex-cardinality problem is (NP-) hard
- can solve globally by branch-and-bound
  - can work for particular problem instances (with some luck)
  - in worst case reduces to checking all (or many of) 2^n sparsity patterns

Boolean LP as convex-cardinality problem

- Boolean LP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b, \quad x_i \in \{0,1\} \end{array}$$

includes many famous (hard) problems, *e.g.*, 3-SAT, traveling salesman

(NP-hard)

- can be expressed as

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b, \quad \mathbf{card}(x) + \mathbf{card}(1-x) \leq n \end{array}$$

since $\mathbf{card}(x) + \mathbf{card}(1-x) \leq n \iff x_i \in \{0,1\}$

- conclusion: general convex-cardinality problem is hard

Sparse design

$$
\begin{array}{ll}
\text{minimize} & \mathbf{card}(x) \\
\text{subject to} & x \in \mathcal{C}
\end{array}
$$

- find sparsest design vector x that satisfies a set of specifications
- zero values of x simplify design, or correspond to components that aren't even needed
- examples:
  - FIR filter design (zero coefficients reduce required hardware)
  - antenna array beamforming (zero coefficients correspond to unneeded antenna elements)
  - truss design (zero coefficients correspond to bars that are not needed)
  - wire sizing (zero coefficients correspond to wires that are not needed)

– deep neural networks pruning

Sparse modeling / regressor selection

fit vector $b \in \mathbf{R}^m$ as a linear combination of k regressors (chosen from n possible regressors)

$$
\begin{array}{ll}
\text{minimize} & \|Ax - b\|_2 \\
\text{subject to} & \mathbf{card}(x) \le k
\end{array}
$$

(loss function)

- gives k-term model
- chooses subset of k regressors that (together) best fit or explain b
- can solve (in principle) by trying all $\binom{n}{k}$ choices

$\underbrace{\binom{n}{0}+\binom{n}{1}+\dots}_{x}+\binom{n}{n}$

- variations:
  - minimize $\mathbf{card}(x)$ subject to $\|Ax - b\|_2 \le \epsilon$
  - minimize $\underbrace{\|Ax - b\|_2}_{\text{loss}} + \lambda \underbrace{\mathbf{card}(x)}_{\text{regularity}}$

lasso

hyperparameter

probalistic model:

Goal: estimate $\Pr(Y \mid X = \underline{x})$

Assumption:

1. There is some underlying determistic linear function h that relates X to Y:

$$h(x; w_0, w) = \underline{w_0 + w^T x} \quad A$$

2. The oberserved values of Y are equal to $h(X; w_0, w)$ perturbed with additive Gaussian noise

$$Y \mid X \sim \underline{\text{normal}}(h(X; w_1, w_0), \beta^{-1})$$

$$(f(x \mid u, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-u)^2}{2\sigma^2}\right)$$

Given this model, we try to find the maximum likelihood estimator (MLE) for w_1, w_0

data set $\{x^{(i)}, y^{(i)}\}_{i=1}^{n}$

$$w_{ML}, w_{0,ML} = \arg\max_{w_0, w} \prod_{i=1}^{n} \Pr(y^{(i)} | x^{(i)}; w, w_0)$$

$$= \arg\max_{w_0, w} \sum_{i=1}^{n} \log \Pr(y^{(i)} | x^{(i)}; w, w_0)$$

$$= \arg\max_{w_0, w} - \sum_{i=1}^{n} (\underbrace{w^T x^{(i)} + w_0}_{\text{linear model}} - y^{(i)})^2$$

Sparse signal reconstruction

- estimate signal x, given
  - noisy measurement $y = Ax + v$, $v \sim \mathcal{N}(0, \sigma^2 I)$ (A is known; v is not)
  - prior information $\mathbf{card}(x) \leq k$

- maximum likelihood estimate $\hat{x}_{\rm ml}$ is solution of

$$\begin{array}{ll} \text{minimize} & \|Ax - y\|_2 \\ \text{subject to} & \mathbf{card}(x) \leq k \end{array}$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad Y|X \sim N(AX, \sigma^2)$$

Estimation with outliers

- we have measurements $y_i = a_i^T x + v_i + w_i$, $i = 1, \ldots, m$
- 1 noises $v_i \sim \mathcal{N}(0, \sigma^2)$ are independent
- 2 only assumption on w is sparsity: $\mathbf{card}(w) \leq k$
- $\mathcal{B} = \{i \mid w_i \neq 0\}$ is set of bad measurements or *outliers*
- maximum likelihood estimate of x found by solving

$$\begin{array}{ll} \underset{x,\ \mathcal{B}}{\text{minimize}} & \sum_{i \notin \mathcal{B}} (y_i - a_i^T x)^2 \\ \text{subject to} & |\mathcal{B}| \leq k \end{array}$$

$i \notin B \Rightarrow w_i = 0$

with variables x and $\mathcal{B} \subseteq \{1, \ldots, m\}$

- equivalent to

$$\begin{array}{ll} \text{minimize} & \|y - Ax - w\|_2^2 \\ \text{subject to} & \mathbf{card}(w) \leq k \end{array}$$

$Y | X, w$

$\sim N(AX + W, \sigma^2)$

Minimum number of violations

- set of convex inequalities

$$f_1(x) \le 0, \; \dots, \; f_m(x) \le 0, \qquad x \in \mathcal{C}$$

- choose x to minimize the number of violated inequalities:

$t_i = 0 \Rightarrow f_i(x) \le 0$

$t_i > 0 \Rightarrow f_i(x) > 0$

$\Downarrow$

$f_i(x) \le 0$ infeasible

$$\begin{array}{ll} \underset{x,t}{\text{minimize}} & \textbf{card}(t) \\ \text{subject to} & f_i(x) \le t_i, \quad i = 1, \dots, m \\ & x \in \mathcal{C}, \quad t \ge 0 \end{array}$$

- determining whether zero inequalities can be violated is (easy) convex feasibility problem

Linear classifier with fewest errors

- given data $(x_1, y_1), \ldots, (x_m, y_m) \in \mathbf{R}^n \times \{-1, 1\}$
- we seek linear (affine) classifier $y \approx \mathbf{sign}(w^T x + v)$
- classification error corresponds to $y_i(w^T x + v) \le 0$
- to find w, v that give fewest classification errors:

$$\begin{array}{ll} \underset{w,v,t}{\text{minimize}} & \mathbf{card}(t) \\ \text{subject to} & y_i(w^T x_i + v) + t_i \ge 1, \quad i = 1, \ldots, m, \quad t_i \geq 0 \end{array}$$

with variables w, v, t (we use homogeneity in w, v here) support vector machine (SVM)

$+1 \quad w^T x + b > 0$

$w^T x + b$ (hyperplane $\underline{h = (b, w)}$)

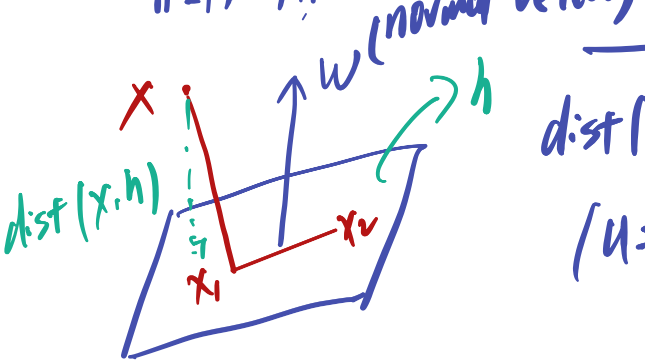
$-1 \quad w^T x + b < 0$

- h seperates the data means:

$$y_n(\underbrace{w^T}_{\text{weights}} x_n + \underbrace{b}_{\text{bias}}) > 0$$

- by re-scaling the weights and bias

$$\min_{n=1,\dots,N} y_n(w^T x_n + b) \underline{= 1}$$

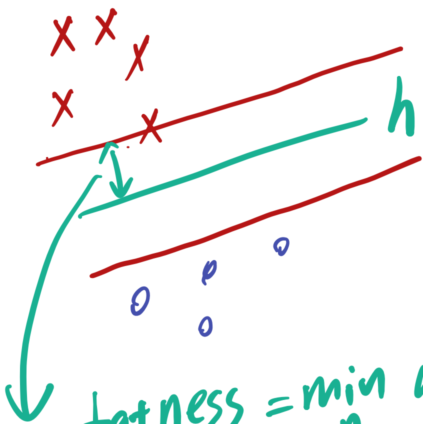


$$dist(x, h) = |u^T(x - x_1)|$$

$$\left(u = \frac{w}{\|w\|}\right)$$

$$= \frac{1}{\|w\|} |w^T x - \underbrace{w^T x_1}_{=b}|$$

$$= \frac{1}{\|w\|} |w^T x + b|$$



$$\mathrm{dist}(x,h) = \frac{1}{\|w\|}\left|w^T x + b\right|$$

$$\left(\text{since } \left|w^T x_n + b\right| = \left|y_n(w^T x_n + b)\right|\right.$$

$$= y_n(w^T x_n + b)\Big)$$

$$\mathrm{dist}(x_n, h) = \frac{1}{\|w\|} y_n(w^T x_n + b)$$

$$\text{fatness} = \min_n \mathrm{dist}(x_n, h)$$

$$= \frac{1}{\|w\|} \underbrace{\min_n y_n(w^T x_n + b)}_{=1}$$

$$= \frac{1}{\|w\|}$$

$$\text{maximize} \quad \frac{1}{\|w\|}$$

$$\text{subject to} \quad \min y_n(w^T x_n + b) = 1$$

$$\Downarrow$$

$$\text{minimize} \quad \frac{1}{2} w^T w$$

$$\text{subject to} \quad y_n(w^T x_n + b) \geq 1, \quad n = 1, \ldots, m$$

SVM

Smallest set of mutually infeasible inequalities

- given a set of mutually infeasible convex inequalities $f_1(x) \leq 0, \ldots, f_m(x) \leq 0$

- find smallest (cardinality) subset of these that is infeasible

- certificate of infeasibility is $g(\lambda) = \inf_x(\sum_{i=1}^m \lambda_i f_i(x)) \geq 1$, $\lambda \succeq 0$

- to find smallest cardinality infeasible subset, we solve

$\lambda_i = 0 \Rightarrow f_i(x) \leq 0$

$\lambda_i > 0 \Rightarrow f_i(x) > 0$

$$\begin{array}{ll} \text{minimize} & \mathbf{card}(\lambda) \\ \text{subject to} & g(\lambda) \geq 1, \quad \lambda \succeq 0 \end{array}$$

(assuming some constraint qualifications)

Portfolio investment with linear and fixed costs

- we use budget B to purchase (dollar) amount $x_i \geq 0$ of stock i
- trading fee is fixed cost plus linear cost: $\beta\,\mathbf{card}(x) + \alpha^T x$
- budget constraint is $\mathbf{1}^T x + \underbrace{\beta\,\mathbf{card}(x) + \alpha^T x}_{\text{trading fee}} \leq B$ trading fee
- mean return on investment is $\mu^T x$; variance is $x^T \Sigma x$

① ②

- minimize investment variance (risk) with mean return $\geq R_{\min}$:

$$
\begin{array}{ll}
\text{minimize} & x^T \Sigma x \quad \text{risk} \\
\text{subject to} & \mu^T x \geq R_{\min}, \quad x \succeq 0 \\
& \mathbf{1}^T x + \beta\,\mathbf{card}(x) + \alpha^T x \leq B
\end{array}
$$

Piecewise constant fitting

- fit corrupted x_{cor} by a piecewise constant signal $\hat{x}$ with k or fewer jumps
- problem is convex once location (indices) of jumps are fixed
- $\hat{x}$ is piecewise constant with $\leq k$ jumps $\Longleftrightarrow$ $\mathbf{card}(D\hat{x}) \leq k$, where

$\Rightarrow x_1 - x_2$

$$\cancel{D =} \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix} \in \mathbf{R}^{(n-1)\times n}$$

$$D\hat{x} = \begin{pmatrix} x_1 - x_2 \\ x_2 - x_3 \\ \vdots \\ x_{n-1} - x_n \end{pmatrix}$$

- as convex-cardinality problem:

$\Rightarrow$ given

$$\begin{array}{ll} \underset{\hat{x}}{\text{minimize}} & \|\hat{x} - x_{\text{cor}}\|_2 \\ \text{subject to} & \mathbf{card}(D\hat{x}) \leq k \end{array}$$

Piecewise linear fitting

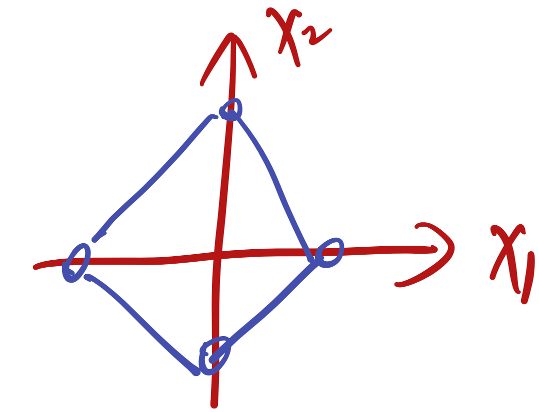
- fit x_{cor} by a piecewise linear signal $\hat{x}$ with k or fewer kinks
- as convex-cardinality problem:

$$\begin{array}{ll} \text{minimize} & \|\hat{x} - x_{\mathrm{cor}}\|_2 \\ \text{subject to} & \mathbf{card}(\nabla^2 \hat{x}) \leq k \end{array}$$

where

$$\nabla^2 = \begin{bmatrix} -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \end{bmatrix}$$

$$\nabla^2 \hat{x} = \begin{pmatrix} 2x_2 - x_1 - x_3 \\ 2x_3 - x_2 - x_4 \\ \vdots \end{pmatrix}$$



ℓ_1-norm heuristic

- replace $\mathbf{card}(z)$ with $\gamma\|z\|_1$, or add regularization term $\gamma\|z\|_1$ to objective

- $\gamma > 0$ is parameter used to achieve desired sparsity (when $\mathbf{card}$ appears in constraint, or as term in objective)

- more sophisticated versions use $\sum_i w_i|z_i|$ or $\sum_i w_i(z_i)_+ + \sum_i v_i(z_i)_-$, where w, v are positive weights

1 2

$$(z)_+ = \begin{cases} z, & \text{if } z > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$(z)_- = \begin{cases} -z, & \text{if } z < 0 \\ 0, & \text{otherwise} \end{cases}$$

Example: Minimum cardinality problem

- start with (hard) minimum cardinality problem

$$\begin{array}{ll} \text{minimize} & \textbf{card}(x) \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

$\|x\|_0 \rightarrow \|x\|_1$

($\mathcal{C}$ convex)

- apply heuristic to get (easy) ℓ_1-norm minimization problem

$$\begin{array}{ll} \text{minimize} & \|x\|_1 \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

Example: Cardinality constrained problem

- start with (hard) cardinality constrained problem (f, $\mathcal{C}$ convex)

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C}, \quad \mathbf{card}(x) \leq k \end{array}$$

- apply heuristic to get (easy) ℓ_1-constrained problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C}, \quad \|x\|_1 \leq \beta \end{array}$$

convex

or ℓ_1-regularized problem

$$\begin{array}{ll} \text{minimize} & f(x) + \gamma \|x\|_1 \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

convex

β, γ adjusted so that $\mathbf{card}(x) \leq k$

R

$\text{support}(\hat{x}) = \{ i \mid \hat{x}_{[i]} \neq 0 \} \Rightarrow \underbrace{\hat{x}_{[1]} \le \hat{x}_{[2]} \le}_{\text{set to } 0} \cdots \le \underbrace{\hat{x}_{[n]}}_{\hat{x}_{[n-k+1]}, \cdots}$

Polishing

- use ℓ_1 heuristic to find $\hat{x}$ with required sparsity
- fix the sparsity pattern of $\hat{x}$

  $\hat{x}_{[1]} \cdots \hat{x}_{[n-k]} = 0$

- re-solve the (convex) optimization problem with this sparsity pattern to obtain final (heuristic) solution

Interpretation as convex relaxation

- start with

$$\begin{array}{ll} \text{minimize} & \mathbf{card}(x) \\ \text{subject to} & x \in \mathcal{C}, \quad \underbrace{\|x\|_\infty \le R}_{} \end{array}$$

$|x_i| \le R, \ i=1,\dots,n$

- equivalent to mixed Boolean convex problem

$$\begin{array}{ll} \underset{x,\,z}{\text{minimize}} & \mathbf{1}^T z \\ \text{subject to} & |x_i| \le R z_i, \quad i = 1, \dots, n \\ & x \in \mathcal{C}, \quad \underbrace{z_i \in \{0, 1\},}_{} \quad i = 1, \dots, n \end{array}$$

with variables x, z

$1^T z = \sum_{i=1}^{n} z_i = \mathrm{card}(x)$

- now relax $z_i \in \{0,1\}$ to $z_i \in [0,1]$ to obtain

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T z \\ \text{subject to} & |x_i| \le R z_i, \quad i = 1,\dots,n \\ & x \in \mathcal{C} \\ & 0 \le z_i \le 1, \quad i = 1,\dots,n \end{array}$$

$\Rightarrow \frac{1}{R}|x_i| \le z_i$

which is equivalent to

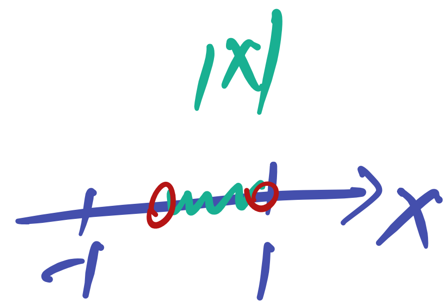
$$\begin{array}{ll} \text{minimize} & (1/R)\|x\|_1 \\ \text{subject to} & x \in \mathcal{C} \\ & \|x\|_\infty \le R \end{array}$$

the ℓ_1 heuristic

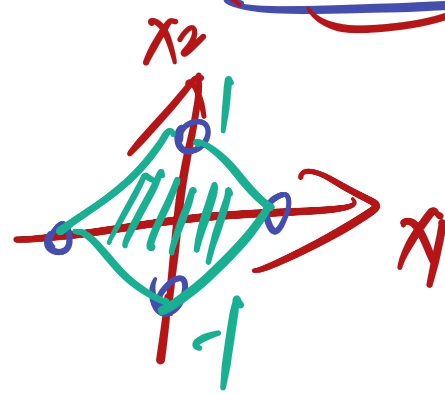
- optimal value of this problem is lower bound on original problem

Interpretation via convex envelope

- convex envelope f^{env} of a function f on set $\mathcal{C}$ is the largest convex function that is an underestimator of f on $\mathcal{C}$



- $\textbf{epi}(f^{\text{env}}) = \textbf{Co}(\textbf{epi}(f))$
- $f^{\text{env}} = (f^*)^*$ (with some technical conditions)
- for x scalar, $|x|$ is the convex envelope of $\textbf{card}(x)$ on $[-1, 1]$
- for $x \in \mathbf{R}^n$ scalar, $(1/R)\|x\|_1$ is convex envelope of $\textbf{card}(x)$ on $\{z \mid \|z\|_\infty \leq R\}$



Weighted and asymmetric ℓ_1 heuristics

$\min_x \ \mathbf{card}(x)$
$\text{s.t. } x \in C$

- ~~minimize $\mathbf{card}(x)$ over convex set $\mathcal{C}$~~
- suppose we know lower and upper bounds on x_i over $\mathcal{C}$

$$x \in \mathcal{C} \implies l_i \le x_i \le u_i$$

(best values for these can be found by solving $2n$ convex problems)

- ~~if $u_i < 0$ or $l_i > 0$, then $\mathbf{card}(x_i) = 1$ (*i.e.*, $x_i \ne 0$) for all $x \in \mathcal{C}$~~
- assuming $l_i < 0$, $u_i > 0$, convex relaxation and convex envelope interpretations suggest using

$$\sum_{i=1}^{n} \left(\frac{(x_i)_+}{u_i} + \frac{(x_i)_-}{-l_i} \right)$$

$\|x\|_p, \ 0 \le p \le 1$

as surrogate (and also lower bound) for $\mathbf{card}(x)$

sparsity-inducing function $\|x\|_1$

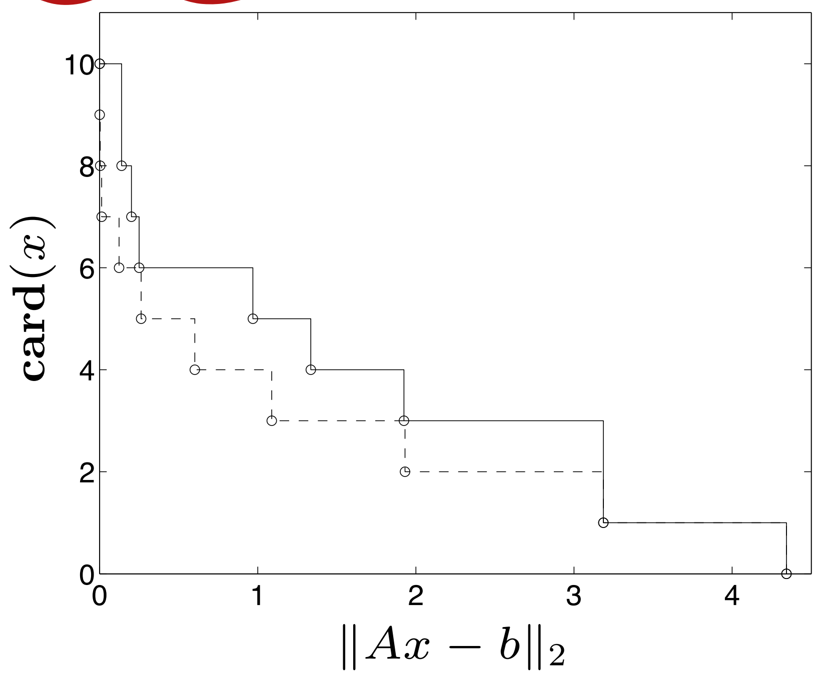
Regressor selection

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2 \\ \text{subject to} & \mathbf{card}(x) \leq k \end{array}$$

- heuristic:
  - minimize $\|Ax - b\|_2 + \gamma\|x\|_1$ lasso
  - find smallest value of γ that gives $\mathbf{card}(x) \leq k$
  - fix associated sparsity pattern ($i.e.$, subset of selected regressors) and find x that minimizes $\|Ax - b\|_2$

Example (6.4 in BV book)

- $A \in \mathbf{R}^{10\times 20}$, $x \in \mathbf{R}^{20}$, $b \in \mathbf{R}^{10}$
- dashed curve: exact optimal (via enumeration)
- solid curve: ℓ_1 heuristic with polishing



Sparse signal reconstruction

- convex-cardinality problem:

$$\begin{array}{ll} \text{minimize} & \|Ax - y\|_2 \\ \text{subject to} & \mathbf{card}(x) \leq k \end{array}$$

- ℓ_1 heuristic:

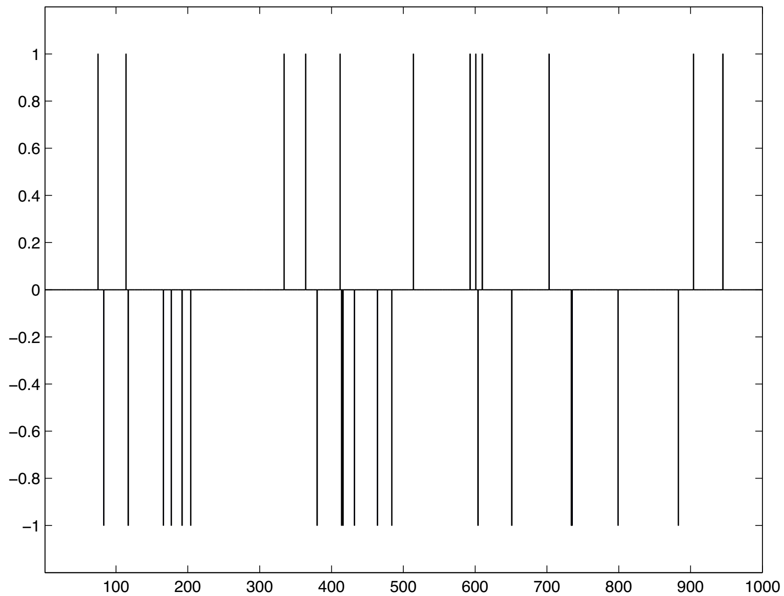
$$\begin{array}{ll} \text{minimize} & \|Ax - y\|_2 \\ \text{subject to} & \|x\|_1 \leq \beta \end{array}$$

  (called LASSO)

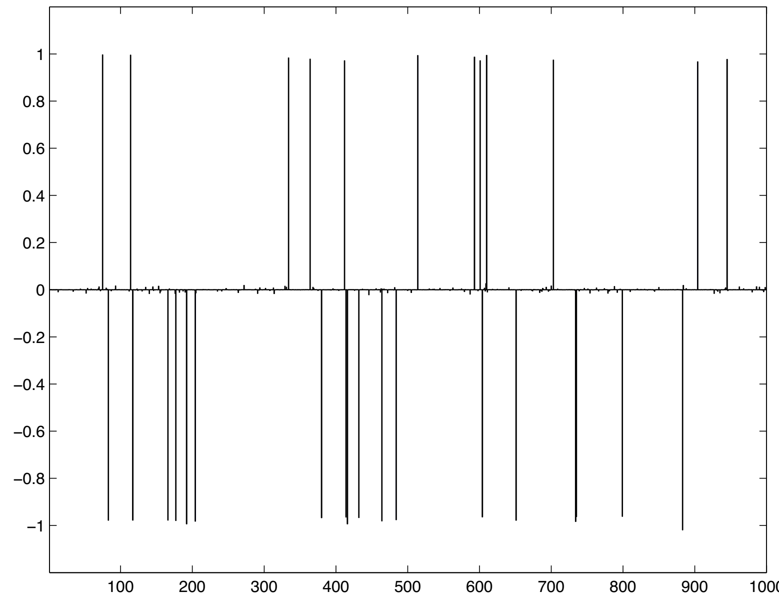
- another form: minimize $\|Ax - y\|_2 + \gamma\|x\|_1$
  (called basis pursuit denoising)

Example

- signal $x \in \mathbf{R}^n$ with $n = 1000$, $\mathbf{card}(x) = 30$
- $m = 200$ (random) noisy measurements: $y = Ax + v$, $v \sim \mathcal{N}(0, \sigma^2 I)$, $A_{ij} \sim \mathcal{N}(0, 1)$
- *left*: original; *right*: ℓ_1 reconstruction with $\gamma = 10^{-3}$



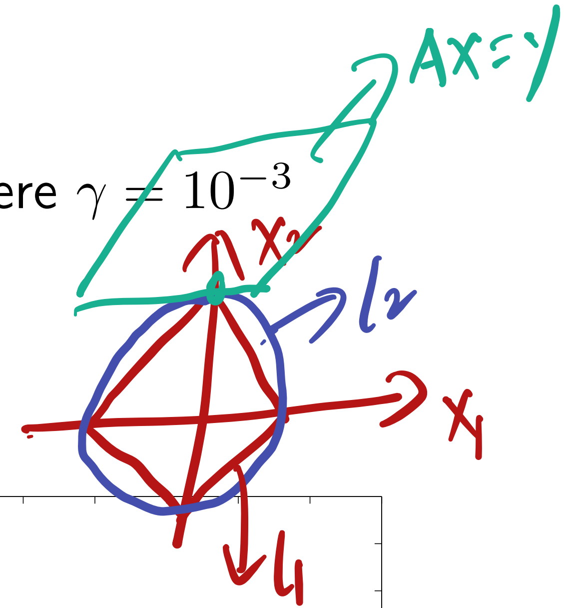
ground truth

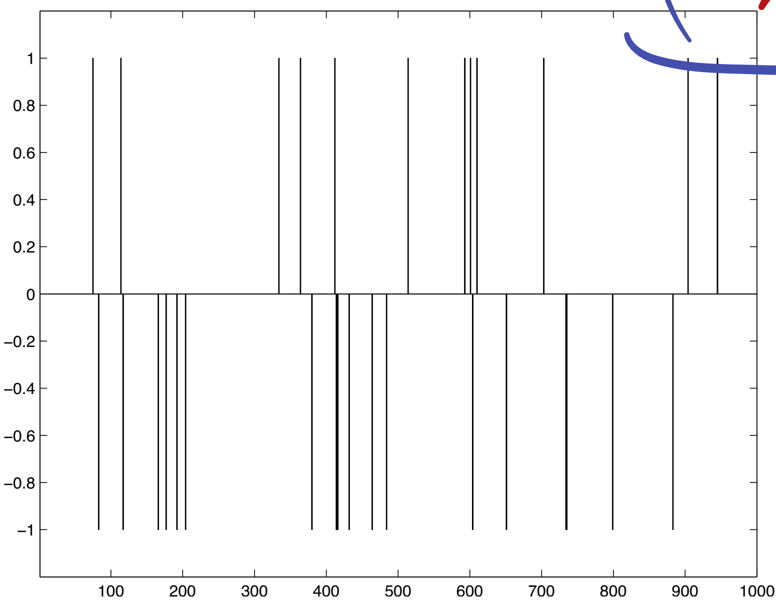


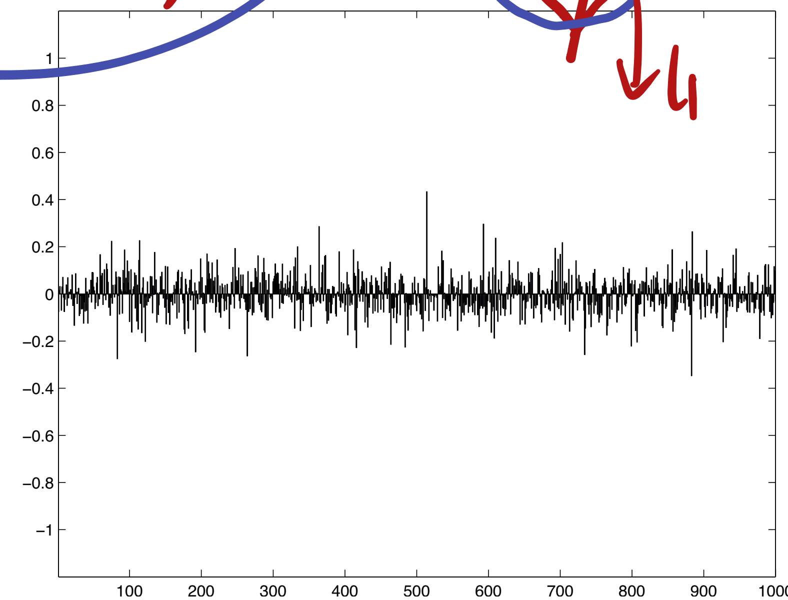
estimate

- $\cancel{\ell_2}$ reconstruction; minimizes $\|Ax - y\|_2 + \gamma\|x\|_2$, where $\gamma = 10^{-3}$
- *left*: original; *right*: ℓ_2 reconstruction

$$\min \|x\|_1 \quad \text{s.t.} \quad Ax = y$$







estimate

Some recent theoretical results

- suppose $y = Ax$, $A \in \mathbf{R}^{m \times n}$, $\mathbf{card}(x) \leq k$
- to reconstruct x, clearly need $m \geq k$
- if $m \geq n$ and A is full rank, we can reconstruct x without cardinality assumption
- when does the ℓ_1 heuristic (minimizing $\|x\|_1$ subject to $Ax = y$) reconstruct x (exactly)?

recent results by Candès, Donoho, Romberg, Tao, . . .

- (for some choices of A) if $m \geq (C \log n)k$, ℓ_1 heuristic reconstructs x exactly, with overwhelming probability

- C is absolute constant; valid A's include
  - $A_{ij} \sim \mathcal{N}(0, \sigma^2)$
  - Ax gives Fourier transform of x at m frequencies, chosen from uniform distribution