Brac University
Department of Electrical and Electronic Engineering
EEE282/ECE282 (V3)
Numerical Techniques
Experiment-03: Numerical Differentiation Techniques

Introduction:

We are familiar with the analytical method of finding the derivative of a function when the functional relation between the dependent variable y and the independent variable x is known. However, in practice, most often functions are defined only by tabulated data, or the values of y for specified values of x can be found experimentally. Also in some cases, it is not possible to find the derivative of a function by analytical method. In such cases, the analytical process of differentiation breaks down and some numerical processes have to be invented. The process of calculating the derivatives of a function by means of a set of given values of that function is called numerical differentiation. This process consists in replacing a complicated or an unknown function by an interpolation polynomial and then differentiating this polynomial as many times as desired.

Forward Difference Formula:

All numerical differentiation are done by expansion of Taylor series

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)h^2}{2} + \frac{f'''(x)h^3}{6} + \ldots\ldots\ldots\ldots\ldots..(1)$$

From (1)

$$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h)\ldots\ldots\ldots\ldots\ldots..(2)$$

Where, $O(h)$ is the truncation error, which consists of terms containing h and higher order in terms of h.

Lab Task 1:

Write the MATLAB code for numerical differentiation using Forward Difference Method.

Given $f(x) = e^x$, find $f'(1)$ using $h=10^{-1}$, 10^{-2},,,, upto 10^{-6}. Find out the error in each case by comparing the calculated value with exact value.

Function generating code for Forward Difference method-

```
function value = forward_difference(f,x,h,o,i)
%f = derivative function
%x = the value at which the derivative has to be evaluated
%h = divided difference
%o = Order of the derivative
%i = takes the values either 1 or 2; 2 denotes more accurate
results

    if(o == 1 && i == 1)
        value = (f(x+h)-f(x))/h;
    elseif(o == 1 && i == 2)
        value = (-f(x + 2*h) + 4*f(x+h) - 3*f(x))/(2*h);
    elseif(o == 2 && i == 1)
        value = (f(x+2*h)-2*f(x+h)+f(x))/(h*h);
    elseif(o == 2 && i == 2)
        value = (-f(x+3*h)+ 4*f(x+2*h)-5*f(x+h)+2*f(x))/(h*h);
    elseif(o == 3 && i == 1)
        value = (f(x+3*h) - 3*f(x+2*h) + 3*f(x+h) - f(x))/(h^3);
    elseif(o == 3 && i == 2)
        value = (-3*f(x+4*h) + 14*f(x+3*h) - 24*f(x+2*h) +
18*f(x+h) - 5*f(x))/(2*(h^3));
    elseif(o == 4 && i == 1)
        value = (f(x+4*h) - 4*f(x+3*h) + 6*f(x+2*h) - 4*f(x+h) +
f(x))/(h^4);
    elseif(o == 4 && i == 2)
        value = (-2*f(x+5*h) + 11*f(x+4*h) - 24*f(x+3*h) +
26*f(x+2*h) - 14*f(x+h) + 3*f(x))/(h^4);
    end

end
```

Central Difference Formula (of order O (h^2)):

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)h^2}{2} + \frac{f'''(c_1)h^3}{6} \ldots\ldots\ldots(3)$$

$$f(x-h) = f(x) - f'(x)h + \frac{f''(x)h^2}{2} - \frac{f'''(c_2)h^3}{6} \ldots\ldots\ldots(4)$$

Using (3) and (4)

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2) \ldots\ldots\ldots(5)$$

Where, $O(h^2)$ is the truncation error, which consists of terms containing h^2 and higher order terms of h.

Lab Task 2:

Write the MATLAB code for numerical differentiation using Central Difference Method.

Given f(x) = sin (cos (1/x)) find $f'(1/\sqrt{2})$ using h=10^{-1}, 10^{-2},,,, up to 10^{-6}. Find out the error in each case by comparing the calculated value with exact value.

Function generating code for Central Difference method-

```
function value = central_difference(f,x,h,o,i)
%f = derivative function
%x = the value at which the derivative has to be evaluated
%h = divided difference
%o = Order of the derivative
%i = takes the values either 1 or 2; 2 denotes more accurate
results

    if(o == 1 && i == 1)
        value = (f(x+h)-f(x-h))/(2*h);
    elseif(o == 1 && i == 2)
        value = (-f(x + 2*h) + 8*f(x+h) - 8*f(x-h) +
f(x-2*h))/(12*h);
    elseif(o == 2 && i == 1)
        value = (f(x+h)-2*f(x)+f(x-h))/(h*h);
```

```
    elseif(o == 2 && i == 2)
        value = (-f(x+2*h) + 16*f(x+h) - 30*f(x) + 16*f(x-h) -
f(x-2*h))/(12*h*h);
    elseif(o == 3 && i == 1)
        value = (f(x+2*h) - 2*f(x+h) + 2*f(x-h) -
f(x-2*h))/(2*(h^3));
    elseif(o == 3 && i == 2)
        value = (-f(x+3*h) + 8*f(x+2*h) - 13*f(x+h) + 13*f(x-h) -
8*f(x-2*h) + f(x-3*h))/(8*(h^3));
    elseif(o == 4 && i == 1)
        value = (f(x+2*h) - 4*f(x+h) + 6*f(x) - 4*f(x-h) +
f(x-2*h))/(h^4);
    elseif(o == 4 && i == 2)
        value = (-f(x+3*h) + 12*f(x+2*h) + 39*f(x+h) + 56*f(x) -
39*f(x-h) + 12*f(x-2*h) + f(x-3*h))/(6*(h^4));
    end

end
```

Central Difference Formula (of order $O(h^4)$):

Using Taylor series expansion it can be shown that

$$f'(x) = \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h} + O(h^4) \text{ ....(6)}$$

Here the truncation error reduces to h^4

Richardson's Extrapolation:

We have seen that

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2)$$

Which can be written as

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h} + Ch^2$$

Or, $f'(x) \approx D_0(h) + Ch^2$ ……………………………….(7)

If step size is converted to 2h

$$f'(x) \approx D_0(2h) + 4Ch^2 \text{ ……………………………….(8)}$$

Using (7) and (8)

$$f'(x) \approx \frac{4D_0(h) - D_0(2h)}{3} = \frac{-f_2 + 8f_1 - 8f_{-1} + f_{-2}}{12h} \text{ ……………….(9)}$$

Equation (9) is same as equation (6)

The method of obtaining a formula for $f'(x)$ of higher order from a formula of lower order is called extrapolation. The general formula for Richardson's extrapolation is

$$f'(x) = D_k(h) + O(h^{2k+2}) = \frac{4^k D_{k-1}(h) - D_{k-1}(2h)}{4^k - 1} + O(h^{2k+2}) \text{ ……(10)}$$

Practice Problems (Experiment - 03)

Problem-1: Derive the expressions of Forward Divided Difference (FDD), Backward Divided Difference (BDD), Central Divided Difference (CDD) of 1st order derivatives.

Problem-2: Write the MATLAB functions for FDD, BDD & CDD of 1st order derivatives using the expressions you derived in the previous problem.

Problem-3: For the given function:

$$f(x) = \frac{2x\sin(3x) + e^{-2x}}{x^{0.5}}$$

Determine the value of the 1st derivative of the function numerically for $x = 3.5$ with best possible accuracy.
Also, compare the numerical results of the derivatives with the analytical (by using the formulae of differentiation) results.

Problem-4: The following data are provided for the *displacement*, s of a moving object as a function of *time*, t:

$s\,(meter)$	0	4	8	12	16	20
$t\,(second)$	0	34.7	61.8	82.8	99.2	112.0

Determine the *velocity*, v of the object for each of the time instants. [Just to recall: $v = \frac{ds}{dt}$]