Brac University
Department of Electrical and Electronic Engineering
EEE282/ECE282 (V3)
Numerical Techniques
Experiment-04: Numerical Integration Techniques

Introduction

There are two cases in which engineers and scientists may require the help of numerical integration techniques. (1) Where experimental data is obtained whose integral may be required and (2) where a closed form formula for integrating a function using calculus is difficult or so complicated as to be almost useless. For example the integral

$$\Phi(t) = \int_0 \frac{t^3}{e^t - 1}\, dt.$$

Since there is no analytic expression for $\Phi(x)$, numerical integration techniques must be used to obtain approximate values of $\Phi(x)$.

Formulae for numerical integration called quadrature are based on fitting a polynomial through a specified set of points (experimental data or function values of the complicated function) and integrating (finding the area under the fitted polynomial) this approximating function. Any one of the interpolation polynomials studied earlier may be used.

Some of the Techniques for Numerical Integration

Trapezoidal Rule

Assume that the values of a function $f(x)$ are given at $x_1, x_1+h, x_1+2h \ldots\ldots x_1+nh$ and it is required to find the integral of $f(x)$ between x_1 and x_1+nh. The simplest technique to use would be to fit **straight lines** through $f(x_1), f(x_1+h) \ldots\ldots$ and to determine the **area** under this approximating function as shown in Fig 7.1.

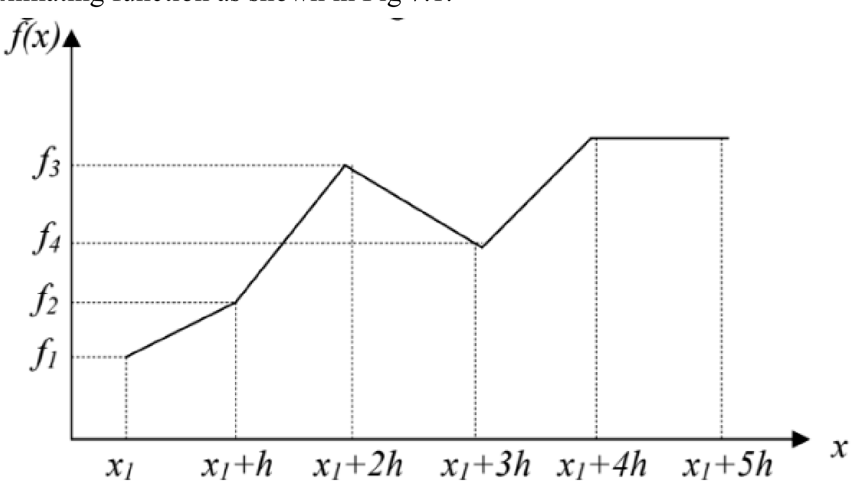


Fig. 7.1 Illustrating trapezoidal rule

For the first two points we can write:

$$\int_{x_1}^{x_1+h} f(x)dx = \frac{h}{2}(f_1 + f_2)$$

This is called first-degree Newton-Cotes formula.
From the above figure it is evident that the result of integration between x_1 *and* x_1+nh is nothing but the sum of areas of some trapezoids. In equation form this can be written as:

$$\int_{x_1}^{x_1+nh} f(x)dx = \sum_{i=1}^{n} \frac{(f_i + f_{i+1})}{2} h$$

The above integration formula is known as *Composite Trapezoidal rule.*
The composite trapezoidal rule can explicitly be written as:

$$\int_{x_1}^{x_1+nh} f(x)dx = \frac{h}{2}(f_1 + 2f_2 + 2f_3 + ...2f_n + f_{n+1})$$

Simpson's 1/3 Rule
This is based on approximating the function *f(x)* by fitting **quadratics** through sets of **three** points. For only three points it can be written as:

$$\int_{x_1}^{x_1+2h} f(x)dx = \frac{h}{3}(f_1 + 4f_2 + f_3)$$

This is called second-degree Newton-Cotes formula.
It is evident that the result of integration between x_1 *and* x_1+nh can be written as

$$\int_{x_1}^{x_1+nh} f(x)dx = \sum_{i=1,3,5,\dots,n-1} \frac{h}{3}(f_i + 4f_{i+1} + f_{i+2})$$

$$= \frac{h}{3}(f_1 + 4f_2 + 2f_3 + 4f_4 + 2f_5 + 4f_6 + \ \dots\ 4f_n + f_{n+1})$$

In using the above formula it is implied that *f* is known at an **odd number of points (*n*+1 is odd,** where n is the no. of subintervals).

Simpson's 3/8 Rule
This is based on approximating the function *f(x)* by fitting **cubic** interpolating polynomial through sets of **four** points. For only four points it can be written as:

$$\int_{x_1}^{x_1+3h} f(x)dx = \frac{3h}{8}(f_1 + 3f_2 + 3f_3 + f_4)$$

This is called third-degree Newton-Cotes formula.
It is evident that the result of integration between x_1 *and* x_1+nh can be written as

$$\int_{x_1}^{x_1+nh} f(x)dx = \sum_{i=1,4,7,\dots,n-2} \frac{h}{3}(f_i + 3f_{i+1} + 3f_{i+2} + f_{i+3})$$

$$= \frac{3h}{8}(f_1 + 3f_2 + 3f_3 + 2f_4 + 3f_5 + 3f_6 + 2f_7 + \dots\ + 2f_{n-2} + 3f_{n-1} + 3f_n + f_{n+1})$$

In using the above formula it is implied that *f* is known at ($n+1$) points where **n is divisible by 3**.
An algorithm for integrating a **tabulated function** using composite trapezoidal rule:
Remarks: $f_1, f_2, \dots\dots, f_{n+1}$ are the tabulated values at $x_1, x_{1+h}, \dots\dots x_{1+nh}$ ($n+1$ points)

1 Read h
2 *for* $i = 1$ to $n+1$ Read f_i *endfor*
3 $sum \leftarrow (f_1 + f_{n+1})/2$

```
4      for j = 2 to n do
5          sum ← sum + f_j
       endfor
6      integral ← h . sum
7  write integral  stop
```

Ex. 1. Integrate the function tabulated in Table 7.1 over the interval from x=1.6 to x=3.8 using composite trapezoidal rule with (a) h=0.2, (b) h=0.4 and (c) h=0.6

Table 7.1

X	*f(x)*	*X*	*f(x)*
1.6	4.953	2.8	16.445
1.8	6.050	3.0	20.086
2.0	7.389	3.2	24.533
2.2	9.025	3.4	29.964
2.4	11.023	3.6	36.598
2.6	13.468	3.8	44.701

The data in Table 7.1 are for $f(x) = e^x$. Find the true value of the integral and compare this with those found in (a), (b) and (c).

Ex. 2. (a) Integrate the function tabulated in Table 7.1 over the interval from x=1.6 to x=3.6 using Simpson's composite 1/3 rule.

(b) Integrate the function tabulated in Table 7.1 over the interval from x=1.6 to x=3.4 using Simpson's composite 3/8 rule.

An algorithm for integrating a **known function** using composite trapezoidal rule:

If *f(x)* is given as a closed form function such as $f(x) = e^{-x}\cos x$ and we are asked to integrate it from x_1 *to* x_2, we should decide first what h should be. Depending on the value of h we will have to evaluate the value of *f(x)* inside the program for $x=x_1+nh$ where n=0,1, 2,….n and $n = (x_2 - x_1)/h$.

```
1     h = (x2 - x1) / n
2     x ← x1
3     sum ← f(x)
4     for i = 2 to n do
5             x ← x + h
6             sum ← sum + 2f(x)
      endfor
7     x ← x2
8     sum ← sum + f(x)
9     integral ← (h/2) . sum
10    write integral
      stop
```

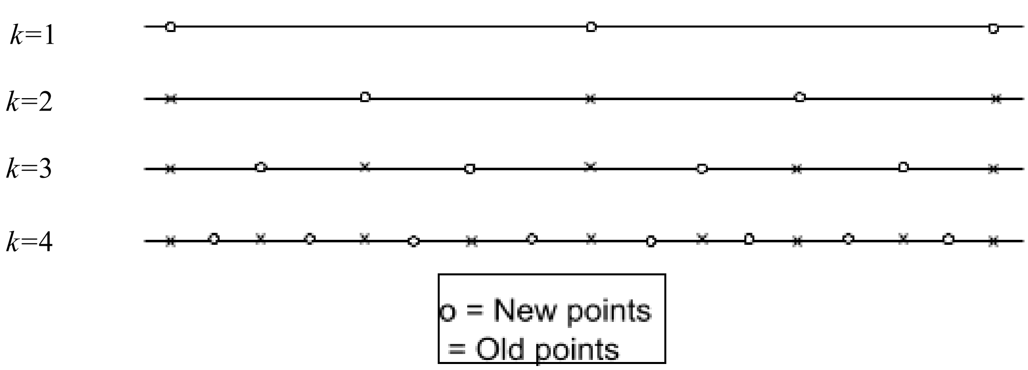
Ex. 3. (a) Find (approximately) each integral given below using the composite trapezoidal rule with $n = 12$.

$$(i)\ \int_{-1}^{1} (1+x^2)^{-1}dx \qquad\qquad (ii)\ \int_{0}^{4} x^2e^{-x}dx$$

Adaptive Integration

When *f(x)* is a known function we can choose the value for *h* arbitrarily. The problem is that we do not know a priori what value to choose for *h* to attain a desired accuracy (for example, for an arbitrary *h* sharp picks of the function might be missed). To overcome this problem, we can start with two subintervals $h = h_1 = (x_2 - x_1) / 2$ and apply either trapezoidal or Simpson's 1/3 rule. Then we let $h_2 = h_1 /2$ and apply the formula again, now with four subintervals and the results are compared. If the new value is sufficiently close, the process is terminated. If the 2nd result is not close enough to the first, *h* is halved again and the procedure is repeated. This is continued until the last result is close enough to its predecessor. This form of numerical integration is termed as adaptive integration.

The no. of computations can be reduced because when *h* is halved, all of the old points at which the function was evaluated appear in the new computation and thus repeating evaluation can be avoided. This is illustrated below.



An algorithm for adaptive integration of a **known function** using trapezoidal rule:

```
    Read x1, x2 , e          Remark: The allowed error in integral is e
    h ← x2 − x1
    S1 ← ( f (x1) + f (x2 )) / 2
    I1 ← h . S1
    i ← 1
    Repeat
        x ← x1 + h /2
        for j = 1 to i do
```

```
        S_1 ← S_1 + f(x)
            x ← x + h
     endfor
     i ← 2i
     h ← h/2
     I_0 ← I_1
     I_1 ← h . S_1
  until I_1 − I_0 ≤ e . I_1
  write I_1, h, i
  stop
```

Ex. 4. Evaluate the integral of xe^{-2x^2} between x=0 and x=2 using a tolerance value sufficiently small as to get an answer within 0.1% of the true answer, 0.249916.

Ex. 5. Evaluate the integral of $\sin^2(16x)$ between $x = 0$ and $x = \pi/2$. Why the result is erroneous? How can this be solved? (The correct result is $\pi/4$)

Practice Problem (Experiment - 04)

Problem-1: Write the MATLAB functions implementing Trapezoidal, Simpson's 1/3 & Simpson's 3/8 Rule to determine the integral result of any function numerically.

Problem-2: Determine the integral of the given function using:

$$\int_{0.5}^{2} \frac{1 + 5\sin(x^2)}{(x+1)^{\frac{3}{2}}}\, dx$$

i. Trapezoidal Rule
ii. Simpson's 1/3 Rule
iii. Simpson's 3/8 Rule

Find the Analytical result of the integral as well. Calculate the Relative Percentage Error for each of the cases.
[You may use calculator to get the analytical value of the integral]

Problem-3: The following data are provided for the $velocity, v$ of a moving object as a function of $time, t$:

$t\ (sec.)$	0	1	2	3	4
$v\ (m/s)$	0	2.15	3.26	4.18	4.82

Determine the $displacement, s$ of the object for the given period of time with best possible accuracy. Which numerical approach would you choose?
[Just to recall: $s = \int v\ dt$]

Problem-4: Consider the data-points once again:

$t\ (sec.)$	0	1	2	3	4	5	6
$v\ (m/s)$	0	2.15	3.26	4.18	4.82	5.61	4.77

Determine the $displacement, s$ again. Which numerical approach would you choose now to have better accuracy?