Lecture 8 : Complicated Patterns and Messy data

Sakib Anwar

AN7914 Data Analytics and Modelling

University of Winchester

2024

Motivation

- Interested in the pattern of association between life expectancy in a country and how rich that country is.
  - Uncovering that pattern is interesting for many reasons: discovery and learning from data.
- Identify countries where people live longer than what we would expect based on their income, or countries where people live shorter lives.
  - Analyzing regression residuals.
  - Getting a good approximation of the $y^E = f(x)$ function is important.

Functional form

- Relationships between y and x are often complicated!
- When and why care about the shape of a regression?
- How can we capture function form better?
  - This class is about transforming variables in a simple linear regression.

Functional form - linear approximation

- Linear regression – linear approximation to a regression of unknown shape:

$$y^E = f(x) \approx \alpha + \beta x$$

.

- Modify the regression to better characterize the nonlinear pattern if,
  - we want to make a prediction or analyze residuals - better fit
  - we want to go beyond the average pattern of association - good reason for complicated patterns
  - all we care about is the average pattern of association, but the linear regression gives a bad approximation to that - linear approximation is bad
- Do Not modify
  - if all we care about is the average pattern of association,
  - if linear regression is good approximation to the average pattern

Functional form - types

There are many types of non-linearities!

- Linearity is one special cases of functional forms.
- We are covering the most commonly used transformations:
  - Natural log transformation (written as $ln(x)$ mathematically for natural log of x))
  - Piecewise linear splines
  - Polynomials - quadratic form
  - Ratios

Functional form: ln transformation

- Frequent nonlinear patterns better approximated with y or x transformed by taking relative differences:
- In cross-sectional data usually there is no natural base for comparison.
- Taking the natural logarithm of a variable is often a good solution in such cases.
- When transformed by taking the natural logarithm, differences in variable values we *approximate relative differences*.
  - Log differences works because differences in natural logs approximate percentage differences!

Logarithmic transformation - interpretation

- $ln(x)$ = the natural logarithm of x
  - Sometimes we just say log x and mean $ln(x)$. Could also mean log of base 10. Here we use $ln(x)$
- x needs to be a positive number
  - $ln(0)$ or $ln(negative\ number)$ do not exist
- Log transformation allows for comparison in relative terms – percentages!

  Claim:

$$\ln(x + \Delta x) - \ln(x) \approx \frac{\Delta x}{x}$$

- The difference between the natural log of two numbers is approximately the relative difference between the two for small differences.

Logarithmic transformation - derivation [Optional]

- From calculus we know:

$$\lim_{x \to x_0} \frac{ln(x) - ln(x_0)}{x - x_0} = \frac{1}{x_0}$$

- By definition it means a small change in x or $\Delta x = x - x_0$. Manipulating the equation, we get:

$$\lim_{\Delta x \to 0} ln(x_0 + \Delta x) - ln(x_0) = \lim_{\Delta x \to 0} \frac{\Delta x}{x_0}$$

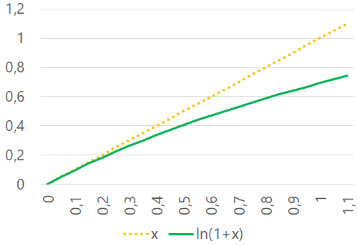
- If Δx is not converging to 0, this is an approximation of percentage changes.

$$ln(x_0 + \Delta x) - ln(x_0) \approx \frac{\Delta x}{x_0}$$

- Numerical examples ($x_0 = 1$):
  - Δx = 0.01 or 1% larger: ln(1+0.01) = ln(1.01) = 0.0099 $\approx$ 0.01

Log approximation: what is considered small?

- Log differences are good approximations for small relative differences!
- When Δx is considered small?
  - Rule of thumb: 0.3 (30% difference) or smaller
- But for larger x, there is a considerable difference,
  - A log difference of +1.0 corresponds to a +170 percentage point difference
  - A log difference of -1.0 corresponds to a -63% percentage point difference
- In case of large differences you may have to calculate percentage change by hand



When to take logs?

- Comparison makes mores sense in relative terms
  - Percentage differences
- Variable is positive value
  - There are some tricks to deal with 0s and negative numbers, but these are not so robust techniques.
- Most important examples:
  - Prices
  - Sales, turnover, GDP
  - Population, employment
  - Capital stock, inventories
- You may take the log for y or x or both!
  - These yield different models!

Interpreting parameters of regressions with log variables

$ln(y)^E = \alpha + \beta x_i$ - 'log-level' regression

- log y, level x
- α is average $\ln(y)$ when x is zero. (Often meaningless.)
- β: y is $\beta * 100$ percent higher, on average for observations with one unit higher x.

Interpreting parameters of regressions with log variables

$ln(y)^E = \alpha + \beta x_i$ - 'log-level' regression

- log y, level x
- α is average $\ln(y)$ when x is zero. (Often meaningless.)
- β: y is $\beta * 100$ percent higher, on average for observations with one unit higher x.

$y^E = \alpha + \beta ln(x_i)$ - 'level-log' regression

- level y, log x
- α is : average y when $\ln(x)$ is zero (and thus x is one).
- β: y is $\beta/100$ units higher, on average, for observations with one percent higher x.

Interpreting parameters of regressions with log variables

$ln(y)^E = \alpha + \beta x_i$ - 'log-level' regression

- log y, level x
- α is average $\ln(y)$ when x is zero. (Often meaningless.)
- β: y is $\beta * 100$ percent higher, on average for observations with one unit higher x.

$y^E = \alpha + \beta ln(x_i)$ - 'level-log' regression

- level y, log x
- α is : average y when $\ln(x)$ is zero (and thus x is one).
- β: y is $\beta/100$ units higher, on average, for observations with one percent higher x.

$ln(y)^E = \alpha + \beta ln(x_i)$ - 'log-log' regression

- log y, log x
- α: is average $\ln(y)$ when $\ln(x)$ is zero. (Often meaningless.)
- β: y is β percent higher on average for observations with one percent higher x.

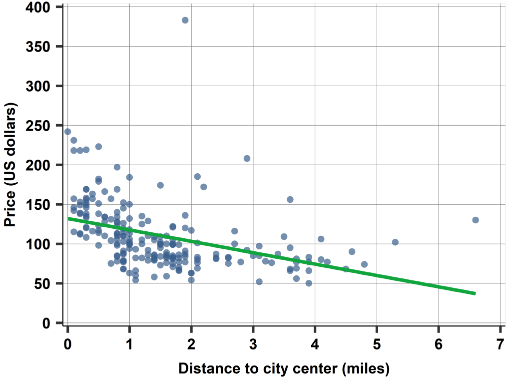
Interpreting parameters of regressions with log variables

- Precise interpretation is key
- The interpretation of the slope (and the intercept) coefficient(s) differs in each case!

- Often verbal comparison is made about a 10% difference in x if using level-log or log-log regression.

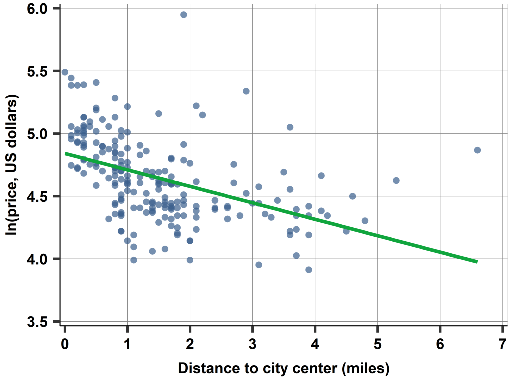
Hotel price-distance regression and functional form

- $price_i = 132.02 - 14.41 * distance_i$
- Issue ?



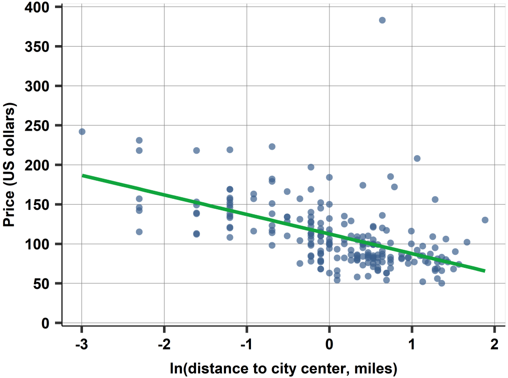
Hotel price-distance regression and functional form - log-level

- $\ln(price_i) = 4.84 - 0.13 * distance_i$
- Better approximation to the average slope of the pattern.
  - Distribution of log price is closer to normal than the distribution of price itself.
  - Scatterplot is more symmetrically distributed around the regression line



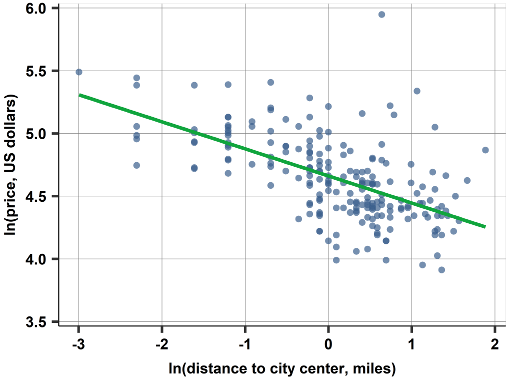
Hotel price-distance regression and functional form - level-log

- $price_i = 116.29 - 28.30 * \ln(distance_i)$
- We now make comparisons in terms percentage difference in distance
  - This transformation focuses on the lower and upper part of the domain in x: smaller values have even smaller log-values, while large values become closer to the average value.



Hotel price-distance regression and functional form - log-log

- $\ln(price_i) = 4.70 - 0.25 * \ln(distance_i)$
- Comparisons relative terms for both price and distance



Comparing different models

Table: Hotel price and distance regressions

Variables	(1) price	(2) ln(price)	(3) price	(4) ln(price)
Distance to city center, miles	-14.41	-0.13		
ln(distance to city center)			-24.77	-0.22
Constant	132.02	4.84	112.42	4.66
Observations	207	207	207	207
R-squared	0.157	0.205	0.280	0.334

Source: `hotels-vienna` dataset. Prices in US dollars, distance in miles.

Hotel price-distance regression interpretations

- price-distance: hotels that are 1 mile farther away from the city center are 14 US dollars less expensive, on average.
- ln(price) - distance: hotels that are 1 mile farther away from the city center are 13 percent less expensive, on average.
- price - ln(distance): hotels that are 10 percent farther away from the city center are 2.477 US dollars less expensive, on average.
- ln(price) - ln(distance): hotels that are 10 percent farther away from the city center are 2.2 percent less expensive, on average.

To Take log or Not to Take log - substantive reason

Decide for substantive reason:

- Take logs if variable is likely affected in multiplicative ways
- Don't take logs if variable is likely affected in additive ways

Decide for statistical reason:

- Linear regression is better at approximating average differences if distribution of *dependent variable* is closer to normal.
- Take logs if skewed distribution with long *right* tail

- Most often the substantive *and* statistical arguments are aligned

Comparing different models - model choice

Table: Hotel price and distance regressions

Variables	(1) price	(2) ln(price)	(3) price	(4) ln(price)
Distance to city center, miles	-14.41	-0.13		
ln(distance to city center)			-24.77	-0.22
Constant	132.02	4.84	112.42	4.66
Observations	207	207	207	207
R-squared	0.157	0.205	0.280	0.334

Source: `hotels-vienna` dataset. Prices in US dollars, distance in miles.

Model choice - substantive reasoning

- It depends on the goal of the analysis!
- Prices
  - We are after a good deal on a single night – absolute price differences are meaningful.
  - Percentage differences in price may remain valid if inflation and seasonal fluctuations affect prices proportionately.
  - Or we are after relative differences - we do not mind about the magnitude that we are paying, we only need the best deal.
- Distance
  - Distance makes more sense in miles than in relative terms – given our purpose is to find a *relatively* cheap hotel.

Model choice - statistical reasoning

- Visual inspection
  - Log price models capture patterns better, this could be preferred.
- Compare fit measure (R^2)
  - Level-level and level-log regression: R-squared of the level-log regression is higher, suggesting a better fit.
  - Log-level and log-log regression: R-squared of the log-log regression is higher, suggesting a better fit.
- Should not compare R-squared of two regressions with *different dependent variables* – compares fit in different units!

Model choice - statistical reasoning

- Visual inspection
  - Log price models capture patterns better, this could be preferred.
- Compare fit measure (R^2)
  - Level-level and level-log regression: R-squared of the level-log regression is higher, suggesting a better fit.
  - Log-level and log-log regression: R-squared of the log-log regression is higher, suggesting a better fit.
- Should not compare R-squared of two regressions with *different dependent variables* – compares fit in different units!
- Final verdict:
  - log-log probably the best choice:
    - can interpret in a meaningful way and
    - gives good prediction as this is the goal!
    - Note: prediction with log dependent variable is tricky.

Polynomials

- Quadratic function of the explanatory variable
  - Allow for a smooth change in the slope
  - Without any further decision from the analyst
- Technically: quadratic function is not a linear function (a parabola, not a line)
  - Handles only nonlinearity, which can be captured by a parabola.
  - Less flexible than a piecewise linear spline, but easier interpretation!

$$y^E = \alpha + \beta_1 x + \beta_2 x^2$$

- Can have higher order polynomials, in practice you may use cubic specification: $y^E = \alpha + \beta_1 x + \beta_2 x^2 + \beta_3 x^3$
- General case

$$y^E = \alpha + \beta_1 x + \beta_2 x^2 + \ldots \beta_n x^n$$

Quadratic form - interpretation I.

$$y^E = \alpha + \beta_1 x + \beta_2 x^2$$

- α is average y when $x = 0$,
- β_1 has no interpretation in itself,
- β_2 shows whether the parabola is
  - U-shaped or convex (if $\beta_2 > 0$)
  - inverted U-shaped or concave (if $\beta_2 < 0$).

Quadratic form - interpretation II.

$$y^E = \alpha + \beta_1 x + \beta_2 x^2$$

- Difference in y, when x is different. This leads to (partial) derivative of y^E w.r.t. x,

$$\frac{\partial y^E}{\partial x} = \beta_1 + 2\beta_2 x$$

- the slope is *different for different values of* x
  - Compare two observations, j and k, that are different in x, by one unit: $x_k = x_j + 1$.
- Units which are one unit larger than x_j are higher by $\beta_1 + 2\beta_2 x_j$ in y on average.
  - Usually we compare to the average of x: $x_j = \bar{x}$.
    - Units which are one unit larger than the average of x are higher by $\gamma = \beta_1 + 2\beta_2\bar{x}$ in y on average.
- Why, higher order polynomial is rather non-parametric method?

Which functional form to choose? - guidelines

Start with deciding whether you care about nonlinear patterns.

- Linear approximation OK if focus is on an average association.
- Transform variables for a better interpretation of the results (e.g. log), and it often makes linear regression better approximate the average association.
- Accommodate a nonlinear pattern if our focus is
  - on prediction,
  - analysis of residuals,
  - about how an association varies beyond its average.
  - Keep in mind - simpler the better!

Which functional form to choose? - practice

To uncover and include a potentially nonlinear pattern in the regression analysis:

1. Check the distribution of your main variables (y and x)
2. Uncover the most important features of the pattern of association by examining a scatterplot or a graph produced by a *nonparametric* regression such as *lowess* or *bin scatter*.
3. Think and check what would be the best transformation!
   3.1 Choose one or more ways to incorporate those features into a linear regression (transformed variables, piecewise linear spline, quadratic, etc.).
   3.2 Remember for some variables log transformation or using ratios is not meaningful!
4. Compare the results across various regression approaches that appear to be good choices. –> *robustness check*.

Data Is Messy

- Clean and neat data exist only in dreams and in some textbooks...
- Data may be messy in many ways!
- Structure, storage type differs from what we want

There are potential issues with the variable(s) itself:

- Some observations are influential
  - How to handle them? Drop them? Probably not but depends on the context.
- Variables measured with (systematic) error
  - When does it lead to biased estimates?

Extreme values vs influential observations

- Extreme values concept:
  - Observations with extreme values for some variable
- Extreme values examples:
- Influential observations
  - Their inclusion or exclusion influences the regression line
  - Influential observations are extreme values
  - But not all extreme values are influential observations!
- Influential observations example

Extreme values and influential observations

- What to do with them?
- Depends on why they are extreme
  - If by mistake: may want to drop them
  - If by nature: don't want to drop them
  - Grey zone: patterns work differently for them for substantive reasons
    - General rule: avoid dropping observations based on value of y variable
- Dropping extreme observations by x variable may be OK
  - May want to drop observations with extreme x if such values are atypical for question analyzed.
  - But often extreme x values are the most valuable as they represent informative and large variation.

Classical Measurement Error

- You want to measure a variable which is not so easy to measure:
  - Quality of the hotels
  - Inflation
  - Other latent variables with proxy measures
- Usually these miss-measurement are present due to
  - Recording errors (mistakes in entering data)
  - Reporting errors in surveys (you do not know the exact value) or administrative data (miss-reporting)
- 'Classical measurement error':
  - One of the most common and 'best' behaving problem – but a problem.
  - It needs to satisfy the followings:
    - It is zero on average (so it does not affect the average of the measured variable)
    - (Mean) independent from all variables.
- There are many other 'non-classical' measurement error, which cause problems in modelling.

Is measurement error in variables a problem?

It depends...

- Prediction: your are predicting *with* the errors - not a particular problem, but need to be addressed when predicting or generalizing.
- Association:
  - Interested in the estimated coefficient value (not just the sign)

Solution?

- Often cannot do anything about it!
  - The problem is with data collection/how data is generated.
- If cannot do anything, what is the consequence of such errors:
  - Does measurement error make a difference in the model parameter estimates?

Two cases for classical Measurement Error

- Classical measurement error in the dependent (y or left-hand-side) variable
  - is not expected to affect the regression coefficients.
- Classical measurement error in the explanatory (x or right-hand-side) variable
  - will affect the regression coefficients.

- We are covering how to mathematically approach this problem.
  - Show general way of thinking about *any* type of measurement error.
  - There are lot of format for measurement errors, you may want to have an idea whether it affects your regression coefficient(s):
    - If yes we call it 'biased' parameter(s).

Classical measurement error in the dependent variable (y) - I.

It means:

$$y = y^* + e$$

Where, $E[e] = 0$ and e is mean independent from x and y ($E[e \mid x, y] = 0$).
Reminder if e is mean independent from x, y, then $Cov[e, x] = 0$, $Cov[e, y] = 0$)

Compare the slope of model with an error-free dependent variable (y^*) to the slope of the same regression where y is measured with error (y).

$$y^* = \alpha^* + \beta^* x + u^*$$
$$y = \alpha + \beta x + u$$

Slope coefficients in the two regression are:

$$\beta^* = \frac{Cov[y^*, x]}{Var[x]}, \qquad \beta = \frac{Cov[y, x]}{Var[x]}$$

Classical measurement error in the dependent variable (y) - II.

Compering the two coefficients we show the two are equal because the measurement error is not correlated with any relevant variable(s), including x so that $Cov\left[e, x\right] = 0$

$$\beta = \frac{Cov\left[y, x\right]}{Var\left[x\right]} = \frac{Cov\left[\left(y^* + e\right), x\right]}{Var\left[x\right]} = \frac{Cov\left[y^*, x\right] + Cov\left[e, x\right]}{Var\left[x\right]} = \frac{Cov\left[y^*, x\right]}{Var\left[x\right]} = \beta^*$$

- Classical measurement error in the dependent (LHS) variable makes the slope coefficient unchanged because the expected value of the error-ridden y is the same as the expected value of the error-free y.
- Consequence: classical measurement error in the dependent variable is not expected to affect the regression coefficients.
  - But it lowers R^2 by increasing the disturbance term $u = u^* + e$.

Classical measurement error in the explanatory variable (x) - I.

It means:

$$x = x^* + e$$

Where, $E[e] = 0$ and e is mean independent from y and x, thus $Cov[e, y] = 0$, $Cov[e, x] = 0$.

Again let us compare the slopes of the two models, where x^* is the error-free explanatory variable x is measured with error.

$$y = \alpha^* + \beta^* x^* + u^*$$
$$y = \alpha + \beta x + u$$

The slope coefficients for the two models are similar to the previous ones:

$$\beta^* = \frac{Cov[y, x^*]}{Var[x^*]}, \qquad \beta = \frac{Cov[y, x]}{Var[x]}$$

Classical measurement error in the explanatory variable (x) - II.

Let us relate β to β^*:

$$\beta = \frac{Cov\left[y, x\right]}{Var\left[x\right]} = \frac{Cov\left[y, (x^* + e)\right]}{Var\left[x^* + e\right]} = \frac{Cov\left[y, x^*\right] + Cov\left[y, e\right]}{Var\left[x^*\right] + Var\left[e\right]} = \frac{Cov\left[y, x^*\right]}{Var\left[x^*\right] + Var\left[e\right]}$$
$$= \frac{Cov\left[y, x^*\right]}{Var\left[x^*\right]} \frac{Var\left[x^*\right]}{Var\left[x^*\right] + Var\left[e\right]}$$
$$= \beta^* \frac{Var\left[x^*\right]}{Var\left[x^*\right] + Var\left[e\right]}$$

- $\beta \neq \beta^*$, thus it is a ‘bias’.
- We call it the ‘*attenuation bias*’, while the error inflates the variance in the explanatory (RHS) variable and makes β closer to zero.

Classical measurement error in the explanatory variable (x) - III.

- Slope coefficients are different in the presence of classical measurement error in the explanatory variable.
  - The slope coefficient in the regression with an error-ridden explanatory (x) variable is smaller in absolute value than the slope coefficient in the corresponding regression with an error-free explanatory variable.

$$\beta = \beta^* \frac{Var\left[x^*\right]}{Var\left[x^*\right] + Var\left[e\right]}$$

  - The sign of the two slopes is the same
  - But the magnitudes differ.
- Consequence: on average β^* is closer to zero than it should be.

Effect of a biased parameter

- Attenuation bias in the slope coefficient:

$$\beta = \beta^* \frac{Var\left[x^*\right]}{Var\left[x^*\right] + Var\left[e\right]}$$

  - So β is smaller in absolute value than β^*
- As a consequence α is also biased

$$\alpha = \bar{y} - \beta\bar{x}$$

- If one parameter is biased the other one usually biased too
  - The value of intercept changes in the opposite direction!
  - β is closer to zero, α is further away from α^*

Classical measurement error in the explanatory variable (x)

- Without measurement error,

$$\alpha^* = \bar{y} - \beta^* \overline{x^*}$$

- With measurement error,

$$\alpha = \bar{y} - \beta \bar{x}$$

Classical measurement error in the explanatory variable (x)

- Classical measurement error leaves expected values (averages) unchanged so we can expect

$$\bar{x} = \overline{x^*}$$

Both regressions go through the same $(\bar{x}, \bar{y})$ point. Can derive that the difference in the two intercepts:

$$\alpha = \bar{y} - \beta\bar{x} = \alpha^* + \beta^*\overline{x^*} - \beta\bar{x} = \alpha^* + \beta^*\bar{x} - \beta\bar{x} = \alpha^* + (\beta^* - \beta)\,\bar{x}$$
$$= \alpha^* + \left(\beta^* - \beta^*\frac{Var\,[x^*]}{Var\,[x^*] + Var\,[e]}\right)\bar{x} = \alpha^* + \beta^*\bar{x}\frac{Var\,[e]}{Var\,[x^*] + Var\,[e]}$$

Review for classical measurement errors

- Classical measurement error in *dependent variable*
  - No bias, but noisier results.
- Classical measurement error in *explanatory variable*
  - Larger variation of x
  - Beta will be biased - attenuation bias
    - closer to zero / smaller in absolute value
  - Consequence:
    - When we compare two observations that are different in x by one unit, the true difference in x^* is likely less than one unit. (Larger variation in x)
    - Therefore we should expect smaller difference in y associated with differences in x, than with differences in the true variable x^*. (Biased parameter)
    - You can interpret your result as a lower (higher) bound of the true parameter if your sign is positive (negative).
- Most often you only speculate about classic measurement error.
  - Looking at how is data collected
  - Infer from what you learn about the sampling process.

Consequences

- Most variables in economic and social data are measured with noise. So what is the practical consequence of knowing the potential bias?
- Estimate magnitude which affects regression estimates.
- Look for the source, think about it's nature and consider impact.
- Super relevant issue for data collection, data quality!

- Have a look at the case study on hotels in Chapter 8!

Summary take-away

- Regression – functional form selection can help better capture relationships
- Several real life data problems may lead to estimation problems.

Essential Reading: Please Read Chapter 8