Debt-Maturity Management with Liquidity Costs: Implementation in Excel

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October 2022

This report documents the implementation in Microsoft Excel of the debt-maturity optimization model in Bigio, Nuño, and Passadore (2022), forthcoming at the Journal of Political Economy: Macroeconomics. The paper characterizes the optimal debt-maturity management problem of a government that issues a continuum of finite-maturity bonds in the presence of liquidity costs.

The paper lays out a continuous-time small open economy model, where a relatively impatient government tries to smooth consumption by choosing to issue or repurchase finite-life bonds within a continuum of maturities. Bonds are auctioned to primary dealers who later resell them to risk-neutral international investors. Liquidity costs appear because these primary dealers need time to liquidate their bond holdings after an auction. The model assumes that the bond market is segmented across maturities and vintages, so the larger the auction, the lower the price.

In this context, the optimal issuance problem can be decentralized: it can be studied as if the government delegates issuances to a continuum of subordinate traders. Each trader is in charge of managing a single maturity, and applies a simple rule to determine how much to issue. The rule indicates that the optimal issuance of a bond of a certain maturity is proportional to the difference between its market price and its domestic valuation, referred to as the relative value gap. When the gap is positive the trader would issue as much debt as possible, but this desire is contained by liquidity costs, which reduce prices in the primary market. The model is extended in the paper to allow for a government that faces income and interest rate risk and can default when the risks materialize, but the Excel implementation explained here is only available for the perfect foresight version.

The document is organized as follows. Part 1 describes the Excel file and how each sheet works. Part 2 illustrates how the model works using Spain's debt profile as the initial debt structure, mainly focusing on the transitional dynamics. Part 3 offers an extension of the model in which the liquidity costs vary for different maturities. This allows to adapt the model to cases in which treasuries can only issue at certain maturities (the cost of issuing at the non-available maturities is infinite), which is usually the case in reality. Finally, part 4 analyzes the transitional dynamics of the model in the presence of shocks to the exogenous paths of income and the international risk-free interest rate.

1 Excel file

The Excel file is organized in five sheets. In *Parameters*, the user inputs the parameters of the model, as well as the initial distribution of debt and the exogenous paths of output and the international risk-free interest rate. The sheet *Steady State* computes the steady state values of the main variables and *Dynamics* computes the transitional dynamics of the model. The sheet *Run* contains a button to run the algorithm that finds the optimal maturity structure. Finally, the sheet *Graphs* contains some plots to visually analyze the transitional dynamics.

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Once the user has input the parameters in the *Parameters* sheet and the initial guess for the path of the domestic interest rate in the *Dynamics* sheet, the optimal debt structure is found with an algorithm that updates the path for the interest rate until convergence. This algorithm is programmed in Visual Basic as a macro, and is run by the use of a button located in the *Run* sheet.

1.1 *Parameters* sheet

In *Parameters* the user can input the parameters of the model. First is a section with "fixed" parameters: the discount factor, ρ, the coupon, δ, the liquidity cost, λ, the relative risk aversion coefficient, σ, the steady state level of the international risk-free interest rate, r^*, the steady state level of output, y, the number of available maturities, T and steps in time, dt. There is then a section with parameters used by the algorithm: the relaxation coefficient, α, the tolerance and the maximum number of iterations.

There is then another section for the paths of the international interest rate $r^*(t)$ and output $y(t)$. These variables follow autoregressive processes according to:

$$y(t) = \alpha_y \Delta t\, y_{ss} + (1 - \alpha_y \Delta t) y(t-1) \tag{1}$$

$$r^*(t) = \alpha_y \Delta t\, r^*_{ss} + (1 - \alpha_{r^*} \Delta t) r^*(t-1) \tag{2}$$

The formulas for the paths of $r^*(t)$ and output $y(t)$ reflect these equations, starting from some initial value to be input in the first row. If these initial values are equal to the steady state values, income and the risk-free rate are always at their steady state. On the other hand, the user can input different values to anaylize the effect of shocks, either in period 1 or at a later period. The reversion coefficients α_y and α_{r^*} can be chosen by the user (in the section of parameters on the left).

To the right, the user can input the initial debt profile of the economy. Finally, there is a section where we can input different liquidity costs for the different maturities, as well as a section to re-scale the liquidity costs when we change the number of available maturities (see part 3).

1.2 *Steady State* sheet

The sheet *Steady State* computes the steady state values for the main variables: the market price of bonds, $\psi_{ss}(\tau)$, the domestic valuation of bonds, $v_{ss}(\tau)$, issuances, $\iota_{ss}(\tau)$, and the outstanding stock of bonds, $f_{ss}(\tau)$, for each maturity τ, and finally consumption, c_{ss}. These values are computed according to:

$$\psi_{ss}(\tau) = \delta \frac{1 - e^{-r^*_{ss}\tau}}{r^*_{ss}} + e^{-r^*_{ss}\tau} \tag{3}$$

$$v_{ss}(\tau) = \delta \frac{1 - e^{-\rho\tau}}{\rho} + e^{-\rho\tau} \tag{4}$$

$$\iota_{ss}(\tau) = \frac{\rho - r^*_{ss}}{\rho\bar{\lambda}} (1 - e^{-\rho\tau}) \tag{5}$$

$$f_{ss}(\tau) = \int_\tau^T \iota_{ss}(s)\,\mathrm{d}s = \frac{\rho - r^*_{ss}}{\rho\bar{\lambda}} \int_\tau^T (1 - e^{-\rho s})\,\mathrm{d}s \tag{6}$$

$$c_{ss} = y_{ss} - f_{ss}(0) + \int_0^T \left[\left(\psi_{ss}(\tau) - \frac{1}{2}\bar{\lambda}\psi_{ss}(\tau)\iota_{ss}(\tau) \right) \iota_{ss}(\tau) - \delta f_{ss}(\tau) \right] \mathrm{d}\tau \tag{7}$$

1.3 *Dynamics* sheet

Dynamics is the main sheet of the file, as it computes the transitional dynamics of the model. The rows correspond to the time dimension (months: t), while the columns (within each variable that is a function of maturity) correspond to maturities (τ), also measured in months.

Going from left to right, the first column (column B) brings the exogenous path of the international risk-free interest rate, $r^*(t)$ from the *Parameters* sheet. Next (column C) is the domestic interest rate, $r(t)$, which is the variable that the algorithm updates to find the optimal debt maturity structure. We usually start by setting the interest rate constant and equal to ρ (steady state value of the interest rate in the paper) and then it is updated by the algorithm. The following block (columns D to II) corresponds to the market price of bonds, $\psi(\tau)$, which is computed according to the formula for the market price (PF) in Table 5 of the Appendix of the paper:

$$\begin{aligned} r^*(t)\psi(\tau,t) &= \delta + \frac{\partial \psi}{\partial t} - \frac{\partial \psi}{\partial \tau} \\ \psi(0,t) &= 1 \, \forall t \end{aligned} \tag{8}$$

To implement this equation in Excel, we have to approximate it by[1]:

$$\begin{aligned} \psi(\tau,t) &= \frac{\delta \Delta t + \psi(\tau,t+1) + \psi(\tau-1,t)}{2 + r^*(t)\Delta t} \\ \psi(0,t) &= 1 \, \forall t \end{aligned} \tag{9}$$

The way to implement this in Excel is to set the last row equal to the vector of steady state market prices, the first column equal to 1 (initial condition) and let the formula run backwards.

The following block (columns IJ to RO) corresponds to the domestic valuation of the bonds, $v(\tau,t)$. This block is computed in the exact same way as the market price of bonds, except using $r(t)$ (the domestic interest rate) instead of $r^*(t)$ (international risk-free rate). The equation can also be found in Table 5 of the Appendix of the paper:

$$\begin{aligned} r^*(t)v(\tau,t) &= \delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau} \\ v(0,t) &= 1 \, \forall t \end{aligned} \tag{10}$$

which can be approximated by:

$$\begin{aligned} v(\tau,t) &= \frac{\delta \Delta t + v(\tau,t+1) + v(\tau-1,t)}{2 + r^*(t)\Delta t} \\ v(0,t) &= 1 \, \forall t \end{aligned} \tag{11}$$

Just as in the case of the market price, this is implemented in Excel by setting the last row equal to the steady state value of the domestic valuation of bonds, setting the first column equal to 1 (initial condition) and letting the formula run backwards.

Next (columns RP to AAU), issuances $\iota(\tau,t)$ are computed according to the issuance rule found in the paper:

$$\iota(\tau,t) = \frac{\psi(\tau,t) - v(\tau,t)}{\psi(\tau,t)} \frac{1}{\bar{\lambda}} \tag{12}$$

Then, columns AAV to AKA compute the outstanding stock of debt using the law of motion of $f(\tau,t)$, which follows a Kolmogorov forward equation:

$$\begin{aligned} \frac{\partial f}{\partial t} &= \iota(\tau,t) + \frac{\partial f}{\partial \tau'} \\ f(I,t) &= 0 \, \forall t \end{aligned} \tag{13}$$

approximated by:

$$\begin{aligned} f(\tau,t+1) &= \frac{\iota(\tau,t+1)\Delta t + f(\tau+1,t+1) + f(\tau,t)}{2} \\ f(I,t) &= 0 \, \forall t \end{aligned} \tag{14}$$

[1] The proof for this approximation and the ones that follow can be found in the Appendix.

where I is the highest maturity. In this case we set the first row equal to the initial debt distribution, the last column equal to 0 (initial condition) and let the formulas run forwards.

Consumption is computed in column AKB using the government's budget constraint. This path for consumption gives the updated domestic interest rate (in column AKC), which is computed according to:

$$r(t) = \rho + \frac{U''(c(t))c(t)}{U'(c(t))} \frac{\dot{c}(t)}{c(t)} \tag{15}$$

Under Constant Relative Risk Aversion (CRRA) utility, $U(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, this equation reduces to:

$$r(t) = \rho + \sigma \frac{\dot{c}(t)}{c(t)} \tag{16}$$

approximated by:

$$r(t) = \rho + \sigma \frac{c(t+1) - c(t)}{c(t)} \tag{17}$$

Finally, column AKD computes the absolute value of the difference between the initial interest rate (column C) and the updated interest rate (column AKC), which the algorithm uses to check for convergence. Then column AKE computes a guess for the interest rate path according to:

$$r^{new}(t) = \alpha r^{u}(t) + (1-\alpha) r(t) \tag{18}$$

where α is the relaxation coefficient defined in the *Parameters* sheet.

1.4 *Run* sheet

Finally, the sheet *Run* contains a button to run the macro with the algorithm to find the optimal debt structure. The **algorithm** works as follows. Starting with an initial path for the domestic interest rate (typically constant and equal to ρ), the sheet *Dynamics* computes the paths for the domestic valuation of the bonds of different maturities, which give the optimal issuances of each maturity and therefore the distribution f. This allows to compute the path for consumption and the updated path for the domestic interest rate. If the updated path for the interest rate, call it $r^u(t)$, satisfies $max|r^u(t) - r(t)| < tolerance$, the algorithm stops. Otherwise, it updates the path of the domestic interest rate. The algorithm runs until the convergence criteria is met or until the maximum number of iterations is reached. Notice that the path for the market valuation of the bonds is left outside the loop because it only depends on the international risk-free rate, which is exogenous.

1.5 *Graphs* sheet

Graphs displays the plots for the transitional dynamics of the domestic interest rate, consumption, issuances and the stock of debt.

2 Illustration for the case of Spain

In this section I use Spanish data and an illustrative calibration to explain how the Excel implementation works. The risk aversion coefficient, σ, is set to 2. The long-run annual international risk-free rate, r^*_{ss}, is set to 4 percent, while long-run income, y_{ss}, is normalized to 1. The coupon δ is set to 4 percent, so the market price, ψ, equals 1 at all maturities. The liquidity cost, $\bar{\lambda}$, is set to 7.08, while the discount factor, ρ, is set to 0.0416. I set the maximum maturity, I, to 20 years because roughly 90 percent of Spanish debt has maturity of less than 20 years. The model is monthly, so the steps, dt, are 1/12, and the number of available maturities, T, is 240. I use a period of 100 years to compute the transitional dynamics. For the algorithm, I use a maximum number of iterations of 1000, a tolerance of 0.00005 and a relaxation coefficient, α, of 0.005.

Finally, to determine the initial debt profile I use data on the maturity structure of outstanding debt of the Spanish government (at 30/06/2019), taken from the Spanish Treasury[2]. To express it in terms of GDP I take Spanish GDP for 2018 from FRED economic data[3]. The available data is total debt maturing per year (and in some cases in groups of two or three years). To divide it into monthly maturities (to fit the model) we distribute it evenly among the corresponding months. The resulting distribution can be seen in Figure 1.

Figure 1: Spanish outstanding debt by maturity

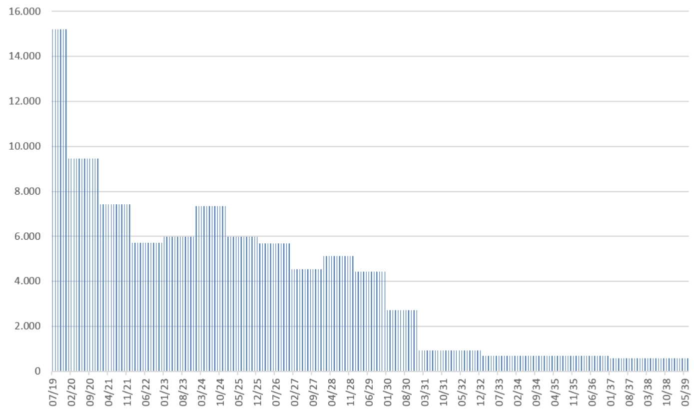


Figure 2 shows the transitional dynamics for the domestic interest rate, consumption, issuances and outstanding debt. To analyze the paths of issuances and the debt profile, I group maturities in 4 categories: up to 1 year, between 1 and 5 years, between 5 and 10 years and between 10 and 20 years. In the graph for the debt profile we can see the optimal transition from the initial debt profile to the steady state, and the issuances graph shows us the path of issuances that lead to this path.

Figure 2: Transitional dynamics of the system

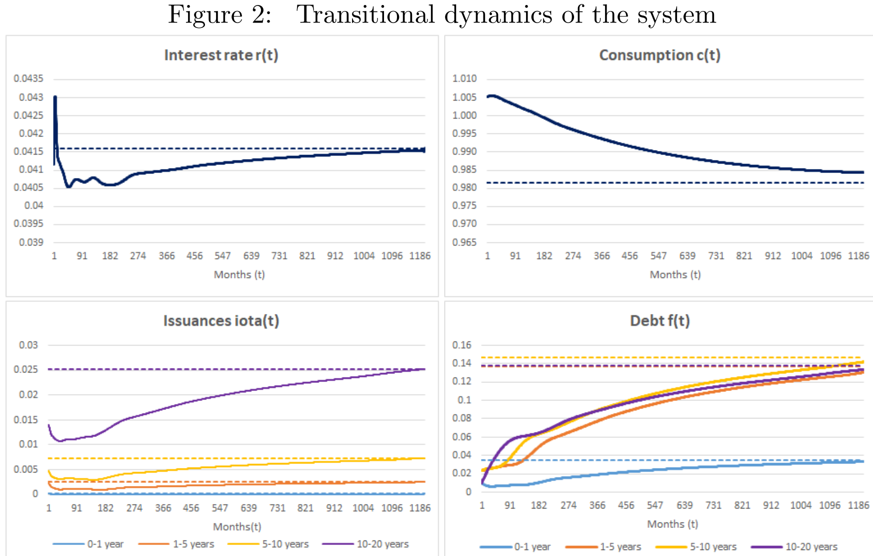


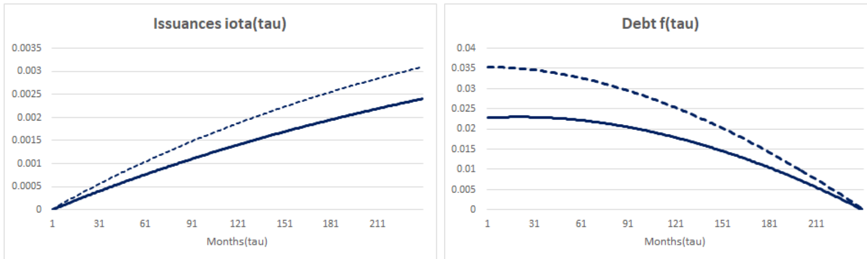
Figure 3 describes the maturity profile of issuances and outstanding debt along the transitional dynamics (average during all periods), as well as the steady state values (dotted lines). In this case, the horizontal axis corresponds to maturities (τ) and not time (t). Issuances are increasing in maturity, as the government

[2] http://www.tesoro.es/sites/default/files/estadisticas/14.pdf for the maturity profile and http://www.tesoro.es/sites/default/files/estadisticas/01.xlsx for the gross total of outstanding debt.

[3] https://fred.stlouisfed.org/series/CLVMNACSCAB1GQES

prefers long-maturity bonds because they have to be rolled over less frequently and therefore imply lower liquidity costs. On the other hand, the distribution of debt is decreasing in maturity, which is trivial since debt is the integral of issuances, so it should be decreasing in maturity as long as issuances are positive.

Figure 3: Maturity distribution of issuances and debt



3 Extension: varying liquidity costs

Bigio et al (2022) propose an extension of the model to account for the fact that countries issue only at a discrete number of maturities, and not in a continuum of them (as in the model). For example, in the case of Spain, between January 2000 and December 2018, the treasury issued bills (zero-coupon bonds) at 3, 6, 12 and 18-month maturities and bonds at 3, 5, 10, 15 and 30-year maturities.

The way to allow for issuances only at a discrete set of points is to let the liquidity coefficient $\bar{\lambda}(N)$ be finite only for a discrete number of available maturities $\{\tau_1, \tau_2, \dots, \tau_N\}$ and arbitrarily high for non-available maturities. As the paper points out, to obtain reasonable comparisons, the coefficient $\bar{\lambda}(N)$ has to be adapted to the total number of available maturities, N. This can be done by keeping the overall costumer flow constant and spreading it equally across maturities. For example, if the overall cost when we have N_1 available securities is $C(N_1) = N_1\lambda_1 = \omega$ (constant), when we change the number of available maturities to N_2, we want to have $C(N_2) = N_2\lambda_2 = \omega$, so we need $\lambda_2 = (N_1\lambda_1)/N_2$. This re-calibration can be done in the corresponding section of the *Parameters* sheet.

Using the Spanish data I solve the model setting the liquidity cost to 0.234 for bonds of maturities of 3, 6, 12, 18, 36, 60, 120 and 180 months, and 1 million for all the others. The liquidity cost 0.234 is obtained by multiplying the liquidity cost calibrated for 240 maturities (7.024) by the factor 8/240 (available maturities divided by previous number of maturities). This means that optimal issuance at non-available maturities is zero. This can be seen in Figure 4, where it is only optimal to issue at the available maturities, and issuances increase with the maturity (as was the case of constant liquidity costs). The maturity distribution of debt is therefore decreasing and step-shaped. Transitional dynamics can be seen in Figure 5.

Figure 4: Maturity distribution of issuances and debt with varying liquidity costs

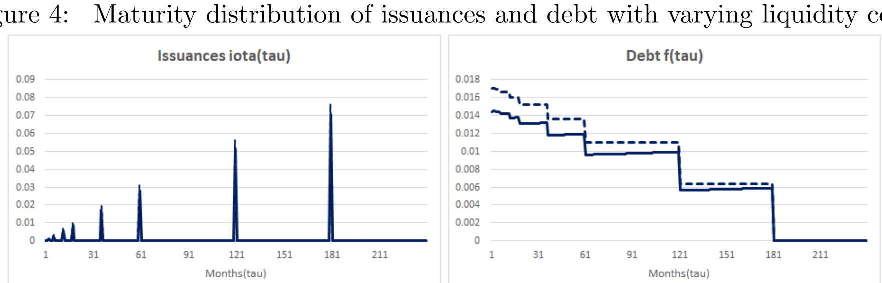
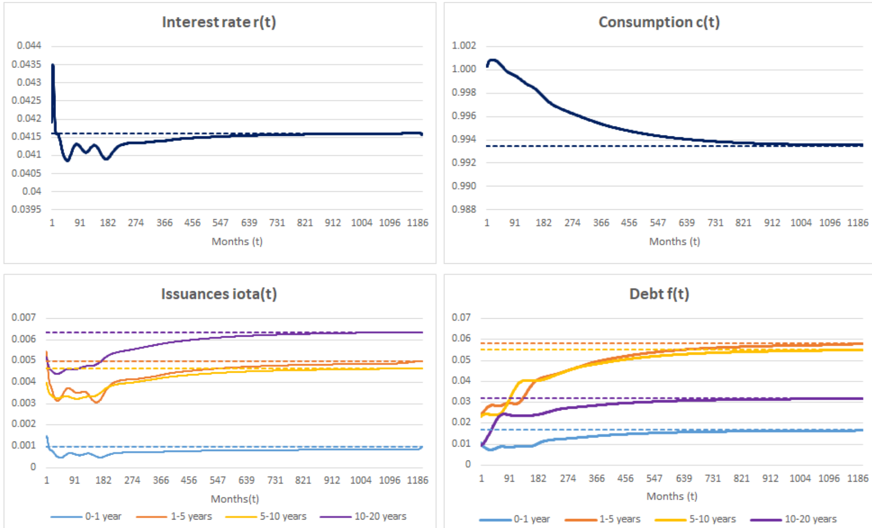


Figure 5: Transitional dynamics of the system with varying liquidity costs



4 The model with unexpected shocks

This section analyzes the effects of unexpected shocks to the exogenous paths of output and the international risk-free interest rate (returning to the case with constant liquidity costs across maturities).

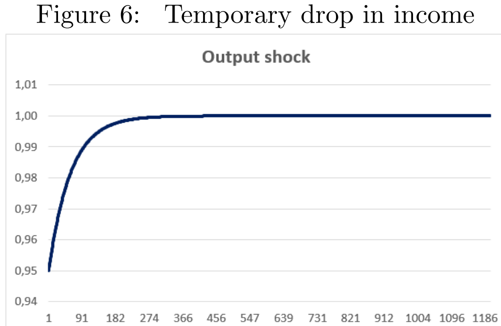
4.1 Income shock

I first consider a 5 percent temporary drop in income, which corresponds to a major recession in Spain. The shock occurs in period 1, and is therefore unexpected, with income later following an autoregressive process according to:

$$y(t) = \alpha_y \Delta t \, y_{ss} + (1 - \alpha_y \Delta t) y(t-1) \tag{19}$$

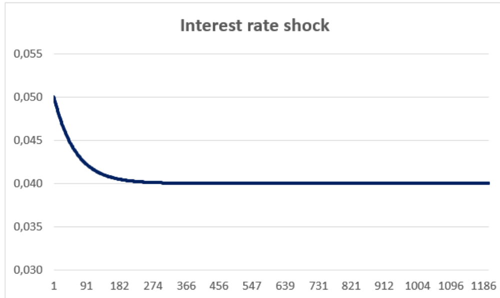
where α_y is set to 0.2 to fit Spanish data. With this parametrization, income converges to its steady state value approximately 20 years after the shock, as can be seen in Figure 6.

Figure 6: Temporary drop in income



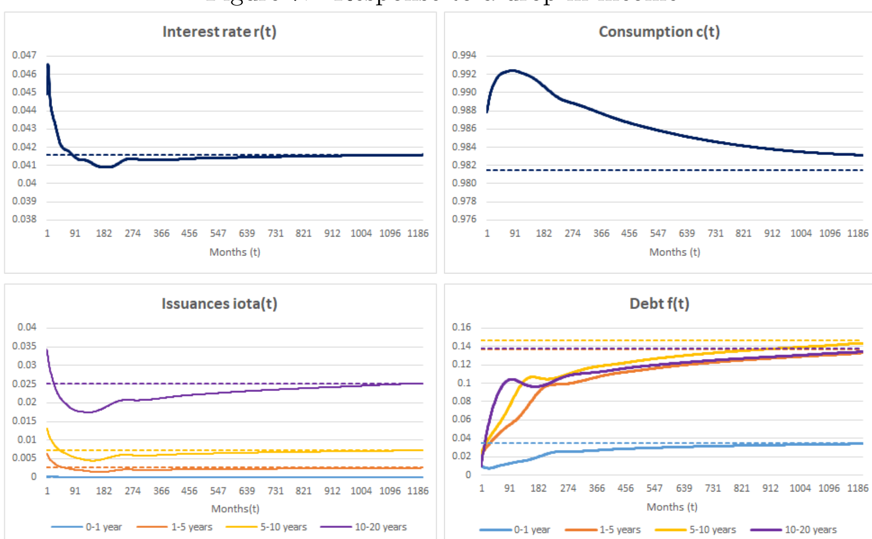
If we compare the transitional dynamics in the case of the shock (Figure 7) with the transitional dynamics without the shock (Figure 2), we can see that the output drop increases the domestic interest rate and reduces consumption on impact. At the same time, we can see that issuances of bonds of all maturities increase at the time of the shock, which means that the levels of debt at each maturity increase more rapidly, converging faster to their steady state values (which are above the initial values). The increase in issuances is more pronounced for long-term bonds (10-20 years), so there is a lengthening in the maturity of the debt

Figure 8: Temporary increase in the international risk-free interest rate



profile. The government responds to the drop in income by issuing more debt in an attempt to smooth consumption. The government also wants to delay the liquidity costs associated with debt rollover because the drop in income (and therefore consumption) is more steep at the beginning of the period, which explains the lengthening in the maturity.

Figure 7: Response to a drop in income



4.2 International interest rate shock

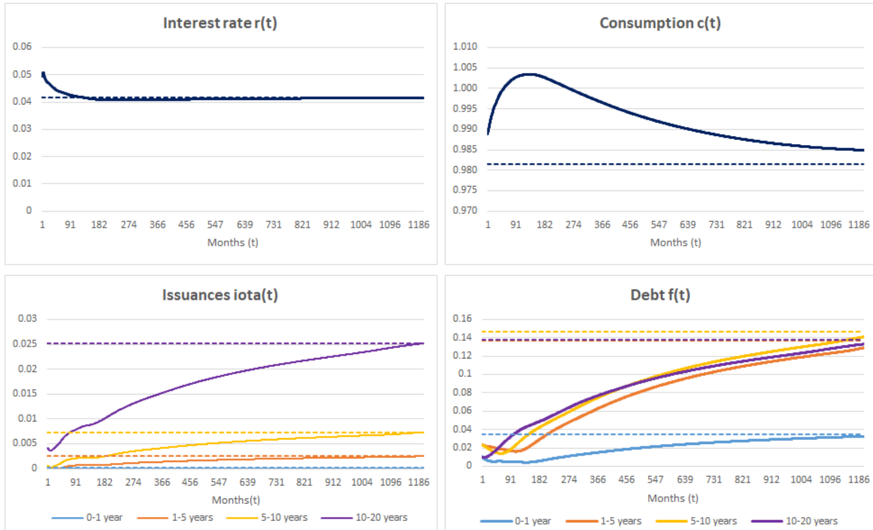
Next, I analyze a 1 percentage point increase in the international risk-free interest rate in period 1. As was the case for output, the risk-free rate evolves according to an autoregressive process:

$$r^*(t) = \alpha_y \Delta t\, r^*_{ss} + (1 - \alpha_{r^*} \Delta t) r^*(t-1) \tag{20}$$

where α_{r^*} is also set to 0.2. The path of the risk-free rate after the shock can be seen in Figure 8.

Figure 9 shows the transitional dynamics of the system after the shock. Comparing with Figure 2, we can see that the domestic interest rate increases on impact and consumption decreases relative to the case without the shock, as was the case with the income shock. On the contrary, the effect on issuances is the opposite: issuances are lower for all maturities at the time of the shock. Similar to what happened with the income shock but with opposite sign, the effect on issuances is stronger for longer maturities, because they are the most affected by the fall in the valuation gap (the difference between the market price and the domestic valuation of bonds, which determines the optimal issuance).

Figure 9: Response to an increase in the risk-free international interest rate



5 References

Bigio, S., Nuño, G., & Passadore, J. (2022). Debt-Maturity Management with Liquidity Costs. *Journal of Political Economy: Macroeconomics, forthcoming.*

Appendix: Proofs

The appendix provides proofs for the approximations of the laws of motion of the market price of bonds, their domestic valuation and the distribution of debt.

<u>Market price of bonds:</u>

$$\begin{aligned}
r^*(t)\psi(\tau,t) &= \delta + \frac{\partial \psi}{\partial t} - \frac{\partial \psi}{\partial \tau} \\
r^*(t)\psi(\tau,t) &= \delta + \frac{1}{\Delta t}[\psi(\tau,t+1) - \psi(\tau,t)] - \frac{1}{\Delta t}[\psi(\tau,t) - \psi(\tau-1,t)] \\
[r^*(t)+2]\psi(\tau,t) &= \delta\Delta t + \psi(\tau,t+1) + \psi(\tau-1,t) \\
\psi(\tau,t) &= \frac{\delta\Delta t + \psi(\tau,t+1) + \psi(\tau-1,t)}{2 + r^*(t)\Delta t}
\end{aligned} \tag{21}$$

<u>Domestic valuation of bonds:</u>

$$\begin{aligned}
r(t)v(\tau,t) &= \delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau} \\
r(t)v(\tau,t) &= \delta + \frac{1}{\Delta t}[v(\tau,t+1) - v(\tau,t)] - \frac{1}{\Delta t}[v(\tau,t) - v(\tau-1,t)] \\
[r(t)+2]v(\tau,t) &= \delta\Delta t + v(\tau,t+1) + v(\tau-1,t) \\
v(\tau,t) &= \frac{\delta\Delta t + v(\tau,t+1) + v(\tau-1,t)}{2 + r(t)\Delta t}
\end{aligned} \tag{22}$$

<u>Distribution of debt:</u>

$$\begin{aligned}
&\frac{\partial f}{\partial t} = \iota(\tau,t) + \frac{\partial f}{\partial \tau'} \\
&\frac{1}{\Delta t}[f(\tau,t+1) - f(\tau,t)] = \iota(\tau,t+1) + \frac{1}{\Delta t}][f(\tau+1,t+1) - f(\tau,t+1)] \\
&2f(\tau,t+1) = \iota(\tau,t+1)\Delta t + f(\tau+1,t+1) + f(\tau,t) \\
&f(\tau,t+1) = \frac{\iota(\tau,t+1)\Delta t + f(\tau+1,t+1) + f(\tau,t)}{2}
\end{aligned} \tag{23}$$