Convenience Yields, Portfolio Choice, and OTC Market Frictions *

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Abstract

We study settlement frictions, stemming from the need to finance negative balances via an over-the-counter (OTC) market. We derive an endogenous convenience yield in closed form, and show how it can be embedded in a canonical portfolio problem. Using this framework, we examine how shifts in settlement frictions influence liquidity premia, the volume of overnight funding, the dispersion of market rates, and optimal portfolio allocations. On the normative front, we show that in the competitive equilibrium investors may either over- or under-invest in liquid assets; moreover, higher risk aversion and tighter aggregate liquidity both increase the likelihood of under-accumulation.

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1. Introduction

The empirical finance literature has established that certain assets command substantial convenience yields—premia not accounted by their cash flows alone. The pattern appears across short-term assets such as U.S. Treasuries (Krishnamurthy and Vissing-Jørgensen, 2012), cash-like instruments (Nagel, 2016), and synthetic dollars in international markets (Jiang, Krishnamurthy and Lustig, 2021; Engel and Wu, 2023). Convenience yields vary substantially across assets and over time, with important implications for the transmission of monetary policy, fiscal capacity, and international capital flows. While convenience yields have become central to understanding asset prices and macroeconomic policy, their theoretical foundations remain incomplete. This paper develops a tractable microfoundation for convenience yields arising from trading frictions in over-the-counter (OTC) financial markets and incorporates it into canonical portfolio theory.

In our theory, investors are subject to payoff risk—stemming from fundamental variations in returns—and liquidity risk. Liquidity risk emerges because cash flows are unpredictable, and financial imbalances must be settled in an OTC market. Examples of financial positions that expose investors to liquidity risk include deposit withdrawals for banks, margin calls for hedge funds, or claim payouts for insurance companies. To meet these obligations, investors hold buffers of liquid assets that can be deployed for settlement. When liquidity needs exceed available buffers, investors must borrow settlement instruments in a frictional OTC market. The interaction between portfolio-induced settlement needs and OTC market frictions generates an endogenous convenience yield that lowers the returns on liquid assets. Moreover, to the extent that settlement needs are correlated with asset returns, this creates an additional a liquidity risk premium in excess of the conventional risk premium. Thus, convenience yields reflect both a first-order effect and the covariance between settlement needs and asset payoffs. Crucially, because these yields endogenously depend on investors' portfolios and OTC market conditions, they are not invariant to policy and vary with market tightness and trading efficiency of the OTC market.

Our analysis yields a tractable derivation of convenience yields that can be easily embedded in standard portfolio problems. To do so, we build on the sequential OTC framework of Afonso and Lagos (2015b), with two key innovations. First, instead of taking settlement positions as given, we endogenize them by modeling the portfolio choices of investors. Second, we model portfolio managers as large institutions who delegate settlement trades to many small traders, following Shi (1997) and the OTC model of Atkeson, Eisfeldt and Weill (2015). Taking limits as trader size vanishes yields closed-form characterizations of the entire equilibrium path of trades and rates.

Settlement frictions manifest as a piecewise linear liquidity yield function $\chi(s)$ that maps settlement positions s for each asset into an additional portfolio return. This function is kinked at zero, reflecting the asymmetry between borrowing costs and lending returns that is characteristic of the OTC market. Its slopes are precisely the expected lending return for investors with surplus and the borrowing costs for those with deficits arising from the OTC market. These convenience-yield coefficients depend on only three objects: an endogenous market tightness (the ratio of aggregate deficits to surpluses), a bargaining parameter, and the matching technology. The asymmetry resulting from the kink induces risk-averse behavior even under risk-neutral preferences and implies that assets that generate volatile settlement needs command higher convenience yields. Moreover, when settlement are correlates with asset returns—as when margin calls intensify in downturns—an additional liquidity risk premium emerges. Thus convenience yields reflect both pure OTC frictions and the interaction between liquidity risk and return risk.

Our analysis reveals several important properties linking OTC markets to convenience yields. First, the choice of matching technology fundamentally shapes market dynamics: under Cobb-Douglas matching, one side of the market can be depleted in finite time, causing trading to cease and convenience yields to reach their bounds; under Leontief matching, the short side, always trades, preventing a complete market freeze. This distinction helps explain a puzzling observation in interbank markets where minimal trading persists even when rates reach the floor (e.g., see Lopez-Salido and Vissing-Jorgenson, 2023; Afonso, Giannone, Spada and Williams, 2023; Lagos and Navarro, 2023).[1] Second, we establish that convenience yields satisfy time dilation (the passage of time is mathematically equivalent to reduced matching efficiency) and symmetry (reversing market tightness and swapping bargaining powers yields identical surplus extraction). Third, our comparative statics yield sharp predictions: market tightness unambiguously increases convenience yields by steepening the liquidity yield function, whereas matching efficiency has non-monotonic effects. The intuition behind this non-monotonicity—perhaps a surprising result—is that higher matching efficiency benefits the short side of the market, raising convenience yields when liquidity is scarce but potentially lowering them when it is abundant. Furthermore, our closed-form solutions can facilitate identification of structural OTC market parameters and shocks from observable data for trading volumes and spreads.

Finally, we show how the interaction between portfolio choice and settlement frictions generates a pecuniary externality that induces inefficient investment in liquid assets. Individual investors, when choosing portfolios, do not internalize how their settlement needs affect aggregate market tightness and thus the liquidity yields faced by all market participants. This creates a wedge between private and social returns to holding liquid assets. We characterize conditions under which competitive equilibria feature over- or under-investment in liquidity. We show that under risk neutrality,

[1]Our analysis also shows that only certain matching functions, those with infinite-rate of decay, can generate near-zero convenience yields with positive trading volumes.

investment in liquid assets is insufficient when the marginal impact of tightness on borrowing costs exceeds its impact on lending returns. This occurs in turn when aggregate surplus exceeds deficits. With risk aversion, the inefficiency is amplified as investors fail to account for how their portfolios affect others' liquidity risk. Strikingly, under balanced markets (equal settlement instrument deficits as surpluses) with symmetric shocks and Cobb-Douglas matching, the competitive equilibrium is constrained efficient under risk neutrality—but exhibits under-provision of liquidity under risk aversion. The findings have implications for liquidity regulation, suggesting that optimal policy depends critically on market conditions.

Literature Review. Our paper is related to a large literature on portfolio choice and asset pricing with liquidity frictions, such as portfolio constraints or transaction costs. Important examples in this literature include Constantinides (1986); Basak and Cuoco (1998); Vayanos and Vila (1999); He and Krishnamurthy (2013); Constantinides and Duffie (1996); Krueger and Lustig (2010); Holmstrom and Tirole (2001); Lagos (2010). Typically, these studies model liquidity frictions as exogenous. Our goal in this paper is to develop a tractable framework where liquidity premia emerges endogenously and explore the implications for asset portfolios.

Following Duffie, Garleanu and Pedersen (2005), a burgeoning literature on OTC markets has studied environments where assets are traded in presence of search frictions.[2] This literature has identified how features of the trading environment such as the speed of transactions and the heterogeneity in the motives for trade affect trading volumes and impact liquidity premia. While the literature began with strong restrictions on portfolio holdings, namely binary holdings, work by Lagos and Rocheteau (2009) allow for arbitrary portfolio holdings bringing this literature closer to standard portfolio theory. Hugonnier, Lester and Weill (2022) consider heterogeneity in asset selling speeds. Uslu (2019) introduce risk-averse behavior into this class of models. In these studies, trading speeds affect asset values because time is discounted, a feature that depresses the option value of selling assets when gains from trade emerge. Silva, Passadore and Kargar (2023) study portfolio problems that explicitly take into account trading times when agents want to modify their portfolios.[3] In the language of Hugonnier, Lester and Weill (2025), these papers study a semi-centralized setup, while we study a purely decentralized setup. .

As mentioned above, the sequential OTC market is very similar to Afonso and Lagos (2015a), where agents are restricted to have $-1, 1$ settlement positions. While the matching structure is different, some of the formulas derived here mirror those in Afonso and Lagos (2015a), which char-

[2]This literature has ran in parallel to the money search literature, pioneered by Kiyotaki and Wright (1993). See Williamson and Wright (2010); Lagos, Rocheteau and Wright (2017) for recent surveys.

[3]In our model, time discounting plays no role, as trading occurs within a single period although the sequence of trades matters because it affects the terms of trade when two investors match.

acterize volumes and terms of trade. Our central contribution is to show that, by working with a large number of traders—as in Shi (1997) and Atkeson, Eisfeldt and Weill (2015)—the OTC block can be readily embedded into portfolio theory. In addition, we provide comparative statics with respect to market tightness and matching efficiency, and show how these outcomes can be mapped to observable empirical moments.

Our normative analysis is related to a broad literature on the welfare properties of competitive equilibrium with financial frictions, in particular, those studies analyzing the efficiency of risk-taking decisions. One branch of the literature more closely related focuses on over or under-investment in liquid assets (Jacklin, 1987; Bhattacharya and Gale, 1987; Farhi, Golosov and Tsyvinski, 2009; Yared, 2013; Geanakoplos and Walsh, 2017). These studies consider a Walrasian interbank market where the risk-free rate affects the degree of enforcement and risk sharing. In contrast, we consider a setting with an OTC market where a pecuniary externality emerges from congestion in the interbank market. The externality we study is related to other congestion externalities in the search and matching literature. In particular, Uslu (2019) identify an externality where fast investors able to capture a private transaction surplus larger than their contribution to surplus creation [4] In addition, Wong and Zhang (2023) show the degree of search and intermediation is inefficient in the OTC market where search intensity is endogenous. Different from these contributions, our approach compares the portfolio choices of individual investors vis a vis a social planner that takes as given the financial arrangements in the OTC market. In this respect, our analysis is closer to Arseneau and Rappoport (2017) who study inefficient liquidity provision in a model where firms issue long-term bonds that are retraded by investors in an imperfectly liquid secondary market.[5]

Finally, we emphasize that while our earlier work (Bianchi and Bigio, 2022) employs the convenience yield function derived here to analyze the transmission and implementation of monetary policy through the banking system, the present paper develops its theoretical foundations.[6] Other applications include Arce, Nuño, Thaler and Thomas (2019) and Bigio and Sannikov (2019), who study optimal reserve policy; Bianchi, Bigio and Engel (2020), who examine exchange rate determination in an international context; Lopez-Salido and Vissing-Jorgenson (2023) who study quantitative tightening, and Bittner, Jamilov and Saidi (2025), who apply the framework to the German interbank market with assortative matching.[7]

[4]In contrast, in a model with exogenous search effort, **?** finds that the equilibrium implements the efficient reallocation of reserves.

[5]A different strand of the literature analyzes pecuniary externalities that can result in excessive leverage (see e.g. Caballero and Krishnamurthy, 2001; Lorenzoni, 2008; Bianchi, 2011; Dávila and Korinek, 2018; Amador and Bianchi, 2024).

[6]An additional contribution relative to Bianchi and Bigio (2022) is the analysis of efficiency of portfolio choices.

[7]In Piazzesi, Rogers and Schneider (2019) and Lenel, Piazzesi and Schneider (2019), settlement risks are used to explain the short-term rate puzzle and the determinacy of interest rate rules.

Roadmap. The paper is organized as follows. Section 2 presents the environment. Section 3 presents the main theoretical results. Section 4 presents the applications and the normative analysis. Section 5 concludes.

2. Environment

We present an infinite-horizon model in which a unit mass of investors take portfolio positions given exogenous returns. Assets and liabilities differ in their payoffs across states and, crucially, in their settlement risks, which gives rise to a liquidity-management problem. Investors are subject to idiosyncratic shocks, and trade overnight loans in an over-the-counter (OTC) market.

Asset structure. There is a collection of assets, $\{a_i\}$ where $i \in \mathbb{I}$=$\{1, 2, ..., I\}$, where an investor takes a positive position when long and a negative position when short. Assets are real, and differ in their realized returns as well as their settlement properties. In addition, there is a special asset, m, that is used to settle all payoffs and represents "cash." Investors may hold a negative position in m during the period but must end each period with a strictly positive position.

2.1 The Portfolio and the Balancing Stage

Each period consists of two stages: a *portfolio stage* and a *balancing stage*. In the portfolio stage, investors take portfolio positions, given possible asset returns and cash-flow shocks. In the balancing stage, investors are subject to cash-flow shocks and may face a shortage or surplus of cash. When the investor has a cash deficit, it can borrow cash in an OTC market or through a lender of last resort. In the case of banks, the lender of last-resort can be a central bank, but more generally, it is a credit line from a cash provider. When it has a surplus, it can lend in the OTC market or or hold cash and earn a short-term rate. We proceed next to describe in detail the two stages.

Portfolio Stage. In this first sub-stage, the individual investor makes portfolio-holding decisions. The investor enters the period with a given initial amount of wealth or equity e_t. Wealth is composed of a portfolio of assets, $\{a^i_t\}_{i\in\mathbb{I}}$, cash, m_t, loans borrowed from other investors (negative when loans are provided to other investors), f_t , and borrowing from the lender of last resort, w_t. That is,

$$e_t \equiv \sum_{i\in\mathbb{I}} R^i_t a^i_t + R^m_t m_t - \bar{R}^f_t f_t - R^w_t w_t, \tag{1}$$

where R denotes realized gross returns, and each return is indexed by X, the aggregate state. We assume that $R^w_t \geq R^m_t$, implying a penalty for borrowing from the lender of last resort.[8] The return $\bar{R}^f_t$, constitutes a weighted average rate of OTC loans to be described below. This is the only return rate determined in equilibrium.

Starting from a given wealth level, the investor chooses the dividend—interpreted as an equity injection if negative—denoted by c_t, along with a new portfolio $\left\{\left\{\tilde{a}^i_{t+1}\right\}_{i\in\mathbb{I}}, \tilde{m}_{t+1}\right\}$. We use *'tilde'* to denote a portfolio variable chosen at the portfolio stage. The budget constraint is:

$$c_t + \sum_{i\in\mathbb{I}} \tilde{a}^i_{t+1} + \tilde{m}_{t+1} = e_t.$$

Balancing stage. Investors enter the balancing stage with the portfolio $\left\{\left\{\tilde{a}^i_{t+1}\right\}_{i\in\mathbb{I}}, \tilde{m}_{t+1}\right\}$. At the beginning of the balancing stage, the investor experiences cash-flow shocks $\{\omega^i_t\}$ to each of the assets in its portfolio

$$a^i_{t+1} = \tilde{a}^i_{t+1}(1+\omega^i_t). \tag{2}$$

where ω^i follows a C.D.F $\Phi^i(\cdot)$.

When $\tilde{a}^i_{t+1}\omega^i > 0$, this represents a *positive funding shock*; conversely, when $\tilde{a}^i_{t+1}\omega^i < 0$, it constitutes a *negative funding shock*. In the case where $a_i < 0$, a negative ω^i shock corresponds to a sudden withdrawal of liabilities, while a positive ω^i shock reflects a sudden inflow of liabilities. Conversely, when $a_i > 0$, a negative ω^i shock can be interpreted as the prepayment of a loan or callable bond, and a positive ω^i shock as a need to invest more in a particular asset. We cover some examples below.

Cash-flow shocks are settled with cash. Taking into account that the investor must end the period with positive cash holdings, the investor's *cash surplus* after the realization of cash-flow shocks is:

$$s\left(\left\{\tilde{a}^i_{t+1}\right\}_{i\in\mathbb{I}}, \tilde{m}^i_{t+1}, \left\{\omega^i_t\right\}_{i\in\mathbb{I}}\right) = \underbrace{\tilde{m}^i_{t+1} + \sum_{i\in\mathbb{I}} \frac{R^i_{t+1}}{R^m_{t+1}} \omega^i_t \tilde{a}^i_{t+1}}_{\text{settlement balance}}. \tag{3}$$

The surplus s depends on the portfolio chosen in the first substage, $\left\{\left\{\tilde{a}^i_{t+1}\right\}_{i\in\mathbb{I}}, \tilde{m}_{t+1}\right\}$, and the shocks $\{\omega^i\}_{i\in\mathbb{I}}$. Equation (3) shows that if the investor faces sufficiently large negative cash-flow shocks, it may end up with a *cash deficit*. Implicit in this accounting of cash-flow is the assumption that when settlement occurs, the return on each asset accrues to its holder, which is taken care by the

[8]This assumption is consistent with monetary policy implementation, where the discount window rate is strictly above the interest on reserve balances.

ratio of returns.

If an investor ends up with a cash deficit, it must either borrow in the OTC market or resort to the lender of last resort. Conversely, if it has a surplus, it may lend in the OTC market or retain the cash. At the end of the period, cash holdings are given by:

$$m_{t+1} = s\left(\left\{\tilde{a}^i_{t+1}\right\}_{i\in\mathbb{I}}, \tilde{m}_{t+1}, \left\{\omega^i_t\right\}_{i\in\mathbb{I}}\right) + f_{t+1} + w_{t+1} \geq 0. \tag{4}$$

The condition $m_{t+1} \geq 0$, combined with the assumption that $R^w \geq R^m$, implies that running a cash deficit is costly. Moreover, since investors may end up with different cash positions, there are gains from trade in the balancing stage. We assume that this money market is frictional, and operates as an *over-the-counter* (OTC) market—a natural assumption given that in practice, the money markets is often decentralized: participants must search for counterparties with the opposite position.

Because this market is OTC, access to funding is not guaranteed. As a result, funding deficits may be only partially covered through the OTC market, with the remainder financed via the lender of last resort. Conversely, investors with a positive surplus may ultimately end up holding excess cash.

The outcome of the OTC market determines that the amounts traded in the OTC market and borrowed from the lender of last resort, $\{f_t, w_t\}$, are given by two endogenous probabilities $\left\{\Psi^+_t, \Psi^-_t\right\}$. In particular, given cash holdings $s_t \equiv s\left(\left\{\tilde{a}^i_{t+1}\right\}_{i\in\mathbb{I}}, \tilde{m}_{t+1}, \{\omega^i_t\}_{i\in\mathbb{I}}\right)$:

$$f_t = \begin{cases} -\Psi^-_t s_t & \text{if } s_t \leq 0 \\ -\Psi^+_t s_t & \text{if } s_t > 0 \end{cases} \quad \text{and} \quad w_t = \begin{cases} -(1-\Psi^-_t)s_t & \text{if } s_t \leq 0 \\ 0 & \text{if } s_t > 0. \end{cases} \tag{5}$$

In turn, the average interest rate of the OTC transactions is where ϕ_t, is also an outcome of the OTC market. Substituting (5) into (4) the $t+1$ equity can be written as:

$$e_{t+1} \equiv \underbrace{\sum_{i\in\mathbb{I}} R^i_{t+1}\tilde{a}^i_{t+1} + R^m_{t+1}\tilde{m}_{t+1}}_{\text{portfolio return}} + \underbrace{\chi_{t+1}\left(s\left(\left\{\tilde{a}^i_{t+1}\right\}_{i\in\mathbb{I}}, \tilde{m}_{t+1}\right)\right)}_{\text{convenience yield}}, \tag{6}$$

where χ_{t+1} is the piecewise linear function

$$\chi_t(s) = \begin{cases} \chi^-_t s & \text{if } s \leq 0 \\ \chi^+_t s & \text{if } s > 0 \end{cases}, \tag{7}$$

where χ^- and χ^+ given by:

$$\chi_t^- = (\overline{R}_t^f - R_t^m)\Psi_t^- + (R_t^w - R_t^m)(1 - \Psi_t^-), \qquad \chi_t^+ = \Psi_t^+(\overline{R}_t^f - R_t^m). \tag{8}$$

Equation 7 shows that the portfolio choice $\left\{\{\tilde{a}_{t+1}^i\}_{i\in\mathbb{I}}, \tilde{m}_{t+1}\right\}$ induces the direct portfolio returns, but also indirect convenience yields associated with settlement frictions. The following section explains how the OTC market determines $\left\{\Psi_t^+, \Psi_t^-, \overline{R}_t^f\right\}$ and thus the convenience yield as an equilibrium outcome. Before we describe how the OTC market operates we discuss several examples of assets that expose investors to liquidity risk in addition to return risk.

2.2 Asset examples.

i) *Deposit Flows:* Consider the case of a bank whose only liabilities are deposits, d, subject to withdrawal shocks and a reserve requirement ρ, as in Bianchi and Bigio (2022) and the assets are composed of a fully liquid asset and a fully illiquid asset. This is a special case of the general framework here in which $d = -\tilde{a}_t^1$, and the settlement position is given by:

$$s(d, m, \omega) \equiv m - d\left(\frac{R^d}{R^m}\omega - \rho(1+\omega)\right).$$

ii) *Credit Lines:* A credit line specifies a multiple ω of an existing loan ℓ to be drawn upon the realization of a future state at a pre-specified rate. The credit line requires cash to settle the position equal to

$$s(\ell, m, \omega) \equiv m - \omega\ell.$$

iii) *Margin Calls:* A margin call specifies that for all states where the future asset return falls in value below k in the settlement stage, the creditor must bring cash up front to support the margin loan, b. That is,

$$s(b, m) \equiv m - \max\left\{k - R_{t+1}^b, 0\right\} b.$$

3. OTC Market

In this section, we provide formulas that map equilibrium objects $\left\{\Psi_t^+, \Psi_t^-, \overline{R}_t^f\right\}$ to one moment of the distribution of cash surplus among investors, the market tightness of the money market.

Definition 1. *Market tightness θ is defined as $\theta \equiv S^-/S^+$ where S^- and S^+ represent the sum of all* *negative and positive positions in the OTC market over the set of investors indexed by j:*

$$S^{-}=-\int_{0}^{1}\min\left\{s^{j},0\right\}dj \quad \textit{and} \quad S^{+}=\int_{0}^{1}\max\left\{s^{j},0\right\}dj.$$

A higher tightness θ implies that there are relatively more investors with a cash deficit. The equilibrium outcome that $\left\{\Psi_t^{+},\Psi_t^{-},\overline{R}_t^{f}\right\}$ are only functions of θ depends on the structure of the OTC market here. In various OTC models, such as xxxAfonso Lagos/Lagos Navarroxxx, trading probabilities and rates are functions of the entire distribution of cash positions.

OTC structure. The market structure consists of a sequence of over-the-counter (OTC) markets, following Afonso and Lagos (2015b). Unlike in Afonso and Lagos (2015b), were investors trade directly, we assume that each investor delegates its trade to a large number of traders, each handling an order of size Δ. Traders close cash deficits and lend surpluses on behalf of investors.

Traders are segmented into borrowers and lenders sides of the market, giving rise to a two-sided matching structure. If $\theta>1$, short-side of the market the side is the surplus of cash. If $\theta<1$, the short side is the side with deficits.

Trading occurs over N rounds, indexed by $n\in\mathcal{N}\equiv 1,2,\ldots,N$. In each round, matches are formed according to a matching function. Once matched, borrowers and lenders bargain over the interest rate of a loan of cash of size Δ. Upon trade, the surplus trader transfers Δ cash to the deficit trader. The settlement assets plus the bargained rate are returned to the lender at the beginning of the next portfolio stage.

If a trader does not match in a given round, it may or may not match in the subsequent round. A trader in deficit who remains unmatched after round N borrows from the lender of last resort at rate R^{w}. Traders in surplus that are unmatched by round N, allocate only get cash rate R^{m}.

We derive our analytical results for the limit case where the order sizes becomes infinitesimally small and N tends to infinity.[9] This limit is technically convenient because it avoids the combinatorial challenges that arise when the identity of the matches matters. Here, the terms of trade vary by round, but not by the identity of counterparties.[10]

[9]The assumption of infinitesimal order size can also be found in Shi (1997); Atkeson et al. (2015). The assumption bears some realism to the extent that financial institutions often feature multiple traders operating their trading desks.

[10]The idea behind the modeling of fixed sized order convention is that if a lender bank places lending orders that exceed his excess reserves, there is a chance it will not be able to have the funds to transfer to the borrowing bank. If a lending bank lacks the funds to transfer to the bank in deficit, the bank would violate a contract and face a large legal cost of default. For sufficiently high costs, no bank will ever place lending orders above the amounts they hold. Assuming that orders are of a fixed size, investors cannot place more than $\eta\left(s,\Delta\right)\equiv\lfloor|s|/\Delta\rfloor$ orders. Here, $\lfloor x\rfloor$ is the floor function understood as the largest integer not greater than x. Because investors can only place integer numbers of orders, typically, there will be a remainder of reserves that cannot be lent or borrowed at the OTC market. These

3.1 OTC market and its equilibrium

Denote $\{S_n^+, S_n^-\}$ the aggregate mass of surplus and deficit positions after the realization of matches in round n and $S_0^+ = S^+$and $S_0^- = S^-$. The number of matches in round $n+1$, m_{n+1}, is given by

$$m_n \equiv \lambda_N G(S_n^+, S_n^-), \quad n \in \mathcal{N}.$$

where λ_N is a parameter capturing the matching efficiency and $G : \mathbb{R}_+^2 \to \mathbb{R}_+$ is the matching function. The subscript N is useful once we take the limit $N \to \infty$.

Assumption 1 (Matching Function)**.** *The matching function G satisfies:*

1. *No disposal.* $G(0,1) = G(1,0) = 0$.
2. *Constant returns to scale.* $G(\cdot,\cdot)$ *is homogeneous of degree one.*
3. *Symmetry.* $G(a,b) = G(b,a)$, $\forall a, b$.
4. *Weak exhaustion:* $\lambda_N G(S_n^+, S_n^-) \leq \min\{S_n^+, S_n^-\}$.
5. *Strictly increasing.* $G_a(a,b) \geq 0, G_b(a,b) \geq 0$.
6. *Weakly Concave.* $G_{aa}(a,b) \leq 0, G_{bb}(a,b) \leq 0$.

No disposal implies that counterparts are needed for a match. Constant returns to scale imply that the number of matches scales proportionally with the size of the market. Symmetry assumes that the number of matches depends on the relative scarcity of borrowers or lenders, independently of which side is shorter. Week exhaustion means that there cannot be more matches than the shortest side of the market. Monotone increasing in each argument implies that adding more participants to either side leads to more matches. Finally, concavity, implies diminishing returns in the number of matches as a function of one side of the market. We normalize $G(1,1) = 1$, without loss of generality—we can rescale the number of matches by λ_N.

Given, $\left\{S_0^+, S_0^-\right\}$ we construct the sequence of matches for each round m_n by tracking the evolution of surplus and deficit positions as follows:

$$S_n^+ \equiv S_{n-1}^+ - m_n \qquad \text{and} \qquad S_n^- \equiv S_{n-1}^- - m_n, \qquad \forall n \in \{1, 2, \ldots, N+1\}.$$

residuals can be borrowed (lent) from the Fed at the discount window rated r^{DW} (excess reserve rate r^{IOR}) directly. Mathematically, this residual is $\phi(x, \Delta) = s - \lfloor |s| / \Delta \rfloor \Delta$.

As matches take in place, the surplus and deficit positions shrink in the following round. The recursion assumes implicitly that all matches result in trade.[11] Accordingly, we define market tightness in round n as

$$\theta_n \equiv \frac{S_n^-}{S_n^+}, \qquad n \in \{0, 1, 2, \ldots, N\}. \tag{9}$$

and matching probabilities $\{\psi_n^+, \psi_n^-\}$ for a trader in surplus and deficit respectively as

$$\psi_n^+ \equiv \frac{m_n}{S_{n-1}^+} = \lambda_N G\left(1, \theta_{n-1}\right), \quad \psi_n^- \equiv \frac{m_n}{S_{n-1}^-} = \lambda_N G\left(\frac{1}{\theta_{n-1}}, 1\right) \qquad n \in \mathcal{N}. \tag{10}$$

By convention, $\psi_{N+1}^+ = \psi_{N+1}^- = 0$, given that the last trading round with matches is N. By constant returns to scale, $\psi_n^+ = \theta_{n-1}\psi_n^-$.

Bargaining. At a given match, traders bargain over an OTC loan rate, R_n^f. Generically, each rate is depends on the wealth of the counterparties and the round. We verify that as the size of trades vanishes, $\Delta \to 0$, rates only depend on the round.

The rate R_n^f is determined through Nash bargaining: The outside options for borrowers and lenders varies by round. In a match at the N-th round, the lender's outside option is R^m and the borrower's outside option is the outside borrowing rate, R^w. In earlier rounds, the outside option is the expected value of entering the next round with an unmatched position.

Now, consider an individual trader that bargains on behalf of investor j. The trader is responsible of closing the order of size Δ. When bargaining, traders must form an expectation of his investor's equity position in each round of the OTC market. These equity position depends on all other trades delegated by the investor. Constructing an expectation of what other traders do is a complex combinatorial problem. However, the limit as $\Delta \to 0$ circumvents that challenge. Before taking that limit, assume that when traders bargain, the trader uses the law of large numbers to estimate the equity position in each round and the average bargained rate $\overline{R}_n^f$ of other traders—a rate we have to solve for.

Under the law of large numbers assumption, if investor j has a cash deficit, $s^j < 0$, the trader estimates that the fraction $\Psi^- \left(s^j + \Delta\right)$ will be borrowed across all rounds at an average rate $\overline{R}^f$ and $\left(1 - \Psi^-\right)\left(s^j + \Delta\right)$ will be borrowed at the discount rate. If the investor has a cash surplus, $s^j > 0$, the trader expects that $\Psi^+ \cdot \left(s^j - \Delta\right)$ funds will be lent at the OTC market and that $\left(1 - \Psi^+\right) \cdot \left(s^j - \Delta\right)$ of the funds will remain idle.

Recall that in 6 we determine that future equity is given by a liquidity-yield function mapping an

[11]By no exhaustion and no disposal, if at any round either side of the market is exhausted—i.e., ends at zero—no further matches are formed.

investor's position s to overall settlement costs through the function $\chi_{t+1}(s)$. In turn, this function depends on the trading probabilities $\{\Psi^{-}, \Psi^{+}\}$ and an average rate $\overline{R}^{f}$. Suppose the trader matches in round n. Thus, using his estimate for $\left\{\Psi^{-}, \Psi^{+}, \overline{R}^{f}\right\}$, the trader estimates his investor's equity to be:

$$e_{\Delta}^{j} = \underbrace{\sum_{i \in \mathbb{I}} a_{t+1}^{i} R_{t+1}^{i} + m_{t+1} R_{t+1}^{m} + \chi_{t+1}(s^{j} - \text{sign}\{s^{j}\}\Delta)}_{\equiv \mathcal{E}_{t+1}^{j}(\Delta)} + \text{sign}\{s^{j}\} \left(R_{n}^{f} - R_{t+1}^{m}\right) \Delta.$$

This is the analogue of e_{t+1} obtained in equation (6) with two modifications: First, the trader applies χ_{t+1} to his investors cash position net of the amount he is delegated trade, $s^{j} - \text{sign}\{s^{j}\}\Delta$. This is because he takes $\left\{\Psi^{-}, \Psi^{+}, \overline{R}^{f}\right\}$ as given for other trades but his. The term $\mathcal{E}_{t+1}^{j}(\Delta)$ defined above is the trader's estimate of equity excluding his own trade. Second, the trader internalizes the effect of his bargained rate R_{n}^{f}. Thus, the payoff $\text{sign}\{s^{j}\} \left(R_{n}^{f} - R_{t+1}^{m}\right) \Delta$ depends on his bargain.

When bargaining, the trader =internalizes his contribution to equity. We suppress the time subscript (and conditioning on the state) in what follows since the matching occurs intra-period. The value of the investor's future equity is $V(e')$. Let $J_{M}^{\text{sign}\left\{s^{j}\right\}}(n;\Delta) \in \left\{J_{M}^{+}(n;\Delta), J_{M}^{-}(n;\Delta)\right\}$ represent the round-n value functions of the surplus and deficit traders conditional on matching, respectively. These values satisfy:

$$J_{M}^{\text{sign}\{s^{j}\}}(n;\Delta) \equiv V(e') = V\left(\mathcal{E}^{j}(\Delta) + (\text{sign}\{s^{j}\})\left(R_{n}^{f} - R^{m}\right)\Delta\right).$$

Analogously, let $J_{U}^{\text{sign}\left\{s^{j}\right\}}(n;\Delta) \in \left\{J_{U}^{+}(n;\Delta), J_{U}^{-}(n;\Delta)\right\}$ represent the corresponding trader's value of conditional on being unmatched in round n. Thes values are written recursively:

$$\begin{aligned} J_{U}^{\text{sign}\{s^{j}\}}(n;\Delta) &\equiv \mathbb{E}\left[V(e') \,|\text{unmatched at round } n\right] \\ &= \mathbb{E}\left[\psi_{n+1}^{\text{sign}\left\{s^{j}\right\}} J_{M}^{\text{sign}\{s^{j}\}}(n+1;\Delta) + \left(1 - \psi_{n+1}^{\text{sign}\{s^{j}\}}\right) J_{U}^{\text{sign}\{s^{j}\}}(n+1;\Delta)\right]. \end{aligned}$$

This expression uses the trading probabilities obtained above. The value of an unmatched trader is the probability of matching in the next round times the value of matching its position, $\psi_{n+1}^{\text{sign}\{s^{j}\}} J_{M}^{\text{sign}\{s^{j}\}}(n+1;\Delta)$, plus the probability of not matching and going to the subsequent round with an unmatched position. If by round N the trader is unmatched, a surplus traders gains nothing (as if $R_{n}^{f} = R^{m}$) whereas a deficit trader pays the difference the external borrowing rate (as if $R_{n}^{f} = R^{w}$):

$$J_{U}^{+}(N;\Delta) \equiv V\left(\mathcal{E}^{j}(\Delta)\right), \quad J_{U}^{-}(N;\Delta) \equiv V\left(\mathcal{E}^{j}(\Delta) - (R^{w} - R^{m})\Delta\right).$$

With these value functions, we can describe the bargaining problem. Let $r \equiv R-1$ be the net rate of any gross return R. Upon a match, an OTC rate solves the following Nash- bargaining problem,

$$
\begin{aligned}
r_n^f(\Delta) &= \arg\max_{r_n} \left\{ \left[\mathcal{S}_n^-(\Delta)\right]^{\eta} \left[\mathcal{S}_n^+(\Delta)\right]^{1-\eta} \right\} \qquad (11)\\
\text{s.t. } \mathcal{S}_n^-(\Delta) &= V\left(\mathcal{E}^j\left(\Delta\right) - \left(r_n^f - r^m\right)\Delta\right) - J_U^-(n;\Delta)\\
\mathcal{S}_n^+(\Delta) &= V\left(\mathcal{E}^j\left(\Delta\right) + \left(r_n^f - r^m\right)\Delta\right) - J_U^+(n;\Delta),
\end{aligned}
$$

where the trader of a deficit position is j and the surplus counterpart is k.

For $\Delta > 0$, the combinatorial probability of who matches with who induces risk making the the investors equity a random variable. In the bargaining problem, the estimate $\mathcal{E}^j\left(\Delta\right)$ is treated as deterministic, so it is only approximation that exploits the law of large numbers. Furthermore, because of the possible curvature of V, the solution to the bargaining problem also depends on the meeting counterparts. Once we take the limit $\Delta \to 0$, $\mathcal{E}^j\left(\Delta\right)$ is not an approximation but an exact estimate and, furthermore, the curvature of V plays no role. We work with that limit next.

Infinitesimal Orders. We now consider the limiting equilibrium as the order size vanishes, $\Delta \to 0$. Consider the following:

Problem 1 (Infinitesimal Trade Bargaining Problem)**.** *The infinitesimal trade bargaining problem is:*

$$
\max_{r_n^f \in \left\{r^m+\chi_n^+, r^m+\chi_n^-\right\}} \left(\chi_n^- - \left(r_n^f - r^m\right)\right)^{\eta} \left(\left(r_n^f - r^m\right) - \chi_n^+\right)^{1-\eta}
$$

where $\{\chi_n^+, \chi_n^-\}$ *solve:*

$$
\chi_n^+ = \left(r_{n+1}^f - r^m\right)\psi_{n+1}^+ + \chi_{n+1}^+\left(1 - \psi_{n+1}^+\right) \qquad (12)
$$

and:

$$
\chi_n^- = \left(r_{n+1}^f - r^m\right)\psi_{n+1}^- + \chi_{n+1}^-\left(1 - \psi_{n+1}^-\right) \qquad (13)
$$

for $n \in \mathcal{N}$ *with terminal conditions* $\chi_{N+1}^+ = 0$ *and* $\chi_{N+1}^- = r^w - r^m$*.*

This problem consists of a sequence bargaining problems. The sequence $\{\psi_n^+, \psi_n^-\}_{n=1}^{N+1}$ is known. In round n, the bargained amount is r_n^f and the borrower's (lender's) outside options is the expected rate paid (received) conditional on going to the following round unmatched χ_n^- (χ_n^+). Notice that χ_n^- and χ_n^+ inherit a recursive structure: Indeed, the expected cost of an unmatched deficit by round n is given by the cost of closing the deficit in the next round $\left(r_{n+1}^f - r^m\right)$ times the probability of matching in round $n+1$, ψ_{n+1}^-, plus the expected cost of heading to the subsequent round unmatched

χ_{n+1}^{-} times the probability of not matching in round $n+1$, $\left(1-\psi_{n+1}^{-}\right)$; and an isomorphic recursion applies to χ_{n}^{+}.

Our first main result states that as $\Delta \rightarrow 0$, the solution to the bargaining problems (11) is indeed the solution to Problem 1. Thus, as long as trades are small, the identity of the counterparts is irrelevant to determining the OTC rate and the only relevant information is the trading round n. This is a convenient property: we do not have to keep track of the evolution of the distribution of surplus positions across round, as do Afonso and Lagos (2015b). This feature simplifies the equilibrium dramatically.

As a result, r_n is obtained by solving a system of linear difference equations.

Proposition 1 (Limit of Bargaining Problems)**.** *As $\Delta \rightarrow 0$, any bargained rate at round n solves Problem 1. That is, $r_{n}^{f}=r^{m}+(1-\eta)\,\chi_{n}^{-}+\eta\chi_{n}^{+}$, for all $n \in \{1,2,...,N\}$.*

The proposition tells us that spread between the OTC rate and the rate on cash, $r_{n}^{f}-r^{m}$, is given by a weighted average of the outside options, $(1-\eta)\,\chi_{n}^{-}+\eta\chi_{n}^{+}$.

A gist of the proof is as follows: we can divide the objective of the bargaining problem (11) by any constant without changing the bargained rate. One such constant is the trade size Δ. Hence, we can write the objective as $[\mathcal{S}_{n}^{-}(\Delta)/\Delta]^{\eta}\,[\mathcal{S}_{n}^{+}(\Delta)/\Delta]^{1-\eta}$. Then, observe that

$$\lim_{\Delta\downarrow 0}\frac{V\left(\mathcal{E}^{j}+\text{sign}\{s^{j}\}\left(r_{n}(\Delta)-r^{m}\right)\Delta\right)-V(\mathcal{E}^{j}+\chi_{n}^{\text{sign}\{s^{j}\}}\Delta)}{\Delta}= \\ V'\left(\mathcal{E}^{j}\right)\text{sign}\{s^{j}\}\left(r_{n}-r^{m}-\chi^{\text{sign}\{s^{j}\}}\right),$$

where the equality follows the definition of the derivative. Using this identity, the original bargaining problem simplifies,

$$\lim_{\Delta\downarrow 0}\left\{\max_{r_{n}^{f}}\left[\mathcal{S}_{n}^{-}(\Delta)/\Delta\right]^{\eta}\left[\mathcal{S}_{n}^{+}(\Delta)/\Delta\right]^{1-\eta}\right\}= \\ V'\left(\mathcal{E}^{j}\right)^{\eta}V'\left(\mathcal{E}^{k}\right)^{1-\eta}\max_{r_{n}^{f}}\left[\chi_{n+1}^{-}-(r_{n}-r^{m})\right]^{\eta}\left[(r_{n}-r^{m})-\chi_{n+1}^{+}\right]^{1-\eta}.$$

Thus, as the transaction size shrinks, the rate is independent of the counterparties. This is because as the trade size vanishes, the trade the influence on the counterparts' wealth vanishes too. Thus, the effects of whether trade takes place or not has a negligible effect on marginal utilities—the effect of the trade is of second order. Yet, traders want to maximize wealth even the size of the trade has a vanishing effect on wealth, a first-order effect. For that reason, each surplus is the difference in expected financing costs and proportional to marginal utilities.

The difference between costs and benefits define the total surplus in round n, which we define as $\Sigma_n \equiv \chi_n^- - \chi_n^+$. This surplus is the difference between the expected borrowing costs for lenders minus the lending benefit for lenders. The solution to the Nash bargaining problems render the conventional surplus splitting rule:

$$\lim_{\Delta \to 0} \mathcal{S}_n^-(\Delta) = \chi_n^- - \left(r_n^f - r^m\right) = \eta \Sigma_n, \quad \lim_{\Delta \to 0} \mathcal{S}_n^+(\Delta) = \left(r_n^f - r^m\right) - \chi_n^+ = (1-\eta)\,\Sigma_n.$$

Algorithm. A by product of the proposition above is an algorithm to solve the interbank rate of each round: First, solve $\{\psi_n^+\}$ and $\{\psi_n^-\}$ forward, taking θ_0 as given. Then, solve $\{\chi_n^+\}$ and $\{\chi_n^-\}$ backwards and obtain r_n^f as the interest via (12) and (13).

The algorithm works because we can confirm that the solutions χ_0^+ and χ_0^-, are indeed the slopes of the liquidity yield function, χ, and furthermore, that $\bar{r}^f$, that defines the liquidity-yield function $\chi(s)$.

Proposition 2 (Convenience-yield function and coefficients)**.** *The probability of closing deficit and surplus positions in the OTC market are given by:*

$$\Psi^- \equiv 1 - \prod_{n=1}^{N}\left(1 - \psi_n^-\right) \text{ and } \Psi^+ \equiv 1 - \prod_{n=1}^{N}\left(1 - \psi_n^+\right).$$

The coefficients of the convenience yield function $\chi(s)$ are solutions to (12) and (13) that satisfy:

$$\begin{aligned} \chi^- &= \Psi^-(\bar{r}^f - r^m) + (1-\Psi^-)(r^w - r^m) = \chi_0^-, \\ \chi^+ &= \Psi^+(\bar{r}^f - r^m) = \chi_0^+. \end{aligned}$$

where $\bar{r}^f$ is the average of $\left\{r_n^f\right\}$ weighted by the trade volume.

The proposition confirms that the convenience-yield function delivers the exact cost or benefit of settling in the OTC market as trade size converges to zero. To construct the convenience-yield function, we equate $\{\chi^-, \chi^+\} = \left\{\chi_0^-, \chi_0^+\right\}$. This step guarantees that the overall matching probabilities $\{\Psi^-, \Psi^+\}$ and the average interbank rate determines delivers of the average settlement costs and benefits for borrowers and lenders respectively.

While we could proceed numerically computing the paths of $\left\{r_n^f, \chi_n^+, \chi_n^-, \psi_n^+, \psi_n^-\right\}$ to obtain χ^- and χ^+ using the algorithm, we proceed to add further structure that produces analytic expressions. To do so, we derive the limit as the number of rounds goes to infinity. This limit is useful because it yields a differential form of $\{\chi^-, \chi^+\}$. In Section 3.3 we show that making specific assumptions about the matching technology, G, we can derive closed-form solutions to the entire OTC market equilibrium.

Infinite Rounds and Continuous-Time Limit. Let $N \to \infty$. As noted above, we define $\lambda_N = \bar{\lambda}/N$, so λ_N vanishes with N, but at a convergent rate $\bar{\lambda}$. This convergence rate allows us to convert the realization of a match into a Poisson process with time-varying intensity. We first derive the evolution of the masses of surpluses and deficits. The time interval between rounds is $1/N$, which will shrink to zero. Also we index the normalized round by $\tau \in \{0, 1/N, 2/N, ..., 1\}$. Thus, as $N \to \infty$, we can associate a round with point in time, $\tau \in [0, 1]$. Thus, from now on, we index all equilibrium variables by τ instead of by n.

Because the matching function is symmetric, we compress the notation by defining the normalized (the intensive form) matching function γ:

$$\gamma(\theta) \equiv G(\theta, 1).$$

As we increase the rounds, probabilities of matching in any given instant τ vanish. However, given the tightness θ of a given round, matching probabilities turn into matching rates (intensities) defined as:

$$\psi^{+}(\theta) = \bar{\lambda}\gamma(\theta) \quad \psi^{-}(\theta) = \bar{\lambda}\gamma\left(\theta^{-1}\right).$$

These intensities satisfy (i) $\psi^{+}(\theta) = \psi^{-}(\theta^{-1})$ and (ii) $\psi^{+}(\theta) = \theta\psi(\theta^{-1})$. The matching rates, in turn, determine the overall matching probabilities:

$$\Psi_{\tau}^{+} = 1 - \exp\left(-\int_{\tau}^{1} \psi_{x}^{+}(\theta)\, dx\right), \quad \Psi_{\tau}^{-} = 1 - \exp\left(-\int_{\tau}^{1} \psi_{x}^{-}(\theta)\, dx\right),$$

translating probability distributions into density functions. In the continuous-time limit, we can dispense the non-exhaustion assumption as $\lambda_N \to 0$. However, we show below that for certain matching functions the shortest side of the market may vanish in finite time. This density is distributed exponential with time varying intensities.

Lemma 1 (ODE for market tightness)**.** *Let the number of trading rounds $N \to \infty$. Then, by the round indexed by τ, the ratio of deficit to surpluses θ_τ satisfies the following first-order homogeneous differential equation:*

$$\dot{\theta}_{\tau} = -\bar{\lambda}\theta_{\tau}\left[\gamma\left(\theta_{\tau}^{-1}\right) - \gamma(\theta_{\tau})\right], \quad \tau \in [0, 1]. \tag{14}$$

where θ_0 given. The corresponding matching intensities are $\psi_{\tau}^{+} = \psi^{+}(\theta_{\tau})$ and $\psi_{\tau}^{-} = \psi^{-}(\theta_{\tau})$.

As the number of rounds increases, θ is translated from a sequence into a function of time, τ. This function satisfies the ordinary-differential equation (ODE) ((14)). Once we solve the ODE for tightness, we can solve for the matching intensities ψ_τ^+ and ψ_τ^- in $\tau \in [0,1]$.

While it is not possible to obtain a formula for θ_τ for any matching function G, we can characterize serveral general properties and obtain solutions for several specific matching functions as we do below. Hence, any analytic expression for θ_τ given a functional form for G translates into analytic expressions for the matching rates.

Observe that the reciprocal tightness, θ^{-1}, satisfies the ODE:

$$\dot{\theta}^{-1} = \bar{\lambda}\theta^{-1}\left[\gamma(\theta) - \gamma\left(\theta^{-1}\right)\right] = -\bar{\lambda}\gamma(\theta_\tau)(1-\theta_\tau),$$

where the second equality follows from symmetry and homogeneity, $\gamma(\theta) = \theta\gamma(\theta^{-1})$. Thus, the ODE for tightness or its reciprocal is the same.[12]

Proposition 3 (Properties of θ)**.** *The evolution of market tightness features the following properties:*

1. *If $\theta_0 = 1$, then $\theta_\tau = 1$. If $\theta_0 > 1$ ($\theta_0 < 1$), then θ_τ is increasing (decreasing) with time.*
2. *If $\theta_0 = 1$, matching rates are $\psi_\tau^+ = \psi_\tau^- = 1$. If $\theta_0 > 1$ ($\theta_0 < 1$), then ψ_τ^+ is increasing (decreasing) and ψ_τ^- is decreasing (increasing) with time.*
3. *Take two matching functions, such that$\gamma(\theta) < \tilde{\gamma}(\theta), \quad \forall\theta$. Then, if $\theta > 1$ ($\theta < 1$), θ rises (falls) faster under $\tilde{\gamma}$.*

The proposition shows that the scarcer side of the market becomes relative more scarce over the rounds, and this reflects on the probabilities. We also know that if a matching function creates more matches, it will result in a tighter market throughout, provided that the deficit side exceeds one initially.

Given ψ_τ^+ and ψ_τ^-, just as in the case with finite rounds, we can solve the differential equation for $\{\chi_\tau^+, \chi_\tau^-\}$ in closed form:

Proposition 4 (Liquidity-Yield Coefficients in the Continuous-Time Limit)**.** *Let the number of trading rounds $N \to \infty$. Then, the solution to $\{\chi_\tau^+, \chi_\tau^-\}$ is:*

$$\chi_\tau^+ = (r^w - r^m)\int_\tau^1 (1-\eta)\,\psi_y^+ \exp\left(\int_y^1 -\left((1-\eta)\,\psi_x^+ + \eta\psi_x^-\right)dx\right)dy$$

[12] The Picard--Lindelöf theoremTheorem guarantees the uniqueness of the solution, as long as γ is Lipschitz continuous. We note that for some matching functions, γ may fail to be Lipschitz continuous at $\theta = 0$ or ∞, but even in that case, uniqueness can be guaranteed by finding the path of θ backward in time to reach the initial condition θ_0 and using the fact that if $\theta_\tau = 0$ for any τit remains at zero for all $\tau' > \tau$.

and for the expected cost of deficits:

$$\chi_\tau^- = (r^w - r^m)\left(1 - \int_\tau^1 \eta\psi_y^- \exp\left(\int_y^1 -\left((1-\eta)\,\psi_x^+ + \eta\psi_x^-\right)dx\right)\right)dy,$$

for all $\tau = [0,1]$*. In turn, the coefficients of the liquidity-yield function are* $\chi^+ = \chi_0^+$ *and* $\chi^- = \chi_0^-$ *and the rate at time* τ *is given by:* $r_\tau^f = r^m + (1-\eta)\,\chi_\tau^- + \eta\chi_\tau^+$.

This proposition delivers $\{\chi_\tau^+, \chi_\tau^-\}$ in terms of the matching intensities. We observe several consistency properties: First, as $\tau \to 1$, the formula yields the terminal conditions $\chi_1^+ = 0$ and χ_1^-. Second, if matching intensities approach zero, each side receives the terminal value $\chi_\tau^+ = \chi_1^+$ or $\chi_\tau^- = \chi_1^-$. Third, χ_τ^- is increasing in time whereas χ_τ^+ is decreasing: as the chances of trading are the future is falling, surplus positions have a lower expected value of lending benefits whereas deficit positions expect a higher average rate. This tells us that regardless of the direction of rates as function of time, matching rates dictate the direction of expected values of trading as a function of time.

The formula in the proposition has the interpretation of being equivalent to a setting where one side of the market extracts all the surplus with a constant probability. To see this, it is convenient to compute the surplus at time τ, which in this case is given by:

$$\begin{aligned}\Sigma_\tau &= (r^w - r^m)\left(1 - \int_\tau^1 \left((1-\eta)\,\psi_y^+ + \eta\psi_y^-\right)\exp\left[\int_y^1 -\left[(1-\eta)\,\psi_x^+ + \eta\psi_x^-\right]dx\right]\right). \\ &= (r^w - r^m)\left(1 - H_\tau^+\right)\left(1 - H_\tau^-\right),\end{aligned}$$

where $H_\tau^+ \equiv 1 - \exp\left(\int_\tau^1 -\left((1-\eta)\,\psi_x^+\right)dx\right)$ $\quad H_\tau^- \equiv 1 - \exp\left(\int_\tau^1 -\left(\eta\psi_x^-\right)dx\right)$.

This representation reveals two things: first, the surplus increases with time because, as time is running out, the option to trade in the future is falling. Second, the surplus at time τ is, proportional to the trading between the surplus in the final round multiplied by the product of two probabilities, $(1 - H_\tau^+)$ and $(1 - H_\tau^-)$. The product $(1 - H_\tau^+)(1 - H_\tau^-)$ is analogous to the probability that neither side extracts any surplus in the future.[13]

[13] We use $\Sigma_\tau = (r^w - r^m)\left(1 - \int_\tau^1 \frac{\partial}{\partial y}\exp\left(\int_y^1 -\left((1-\eta)\,\psi_x^+ + \eta\psi_x^-\right)dx\right)\right)dy = (r^w - r^m)\exp\left(\int_\tau^1 -\left((1-\eta)\,\psi_x^+ + \eta\psi_x^-\right)dx\right)$. These probabilities correspond to the probability that an (homogeneous) Poisson event is *not* realized between time τ and time 1. Indeed, H_τ^+ and H_τ^- are the probabilities of two independent Poisson events realizing with respective intensities $(1-\eta)\,\psi_x^+$ and $\eta\psi_x^-$. Thus, H_τ^+ and H_τ^- correspond to a compound processes in which borrowers and lenders extract all the terminal surplus $(r^w - r^m)$ with probability $1-\eta$ and η respectively once matching.

We further simplify the formulas for χ_τ^- and χ_τ^+ as time-varying fractions of the total surplus:

$$\chi_\tau^+ = \int_\tau^1 (1-\eta)\,\psi_y^+ \Sigma_y dy, \quad \chi_\tau^- = (r^w - r^m) - \int_\tau^1 \eta \psi_y^- \Sigma_y dy.$$

Thus, the outside options satisfy a dynamic surplus splitting rule and are obtained as the expected probability of extracting all the surplus in a future match.

Examples. Figure 1 presents an example of the OTC market using a Leontief matching function, for various initial conditions θ_0. The figure reports the movement of the OTC market rate through time, plotted together with the outside options=. As the trading session ends, the outside options for the surplus (deficit) side = collapses to zero $(r^w - r^m)$. The outside options at the beginning of the trading session yield theliquidity yield coefficients. Figure 1 presents an analog for Cobb-Douglas matching. While the patterns within a trading day seem similar, comparative statics differ.

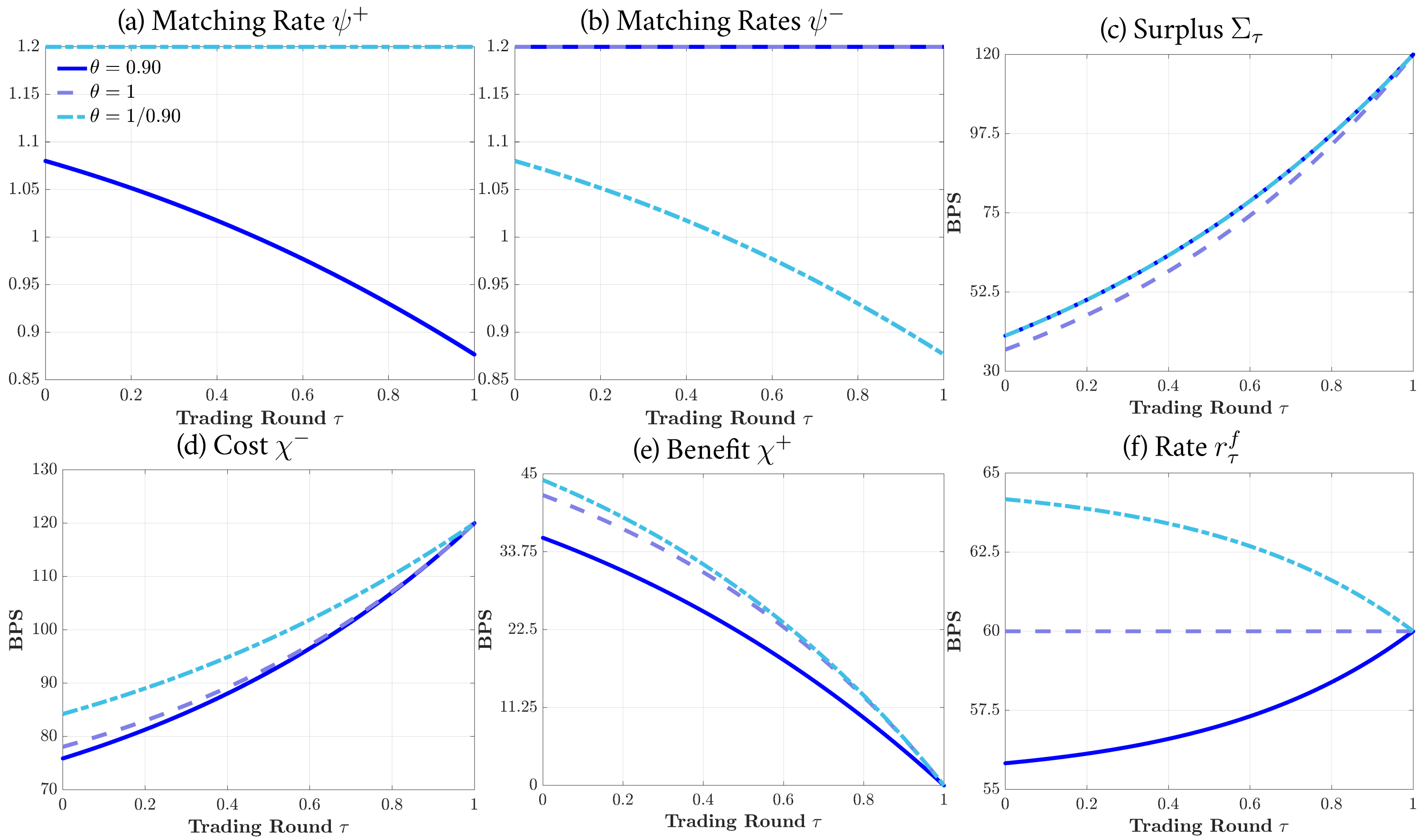


Figure 1: **Leontief Example:** Trading at various rounds.

Note: $\theta_0 = \theta$ is the initial market tightness defined as ratio of the the initial aggregate deficit and initial aggregate surplus. The example is calibrated with $\eta = 0.5$, $\bar{\lambda} = 1.2$, $r^w - r^m = 120$bps. See Equation 9.

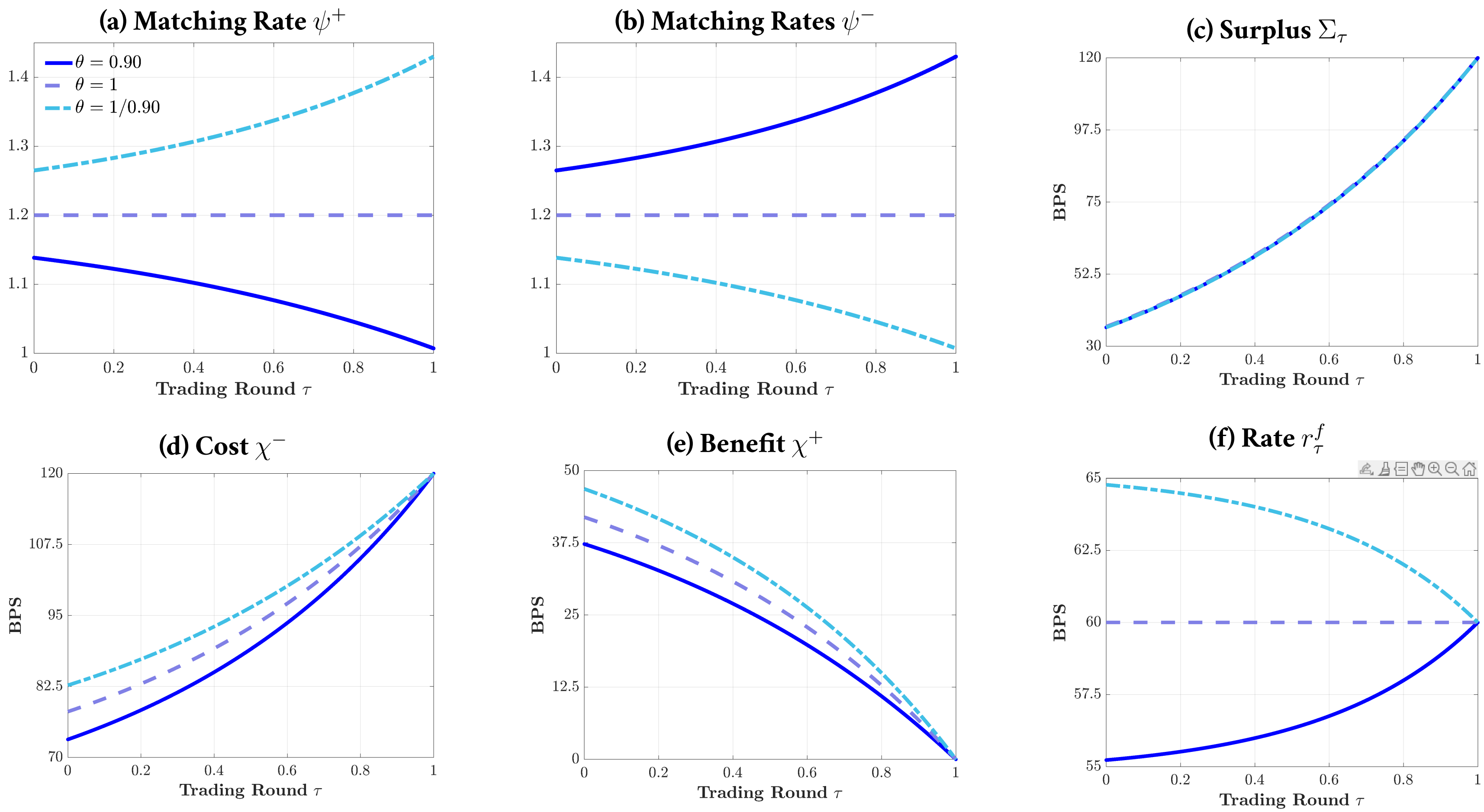


Figure 2: **Cobb-Douglas Example: Trading at various rounds**

Note: $\theta_0 = \theta$ is the initial market tightness defined as ratio of the initial aggregate deficit and initial aggregate surplus. The example is calibrated with $\eta = 0.5$, $\bar{\lambda} = 1.2$, $r^w - r^m = 120$bps. See Equation 9.

3.2 General Properties

In this section, we present general properties of the OTC market and liquidity-yield coefficients and then present some closed form examples.

Balanced Market Solution. Recall that when the OTC market is initially balanced, $\theta_0 = 1$, the market remains balanced throughout and, thus, matching rates are equalized: $\psi_\tau^+ = \psi_\tau^- = (1-\eta)\,\psi_\tau^+ + \eta\psi_\tau^- = \bar{\lambda}$. The following result follows from that observation:

Corollary 1 (Balanced Market)**.** *For $\theta_0 = 1$, $\Sigma_\tau = (r^w - r^m)\, e^{-\bar{\lambda}(1-\tau)}$, $r_\tau^f - r^m = (1-\eta)\,(r^w - r^m)$, and $\{\chi_\tau^+, \chi_\tau^-\} = \{(1-\eta)\,((r^w - r^m) - \Sigma_\tau)\,, \quad r^w - r^m - \eta\,((r^w - r^m) - \Sigma_\tau)\}$.*

When the market is balanced, the trading surplus is the terminal surplus, $(r^w - r^m)$, scaled by the probability of no matches in the remaining time $(1-\tau)$.[14] The yield coefficients are affine in the surplus, with the average cost of deficits increasing as time is running out—the opposite for the surplus benefit. The yield coefficients balance exactly such that the average negotiated rates equals the one obtained under static bargaining.

Time Dilation. Another property of the OTC market equilibrium is time dilation: Let $\theta\left(\tau, \theta_0, \bar{\lambda}\right)$ denote the value of market tightness θ at time τ given an initial condition θ_0 and efficiency $\bar{\lambda}$: the function representing the solution (14) as function of time, initial condition and parameters; with the same notation for $\left\{\gamma_\tau^+, \gamma_\tau^-, r_\tau^f, \chi_\tau^-, \chi_\tau^+, \Sigma_\tau\right\}$. Time dilation refers to the property that the passage of time is equivalent to a reduction in efficiency:

Proposition 5 (Time Dilation)**.** *Fix $\tau, \tau' \in [0,1]$ such that $\tau' > \tau$. Then,*

$$\theta\left(\tau', \theta_0, \bar{\lambda}\right) = \theta\left(\frac{\tau' - \tau}{1-\tau}, \theta\left(\tau, \theta_0, \bar{\lambda}\right), \bar{\lambda}\,(1-\tau)\right).$$

The same property holds for all the equilibrium objects $\left\{\gamma^+, \gamma^-, \chi^+, \chi^-, r^f\right\}$.

Time dilation allows to obtain the value of market tightness, at an instant τ', by computing first the value of market tightness at a prior instant τ: we can obtain the value at τ' by (i) solving the equilibrium renormalizing time by the remaining time $(1-\tau)$, (ii) scaling efficiency by the remaining time, and (iii) setting the the initial condition to θ_τ. This property is useful because it tells us that any property (e.g., monotonicity, concavity, etc.) of the equilibrium functions at some moment in time generalizes to all times. It also reveals the recursive nature of the market and shows that the normalization of time to the interval $[0,1]$ is inconsequential.

[14] This probability is an exponential with intensity $\bar{\lambda}$.

Symmetry. We established symmetry for the trading intensities from the symmetry in the matching function. These property carries through to the market rates and the yield coefficients:

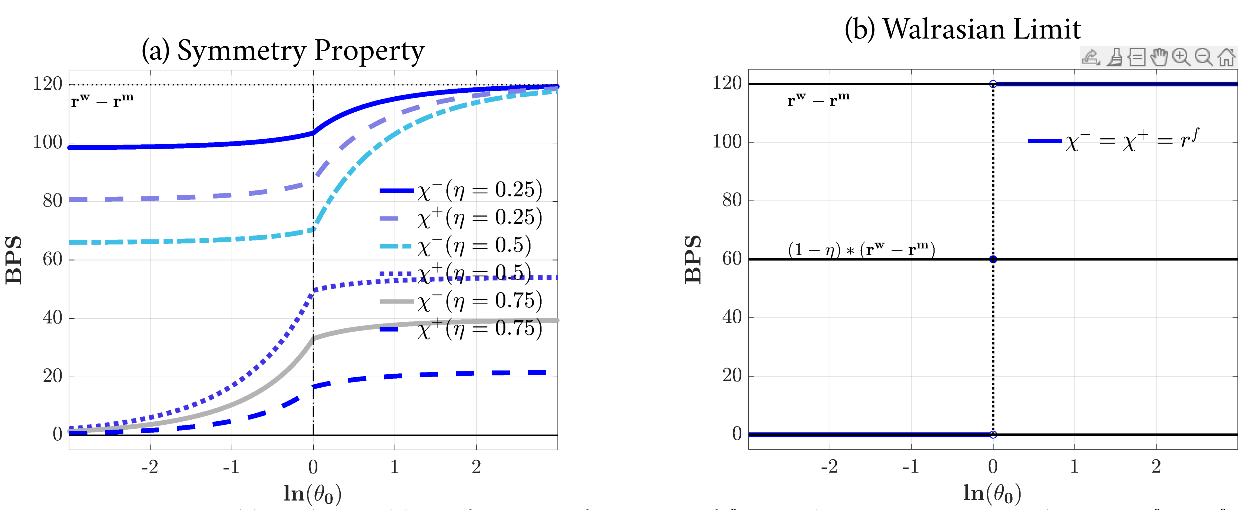
Proposition 6 (Symmetry)**.** *The OTC market satisfies:*

$$\Sigma\left(\tau,\theta,\eta,\bar{\lambda}\right)=\Sigma\left(\tau,\theta^{-1},1-\eta,\bar{\lambda}\right),\quad r^{f}\left(\tau,\theta,\eta,\bar{\lambda}\right)=\left(r^{w}-r^{m}\right)-r^{f}\left(\tau\theta^{-1},1-\eta,\bar{\lambda}\right),\ \textit{and}$$

$$\chi^{-}\left(\tau,\theta,\eta,\bar{\lambda}\right)=(r^{w}-r^{m})-\chi^{+}\left(\tau,\theta^{-1},1-\eta,\bar{\lambda}\right),\chi^{+}\left(\tau,\theta,\eta,\bar{\lambda}\right)=(r^{w}-r^{m})-\chi^{-}\left(\tau,\theta^{-1},1-\eta,\bar{\lambda}\right).$$

This symmetry states that if we can reverse the market tightness and exchange the bargaining power of surplus and deficit sides, and the surplus function is the same. For the expect cost of being in deficit, the benefit of a surplus, and the OTC rate the same symmetry holds relative to $r^w - r^m$. Panel (a) of Figure 3 plots the liquidity-yield coefficients for various values of η and $\log\theta_0$ in the x axis. A rotation of 180 degrees around the center of the figure gives the same figure once we swap η for $(1-\eta)$. A similar pattern holds for the the average interbank rate. This property reveals that any asymmetry in outcomes follows from asymmetries in assumed bargaining powers or the tightness. This symmetry property implies that we can solve the model for $\theta < 1$ and obtain solutions for $\theta > 1$, immediately. Furthermore, as we will see, many properties change exactly at $\theta = 1$, due to this property.

Figure 3: **Symmetry and Walrasian Limit Properties:**



Note: OTC rate and liquidity yield coefficients as functions of θ_0. Trading at various rounds. Note: $\theta_0 = \theta$ is the initial market tightness defined as ratio of the the initial aggregate deficit and initial aggregate surplus. The example in panel (a) is calibrated using $\bar{\lambda} = 0.8$, $r^w - r^m = 120$bps. Panel (a) is calibrated with $\eta = 0.5$, $r^w - r^m = 120$ bps.

Bargaining Power. Next, we establish how the borrower's bargaining affect outcomes.

Proposition 7 (Role of Bargaining Power)**.** *The equilibrium objects $\left\{\chi^{+},\chi^{-},\bar{r}^{f}\right\}$ are decreasing in η. In addition, at the extremes, we have that*

i) $\eta=1$*:* $\bar{r}^{f}=r^{m},\ \left\{\chi^{+},\chi^{-}\right\}=\left\{0,(1-\Psi^{-})(r^{w}-r^{m})\right\},$

ii) $\eta=0:\bar{r}^{f}=r^{w},\ \left\{\chi^{+},\chi^{-}\right\}=\left\{\Psi^{+}(r^{w}-r^{m}),(r^{w}-r^{m})\right\}.$

Thus, as η increases giving more bargaining power to borrowers, rates at every round fall. Thus, liquidity yields are decreasing. Regarding extrema, as borrowers extract all the matching surplus, rates go to the reserve rate and average cost of deficits are the penalty rates multiplied by the probability of matching in any round, $\chi^{-}=(1-\Psi^{-})(r^{w}-r^{m})$; the opposite extreme follows by symmetry.

Efficiency Extrema. We also derive the limiting properties as the market efficiency is taken to its extrema.

Proposition 8 (Efficiency Limits)**.** *The OTC market equilibrium satisfies:*

Walrasian: *As $\bar{\lambda}\to\infty$, the OTC market converges to its Walrasian market:*

i) $\theta=1$*:* $\Psi^{+}=\Psi^{-}=1$ *and*

$$\left\{\chi^{+},\chi^{-}\right\}=\left\{\left(r^{w}-r^{m}\right)(1-\eta),\left(r^{w}-r^{m}\right)(1-\eta)\right\},\bar{r}^{f}=\ r^{m}+\left(r^{w}-r^{m}\right)(1-\eta).$$

ii) $\theta>1$*:* $\Psi^{+}=1,\Psi^{-}=\theta^{-1}$ *and* $\chi^{+}=\chi^{-}=\left(r^{w}-r^{m}\right),\quad \bar{r}^{f}=r^{w}.$

iii) $\theta<1$*:* $\Psi^{+}=\theta,\Psi^{-}=1$ *and* $\chi^{+}=\chi^{-}=0,\quad \bar{r}^{f}=r^{m}.$

Static: *As $\bar{\lambda}\to 0$, the OTC market converges to a static bargaining:* $\Psi^{+}=\Psi^{-}=0$ *and*

$$\left\{\chi^{+},\chi^{-}\right\}=\left\{0,\left(r^{w}-r^{m}\right)\right\},\bar{r}^{f}=r^{m}+\left(r^{w}-r^{m}\right)(1-\eta).$$

As efficiency increases, the OTC market rate approaches a Walrasian limit: In this limit, if the market features an aggregate cash (scarcity of funds) deficit, $\theta>1$, average OTC market rate converges to the borrowers outside option, r^{w}. In the opposite case, as $\theta<1$, the rate converges to r^{m}. Likewise, the liquidity yields converge to the terminal trading surplus (in the case of an aggregate cash deficit) and zero (otherwise). Panel (b) of Figure 3 plots the rates and liquidity yields at the Walrasian limit.[15] Finally, when efficiency approaches to zero, rates and yields converge to those of a static bargaining and trade volume vanishes.

[15]In the knife-edge case where $\theta=1$, the average rate is an average of outside options weighted by the bargaining power. In either case, the trading probabilities are one for the shortest side of the market.

Market Tightness. Another property of interest regard how the yield coefficients and average rate vary with the initial market tightness. A Corollary to Proposition 3 the monotonicity of $\left\{\chi^{+},\chi^{-},\overline{r}^{f}\right\}$.

Corollary 2 (Market Tightness Effect on Yields and Rates)**.** *In any OTC market equilibrium:*

(monotonicity) $\left\{\chi^{+},\chi^{-},\overline{r}^{f}\right\}$ *are increasing in θ.*

The monotonicity of $\left\{\chi^{+},\chi^{-},\overline{r}^{f}\right\}$ is intuitive. Market tightness captures the relative size of settlement deficits. As deficits increase, lenders charge more for their funds and can match with greater probability. Borrowers pay higher interests on their OTC borrowings and are more likely to borrow from the last resource. The change from concavity to convexity around $\theta = 1$ follows immediately from the symmetry properties described above.

Next, we investigate the limit as market tightness approaches its extrema. By symmetry, it is sufficient to discuss the limit $\theta_0 \to 0$. Understanding this limit is important because it tells how the market behaves as the *aggregate* cash deficit vanishes. This limit reveals some mathematical subtleties: a naive intuition tells us that as one side of the market vanishes, say when $\theta_0 \to 0$, the short side (i.e., borrowers) should extract all the surplus and, hence, rates and yield coefficients should approach zero. The limiting behavior is far from obvious and, in fact, that naive intuition is true only in special cases.

Whether the shortest side of the market extracts all surplus depends on whether the intensive form of the matching function $\gamma(\cdot)$ is bounded above. Let $\bar{\gamma} \equiv \lim_{\theta\to 0}\gamma\left(\theta^{-1}\right)$. Clearly, $\bar{\gamma}$ is bounded for some matching functions, e.g., for the harmonic mean, $G(a,b) = \left(\frac{1}{2}a^{-1}+\frac{1}{2}b^{-1}\right)^{-b}$ but not for others, e.g., the Cobb-Douglas, $G(a,b) = a^{1/2}b^{1/2}$. This bound is critical in determining the decay rate of θ as θ vanishes:

$$\frac{\dot{\theta}}{\theta} = -\bar{\lambda}\bar{\gamma} \quad \text{as} \quad \theta \to 0.$$

When the decay rate is finite, market tightness behaves as a exponentially decaying function for θ close to zero. That is, the deficit side of the market approaches zero, but is positive for all θ as $\theta \to 0$. By contrast, if $\bar{\gamma}$ is unbounded, the decay rate explodes, so the market tightness approaches zero extremely fast as $\theta \to 0$, leading to a singularity point in the ODE where θ actually reaches zero. Thus, θ_τ in that case may approach zero in finite time. This rate of decay, in turn, governs the behavior of they OTC rates at the extrema.

Proposition 9 (Tightness Limits)**.** *The the liquidity-yield function has the following limiting behavior:*

- $\theta \to 0$*:* $\left\{\Psi^{+},\Psi^{-}\right\} = \left\{0, 1-e^{-\bar{\lambda}\bar{\gamma}}\right\}$ *and* $\left\{\chi^{+},\chi^{-}\right\} = \left\{0, \left(r^{w}-r^{m}\right)e^{-\bar{\lambda}\bar{\gamma}\eta}\right\}.$

- $\theta \to \infty$: $\{\Psi^{+}, \Psi^{-}\} = \left\{1 - e^{-\bar{\lambda}\bar{\gamma}}, 0\right\}$ *and* $\{\chi^{+}, \chi^{-}\} = \left\{(r^{w} - r^{m})\left(1 - e^{-(1-\eta)\bar{\lambda}\bar{\gamma}}\right), r^{w} - r^{m}\right\}$.

As tightness approaches zero, the lenders matching probability approaches zero. Thus, $\chi^{+} \to 0$. However, for the deficit side, the short side of the market, whether the overall probability of matching in the remaining rounds approaches one or remains less than one, depends on the asymptotic decay rate, $\bar{\gamma}$. If this decay rate is finite, although the deficit side is negligible relative to the surplus side, the matching probability converges, $1 - e^{-\bar{\lambda}\bar{\gamma}}$, a number strictly less than one. As a result χ^{-} does not vanisth. If the asymptotic decay rate is unbounded, the matching probability does converge to one and, in that case, χ^{-} will vanish. By symmetry, the opposite occurs in the limit as $\theta \to \infty$. These properties reflect on the OTC market rate.

Corollary 3 (Rates of Decay)**.** *If the asymptotic decay rate is unbounded, $\bar{r}^{f} \to r^{f}$ ($\bar{r}^{f} \to r^{w}$) as $\theta \to 0$ ($\theta \to \infty$). Otherwise,*

$$\lim_{\theta \to 0} \bar{r}^{f} = r^{m} + (r^{w} - r^{m})\left(\frac{e^{(1-\eta)\bar{\lambda}\bar{\gamma}} - 1}{e^{\bar{\lambda}\bar{\gamma}} - 1}\right), \quad \lim_{\theta \to \infty} \bar{r}^{f} = r^{m} + (r^{w} - r^{m})\left(\frac{e^{\bar{\lambda}\bar{\gamma}} - e^{\eta\bar{\lambda}\bar{\gamma}}}{e^{\bar{\lambda}\bar{\gamma}} - 1}\right).$$

This corollary asserts that if the asymptotic decay rate is infinite, as market tightness vanishes, the average OTC rate approaches zero akin to giving all the bargaining power to the deficit side. However, if the asymptotic decay rate is finite, the OTC rate remains positive. Instead, if the asymptotic decay rate is finite, a trader with surplus is able to extract some surplus in the very unlikely event that he gets to match. This occurs because the deficit side has a positive probability of not finding matches when $\bar{\gamma}$ is bounded. By symmetry, the opposite occurs as the market tightness explodes.

Corollary (3) is relevant because it shows that even as all investors approach cash satiation, a situation where no investor is in deficit, OTC rates may exceed the rate on cash. This situation does not occur when the matching function assumed has an unbounded asymptotic decay rate.

CES (Generalized Means) Matching Functions. The properties above apply to all matching functions. We now specialize and analyze the constant-elasticity of substitution (CES) class (also known as generalized means in mathematics).[16] Thus, we specialize to:

$$G(a, b; p) = \left(\frac{1}{2}a^{p} + \frac{1}{2}b^{p}\right)^{1/p},$$

[16]A Theorem by Kolmogorov states that the only functions that satisfy symmetry and monotonicity satisfy that $G(a, b) = g^{-1}\left(\frac{1}{2}g(a) + \frac{1}{2}g(b)\right)$. A special case of such functions is the CES class here.

which by concavity requires $p \leq 0$.[17] It is known that for $p = 0$ the matching function converges to a Cobb-Douglas (geometric mean): $G(a, b; p) = a^{1/2} b^{1/2}$ whereas for $p = -\infty$ converges to the Leontief matching function, $\min\{a, b\}$.[18]by Proposition 17, for any $\theta < 1$ ($\theta > 1$), market tightness falls (rises) faster as we increase p. We furthermore have the following application of Proposition (9).

Corollary 4. *Let $\theta_0 < 1$. Then, θ_τ may reach zero in some finite time $\tau < 1$ if and only if $p = 0$ (the matching function is Cobb-Douglas).*

This corollary states that the Cobb-Douglas case is a knife-edge within the CES matching function satisfying Assumption 1. That is, only for the Cobb-Douglas case, the OTC rates and yields can reach zero for a sufficiently small tightness θ_0. As we noted above, this property is that only for the Cobb-Douglas the market will efficiently all allocate cash surpluses to the deficit side provided that the initial deficit is not excessive. This provides a notion of cash-satiation that differs from a situation where all investors are indeed in surplus, i.e., $\theta_0 = 0$.

Figure 4 presents a comparison across several CES matching functions. Panel (a) plots $\dot{\theta}/\theta$ given θ for various matching functions. Observe that in all cases except the Cobb-Douglas, the growth rates stabilize to a finite number, consistent with finite rates of decay. Panel (b) shows the evolution of θ_τ over time $\tau$for the same initial condition θ_0. Again, notice how in this example, only for the Cobb-Douglas case does θ_τ reach zero whereas for all other matching functions θ decay at a finite rate as it approaches zero. Panel's (c) and (d) plot the corresponding values of the liquidity yields as a function of the initial condition $\log(\theta_0)$. Only for the Cobb-Douglas case, the coefficients can be zero (or $r^w - r^m$) provided that the initial tightness is sufficiently far from zero.

3.3 Analytic Formulas: the Cobb-Douglas and Leontief Cases

As we explained above, the market tightness that solves (3) has analytic solutions only for a limited number of cases of p. Moreover, in many cases, e.g. $p = -1$, these expressions are not convenient even though closed-form expression can be found. However, the two polar cases of the CES class, the Cobb-Douglas ($p = 0$) and Leontief ($p = -\infty$), these expressions are rather simple.

Tightness. The following table presents the solution for the market tightness in the Cobb-Douglas and Leontief cases, respectively.

[17] The coefficient relates to the elasticity of substitution $\rho \geq 0$ in production via $p = 1/(1-\rho)$. For $p \leq 0$, we have $2^{1/p} G(a, b; p) \leq \min(a, b)$, so $\lambda_N < 2^{1/p}$ guarantees weak exhaustion in the finite rounds case.

[18] Other common values include, $p = -1$, so that the matching function becomes the harmonic mean: $G(a, b; p) = 2\left(\frac{1}{a^{-1}+b^{-1}}\right)$. It is also known that for $q > p$, $G(a, b; p) < G(a, b; q)$.Thus,

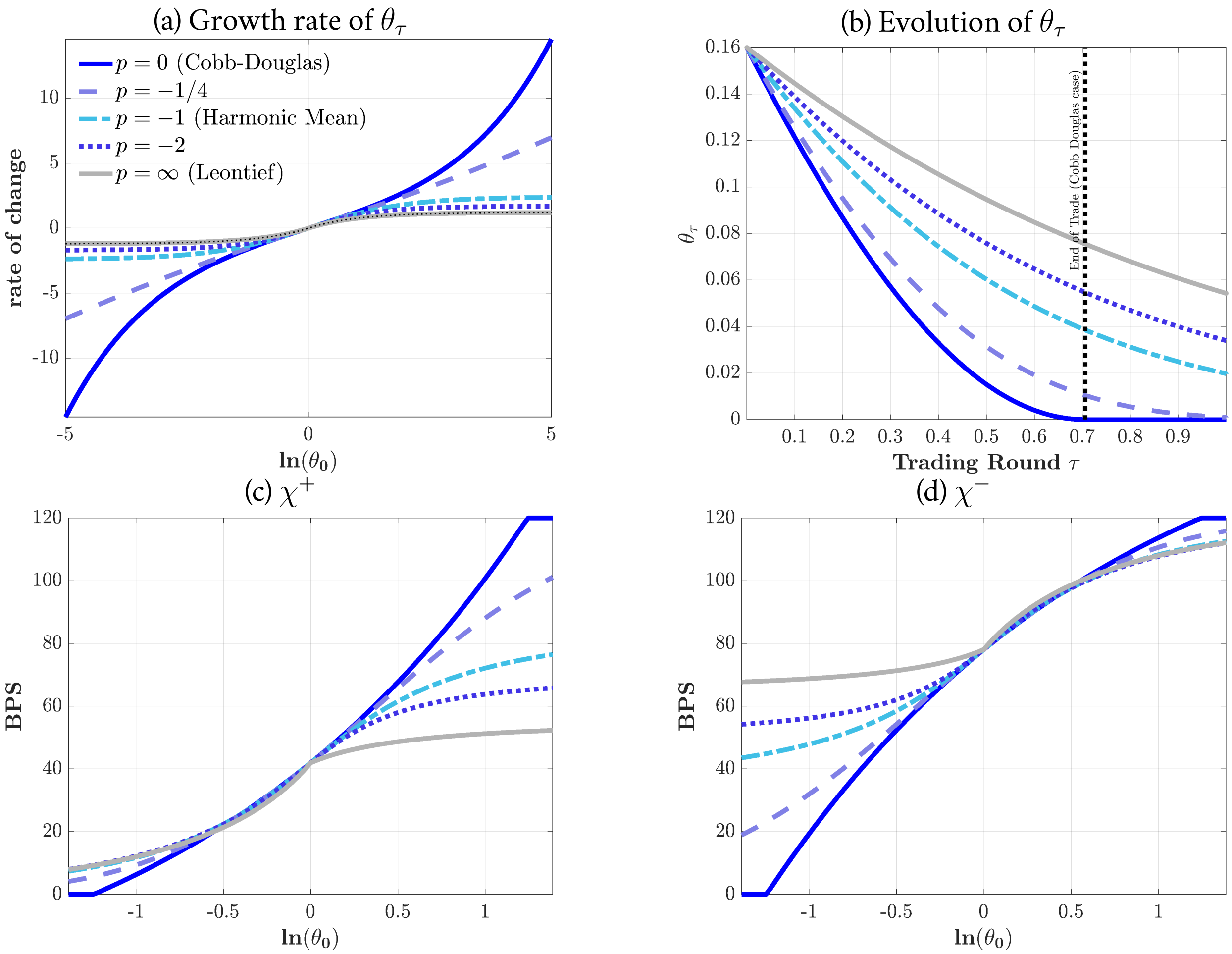


Figure 4: **Comparison Across CES matching functions.**

Note: $\theta = \theta_0$ is the initial market tightness. The example is calibrated with $\eta = 0.5$, $\bar{\lambda} = 1.2$, $r^w - r^m =$ 120bps in all cases.

Matching	Cobb-Douglas ($p = 0$)	Leontief ($p = -\infty$)
$\boldsymbol{\theta(\tau)}, \quad \boldsymbol{\tau \in [0, T]}$	$\left(\frac{(1+\sqrt{\theta_0})\,e^{-\bar{\lambda}t} - (1-\sqrt{\theta_0})}{(1+\sqrt{\theta_0})\,e^{-\bar{\lambda}t} + (1-\sqrt{\theta_0})}\right)^2$	$\begin{cases} 1 + (\theta - 1)\,e^{\bar{\lambda}\tau}, & \theta > 1 \\ \frac{\theta_0}{\theta_0 + (1-\theta_0)e^{\bar{\lambda}\tau}}, & \theta < 1 \end{cases}$
Stop Time T	$T = \min\left\{\frac{1}{\bar{\lambda}} \log\left(\left\|\frac{1+\sqrt{\theta_0}}{1-\sqrt{\theta_0}}\right\|\right), 1\right\}$	$T = \infty$
$\boldsymbol{\Psi^+}$	$1 - e^{-\bar{\lambda}T}\left(\frac{(1+\sqrt{\theta_0}) + (1-\sqrt{\theta_0})\,e^{\bar{\lambda}T}}{(1+\sqrt{\theta_0}) + (1-\sqrt{\theta_0})}\right)^2$	$\begin{cases} 1 - e^{-\bar{\lambda}}, & \theta_0 \geq 1 \\ \theta_0(1 - e^{-\bar{\lambda}}), & \theta_0 < 1 \end{cases}$
$\boldsymbol{\Psi^-}$	$1 - e^{-\bar{\lambda}T}\left(\frac{(1+\sqrt{\theta_0}) - (1-\sqrt{\theta_0})\,e^{\bar{\lambda}T}}{(1+\sqrt{\theta_0}) - (1-\sqrt{\theta_0})}\right)^2$	$\begin{cases} (1 - e^{-\bar{\lambda}})\theta_0^{-1}, & \theta_0 > 1 \\ 1 - e^{-\bar{\lambda}}, & \theta_0 \leq 1 \end{cases}$

The Cobb-Douglas case and Leontief cases exhibit qualitatively different behavior, as is clear from Figure 4. In the Leontief case, the short side of the market features trading probabilities that are exponential distributed with parameter $\bar{\lambda}$. Thus, in the Leontief case, matching probabilities do not depend on the initial market tightness. As a result, θ_τ follows a logistic formula. Trading never stops in the Leontief case.

In the Cobb-Douglas case, the the short-side of the market may vanish before the trading time is over, if the stopping time T in the table is $T < 1$.[19] Indeed, evaluating the formula for θ_τ at the stopping time, $\tau = T$, yields zero or infinity, which is consistent with the end of trade. Hence, in the Cobb-Douglas case, trade stops if the initial tightness is sufficiently unbalanced:[20]

$$\theta_0 \notin \left[\left(\frac{\exp\left(\bar{\lambda}\right)-1}{\exp\left(\bar{\lambda}\right)+1}\right)^2, \left(\frac{\exp\left(\bar{\lambda}\right)+1}{\exp\left(\bar{\lambda}\right)-1}\right)^2\right].$$

Liquidity Yields and Rates. While the solution for market tightness is different in the Leontief and Cobb-Douglas cases, the yield coefficients have the same solution as a function of θ:

Theorem 1. *Fix, $\theta = \theta_0$ and compute $\bar{\theta} = \theta_1$ under the Cobb-Douglas and Leontief cases. In either case, the yield coefficients are:*

$$\{\chi^+, \chi^-\} = \left\{(r^w - r^m)\left(\frac{\bar{\theta} - \bar{\theta}^\eta \theta^{1-\eta}}{\bar{\theta}-1}\right), (r^w - r^m)\left(\frac{\bar{\theta} - \bar{\theta}^\eta \theta^{-\eta}}{\bar{\theta}-1}\right)\right\} \in [0, r^w - r^m].$$

The solution to $\{\chi^+, \chi^-\}$ is a continuous function of the initial tightness and the terminal tightness, θ and $\bar{\theta}$. $\{\chi^+, \chi^-\}$ are always positive and bounded by the terminal surplus, $r^w - r^m$, as expected.[21] From the yield coefficients, we furthermore obtain the OTC rate as a weighted average of the cash rate and discount rate:

$$\bar{r}^f = \phi(\theta)\, r^m + (1 - \phi(\theta))r^w, \text{ where } \phi(\theta) = \frac{\left(\bar{\theta}/\theta\right)^\eta - \theta}{\bar{\theta}/\theta - 1}\phi(\theta) \in [0,1]. \tag{15}$$

[19] The formula for θ_τ is actually associated with the hyperbolic tangent function.

[20] Likewise, we can invert this expression to also find a value of $\bar{\lambda}$ such that for any θ_0, the short side of the market vanishes: Given θ_0, trading vanishes in finite time if $\bar{\lambda} > 2\log\left(\left|\frac{1+\sqrt{\theta_0}}{1-\sqrt{\theta_0}}\right|\right)$.

[21] The numerator and denominator of the expressions that multiply $(r^w - r^m)$ positive terms that are bounded by one. Moreover, the initial trading surplus is:

$$\Sigma = (r^w - r^m)\left(\frac{\bar{\theta}^\eta \theta^{1-\eta} - \bar{\theta}^\eta \theta^{-\eta}}{\bar{\theta}-1}\right),$$

The term $\phi(\theta)$ is acts as an endogenous bargaining rate: unlike static bargaining, this bargaining rate depends on the evolution of outside options. $\phi(\theta)$ compactly captures the information from the matching function and the bargaining process. We further verify that $\lim_{\theta \to 1} \phi(\theta) = \eta$, consistent with the balanced-market outcome above.

Whereas Theorem 1 applies to both the Cobb-Douglas and Leontief cases, the formula is not valid in general, not even for other values of p that yield closed forms. It is remarkable that it holds for the extrema of p, Cobb-Douglas and Leontief, even though the evolution of market tightness evolves differs the most among those to cases.

The behavior of the yield coefficients and the average rate as a function of $\log(\theta_0)$ is depicted in Figure 5. Panels (a) and (b) depict the outcomes as functions of the initial condition for the Leontief and Cobb-Douglas cases. Both figures have sigmoid-like patterns, but the differences regarding concavity, and the limits as θ moves to its extreme are clear.

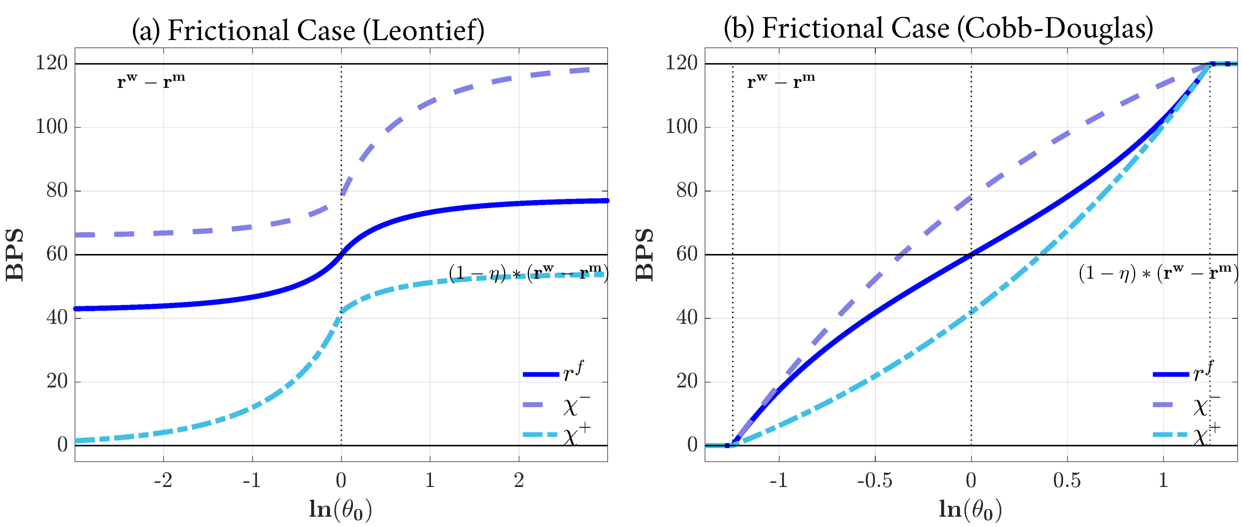


Figure 5: **Analytic Solutions:** OTC rates and yield coefficients for Leontief and Cobb-Douglas Matching Functions.

Note: The OTC rate and liquidity yield coefficients are plotted as functions of θ_0. $\theta_0 = \theta$ is the initial market tightness defined as ratio of the the initial aggregate deficit and initial aggregate surplus. Both panels are calibrated using $\eta = 0.5$,$\bar{\lambda} = 1.2$, $r^w - r^m = 120$bps.

4. Applications

4.1 Portfolio choices and convenience yields

We now study optimal portfolios in presence of settlement risk. We maintain the assumption that all returns are exogenous, except for the endogenous OTC rate $\bar{R}^f$. Thus, we abstract away from additional price effects on assets. Investor preferences are represented by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{16}$$

where $\beta < 1$ is the time discount factor, and $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$ is the utility function over the consumption good with $\gamma \geq 0$. As described in Section 2, each period, investors start with an initial wealth e, which depends on previous portfolio choices and the realization of returns, and choose their portfolio decisions subject to the budget constraint (1) We present below the problem of the investor in recursive form. We use X to denote an aggregate state.

Problem 2 (Investor's Problem)**.** *The savings-portfolio problem is:*

$$V_t(e, X) = \max_{\left\{c, \left\{\tilde{a}^i_{t+1}\right\}_{i\in\mathbb{I}}, m\tilde{_{t+1}}\right\}} u(c) + \beta \mathbb{E}\left[V_{t+1}(e_{t+1}, X')\right], \tag{17}$$

subject to: (1), (3), and (6).

In the investor's problem $\mathbb{E}$ denotes the expectation operator with respect to the idiosyncratic liquidity shock in the settlement stage in the current period and the next period's realization of returns. Because the objective is homogenous and the budget constraint and returns are linear, the problem admits portfolio separation: dividends can be chosen independent of the optimal portfolio weights and, furthermore, portfolio weights are independent of equity after dividends.[22] In turn, the optimal portfolio weights consists of choosing portfolio weights on assets $\{a^i\}$ and the cash asset m to maximize the risk-adjusted return on equity:[23]

$$\max_{m, \left\{a^i\right\}_{i\in\mathbb{I}}} \left(\mathbb{E}\left[\sum_{i\in\mathbb{I}} R^i_{t+1}\left(X'\right) a^i + R^m_{t+1}(X')m + \chi_{t+1}\left(s\left(\left\{a^i\right\}_{i\in\mathbb{I}}, m, \left\{\omega^i_{t+1}\right\}_{i\in\mathbb{I}}\right); \theta_t\right)\right]^{1-\gamma}\right)^{\frac{1}{1-\gamma}} \tag{18}$$

subject to:

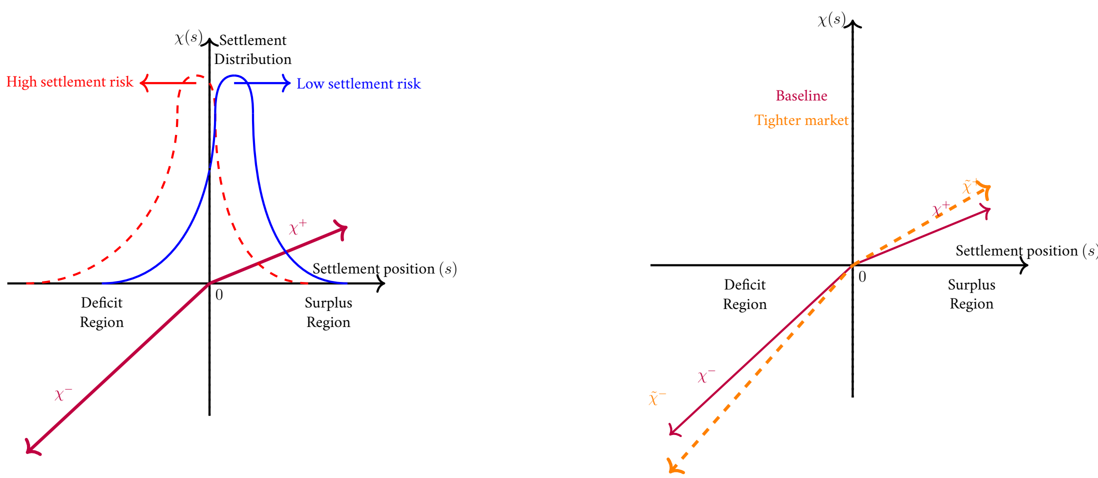
$$\sum_{i\in\mathbb{I}} a^i + m = 1$$

[22] See Proposition 3 of Bianchi and Bigio (2022) for a full characterization.

[23] The weights are defined as portfolio holdings relative to equity. With abuse of notation, we denote the weights with the same notation as the holdings.

here we make explicit the dependence of the convenience yield on market tightness, with a slight abuse of notation.

Crucially, the return on the portfolio depend on the idiosyncratic shock ω, its liquidity holdings, and the market tightness in the OTC market. It is worth visualizing how this portfolio problem differs from standard portfolio problems. Panel (a) of Figure 6 illustrates how portfolio choice affects settlement risk through the liquidity yield function$\chi\left(s;\theta\right)$. The purple kinked function represents $\chi\left(s;\theta\right)$, which maps settlement positions s additional payoffs. When $s<0$ (settlement deficit), investors face average borrowing costs χ^{-}; when $s>0$ (surplus), they earn lending returns with slope χ^{+}. The asymmetry in the yield coefficients reflects higher penalty rates for emergency borrowing than returns on surplus lending. The panel shows two settlement distributions arising from different portfolio choices. The high-risk portfolio (red, dashed) generates a wide distribution of settlement needs, with significant probability mass in deficit regions. The low-risk portfolio (blue, solid) concentrates settlement positions at higher values. Due to the kink at zero, the high-settlement risk portfolio incurs expected losses even when distributions have zero mean. This illustrates a key insight: assets that generate volatile settlement needs command convenience yields to compensate for these expected losses, even for risk-neutral investors. Panel (b) of Figure 6 illustrates shows how individual portfolio choices affects the liquidity yield function$\chi\left(s;\theta\right)$ of others. Next, we derive the optimal portfolio choice bearing this idea in mind.



(a) Portfolio choice and liquidity yields

(b) Market tightness and liquidity yields

Figure 6: Liquidity yield function $\chi(s)$ and portfolio choice. Panel (a) shows how different portfolio volatilities interact with the kinked liquidity yield function. Panel (b) illustrates how market tightness (driven by others' portfolio choices) rotates the liquidity yield function, increasing both borrowing costs and lending returns.

Convenience yields. Let us define $\chi_s \equiv \frac{\partial \chi}{\partial s}$ which is equal to χ^+ if $s \geq 0$ and χ^-otherwise. Taking first-order conditions in the portfolio problem (18), assuming strictly interior portfolio positions, we obtain

$$\underbrace{\mathbb{E}_X\left[R^i\right] - R^m}_{\text{premium}} = \underbrace{\underbrace{\mathbb{E}_{X,\omega}\left[\chi_s \frac{\partial s}{\partial m} - \chi_s \frac{\partial s}{\partial a^i}\right]}_{\text{first-order liquidity premium}} - \underbrace{\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma}, \chi_s \frac{\partial s}{\partial m} - \chi_s \frac{\partial s}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}}_{\text{liquidity risk premium}}}_{\text{liquidity premium}}$$
$$- \underbrace{\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma}, R^i\left(X\right)\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}}_{\text{conventional risk premium}} \tag{19}$$
$$= \underbrace{-\mathbb{E}_{X,\omega}\left[\chi_s \frac{\partial s}{\partial m} - \chi_s \frac{\partial s}{\partial a^i}\right]}_{\text{first-order liquidity yield}} - \underbrace{\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma}, R^i\left(X\right) + \chi_s \frac{\partial s}{\partial m} - \chi_s \frac{\partial s}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}}_{\text{total risk premium}} \tag{20}$$

The left-hand side is the difference in the expected return of asset i relative to cash. At an optimal solution, the representation in (19) shows that this asset's premium equals the sum of a convenience yield (liquidity premium) and a conventional risk premium. Thus, the equation captures a trade-off between expected return differentials against settlement risk and a conventional risk premium. The former emerges because asset a^i exposes the investor to settlement risk whereas a larger m provides liquidity. In particular, in the case of a negative settlement shock, the investor obtains a higher return on cash compared to a less liquid asset and vice-versa. Assets that induce greater liquidity risk command a greater premium.

The liquidity premium term can be further unpacked into two terms: The first term captures how changes in the portfolio affect the expected settlement costs—note that χ is differentiable almost everywhere.[24] This term is present under risk neutrality. The terms is negative even when the cash position is zero in expectation because the liquidity yield function is a concave function. In turn, the second term captures the covariance term between liquidity payoffs and the discount factor: when the investor is risk-averse, states with negative returns penalize additional settlement costs. The second equation, (20), consolidates the two terms involving covariances, thus capturing an overall risk premia.

There are some important lessons here: First, settlement risks induce determinate portfolios even among risk-neutral investors. Unlike standard portfolio problems where the riskiness of assets is

[24]Note that while changing the portfolio affects at the margin the probability of being short or long, this does not enter into the optimality condition because the bank obtains the same marginal payoff (i.e., zero) evaluated at the threshold ω^*. Using Leibniz's, we can show that the effect disappears given that $s = 0$ when the shock is ω^*.

given, here, by choosing its cash position, investors control the amount of risk. Given the concavity of χ, investors must be compensated with returns in order to hold portfolios that are more exposed to settlement risk. Second, the standard risk-adjustment for \is insufficient to price the risk associated with an asset. Crucially, without considering the associated settlement needs created by an asset, the return in payoffs is not enough to capture the full extent of risk. Conversely, convenience yields cannot be treated as pricing factors independent of risk, because the correlation between the riskiness of the asset and the liquidity needs must be considered. Another possible application of the theory is to investigate how the equity premium puzzle (Merha and Prescott, 1985) is impacted by the presence of convenience yields.

Banking Example. Next, we present a next a concrete example deriving convenience yields in example i) in Section 2 regarding example i) on deposit funding risks. Thus, we specialize to the case where there in addition to m, there is an illiquid asset b, and a liability (deposits) d subject to withdrawal risk:

$$s\left(\{b,d\},m\right)\equiv m+\left(\frac{R^{d}}{R^{m}}\omega-\rho(1+\omega)\right)d, \tag{21}$$

and their budget constraint is: $b+m=1+d$ where ω is distributed F with mean zero. The investor's problem, after substituting his budget constraint, thus becomes:

Problem 3 (Withdrawal Risk Problem)**.** *The investor's problem in example i) is:*

$$\max_{m\geq 0,d\geq 0}\left\{\mathbb{E}_{\omega}\left[R^{b}+R^{b}(d-m)+\ R^{m}m-\ R^{d}d+\chi\left(s\left(d,m\right);\theta\right)\right]^{1-\gamma}\right\}^{\frac{1}{1-\gamma}},$$

In this simple example, the optimality condition (20), is simple and links the portfolio to a convenience yield, a spread, between illiquid assets and cash:

$$\underbrace{R^{b}-R^{m}}_{\text{liquidity premium}}=\chi^{+}\left(1-\tilde{\Phi}\left(\omega^{*}\right)\right)+\chi^{-}\tilde{\Phi}\left(\omega^{*}\right)=\chi^{+}+\Sigma\left(\theta\right)\tilde{\Phi}\left(\omega^{*}\right)\cdot \tag{22}$$

where

$$\tilde{\Phi}\left(\omega^{*}\right)=\underbrace{\underbrace{\Phi\left(\omega^{*}\right)}_{\text{deficit prob}}\cdot\underbrace{\frac{\mathbb{E}_{\omega}\left[R^{e}(\omega)^{-\gamma}|\omega<\omega^{*}\right]}{\mathbb{E}_{\omega}\left[R^{e}(\omega)^{-\gamma}\right]}}_{\text{risk-aversion correction}}}_{\text{risk-adjusted deficit probability}},\quad \omega^{*}\equiv\frac{\rho-\dfrac{m}{d}}{\dfrac{R^{d}}{R^{m}}-\rho}.$$

Thus, the liquidity premium is a weighted average of the liquidity yield coefficients, $\chi^{+}\left(1-\tilde{\Phi}\left(\omega^{*}\right)\right)+$

$\chi^{-}\tilde{\Phi}(\omega^{*})$. In the expression, $\tilde{\Phi}(\omega^{*})$ is the risk-adjusted probability of falling in a cash deficit.[25] The probability of a deficit is $\Phi(\omega^{*})$ where ω^{*} is the threshold shock that puts banks in deficit—recall that lower values of ω represent an outflow of funds. Thus, the liquidity premium of cash over illiquid assets is their risk-adjusted expected return of coming in to the OTC market with an extra unit of cash.

In the special case of risk-neutrality, we have that (22) becomes:

$$\underbrace{R^{b}-R^{m}}_{\text{liquidity premium}}=\chi^{+}+\left(R^{w}-R^{m}\right)\left(1-\Psi^{-}\left(\theta\right)\right)\Phi\left(\omega^{*}\right).$$

The corresponding difference between illiquid assets and deposit is an interest margin associated with the liquidity risk of deposits:

$$\underbrace{R^{b}-R^{d}}_{\text{liquidity premium}}=\chi^{+}+\left(\chi^{-}-\chi^{+}\right)\tilde{\Phi}\left(\omega^{*}\right)\left(\left(\frac{R^{d}}{R^{m}}-\rho\right)\mathbb{E}_{\omega}\left[\omega R^{e}(\omega)^{-\gamma}|\omega<\omega^{*}\right]-\rho\right)\cdot \tag{23}$$

This is a similar expression to (22) but one that considers the riskiness of the left tail of deposits. Settings where deposits are riskier, require a greater compensation for risk to maintain the same portfolio.

Next, we discuss how features of the OTC market affect the liquidity yield coefficients and, consequently, the return premia. In turn, we discuss how OTC market frictions can be inferred from convenience yields or OTC market data to discipline the theory.

4.2 Parameter Identification

Having derived closed-form expressions for convenience yields (20) and OTC rates (15), we now turn to a crucial empirical question: how can we identify the underlying OTC market parameters and settlement risk from observable data? Our analytical formulas reveal that the average OTC rate $\bar{r}^{f}$ and the liquidity yield coefficients $\{\chi^{+},\chi^{-}\}$ depend on three key elements: market tightness θ (which, given portfolios, reflects the distribution of unobservable settlement shocks), matching efficiency $\bar{\lambda}$, and bargaining power η. However, these parameters are not directly observable. This section develops an approach to infer them from two potential sources of data: (i) convenience yields on liquid assets, which reveal the shadow value of liquidity, and (ii) OTC market outcomes, particularly rate dispersion and trading volumes.

An estimation of these structural parameters is important for three reasons. First, it allows to

[25] The risk adjustment scales probabilities by marginal utilities of wealth.

quantify the sources of convenience yields—distinguishing whether high yields reflect scarce liquidity or inefficient trading. Second, it enables policy counterfactuals by revealing how market interventions affect the underlying parameters. Third, it provides a structural interpretation of time-series and cross-sectional variation in liquidity premia, linking observed yield movements to changes in market conditions.

We focus on the Leontief matching function, which yields particularly sharp identification results. We proceed first by studying comparative statics.

Market tightness. It is worth recalling a key result from our earlier analysis. As established in Corollary 2, the liquidity yield coefficients and the average OTC rate $\left\{\chi^{+}, \chi^{-}, \bar{r}^{f}\right\}$ are all monotonically increasing in market tightness θ. This monotonicity enables identification of θ from observable data. For instance, the convenience yield on illiquid bonds (22) depends on a weighted sum of the liquidity coefficients, which is also monotonic in θ given portfolios. Similarly, the average OTC rate increases monotonically with θ. Once we identify θ from these observables, we can back out parameters of the unobserved settlement shock distribution, such as the variance of withdrawal risk or the frequency of margin calls. Note that the net interest margin in equation (23) depends on a weighted difference of the yield coefficients, which need not be monotonic in θ.[26]

Matching efficiency. Let us now turn to the comparative static with respect to the efficiency parameter $\bar{\lambda}$.

Proposition 10 (Comparative Statics: Efficiency and Yield Coeficients)**.** *In the Leontief case:*

I) Let $\theta < 1$, then χ^{-} is decreasing in $\bar{\lambda}$, $\bar{r}^{f}$ is decreasing in $\bar{\lambda}$, and χ^{+} is non-montonic in $\bar{\lambda}$.

II) Let $\theta > 1$, then χ^{+} is increasing in $\bar{\lambda}$, $\bar{r}^{f}$ is increasing in $\bar{\lambda}$, and χ^{-} is non-montonic in $\bar{\lambda}$.

III) Let $\theta = 1$, then

$$\frac{\partial \chi^{+}}{\partial \bar{\lambda}} = \left(r^{w} - r^{m}\right)(1-\eta)\frac{\partial \Psi^{+}}{\partial \bar{\lambda}} > 0, \qquad \frac{\partial \chi^{-}}{\partial \bar{\lambda}} = -(r^{w} - r^{m})\eta\frac{\partial \Psi^{-}}{\partial \bar{\lambda}} < 0.$$

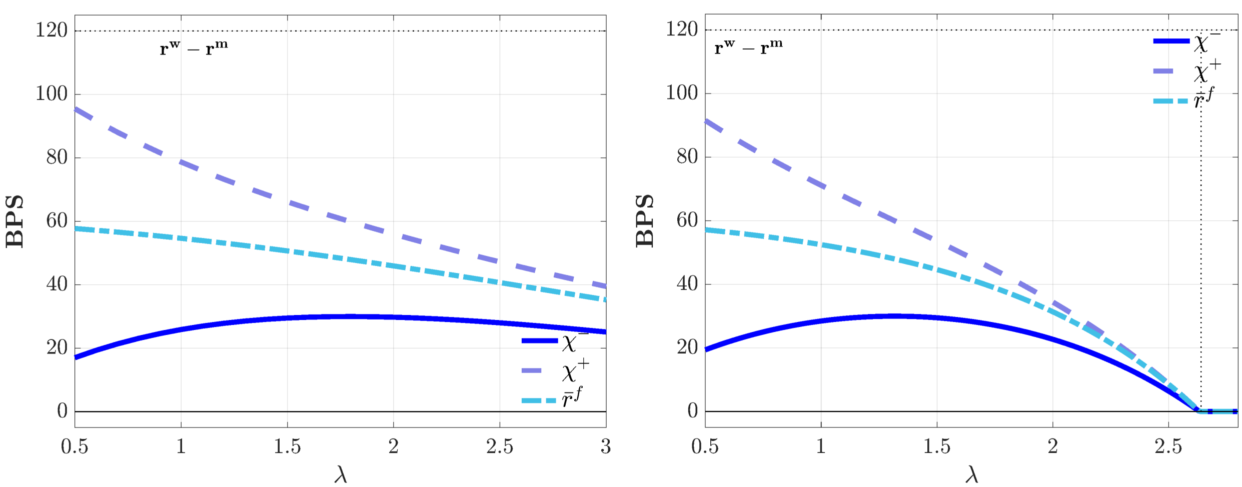
The non-monotonic relationship between matching efficiency and convenience yields reveals a subtle but important insight. When markets are tight ($\theta > 1$), improving matching efficiency has two opposing effects: it increases the probability that deficit investors find lenders (reducing their borrowing costs), but it also strengthens lenders' bargaining position, raising equilibrium rates. The net effect on χ^{-} depends on which force dominates. This non-monotonicity means that higher

[26] The curvature properties of the yield coefficients—whether convex or concave—may provide additional information.

convenience yields need not signal market dysfunction—they could reflect improved matching that benefits the scarce side of the market.

Consider the bond liquidity premium in equation (22). An increase in this premium, given observed portfolios, could indicate that market efficiency has improved, raising the option value of holding cash for potential lending opportunities. This occurs when the increase in χ^+ (the benefit of lending) more than offsets any decline in χ^- (the cost of borrowing). The same non-monotonic pattern characterizes the average OTC rate $\bar{r}^f$. Figure 7 illustrates these relationships: Panel (a) confirms our theoretical results for the Leontief case, while Panel (b) demonstrates that this pattern extends to the Cobb-Douglas matching function, suggesting it is a robust feature of OTC markets for various matching functions.

OTC Rate Dispersion. The dispersion of OTC rates within each trading day—the difference between rates at the beginning and end of the trading session—provides another observable moment for parameter identification. Recall from Proposition (15) that the equilibrium rate varies during the trading session according to market evolution of the tightness θ_τ. Since rates change monotonically throughout the day, increasing (decreasing) with time if $\theta > 1$ ($\theta < 1$), we can measure dispersion in rates :



(a) Leontief Matching

(b) Cobb-Dogulas Matching

Figure 7: **Effects of Matching Efficiency:**

Note: Rates and Yield Coefficients as functions of λ for Leontief and Cobb-Douglas Matching Functions. Note: The OTC rate and liquidity yield coefficients are plotted as functions of $\bar{\lambda}$. Both panels are calibrated using $\eta = 0.5$,$\theta_0 = 0.75$, $r^w - r^m = 120$bps.

$$Q \equiv \left| r_1^f - r_0^f \right|$$

We obtain the following relationship between parameters and the dispersion:

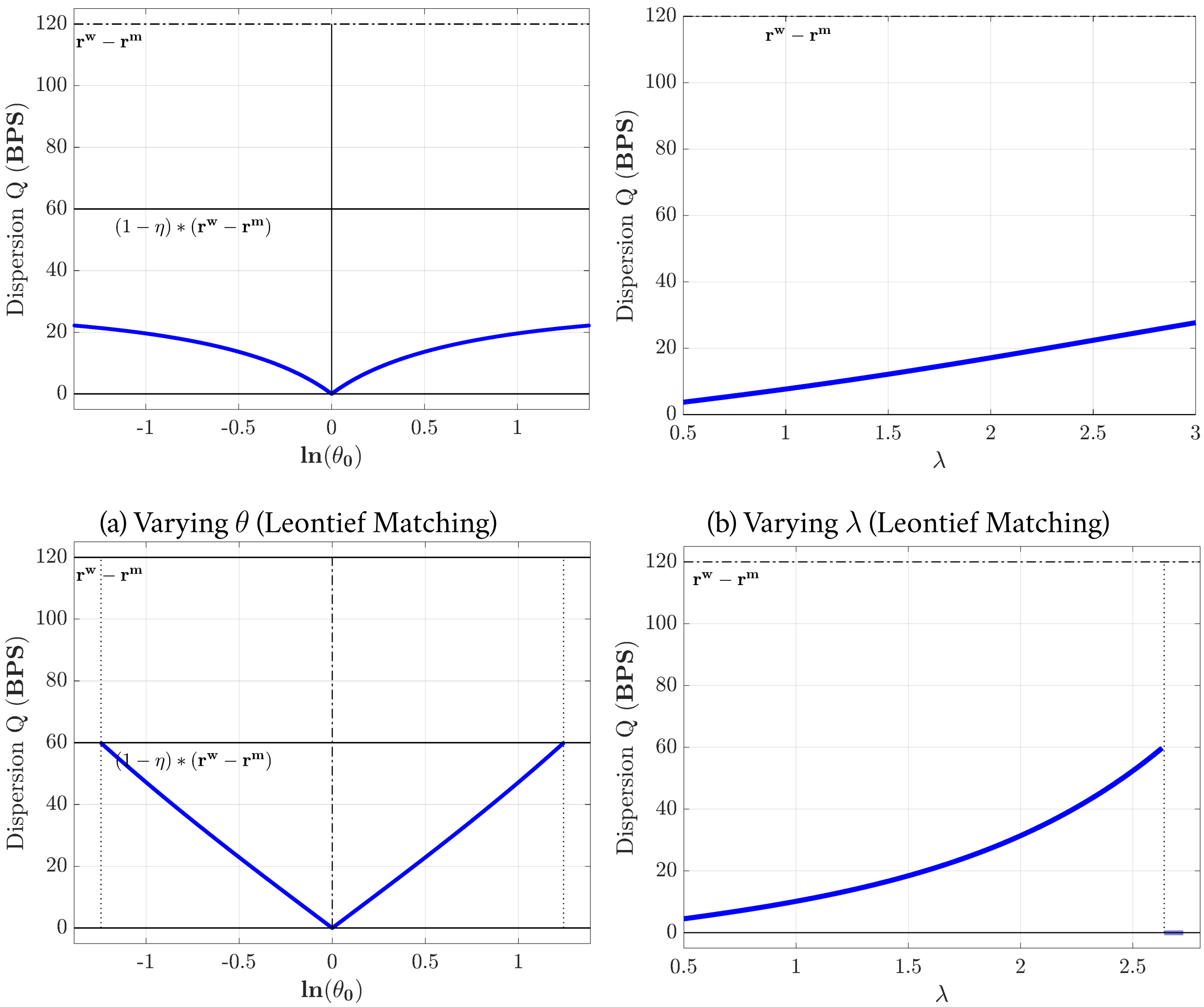
Proposition 11 (Comparative Statics: Rate Dispersion)**.** *Rate dispersion features the following comparative statics:*

I) Let $\theta \leq 1$ ($\theta \geq 1$), then $\frac{\partial Q}{\partial \theta} \leq 0$ ($\frac{\partial Q}{\partial \theta} \geq 0$)—with equality if and only if $\theta = 1$.

II) $\frac{\partial Q}{\partial \bar{\lambda}} \geq 0$—equality if and only if $\theta = 1$.

The intuition is straightforward. Rate dispersion reflects how much the market "unwinds" during the trading session. When the market is unbalanced ($\theta \neq 1$), the scarce side gets progressively better terms as trading proceeds, creating larger rate movements. The more unbalanced the market—in either direction—the greater the dispersion. Similarly, higher matching efficiency $\bar{\lambda}$ accelerates this unwinding process, amplifying dispersion. At $\theta = 1$, the market is balanced and rates remain constant throughout the session, yielding $Q = 0$.

Figure 8 illustrates these patterns. Panel (a) shows the symmetric U-shape of dispersion around $\theta = 1$ for the, while panel (b) demonstrates how dispersion increases with matching efficiency. Panels (c) and (d) show the corresponding mappings for the Cobb-Douglas case, demonstrating that these patterns are robust across matching technologies. These relationships provide additional moments for identification: observing both the level of OTC rates (which pins down θ) and their intraday dispersion Q (which helps identify $\bar{\lambda}$) can jointly identify OTC market parameters.



(a) Varying θ (Leontief Matching)

(b) Varying λ (Leontief Matching)

(c) Varying θ (Cobb-Douglas Matching)

(d) Varying λ (Cobb-Douglas Case)

Figure 8: **Dispersion of OTC Rates as Function of** $\{\lambda, \theta_0\}$**.**

Note: The dispersion of the OTC rate is plotted as a function of $\bar{\lambda}$ and θ_0 for the Leontief and Cobb-Douglas matching functions. Both panels are calibrated using $\eta = 0.5$, $r^w - r^m = 120$bps. When we vary $\bar{\lambda}$, we set $\theta_0 = 0.75$. When we vary θ_0 we set $\bar{\lambda} = 1.2$.

Trading volumes. Trading volumes provide a third observable moment for parameter identification. Unlike convenience yields and rate dispersion, the volume of discount window borrowing relative to OTC trading offers a particularly clean identification strategy.

Define the relative volume as the ratio of lender-of-last-resort borrowing to OTC market borrowing:

$$I(\theta) \equiv \frac{1-\Psi^{-}(\theta)}{\Psi^{-}(\theta)} = \begin{cases} \frac{e^{-\bar{\lambda}}}{1-e^{-\bar{\lambda}}} & \theta \leq 1 \\ \\ \frac{1-\left(1-e^{-\bar{\lambda}}\right)\theta^{-1}}{\left(1-e^{-\bar{\lambda}}\right)\theta^{-1}} & \theta > 1. \end{cases}$$

This ratio captures market efficiency in reallocating liquidity: if $I(\theta) = 0$ indicates perfect reallocation through the OTC market whereas $I(\theta) = \infty$ indicates no OTC volume at all. We call $I(\theta)$ the relative volume.

Proposition 12 (Comparative Statics: Relative Volume)**.** *The relative volume features the following comparative statics:*

(Tightness): The relative volume increases (remains constant) with the tightness if $\theta > 1$ ($\theta < 1$):

$$\frac{\partial I(\theta)}{\partial \theta}\frac{\theta}{I(\theta)} = \frac{\theta}{\theta - 1 + e^{-\bar{\lambda}}}\mathbb{I}_{[\theta>1]} \geq 0.$$

(Effiency): The relative volume increases (remains constant) with the tightness if $\theta > 1$ ($\theta < 1$):

$$\frac{\partial I(\theta)}{\partial \bar{\lambda}}\frac{\bar{\lambda}}{I(\theta)} = -\bar{\lambda}^{2}\left(\frac{e^{-\bar{\lambda}}}{1-e^{-\bar{\lambda}}} + \frac{(\theta-1)\,\mathbb{I}_{[\theta>1]}+1}{e^{-\bar{\lambda}}}\right) < 0.$$

These results reveal a key identification advantage: relative volume decreases monotonically with matching efficiency $\bar{\lambda}$ regardless of market tightness. This contrasts sharply with convenience yields, which can be non-monotonic in $\bar{\lambda}$. Moreover, when markets have excess liquidity ($\theta < 1$), relative volume depends only on $\bar{\lambda}$ and is independent of θ in the Leontief case. Combined with our earlier results, this suggests that relative volumes is convenient to pin down $\bar{\lambda}$.

Figure 9 illustrates these patterns for the Leontief case (panels a and b). Panels (c and d) demonstrate a similar pattern for the Cobb-Douglas case, demonstrating again that patterns are robust across matching technologies.

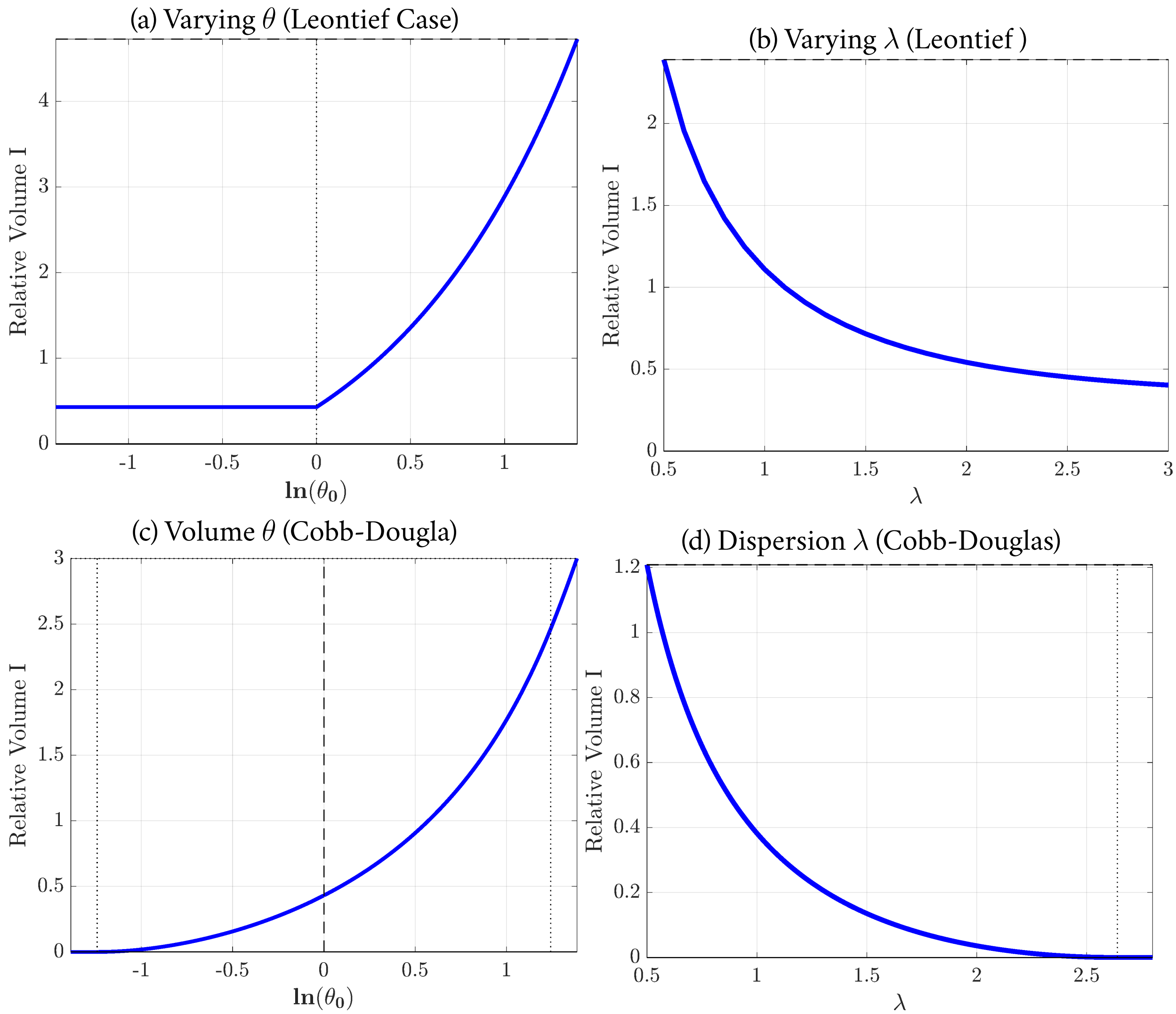


Figure 9: **Relative Volume as Function of $\{\lambda, \theta_0\}$.**

Note: The relative trading volume is plotted as a function of $\bar{\lambda}$ and θ_0 for the Leontief and Cobb-Douglas matching functions. Both panels are calibrated using $\eta = 0.5$, $r^w - r^m = 120$bps. When we vary $\bar{\lambda}$, we set $\theta_0 = 0.75$. When we vary θ_0 we set $\bar{\lambda} = 1.2$.

Identification Strategy: Putting It All Together. Our comparative statics results suggest practical approaches for identifying the three key parameters of the model $\{\theta, \bar{\lambda}, \eta\}$ from observable data. The key insight is to exploit the different monotonicity properties to achieve robust identification: relative volume $I(\theta)$ appears to provide a clean first-stage identification of $\bar{\lambda}$ because it decreases monotonically with matching efficiency regardless of market conditions. Once $\bar{\lambda}$ is pinned down, the monotonic relationship between convenience yields and θ allows for second-stage identification. Crucially, since θ depends on both portfolios and the distribution of settlement shocks, observing portfolios allows us to infer the unobservable shock distribution Φ.

For empirical work, this framework suggests focusing data collection on: (i) portfolio compositions across institutions, (ii) spreads between liquid and illiquid assets, (iii) the level and intraday range of OTC rates, and (iv) the relative use of emergency lending facilities, if available. Together, these moments may sufficiently identify both the market microstructure driving convenience yields and the underlying distribution of liquidity shocks. We provide a sketch of an identification strategy next.

4.3 Efficiency of Portfolio Management

We now study the efficiency of portfolio choices. We do so comparing the decentralized choices vis-a-vis a social planner's portfolio choices. The goal is to analyze externalities among investors, in isolation from the effects on supply and demand effects on assets nor the revenues of the lender of last resort. Potential inefficiencies arise because portfolio decisions affect the tightness in the OTC market inducing possible congestion externalities. In particular, when the planner chooses a higher level of liquid assets, this increases the likelihood that an investor in deficit will be able match. Conversely, if the planner chooses a lower level of liquid assets, it increases the likelihood of a surplus investor finding a match. Moreover, as we show below, the fact that the convenience yield is asymmetric implies that risk aversion plays an important role, as the amount of aggregate liquidity affect the degree of risk faced by investors.

To illustrate the externality, we do so in the context of example i) again. We assume the planner chooses the portfolio shares (d, b, m) on behalf of investors—investors choose their consumption efficiently given returns. The key difference is that the planner takes the OTC frictions as given. Thus, the planner takes actions in the portfolio stage but lets banks trade in the OTC market. Moreover, we focus on the problem of a representative bank, assuming away from distributional concerns—this is akin to redistributing any idiosyncratic risk.

Problem 4. *The planner's problem in example i) is:*

$$\max_{m\geq 0, d\geq 0}\left\{\mathbb{E}_{\omega}\left[R^{b}(d-m)+\ R^{m}m-\ R^{d}d+\chi\left(s\left(\omega,m,d\right),\theta(m,d)\right)\right]^{1-\gamma}\right\}^{\frac{1}{1-\gamma}},$$

subject to (21) and the market tightness

$$\theta\left(m,d\right)\equiv\frac{\int_{-1}^{\omega^{*}}s\left(\omega,m,d\right)\Phi(d\omega)}{\int_{\omega^{*}}^{\infty}s\left(\omega,m,d\right)\Phi(d\omega)},\quad \omega^{*}\equiv\frac{\rho-\dfrac{m}{d}}{\dfrac{R^{d}}{R^{m}}-\rho}.$$

The key distinction is that the planner considers how the portfolio determines θ and how in turn this affects the convenience yield function χ. The planner's first-order conditions with respect to m and b yields the following first-order condition:

$$\underbrace{R^b - R^m}_{\text{asset premium}} = \chi^+ + \left(\chi^- - \chi^+\right) \cdot \tilde{\Phi}\left(\omega^*\right) + H,$$

where H represents the externality and is given by:

$$\begin{aligned} H \quad \equiv \quad & \frac{\partial\theta}{\partial m}\frac{\partial\chi^+(\theta)}{\partial\theta} \cdot (1 - \tilde{\Phi}(\omega^*)) \cdot \mathbb{E}\left[s \cdot \frac{R^e(\omega)^{-\gamma}}{\mathbb{E}\left[R^e(\omega)^{-\gamma} \mid \omega > \omega^*\right]} \mid \omega > \omega^*\right] \\ & + \frac{\partial\theta}{\partial m}\frac{\partial\chi^-(\theta)}{\partial\theta} \cdot \tilde{\Phi}(\omega^*) \cdot \mathbb{E}\left[s \cdot \frac{R^e(\omega)^{-\gamma}}{\mathbb{E}\left[R^e(\omega)^{-\gamma} \mid \omega < \omega^*\right]} \mid \omega < \omega^*\right]. \end{aligned}$$

Just like individual investors, the planner trades off the higher return on loans with the liquidity benefits of cash considering the uses of cash in the OTC market. However, the planner internalizes the pecuniary externality that emerges because it understands how portfolio choices will affect trading probabilities, as encoded in the convenience yield χ.

A key insight is that the sign of the externality is ambiguous: it depends on the derivatives of the convenience-yield coefficients. To understand whether investors over or under invest in liquid assets, it is useful to consider first the limiting case with risk neutral investors. With $\gamma \to 0$, given that $\frac{\partial\theta}{\partial m} < 0, \frac{\partial\chi^+(\theta)}{\partial\theta} > 0, \frac{\partial\chi^-(\theta)}{\partial\theta} > 0$, we have that the *planner values cash more* than individual investors if and only if

$$\frac{\partial\chi^+(\theta)}{\partial\theta} \cdot S^+ > \frac{\partial\chi^-(\theta)}{\partial\theta} \cdot S^-$$

This inequality underscores that there is under-accumulation of liquid assets when the planner perceives that higher cash holdings (lower market tightness) raises more the marginal return on liquid assets when in deficit compared to the case in surplus. That is, when market tightness goes up, this favors investors that are in deficit (by allowing them to borrow at a lower rate and by raising the probability of a match) relative to investors that are in surplus (as they now must lend at a lower rate and face a lower probability of matching).

Suppose that the planner picks a portfolio with $\theta = 1$. Under the case with Cobb-Douglas matching, we know that $\frac{\partial\chi^+(\theta)}{\partial\theta} = \frac{\partial\chi^-(\theta)}{\partial\theta}$. If, in addition, the shock is symmetric $F(\omega^*) = 0.5$ and $\mathbb{E}\left[-s \mid \omega < \omega^*\right] = \mathbb{E}\left[s \mid \omega \geq \omega^*\right]$.[27] In this case, it follows that there is neither over nor under-accumulation of liquid assets. However, if investors were risk averse, the risk adjustment correction would imply that by accumulating more liquid assets, the planner would effectively provide more insurance. Because individual investors to not internalize these benefits, the competitive equilibrium

[27]To see that $\frac{\partial\chi^+(\theta)}{\partial\theta}|_{\theta=1} = \frac{\partial\chi^-(\theta)}{\partial\theta}|_{\theta=1}$ under Cobb Douglas we can exploit the symmetry property of derivatives.

would feature under-accumulation of liquid assets.In addition, an allocation with a higher probability of being in deficit $\Phi(\omega^*)$ or with a more sensitive liquidity yield $\frac{\partial\chi^-(\theta)}{\partial\theta}$ implies that the planner perceives a higher value from higher liquid holdings.

It is also interesting to discuss the case with Leontief matching function. In this case, while matching probabilities do not change with market tightness on the short-side of the market at the margin, aggregate cash holdings affect endogenously the outside options of investors, the OTC rate does change, and so does χ^+ and χ^-. Notice that if the OTC market were static, the OTC rate would be fixed, and so in this case, it would suffice to know whether the market features excess surplus or deficit to trace the sign of the inefficiency, under risk neutrality. In particular, if the market had on average excess surplus, the planner would value less cash than individual investors at the margin. Conversely, if the market had on average excess deficits, the planner would value more cash than individual investors at the margin. This follows because in the former case $\frac{\partial\chi^-(\theta)}{\partial\theta} = 0$ and $\frac{\partial\chi^+(\theta)}{\partial\theta} > 0$ while in the latter $\frac{\partial\chi^+(\theta)}{\partial\theta} = 0$ and $\frac{\partial\chi^-(\theta)}{\partial\theta} > 0$.

5. Conclusions

We develop a tractable microfoundation for convenience yields arising from trading frictions in OTC financial markets and show how it can be readily introduced into a canonical portfolio problem. We characterize how the liquidity yield function depends on market tightness, bargaining power, and matching efficiency, and show that convenience yields reflect both direct OTC frictions and the interaction between liquidity and return risk. The framework generates closed-form expressions for rates and spreads that facilitate comparative statics and quantitative analysis. Finally, we show that individual investors fail to internalize how their portfolio choices influence aggregate market tightness, which leads to over- or under-investment in liquid assets depending on the level of tightness and the degree of risk aversion.

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A. Proofs of Proposition 1 and Proposition 2

The proof of both propositions is presented in sequence. We begin with an auxiliary Lemma.

Market tightness follows a differential equation as described by the lemma below. With the market tightness, we obtain the matching probabilities at each round:

Lemma 2. *Let θ_0 be the initial market tightness. Then, the ratio $\{\theta_n\}$ features the following law of motion:*

$$\theta_n = \theta_{n-1} \frac{(1 - \lambda(N) G(1/\theta_{n-1}, 1))}{(1 - \lambda(N) G(1, \theta_{n-1}))} \qquad \forall n \in \{1, 2, \ldots, N\}.$$

and the matching probabilities can be expressed in terms of the ratio via:

$$\psi_n^+ = \lambda(N) G(1, \theta_{n-1}) \text{ and } \psi_n^- = \lambda(N) G(1/\theta_{n-1}, 1).$$

The proof is simple. By definition and homogeneity:

$$\theta_n = \frac{S_n^-}{S_n^+} = \frac{S_{n-1}^- - g_n}{S_{n-1}^+ - g_n} = \theta_{n-1} \frac{(1 - \lambda(N) G(1/\theta_{n-1}, 1))}{(1 - \lambda(N) G(1, \theta_{n-1}))}, \qquad \forall n \in \{1, 2, \ldots, N\}.$$

where the second equality follow from the definition of g_n and uses its homogeneity property. Hence, the Lemma.

The lemma shows how we can track matching probabilities in terms of the initial market tightness. It also shows that the these probabilities are scale invariant. We use this observations in what follows.

Auxiliary Calculations Next, we describe the limit of the bargaining problem as $\Delta \to 0$. We begin with some observations:

$$\lim_{\Delta\to 0}\left\{\frac{J_U^+(N;\Delta)-V\left(\mathcal{E}^j\right)}{\Delta}\right\}=\lim_{\Delta\to 0}\left\{\frac{V\left(\mathcal{E}^j-\chi^+\Delta\right)-V\left(\mathcal{E}^j\right)}{\Delta}\right\}$$
$$=-\chi^+\lim_{\Delta\to 0}\left\{\frac{V\left(\mathcal{E}^j-\chi^+\Delta\right)-V\left(\mathcal{E}^j\right)}{-\chi^+\Delta}\right\}$$
$$=-\chi^+V'\left(\mathcal{E}^j\right);$$
$$\lim_{\Delta\to 0}\left\{\frac{J_U^+(n;\Delta)-V\left(\mathcal{E}^j\right)}{\Delta}\right\}=\lim_{\Delta\to 0}\left\{\frac{\psi_{n+1}^+J_M^+(n+1;\Delta)+\left(1-\psi_{n+1}^+\right)J_U^+(n+1;\Delta)-V\left(\mathcal{E}^j\right)}{\Delta}\right\}$$
$$=\lim_{\Delta\to 0}\left\{\frac{\psi_{n+1}^+\left(J_M^+(n+1;\Delta)-V\left(\mathcal{E}^j\right)\right)+\left(1-\psi_{n+1}^+\right)\left(J_U^+(n+1;\Delta)-V\left(\mathcal{E}^j\right)\right)}{\Delta}\right\}$$
$$=\psi_{n+1}^+\left(i_{n+1}-r^m-\chi^+\right)V'\left(\mathcal{E}^j\right)+\left(1-\psi_{n+1}^+\right)\lim_{\Delta\to 0}\left\{\frac{J_U^+(n+1;\Delta)-V\left(\mathcal{E}^j\right)}{\Delta}\right\};$$
$$\lim_{\Delta\to 0}\left\{\frac{J_M^+(n;\Delta)-V\left(\mathcal{E}^j\right)}{\Delta}\right\}=\lim_{\Delta\to 0}\left\{\frac{V\left(\mathcal{E}^j+\left(r_n(\Delta)-r^m-\chi^+\right)\Delta\right)-V\left(\mathcal{E}^j\right)}{\Delta}\right\}$$
$$=\lim_{\Delta\to 0}\left\{r_n(\Delta)-r^m-\chi^+\right\}\lim_{\Delta\to 0}\left\{\frac{V\left(\mathcal{E}^j+\left(r_n-r^m-\chi^+\right)\Delta\right)-V\left(\mathcal{E}^j\right)}{\left(r_n(\Delta)-r^m-\chi^+\right)\Delta}\right\}$$
$$=\left(r_n-r^m-\chi^+\right)V'\left(\mathcal{E}^j\right).$$

Similarly with the same steps we show that,

$$\lim_{\Delta\to 0}\left\{\frac{J_M^-(n;\Delta)-V\left(\mathcal{E}^j\right)}{\Delta}\right\}=-\left(r_n-r^m-\chi^-\right)V'\left(\mathcal{E}^j\right);$$
$$\lim_{\Delta\to 0}\left\{\frac{J_U^-(n;\Delta)-V\left(\mathcal{E}^j\right)}{\Delta}\right\}=-\psi_{n+1}^-\left(i_{n+1}-r^m-\chi^-\right)V'\left(\mathcal{E}^j\right)+\left(1-\psi_{n+1}^-\right)\lim_{\Delta\to 0}\left\{\frac{J_U^-(n+1;\Delta)-V\left(\mathcal{E}^j\right)}{\Delta}\right\};$$
$$\lim_{\Delta\to 0}\left\{\frac{J_U^-(N;\Delta)-V\left(\mathcal{E}^j\right)}{\Delta}\right\}=-\left(r^w-r^m-\chi^-\right)V'\left(\mathcal{E}^j\right).$$

Now, let's consider the interbank rate at this limit. Consider the bargaining problem given by (11):

$$r_n^f(\Delta)=\arg\max_{r_n}\left\{\left[\mathcal{S}_n^-(\Delta)\right]^\eta\left[\mathcal{S}_n^+(\Delta)\right]^{1-\eta}\right\}$$
$$\text{s.t. } \mathcal{S}_n^-(\Delta)=V\left(\mathcal{E}^j-\left(r_n(\Delta)-r^m-\chi^-\right)\Delta\right)-J_U^-(n;\Delta)$$
$$\mathcal{S}_n^+(\Delta)=V\left(\mathcal{E}^k+\left(r_n(\Delta)-r^m-\chi^+\right)\Delta\right)-J_U^+(n;\Delta)$$

The solution to the OTC market rate doesn't change if we multiply the right-hand side of (11) by Δ. Thus, the limiting rate for round n satisfies:

$$r_n^f=\lim_{\Delta\to 0}\left\{r_n^f(\Delta)\right\}=\lim_{\Delta\to 0}\left\{\arg\max_{r_n}\left\{\frac{\left[\mathcal{S}_n^-(\Delta)\right]^\eta\left[\mathcal{S}_n^+(\Delta)\right]^{1-\eta}}{\Delta}\right\}\right\}=\arg\max_{r_n}\left\{\left[\lim_{\Delta\to 0}\left\{\frac{\mathcal{S}_n^-(\Delta)}{\Delta}\right\}\right]^\eta\left[\lim_{\Delta\to 0}\left\{\frac{\mathcal{S}_n^+(\Delta)}{\Delta}\right\}\right]^{1-\eta}\right\},$$

where

$$
\begin{aligned}
\mathcal{S}_n^-(\Delta) &= V\left(\mathcal{E}^j - \left(r_n(\Delta) - r^m - \chi^-\right)\Delta\right) - J_U^-(n;\Delta) \\
\text{and} \quad \mathcal{S}_n^+(\Delta) &= V\left(\mathcal{E}^k + \left(r_n(\Delta) - r^m - \chi^+\right)\Delta\right) - J_U^+(n;\Delta).
\end{aligned}
$$

Since, the solution belongs to a compact space, namely $i \in [r^m, r^w]$, this problem satisfies the conditions for the Maximum Theorem, so the continuity of the solution is guaranteed. This means that we can take limits as Δ converges to zero. Next, we by backward induction: first obtaining a solution at round N, at round $N-1$ and so forth. We do this in three steps:

Step 1: Round N-th of Matching Process Let us start in the last period of the matching process, the N-th round. In this case, the outside option limit identities of an atomistic bank in a deficit position j and an atomistic bank in a surplus position k are

$$
\lim_{\Delta\to 0}\left\{\frac{J_U^-(N;\Delta) - V\left(\mathcal{E}^j\right)}{\Delta}\right\} = -\left(r^w - r^m - \chi^-\right)V'\left(\mathcal{E}^j\right) \quad \text{and} \quad \lim_{\Delta\to 0}\left\{\frac{J_U^+(N;\Delta) - V\left(\mathcal{E}^k\right)}{\Delta}\right\} = -\chi^+ V'\left(\mathcal{E}^k\right).
$$

Thus, the limit surpluses can be described as,

$$
\begin{aligned}
\lim_{\Delta\to 0}\left\{\frac{\mathcal{S}_N^-(\Delta)}{\Delta}\right\} &= \lim_{\Delta\to 0}\left\{\frac{V\left(\mathcal{E}^j - \left(r_n(\Delta) - r^m - \chi^-\right)\Delta\right) - J_U^-(N;\Delta)}{\Delta}\right\} \\
&= \lim_{\Delta\to 0}\left\{\frac{V\left(\mathcal{E}^j - \left(r_n(\Delta) - r^m - \chi^-\right)\Delta\right) - V\left(\mathcal{E}^j\right)}{\Delta}\right\} \\
&- \lim_{\Delta\to 0}\left\{\frac{J_U^-(N;\Delta) - V\left(\mathcal{E}^j\right)}{\Delta}\right\} \\
&= \lim_{\Delta\to 0}\left\{-\left(r_n(\Delta) - r^m - \chi^-\right)\right\}\lim_{\Delta\to 0}\left\{\frac{V\left(\mathcal{E}^j - \left(r_n(\Delta) - r^m - \chi^-\right)\Delta\right) - V\left(\mathcal{E}^j\right)}{-\left(r_n(\Delta) - r^m - \chi^-\right)\Delta}\right\}\dots \\
&- \lim_{\Delta\to 0}\left\{\frac{J_U^-(N;\Delta) - V\left(\mathcal{E}^j\right)}{\Delta}\right\} \\
&= -\left(r_n - r^m - \chi^-\right)V'\left(\mathcal{E}^j\right) + \left(r^w - r^m - \chi^-\right)V'\left(\mathcal{E}^j\right) \\
&= \left(r^w - r_n\right)V'\left(\mathcal{E}^j\right);
\end{aligned}
$$

and similarly,

$$
\lim_{\Delta\to 0}\left\{\frac{\mathcal{S}_N^+(\Delta)}{\Delta}\right\} = \left(r_n - r^m\right)V'\left(\mathcal{E}^k\right).
$$

Therefore, the bargaining problem for an infinitesimal size transaction in the N-th matching round can be described as

$$\begin{aligned} r_N^f &= \arg\max_{r_n} \left\{ \left[\lim_{\Delta\to 0} \left\{ \frac{\mathcal{S}_N^-(\Delta)}{\Delta} \right\} \right]^{\eta} \left[\lim_{\Delta\to 0} \left\{ \frac{\mathcal{S}_N^+(\Delta)}{\Delta} \right\} \right]^{1-\eta} \right\} \\ &= \arg\max_{r_n} \left\{ \left[(r^w - r_n) V'\left(\mathcal{E}^j\right) \right]^{\eta} \left[(r_n - r^m) V'\left(\mathcal{E}^k\right) \right]^{1-\eta} \right\} \\ &= \arg\max_{r_n} \left\{ \left[V'\left(\mathcal{E}^j\right) \right]^{\eta} \left[V'\left(\mathcal{E}^k\right) \right]^{1-\eta} \left[r^w - r_n \right]^{\eta} \left[r_n - r^m \right]^{1-\eta} \right\} \\ &= \arg\max_{r_n} \left\{ \left[r^w - r_n \right]^{\eta} \left[r_n - r^m \right]^{1-\eta} \right\}. \end{aligned}$$

Taking the first order conditions, we get

$$\eta \left(\frac{r_N^f - r^m}{r^w - r_N^f} \right)^{1-\eta} = (1-\eta) \left(\frac{r^w - r_N^f}{r_N^f - r^m} \right)^{\eta}.$$

Thus, we get to the optimal interest rate

$$r_N^f = r^m + (1-\eta)\left(r^w - r^m\right).$$

Finally, define $\chi_N^+ \equiv 0$ and $\chi_N^- \equiv r^w - r^m$. From here, we conclude that,

$$r_N^f = r^m + (1-\eta)\chi_N^- + \eta\chi_N^+.$$

Step 2: Round $\{N-1\}$-th of Matching Process Let now obtain a similar equations for the $\{N-1\}$-th round. Following the same steps as for the N-th round we have:

$$\begin{aligned} \lim_{\Delta\to 0} \left\{ \frac{J_U^-(N-1;\Delta) - V\left(\mathcal{E}^j\right)}{\Delta} \right\} &= -\psi_N^- \left(r_N^f - r^m - \chi^- \right) V'\left(\mathcal{E}^j\right) + \left(1-\psi_N^-\right) \lim_{\Delta\to 0} \left\{ \frac{J_U^-(N;\Delta) - V\left(\mathcal{E}^j\right)}{\Delta} \right\} \\ &= -\psi_N^- \left(r_N^f - r^m - \chi^- \right) V'\left(\mathcal{E}^j\right) - \left(1-\psi_N^-\right)\left(r^w - r^m - \chi^-\right) V'\left(\mathcal{E}^j\right) \\ &= -\left(\psi_N^- \left(r_N^f - r^m - \chi^- \right) + \left(1-\psi_N^-\right)\left(r^w - r^m - \chi^-\right) \right) V'\left(\mathcal{E}^j\right) \\ &= -\left(\psi_N^- \left(r_N^f - r^m \right) + \left(1-\psi_N^-\right)\left(r^w - r^m\right) - \chi^- \right) V'\left(\mathcal{E}^j\right) \\ &= -\left(\psi_N^- \left(r_N^f - r^m \right) + \left(1-\psi_N^-\right)\chi_N^- - \chi^- \right) V'\left(\mathcal{E}^j\right), \end{aligned}$$

and through similar steps:

$$\lim_{\Delta\to 0} \left\{ \frac{J_U^+(N-1;\Delta) - V\left(\mathcal{E}^k\right)}{\Delta} \right\} = \left(\psi_N^+ \left(r_N^f - r^m \right) + \left(1-\psi_N^+\right)\chi_N^+ - \chi^+ \right) V'\left(\mathcal{E}^k\right).$$

Define $\chi_{N-1}^{-} \equiv \psi_N^{-}\left(r_N^f - r^m\right) + \left(1-\psi_N^{-}\right)\chi_N^{-}$ and $\chi_{N-1}^{+} \equiv \psi_N^{+}\left(r_N^f - r^m\right) + \left(1-\psi_N^{+}\right)\chi_N^{+}$ so

$$
\begin{aligned}
\lim_{\Delta\to 0}\left\{\frac{J_U^{-}(N-1;\Delta) - V\left(\mathcal{E}^j\right)}{\Delta}\right\} &= -\left(\chi_{N-1}^{-} - \chi^{-}\right)V'\left(\mathcal{E}^j\right) \\
\text{and} \qquad \lim_{\Delta\to 0}\left\{\frac{J_U^{+}(N-1;\Delta) - V\left(\mathcal{E}^k\right)}{\Delta}\right\} &= \left(\chi_{N-1}^{+} - \chi^{+}\right)V'\left(\mathcal{E}^k\right).
\end{aligned}
$$

Thus, the limit surpluses can be described as,

$$
\begin{aligned}
\lim_{\Delta\to 0}\left\{\frac{\mathcal{S}_{N-1}^{-}(\Delta)}{\Delta}\right\} &= \lim_{\Delta\to 0}\left\{\frac{V\left(\mathcal{E}^j - \left(i_{N-1}(\Delta) - r^m - \chi^{-}\right)\Delta\right) - J_U^{-}(N-1;\Delta)}{\Delta}\right\} \\
&= \lim_{\Delta\to 0}\left\{\frac{V\left(\mathcal{E}^j - \left(i_{N-1}(\Delta) - r^m - \chi^{-}\right)\Delta\right) - V\left(\mathcal{E}^j\right)}{\Delta}\right\} \\
&- \lim_{\Delta\to 0}\left\{\frac{J_U^{-}(N-1;\Delta) - V\left(\mathcal{E}^j\right)}{\Delta}\right\} \\
&= \lim_{\Delta\to 0}\left\{-\left(i_{N-1}(\Delta) - r^m - \chi^{-}\right)\right\}\lim_{\Delta\to 0}\left\{\frac{V\left(\mathcal{E}^j - \left(i_{N-1}(\Delta) - r^m - \chi^{-}\right)\Delta\right) - V\left(\mathcal{E}^j\right)}{-\left(i_{N-1}(\Delta) - r^m - \chi^{-}\right)\Delta}\right\} \\
&- \lim_{\Delta\to 0}\left\{\frac{J_U^{-}(N-1;\Delta) - V\left(\mathcal{E}^j\right)}{\Delta}\right\} \\
&= -\left(i_{N-1} - r^m - \chi^{-}\right)V'\left(\mathcal{E}^j\right) + \left(\chi_{N-1}^{-} - \chi^{-}\right)V'\left(\mathcal{E}^j\right) \\
&= \left(\chi_{N-1}^{-} - \left(i_{N-1} - r^m\right)\right)V'\left(\mathcal{E}^j\right);
\end{aligned}
$$

and, thus,

$$
\lim_{\Delta\to 0}\left\{\frac{\mathcal{S}_{N-1}^{+}(\Delta)}{\Delta}\right\} = \left(\left(i_{N-1} - r^m\right) - \chi_{N-1}^{+}\right)V'\left(\mathcal{E}^k\right).
$$

Therefore, the bargaining problem for an infinitesimal size transaction in the $\{N-1\}$-th matching round can be described as

$$
\begin{aligned}
i_{N-1}^f &= \arg\max_{i_{N-1}}\left\{\left[\lim_{\Delta\to 0}\left\{\frac{\mathcal{S}_{N-1}^{-}(\Delta)}{\Delta}\right\}\right]^{\eta}\left[\lim_{\Delta\to 0}\left\{\frac{\mathcal{S}_{N-1}^{+}(\Delta)}{\Delta}\right\}\right]^{1-\eta}\right\} \\
&= \arg\max_{i_{N-1}}\left\{\left[\left(\chi_{N-1}^{-} - \left(i_{N-1} - r^m\right)\right)V'\left(\mathcal{E}^j\right)\right]^{\eta}\left[\left(\left(i_{N-1} - r^m\right) - \chi_{N-1}^{+}\right)V'\left(\mathcal{E}^k\right)\right]^{1-\eta}\right\} \\
&= \arg\max_{i_{N-1}}\left\{\left[V'\left(\mathcal{E}^j\right)\right]^{\eta}\left[V'\left(\mathcal{E}^k\right)\right]^{1-\eta}\left[\chi_{N-1}^{-} - \left(i_{N-1} - r^m\right)\right]^{\eta}\left[\left(i_{N-1} - r^m\right) - \chi_{N-1}^{+}\right]^{1-\eta}\right\} \\
&= \arg\max_{i_{N-1}}\left\{\left[\chi_{N-1}^{-} - \left(i_{N-1} - r^m\right)\right]^{\eta}\left[\left(i_{N-1} - r^m\right) - \chi_{N-1}^{+}\right]^{1-\eta}\right\}.
\end{aligned}
$$

Taking the first-order conditions, we get

$$
\eta\left(\frac{i_{N-1} - r^m - \chi_{N-1}^{+}}{\chi_{N-1}^{-} - i_{N-1} - r^m}\right)^{1-\eta} = (1-\eta)\left(\frac{\chi_{N-1}^{-} - i_{N-1} - r^m}{i_{N-1} - r^m - \chi_{N-1}^{+}}\right)^{\eta}.
$$

Finally, the solution to the interest rate is:

$$i^f_{N-1} = r^m + (1-\eta)\chi^-_{N-1} + \eta\chi^+_{N-1}.$$

Step 3: Round $\{N-2\}$-th of Matching Process Let's now study the matching process, at the $\{N-2\}$-th round. In this case, the outside option limit identities of an atomistic bank in a deficit position j and an atomistic bank in a surplus position k are

$$\lim_{\Delta\to 0}\left\{\frac{J^-_U(N-2;\Delta) - V\left(\mathcal{E}^j\right)}{\Delta}\right\} = -\left(\psi^-_{N-1}\left(i^f_{N-1} - r^m\right) + \left(1-\psi^-_{N-1}\right)\chi^-_{N-1} - \chi^-\right)V'\left(\mathcal{E}^j\right),$$

and

$$\lim_{\Delta\to 0}\left\{\frac{J^+_U(N-2;\Delta) - V\left(\mathcal{E}^k\right)}{\Delta}\right\} = \left(\psi^+_{N-1}\left(i^f_{N-1} - r^m\right) + \left(1-\psi^+_{N-1}\right)\chi^+_{N-1} - \chi^+\right)V'\left(\mathcal{E}^k\right).$$

Define $\chi^-_{N-2} \equiv \psi^-_{N-1}\left(i^f_{N-1} - r^m\right) + \left(1-\psi^-_{N-1}\right)\chi^-_{N-1}$ and $\chi^+_{N-2} \equiv \psi^+_{N-1}\left(i^f_{N-1} - r^m\right) + \left(1-\psi^-_{N-1}\right)\chi^+_{N-1}$ so

$$\begin{aligned}
\lim_{\Delta\to 0}\left\{\frac{J^-_U(N-2;\Delta) - V\left(\mathcal{E}^j\right)}{\Delta}\right\} &= -\left(\chi^-_{N-2} - \chi^-\right)V'\left(\mathcal{E}^j\right) \\
\text{and} \quad \lim_{\Delta\to 0}\left\{\frac{J^+_U(N-2;\Delta) - V\left(\mathcal{E}^k\right)}{\Delta}\right\} &= \left(\chi^+_{N-2} - \chi^+\right)V'\left(\mathcal{E}^k\right).
\end{aligned}$$

Thus, the limit surpluses can be described as,

$$\lim_{\Delta\to 0}\left\{\frac{\mathcal{S}^-_{N-2}(\Delta)}{\Delta}\right\} = \left(\chi^-_{N-2} - \left(i_{N-2} - r^m\right)\right)V'\left(\mathcal{E}^j\right);$$

and

$$\lim_{\Delta\to 0}\left\{\frac{\mathcal{S}^+_{N-2}(\Delta)}{\Delta}\right\} = \left(\left(i_{N-2} - r^m\right) - \chi^+_{N-2}\right)V'\left(\mathcal{E}^k\right).$$

Therefore, the bargaining problem in round $\{N-2\}$-th is,

$$\begin{aligned}
i^f_{N-2} &= \arg\max_{i_{N-2}}\left\{\left[\lim_{\Delta\to 0}\left\{\frac{\mathcal{S}^-_{N-2}(\Delta)}{\Delta}\right\}\right]^\eta\left[\lim_{\Delta\to 0}\left\{\frac{\mathcal{S}^+_{N-2}(\Delta)}{\Delta}\right\}\right]^{1-\eta}\right\} \\
&= \arg\max_{i_{N-2}}\left\{\left[\chi^-_{N-2} - \left(i_{N-2} - r^m\right)\right]^\eta\left[\left(i_{N-2} - r^m\right) - \chi^+_{N-2}\right]^{1-\eta}\right\}.
\end{aligned}$$

Taking the first-order conditions, we obtain:

$$i^f_{N-2} = r^m + (1-\eta)\chi^-_{N-2} + \eta\chi^+_{N-2}.$$

Step 4: Round n-th of Matching Process We continue by induction, to obtain:

$$\chi_n^- = \psi_{n+1}^- \left(i_{n+1}^f - r^m\right) + \left(1 - \psi_{n+1}^-\right) \chi_{n+1}^- \qquad \text{and} \qquad \chi_n^+ = \psi_{n+1}^+ \left(i_{n+1}^f - r^m\right) + \left(1 - \psi_{n+1}^+\right) \chi_{n+1}^+. \tag{24}$$

Furthermore, the optimal interbank interest rates that solve the bargaining problem at the n-th matching round is

$$r_n^f = r^m + (1-\eta)\chi_n^- + \eta\chi_n^+.$$

This, and the previous recursions are the expressions in Proposition 1. Next, we verify the consistency of the solution.

Proof of Proposition 2. The probability of matching in one of the N matching rounds for individual surplus and deficit traders are:

$$\Psi^+ = \sum_{n=1}^{N} \psi_n^+ \left[\prod_{m=1}^{n-1} \left(1-\psi_m^+\right)\right] = 1 - \left[\prod_{m=1}^{N} \left(1-\psi_m^+\right)\right] \qquad \text{and} \qquad \Psi^- = \sum_{n=1}^{N} \psi_n^- \left[\prod_{m=1}^{n-1} \left(1-\psi_m^-\right)\right] = 1 - \left[\prod_{m=1}^{N} \left(1-\psi_m^-\right)\right],$$

where $\psi_0^+ = \psi_0^- = 0$. Define the weights $\{\varkappa_n\}_{n=1}^N$ as the distribution of matching in round n conditional on matching,

$$\varkappa_n^+ \equiv \frac{\psi_n^+ \left[\prod_{m=1}^{n-1} \left(1-\psi_m^+\right)\right]}{\Psi^+} \qquad \text{and} \qquad \varkappa_n^- \equiv \frac{\psi_n^- \left[\prod_{m=1}^{n-1} \left(1-\psi_m^-\right)\right]}{\Psi^-}.$$

The numerator corresponds to the unconditional probability of a match at round n and the denominator is the probability of matching at all. By the law of large numbers, this is proportional to the volume at that round. Clearly the weights sum to one. Next, we show that conditional distributions are the same for deficits and surpluses:

$$\begin{aligned}
\varkappa_n^+ &\equiv \frac{\psi_n^+ \left[\prod_{m=1}^{n-1} \left(1-\psi_m^+\right)\right]}{\Psi^+} = \frac{\psi_n^+ \left[\prod_{m=1}^{n-1} \left(1-\lambda\left(N\right)G\left(1/\theta_m, 1\right)\right)\right]}{\Psi^+} \\
&= \frac{\psi_n^+ \left[\prod_{m=1}^{n-1} \frac{\theta_{m=1}}{\theta_m} \left(1-\lambda\left(N\right)G\left(1, \theta_m\right)\right)\right]}{\Psi^+} = \frac{\theta_{n-1}^{-1}\psi_n^+ \left[\prod_{m=1}^{n-1} \left(1-\lambda\left(N\right)G\left(1, \theta_m\right)\right)\right]}{\theta_0^- \Psi^+} \\
&= \frac{\psi_n^- \left[\prod_{m=1}^{n-1} \left(1-\psi_n^-\right)\right]}{\Psi^-} = \varkappa_n^-.
\end{aligned}$$

where we used 2 and the definition of ψ_n^+ and ψ_n^-.

Thus, the average interbank-interest rate is the weighted average of the interbank interest rates of each round,

$$\begin{aligned}
\overline{r}^f &= \sum_{n=1}^{N} \varkappa_n^+ r_n^f \\
&= \sum_{n=1}^{N} \varkappa_n^+ \left(r^m + (1-\eta)\chi_n^- + \eta\chi_n^+\right) \\
&= \left[\sum_{n=1}^{N} \varkappa_n^+\right] r^m + \left[\sum_{n=1}^{N} \varkappa_n^+ \left((1-\eta)\chi_n^- + \eta\chi_n^+\right)\right] \\
&= r^m + \left[\frac{\sum_{n=1}^{N} \psi_n^+ \left[\prod_{m=1}^{n-1} (1-\psi_m^+)\right] \left((1-\eta)\chi_n^- + \eta\chi_n^+\right)}{\Psi^+}\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Psi^+ \left(\overline{r}^f - r^m\right) &= \sum_{n=1}^{N} \psi_n^+ \left[\prod_{m=1}^{n-1} \left(1-\psi_m^+\right)\right] \left((1-\eta)\chi_n^- + \eta\chi_n^+\right) \\
&= \sum_{n=1}^{N} \psi_n^+ \left[\prod_{m=1}^{n-1} \left(1-\psi_m^+\right)\right] \left(r_n^f - r^m\right) \\
&= \chi_0^+.
\end{aligned}$$

where the last line follows from solving χ_0^+ in (24) forward. This verifies that $\chi_0^+ = \chi^+$.

Similarly, if we follow the same steps we have that:

$$\begin{aligned}
\overline{r}^f &= \sum_{n=1}^{N} \varkappa_n^- r_n^f \\
&= r^m + \left[\frac{\sum_{n=1}^{N} \psi_n^- \left[\prod_{m=1}^{n-1} (1-\psi_m^-)\right] \left((1-\eta)\chi_n^- + \eta\chi_n^+\right)}{\Psi^-}\right].
\end{aligned}$$

And then solving forward we arrive at:

$$\chi_0^- = \Psi^- \left(\overline{r}^f - r^m\right) + \left(1-\Psi^-\right)\left(r^w - r^m\right).$$

which verifies that $\chi_0^- = \chi^-$. Given that $\varkappa_n^+ = \varkappa_n^-$, $\overline{r}^f$ is the same in both calculations. This concludes the proof of Proposition 2.

B. Proofs for the Infinite-Rounds Limit

B.1 Proof of Lemma 1

Here we solve for the limit as $N \to \infty$. First, let us define a matching round size step subscript as $\Delta \equiv \frac{1}{N}$. Let us abuse in notation and given that we drop the time subscript, call τ the matching round subscript in the domain $[0,1]$. Thus, realise that by definition,

$$S^+_{n+1} - S^+_n = -\lambda(N)G\left(S^+_n, S^-_n\right) \qquad \Leftrightarrow \qquad S^+_{\tau+\Delta} - S^+_\tau = -\lambda\left(\frac{1}{\Delta}\right)G\left(S^+_\tau, S^-_\tau\right)$$
$$S^-_{n+1} - S^+_n = -\lambda(N)G\left(S^+_n, S^-_n\right) \qquad \Leftrightarrow \qquad S^-_{\tau+\Delta} - S^-_\tau = -\lambda\left(\frac{1}{\Delta}\right)G\left(S^+_\tau, S^-_\tau\right);$$

where $\tau \equiv \frac{n}{N}$ for $n \in \{0, 1, 2, \ldots, N-1\}$. Next, observe that:

$$\dot{S^+_\tau} = \lim_{\Delta\to 0}\left\{\frac{S^+_{\tau+\Delta} - S^+_\tau}{\Delta}\right\} = \lim_{\Delta\to 0}\left\{\frac{-\lambda\left(\frac{1}{\Delta}\right)G\left(S^+_\tau, S^-_\tau\right)}{\Delta}\right\} = -\left[\lim_{\Delta\to 0}\left\{\frac{\lambda\left(\frac{1}{\Delta}\right)}{\Delta}\right\}\right]G\left(S^+_\tau, S^-_\tau\right)$$
$$= -\left[\lim_{N\to\infty}\{N\lambda(N)\}\right]G\left(S^+_\tau, S^-_\tau\right) = -\bar{\lambda}G\left(S^+_\tau, S^-_\tau\right),$$

and similarly:

$$\dot{S^-_\tau} = -\bar{\lambda}G\left(S^+_\tau, S^-_\tau\right).$$

Therefore:

$$\frac{\dot{S^+_\tau}}{S^+_\tau} = \frac{-\bar{\lambda}G\left(S^+_\tau, S^-_\tau\right)}{S^+_\tau} = -\bar{\lambda}G\left(1, \theta_\tau\right) \qquad \text{and} \qquad \frac{\dot{S^-_\tau}}{S^-_\tau} = \frac{-\bar{\lambda}G\left(S^+_\tau, S^-_\tau\right)}{S^-_\tau} = -\bar{\lambda}G\left(\frac{1}{\theta_\tau}, 1\right).$$

Now, since $\theta_\tau = \frac{S^-_\tau}{S^+_\tau}$,

$$\ln\left(\theta_\tau\right) = \ln\left(\frac{S^-_\tau}{S^+_\tau}\right) = \ln\left(S^-_\tau\right) - \ln\left(S^+_\tau\right).$$

Differentiating with respect of τ, then

$$\frac{\dot{\theta_\tau}}{\theta_\tau} = \frac{\dot{S^-_\tau}}{S^-_\tau} - \frac{\dot{S^+_\tau}}{S^+_\tau} = \bar{\lambda}G\left(1, \theta_\tau\right) - \bar{\lambda}G\left(\frac{1}{\theta_\tau}, 1\right).$$

Hence,

$$\dot{\theta_\tau} = \bar{\lambda}\theta_\tau\left(G\left(1, \theta_\tau\right) - G\left(\frac{1}{\theta_\tau}, 1\right)\right).$$

Matching probabilities converge to:

$$\psi^+_\tau = \bar{\lambda}G\left(1, \theta_\tau\right) \qquad \text{and} \qquad \psi^-_\tau = \bar{\lambda}G\left(\frac{1}{\theta_\tau}, 1\right) = \frac{\psi^+_\tau}{\theta_\tau}.$$

This concludes the proof of this proposition.

B.2 Auxiliary Definitions for Proposition 4

Define the probability of matching at round n for both sides of the market:

$$f^+(n) = \psi_n^+ \left[\prod_{m=1}^{n-1} \left(1-\psi_m^+\right)\right] \quad \text{and} \quad f^-(n) = \psi_n^- \left[\prod_{m=1}^{n-1} \left(1-\psi_m^-\right)\right], \quad \forall n \in \{1,2,\ldots,N\}.$$

Then, observe that:

$$\begin{aligned} &f^+(1) = \psi_1^+,\ f^+(2) = \psi_2^+\left(1-\psi_1^+\right),\ f^+(3) = \psi_3^+\left(1-\psi_2^+\right)\left(1-\psi_1^+\right), \ldots \\ &f^+(N) = \psi_N^+ \left[\prod_{m=1}^{N-1} \left(1-\psi_m^+\right)\right]; \end{aligned}$$

Thus, we can write:

$$f^+(n) = \psi_n^+ \left(1-F^+(n-1)\right) \quad \text{and} \quad f^-(n) = \psi_n^- \left(1-F^-(n-1)\right), \quad \forall n \in \{1,2,\ldots,N\}.$$

When as the number of rounds tends to infinity and we transform the support from n to τ, we arrive to

$$\begin{aligned} \psi_\tau^+ &= \lim_{N\to\infty}\left\{\frac{f^+\left(\frac{n}{N}\right)}{1-F^+\left(\frac{n-1}{N}\right)}\right\} = \lim_{\Delta\to 0}\left\{\frac{f^+(\tau)}{1-F^+(\tau-\Delta)}\right\} = \frac{f^+(\tau)}{1-F^+(\tau)} \quad \text{and} \\ \psi_\tau^- &= \lim_{N\to\infty}\left\{\frac{f^-\left(\frac{n}{N}\right)}{1-F^-\left(\frac{n-1}{N}\right)}\right\} = \lim_{\Delta\to 0}\left\{\frac{f^-(\tau)}{1-F^-(\tau-\Delta)}\right\} = \frac{f^-(\tau)}{1-F^-(\tau)}, \quad \forall \tau \in [0,1]. \end{aligned}$$

Therefore, the cumulative distribution function conditional there is a match satisfies the following ordinary differential equation

$$\dot{F}^+(\tau) = \psi_\tau^+\left(1-F^+(\tau)\right) \quad \text{and} \quad \dot{F}^-(\tau) = \psi_\tau^-\left(1-F^-(\tau)\right).$$

Thus, solving the differential equation we arrive to

$$F^+(\tau) = 1-e^{-\int_0^\tau \psi_s^+ ds} = 1-e^{-\bar{\lambda}\int_0^\tau G(1,\theta_s)ds} \quad \text{and} \quad F^-(\tau) = 1-e^{-\bar{\lambda}\int_0^\tau G(1/\theta_s,1)ds}, \quad \forall \tau \in [0,1].$$

Moreover,

$$f^+(\tau) = \bar{\lambda} G\left(1,\theta_\tau\right) e^{-\bar{\lambda}\int_0^\tau G(1,\theta_s)ds} \quad \text{and} \quad f^-(\tau) = \bar{\lambda} G\left(\frac{1}{\theta_\tau},1\right) e^{-\bar{\lambda}\int_0^\tau G(1/\theta_s,1)ds}, \quad \forall \tau \in [0,1].$$

Hence, by construction, the probability of having a match for the atomistic investors in surplus and deficit position will be

$$\Psi^+ = F^+(1) = 1-e^{-\bar{\lambda}\int_0^1 G(1,\theta_s)ds} \quad \text{and} \quad \Psi^- = F^-(1) = 1-e^{-\bar{\lambda}\int_0^1 G\left(\frac{1}{\theta_s},1\right)ds}.$$

Define the weights $\{\varkappa_\tau^+, \varkappa_\tau^-\}_{\tau\in[0,1]}$ as the probability weight of having a match in the interbank round process given there is a match in the settlement stage,

$$\varkappa_\tau^+ \equiv \frac{f^+(\tau)}{F^+(1)} \quad \text{and} \quad \varkappa_\tau^- \equiv \frac{f^-(\tau)}{F^-(1)}, \quad \forall \tau \in [0,1].$$

B.3 Proof of Proposition 4

Now, consider the discrete round recursion for the value of χ_n^-

$$
\begin{aligned}
\chi_n^- &= \psi_{n+1}^-\left(i_{n+1}^f - r^m\right) + \left(1-\psi_{n+1}^-\right)\chi_{n+1}^- \rightarrow \\
\chi_\tau^- &= \psi_{\tau+\Delta}^-\left(i_{\tau+\Delta}^f - r^m\right) + \left(1-\psi_{\tau+\Delta}^-\right)\chi_{\tau+\Delta}^- \rightarrow \\
\chi_{\tau+\Delta}^- - \chi_\tau^- &= -\psi_{\tau+\Delta}^-\left(i_{\tau+\Delta}^f - r^m\right) + \psi_{\tau+\Delta}^-\chi_{\tau+\Delta}^-.
\end{aligned}
$$

Divide both sides by Δ, and take the limit $\Delta \to 0$, to obtain the law of motion for χ_τ^-

$$\dot{\chi}_\tau^- = \psi_\tau^-\chi_\tau^- - \psi_\tau^-\left(i_\tau^f - r^m\right). \tag{25}$$

Similarly, we follow the same operations to obtain the law of motion for χ_τ^+ :

$$\dot{\chi}_\tau^+ = \psi_\tau^+\chi_\tau^+ - \psi_\tau^+\left(i_\tau^f - r^m\right). \tag{26}$$

The terminal conditions in both cases are $\chi_\tau^- = (r^w - r^m)$ and $\chi_\tau^+ = 0$. Acknowledge that the average interbank interest rate in continuous time can be computed as

$$r_\tau^f = r^m + (1-\eta)\chi_\tau^- + \eta\chi_\tau^+.$$

Thus, substituting we get

$$\dot{\chi}_\tau^+ = -(1-\eta)\psi_\tau^+\left(\chi_\tau^- - \chi_\tau^+\right) \qquad \text{and} \qquad \dot{\chi}_\tau^- = \eta\psi_\tau^-\left(\chi_\tau^- - \chi_\tau^+\right).$$

Doing a change of variable, $z_\tau = \chi_\tau^- - \chi_\tau^+$. Thus,

$$\dot{z}_\tau = \left[\eta\psi_\tau^- + (1-\eta)\psi_\tau^+\right] z_\tau.$$

Define,

$$\Psi_\tau^+ \equiv \int_\tau^1 \psi_s^+ ds \qquad \text{and} \qquad \Psi_\tau^- \equiv \int_\tau^1 \psi_s^- ds.$$

Using the boundary condition $z_1 = r^w - r^m$

$$\ln\left(\frac{z_\tau}{z_1}\right) = -\int_\tau^1 \eta\psi_s^- + (1-\eta)\psi_s^+ ds = -\left[\eta\Psi_\tau^- + (1-\eta)\Psi_\tau^+\right].$$

Therefore,

$$\chi_\tau^- - \chi_\tau^+ = \left(r^w - r^m\right) e^{-\left[\eta\Psi_\tau^- + (1-\eta)\Psi_\tau^+\right]}$$

Substituting the solution to z_τ into (25) and (26) we obtain:

$$\dot{\chi}_\tau^+ = -(1-\eta)\psi_\tau^+\left(r^w - r^m\right) e^{-\left[\eta\Psi_\tau^- + (1-\eta)\Psi_\tau^+\right]} \qquad \text{and} \qquad \dot{\chi}_\tau^- = \eta\psi_\tau^-\left(r^w - r^m\right) e^{-\left[\eta\Psi_\tau^- + (1-\eta)\Psi_\tau^+\right]}.$$

Define the residual bargained probability of match as:

$$\mathbb{P}_\tau^+ \equiv \int_\tau^1 \psi_s^+ e^{-\left[\eta\Psi_s^- + (1-\eta)\Psi_s^+\right]} ds \qquad \text{and} \qquad \mathbb{P}_\tau^- \equiv \int_\tau^1 \psi_s^- e^{-\left[\eta\Psi_s^- + (1-\eta)\Psi_s^+\right]} ds. \tag{27}$$

Hence, solving the ordinary differential equations and applying the boundary conditions we get

$$\chi_\tau^+ = (1-\eta)\left(r^w - r^m\right)\mathbb{P}_\tau^+ \qquad \text{and} \qquad \chi_\tau^- = \left(r^w - r^m\right)\left(1 - \eta\mathbb{P}_\tau^-\right). \tag{28}$$

This, in fact, is the closed form solution presented in Proposition 4.

B.4 Consistency of Solution

The final step of the proof is to verify that for the continuous-time limit, it also holds that , χ_0^- is indeed χ^-. For that, let:

$$\left(1 - F^-(\tau)\right)\dot{\chi}_\tau^- = \left(1 - F^-(\tau)\right)\psi_\tau^-\chi_\tau^- - \left(1 - F^-(\tau)\right)\psi_\tau^-\left(r_\tau^f - r^m\right),$$

where $F^-(\tau)$ is cdf of the time distribution of matches. Rearranging terms yields:

$$\Psi^-\varkappa_\tau^-\left(r_\tau^f - r^m\right) = \left(1 - F^-(\tau)\right)\left[\psi_\tau^-\chi_\tau^- - \dot{\chi}_\tau^-\right].$$

Integrating both sides over the rounds support,

$$\int_0^1 \Psi^-\varkappa_\tau^-\left(r_\tau^f - r^m\right)d\tau = \int_0^1 \left(1 - F^-(\tau)\right)\left[\psi_\tau^-\chi_\tau^- - \dot{\chi}_\tau^-\right]d\tau.$$

The left-hand side yields,

$$\int_0^1 \Psi^-\varkappa_\tau^-\left(r_\tau^f - r^m\right)d\tau = \Psi^-\left[\int_0^1 \varkappa_\tau^-\left(r_\tau^f - r^m\right)dt\right] = \Psi^-\left(\bar{r}^f - r^m\right).$$

The right-hand side yields

$$\begin{aligned}
\int_0^1 \left(1 - F^-(\tau)\right)\left[\psi_\tau^-\chi_\tau^- - \dot{\chi}_\tau^-\right]d\tau &= \int_0^1 \left(1 - F^-(\tau)\right)\psi_\tau^-\chi_\tau^- - \left(1 - F^-(\tau)\right)\dot{\chi}_\tau^- d\tau \\
&= \int_0^1 f^-(\tau)\chi_\tau^- - \left(1 - F^-(\tau)\right)\dot{\chi}_\tau^- d\tau \\
&= -\int_0^1 \frac{d}{d\tau}\left(\left(1 - F^-(\tau)\right)\chi_\tau^-\right)d\tau \\
&= -\left[\left(1 - F^-(\tau)\right)\chi_\tau^-\right]_0^1 \\
&= \left(1 - F^-(0)\right)\chi_0^- - \left(1 - F^-(1)\right)\chi_1^- \\
&= \chi_0^- - \left(1 - \Psi^-\right)\left(r^w - r^m\right).
\end{aligned}$$

Finally, joining these two expressions, the average interbank interest rate is:

$$\bar{r}^f = r^m + \left(\frac{\chi_0^- - \left(1 - \Psi^-\right)\left(r^w - r^m\right)}{\Psi^-}\right).$$

This verifies that χ_0^- is indeed χ^-. Similar steps prove that χ_0^+ is indeed χ^+.

B.5 Proof of Proposition ??

Assume a Leontief matching function so that $G(a,b) = \min\{a,b\}$. By Proposition 1, you obtain:

$$\frac{\dot{\theta}_\tau}{\theta_\tau} = \psi_\tau^+ - \psi_\tau^- = \psi_\tau^+ - \frac{\psi_\tau^+}{\theta_\tau} = \left(\frac{\theta_\tau - 1}{\theta_\tau}\right)\psi_\tau^+.$$

Thus,

$$\dot{\theta}_\tau = (\theta_\tau - 1)\,\psi_\tau^+ = (\theta_\tau - 1)\,\bar{\lambda}G(1,\theta_\tau) = (\theta_\tau - 1)\,\bar{\lambda}\min\{1,\theta_\tau\}.$$

Thus, we have that:

$$\begin{aligned}
\theta_0 > 1 \quad &\Rightarrow \quad \theta_\tau > 1 \qquad \forall\tau \in [0,1],\\
\theta_0 = 1 \quad &\Rightarrow \quad \theta_\tau = 1 \qquad \forall\tau \in [0,1],\\
\theta_0 < 1 \quad &\Rightarrow \quad \theta_\tau < 1 \qquad \forall\tau \in [0,1].
\end{aligned}$$

There are three possible cases that determine the solutions to the ODE's (25) and (26):

Case $\theta_0 = 1$.

In this case, we have that

$$\dot{\theta}_\tau = 0 \quad \Rightarrow \quad \theta_\tau = 1, \qquad \forall\tau \in [0,1].$$

Thus,

$$\psi_\tau^+ = \bar{\lambda} \quad \text{and} \quad \psi_\tau^- = \bar{\lambda}, \qquad \forall\tau \in [0,1].$$

Also,

$$\Psi_t^+ = \int_t^1 \psi_\tau^+ d\tau = \int_t^1 \bar{\lambda} d\tau = \bar{\lambda}(1-t) \quad \text{and} \quad \Psi_t^- = \int_t^1 \psi_\tau^- d\tau = \int_t^1 \bar{\lambda} d\tau = \bar{\lambda}(1-t); \qquad \forall t \in [0,1].$$

Then:

$$\begin{aligned}
\mathbb{P}_t^+ &= \int_t^1 \bar{\lambda}e^{-\bar{\lambda}(1-\tau)}d\tau = \left[e^{-\bar{\lambda}(1-\tau)}\right]_t^1 = 1 - e^{-\bar{\lambda}(1-t)} \text{ and}\\
\mathbb{P}_t^- &= \int_t^1 \bar{\lambda}e^{-\bar{\lambda}(1-\tau)}d\tau = \left[e^{-\bar{\lambda}(1-\tau)}\right]_t^1 = 1 - e^{-\bar{\lambda}(1-t)}.
\end{aligned}$$

Therefore, the solution to the average payments are:

$$\chi_\tau^+ = (1-\eta)\left(r^w - r^m\right)\left(1 - e^{-\bar{\lambda}(1-\tau)}\right) \qquad \text{and} \qquad \chi_\tau^- = \left(r^w - r^m\right)\left(1 - \eta\left(1 - e^{-\bar{\lambda}(1-\tau)}\right)\right)$$

This allows us to arrive to the interbank-reserves interest rate spread:

$$r_\tau^f - r^m = (1-\eta)\left(r^w - r^m\right).$$

This implies that $\overline{r}^f = r^m + (1-\eta)(r^w - r^m)$ as in the statement of the Proposition. For $\eta \to 0$, we arrive to the intuitive result of $\overline{r}^f = r^w$ and for $\eta \to 1, \overline{r}^f = r^m$.

Case $\theta_0 > 1$. In this case, we have that $\theta_\tau > 1$ for every $\tau \in [0,1]$. Thus, it follows that the law of motion for tightness is

$$\dot{\theta}_\tau = \bar{\lambda}(\theta_\tau - 1) \quad \Rightarrow \quad \theta_\tau = 1 + (\theta_0 - 1)e^{\bar{\lambda}\tau}, \qquad \forall \tau \in [0,1].$$

$$\psi_\tau^+ = \bar{\lambda} \quad \text{and} \quad \psi_\tau^- = \frac{\bar{\lambda}}{1 + (\theta_0 - 1)e^{\bar{\lambda}\tau}}, \qquad \forall \tau \in [0,1].$$

Applying this results:

$$\Psi_t^+ = \int_t^1 \psi_\tau^+ d\tau = \int_t^1 \bar{\lambda} d\tau = \bar{\lambda}(1-t)$$

$$\Psi_t^- = \int_t^1 \psi_\tau^- d\tau = \int_t^1 \frac{\bar{\lambda}}{1 + (\theta_0 - 1)e^{\bar{\lambda}\tau}} d\tau = \left[\bar{\lambda}\tau - \ln\left(1 + (\theta_0 - 1)e^{\bar{\lambda}\tau}\right)\right]_t^1 = \bar{\lambda}(1-t) - \ln\left(\frac{1 + (\theta_0 - 1)e^{\bar{\lambda}}}{1 + (\theta_0 - 1)e^{\bar{\lambda}t}}\right).$$

Following,

$$\begin{aligned}
\mathbb{P}_t^+ &= \int_t^1 \psi_\tau^+ e^{-\left[\eta\Psi_\tau^- + (1-\eta)\Psi_\tau^+\right]} d\tau = \int_t^1 \bar{\lambda} e^{-\bar{\lambda}(1-\tau) + \eta \ln\left(\frac{1+(\theta_0-1)e^{\bar{\lambda}}}{1+(\theta_0-1)e^{\bar{\lambda}\tau}}\right)} d\tau \\
&= \int_t^1 \left(\frac{1 + (\theta_0 - 1)e^{\bar{\lambda}}}{1 + (\theta_0 - 1)e^{\bar{\lambda}\tau}}\right)^\eta \bar{\lambda} e^{-\bar{\lambda}(1-\tau)} d\tau = \left[\left(\frac{1 + (\theta_0 - 1)e^{\bar{\lambda}\tau}}{(1-\eta)(\theta_0 - 1)e^{\bar{\lambda}}}\right)\left(\frac{1 + (\theta_0 - 1)e^{\bar{\lambda}}}{1 + (\theta_0 - 1)e^{\bar{\lambda}\tau}}\right)^\eta\right]_t^1 \\
&= \left(\frac{1 + (\theta_0 - 1)e^{\bar{\lambda}}}{(1-\eta)(\theta_0 - 1)e^{\bar{\lambda}}}\right) - \left(\frac{1 + (\theta_0 - 1)e^{\bar{\lambda}t}}{(1-\eta)(\theta_0 - 1)e^{\bar{\lambda}}}\right)\left(\frac{1 + (\theta_0 - 1)e^{\bar{\lambda}}}{1 + (\theta_0 - 1)e^{\bar{\lambda}t}}\right)^\eta \\
&= \frac{\theta_1\left(1 - \left(\frac{\theta_\tau}{\theta_1}\right)^{1-\eta}\right)}{(1-\eta)(\theta_1 - 1)} = \left(\frac{1}{1-\eta}\right)\left(\frac{\theta_1 - \theta_\tau^{1-\eta}\theta_1^\eta}{\theta_1 - 1}\right) = \left(\frac{1}{1-\eta}\right)\left(\frac{\bar{\theta}}{\theta}\right)^\eta\left(\frac{\theta^\eta\bar{\theta}^{1-\eta} - \theta}{\bar{\theta} - 1}\right).
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}_t^- &= \int_t^1 \psi_\tau^- e^{-\left[\eta\Psi_\tau^- + (1-\eta)\Psi_\tau^+\right]} d\tau = \int_t^1 \left(\frac{\bar{\lambda}}{1 + (\theta_0 - 1)e^{\bar{\lambda}\tau}}\right) e^{-\bar{\lambda}(1-\tau) + \eta \ln\left(\frac{1+(\theta_0-1)e^{\bar{\lambda}}}{1+(\theta_0-1)e^{\bar{\lambda}\tau}}\right)} d\tau \\
&= \int_t^1 \left(\frac{1 + (\theta_0 - 1)e^{\bar{\lambda}}}{1 + (\theta_0 - 1)e^{\bar{\lambda}\tau}}\right)^\eta \left(\frac{\bar{\lambda}e^{-\bar{\lambda}(1-\tau)}}{1 + (\theta_0 - 1)e^{\bar{\lambda}\tau}}\right) d\tau = \left[\left(\frac{-1}{\eta(\theta_0 - 1)e^{\bar{\lambda}}}\right)\left(\frac{1 + (\theta_0 - 1)e^{\bar{\lambda}}}{1 + (\theta_0 - 1)e^{\bar{\lambda}\tau}}\right)^\eta\right]_t^1 \\
&= \left(\frac{1}{\eta(\theta_0 - 1)e^{\bar{\lambda}}}\right)\left(\frac{1 + (\theta_0 - 1)e^{\bar{\lambda}}}{1 + (\theta_0 - 1)e^{\bar{\lambda}t}}\right)^\eta - \left(\frac{1}{\eta(\theta_0 - 1)e^{\bar{\lambda}}}\right) \\
&= \frac{\left(\frac{\theta_1}{\theta_\tau}\right)^\eta - 1}{\eta(\theta_1 - 1)}
\end{aligned}$$

Therefore, using (28), we have that:

$$\chi_\tau^+ = (1-\eta)(r^w - r^m)\mathbb{P}_\tau^+ \qquad \text{and} \qquad \chi_\tau^- = (r^w - r^m)\left(1 - \eta\mathbb{P}_\tau^-\right),$$

which implies:

$$\chi_{\tau}^{+}=\left(r^{w}-r^{m}\right)\left(\frac{\theta_{1}}{\theta_{\tau}}\right)^{\eta}\left(\frac{\theta_{\tau}^{\eta}\theta_{1}^{1-\eta}-\theta_{\tau}}{\theta_{1}-1}\right)\qquad\text{and}\qquad\chi_{\tau}^{-}=\left(r^{w}-r^{m}\right)\left(\frac{\theta_{1}}{\theta_{\tau}}\right)^{\eta}\left(\frac{\theta_{\tau}^{\eta}\theta_{1}^{1-\eta}-1}{\theta_{1}-1}\right).$$

From here, substitute for $\tau=0$ and obtain $\left\{\chi_{0}^{+},\chi_{0}^{-}\right\}$.

Recall that the interbank rates satisfy:

$$r_{n}^{f}\equiv r^{m}+\left(1-\eta\right)\chi_{n+1}^{-}+\eta\chi_{n+1}^{+}.$$

Now from here we obtain the interbank-reserves interest rate spread,

$$\begin{aligned}r_{\tau}^{f}-r^{m} &=\\ &=\left(r^{w}-r^{m}\right)\left(\frac{\theta_{1}}{\theta_{\tau}}\right)^{\eta}\left(\frac{\theta_{\tau}^{\eta}\theta_{1}^{1-\eta}-1}{\theta_{1}-1}-\eta\left(\frac{\theta_{\tau}-1}{\theta_{1}-1}\right)\right).\end{aligned}$$

The matching probabilities are:

$$\psi_{\tau}^{+}=1-e^{-\bar{\lambda}}\qquad\text{and}\qquad\psi^{-}=1-e^{-\bar{\lambda}\int_{0}^{1}\frac{1}{\theta_{\tau}}d\tau}=1-e^{-\bar{\lambda}\int_{0}^{1}\frac{1}{1+(\theta_{0}-1)e^{\bar{\lambda}\tau}}d\tau}=1-\left(\frac{(\theta_{0}-1)e^{\bar{\lambda}}+1}{\theta_{0}}\right)e^{-\bar{\lambda}}=\frac{1-e^{-\bar{\lambda}}}{\theta_{0}}.$$

Finally, let us compute the average interbank interest rate

$$\begin{aligned}\bar{r}^{f} &=r^{m}+\left(\frac{\chi_{0}^{-}-\left(1-\psi^{-}\right)\left(r^{w}-r^{m}\right)}{\psi^{-}}\right)\\ &=r^{m}+\left(\frac{\chi_{0}^{-}}{\psi^{-}}\right)-\left(\frac{1-\psi^{-}}{\psi^{-}}\right)\left(r^{w}-r^{m}\right)\\ &=r^{m}+\left(\frac{1}{\psi^{-}}\right)\left(\frac{\theta_{1}}{\theta_{0}}\right)^{\eta}\left(\frac{\theta_{0}^{\eta}\theta_{1}^{1-\eta}-1}{\theta_{1}-1}\right)\left(r^{w}-r^{m}\right)+\left(1-\left(\frac{1}{\psi^{-}}\right)\right)\left(r^{w}-r^{m}\right)\\ &=r^{w}+\left(\frac{r^{w}-r^{m}}{\psi^{-}}\right)\left[\left(\frac{\theta_{1}}{\theta_{0}}\right)^{\eta}\left(\frac{\theta_{0}^{\eta}\theta_{1}^{1-\eta}-1}{\theta_{1}-1}\right)-1\right]\\ &=r^{w}-\left(\frac{r^{w}-r^{m}}{\psi^{-}}\right)\left(\frac{\left(\frac{\theta_{1}}{\theta_{0}}\right)^{\eta}-1}{\theta_{1}-1}\right)\\ &=r^{w}-\left(r^{w}-r^{m}\right)\left(\frac{\theta_{0}}{\theta_{0}-1}\right)\left(\left(\frac{\theta_{1}}{\theta_{0}}\right)^{\eta}-1\right)\left(\frac{1}{e^{\bar{\lambda}}-1}\right).\end{aligned}$$

Which is the formula in the expression.Notice that, if $\eta\rightarrow 0$, we arrive to the intuitive result of $\bar{r}^{f}=r^{w}$ and $\eta\rightarrow 1$, substitute for θ_{1} in terms of θ_{0} and this shows that at this limit $\bar{r}^{f}=r^{m}$.

Case $\theta_{0}<1$. In this case, we have that $\theta_{\tau}<1$ for every $\tau\in[0,1]$ and thus:

$$\dot{\theta}_{\tau}=\bar{\lambda}\left(\theta_{\tau}-1\right)\theta_{\tau}\quad\Rightarrow\quad\theta_{\tau}=\frac{1}{1+\left(\frac{1-\theta_{0}}{\theta_{0}}\right)e^{\bar{\lambda}\tau}},\qquad\forall\tau\in[0,1].$$

$$\psi_\tau^+ = \frac{\bar\lambda}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda t}} \quad \text{and} \quad \psi_\tau^- = \bar\lambda, \qquad \forall \tau \in [0,1].$$

Then,

$$\Psi_t^+ = \int_t^1 \psi_\tau^+ d\tau = \int_t^1 \frac{\bar\lambda}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda\tau}} d\tau = \left[\bar\lambda\tau - \ln\left(1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda\tau}\right)\right]_t^1 = \bar\lambda(1-t) - \ln\left(\frac{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda}}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda t}}\right)$$

$$\Psi_t^- = \int_t^1 \psi_\tau^- d\tau = \int_t^1 \bar\lambda d\tau = \bar\lambda(1-t).$$

Following,

$$\begin{aligned}
\mathbb{P}_t^+ &= \int_t^1 \psi_\tau^+ e^{-\left[\eta\Psi_\tau^- + (1-\eta)\Psi_\tau^+\right]} d\tau = \int_t^1 \left(\frac{\bar\lambda}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda\tau}}\right) e^{-\bar\lambda(1-\tau)+(1-\eta)\ln\left(\frac{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda}}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda\tau}}\right)} d\tau \\
&= \int_t^1 \left(\frac{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda}}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda\tau}}\right)^{1-\eta} \left(\frac{\bar\lambda e^{-\bar\lambda(1-\tau)}}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda\tau}}\right) d\tau = \left[\left(\frac{-1}{(1-\eta)\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda}}\right)\left(\frac{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda}}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda\tau}}\right)^{1-\eta}\right]_t^1 \\
&= \left(\frac{1}{(1-\eta)\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda}}\right)\left(\frac{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda}}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda t}}\right)^{1-\eta} - \left(\frac{1}{(1-\eta)\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda}}\right) \\
&= \left(\frac{1}{1-\eta}\right)\left(\frac{\theta_1}{1-\theta_1}\right)\left(\left(\frac{\theta_t}{\theta_1}\right)^{1-\eta} - 1\right) \\
&= \left(\frac{1}{1-\eta}\right)\left(\frac{\theta_t^{1-\eta}\theta_1^{\eta} - \theta_1}{1-\theta_1}\right).
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}_t^- &= \int_t^1 \psi_\tau^- e^{-\left[\eta\Psi_\tau^- + (1-\eta)\Psi_\tau^+\right]} d\tau = \int_t^1 \bar\lambda e^{-\bar\lambda(1-\tau)+(1-\eta)\ln\left(\frac{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda}}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda\tau}}\right)} d\tau \\
&= \int_t^1 \left(\frac{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda}}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda\tau}}\right)^{1-\eta} \bar\lambda e^{-\bar\lambda(1-\tau)} d\tau \\
&= \left[\left(\frac{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda}}{\eta\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda}}\right)\left(\frac{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda\tau}}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda}}\right)^{\eta}\right]_t^1 \\
&= \left(\frac{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda}}{\eta\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda}}\right) - \left(\frac{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda}}{\eta\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda}}\right)\left(\frac{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda t}}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar\lambda}}\right)^{\eta} \\
&= \left(\frac{1}{\eta}\right)\left(\frac{1-\left(\frac{\theta_1}{\theta_t}\right)^{\eta}}{1-\theta_1}\right).
\end{aligned}$$

Therefore,

$$\chi_\tau^+ = (1-\eta)\left(r^w - r^m\right)\mathbb{P}_\tau^+ \qquad \text{and} \qquad \chi_\tau^- = \left(r^w - r^m\right)\left(1-\eta\mathbb{P}_\tau^-\right).$$

Thus:

$$\chi_\tau^+ = \left(r^w - r^m\right)\left(\frac{\theta_\tau^{1-\eta}\theta_1^\eta - \theta_1}{1-\theta_1}\right) = \left(r^w - r^m\right)\left(\frac{\theta_1}{\theta_\tau}\right)^\eta\left(\frac{\theta_1^{1-\eta}\theta_\tau^\eta - \theta_\tau}{\theta_1 - 1}\right)$$

and

$$\chi_\tau^- = \left(r^w - r^m\right)\left(1-\left(\frac{1-\left(\frac{\theta_1}{\theta_\tau}\right)^\eta}{1-\theta_1}\right)\right) = \left(r^w - r^m\right)\left(\frac{\theta_1}{\theta_\tau}\right)^\eta\left(\frac{\theta_1^{1-\eta}\theta_\tau^\eta - 1}{\theta_1 - 1}\right).$$

Hence, in summary we obtain:

$$\chi_\tau^+ = \left(r^w - r^m\right)\left(\frac{\theta_1}{\theta_\tau}\right)^\eta\left(\frac{\theta_1^{1-\eta}\theta_\tau^\eta - \theta_\tau}{\theta_1 - 1}\right) \qquad \text{and} \qquad \chi_\tau^- = \left(r^w - r^m\right)\left(\frac{\theta_1}{\theta_\tau}\right)^\eta\left(\frac{\theta_\tau^\eta\theta_1^{1-\eta} - 1}{\theta_1 - 1}\right).$$

If we substitute for $t = 0$, we arrive to the expressions for $\left\{\chi_0^+, \chi_0^-\right\}$ employed in the proposition.

Recall that the interbank rates satisfy:

$$r_n^f \equiv r^m + (1-\eta)\,\chi_{n+1}^- + \eta\chi_{n+1}^+.$$

Now from here we obtain the interbank-reserves interest rate spread,

$$\begin{aligned} r_\tau^f - r^m &= \\ &= \left(r^w - r^m\right)\left(\frac{\theta_1}{\theta_\tau}\right)^\eta\left(\frac{\theta_\tau^\eta\theta_1^{1-\eta} - 1}{\theta_1 - 1} - \eta\left(\frac{\theta_\tau - 1}{\theta_1 - 1}\right)\right). \end{aligned}$$

Now the probability of finding a match during the interbank matching process can be computed as,

$$\psi_\tau^+ = 1 - e^{-\bar{\lambda}\int_0^1 \theta_\tau d\tau} = 1 - e^{-\bar{\lambda}\int_0^1 \frac{1}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}\tau}}d\tau} = 1 - \theta_0 e^{-\bar{\lambda}}\left(1 + \left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}}\right) = \theta_0\left(1 - e^{-\bar{\lambda}}\right)$$

and

$$\psi^- = 1 - e^{-\bar{\lambda}}.$$

From here, we compute the average interbank interest rate:

$$\begin{aligned}
\overline{r}^{f} &= r^{m} + \left(\frac{\chi_{0}^{-} - \left(1-\psi^{-}\right)\left(r^{w}-r^{m}\right)}{\psi^{-}}\right) \\
&= r^{m} + \left(\frac{\chi_{0}^{-}}{\psi^{-}}\right) - \left(\frac{1-\psi^{-}}{\psi^{-}}\right)\left(r^{w}-r^{m}\right) \\
&= r^{m} + \left(\frac{1}{\psi^{-}}\right)\left(\frac{\theta_{1}}{\theta_{0}}\right)^{\eta}\left(\frac{1-\theta_{0}^{\eta}\theta_{1}^{1-\eta}}{1-\theta_{1}}\right)\left(r^{w}-r^{m}\right) + \left(1-\left(\frac{1}{\psi^{-}}\right)\right)\left(r^{w}-r^{m}\right) \\
&= r^{w} + \left(\frac{r^{w}-r^{m}}{\psi^{-}}\right)\left[\left(\frac{\theta_{1}}{\theta_{0}}\right)^{\eta}\left(\frac{1-\theta_{0}^{\eta}\theta_{1}^{1-\eta}}{1-\theta_{1}}\right)-1\right] \\
&= r^{w} - \left(\frac{r^{w}-r^{m}}{\psi^{-}}\right)\left(\frac{1-\left(\frac{\theta_{1}}{\theta_{0}}\right)^{\eta}}{1-\theta_{1}}\right) \\
&= r^{w} - \left(1-\left(\frac{\theta_{1}}{\theta_{0}}\right)^{\eta}\right)\left(\left(\frac{\theta_{0}}{\theta_{0}-1}\right)+e^{\bar{\lambda}}\right)\left(\frac{r^{w}-r^{m}}{e^{\bar{\lambda}}-1}\right) \\
&= r^{w} - \left(1-\left(\frac{\theta_{1}}{\theta_{0}}\right)^{\eta}\right)\left(\left(\frac{\theta_{0}}{1-\theta_{0}}\right)+e^{\bar{\lambda}}\right)\left(\frac{r^{w}-r^{m}}{e^{\bar{\lambda}}-1}\right) \\
&= r^{w} - \left(1-\left(\frac{\theta_{1}}{\theta_{0}}\right)^{\eta}\right)\left(\left(\frac{\theta_{0}+e^{\bar{\lambda}}\left(1-\theta_{0}\right)}{1-\theta_{0}}\right)\right)\left(\frac{r^{w}-r^{m}}{e^{\bar{\lambda}}-1}\right).
\end{aligned}$$

Notice that, if $\eta \to 0$, we arrive to the intuitive result of $\overline{r}^{f} = r^{w}$. Similarly, we obtain that $\eta \to 1$, leads to $\overline{r}^{f} = r^{m}$.

Then, we have the relationship:

$$\left(\theta_{1}\right)^{-1}\theta_{0} = \theta_{0} + \left(1-\theta_{0}\right)e^{\bar{\lambda}}$$

in which case:

$$\begin{aligned}
\overline{r}^{f} &= r^{w} - \left(1-\left(\frac{\theta_{1}}{\theta_{0}}\right)^{\eta}\right)\left(\left(\frac{\theta_{0}+\left(\left(\theta_{1}\right)^{-1}-1\right)\theta_{0}}{1-\theta_{0}}\right)\right)\left(\frac{r^{w}-r^{m}}{e^{\bar{\lambda}}-1}\right) \\
&= r^{w} - \left(\left(\frac{\theta_{1}}{\theta_{0}}\right)^{\eta}-1\right)\left(\frac{1}{\theta_{1}}\frac{\theta_{0}}{\theta_{0}-1}\right)\left(\frac{r^{w}-r^{m}}{e^{\bar{\lambda}}-1}\right).
\end{aligned}$$

Alternative representation. The statement in the proposition uses the following alternative representation.

$$\chi_{\tau}^{+} = \left(r^{w}-r^{m}\right)\left(\frac{\theta_{1}}{\theta_{\tau}}\right)^{\eta}\left(\frac{\theta_{\tau}^{\eta}\theta_{1}^{1-\eta}-\theta_{\tau}}{\theta_{1}-1}\right) = \left(r^{w}-r^{m}\right)\left(\frac{\theta_{1}-\theta_{1}^{\eta}\theta_{\tau}^{1-\eta}}{\theta_{1}-1}\right)$$

and

$$\chi_{\tau}^{-} = \left(r^{w}-r^{m}\right)\left(\frac{\theta_{1}}{\theta_{\tau}}\right)^{\eta}\left(\frac{\theta_{\tau}^{\eta}\theta_{1}^{1-\eta}-1}{\theta_{1}-1}\right) = \left(r^{w}-r^{m}\right)\left(\frac{\theta_{1}-\theta_{1}^{\eta}\theta_{\tau}^{-\eta}}{\theta_{1}-1}\right).$$

Then, using that:

$$r_{n}^{f} - r^{m} \equiv \left(1-\eta\right)\chi_{n+1}^{-} + \eta\chi_{n+1}^{+} = \left(r^{w}-r^{m}\right)\left(\frac{\theta_{1}-\eta\theta_{1}^{\eta}\theta_{\tau}^{1-\eta}-\left(1-\eta\right)\theta_{1}^{\eta}\theta_{\tau}^{-\eta}}{\theta_{1}-1}\right).$$

Thus, letting ϕ_τ denote the endogenous bargaining power of the corresponding round, we obtain:

$$\phi_\tau = 1 - \left(\frac{\theta_1 - \eta\theta_1^\eta\theta_\tau^{1-\eta} - (1-\eta)\,\theta_1^\eta\theta_\tau^{-\eta}}{\theta_1 - 1}\right) = \left(\frac{\left(\frac{\theta_1}{\theta_\tau}\right)^\eta (\eta\theta_\tau + (1-\eta)) - 1}{\theta_1 - 1}\right).$$

B.6 Proof of Corollary ??

The average interbank rate is:

$$\bar{R}^f = R^m + \frac{\chi^+}{\Psi^+}.$$

Thus, we obtain that:

$$\begin{aligned}\bar{R}^f &= R^m + \frac{(R^w - R^m)\left(\frac{\bar{\theta}}{\theta}\right)^\eta \left(\frac{\theta^\eta\bar{\theta}^{1-\eta} - \theta}{\bar{\theta} - 1}\right)}{\Psi^+} \\ &= (1 - \phi(\theta))\, R^w + \phi(\theta)\, R^m,\end{aligned}$$

where

$$\phi(\theta) = 1 - \frac{\left(\frac{\bar{\theta}}{\theta}\right)^\eta \left(\frac{\theta^\eta\bar{\theta}^{1-\eta} - \theta}{\bar{\theta} - 1}\right)}{\Psi^+} = 1 - \frac{\bar{\theta} - \theta^{1-\eta}\bar{\theta}^\eta}{\left(\bar{\theta} - 1\right)\Psi^+}.$$

Recall that

$$\Psi^+ = \begin{cases} 1 - e^{-\bar{\lambda}} & \text{if } \theta \geq 1 \\ \theta\left(1 - e^{-\bar{\lambda}}\right) & \text{if } \theta < 1 \end{cases},$$

and that:

$$\bar{\theta} = \begin{cases} 1 + (\theta - 1)\exp\left(\bar{\lambda}\right) & \text{if } \theta \geq 1 \\ \left(1 + \left(\theta^{-1} - 1\right)\exp\left(\bar{\lambda}\right)\right)^{-1} & \text{if } \theta < 1 \end{cases}.$$

Thus, we have the following two cases:

- If $\theta > 1$ we have that:

$$\begin{aligned}1 - \phi(\theta) &= 1 - \frac{\bar{\theta} - \theta^{1-\eta}\bar{\theta}^\eta}{(\theta - 1)\exp\left(\bar{\lambda}\right)\left(1 - \exp\left(-\bar{\lambda}\right)\right)} \\ &= 1 - \frac{\bar{\theta} - \theta^{1-\eta}\bar{\theta}^\eta}{(\theta - 1)\exp\left(\bar{\lambda}\right) + 1 - \theta} \\ &= 1 - \frac{\bar{\theta} - \theta^{1-\eta}\bar{\theta}^\eta}{1 + (\theta - 1)\exp\left(\bar{\lambda}\right) - \theta} \\ &= 1 - \frac{\bar{\theta} - \theta^{1-\eta}\bar{\theta}^\eta}{\bar{\theta} - \theta}.\end{aligned}$$

  Hence:

$$\phi(\theta) = \frac{\bar{\theta} - \theta^{1-\eta}\bar{\theta}^\eta}{\bar{\theta} - \theta}.$$

- This number is between zero and one since $\bar{\theta} > \theta \to \bar{\theta} > \theta^{1-\eta}\bar{\theta}^{\eta} > \theta$.

- If $\theta < 1$ we have that:

$$\begin{aligned}
1-\phi(\theta) &= 1-\frac{\bar{\theta}-\theta^{1-\eta}\bar{\theta}^{\eta}}{\left(\bar{\theta}-1\right)\theta\left(1-e^{-\bar{\lambda}}\right)} \\
&= 1-\frac{\theta^{-1}-\theta^{-\eta}\bar{\theta}^{\eta-1}}{\left(1-\bar{\theta}^{-1}\right)\left(1-e^{-\bar{\lambda}}\right)} \\
&= 1-\frac{\theta^{-1}-\theta^{-\eta}\left(1+\left(\theta^{-1}-1\right)\exp\left(\bar{\lambda}\right)\right)^{(1-\eta)}}{\left(\theta^{-1}-1\right)\left(\exp\left(\bar{\lambda}\right)-1\right)} \\
&= 1-\frac{\theta^{-1}-\theta^{-\eta}\left(\bar{\theta}^{-1}\right)^{(1-\eta)}}{\bar{\theta}^{-1}-\theta^{-1}}.
\end{aligned}$$

  Hence, we have:

$$\phi(\theta) = 1-\frac{\bar{\theta}^{-1}-\theta^{-\eta}\left(\bar{\theta}^{-1}\right)^{(1-\eta)}}{\bar{\theta}^{-1}-\theta^{-1}}.$$

  This shows the symmetry property. The bargaining power falls between zero and one since $\bar{\theta}^{-1} > \theta^{-1} \to \bar{\theta}^{-1} > \theta^{-(1-\eta)}\bar{\theta}^{-\eta} > \theta^{-1}$.

B.7 Symmetry

So far, we have show that given θ, the after-trade tightness is given by:

$$\bar{\theta} = \theta_1 = \begin{cases} 1+(\theta-1)\exp\left(\bar{\lambda}\right) & \text{if } \theta > 1 \\ 1 & \text{if } \theta = 1 \\ \left(1+\left(\theta^{-1}-1\right)\exp\left(\bar{\lambda}\right)\right)^{-1} & \text{if } \theta < 1 \end{cases}.$$

Trading probabilities are given by:

$$\Psi^{+} = \begin{cases} 1-e^{-\bar{\lambda}} & \text{if } \theta \geq 1 \\ \theta\left(1-e^{-\bar{\lambda}}\right) & \text{if } \theta < 1 \end{cases}, \qquad \Psi^{-} = \begin{cases} \left(1-e^{-\bar{\lambda}}\right)\theta^{-1} & \text{if } \theta > 1 \\ 1-e^{-\bar{\lambda}} & \text{if } \theta \leq 1 \end{cases}.$$

Thus, the slopes of the liquidity-yield function are given by:

$$\chi^{+} = (r^{w}-r^{m})\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\left(\frac{\theta^{\eta}\bar{\theta}^{1-\eta}-\theta}{\bar{\theta}-1}\right) \text{ and } \chi^{-} = (r^{w}-r^{m})\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\left(\frac{\theta^{\eta}\bar{\theta}^{1-\eta}-1}{\bar{\theta}-1}\right).$$

Next, we simplify the solution.

Case $\theta < 1$.

In this case:

$$\bar{\theta} = \left(1+\left(\theta^{-1}-1\right)\exp(\lambda)\right)^{-1}.$$

The following calculations are useful. Observe that,

$$\frac{\bar{\theta}}{\theta}=\frac{\theta}{\theta\left(\theta+(1-\theta)\exp\left(\lambda\right)\right)}=\frac{1}{\theta+(1-\theta)\exp\left(\lambda\right)}$$

And from here, that,

$$1-\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}=1-\left(\frac{1}{\theta+(1-\theta)\exp\left(\lambda\right)}\right)^{\eta}$$

and,

$$1-\bar{\theta}=1-\frac{1}{\left(1+\left(\theta^{-1}-1\right)\exp\left(\lambda\right)\right)}=\frac{(1-\theta)\exp\left(\lambda\right)}{\theta+(1-\theta)\exp\left(\lambda\right)}.$$

Thus, the "Cobb-Douglas" term, satisfies:

$$\theta^{\eta}\bar{\theta}^{1-\eta}=\frac{\theta^{\eta}\left(1+\left(\theta^{-1}-1\right)\exp\left(\lambda\right)\right)^{\eta}}{\left(1+\left(\theta^{-1}-1\right)\exp\left(\lambda\right)\right)}=\frac{\left(\theta+(1-\theta)\exp\left(\lambda\right)\right)^{\eta}}{\left(1+\left(\theta^{-1}-1\right)\exp\left(\lambda\right)\right)}=\theta\frac{\left(\theta+(1-\theta)\exp\left(\lambda\right)\right)^{\eta}}{\left(\theta+(1-\theta)\exp\left(\lambda\right)\right)}.$$

Now, define:

$$h^{+}\equiv\theta-\theta^{\eta}\bar{\theta}^{1-\eta}=\theta\left(1-\frac{\left(\theta+(1-\theta)\exp\left(\lambda\right)\right)^{\eta}}{\left(\theta+(1-\theta)\exp\left(\lambda\right)\right)}\right)$$

and

$$h^{-}\equiv1-\theta^{\eta}\bar{\theta}^{1-\eta}=\left(1-\theta\frac{\left(\theta+(1-\theta)\exp\left(\lambda\right)\right)^{\eta}}{\left(\theta+(1-\theta)\exp\left(\lambda\right)\right)}\right).$$

With these definitions, we obtain the following expression for the slopes of the liquidity yield:

$$\chi^{+}=\left(r^{w}-r^{m}\right)\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}h^{+}\left(\frac{1}{1-\bar{\theta}}\right)=\left(r^{w}-r^{m}\right)\left(\frac{1}{\theta+(1-\theta)\exp\left(\lambda\right)}\right)^{\eta}\frac{\theta\left(1-\frac{\left(\theta+(1-\theta)\exp(\lambda)\right)^{\eta}}{\left(\theta+(1-\theta)\exp(\lambda)\right)}\right)}{\frac{(1-\theta)\exp(\lambda)}{\theta+(1-\theta)\exp(\lambda)}}$$

and

$$\chi^{-}=\left(r^{w}-r^{m}\right)\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}h^{-}\left(\frac{1}{1-\bar{\theta}}\right)=\left(r^{w}-r^{m}\right)\left(\frac{1}{\theta+(1-\theta)\exp\left(\lambda\right)}\right)^{\eta}\frac{\left(1-\theta\frac{\left(\theta+(1-\theta)\exp(\lambda)\right)^{\eta}}{\left(\theta+(1-\theta)\exp(\lambda)\right)}\right)}{\frac{(1-\theta)\exp(\lambda)}{\theta+(1-\theta)\exp(\lambda)}}.$$

Define:

$$\rho\equiv(1-\theta)\exp\left(\lambda\right)$$

Then

$$
\begin{aligned}
\chi^{+} &= (r^{w}-r^{m})\left(\frac{1}{\theta+\rho}\right)^{\eta}\frac{\theta\left(1-\frac{(\theta+\rho)^{\eta}}{\theta+\rho}\right)}{\frac{\rho}{\theta+\rho}} = (r^{w}-r^{m})\,\frac{\theta}{\rho}\,(\theta+\rho)^{-\eta}\left((\theta+\rho)-(\theta+\rho)^{\eta}\right)\\
&= (r^{w}-r^{m})\,\theta\frac{\left((\theta+(1-\theta)\exp(\lambda))^{1-\eta}-1\right)}{(1-\theta)\exp(\lambda)},
\end{aligned}
$$

and

$$
\begin{aligned}
\chi^{-} &= (r^{w}-r^{m})\left(\frac{1}{\theta+\rho}\right)^{\eta}\frac{\left(1-\frac{(\theta+\rho)^{\eta}}{\theta+\rho}\right)}{\frac{\rho}{\theta+\rho}} = (r^{w}-r^{m})\left(\frac{1}{\theta+\rho}\right)^{\eta}\frac{1}{\rho}\,(\theta+\rho)^{-\eta}\left((\theta+\rho)-\theta\,(\theta+\rho)^{\eta}\right)\\
&= (r^{w}-r^{m})\,\frac{\left((\theta+(1-\theta)\exp(\lambda))^{1-\eta}-\theta\right)}{(1-\theta)\exp(\lambda)}.
\end{aligned}
$$

Therefore, in summary:

$$
\chi^{-} = (r^{w}-r^{m})\,\frac{(\theta+(1-\theta)\exp(\lambda))^{1-\eta}-\theta}{(1-\theta)\exp(\lambda)},
$$

and

$$
\chi^{+} = (r^{w}-r^{m})\,\frac{\theta\,(\theta+(1-\theta)\exp(\lambda))^{1-\eta}-\theta}{(1-\theta)\exp(\lambda)}.
$$

The resulting average OTC market rate is determined by the average of Nash bargaining over the positions and is given by:

$$
\bar{r}^{f}(\theta, r^{m}, r^{w}) \equiv r^{m} + (r^{w}-r^{m})\,\frac{(\theta+(1-\theta)\exp(\lambda))^{1-\eta}-1}{1-\exp(\lambda)} \tag{29}
$$

Case $\theta > 1$. In this case:

$$
\bar{\theta} = 1 + (\theta-1)\exp\left(\bar{\lambda}\right).
$$

The following calculations are useful. Observe that,

$$
\frac{\bar{\theta}}{\theta} = \frac{1+(\theta-1)\exp\left(\bar{\lambda}\right)}{\theta} = \theta^{-1} + \left(1-\theta^{-1}\right)\exp\left(\bar{\lambda}\right).
$$

And from here, that,

$$
1-\left(\frac{\bar{\theta}}{\theta}\right)^{\eta} = 1-\left(\theta^{-1}+\left(1-\theta^{-1}\right)\exp\left(\bar{\lambda}\right)\right)^{\eta}
$$

and,

$$
1-\bar{\theta} = 1-\theta^{-1}-\left(1-\theta^{-1}\right)\exp\left(\bar{\lambda}\right) = \left(1-\theta^{-1}\right)\left(1-\exp(\lambda)\right).
$$

Thus, :

$$
\theta^{\eta}\bar{\theta}^{1-\eta} = \theta\left(\theta^{-1}+\left(1-\theta^{-1}\right)\exp\left(\bar{\lambda}\right)\right)^{1-\eta}.
$$

Now, define:

$$h^{+} \equiv \theta - \theta^{\eta}\bar{\theta}^{1-\eta} = \theta\left(1 - \left(\theta^{-1} + \left(1-\theta^{-1}\right)\exp\left(\bar{\lambda}\right)\right)^{1-\eta}\right)$$

and

$$h^{-} \equiv 1 - \theta^{\eta}\bar{\theta}^{1-\eta} = \left(1 - \theta\left(\theta^{-1} + \left(1-\theta^{-1}\right)\exp\left(\bar{\lambda}\right)\right)^{1-\eta}\right).$$

With these definitions, we obtain the following expression for the slopes of the liquidity yield:

$$\begin{aligned}
\chi^{+} &= \left(r^{w} - r^{m}\right)\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\frac{h^{+}}{1-\bar{\theta}} \\
&= \left(r^{w} - r^{m}\right)\left(\theta^{-1} + \left(1-\theta^{-1}\right)\exp\left(\bar{\lambda}\right)\right)^{\eta}\frac{\left(\left(\theta^{-1} + \left(1-\theta^{-1}\right)\exp\left(\bar{\lambda}\right)\right)^{1-\eta} - 1\right)}{\left(1-\theta^{-1}\right)\exp\left(\bar{\lambda}\right)} \\
&= \frac{\left(\theta^{-1} + \left(1-\theta^{-1}\right)\exp\left(\lambda\right)\right) - \left(\theta^{-1} + \left(1-\theta^{-1}\right)\exp\left(\lambda\right)\right)^{\eta}}{\left(1-\theta^{-1}\right)\exp\left(\bar{\lambda}\right)}.
\end{aligned}$$

Likewise, we obtain that:

$$\begin{aligned}
\chi^{-} &= \left(r^{w} - r^{m}\right)\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\frac{h^{-}}{1-\bar{\theta}} \\
&= \left(r^{w} - r^{m}\right)\left(\theta^{-1} + \left(1-\theta^{-1}\right)\exp\left(\bar{\lambda}\right)\right)^{\eta}\frac{\left(\left(\theta^{-1} + \left(1-\theta^{-1}\right)\exp\left(\bar{\lambda}\right)\right)^{1-\eta} - \theta^{-1}\right)}{\left(1-\theta^{-1}\right)\exp\left(\bar{\lambda}\right)} \\
&= \frac{\left(\theta^{-1} + \left(1-\theta^{-1}\right)\exp\left(\lambda\right)\right) - \theta^{-1}\left(\theta^{-1} + \left(1-\theta^{-1}\right)\exp\left(\lambda\right)\right)^{\eta}}{\left(1-\theta^{-1}\right)\exp\left(\bar{\lambda}\right)}.
\end{aligned}$$

Next, we prove the symmetry:

$$\chi^{-}\left(\theta,\eta\right) = \Delta - \chi^{+}\left(\theta^{-1}, 1-\eta\right)$$

Using the previous formulas:

$$\begin{aligned}
\chi^{-}\left(\theta,\eta\right) &= \left(r^{w} - r^{m}\right)\left(1 - \frac{\theta^{-1}\left(\theta^{-1} + \left(1-\theta^{-1}\right)\exp\left(\lambda\right)\right)^{\eta} - \theta^{-1}}{\left(1-\theta^{-1}\right)\exp\left(\lambda\right)}\right) \\
&= \left(r^{w} - r^{m}\right)\frac{\left(\theta^{-1} + \left(1-\theta^{-1}\right)\exp\left(\lambda\right)\right) - \theta^{-1}\left(\theta^{-1} + \left(1-\theta^{-1}\right)\exp\left(\lambda\right)\right)^{\eta}}{\left(1-\theta^{-1}\right)\exp\left(\lambda\right)}.
\end{aligned}$$

Likewise, we have that:

$$\chi^{+}\left(\theta,\eta\right) = \Delta - \chi^{-}\left(\theta^{-1}, 1-\eta\right)$$

Using the previous formulas,

$$\begin{aligned}
\chi^{+}\left(\theta,\eta\right) &= \left(r^{w} - r^{m}\right)\left(1 - \frac{\left(\theta^{-1} + \left(1-\theta^{-1}\right)\exp\left(\lambda\right)\right)^{\eta} - \theta^{-1}}{\left(1-\theta^{-1}\right)\exp\left(\lambda\right)}\right) \\
&= \left(r^{w} - r^{m}\right)\left(\frac{\left(\theta^{-1} + \left(1-\theta^{-1}\right)\exp\left(\lambda\right)\right) - \left(\theta^{-1} + \left(1-\theta^{-1}\right)\exp\left(\lambda\right)\right)^{\eta}}{\left(1-\theta^{-1}\right)\exp\left(\lambda\right)}\right).
\end{aligned}$$

Summary. [TBA]

B.8 Volume Distribution

Fix a partiucular date t and let S be the shorter side of the market, that is $S = \min\{S^-, S^+\}$. Since $\theta > 1$ implies that $\theta_\tau > 1$ at all trading sessions, we know that the shorter side of the market remains the shorter side at all trading sessions. Hence, the continuous-time limit of equation (**??**) yields a law of motion for the shorter side:

$$\dot{S} = -\lambda S.$$

Thus, we have that:

$$S_\tau = S_0 \exp\left[-\lambda \tau\right].$$

As a result, we know that the total volume of trade is:

$$V = S_0 - S_1 = S_0(1 - \exp\left[-\lambda\right]).$$

Moreover, the transactions per instant of time are:

$$g_\tau = \lambda G\left[S_\tau^+, S_\tau^-\right] = \lambda S_\tau \min\left[\frac{S_\tau^+}{\min\{S^-, S^+\}}, \frac{S_\tau^-}{\min\{S^-, S^+\}}\right] = \lambda S_\tau.$$

Hence, the volume distribution in the interbank market is:

$$v_\tau = \frac{g_\tau}{V} = \frac{\lambda \exp\left[-\lambda * \tau\right]}{1 - \exp\left(\lambda\right)}.$$

The volume of discount-window loans is given by:

$$W = S^- - V = \begin{cases} S^+\left(\exp\left[-\lambda\right] + (\theta - 1)\right) & \text{if } \theta > 1 \\ \\ S^- \exp\left[-\lambda\right] & \text{if } \theta \leq 1. \end{cases}$$

Thus, the volume of discount loans to overall interbank loans is given by:

$$\frac{W}{V} = \frac{\exp\left[-\lambda\right] + (\theta - 1)\,\mathbb{I}_{[\theta > 1]}}{1 - \exp\left[-\lambda\right]}.$$

B.9 Dispersion

We can produce different metrics of dispersion in the interbank market.

- For $\theta = 1$

$$r_\tau^f = r^m + (1 - \eta)\left(r^w - r^m\right).$$

- For $\theta > 1$
$$r_\tau^f = r^m + (r^w - r^m)\left(\frac{\bar{\theta}}{\theta_\tau}\right)^\eta \left(\frac{\theta_\tau^\eta \bar{\theta}^{1-\eta} - 1}{\bar{\theta} - 1} - \eta\left(\frac{\theta_\tau - 1}{\bar{\theta} - 1}\right)\right).$$
- For $\theta < 1$
$$r_\tau^f = r^m + (r^w - r^m)\left(\frac{\bar{\theta}}{\theta_\tau}\right)^\eta \left(\frac{1 - \theta_\tau^\eta \bar{\theta}^{1-\eta}}{1 - \bar{\theta}} - \eta\left(\frac{1 - \theta_\tau}{1 - \bar{\theta}}\right)\right).$$

Clearly, for $\tau = 1$ we have that:
$$r_1^f = r^m + (1 - \eta)(r^w - r^m).$$

Next, we have that:

- For $\theta = 1$
$$r_1^f - r_\tau^f = 0.$$
Hence, the max-min and the standard deviation of interbank rates is constant.
- For $\theta > 0$, we have that:
$$\begin{aligned} r_1^f - r_\tau^f &= (r^w - r^m)\left[(1 - \eta) + \left(\frac{\bar{\theta}}{\theta_\tau}\right)^\eta \left(\frac{(1 - \eta) - \left(\theta_\tau^\eta \bar{\theta}^{1-\eta} - \eta\theta_\tau\right)}{\bar{\theta} - 1}\right)\right] \\ &= (r^w - r^m)\left[(1 - \eta) + \left(\frac{\bar{\theta}}{\theta_\tau}\right)^\eta \left(\frac{(1 - \eta) - \theta_\tau\left(\left(\frac{\bar{\theta}}{\theta_\tau}\right)^{1-\eta} - \eta\right)}{\bar{\theta} - 1}\right)\right] \end{aligned}$$
- For $\theta < 0$, we have that:
$$\begin{aligned} r_1^f - r_\tau^f &= (r^w - r^m)\left[(1 - \eta) + \left(\frac{\bar{\theta}}{\theta_\tau}\right)^\eta \left(\frac{(1 - \eta) - \left(\theta_\tau^\eta \bar{\theta}^{1-\eta} - \eta\theta_\tau\right)}{\bar{\theta} - 1}\right)\right] \\ &= (r^w - r^m)\left[(1 - \eta) + \left(\frac{\bar{\theta}}{\theta_\tau}\right)^\eta \left(\frac{(1 - \eta) - \theta_\tau\left(\left(\frac{\bar{\theta}}{\theta_\tau}\right)^{1-\eta} - \eta\right)}{\bar{\theta} - 1}\right)\right]. \end{aligned}$$

Clearly, for $\tau = 1$ we have that:

- For $\theta > 1$
$$\rho \equiv \frac{\bar{\theta}}{\theta} = \theta^{-1} + \left(1 - \theta^{-1}\right)\exp\left(\bar{\lambda}\right) > 1.$$
The derivatives of this ratio are:
$$\rho_\theta = -\frac{\exp\left(\bar{\lambda}\right) - 1}{\theta^2} < 1$$
and
$$\rho_\lambda = \left(1 - \theta^{-1}\right)\exp\left(\bar{\lambda}\right) > 0.$$

- For $\theta < 1$

$$\frac{\bar{\theta}}{\theta} = \left(\theta + (1-\theta)\exp\left(\bar{\lambda}\right)\right)^{-1} < 1.$$

The derivatives of this ratio are:

$$\rho_\theta = -\frac{\exp\left(\bar{\lambda}\right) - 1}{\left(\theta + (1-\theta)\exp\left(\bar{\lambda}\right)\right)^2} < 1$$

and

$$\rho_\lambda = \frac{\left(1-\theta^{-1}\right)\exp\left(\bar{\lambda}\right)}{\left(\theta + (1-\theta)\exp\left(\bar{\lambda}\right)\right)^2} > 0.$$

- Then, we have that:

$$r_1^f - r_\tau^f \quad = \quad \left(r^w - r^m\right)\left[(1-\eta) + \rho^\eta\left(\frac{(1-\eta) - \theta_\tau\left(\rho^{1-\eta} - \eta\right)}{\bar{\theta} - 1}\right)\right]$$

Then, taking total derivatives with respect to λ and θ we obtain:

$$\eta\frac{d\rho}{\rho}\left[\rho^\eta\left(\frac{(1-\eta) - \theta_\tau\left(\rho^{1-\eta} - \eta\right)}{\bar{\theta} - 1}\right)\right] - \frac{d\theta_\tau}{\theta_\tau}\left[\rho^\eta\left(\frac{\theta_\tau\left(\rho^{1-\eta} - \eta\right)}{\bar{\theta} - 1}\right)\right] - (1-\eta)\frac{d\rho}{\rho}\left[\rho^\eta\left(\frac{\theta_\tau\rho^{1-\eta}}{\bar{\theta} - 1}\right)\right] - \frac{d\left(\bar{\theta} - 1\right)}{\left(\bar{\theta} - 1\right)}\left[\rho^\eta\frac{(1-\eta) - \theta_\tau\left(\rho^{1-\eta} - \eta\right)}{\bar{\theta} - 1}\right].$$

Grouping terms:

$$\left[\eta\frac{d\rho}{\rho} - \frac{d\left(\bar{\theta} - 1\right)}{\left(\bar{\theta} - 1\right)}\right]\left[\rho^\eta\left(\frac{(1-\eta) - \theta_\tau\left(\rho^{1-\eta} - \eta\right)}{\bar{\theta} - 1}\right)\right] - \frac{d\theta_\tau\left(\rho - \eta\rho^\eta\right) + d\rho(1-\eta)\theta_\tau}{\bar{\theta} - 1}.$$

For $\theta > 1$, The term $\frac{d\rho}{\rho} < 0$. In turn, $d\bar{\theta} - 1 > 0$

C. Proof of Derivatives with respect to θ

Case $\theta < 1$. Trading probabilities are given by:

$$\Psi^{+} = \begin{cases} 1-e^{-\bar{\lambda}} & \text{if } \theta \geq 1 \\ \theta\left(1-e^{-\bar{\lambda}}\right) & \text{if } \theta < 1 \end{cases}, \qquad \Psi^{-} = \begin{cases} \left(1-e^{-\bar{\lambda}}\right)\theta^{-1} & \text{if } \theta > 1 \\ 1-e^{-\bar{\lambda}} & \text{if } \theta \leq 1 \end{cases}.$$

Next, we explore the derivatives of the liquidity yield, varying the market tightness. In the special case where $\theta < 1$. We have the following:

$$\overline{R}^{f}_{\theta} \equiv \left(R^{w}-R^{m}\right)\frac{(1-\eta)}{(\theta+(1-\theta)\exp(\lambda))^{\eta}} \in [0,(1-\eta)].$$

The second derivative in turn satisfies:

$$\overline{R}^{f}_{\theta\theta} > \left(R^{w}-R^{m}\right)\frac{\eta(1-\eta)(\exp(\lambda)-1)}{(\theta+(1-\theta)\exp(\lambda))^{1+\eta}} > 0.$$

Thus, the $\overline{R}^{f}$ is convex in θ when $\theta < 1$.

Using (8), we obtain:

$$\chi^{+}_{\theta} = \left(1-e^{-\bar{\lambda}}\right)\left(\overline{R}^{f}-R^{m}\right)+\theta\left(1-e^{-\bar{\lambda}}\right)\overline{R}^{f}_{\theta} \geq 0$$

and taking a second derivative shows that:

$$\chi^{+}_{\theta\theta} = 2\left(1-e^{-\bar{\lambda}}\right)\left(\overline{R}^{f}_{\theta}\right)+\theta\left(1-e^{-\bar{\lambda}}\right)\overline{R}^{f}_{\theta\theta} \geq 0,$$

which shows that the function is convex as well.

Likewise, using (8), we have that:

$$\chi^{-}_{\theta} = \left(1-e^{-\bar{\lambda}}\right)\overline{R}^{f}_{\theta} \geq 0,$$

and furthermore:

$$\chi^{-}_{\theta\theta} = \left(1-e^{-\bar{\lambda}}\right)\overline{R}^{f}_{\theta\theta} \geq 0.$$

In turn, the spread $\Sigma \equiv \chi^{-}-\chi^{+}$ satisfies:

$$\Sigma \equiv \left(\Psi^{-}-\Psi^{+}\right)\left(\overline{R}^{f}-R^{m}\right)+\left(1-\Psi^{-}\right)\left(R^{w}-R^{m}\right)$$

$$= \left(R^{w}-R^{m}\right)\left[\frac{(\theta+(1-\theta)\exp(\lambda))^{1-\eta}-\theta}{\exp(\lambda)}\right].$$

Thus,

$$\Sigma_{\theta} = -\left[\frac{(1-\eta)(\theta+(1-\theta)\exp(\lambda))^{-\eta}(\exp(\lambda)-1)+1}{\exp(\lambda)}\right] < 0,$$

although:

$$\Sigma_{\theta\theta} = -\left[\frac{\eta(1-\eta)(\theta+(1-\theta)\exp(\lambda))^{-(\eta+1)}(\exp(\lambda)-1)^{2}}{\exp(\lambda)}\right] < 0.$$

Thus, the spread falls and is concave as $\theta < 1$. The result for $\theta > 1$ follows by symmetry.

D. Proof of Derivatives with respect to $\bar{\lambda}$

Recall that:

$$\begin{aligned} \chi^{+} &= \left(r^{w}-r^{m}\right)\left(\frac{\bar{\theta}-\bar{\theta}^{\eta}\theta^{1-\eta}}{\bar{\theta}-1}\right)\in\left[0,r^{w}-r^{m}\right] \\ \text{and } \chi^{-} &= \left(r^{w}-r^{m}\right)\left(\frac{\bar{\theta}-\bar{\theta}^{\eta}\theta^{-\eta}}{\bar{\theta}-1}\right)\in\left[0,r^{w}-r^{m}\right]. \end{aligned}$$

The parameter $\bar{\lambda}$ only enters in $\bar{\theta}$. Thus, we first obtain the derivatives with respect to $\bar{\theta}$. For that, define

$$q^{+}\left(\bar{\theta}\right)\equiv\left(\frac{\bar{\theta}-\bar{\theta}^{\eta}\theta^{1-\eta}}{\bar{\theta}-1}\right)\in[0,1]$$

and

$$q^{-}\left(\bar{\theta}\right)\equiv\left(\frac{\bar{\theta}-\bar{\theta}^{\eta}\theta^{-\eta}}{\bar{\theta}-1}\right)\in[0,1].$$

We have that:

$$\begin{aligned} q_{\bar{\theta}}^{+}\left(\bar{\theta}\right) &= q^{+}\left(\bar{\theta}\right)\left(\frac{1-\eta\bar{\theta}^{\eta-1}\theta^{1-\eta}}{\bar{\theta}-\bar{\theta}^{\eta}\theta^{1-\eta}}-\frac{1}{\bar{\theta}-1}\right) \\ &= q^{+}\left(\bar{\theta}\right)\left(\frac{\bar{\theta}-\eta\bar{\theta}^{\eta}\theta^{1-\eta}-1+\eta\bar{\theta}^{\eta-1}\theta^{1-\eta}-\left(\bar{\theta}-\bar{\theta}^{\eta}\theta^{1-\eta}\right)}{\left(\bar{\theta}-\bar{\theta}^{\eta}\theta^{1-\eta}\right)\left(\bar{\theta}-1\right)}\right) \\ &= q^{+}\left(\bar{\theta}\right)\left(\frac{\eta\bar{\theta}^{\eta-1}\theta^{1-\eta}+\left(1-\eta\right)\bar{\theta}^{\eta}\theta^{1-\eta}-1}{\left(\bar{\theta}-\bar{\theta}^{\eta}\theta^{1-\eta}\right)\left(\bar{\theta}-1\right)}\right). \end{aligned}$$

The denominator is always positive. Hence the sign inherits the sign of the numerator. The numerator satisfies:

$$\eta\bar{\theta}^{\eta-1}\theta^{1-\eta}+\left(1-\eta\right)\bar{\theta}^{\eta}\theta^{1-\eta}-1 = \bar{\theta}^{\eta}\theta^{1-\eta}\left(1-\eta\left(1-\bar{\theta}^{-1}\right)\right)-1.$$

Thus, the numerator is positive if:

$$\theta^{1-\eta}\left(\left(1-\eta\right)\bar{\theta}+\eta\right)>\bar{\theta}^{1-\eta}.$$

Likewise:

$$\begin{aligned} q_{\bar{\theta}}^{-}\left(\bar{\theta}\right) &= q^{-}\left(\bar{\theta}\right)\left(\frac{1-\eta\bar{\theta}^{\eta-1}\theta^{-\eta}}{\bar{\theta}-\bar{\theta}^{\eta}\theta^{-\eta}}-\frac{1}{\bar{\theta}-1}\right) \\ &= q^{-}\left(\bar{\theta}\right)\left(\frac{\bar{\theta}-\eta\bar{\theta}^{\eta}\theta^{-\eta}-1+\eta\bar{\theta}^{\eta-1}\theta^{-\eta}-\left(\bar{\theta}-\bar{\theta}^{\eta}\theta^{-\eta}\right)}{\left(\bar{\theta}-\bar{\theta}^{\eta}\theta^{-\eta}\right)\left(\bar{\theta}-1\right)}\right) \\ &= q^{-}\left(\bar{\theta}\right)\left(\frac{\eta\bar{\theta}^{\eta-1}\theta^{-\eta}+\left(1-\eta\right)\bar{\theta}^{\eta}\theta^{-\eta}-1}{\left(\bar{\theta}-\bar{\theta}^{\eta}\theta^{-\eta}\right)\left(\bar{\theta}-1\right)}\right). \end{aligned}$$

The numerator satisfies:

$$\eta\bar{\theta}^{\eta-1}\theta^{-\eta}+\left(1-\eta\right)\bar{\theta}^{\eta}\theta^{-\eta}-1 = \bar{\theta}^{\eta}\theta^{-\eta}\left(1-\eta\left(1-\bar{\theta}^{-1}\right)\right)-1.$$

The denominator is always positive. Hence the sign inherits the sign of the numerator. The numerator is positive if:

$$\theta^{-\eta}\left(\left(1-\eta\right)\bar{\theta}+\eta\right)>\bar{\theta}^{1-\eta}.$$

We must inspect the validity of the following conditions:

- $\theta^{1-\eta}\left(\left(1-\eta\right)\bar{\theta}+\eta\right)-\bar{\theta}^{1-\eta}>0$
- $\theta^{-\eta}\left(\left(1-\eta\right)\bar{\theta}+\eta\right)-\bar{\theta}^{1-\eta}>0$

Again, we break the analysis into cases:

Case $\theta < 1$. When $\theta < 1$, $\bar{\theta} \in [0, \theta]$. Consider the first condition. Then, at $\bar{\theta} = 0$, the condition is satisfied. However, it is violated at $\bar{\theta} = \theta$. By continuity, the derivative changes for some $\bar{\theta}$. This implies that $q^{+}_{\bar{\theta}}\left(\bar{\theta}\right)$ switches sign for some $\bar{\theta}$ when $\theta < 1$. Hence, $\chi^{+}_{\bar{\lambda}}$ is not monotone.

Now consider the second condition. Then, at $\bar{\theta} = 0$, the condition is satisfied. At $\bar{\theta} = \theta$, the condition is also satisfied since:

$$\eta\left(1-\theta\right)>0.$$

The derivative of the term in the left is:

$$\left(1-\eta\right)\left(\theta^{-\eta}-\bar{\theta}^{-\eta}\right)<0,$$

thus, the condition is satisfied for all values. This implies that $q^{-}_{\bar{\theta}}\left(\bar{\theta}\right)>0$ for $\theta < 1$. Hence, $\chi^{-}_{\bar{\lambda}}$ is monotone. Since:

$$\chi^{-}_{\bar{\lambda}}=q^{-}_{\bar{\theta}}\left(\bar{\theta}\right)\times\bar{\theta}_{\bar{\lambda}},$$

we have that $\chi^{-}_{\bar{\lambda}}$ is monotone decreasing because $\bar{\theta}_{\bar{\lambda}} < 0$.

Case $\theta > 1$. When $\theta > 1$, $\bar{\theta} \in [\theta, \infty]$. Consider the first condition. Then, at $\bar{\theta} = \theta$, the condition is satisfied. The derivative of the term in the left is:

$$\left(1-\eta\right)\left(\theta^{1-\eta}-\bar{\theta}^{-\eta}\right)>0.$$

Hence, the condition is always satisfied. Then, this implies that $q^{+}_{\bar{\theta}}\left(\bar{\theta}\right)>0$ for $\theta > 1$. Hence, $\chi^{-}_{\bar{\lambda}}$ is monotone. Since:

$$\chi^{+}_{\bar{\lambda}}=q^{+}_{\bar{\theta}}\left(\bar{\theta}\right)\times\bar{\theta}_{\bar{\lambda}},$$

we have that $\chi^{+}_{\bar{\lambda}}$ is monotone increasing because $\bar{\theta}_{\bar{\lambda}} > 0$.

Now consider the second condition. Then, at $\bar{\theta} = \theta$, the condition is violated. The derivative of the term in the left is:

$$\left(1-\eta\right)\left(\theta^{-\eta}-\bar{\theta}^{-\eta}\right)>0,$$

thus, the condition must be satisfied for some values. By continuity, the derivative changes for some $\bar{\theta}$. This implies that $q^{-}_{\bar{\theta}}\left(\bar{\theta}\right)$ switches sign for some $\bar{\theta}$ when $\theta > 1$. Hence, $\chi^{-}_{\bar{\lambda}}$ is not monotone.

Case $\theta = 1$. In this case, λ does not have an effect on $\bar{R}^f = (1-\eta)R^w + \eta R^m$. However, note that:

$$\chi^- = (R^w - R^m) - \Psi^-(R^w - \overline{R}^f), \qquad \chi^+ = \Psi^+(\overline{R}^f - R^m). \tag{30}$$

Thus,

$$\chi^-_{\bar{\lambda}} = -\Psi^-_{\bar{\lambda}}(R^w - \overline{R}^f) = -\eta\Psi^-_{\bar{\lambda}}(R^w - R^m) < 0, \qquad \chi^+_{\bar{\lambda}} = \Psi^+_{\bar{\lambda}}(\overline{R}^f - R^m) = (1-\eta)\,\Psi^+_{\bar{\lambda}}(R^w - R^m) > 0.$$

Average Interest Rate. We now consider the derivative of the average interest rate. Recall that the endogenous bargaining power is:

$$\phi(\theta) = \begin{cases} 1 - \frac{\bar{\theta} - \theta^{1-\eta}\bar{\theta}^{\eta}}{\bar{\theta} - \theta} & \text{if } \theta > 1 \\ \eta & \text{if } \theta = 1 \\ 1 - \frac{\theta^{-1} - \theta^{-\eta}\bar{\theta}^{\eta-1}}{\bar{\theta}^{-1} - \theta^{-1}} & \text{if } \theta < 1. \end{cases}$$

For $\theta > 1$, we have that:

$$\phi(\theta) = 1 - q^+\left(\bar{\theta}\right).$$

Thus, in this case, we have shown above that:

$$q^+_{\bar{\theta}}\left(\bar{\theta}\right) > 0.$$

This implies that

$$\phi_\lambda(\theta) = -\frac{\partial}{\partial\bar{\theta}}\left[q^+\left(\bar{\theta}\right)\right] \cdot \frac{\partial}{\partial\lambda}\left[\bar{\theta}\right] < 0.$$

For $\theta < 1$, we exploit the symmetry property:

$$\phi(\theta) = 1 - \phi\left(\theta^{-1}\right)$$

Thus,

$$\phi_\lambda(\theta) > 0.$$

E. Proof of Proposition (11)

Derivative of the dispersion of the interbank market rate w.r.t. θ:

Case $\theta > 1$.

$$
\begin{aligned}
Q_{min}^{max} &= max_\tau \left\{ r_\tau^f \right\} - min \left\{ r_\tau^f \right\} = r_1^f - r_0^f = -((-R^m + R^w)(1-\eta)) \\
&+ \frac{e^{-\lambda}(-R^m + R^w)\eta \left(1 + e^\lambda(-1+\theta) - \left(1 + e^\lambda(-1+\theta)\right)^\eta \theta^{1-\eta}\right)}{-1+\theta} \\
&+ \frac{e^{-\lambda}(-R^m + R^w)(1-\eta) \left(1 + e^\lambda(-1+\theta) - \left(1 + e^\lambda(-1+\theta)\right)^\eta \theta^{-\eta}\right)}{-1+\theta} \\
&= \frac{-e^{-\lambda}(R^w - R^m)\theta^{-\eta} \left(-\left((-1+\eta)\left(1+e^\lambda(-1+\theta)\right)^\eta\right) + \eta\left(1+e^\lambda(-1+\theta)\right)^\eta \theta + \left(-1+e^\lambda \eta\right)\theta^\eta - e^\lambda \eta \theta^{1+\eta}\right)}{-1+\theta} \\
&= \frac{e^{-\lambda}(R^w - R^m)\theta^{-\eta} \left(\left((-1+\eta)\left(1+e^\lambda(-1+\theta)\right)^\eta\right) - \eta\left(1+e^\lambda(-1+\theta)\right)^\eta \theta - \left(-1+e^\lambda \eta\right)\theta^\eta + e^\lambda \eta \theta^{1+\eta}\right)}{\theta - 1}
\end{aligned}
$$

And:

$$
\begin{aligned}
\frac{\partial Q_{\text{min}}^{\text{max}}}{\partial \theta} &= -\frac{e^{-\lambda}(-R^m + R^w)\eta \left(1 + e^\lambda(-1+\theta) - \left(1 + e^\lambda(-1+\theta)\right)^\eta \theta^{1-\eta}\right)}{(-1+\theta)^2} \\
&+ \frac{e^{-\lambda}(-R^m + R^w)(1-\eta) \left(e^\lambda + \eta\left(1 + e^\lambda(-1+\theta)\right)^\eta \theta^{-1-\eta} - e^\lambda \eta \left(1 + e^\lambda(-1+\theta)\right)^{-1+\eta} \theta^{-\eta}\right)}{-1+\theta} \\
&- \frac{e^{-\lambda}(-R^m + R^w)(1-\eta) \left(1 + e^\lambda(-1+\theta) - \left(1 + e^\lambda(-1+\theta)\right)^\eta \theta^{-\eta}\right)}{(-1+\theta)^2} \\
&+ \frac{e^{-\lambda}(-R^m + R^w)\eta \left(e^\lambda - e^\lambda \eta \left(1 + e^\lambda(-1+\theta)\right)^{-1+\eta} \theta^{1-\eta} - (1-\eta)\left(1 + e^\lambda(-1+\theta)\right)^\eta \theta^{-\eta}\right)}{-1+\theta}
\end{aligned}
$$

Since $\theta > 1$, $\lambda \in [0,1]$, $\eta \in [0,1]$ and $(R^w - R^m) > 0$, we have:

$$
\frac{\partial Q_{\text{min}}^{\text{max}}}{\partial \theta} = \frac{\underbrace{e^{-\lambda}(R^m - R^w)\theta^{-1-\eta}\left(1 + e^\lambda(-1+\theta)\right)}_{<0} F(\theta,\lambda,\eta)}{\underbrace{\left(1 + e^\lambda(-1+\theta)\right)(-1+\theta)^2}_{>0}}
$$

Where:

$$
F(\theta,\lambda,\eta) = \theta^{1+\eta} + \left(1 + e^\lambda(-1+\theta)\right)^{\eta-1} \left(-\theta + (-1+\theta)\left(e^\lambda(-1+\eta)(\eta(-1+\theta)+\theta) + \eta(-1+\eta-\eta\theta)\right)\right)
$$

And it's important to mention that this function $F(\theta,\lambda,\eta)$ is increasing in η, so if we take the following limit $\eta \to 1$:

$$
F(\theta,\lambda,\eta) < \max F(\theta,\lambda,\eta) \approx \lim_{\eta \to 1} F(\theta,\lambda,\eta) = \theta^2 + (-\theta + (-1+\theta)(-\theta)) = \theta^2 - \theta^2 = 0
$$

And therefore:

$$
\frac{\partial Q_{\text{min}}^{\text{max}}}{\partial \theta} = \frac{e^{-\lambda}(R^m - R^w)\theta^{-1-\eta}\left(1+e^\lambda(-1+\theta)\right)F(\theta,\lambda,\eta)}{\left(1+e^\lambda(-1+\theta)\right)(-1+\theta)^2} > 0
$$

Case $\theta < 1$.

$$Q_{min}^{max} = max_{\tau}\left\{r_{\tau}^{f}\right\} - min\left\{r_{\tau}^{f}\right\} = r_0^f - r_1^f = (-R^m + R^w)(1-\eta) - \frac{(-R^m+R^w)\eta\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}} - \left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^{\eta}\theta^{1-\eta}\right)}{-1+\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}}$$

$$-\frac{(-R^m+R^w)(1-\eta)\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}} - \left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^{\eta}\theta^{-\eta}\right)}{-1+\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}}$$

Where:

$$\frac{\partial Q_{\min}^{\max}}{\partial \theta} = -\frac{(-R^m+R^w)\eta\left(-\frac{e^{\lambda}(1-\theta)}{\theta^2} - \frac{e^{\lambda}}{\theta}\right)\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}} - \left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^{\eta}\theta^{1-\eta}\right)}{\left(-1+\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^2\left(1+\frac{e^{\lambda}(1-\theta)}{\theta}\right)^2}$$

$$-\frac{(-R^m+R^w)(1-\eta)\left(-\frac{e^{\lambda}(1-\theta)}{\theta^2} - \frac{e^{\lambda}}{\theta}\right)\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}} - \left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^{\eta}\theta^{-\eta}\right)}{\left(-1+\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^2\left(1+\frac{e^{\lambda}(1-\theta)}{\theta}\right)^2}$$

$$-\frac{(-R^m+R^w)\eta\left(-\frac{-\frac{e^{\lambda}(1-\theta)}{\theta^2}-\frac{e^{\lambda}}{\theta}}{\left(1+\frac{e^{\lambda}(1-\theta)}{\theta}\right)^2} + \eta\left(-\frac{e^{\lambda}(1-\theta)}{\theta^2} - \frac{e^{\lambda}}{\theta}\right)\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^{1+\eta}\theta^{1-\eta} - (1-\eta)\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^{\eta}\theta^{-\eta}\right)}{-1+\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}}$$

$$-\frac{(-R^m+R^w)(1-\eta)\left(-\frac{-\frac{e^{\lambda}(1-\theta)}{\theta^2}-\frac{e^{\lambda}}{\theta}}{\left(1+\frac{e^{\lambda}(1-\theta)}{\theta}\right)^2} + \eta\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^{\eta}\theta^{-1-\eta} + \eta\left(-\frac{e^{\lambda}(1-\theta)}{\theta^2} - \frac{e^{\lambda}}{\theta}\right)\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^{1+\eta}\theta^{-\eta}\right)}{-1+\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}}$$

$$= -\frac{e^{-\lambda}(R^m-R^w)\theta^{-\eta}\left(-\left(\frac{1}{1+e^{\lambda}\left(-1+\frac{1}{\theta}\right)}\right)^{\eta} + \left(-1+e^{\lambda}\right)\eta\left(\frac{1}{1+e^{\lambda}\left(-1+\frac{1}{\theta}\right)}\right)^{\eta}(2+\eta(-1+\theta)-\theta)(-1+\theta)+\theta^{\eta}\right)}{(-1+\theta)^2}$$

Since $\theta < 1$, $\lambda \in [0,1]$, $\eta \in [0,1]$ and $(R^w - R^m) > 0$, we have:

$$\frac{\partial Q_{\min}^{\max}}{\partial \theta} = \frac{\underbrace{e^{-\lambda}(R^w-R^m)\theta^{-\eta}\left(\frac{1}{1+e^{\lambda}\left(-1+\frac{1}{\theta}\right)}\right)^{\eta}}_{>0}\left(-1+\left(-1+e^{\lambda}\right)\eta(2+\eta(-1+\theta)-\theta)(-1+\theta)+\theta^{\eta}\right)}{\underbrace{(-1+\theta)^2}_{>0}}$$

We need to analyze the second factor of the numerator. So we can begin with:

$$0 < \min(2-\eta+\theta\eta-\theta) = 1 < (2-\eta+\theta\eta-\theta)$$

Also we know that $\left(-1+e^{\lambda}\right)\eta > 0$ and $-1+\theta < 0$. So we have:

$$\left(-1+e^{\lambda}\right)\eta(2-\eta+\theta\eta-\theta)(-1+\theta) < 0$$

$$\left(-1+e^{\lambda}\right)\eta(2+\eta(-1+\theta)-\theta)(-1+\theta) + \left(\underbrace{\theta^{\eta}-1}_{<0}\right) < \left(-1+e^{\lambda}\right)\eta(2+\eta(-1+\theta)-\theta)(-1+\theta) < 0$$

And therefore the second factor of the numerator will be:

$$-1+\left(-1+e^{\lambda}\right)\eta(2+\eta(-1+\theta)-\theta)(-1+\theta)+\theta^{\eta} < 0$$

Finally, we have:

$$\frac{\partial Q_{\min}^{\max}}{\partial\theta} = \frac{e^{-\lambda}(R^w-R^m)\theta^{-\eta}\left(\frac{1}{1+e^{\lambda}\left(-1+\frac{1}{\theta}\right)}\right)^{\eta}\left(-1+\left(-1+e^{\lambda}\right)\eta(2+\eta(-1+\theta)-\theta)(-1+\theta)+\theta^{\eta}\right)}{(-1+\theta)^2} < 0$$

Case $\theta = 1$. As we can see in the last two cases, if we take the following limit $\theta \to 1$, we have that $\frac{\partial Q_{\min}^{\max}}{\partial\theta} \to 0$. This is because when the market tightness disappears, we have that the interbank rate is always the rate on reserves (a constant).

Derivative of the dispersion of the interbank market rate w.r.t. λ:

Case $\theta > 1$.

$$\begin{aligned}
\frac{\partial Q_{\min}^{\max}}{\partial\lambda} &= -\frac{e^{-\lambda}(-R^m+R^w)\eta\left(1+e^{\lambda}(-1+\theta)-\left(1+e^{\lambda}(-1+\theta)\right)^{\eta}\theta^{1-\eta}\right)}{-1+\theta} \\
&+ \frac{e^{-\lambda}(-R^m+R^w)\eta\left(e^{\lambda}(-1+\theta)-e^{\lambda}\eta\left(1+e^{\lambda}(-1+\theta)\right)^{-1+\eta}(-1+\theta)\theta^{1-\eta}\right)}{-1+\theta} \\
&- \frac{e^{-\lambda}(-R^m+R^w)(1-\eta)\left(1+e^{\lambda}(-1+\theta)-\left(1+e^{\lambda}(-1+\theta)\right)^{\eta}\theta^{-\eta}\right)}{-1+\theta} \\
&+ \frac{e^{-\lambda}(-R^m+R^w)(1-\eta)\left(e^{\lambda}(-1+\theta)-e^{\lambda}\eta\left(1+e^{\lambda}(-1+\theta)\right)^{-1+\eta}(-1+\theta)\theta^{-\eta}\right)}{-1+\theta} \\
&= \frac{e^{-\lambda}(R^m-R^w)\theta^{-\eta}\left(\left(1+e^{\lambda}(-1+\theta)\right)^{\eta}\left(-1+e^{\lambda}(-1+\eta)(-1+\theta)\right)(1+\eta(-1+\theta))+\left(1+e^{\lambda}(-1+\theta)\right)\theta^{\eta}\right)}{-1+e^{\lambda}(-1+\theta)^2+\theta}
\end{aligned}$$

Since $\theta > 1$, $\lambda \in [0,1]$, $\eta \in [0,1]$ and $(R^w - R^m) > 0$, we have:

$$\frac{\partial Q_{\min}^{\max}}{\partial\lambda} = \frac{\underbrace{e^{-\lambda}(R^m-R^w)\theta^{-\eta}\left(1+e^{\lambda}(-1+\theta)\right)}_{<0}\left(\left(1+e^{\lambda}(-1+\theta)\right)^{\eta-1}\left(-1+e^{\lambda}(-1+\eta)(-1+\theta)\right)(1+\eta(-1+\theta))+\theta^{\eta}\right)}{\underbrace{-1+e^{\lambda}(-1+\theta)^2+\theta}_{>0}}$$

Where:

$$G(\theta,\lambda,\eta)=\left(1+e^{\lambda}(-1+\theta)\right)^{\eta-1}\left(-1+e^{\lambda}(-1+\eta)(-1+\theta)\right)(1+\eta(-1+\theta))+\theta^{\eta}$$

And it's important to mention that the last function is increasing in η, so if we take the following limit $\eta\to 1$:

$$G(\theta,\lambda,\eta)<\max G(\theta,\lambda,\eta)\approx\lim_{\eta\to 1}G(\theta,\lambda,\eta)=\left(1+e^{\lambda}(-1+\theta)\right)^{0}(-1)(\theta)+\theta=-\theta+\theta=0$$

And therefore:

$$\frac{\partial Q_{\min}^{\max}}{\partial\lambda}=\frac{e^{-\lambda}(R^m-R^w)\theta^{-\eta}\left(1+e^{\lambda}(-1+\theta)\right)G(\theta,\lambda,\eta)}{-1+e^{\lambda}(-1+\theta)^2+\theta}>0$$

Case $\theta<1$.

$$\begin{aligned}
\frac{\partial Q_{\min}^{\max}}{\partial\lambda}=&-\frac{(-R^m+R^w)(1-\eta)\left(-\frac{e^{\lambda}(1-\theta)}{\left(1+\frac{e^{\lambda}(1-\theta)}{\theta}\right)^2\theta}+e^{\lambda}\eta\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^{1+\eta}(1-\theta)\theta^{-1-\eta}\right)}{-1+\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}}\\
&-\frac{e^{\lambda}(-R^m+R^w)\eta(1-\theta)\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}-\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^{\eta}\theta^{1-\eta}\right)}{\left(-1+\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^2\left(1+\frac{e^{\lambda}(1-\theta)}{\theta}\right)^2\theta}\\
&-\frac{e^{\lambda}(-R^m+R^w)(1-\eta)(1-\theta)\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}-\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^{\eta}\theta^{-\eta}\right)}{\left(-1+\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^2\left(1+\frac{e^{\lambda}(1-\theta)}{\theta}\right)^2\theta}\\
&-\frac{(-R^m+R^w)\eta\left(-\frac{e^{\lambda}(1-\theta)}{\left(1+\frac{e^{\lambda}(1-\theta)}{\theta}\right)^2\theta}+e^{\lambda}\eta\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^{1+\eta}(1-\theta)\theta^{-\eta}\right)}{-1+\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}}\\
=&\frac{e^{-\lambda}(R^m-R^w)\theta^{-\eta}\left(-\left(\frac{1}{1+e^{\lambda}\left(-1+\frac{1}{\theta}\right)}\right)^{\eta}(1+\eta(-1+\theta))\left(e^{\lambda}\eta(-1+\theta)-\theta\right)-\theta^{1+\eta}\right)}{-1+\theta}
\end{aligned}$$

Since $\theta<1$, $\lambda\in[0,1]$, $\eta\in[0,1]$ and $(R^w-R^m)>0$, we have:

$$\frac{\partial Q_{\min}^{\max}}{\partial\lambda}=\frac{\underbrace{e^{-\lambda}(R^m-R^w)\theta^{-\eta}}_{<0}\left(-\left(\frac{1}{1+e^{\lambda}\left(-1+\frac{1}{\theta}\right)}\right)^{\eta}(1+\eta(-1+\theta))\left(e^{\lambda}\eta(-1+\theta)-\theta\right)-\theta^{1+\eta}\right)}{\underbrace{-1+\theta}_{<0}}$$

Where:

$$H(\theta,\lambda,\eta)=\underbrace{-\left(\frac{1}{1+e^{\lambda}\left(-1+\frac{1}{\theta}\right)}\right)^{\eta}}_{\in(-1,0)}\underbrace{(1+\eta(-1+\theta))}_{\in(0,1)}\underbrace{\left(e^{\lambda}\eta(-1+\theta)-\theta\right)}_{<0}-\underbrace{\theta^{1+\eta}}_{\in(0,1)}$$

All the factors of the first term in the last equation are decreasing w.r.t. η and the second term is also decreasing w.r.t. η. So all the terms are decreasing and bounded, and therefore the limit w.r.t.η is:

$$H(\theta, \lambda, \eta) < \min H(\theta, \lambda, \eta) \approx \lim_{\eta \to 1} H(\theta, \lambda, \eta) = 0$$

And therefore:

$$\frac{\partial Q_{\min}^{\max}}{\partial \lambda} = \frac{e^{-\lambda}(R^m - R^w)\theta^{-\eta} H(\theta,\lambda,\eta)}{-1+\theta} > 0$$

Case $\theta = 1$. As we can see in the last two cases, if we take the following limit $\theta \to 1$, we have that $\frac{\partial Q_{\min}^{\max}}{\partial \lambda} \to 0$.This is because when the market tightness disappears, we have that the interbank rate is always the rate on reserves (a constant).

F. Proof of Proposition ??

Let us study the three possible cases,

As $\theta \to 1$:

Because $\theta = 1$, observe that

$$\Psi^{+} = 1 - e^{-\bar{\lambda}} \qquad \text{and} \qquad \Psi^{-} = 1 - e^{-\bar{\lambda}}.$$

Therefore,

$$\lim_{\theta \to 1} \left\{ \chi^{+} \right\} = \lim_{\theta \to 1} \left\{ (1-\eta)\left(r^{w} - r^{m}\right)\left(1 - e^{-\bar{\lambda}}\right) \right\} = (1-\eta)\left(1 - e^{-\bar{\lambda}}\right)\left(r^{w} - r^{m}\right).$$

Also,

$$\lim_{\theta \to 1} \left\{ \chi^{-} \right\} = \lim_{\theta \to 1} \left\{ \left(r^{w} - r^{m}\right)\left(1 - \eta\left(1 - e^{-\bar{\lambda}}\right)\right) \right\} = \left(1 - \eta\left(1 - e^{-\bar{\lambda}}\right)\right)\left(r^{w} - r^{m}\right).$$

Finally,

$$\lim_{\theta \to 1} \left\{ \bar{r}^{f} \right\} = \lim_{\theta \to 1} \left\{ (1-\eta) r^{w} + \eta r^{m} \right\} = (1-\eta) r^{w} + \eta r^{m}.$$

As $\theta \to \infty$:

Because $\theta > 1$, observe that

$$\Psi^{+} = 1 - e^{-\bar{\lambda}} \qquad \text{and} \qquad \Psi^{-} = 0.$$

Now, before proceeding, realise that

$$\lim_{\theta \to \infty} \left\{ \frac{\bar{\theta}}{\theta} \right\} = \lim_{\theta \to \infty} \left\{ \frac{1 + (\theta - 1) e^{\bar{\lambda}}}{\theta} \right\} = \lim_{\theta \to \infty} \left\{ \frac{1}{\theta} + \left(1 - \frac{1}{\theta}\right) e^{\bar{\lambda}} \right\} = e^{\bar{\lambda}}.$$

Therefore,

$$
\begin{aligned}
\lim_{\theta\to\infty}\left\{\chi^{+}\right\} &= \lim_{\theta\to\infty}\left\{(r^{w}-r^{m})\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\left(\frac{\theta^{\eta}\bar{\theta}^{1-\eta}-\theta}{\bar{\theta}-1}\right)\right\}\\
&= \lim_{\theta\to\infty}\left\{(r^{w}-r^{m})\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\left(\frac{\left(\frac{\bar{\theta}}{\theta}\right)^{1-\eta}-1}{\frac{\bar{\theta}}{\theta}-\frac{1}{\theta}}\right)\right\}\\
&= (r^{w}-r^{m})\left(\lim_{\theta\to\infty}\left\{\frac{\bar{\theta}}{\theta}\right\}\right)^{\eta}\left(\frac{\left(\lim_{\theta\to\infty}\left\{\frac{\bar{\theta}}{\theta}\right\}\right)^{1-\eta}-1}{\lim_{\theta\to\infty}\left\{\frac{\bar{\theta}}{\theta}\right\}-\lim_{\theta\to\infty}\left\{\frac{1}{\theta}\right\}}\right)\\
&= (r^{w}-r^{m})\,e^{\eta\bar{\lambda}}\left(\frac{e^{\bar{\lambda}(1-\eta)}-1}{e^{\bar{\lambda}}}\right)\\
&= (r^{w}-r^{m})\left(1-e^{-\bar{\lambda}(1-\eta)}\right).
\end{aligned}
$$

Also,

$$
\begin{aligned}
\lim_{\theta\to\infty}\left\{\chi^{-}\right\} &= \lim_{\theta\to\infty}\left\{(r^{w}-r^{m})\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\left(\frac{\theta^{\eta}\bar{\theta}^{1-\eta}-1}{\bar{\theta}-1}\right)\right\}\\
&= \lim_{\theta\to\infty}\left\{(r^{w}-r^{m})\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\left(\frac{\left(\frac{\bar{\theta}}{\theta}\right)^{1-\eta}-\frac{1}{\theta}}{\frac{\bar{\theta}}{\theta}-\frac{1}{\theta}}\right)\right\}\\
&= (r^{w}-r^{m})\left(\lim_{\theta\to\infty}\left\{\frac{\bar{\theta}}{\theta}\right\}\right)^{\eta}\left(\frac{\left(\lim_{\theta\to\infty}\left\{\frac{\bar{\theta}}{\theta}\right\}\right)^{1-\eta}-\lim_{\theta\to\infty}\left\{\frac{1}{\theta}\right\}}{\lim_{\theta\to\infty}\left\{\frac{\bar{\theta}}{\theta}\right\}-\lim_{\theta\to\infty}\left\{\frac{1}{\theta}\right\}}\right)\\
&= (r^{w}-r^{m})\,e^{\eta\bar{\lambda}}\left(\frac{e^{\bar{\lambda}(1-\eta)}}{e^{\bar{\lambda}}}\right)\\
&= r^{w}-r^{m}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\lim_{\theta\to\infty}\left\{\bar{r}^{f}\right\} &= \lim_{\theta\to\infty}\left\{r^{w}-\left(\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}-1\right)\left(\frac{\theta}{\theta-1}\right)\left(\frac{r^{w}-r^{m}}{e^{\bar{\lambda}}-1}\right)\right\}\\
&= r^{w}-\left(\left(\lim_{\theta\to\infty}\left\{\frac{\bar{\theta}}{\theta}\right\}\right)^{\eta}-1\right)\left(\frac{1}{1-\lim_{\theta\to\infty}\left\{\frac{1}{\theta}\right\}}\right)\left(\frac{r^{w}-r^{m}}{e^{\bar{\lambda}}-1}\right)\\
&= r^{w}-\left(e^{\bar{\lambda}\eta}-1\right)\left(\frac{r^{w}-r^{m}}{e^{\bar{\lambda}}-1}\right)\\
&= \left(\frac{e^{\bar{\lambda}}-e^{\bar{\lambda}\eta}}{e^{\bar{\lambda}}-1}\right)r^{w}+\left(\frac{e^{\bar{\lambda}\eta}-1}{e^{\bar{\lambda}}-1}\right)r^{m}.
\end{aligned}
$$

As $\theta \to 0:$

Because $\theta < 1$, observe that

$$\Psi^{+} = 0 \qquad \text{and} \qquad \Psi^{-} = 1 - e^{-\bar{\lambda}}.$$

Now, before proceeding, realise that

$$\lim_{\theta \to 0} \left\{ \frac{\bar{\theta}}{\theta} \right\} = \lim_{\theta \to 0} \left\{ \frac{\frac{\theta}{\theta + (1-\theta) e^{\bar{\lambda}}}}{\theta} \right\} = \lim_{\theta \to 0} \left\{ \frac{1}{\theta + (1-\theta) e^{\bar{\lambda}}} \right\} = e^{-\bar{\lambda}}.$$

Therefore,

$$\begin{aligned}
\lim_{\theta \to 0} \left\{ \chi^{+} \right\} &= \lim_{\theta \to 0} \left\{ (r^{w} - r^{m}) \left(\frac{\bar{\theta}}{\theta} \right)^{\eta} \left(\frac{\theta^{\eta} \bar{\theta}^{1-\eta} - \theta}{\bar{\theta} - 1} \right) \right\} \\
&= (r^{w} - r^{m}) \left(\lim_{\theta \to 0} \left\{ \frac{\bar{\theta}}{\theta} \right\} \right)^{\eta} \left(\frac{\lim_{\theta \to 0} \left\{ \theta^{\eta} \bar{\theta}^{1-\eta} \right\} - \lim_{\theta \to 0} \left\{ \theta \right\}}{\lim_{\theta \to 0} \left\{ \bar{\theta} \right\} - 1} \right) \\
&= 0.
\end{aligned}$$

Also,

$$\begin{aligned}
\lim_{\theta \to 0} \left\{ \chi^{-} \right\} &= \lim_{\theta \to 0} \left\{ (r^{w} - r^{m}) \left(\frac{\bar{\theta}}{\theta} \right)^{\eta} \left(\frac{\theta^{\eta} \bar{\theta}^{1-\eta} - 1}{\bar{\theta} - 1} \right) \right\} \\
&= (r^{w} - r^{m}) \left(\lim_{\theta \to 0} \left\{ \frac{\bar{\theta}}{\theta} \right\} \right)^{\eta} \left(\frac{\lim_{\theta \to 0} \left\{ \theta^{\eta} \bar{\theta}^{1-\eta} \right\} - 1}{\lim_{\theta \to 0} \left\{ \bar{\theta} \right\} - 1} \right) \\
&= (r^{w} - r^{m}) \, e^{-\bar{\lambda} \eta}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\lim_{\theta \to 0} \left\{ \overline{r}^{f} \right\} &= \lim_{\theta \to 0} \left\{ r^{w} - \left(\frac{1}{1-\theta} \right) \left(\frac{\theta}{\bar{\theta}} \right) \left(1 - \left(\frac{\bar{\theta}}{\theta} \right)^{\eta} \right) \left(\frac{r^{w} - r^{m}}{e^{\bar{\lambda}} - 1} \right) \right\} \\
&= r^{w} - \left(\frac{1}{1 - \lim_{\theta \to 0} \left\{ \theta \right\}} \right) \left(\lim_{\theta \to 0} \left\{ \frac{\theta}{\bar{\theta}} \right\} \right) \left(1 - \left(\lim_{\theta \to 0} \left\{ \frac{\bar{\theta}}{\theta} \right\} \right)^{\eta} \right) \left(\frac{r^{w} - r^{m}}{e^{\bar{\lambda}} - 1} \right) \\
&= r^{w} - e^{\bar{\lambda}} \left(1 - e^{-\bar{\lambda} \eta} \right) \left(\frac{r^{w} - r^{m}}{e^{\bar{\lambda}} - 1} \right) \\
&= r^{w} - \left(e^{\bar{\lambda}} - e^{\bar{\lambda}(1-\eta)} \right) \left(\frac{r^{w} - r^{m}}{e^{\bar{\lambda}} - 1} \right) \\
&= \left(\frac{e^{\bar{\lambda}(1-\eta)} - 1}{e^{\bar{\lambda}} - 1} \right) r^{w} + \left(\frac{e^{\bar{\lambda}} - e^{\bar{\lambda}(1-\eta)}}{e^{\bar{\lambda}} - 1} \right) r^{m}.
\end{aligned}$$

G. Proof of Proposition ??

G.1 Matching Efficiency

Let us first start with the limiting properties of the matching efficiency parameter.

Let us study the three possible cases,

$\underline{\theta = 1}$:

Because $\theta = 1$, observe that

$$\lim_{\bar{\lambda}\to\infty}\left\{\Psi^{+}\right\} = \lim_{\bar{\lambda}\to\infty}\left\{1-e^{-\bar{\lambda}}\right\} = 1, \qquad \text{and}$$
$$\lim_{\bar{\lambda}\to\infty}\left\{\Psi^{-}\right\} = \lim_{\bar{\lambda}\to\infty}\left\{1-e^{-\bar{\lambda}}\right\} = 1.$$

Therefore,

$$\begin{aligned}\lim_{\bar{\lambda}\to\infty}\left\{\chi^{+}\right\} &= \lim_{\theta\to\infty}\left\{(1-\eta)\left(1-e^{-\bar{\lambda}}\right)(r^{w}-r^{m})\right\}\\ &= (1-\eta)\left(1-\lim_{\bar{\lambda}\to\infty}\left\{e^{-\bar{\lambda}}\right\}\right)(r^{w}-r^{m})\\ &= (r^{w}-r^{m})(1-\eta).\end{aligned}$$

Also,

$$\begin{aligned}\lim_{\bar{\lambda}\to\infty}\left\{\chi^{-}\right\} &= \lim_{\theta\to\infty}\left\{\left(1-\eta\left(1-e^{-\bar{\lambda}}\right)\right)(r^{w}-r^{m})\right\}\\ &= \left(1-\eta\left(1-\lim_{\bar{\lambda}\to\infty}\left\{e^{-\bar{\lambda}}\right\}\right)\right)(r^{w}-r^{m})\\ &= (r^{w}-r^{m})(1-\eta).\end{aligned}$$

Finally,

$$\lim_{\bar{\lambda}\to\infty}\left\{\bar{r}^{f}\right\} = \lim_{\bar{\lambda}\to\infty}\left\{(1-\eta)r^{w}+\eta r^{m}\right\} = (1-\eta)r^{w}+\eta r^{m}.$$

$\underline{\theta > 1}$:

Because $\theta > 1$, observe that

$$\lim_{\bar{\lambda}\to\infty}\left\{\Psi^{+}\right\} = \lim_{\bar{\lambda}\to\infty}\left\{1-e^{-\bar{\lambda}}\right\} = 1, \qquad \text{and}$$
$$\lim_{\bar{\lambda}\to\infty}\left\{\Psi^{-}\right\} = \lim_{\bar{\lambda}\to\infty}\left\{\frac{1-e^{\bar{-}\lambda}}{\theta}\right\} = \frac{1}{\theta}.$$

Now, before proceeding, realise that

$$\lim_{\bar{\lambda}\to\infty}\left\{\frac{\theta}{\bar{\bar{\theta}}}\right\} = \lim_{\bar{\lambda}\to\infty}\left\{\frac{\theta}{1+(\theta-1)e^{\bar{\lambda}}}\right\} = 0.$$

Therefore,

$$
\begin{aligned}
\lim_{\bar{\lambda}\to\infty}\left\{\chi^{+}\right\} &= \lim_{\bar{\lambda}\to\infty}\left\{(r^{w}-r^{m})\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\left(\frac{\theta^{\eta}\bar{\theta}^{1-\eta}-\theta}{\bar{\theta}-1}\right)\right\} \\
&= \lim_{\bar{\lambda}\to\infty}\left\{(r^{w}-r^{m})\left(\frac{1-\left(\frac{\theta}{\bar{\theta}}\right)^{1-\eta}}{1-\frac{1}{\bar{\theta}}}\right)\right\} \\
&= (r^{w}-r^{m})\left(\frac{1-\left(\lim_{\bar{\lambda}\to\infty}\left\{\frac{\theta}{\bar{\theta}}\right\}\right)^{1-\eta}}{1-\lim_{\bar{\lambda}\to\infty}\left\{\frac{1}{\bar{\theta}}\right\}}\right) \\
&= r^{w}-r^{m}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\lim_{\bar{\lambda}\to\infty}\left\{\chi^{-}\right\} &= \lim_{\bar{\lambda}\to\infty}\left\{(r^{w}-r^{m})\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\left(\frac{\theta^{\eta}\bar{\theta}^{1-\eta}-1}{\bar{\theta}-1}\right)\right\} \\
&= \lim_{\bar{\lambda}\to\infty}\left\{(r^{w}-r^{m})\left(\frac{1-\frac{1}{\theta^{\eta}\bar{\theta}^{1-\eta}}}{1-\frac{1}{\bar{\theta}}}\right)\right\} \\
&= (r^{w}-r^{m})\left(\frac{1-\lim_{\bar{\lambda}\to\infty}\left\{\frac{1}{\theta^{\eta}\bar{\theta}^{1-\eta}}\right\}}{1-\lim_{\bar{\lambda}\to\infty}\left\{\frac{1}{\bar{\theta}}\right\}}\right) \\
&= r^{w}-r^{m}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\lim_{\bar{\lambda}\to\infty}\left\{\bar{r}^{f}\right\} &= \lim_{\bar{\lambda}\to\infty}\left\{r^{w}-\left(\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}-1\right)\left(\frac{\theta}{\theta-1}\right)\left(\frac{r^{w}-r^{m}}{e^{\bar{\lambda}}-1}\right)\right\} \\
&= \lim_{\bar{\lambda}\to\infty}\left\{r^{w}-(r^{w}-r^{m})\left(\frac{\theta}{\theta-1}\right)\left(\frac{\left(\frac{1+(\theta-1)e^{\bar{\lambda}}}{\theta}\right)^{\eta}-1}{e^{\bar{\lambda}}-1}\right)\right\} \\
&= r^{w}-(r^{w}-r^{m})\left(\frac{\theta}{\theta-1}\right)\left(\lim_{\bar{\lambda}\to\infty}\left\{\frac{\left(\frac{1+(\theta-1)e^{\bar{\lambda}}}{\theta}\right)^{\eta}-1}{e^{\bar{\lambda}}-1}\right\}\right) \\
&= r^{w}-(r^{w}-r^{m})\left(\frac{\theta}{\theta-1}\right)\left(\lim_{\bar{\lambda}\to\infty}\left\{\eta\left(\frac{\theta-1}{\theta}\right)\left(\frac{\theta}{1+(\theta-1)e^{\bar{\lambda}}}\right)^{1-\eta}\right\}\right) \\
&= r^{w}.
\end{aligned}
$$

<u>$\theta < 1$</u>:

Because $\theta > 1$, observe that

$$
\begin{aligned}
\lim_{\bar{\lambda}\to\infty}\left\{\Psi^{+}\right\} &= \lim_{\bar{\lambda}\to\infty}\left\{\theta\left(1-e^{-\bar{\lambda}}\right)\right\} = \theta, \qquad \text{and} \\
\lim_{\bar{\lambda}\to\infty}\left\{\Psi^{-}\right\} &= \lim_{\bar{\lambda}\to\infty}\left\{1-e^{-\bar{\lambda}}\right\} = 1.
\end{aligned}
$$

Now, before proceeding, realise that

$$\lim_{\bar{\lambda}\to\infty}\left\{\frac{\bar{\theta}}{\theta}\right\} = \lim_{\bar{\lambda}\to\infty}\left\{\frac{\frac{1}{1+\left(\frac{1-\theta}{\theta}\right)e^{\bar{\lambda}}}}{\theta}\right\} = \lim_{\bar{\lambda}\to\infty}\left\{\frac{1}{\theta+(1-\theta)e^{\bar{\lambda}}}\right\} = 0.$$

Therefore,

$$\begin{aligned}
\lim_{\bar{\lambda}\to\infty}\left\{\chi^{+}\right\} &= \lim_{\bar{\lambda}\to\infty}\left\{(r^{w}-r^{m})\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\left(\frac{\theta^{\eta}\bar{\theta}^{1-\eta}-\theta}{\bar{\theta}-1}\right)\right\} \\
&= \lim_{\bar{\lambda}\to\infty}\left\{(r^{w}-r^{m})\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\left(\frac{\left(\frac{\bar{\theta}}{\theta}\right)^{1-\eta}-1}{\frac{\bar{\theta}}{\theta}-\frac{1}{\theta}}\right)\right\} \\
&= (r^{w}-r^{m})\left(\lim_{\bar{\lambda}\to\infty}\left\{\frac{\bar{\theta}}{\theta}\right\}\right)^{\eta}\left(\frac{\left(\lim_{\bar{\lambda}\to\infty}\left\{\frac{\bar{\theta}}{\theta}\right\}\right)^{1-\eta}-1}{\lim_{\bar{\lambda}\to\infty}\left\{\frac{\bar{\theta}}{\theta}\right\}-\frac{1}{\theta}}\right) \\
&= 0.
\end{aligned}$$

Also,

$$\begin{aligned}
\lim_{\bar{\lambda}\to\infty}\left\{\chi^{-}\right\} &= \lim_{\bar{\lambda}\to\infty}\left\{(r^{w}-r^{m})\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\left(\frac{\theta^{\eta}\bar{\theta}^{1-\eta}-1}{\bar{\theta}-1}\right)\right\} \\
&= \lim_{\bar{\lambda}\to\infty}\left\{(r^{w}-r^{m})\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\left(\frac{\left(\frac{\bar{\theta}}{\theta}\right)^{1-\eta}-\frac{1}{\theta}}{\frac{\bar{\theta}}{\theta}-\frac{1}{\theta}}\right)\right\} \\
&= (r^{w}-r^{m})\left(\lim_{\bar{\lambda}\to\infty}\left\{\frac{\bar{\theta}}{\theta}\right\}\right)^{\eta}\left(\frac{\left(\lim_{\bar{\lambda}\to\infty}\left\{\frac{\bar{\theta}}{\theta}\right\}\right)^{1-\eta}-\frac{1}{\theta}}{\lim_{\bar{\lambda}\to\infty}\left\{\frac{\bar{\theta}}{\theta}\right\}-\frac{1}{\theta}}\right) \\
&= 0.
\end{aligned}$$

Finally,

$$\begin{aligned}
\lim_{\bar{\lambda}\to\infty}\left\{\bar{r}^{f}\right\} &= \lim_{\bar{\lambda}\to\infty}\left\{r^{w}-\left(\frac{1}{1-\theta}\right)\left(\frac{\theta}{\bar{\bar{\theta}}}\right)\left(1-\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\right)\left(\frac{r^{w}-r^{m}}{e^{\bar{\lambda}}-1}\right)\right\} \\
&= \lim_{\bar{\lambda}\to\infty}\left\{r^{w}-\left(\frac{r^{w}-r^{m}}{1-\theta}\right)\left(1-\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\right)\left(\frac{\theta+(1-\theta)e^{\bar{\lambda}}}{e^{\bar{\lambda}}-1}\right)\right\} \\
&= r^{w}-\left(\frac{r^{w}-r^{m}}{1-\theta}\right)\left(1-\left(\lim_{\bar{\lambda}\to\infty}\left\{\frac{\bar{\theta}}{\theta}\right\}\right)^{\eta}\right)\left(\lim_{\bar{\lambda}\to\infty}\left\{\frac{\theta+(1-\theta)e^{\bar{\lambda}}}{e^{\bar{\lambda}}-1}\right\}\right) \\
&= r^{w}-\left(\frac{r^{w}-r^{m}}{1-\theta}\right)\left(\lim_{\bar{\lambda}\to\infty}\left\{1-\theta\right\}\right) \\
&= r^{m}.
\end{aligned}$$

G.2 Bargaining Power

Now, let us continue with the limiting properties of the bargaining power parameter.

Let us study the two possible cases,

As $\eta \to 1$ **:**

Firstly, realise that

$$\begin{aligned}
\lim_{\eta \to 1}\left\{\chi^{+}\right\} &= \begin{cases} \lim_{\eta\to 1}\left\{(1-\eta)\left(r^{w}-r^{m}\right)\left(1-e^{-\bar{\lambda}}\right)\right\} & \text{if } \theta = 1 \\ \lim_{\eta\to 1}\left\{\left(r^{w}-r^{m}\right)\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\left(\frac{\theta^{\eta}\bar{\theta}^{1-\eta}-\theta}{\bar{\theta}-1}\right)\right\} & \text{if } \theta \neq 1 \end{cases} \\
&= 0.
\end{aligned}$$

Also,

$$\begin{aligned}
\lim_{\eta \to 1}\left\{\chi^{-}\right\} &= \begin{cases} \lim_{\eta\to 1}\left\{\left(r^{w}-r^{m}\right)\left(1-\eta\left(1-e^{-\bar{\lambda}}\right)\right)\right\} & \text{if } \theta = 1 \\ \lim_{\eta\to 1}\left\{\left(r^{w}-r^{m}\right)\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\left(\frac{\theta^{\eta}\bar{\theta}^{1-\eta}-1}{\bar{\theta}-1}\right)\right\} & \text{if } \theta \neq 1 \end{cases} \\
&= \begin{cases} \left(r^{w}-r^{m}\right)e^{\bar{\lambda}} & \text{if } \theta = 1 \\ \left(r^{w}-r^{m}\right)\left(\frac{\bar{\theta}}{\theta}\right)\left(\frac{\theta-1}{\bar{\theta}-1}\right) & \text{if } \theta \neq 1 \end{cases} \\
&= \begin{cases} \left(r^{w}-r^{m}\right)\left(1-\left(1-e^{\bar{\lambda}}\right)\right) & \text{if } \theta = 1 \\ \left(r^{w}-r^{m}\right)\left(1-\left(\frac{1-e^{-\bar{\lambda}}}{\theta}\right)\right) & \text{if } \theta > 1 \\ \left(r^{w}-r^{m}\right)\left(1-\left(1-e^{-\bar{\lambda}}\right)\right) & \text{if } \theta < 1 \end{cases} \\
&= \left(1-\Psi^{-}\right)\left(r^{w}-r^{m}\right).
\end{aligned}$$

Finally,

$$\lim_{\eta\to 1}\left\{\overline{r}^f\right\} = \begin{cases} \lim_{\eta\to 1}\left\{(1-\eta)r^w - \eta r^m\right\} & \text{if } \theta = 1 \\ \lim_{\eta\to 1}\left\{r^w - \left(\left(\frac{\bar{\theta}}{\theta}\right)^{\eta} - 1\right)\left(\frac{\theta}{\theta-1}\right)\left(\frac{r^w - r^m}{e^{\bar{\lambda}}-1}\right)\right\} & \text{if } \theta > 1 \\ \lim_{\eta\to 1}\left\{r^w - \left(\frac{1}{1-\theta}\right)\left(\frac{\theta}{\bar{\theta}}\right)\left(1 - \left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\right)\left(\frac{r^w - r^m}{e^{\bar{\lambda}}-1}\right)\right\} & \text{if } \theta < 1 \end{cases}$$

$$= \begin{cases} r^m & \text{if } \theta = 1 \\ r^w - \left(\frac{\bar{\theta}-\theta}{\theta}\right)\left(\frac{\theta}{\theta-1}\right)\left(\frac{r^w - r^m}{e^{\bar{\lambda}}-1}\right) & \text{if } \theta > 1 \\ r^w - \left(\frac{1}{1-\theta}\right)\left(\frac{\theta}{\bar{\theta}}\right)\left(\frac{\theta-\bar{\theta}}{\theta}\right)\left(\frac{r^w - r^m}{e^{\bar{\lambda}}-1}\right) & \text{if } \theta < 1 \end{cases}$$

$$= \begin{cases} r^m & \text{if } \theta = 1 \\ r^w - \left(\frac{\bar{\theta}-\theta}{\theta-1}\right)\left(\frac{r^w - r^m}{e^{\bar{\lambda}}-1}\right) & \text{if } \theta > 1 \\ r^w - \left(\frac{1}{\bar{\theta}}\right)\left(\frac{\theta-\bar{\theta}}{1-\theta}\right)\left(\frac{r^w - r^m}{e^{\bar{\lambda}}-1}\right) & \text{if } \theta < 1 \end{cases}$$

$$= \begin{cases} r^m & \text{if } \theta = 1 \\ r^w - \left(\frac{1+(\theta-1)e^{\bar{\lambda}}-\theta}{\theta-1}\right)\left(\frac{r^w - r^m}{e^{\bar{\lambda}}-1}\right) & \text{if } \theta > 1 \\ r^w - \left(\frac{\theta}{\bar{\theta}}\right)\left(\frac{(\theta+(1-\theta)e^{\bar{\lambda}}-1)}{(1-\theta)\left(\theta+(1-\theta)e^{\bar{\lambda}}\right)}\right)\left(\frac{r^w - r^m}{e^{\bar{\lambda}}-1}\right) & \text{if } \theta < 1 \end{cases}$$

$$= r^m.$$

As $\eta \to 0$:

Firstly, realize that

$$\lim_{\eta\to 0}\left\{\chi^+\right\} = \begin{cases} \lim_{\eta\to 0}\left\{(1-\eta)\left(r^w - r^m\right)\left(1 - e^{-\bar{\lambda}}\right)\right\} & \text{if } \theta = 1 \\ \lim_{\eta\to 0}\left\{\left(r^w - r^m\right)\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\left(\frac{\theta^{\eta}\bar{\theta}^{1-\eta}-\theta}{\bar{\theta}-1}\right)\right\} & \text{if } \theta \neq 1 \end{cases}$$

$$= \Psi^+\left(r^w - r^m\right).$$

Also,

$$\lim_{\eta\to 0}\left\{\chi^-\right\} = \begin{cases} \lim_{\eta\to 0}\left\{\left(r^w - r^m\right)\left(1 - \eta\left(1 - e^{-\bar{\lambda}}\right)\right)\right\} & \text{if } \theta = 1 \\ \lim_{\eta\to 0}\left\{\left(r^w - r^m\right)\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\left(\frac{\theta^{\eta}\bar{\theta}^{1-\eta}-1}{\bar{\theta}-1}\right)\right\} & \text{if } \theta \neq 1 \end{cases}$$

$$= r^w - r^m.$$

Finally,

$$\lim_{\eta\to 0}\left\{\overline{r}^f\right\} = \begin{cases} \lim_{\eta\to 0}\left\{(1-\eta)r^w - \eta r^m\right\} & \text{if } \theta = 1 \\ \lim_{\eta\to 0}\left\{r^w - \left(\left(\frac{\bar{\theta}}{\theta}\right)^{\eta} - 1\right)\left(\frac{\theta}{\theta-1}\right)\left(\frac{r^w - r^m}{e^{\bar{\lambda}}-1}\right)\right\} & \text{if } \theta > 1 \\ \lim_{\eta\to 0}\left\{r^w - \left(\frac{1}{1-\theta}\right)\left(\frac{\theta}{\bar{\theta}}\right)\left(1 - \left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\right)\left(\frac{r^w - r^m}{e^{\bar{\lambda}}-1}\right)\right\} & \text{if } \theta < 1 \end{cases}$$

$$= r^w.$$

H. Solution of the Portfolio Problem

H.1 Pricing Conditions

H.1.1 Risk Aversion Case

We have the following problem:

$$\max_A \left(\mathbb{E}_{X,\omega} \left[\left(R^m e + \sum_{i\in\mathbb{I}} \underbrace{\left(R^i\left(X_t\right) - R^m\right)}_{\text{Liquidity Premium}} \bar{a}^i_t + \underbrace{\chi_{t+1}\left(s\left(\{\bar{a}\}_{i\in\mathbb{I}}, e - \sum_{i\in\mathbb{I}} \bar{a}^i_t \right)\right)}_{\text{Liquidity Yield}} \right)^{1-\gamma} \right] \right)^{\frac{1}{1-\gamma}}$$

subject to $\Gamma_t \cdot A_t \geq 0$. So the first order condition is:

$$a^i : \mathbb{E}_{X,\omega}\left[(1-\gamma)\, \mathbb{E}_{X,\omega}\left[(R^e)^{-\gamma} \right] \left(\mathbb{E}_X \left[R^i\left(X_t\right) - R^m \right] + \mathbb{E}_{X,\omega}\left[\chi_s \frac{\partial s}{\partial a^i} \right] \right) \right] = 0$$

$$\mathbb{E}_{X,\omega}\left[\mathbb{E}_{X,\omega}\left[(R^e)^{-\gamma} \right] \left(\mathbb{E}_X \left[R^i\left(X_t\right) - R^m \right] + \mathbb{E}_{X,\omega}\left[\chi_s \frac{\partial s}{\partial a^i} \right] \right) \right] = 0$$

Taking the second term of the expression that is in parentheses, to the right hand side:

$$\mathbb{E}_{X,\omega}\left[\mathbb{E}_{X,\omega}\left[(R^e)^{-\gamma} \right] \mathbb{E}_X \left[R^i\left(X_t\right) - R^m \right] \right] = -\mathbb{E}_{X,\omega}\left[\mathbb{E}_{X,\omega}\left[(R^e)^{-\gamma} \right] \mathbb{E}_{X,\omega}\left[\chi_s \frac{\partial s}{\partial a^i} \right] \right]$$

If we take into account the covariance formula, we have:

$$\begin{aligned} \mathbb{E}_{X,\omega}\left[(R^e)^{-\gamma} \right] \mathbb{E}_X \left[R^i\left(X_t\right) - R^m \right] + \mathbb{COV}_{X,\omega}\left[(R^e)^{-\gamma}, R^i\left(X\right) - R^m \right] = &-\mathbb{E}_{X,\omega}\left[(R^e)^{-\gamma} \right] \mathbb{E}_{X,\omega}\left[\chi_s \frac{\partial s}{\partial a^i} \right] \\ &-\mathbb{COV}_{X,\omega}\left[(R^e)^{-\gamma}, \chi_s \frac{\partial s}{\partial a^i} \right] \end{aligned}$$

$$\begin{aligned} \mathbb{E}_{X,\omega}\left[(R^e)^{-\gamma} \right] \mathbb{E}_X \left[R^i\left(X_t\right) - R^m \right] = &-\mathbb{E}_{X,\omega}\left[(R^e)^{-\gamma} \right] \mathbb{E}_{X,\omega}\left[\chi_s \frac{\partial s}{\partial a^i} \right] - \mathbb{COV}_{X,\omega}\left[(R^e)^{-\gamma}, \chi_s \frac{\partial s}{\partial a^i} \right] \\ &- \mathbb{COV}_{X,\omega}\left[(R^e)^{-\gamma}, R^i\left(X\right) - R^m \right] \end{aligned}$$

Now, we can obtain the asset premium:

$$\mathbb{E}_X\left[R^i\left(X_t\right)-R^m\right]=-\frac{-\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},\chi_s\frac{\partial s}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$
$$-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},R^i\left(X\right)-R^m\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$
$$\mathbb{E}_X\left[R^i\left(X_t\right)\right]-R^m=-\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},\chi_s\frac{\partial s}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},R^i\left(X\right)-R^m\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$
$$\mathbb{E}_X\left[R^i\left(X_t\right)\right]-R^m=-\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},\chi_s\frac{\partial s}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},R^i\left(X\right)\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$
$$\mathbb{E}_X\left[R^i\left(X_t\right)\right]-R^m=-\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},R^i\left(X\right)+\chi_s\frac{\partial s}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$

H.1.2 Risk Neutral Case

We have the same problem as the previous case, but considering $\gamma=0$:

$$\max_A\left(\mathbb{E}_{X,\omega}\left[R^m e+\sum_{i\in\mathbb{I}}\underbrace{\left(R^i\left(X_t\right)-R^m\right)}_{\text{Liquidity Premium}}\bar{a}_t^i+\underbrace{\chi_{t+1}\left(s\left(\{\bar{a}\}_{i\in\mathbb{I}},e-\sum_{i\in\mathbb{I}}\bar{a}_t^i\right)\right)}_{\text{Liquidity Yield}}\right]\right)$$

subject to $\Gamma_t\cdot A_t\geq 0$. So the first order condition is:

$$a^i:\mathbb{E}_{X,\omega}\left[\mathbb{E}_X\left[R^i\left(X_t\right)-R^m\right]+\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]\right]=0$$
$$\mathbb{E}_{X,\omega}\left[\mathbb{E}_X\left[R^i\left(X_t\right)-R^m\right]\right]+\mathbb{E}_{X,\omega}\left[\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]\right]=0$$

So, now we have the asset premium:

$$\mathbb{E}_X\left[R^i\left(X_t\right)-R^m\right]=-\mathbb{E}_{X,\omega}\left[\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]\right]$$
$$\mathbb{E}_X\left[R^i\left(X_t\right)-R^m\right]=-\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]$$
$$\mathbb{E}_X\left[R^i\left(X_t\right)\right]-R^m=-\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]$$

H.2 Efficiency

H.2.1 Risk Aversion Case

We have the same problem as the pricing conditions case, so we will follow the same steps to solve it:

$$\max_A \left(\mathbb{E}_{X,\omega}\left[\left(R^m e + \sum_{i\in\mathbb{I}} \underbrace{\left(R^i\left(X_t\right) - R^m\right)}_{\text{Liquidity Premium}} \bar{a}_t^i + \underbrace{\chi_{t+1}\left(s\left(\left\{\bar{a}\right\}_{i\in\mathbb{I}}, e - \sum_{i\in\mathbb{I}} \bar{a}_t^i \right), \theta(\left\{\bar{a}\right\}_{i\in\mathbb{I}}) \right)}_{\text{Liquidity Yield}} \right)^{1-\gamma} \right] \right)^{\frac{1}{1-\gamma}}$$

subject to $\Gamma_t \cdot A_t \geq 0$. The first order condition is:

$$a^i : \mathbb{E}_{X,\omega}\left[(1-\gamma)\,\mathbb{E}_{X,\omega}\left[(R^e)^{-\gamma} \right] \left(\mathbb{E}_X\left[R^i\left(X_t\right) - R^m \right] + \mathbb{E}_{X,\omega}\left[\chi_s \frac{\partial s}{\partial a^i} + \chi_\theta \frac{\partial \theta}{\partial a^i} \right] \right) \right] = 0$$
$$\mathbb{E}_{X,\omega}\left[\mathbb{E}_{X,\omega}\left[(R^e)^{-\gamma} \right] \left(\mathbb{E}_X\left[R^i\left(X_t\right) - R^m \right] + \mathbb{E}_{X,\omega}\left[\chi_s \frac{\partial s}{\partial a^i} + \chi_\theta \frac{\partial \theta}{\partial a^i} \right] \right) \right] = 0$$

Reordering the last expression:

$$\mathbb{E}_{X,\omega}\left[\mathbb{E}_{X,\omega}\left[(R^e)^{-\gamma} \right] \mathbb{E}_X\left[R^i\left(X_t\right) - R^m \right] \right] = -\mathbb{E}_{X,\omega}\left[\mathbb{E}_{X,\omega}\left[(R^e)^{-\gamma} \left(\chi_s \tfrac{\partial s}{\partial a^i} + \chi_\theta \tfrac{\partial \theta}{\partial a^i} \right) \right] \right]$$

And using the covariance formula:

$$\begin{aligned}\mathbb{E}_{X,\omega}\left[(R^e)^{-\gamma} \right] \mathbb{E}_X\left[R^i\left(X_t\right) - R^m \right] + \mathbb{COV}_{X,\omega}\left[(R^e)^{-\gamma}, R^i\left(X\right) - R^m \right] = {} & -\mathbb{E}_{X,\omega}\left[(R^e)^{-\gamma} \right] \mathbb{E}_{X,\omega}\left[\chi_s \frac{\partial s}{\partial a^i} + \chi_\theta \frac{\partial \theta}{\partial a^i} \right] \\ & -\mathbb{COV}_{X,\omega}\left[(R^e)^{-\gamma}, \chi_s \frac{\partial s}{\partial a^i} + \chi_\theta \frac{\partial \theta}{\partial a^i} \right]\end{aligned}$$

$$\begin{aligned}\mathbb{E}_{X,\omega}\left[(R^e)^{-\gamma} \right] \mathbb{E}_X\left[R^i\left(X_t\right) - R^m \right] = {} & -\mathbb{E}_{X,\omega}\left[(R^e)^{-\gamma} \right] \mathbb{E}_{X,\omega}\left[\chi_s \frac{\partial s}{\partial a^i} + \chi_\theta \frac{\partial \theta}{\partial a^i} \right] - \mathbb{COV}_{X,\omega}\left[(R^e)^{-\gamma}, \chi_s \frac{\partial s}{\partial a^i} + \chi_\theta \frac{\partial \theta}{\partial a^i} \right] \\ & - \mathbb{COV}_{X,\omega}\left[(R^e)^{-\gamma}, R^i\left(X\right) - R^m \right]\end{aligned}$$

Finally, we can obtain the asset premium:

$$\mathbb{E}_X\left[R^i\left(X_t\right)-R^m\right]=-\frac{-\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}+\chi_\theta\frac{\partial\theta}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},\chi_s\frac{\partial s}{\partial a^i}+\chi_\theta\frac{\partial\theta}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$
$$-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},R^i\left(X\right)-R^m\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$

$$\mathbb{E}_X\left[R^i\left(X_t\right)\right]-R^m=-\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}+\chi_\theta\frac{\partial\theta}{\partial a^i}\right]-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},\chi_s\frac{\partial s}{\partial a^i}+\chi_\theta\frac{\partial\theta}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$
$$-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},R^i\left(X\right)-R^m\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$

$$\mathbb{E}_X\left[R^i\left(X_t\right)\right]-R^m=-\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]-\mathbb{E}_{X,\omega}\left[\chi_\theta\frac{\partial\theta}{\partial a^i}\right]-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},\chi_s\frac{\partial s}{\partial a^i}+\chi_\theta\frac{\partial\theta}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$
$$-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},R^i\left(X\right)\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$

$$\mathbb{E}_X\left[R^i\left(X_t\right)\right]-R^m=-\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]-\mathbb{E}_{X,\omega}\left[\chi_\theta\frac{\partial\theta}{\partial a^i}\right]-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},R^i\left(X\right)+\chi_s\frac{\partial s}{\partial a^i}+\chi_\theta\frac{\partial\theta}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$

$$\mathbb{E}_X\left[R^i\left(X_t\right)\right]-R^m=-\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]-\mathbb{E}_{X,\omega}\left[\chi_\theta\frac{\partial\theta}{\partial a^i}\right]-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},R^i\left(X\right)+\chi_s\frac{\partial s}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$
$$-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},\chi_\theta\frac{\partial\theta}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$

$$\mathbb{E}_X\left[R^i\left(X_t\right)\right]-R^m=-\left(\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]+\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},R^i\left(X\right)+\chi_s\frac{\partial s}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}\right)$$
$$-\left(\mathbb{E}_{X,\omega}\left[\chi_\theta\frac{\partial\theta}{\partial a^i}\right]+\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},\chi_\theta\frac{\partial\theta}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}\right)$$

H.2.2 Risk Neutral Case

Now we are going to solve the las problem considering $\gamma=0$:

$$\max_A\left(\mathbb{E}_{X,\omega}\left[R^m e+\sum_{i\in\mathbb{I}}\underbrace{\left(R^i\left(X_t\right)-R^m\right)}_{\text{Liquidity Premium}}\bar{a}_t^i+\underbrace{\chi_{t+1}\left(s\left(\left\{\bar{a}\right\}_{i\in\mathbb{I}},e-\sum_{i\in\mathbb{I}}\bar{a}_t^i\right),\theta(\left\{\bar{a}\right\}_{i\in\mathbb{I}})\right)}_{\text{Liquidity Yield}}\right]\right)$$

subject to $\Gamma_t \cdot A_t \geq 0$. The first order condition is:

$$
\begin{aligned}
a^i : \mathbb{E}_{X,\omega}\left[\mathbb{E}_X\left[R^i\left(X_t\right) - R^m\right] + \mathbb{E}_{X,\omega}\left[\chi_s \frac{\partial s}{\partial a^i} + \chi_\theta \frac{\partial \theta}{\partial a^i}\right]\right] = 0 \\
\mathbb{E}_{X,\omega}\left[\mathbb{E}_X\left[R^i\left(X_t\right) - R^m\right]\right] + \mathbb{E}_{X,\omega}\left[\mathbb{E}_{X,\omega}\left[\chi_s \frac{\partial s}{\partial a^i} + \chi_\theta \frac{\partial \theta}{\partial a^i}\right]\right] = 0
\end{aligned}
$$

And finally we have the asset premium:

$$
\begin{aligned}
\mathbb{E}_X\left[R^i\left(X_t\right) - R^m\right] &= -\mathbb{E}_{X,\omega}\left[\mathbb{E}_{X,\omega}\left[\chi_s \frac{\partial s}{\partial a^i} + \chi_\theta \frac{\partial \theta}{\partial a^i}\right]\right] \\
\mathbb{E}_X\left[R^i\left(X_t\right) - R^m\right] &= -\mathbb{E}_{X,\omega}\left[\chi_s \frac{\partial s}{\partial a^i} + \chi_\theta \frac{\partial \theta}{\partial a^i}\right] \\
\mathbb{E}_X\left[R^i\left(X_t\right)\right] - R^m &= -\mathbb{E}_{X,\omega}\left[\chi_s \frac{\partial s}{\partial a^i}\right] - \mathbb{E}_{X,\omega}\left[\chi_\theta \frac{\partial \theta}{\partial a^i}\right]
\end{aligned}
$$