Title:
The Curvature–Transport Correspondence: A Unified Structural Framework for Field Curvature, Mass, and Transport Flux

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The Curvature–Transport Correspondence (CTC): From Quantum Effective Mass to Cosmological Dark Matter

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Abstract

We introduce the Curvature–Transport Correspondence (CTC), a simple but unifying framework that identifies a shared structural role for mass across physical theories. In quantum mechanics, galactic dynamics, and cosmology, the curvature of a fundamental field, whether wavefunction, gravitational potential, or angular momentum, is governed by the divergence of an associated transport flux. Under CTC, quantities conventionally interpreted as inertial mass, effective density, or stress energy are reframed as curvature response coefficients. They measure the energetic cost of field curvature generated by transport processes. This principle recasts dark matter not as unseen substance but as the curvature response parameter required by the observed gravitational field. Without altering the equations of quantum mechanics or general relativity, CTC provides a falsifiable geometric interpretation of mass, shifting the conceptual foundation from material content to transport driven curvature response.

1. Introduction

What is mass? Across physics, this foundational quantity appears in very different roles. In quantum mechanics, it is the coefficient that sets the curvature of the wavefunction. In cosmology, it enters as the density that sources gravitational potential. In galactic dynamics, it is the inferred "missing mass'' required to sustain observed rotation curves. Despite the diversity of these contexts, we propose that a single geometric structure underlies them all: mass, in any domain, functions as a curvature-response coefficient.

Most physical field equations relate the curvature of a field to a source term. The Schrödinger equation ties wavefunction curvature to inertial mass. The Poisson equation links gravitational curvature to mass density. The angular-momentum equation in collisionless disks connects the curvature of the momentum profile to the divergence of a torque flux. These equations share a common form: curvature is generated by the divergence of an associated transport process.

Traditionally, the source terms in these equations—mass, density, stress—are interpreted as intrinsic properties or as independent substances. The Curvature–Transport Correspondence (CTC) advances a different view. We show that these quantities are more fundamentally understood as curvature-response parameters that arise from an underlying transport flux. The unifying structure is

$$\textbf{Curvature of Field } \boldsymbol{X} = \boldsymbol{\nabla} \cdot (\textbf{Transport Flux } \boldsymbol{F_X}) \quad (1)$$

Under the CTC, the inertial mass in quantum mechanics, the torque-flux divergence in galactic dynamics, and the effective density inferred in cosmology are not separate constructs. They are different manifestations of one physical principle: each quantifies how strongly a system resists curvature generated by the divergence of a transport flux.

This paper develops that correspondence in detail. In Section 2, we examine the curvature structure of the Schrödinger equation, where mass explicitly sets the cost of wavefunction curvature. Section 3 derives how torque-flux divergence governs angular-momentum curvature in collisionless stellar systems. Section 4 shows the structural equivalence to the Poisson equation and to tidal-torque theory in cosmology. Section 5 states the general CTC principle and presents a cross-domain correspondence table. Sections 6 and 7 discuss the implications of this reframing and outline falsifiable predictions that distinguish the CTC from traditional, substance-based interpretations of mass.

The CTC does not modify the mathematics of quantum mechanics, general relativity, or galactic dynamics. It provides a new interpretive lens: one that links curvature directly to measurable transport processes and offers a unified, falsifiable account of mass from the quantum scale to the cosmic scale.

Perspective from Materials Physics.

The viewpoint developed in this work arises naturally from materials physics, where quantities such as effective mass, mobility, and conductivity are routinely understood as geometric or transport-derived response coefficients rather than intrinsic substances. In solid-state theory the effective mass of an electron, for example, reflects the curvature of the band structure and does not represent a distinct particle or material component. By adopting this established operational perspective, the CTC interprets the source terms appearing in quantum, galactic, and cosmological curvature equations as response coefficients generated by transport-flux divergence, restoring interpretational consistency without altering the underlying physics.

2. Curvature-Source Structure in Quantum Mechanics

The role of mass as a curvature-response coefficient finds its most explicit and operational definition in quantum mechanics. The stationary Schrödinger equation for a particle provides the canonical starting point[1]:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi. \quad (2)$$

Here, the Laplacian $\nabla^2\psi$ mathematically represents the spatial curvature of the wavefunction ψ. The coefficient preceding it, $\frac{\hbar^2}{2m}$, is the crucial parameter that converts this geometric property—curvature—into a dynamical quantity: kinetic energy. Within this structure, the *inertial mass m* is the curvature-response coefficient. It quantifies the wavefunction's resistance to spatial bending: a large mass strongly suppresses curvature, resulting in a slowly varying ψ, while a small mass permits rapid spatial variations.

This interpretation extends seamlessly into solid-state physics via the concept of effective mass m^*, defined by the curvature of the electronic energy bands[2]:

$$\frac{1}{m^*} = \frac{1}{\hbar^2}\frac{d^2E}{dk^2}. \quad (3)$$

In this formulation, m^* is literally the parameter that governs how a Bloch electron's state responds to curvature in momentum space, directly determining its acceleration under an applied force.

Thus, quantum mechanics provides a precise, non-metaphysical definition of mass*:* Inertial or effective mass *is the* curvature-response coefficient of the quantum state.

It sets the energetic cost incurred by a curved wavefunction. This perspective reveals the quantum mechanical "mass" not as a primary substance, but as a parameter mediating between geometry (curvature) and dynamics (energy). This is the foundational analogy for the Curvature–Transport Correspondence, establishing a template for identifying analogous curvature-response coefficients in galactic and cosmological systems.

2.1 The Case of the Massless Photon

The interpretation of mass as a curvature-response coefficient extends naturally to the case of fundamentally massless particles within a medium. A photon in a vacuum, obeying the wave equation

$$\nabla^2 \mathrm{A} - \frac{1}{c^2}\frac{\partial^2 \mathrm{A}}{\partial t^2} = 0, \qquad (4)$$

has zero inertial mass. This corresponds to an infinite susceptibility to spacetime curvature—the photon's wavefunction can vary arbitrarily without energetic cost.

However, inside a plasma or dielectric, the photon acquires an effective mass m_γ. In a plasma with frequency ω_p, the dispersion relation becomes

$$\omega^2 = \omega_p^2 + c^2 k^2, \qquad (5)$$

yielding an effective mass of $m_\gamma = \frac{\hbar\omega_p}{c^2}$. This effective mass emerges directly from the material's interaction with the electromagnetic field, and it governs the photon's response to spatial curvature in momentum space, defined by

$$\frac{1}{m_\gamma^*} = \frac{1}{\hbar^2}\frac{d^2E}{dk^2}. \qquad (6)$$

Here, just as for the electron in a solid, the photon's effective mass is not an intrinsic substance but a curvature-response coefficient. It quantifies how the photon-quasiparticle resists variations in its k-space profile due to interactions with the medium. This example reinforces the central CTC principle: what we call "mass" is a measure of a system's response to curvature generated by transport processes, even for a particle whose fundamental mass is zero.

2.2 The Case of the Emergent Phonon

The interpretation of mass as a curvature-response coefficient finds another well-known illustration in the behavior of collective excitations, most notably the phonon. In a crystal lattice, atomic vibrations are quantized into phonons, which can be viewed as emergent quasiparticles that are not fundamental substances but quantized modes of lattice displacement.

The phonon dispersion relation $\omega(k)$ provides the fundamental link. The curvature of this relation in momentum space directly defines the phonon's effective mass:

$$\frac{1}{m_{\text{phonon}}^*} = \frac{1}{\hbar^2}\frac{d^2E}{dk^2}, \text{ where } E = \hbar\omega(k). \qquad (7)$$

This is formally identical to the effective mass definitions for electrons (Eq. 3) and photons in a plasma (Eq. 6).

The effective mass of a phonon arises directly from the curvature of its dispersion relation. Physically, this effective mass governs how a phonon wave packet accelerates under external forces and how it transports energy and momentum through the lattice. Classic treatments of electron–phonon interaction, for example in the works of Ashcroft and Mermin[3] and in the many body theory of Mahan[4], show that lattice vibrations not only possess their own curvature response coefficients but also reshape those of other excitations.

A foundational precursor to this idea appears in Einstein's explanation of the photoelectric effect[5]. When a photon transfers discrete amounts of energy and momentum to an electron, the electron's intrinsic mass does not change, but the event illustrates a deeper principle central to the CTC: transport flux can modify a particle's dynamical response even when it does not alter its rest mass. Fröhlich identified the continuous version of this mechanism. In an ionic crystal, an electron couples to longitudinal optical phonons through the long range interaction he first formulated[6]. The electron becomes dressed by a cloud of virtual phonons and its observed mass increases. Modern analyses, such as the review by Devreese and Alexandrov[7], show that this mass enhancement follows directly from the curvature of the combined electron phonon energy surface.

In the curvature transport interpretation, this renormalization does not arise because one form of matter adds mass to another. Instead, one transport channel, the lattice vibrational flux, modifies the curvature structure of another transport channel, the electronic band. Einstein's discrete momentum transfer and Fröhlich's continuous dressing are two limits of the same principle: the curvature of a dynamical field is shaped by the transport flux acting upon it. The observed mass is the curvature response coefficient of the composite system.

The polaron therefore provides a clear and experimentally verified case where mass arises entirely from transport induced curvature, reinforcing the universality of the CTC principle.

2.3. The Case of the Boson

The curvature response interpretation of mass applies directly to bosonic fields. For a scalar field ϕ, the Klein Gordon equation gives the standard relativistic description of a spin zero boson [8]:

$$(\Box + \frac{m^2c^2}{\hbar^2})\phi = 0, \qquad (8)$$

where □ is the d Alembert operator that contains the spacetime curvature of the field. The term $m^2c^2/\hbar^2$ sets the energetic cost of bending the scalar field in spacetime. In this sense the mass of the boson acts as a curvature response coefficient, measuring the resistance of the field to spacetime curvature.

This viewpoint extends naturally to gauge bosons. In the electroweak theory the W and Z bosons acquire mass through the Higgs mechanism [9]. The Higgs field introduces an additional curvature term in the gauge sector, and the resulting boson masses follow from the vacuum expectation value of the Higgs condensate:

$$m_{W,Z} = \frac{1}{2}gv. \qquad (9)$$

Although this is the standard result of the electroweak theory, the CTC interpretation treats these masses as response coefficients that reflect how the gauge field reacts to the transport structure defined by the Higgs condensate.

Bosonic excitations in condensed matter systems express the same structure. Quasiparticles such as phonons, magnons, and excitons obey dispersion relations of the form $E(k) = \hbar\omega(k)$, and their effective mass is determined by the curvature of the dispersion relation[4], [10]:

$$\frac{1}{m^*_{\text{boson}}} = \frac{1}{\hbar^2}\frac{d^2E}{dk^2}. \qquad (10)$$

Here the effective mass is again a measure of how the excitation energy responds to curvature in the underlying transport field.

These examples illustrate that bosonic systems, whether fundamental fields or emergent quasiparticles, fit naturally within the curvature transport correspondence. Their masses arise as curvature response coefficients determined by the behaviour of the relevant transport flux. This places bosons alongside electrons, photons, nucleons, phonons, and strings in the unified set of systems that display curvature driven energetic response[11], [12].

2.4. The Case of the Nucleon and Quark Confinement

The curvature-response interpretation of mass extends naturally into nuclear physics. Protons and neutrons, though often treated as fundamental carriers of mass, acquire their effective mass from the dynamics of the quantum chromodynamic (QCD) field. Within QCD, quarks are bound by gluon exchange, and the confinement potential defines the curvature of the quark wavefunction. The nucleon mass emerges as the energetic cost of resisting this curvature, rather than as an intrinsic substance[13].

Formally, the dispersion relation for quarks inside a nucleon can be expressed as $E(k)$, where $k$is the quark momentum. Expanding near equilibrium yields:

$$E(k) \approx E_0 + \frac{\hbar^2 k^2}{2m_N^*}, \qquad (11)$$

with the effective nucleon mass defined by

$$\frac{1}{m_N^*} = \frac{1}{\hbar^2}\frac{d^2E}{dk^2}. \qquad (12)$$

Here, m_N^*quantifies the resistance of the bound quark state to curvature in the QCD potential. Just as the electron effective mass in solids (Eq. 3) reflects band curvature, the nucleon mass reflects the curvature of the QCD energy spectrum.

This perspective reframes the nucleon not as a particle with intrinsic mass, but as a transport-derived excitation. The gluon flux acts as the transport process, and its divergence defines the curvature of the quark field. The nucleon mass is therefore the curvature-response coefficient of the QCD system.

To illustrate, consider the MIT bag model[14], where quarks are confined in a finite region of space. The energy of the system can be written as

$$E(R) = \frac{Z}{R} + \frac{4}{3}\pi R^3 B, \quad (13)$$

where R is the bag radius, Z encodes quark kinetic energy, and B is the bag constant representing gluon pressure. Differentiating twice with respect to R gives the curvature of the confinement energy, and the effective nucleon mass emerges as the coefficient that balances this curvature against transport flux. Thus, even at the nuclear scale, "mass" is revealed as a curvature-response coefficient: it measures the geometric rigidity of quark states under gluon transport. This further aligns with the broader concept of nucleon effective mass in nuclear matter[15]. This reinforces the universality of the Curvature–Transport Correspondence, showing that the same principle applies from condensed-matter electrons to nucleons in QCD.

2.5. The Case of String Excitations

The curvature-response interpretation of mass extends naturally to the framework of string theory[16]. The aim here is not to reinterpret the fundamental theory but simply to note the structural parallel between the dispersion curvature of string vibrational modes and the curvature-based effective-mass viewpoint advanced in this work.

In standard string theory, particles arise as quantized vibrational modes of an extended one-dimensional object. Each mode has an energy spectrum

$$E_n(k) = \hbar\, \omega_n(k), \quad (14)$$

with k the momentum along the string. Expanding near a minimum of the dispersion relation yields

$$E_n(k) \approx E_{n,0} + \frac{\hbar^2 k^2}{2m_n^*}, \quad (15)$$

where the effective mass follows from the curvature of the spectrum,

$$\frac{1}{m_n^*} = \frac{1}{\hbar^2}\frac{d^2 E_n}{dk^2}. \quad (16)$$

This relation, presented in its standard form by Green, Schwarz, and Witten [16], shows that the mass level of a string excitation is determined by the curvature of its vibrational energy profile in the compactified directions. Polchinski's conformal field theory formulation [17] formalizes the same structure by deriving the mode spectrum from the Virasoro constraints and the conformal symmetry of the world sheet.

Within the CTC interpretation, these results simply exemplify the central idea: the observed mass level functions as a curvature-response coefficient. It quantifies the energetic cost of deforming a vibrational mode under the divergence of its associated transport flux. This places string excitations in continuity with the earlier examples—photons in plasmas, phonons, bosons, and nucleons—each of which acquires mass from the curvature induced by an underlying transport process.

Physically, transport along the string is carried by oscillatory energy flow residing on the world-sheet. The divergence of this vibrational flux determines the curvature of the mode profile, and the corresponding mass parameter measures the resistance of that profile to deformation in the compactified directions. This mirrors the earlier cases: photons in a plasma acquire effective mass from electromagnetic transport (2.1), phonons from lattice vibrational curvature (2.2), bosons from curvature of their scalar or gauge fields (2.3), and nucleons from curvature imposed by gluon confinement (2.4). String modes simply extend this same geometric principle into higher-dimensional settings, where vibrational flux divergence defines the curvature-response coefficient. In this respect, the string-theoretic mass spectrum fits naturally within the CTC catalogue of systems whose effective mass arises from curvature generated by transport.

2.6. Relation to String Theory and Independence from Supersymmetry

The CTC incorporates string excitations through a structural feature of string theory that does not depend on supersymmetry or on any specific high energy particle content. The mass of a string mode is fixed by the curvature of its vibrational dispersion, a relation derived directly from the world sheet conformal field theory of the string spectrum[16], [17]. This curvature reflects the divergence of the transport flux that propagates along the one dimensional string, a point emphasized in standard treatments of string vibrational modes[18]. It is therefore a kinematic property of the vibrational system rather than a supersymmetry dependent constraint.

In the CTC framework, a string mode is interpreted as the one-dimensional member of a broader class of curvature transport systems. Quasiparticles in solids, phonons, gluon confined nucleons, and orbit averaged angular momentum transport all share the same geometric structure: an extended or collective transport channel whose flux divergence determines the curvature of the excitation spectrum. A fundamental string fits naturally into this family because its vibrational flux plays the same geometric role as the transport fluxes that set curvature in the other systems.

This structural viewpoint also clarifies the relevance of low dimensional condensed matter systems. Quantum Hall edges support chiral one dimensional modes that carry a well-defined transport flux[19]. Spin chains and other one-dimensional quantum fluids exhibit collective excitations whose dispersion curvature follows directly from the internal transport structure[20], [21]. Topological phases contain extended degrees of freedom and stringlike operators whose excitations propagate through a quantized transport channel[22], [23]. In all these systems, the curvature of the vibrational spectrum is determined by the **divergence of the underlying flux** in the same geometric sense as in a fundamental string mode. They therefore provide (potentially) experimentally accessible realizations of the curvature transport relation that governs the string spectrum, without requiring any attempt to reproduce the full microscopic framework of string theory.

From this perspective, supersymmetry remains a consistent and powerful mechanism within string theory, but its role is understood differently. Supersymmetry manages the curvature structure of the spectrum by removing or regulating specific contributions. It is not required for the curvature transport relation itself, which arises directly from the world sheet dynamics of the string. The CTC therefore treats the string spectrum as fully compatible with its geometric framework while remaining independent of supersymmetry and other high energy assumptions.

2.7. Unified Interpretation of Mass Across Physical Scales

Taken together, these examples reveal a common geometric structure underlying the appearance of mass in diverse physical settings. In each case, the quantity called "mass" acts as the curvature-response coefficient associated with an underlying transport process:

- **Electrons in solids:** the effective mass arises from the curvature of electronic bands and the associated crystal-momentum transport.
- **Photons in plasmas:** an effective mass appears through dielectric transport, encoded in the curvature of the plasma dispersion relation.
- **Phonons in lattices:** vibrational modes acquire effective mass from the curvature of the lattice dispersion spectrum.
- **Nucleons in QCD:** the nucleon mass reflects the curvature imposed by gluon-confinement flux in the strongly coupled regime.
- **Bosons in scalar and gauge sectors:** the Higgs field and related curvature terms generate mass as a response to field-transport structure.
- **Strings in higher dimensions:** vibrational excitations acquire mass from the curvature of their mode spectra, determined by the divergence of oscillatory flux along the string.

Across these domains, mass is not a primitive intrinsic quantity but the response coefficient that quantifies how strongly a system resists curvature generated by a transport flux. This recurring structure is the unifying theme of the Curvature–Transport Correspondence.

3. Transport as Curvature Sourcing in Collisionless Galactic Dynamics

Collisionless stellar systems also provide a realization of the curvature transport structure. A complete development of this framework, including the role of non-local torque coupling and its use in reconstructing rotation curves, will appear in a forthcoming study. For completeness, the essential derivation is summarized in the Supplementary Materials. The present section focuses only on the geometric result.

The dynamics begin with the collisionless Boltzmann (Vlasov) equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f - \nabla \Phi \cdot \nabla_{\mathbf{v}} f = 0. \qquad (17)$$

Taking its moment with respect to the specific angular momentum $\ell = R v_\phi$ yields the exact evolution equation for the angular-momentum surface density $L(R, t)$:

$$\frac{\partial L}{\partial t} = -\frac{1}{R}\frac{\partial}{\partial R}(R F_L), \qquad (18)$$

where the radial flux of angular momentum is

$$F_L(R, t) = \int v_R \, \ell \, f \, d^3 v. \qquad (19)$$

This result has a profound interpretation: the temporal curvature (evolution) of the angular momentum field, $\partial L / \partial t$, is sourced explicitly by the negative divergence of the angular momentum torque flux, F_L. This is the direct dynamical analogue of the gravitational Poisson equation,

$$\nabla^2 \Phi = 4\pi G \rho_{\text{eff}}, \qquad (20)$$

where the spatial curvature of the potential is sourced by an effective density. Both equations share the universal CTC form: curvature = divergence of a flux.

This geometric equivalence reframes the physical interpretation: just as F_L is the explicit transport flux sourcing angular momentum curvature, the source of gravitational curvature ($\nabla^2\Phi$) must be an effective density, ρ_{eff}, that functions as a curvature-response coefficient emerging from an underlying gravitational transport flux, F_G. This identity positions ρ_{eff} not as an independent substance, but as a measure of the field's energetic resistance to transport-induced curvature. The cosmological role of ρ_{eff} is developed in Section 4, and the unified CTC principle is formalized in Section 5.

4. Cosmological Curvature from Effective Density and Tidal Flux

4.1. The Effective Density as a Curvature-Response Coefficient

The geometric foundation of cosmological gravity is the Poisson equation, where curvature in the gravitational potential, $\nabla^2\Phi$, is sourced by an effective density in eq. 20. In the standard ΛCDM model, this density is a sum of components:

$$\rho_{\text{eff}} = \rho_{\text{dm}} + \rho_{\text{b}} + \rho_{\text{rad}} + \rho_{\text{vel/stress}}, \qquad (21)$$

encompassing dark matter, baryons, radiation, and velocity-stress corrections.

The CTC reframes the interpretation of this fundamental equation. Observables such as CMB anisotropies, gravitational lensing, and large-scale structure are direct measurements of **gravitational curvature** $\nabla^2\Phi$. The inferred ρ_{eff} is the parameter required to satisfy the Poisson equation given this observed geometry.

Therefore, under the CTC, the effective density ρ_{eff} is the curvature-response coefficient for the gravitational potential. It quantifies the collective response of the cosmic medium to the observed curvature, rather than necessarily representing a sum of distinct particulate substances. This perspective aligns with the structure identified in Section 3: just as the curvature of the angular momentum field is sourced by torque-flux divergence, the curvature of the potential is sourced by an effective density that can be understood as emerging from an underlying gravitational transport flux, F_G.

This perspective remains fully consistent with the ΛCDM model and all its observational successes. The distinction is one of interpretation and focus: whereas the standard view concentrates on the *concentrated entity* of density (e.g., a dark matter particle density), the CTC framework shifts the focus to the *divergence process* from which the effective density emerges. The value of ρ_{eff} is the same, but its physical meaning is reframed from a measure of substance to a measure of curvature-response generated by transport-flux divergence.

4.2. Tidal Torque Theory as a Flux-Divergence Law

The dynamical growth of cosmic structure provides a complementary view through tidal torque theory (TTT). The standard formulation describes the generation of halo spin:

$$\frac{dL_i}{dt} = \epsilon_{ijk} I_{jl} T_{kl}, \qquad (22)$$

where the inertia tensor I_{jl} couples to the tidal tensor T_{kl}.

To reveal the transport-curvature structure, this equation can be recast into a flux-divergence form:

$$\frac{dL}{dt} = \int \nabla \cdot (\rho \Phi T) d^3x. \qquad (23)$$

In this formulation, the evolution (temporal curvature) of the halo's angular momentum field is explicitly governed by the divergence of a **tidal torque flux**. This completes a consistent, unified picture across cosmology:

- The static curvature of the gravitational potential ($\nabla^2\Phi$) is governed by an effective density, a curvature-response coefficient.
- The dynamic curvature of the halo spin field (dL/dt) is governed by tidal-flux divergence, a direct transport source.

Both phenomena are manifestations of the same CTC principle, demonstrating that curvature—whether static or dynamic—is sourced by the divergence of a transport flux.

5. Curvature–Transport Correspondence in General Relativity

The correspondence reaches its fullest expression in General Relativity. The Einstein field equations

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu} \qquad (24)$$

relate spacetime curvature to the stress–energy tensor. Under the CTC interpretation, this becomes a flux–curvature correspondence: there exists a gravitational transport flux ${F_G}^{\mu}{}_{\nu}$ whose covariant divergence reproduces the effective stress–energy content of spacetime,

$$\text{“Effective Curvature”} \propto \nabla_\mu {F_G}^{\mu}{}_{\nu}. \qquad (25)$$

This flux is not a new field and does not alter Einstein's equations. It is a geometric construct already implicit in GR—analogous to probability current in quantum mechanics or torque flux in the Vlasov formalism. Its presence is reflected in three standard formulations of GR, each of which naturally displays a flux–divergence structure.

Local Conservation of Stress–Energy

GR enforces covariant conservation of stress–energy[24],

$$\nabla_\mu T^{\mu\nu} = 0. \qquad (26)$$

This is formally a continuity equation: the matter stress–energy tensor behaves as the divergence of an underlying transport structure. In the CTC view, the total transport of energy–momentum contains two complementary pieces:

- a material transport flux encoded in $T^{\mu\nu}$, and
- a gravitational transport flux encoded implicitly in ${F_G}^{\mu}{}_{\nu}$.

Their divergences must balance so that the combined (matter + gravitational) transport is conserved. Thus, at the most basic level, GR already has a flux–divergence origin for curvature.

ADM Formalism: Gravitational Momentum Flux in 3+1 Dimensions

In the ADM 3+1 decomposition, Einstein's equations split into the momentum and Hamiltonian constraints[25],

$$D_j(K^{ij} - \gamma^{ij}K) = 8\pi G\, S^i, \qquad (27)$$
$$R + K^2 - K_{ij}K^{ij} = 16\pi G\, \rho, \qquad (28)$$

where $D_j$is the spatial covariant derivative. The key observation is that

$$D_j(K^{ij} - \gamma^{ij}K) \qquad (29)$$

is explicitly a spatial divergence of a geometric quantity. The combination $(K^{ij} - \gamma^{ij}K)$plays the role of gravitational momentum flux through the hypersurface. Therefore, the ADM momentum constraint takes the flux–divergence form

$$\nabla \cdot F_G = 8\pi G\, S^i, \qquad (30)$$

identifying the matter momentum density S^i as the curvature-response coefficient balancing gravitational flux divergence. The Hamiltonian constraint performs the same role for the effective energy density ρ. Thus, ADM provides a standard, widely used setting where curvature arises from the divergence of a gravitational flux—precisely as predicted by the CTC.

Bel–Robinson Tensor: Super-Energy Flux in Vacuum

In vacuum, curvature evolution is governed by the Bel–Robinson tensor $T_{\alpha\beta\mu\nu}$, which represents the "super-energy" and "super-momentum'' of the gravitational field [26]. It satisfies the covariant conservation law

$$\nabla^\alpha T_{\alpha\beta\mu\nu} = 0. \qquad (31)$$

This relation defines a purely gravitational energy–momentum flux whose divergence controls the propagation of curvature in vacuum, including gravitational waves and tidal interactions. In the CTC interpretation, the Bel–Robinson tensor provides a concrete example of a gravitational transport flux $F_G$whose divergence governs curvature even when $T_{\mu\nu} = 0$.

Unifying Principle

Across these three frameworks—local conservation laws, the ADM decomposition, and the Bel–Robinson tensor—the same geometric structure appears: curvature in GR is sourced by flux divergences of geometric transport fields. This does not modify Einstein's theory; it sharpens its physical interpretation. Just as effective mass, torque-flux divergence, and tidal flux generate curvature in quantum mechanics, galactic dynamics, and cosmology, the GR stress–energy tensor $T_{\mu\nu}$acts as the curvature-response coefficient emerging from the divergence of the gravitational transport flux F_G. This is the Curvature–Transport Correspondence in the fully relativistic regime.

Signed Behavior: Dark Matter and Dark Energy as Flux Divergence

A crucial feature is that the gravitational flux divergence is signed and dynamical. Its value is fixed by local gravitational evolution:

- **Collapsing regions (proto-halos):**
  $\nabla \cdot F_G > 0$, producing positive curvature that appears observationally as effective dark matter.
- **Expanding voids:**
  $\nabla \cdot F_G < 0$, producing an effective negative pressure that appears observationally as dark energy.

Thus, both dark matter and dark energy arise as opposite-sign manifestations of the same geometric mechanism: gravitational transport whose divergence generates curvature. This moves beyond monotonic back-reaction models and offers a unified origin for the dark sector rooted in the transport–curvature structure already present within GR itself.

Under this interpretation, dark matter and dark energy are opposite-sign manifestations of the same geometric mechanism: curvature generated by the divergence of gravitational transport. This view extends beyond monotonic back-reaction models and offers a unified geometric origin for the dark sector directly within the framework of General Relativity.

6. The Curvature-Transport Correspondence (CTC) Principle

The examples from quantum mechanics (§2), galactic dynamics (§3), and cosmology (§4) reveal a unified structural principle. In each case, the curvature of a physical field is generated by the divergence of an associated transport flux. The coefficients conventionally interpreted as mass, density, or stress are, more fundamentally, the parameters that quantify this response.

We formalize this as the Curvature–Transport Correspondence (CTC):

$$C[X] = \nabla \cdot F_X, \qquad (32)$$

where $C[X]$denotes the curvature of the field X, and $F_X$is the associated transport flux. This formulation reinterprets traditional "source terms" as curvature-response coefficients emerging from an underlying flux.

On the Physical Content of the Correspondence

At first glance, the CTC might appear to be a purely mathematical reformulation, since any source term can formally be expressed as a flux divergence. However, such a perspective overlooks the core physical insight of the framework. The CTC is not merely a change of variables, but a substantive hypothesis regarding the **origin** of conventional source terms.

The central claim is that the fluxes $F_X$correspond to fundamental physical transport processes — the flow of probability density in quantum mechanics, the transport of angular momentum in galactic disks, the divergence of gravitational stress-energy in cosmology, or the confinement flux in nuclear QCD. In this view, quantities like mass and effective density are not primitive substances, but emergent measures of how a system resists curvature induced by these underlying fluxes.

The falsifiability of the CTC therefore resides not in the general form of Eq. (15), but in the **specific identification of fluxes** and the distinct empirical predictions that follow (see §7.6). It is through these identifications and their measurable consequences that the CTC extends beyond mathematical repackaging to offer a new interpretive framework rooted in transport dynamics.

Table 1: Cross-domain correspondence under CTC

Domain	Governing Equation	Curvature Source	Interpretation under CTC
Quantum Mechanics	$\nabla^2\psi + \frac{2m}{\hbar^2}(E - V)\psi = 0$	Inertial/effective mass	Mass as curvature-response coefficient
Galactic Dynamics	$\partial_t L = -\frac{1}{R}\partial_R(RF_L)$	Torque-flux divergence	Angular-momentum curvature from transport
Cosmology (Poisson)	$\nabla^2\Phi = 4\pi G\rho_{\text{eff}}$	Effective density (dark matter, baryons, stresses)	Density as curvature-response coefficient
Cosmology (TTT)	$dL/dt = \epsilon_{ijk}I_{jl}T_{kl}$	Tidal-flux divergence	Halo spin curvature from flux transport
General Relativity	$G_{\mu\nu} = 8\pi G T_{\mu\nu}$	Stress-energy tensor	Back-reaction from the divergence of the gravitational transport flux, $\nabla \cdot F_G$.

Structural Unification Across Domains:

- **Quantum Mechanics**:

$$C[\psi] = \nabla \cdot (\frac{\hbar^2}{2m}\nabla\psi), \qquad (33)$$

identifying m^{-1}as the key curvature-response parameter.

- **Nuclear QCD**:

$$C[\psi_q] = \nabla \cdot F_{\text{gluon}}, \qquad (34)$$

where nucleon mass emerges as the curvature-response coefficient to gluon confinement flux.

- **String Excitations**:

The mass spectrum of string modes corresponds to curvature of their vibrational transport flux,

$$C[\psi_{\text{string}}] = \nabla \cdot F_{\text{vib}}, \qquad (35)$$

consistent with the dispersion-curve interpretation of string mass.

- **Cosmology (Tidal Torque Theory)**:

$$C[L_{\text{halo}}] = \nabla \cdot (\rho\Phi T), \qquad (36)$$

sourced by tidal-flux divergence.

- **Galactic Dynamics (CRT)**:

$$C[L] = -\frac{1}{R}\frac{\partial}{\partial R}(RF_L), \qquad (37)$$

where the torque-flux divergence is the direct curvature source.

- **Gravity (General Relativity):**

$$C[\Phi] = \nabla^2\Phi = 4\pi G\rho_{\text{eff}}, \qquad (38)$$

reframing ρ_{eff} as an effective curvature-response coefficient.

Table 1 above listed catalogs this unified structure, demonstrating that inertial mass, torque flux, effective density, and the back-reaction from F_G all function as curvature-response coefficients within the same flux-divergence principle.

Across these domains, from the quantum to the cosmological, curvature is not an independent ingredient but a natural and universal consequence of transport dynamics.

The CTC framework clarifies the conceptual role of “mass” across multiple physical theories. Rather than representing a universal substance, mass quantifies how strongly a field responds to curvature. This unified interpretation manifests differently across contexts:

- **Quantum mechanics.**
  The inertial mass determines the energetic cost of wavefunction curvature; heavier particles resist curvature more strongly, producing a more slowly varying ψ.
- **Cosmology.**
  Dark matter serves as a curvature-response parameter for the gravitational potential Φ. Its role is to specify the curvature required by the observed gravitational field, not to provide direct evidence for particulate content.
- **Galactic CRT.**
  Torque-flux divergence quantifies the curvature of the angular-momentum distribution $L(R)$, establishing rotational support through radial transport of kinetic energy.
- **General relativity.**
  The stress–energy tensor quantifies spacetime curvature, serving as the flux-divergence source for the Einstein field equations.

Across these cases, curvature is the measurable geometric quantity, while “mass” (or the effective density) is the parameter mediating the geometric response. Thus,

Mass is fundamentally an energy–curvature response coefficient, not intrinsic material content.

This reinterpretation unifies quantum effective mass, gravitational effective mass, and torque-driven curvature within a single conceptual and mathematical structure.

7. General Implications and Falsifiable Predictions

7.1. The Central Shift: From Substance to Process

The Curvature-Transport Correspondence (CTC) reframes the source terms in our fundamental equations as *curvature-response coefficients* emerging from underlying transport processes. This shift moves the conceptual foundation of physics from a catalogue of substances—inertial mass, dark matter, dark energy—to a unified understanding of dynamic, geometric processes. The central question changes from "What is the unseen substance?" to "What transport process is generating the observed curvature?"

7.2. Falsifiable Predictions

The Curvature-Transport Correspondence (CTC) transitions from a conceptual framework to a testable physical theory through specific, empirical predictions. These predictions deviate fundamentally from substance-based, "density-only" models because they arise from the **geometry of flux divergence**, not from the distribution of unseen particulate matter. Each identifies an observable signature that would be unexpected, or even impossible, under the standard interpretation.

(a) Halo Spin Correlations from Anisotropic Tidal-Flux Divergence

- **Standard View:** Halo spin is determined by the inertia tensor coupling to the *isotropic* tidal field.
- **CTC Prediction:** Protohalos in environments with strong, *anisotropic tidal-flux divergence* will exhibit systematic deviations in their spin alignments. The correlation between halo spin and the large-scale structure will be stronger with the *divergence* of the tidal field than with the tidal field itself. This provides a direct signature of the flux-divergence mechanism shaping curvature, distinguishable from purely density-derived tidal effects.

(b) Rotation Curve Systematics from Baryonic Torque Transport

- **Standard View:** Rotation curves are fit by adding a spherically-symmetric, pressureless dark matter halo with a phenomenologically determined density profile (e.g., NFW).
- **CTC Prediction:** The detailed features of galactic rotation curves, including local slope changes, mild asymmetries, and the persistent flattening in the outer disk, can be reproduced by models that explicitly calculate the radial **torque-flux divergence ($\nabla \cdot F_L$)** generated by the observed baryonic distribution (e.g., spiral arms, bars, asymmetries). A successful match between the calculated flux divergence and the inferred gravitational curvature would provide direct confirmation of transport-driven dynamics.

(c) Cross-Scale Coherence in Curvature-Response Formalism

- **CTC Prediction:** The mathematical formalism describing curvature-response will show a quantifiable homology across physical scales. The function used to extract the **effective mass (m^*)** from the curvature of an electronic band structure (d^2E/dk^2) will share an identical structural form with the function determining the **effective gravitational mass** from the curvature of a galactic angular momentum profile. This cross-domain parallelism in how "mass" is mathematically derived from curvature would be a hallmark of the CTC's universal principle.

(d) A Signed Signature for the Dark Sectors in Cosmological Parameters

- **CTC Prediction:** A specific, signed discrepancy will appear between cosmological parameters inferred from different probes. The global value of the dark energy equation of state (w) measured from the homogeneous expansion history (e.g., supernovae, CMB) will differ systematically from the value *effectively* governing the growth of structure and void profiles. This arises because the global w averages over both collapsing ($\nabla \cdot F_G > 0$, DM-like) and expanding ($\nabla \cdot F_G < 0$, DE-like) regions, while structure growth is dominated by the former. This signature is a direct consequence of the signed nature of gravitational flux divergence.

(e) Dynamical Nucleon Mass from Gluon Confinement Flux

- **CTC Prediction:** In deep inelastic scattering experiments, the **nucleon's effective mass (m_N^*)** will not be a constant but will vary with the momentum transfer (Q^2), reflecting its role as a dynamic curvature-response parameter. This measured energy-dependent mass will correlate with calculations of the **gluon field's transport flux divergence** during the confinement process. Such a correlation demonstrates that the nucleon's inertia is a response to QCD transport dynamics, not a fixed intrinsic property.

(f) The Low-Energy Pathway to Unification via Curvature-Transport Algebra

- **CTC Prediction:** Emergent, string-like excitations in low-dimensional condensed-matter systems (e.g., quantum Hall edges, spin chains, topological phases) will obey the same **curvature-transport algebra** $C[X] = \nabla \cdot F_X$ that defines the effective mass spectrum of fundamental strings. The key test is not whether these quasiparticles replicate high-energy microphysics, but whether their dispersion relations reveal an effective mass determined by the divergence of a measurable transport flux (e.g., energy current) in the material. Confirmation of this shared geometric skeleton in laboratory systems would provide a practical, low-energy path to validating the unified principles underlying quantum gravity.

Together, these predictions provide a direct empirical testbed for the CTC framework. They identify signatures that arise specifically from the divergence of transport fluxes and cannot be naturally reproduced by modifying or tuning unseen mass densities. Verification or refutation of these predictions will therefore decisively distinguish the CTC from all substance-based interpretations.

7.3. A Program of Curvature Diagnostics

The CTC reframes cosmological and galactic research as a program of curvature diagnostics. Just as condensed-matter physics progresses by mapping the curvature of electronic band structures to define effective mass, gravitational physics should prioritize the precise mapping of curvature itself—rotation curves, tidal fields, void expansion profiles.

Within this framework, the so-called dark components are interpreted as response coefficients that encode how transport drives curvature. This perspective suggests a systematic methodology: treat gravitational measurements analogously to band-structure mapping in solids. Deviations in halo spin alignments, rotation-curve slopes, or void dynamics then become empirical tests of transport-based curvature generation, moving beyond the inference of hidden substances.

7.4 On the Quantum Nature of Transport

The transport fluxes that appear in the CTC framework are collective and emergent quantities. Their origins lie in microscopic quantum dynamics, yet the fluxes themselves act as classical fields. Examples include the probability flux derived from the wavefunction, the torque flux generated by ensembles of stellar orbits, the mass flux associated with the classical Higgs condensate, and the effective potentials that govern electron and phonon motion in solids. In each case the flux represents a coarse-grained description of many underlying degrees of freedom.

For this reason, there is no requirement and no clear physical meaning in attempting to quantize the flux itself. The situation is closely analogous to hydrodynamics, where fluid flow arises from molecular motion yet the macroscopic fields in the Navier Stokes equations are not quantized. The CTC predictions concern this emergent and classical level of description, where curvature is generated by the divergence of a transport field.

The deeper quantum question therefore shifts from the quantization of the flux to the origin of the structure it expresses. The central issue becomes the following: which classes of quantum dynamics naturally produce the curvature transport relation that is seen in systems ranging from solids to plasmas to galactic disks and the large-scale structure of the universe? The answer to this question defines the direction for a future quantum theory that recovers the CTC structure in the appropriate classical limit.

8. Summary

Quantum mechanics, galactic dynamics, and cosmological gravity share a common transport curvature structure: in every domain the curvature of a physical field is generated by the divergence of an associated transport flux. Quantities usually interpreted as mass, density, or dark matter therefore take the role of curvature response coefficients rather than fundamental substances. The Curvature Transport Correspondence (CTC) formalizes this idea by placing inertial mass, torque flux divergence in stellar disks, effective density in cosmology, and the stress energy tensor in general relativity within a single geometric framework.

By treating these source terms as energetic responses to underlying transport, the CTC provides a unified language that links quantum effective mass, angular momentum transport in galaxies, and tidal curvature in cosmology. This approach preserves all established observational results while broadening the geometric interpretation of the quantities inferred from them. In this way the CTC complements standard theories and offers a coherent and empirically grounded perspective on curvature across scales, from the microscopic to the cosmic.

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Supplementary Materials: Angular-Momentum Transport in the Vlasov–Poisson Framework

This note summarizes the minimal derivation required to obtain the angular-momentum transport law used in the main text in Section 3. The full version will appear in the upcoming paper; here we present a compressed form, retaining only the steps needed to show that torque transport arises directly from the Vlasov–Poisson system.

1. Vlasov–Poisson in Axisymmetric Cylindrical Coordinates

A collisionless stellar disk is governed by the Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f - (\nabla\Phi) \cdot \nabla_{\mathbf{v}} f = 0, \quad \text{(S1)}$$

with potential determined via

$$\nabla^2\Phi = 4\pi G\, \rho, \rho = \int f\, d^3v. \qquad \text{(S2)}$$

In cylindrical coordinates $(R, \phi, z)$with velocities (v_R, v_ϕ, v_z), the advective form becomes

$$\frac{\partial f}{\partial t} + v_R \frac{\partial f}{\partial R} + v_z \frac{\partial f}{\partial z} + A_R \frac{\partial f}{\partial v_R} + A_\phi \frac{\partial f}{\partial v_\phi} + A_z \frac{\partial f}{\partial v_z} = 0, \qquad \text{(S3)}$$

where $A_i = -\partial_i\Phi$. Axisymmetry gives $\partial_\phi f = 0$ and $A_\phi = 0$.

To expose transport explicitly, this equation must be written in conservative form, where each term becomes a divergence.
Using

$$v_R\, \partial_R f = \frac{1}{R}\partial_R(R v_R f), A_i\, \partial_{v_i} f = \partial_{v_i}(A_i f), \qquad \text{(S4)}$$

(the velocity derivatives of $A_i$vanish), the conservative Vlasov equation reduces to

$$\frac{\partial f}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R v_R f) + \partial_{v_R}(A_R f) + \partial_{v_\phi}(A_\phi f) + \partial_{v_z}(A_z f) = 0. \qquad \text{(S5)}$$

This is the only form needed for the moment calculation.

2. Angular-Momentum Surface Density

For axisymmetric disks the specific angular momentum is $l = R v_\phi$. The angular-momentum surface density is the azimuthal moment of f:

$$L(R, t) = \int R v_\phi f\, d^3v\, dz. \qquad \text{(S6)}$$

Using the standard expression for mean rotation $\Omega = \langle v_\phi \rangle / R$,

$$L(R, t) = \Sigma(R, t)\, R^2 \Omega(R, t), \qquad \text{(S7)}$$

so L is the rotational support of the annulus at radius R.

3. Multiplying the Vlasov Equation by $\boldsymbol{l = Rv_\phi}$

Multiply Eq. (S1) by Rv_ϕ and integrate over all velocities and z:

$$\int Rv_\phi\ \partial_t f\, d^3v\, dz + \int Rv_\phi\ \frac{1}{R}\partial_R(Rv_R f)\, d^3v\, dz + \sum_i \int Rv_\phi\ \partial_{v_i}(A_i f)\, d^3v\, dz = 0. \qquad \text{(S8)}$$

We evaluate the three terms:

(1) Time derivative

$$\int Rv_\phi\ \partial_t f\, d^3v dz = \frac{\partial L}{\partial t}. \qquad \text{(S9)}$$

(2) Radial transport

$$\int Rv_\phi\ \frac{1}{R}\partial_R(Rv_R f)\, d^3v\, dz = \frac{1}{R}\frac{\partial}{\partial R}[R\, F_L(R,t)], \qquad \text{(S10)}$$

where the **angular-momentum flux** is defined as

$$F_L(R,t) = \int Rv_\phi v_R f\, d^3v\, dz. \qquad \text{(S11)}$$

This is the flux of angular momentum carried across radius $R$by stars with nonzero v_R.

(3) Velocity-space terms

Integrating by parts gives velocity-space boundary terms that vanish because $f \to 0$as $|\ \mathbf{v}\ | \to \infty$.
Derivative terms vanish because:

- $\partial_{v_R}(Rv_\phi) = 0$
- $\partial_{v_z}(Rv_\phi) = 0$
- $\partial_{v_\phi}(Rv_\phi) = R$, but multiplies $A_\phi = 0$under axisymmetry.

Thus,

$$\sum_i \int Rv_\phi\ \partial_{v_i}(A_i f)\, d^3v\, dz = 0. \qquad \text{(S12)}$$

4. Exact Angular-Momentum Transport Equation

Collecting S8-S11 give the exact conservation law:

$$\frac{\partial L}{\partial t} = -\frac{1}{R}\frac{\partial}{\partial R}[R\, F_L(R,t)] \qquad \text{(S13)}$$

This result follows solely from the collisionless Vlasov equation, meaning no fluid closure, no approximations, and no special orbit assumptions.

It is the precise torque law used in the main text: the curvature of angular-momentum support is the divergence of a radial torque flux.

5. Significance

- If $F_L = 0$, each annulus evolves independently → the enclosed-mass rotation-curve formula is valid.
- If $F_L \neq 0$, inner regions export angular momentum → outer regions maintain elevated rotation → flat or rising rotation curves arise without additional mass.

Thus, the enclosed-mass method corresponds to a highly restricted, torque-free subset of Vlasov solutions. The general case, seen in all real disks, requires the full transport law from the equation S13.