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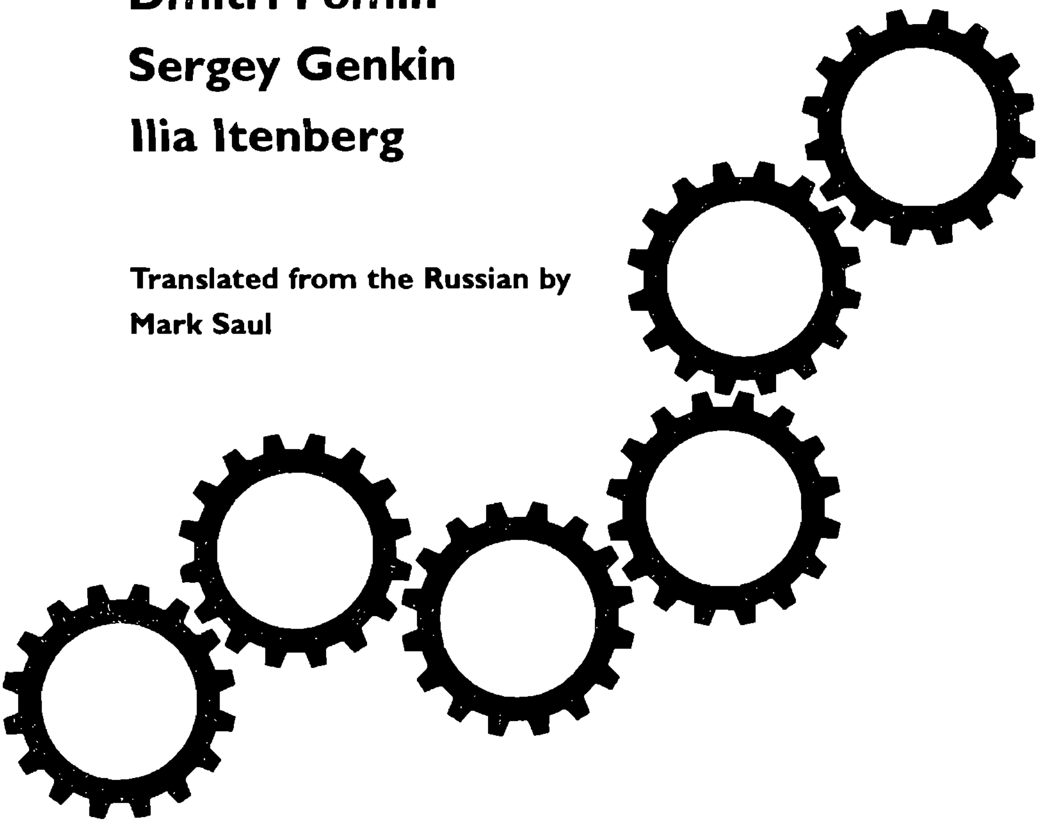
Mathematical World

Mathematical Circles
(Russian Experience)

Dmitri Fomin
Sergey Genkin
Ilia Itenberg

Translated from the Russian by
Mark Saul



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Contents

Foreword

This is not a textbook. It is not a contest booklet. It is not a set of lessons for classroom instruction. It does not give a series of projects for students, nor does it offer a development of parts of mathematics for self-instruction.

So what kind of book is this? It is a book produced by a remarkable cultural circumstance, which fostered the creation of groups of students, teachers, and mathematicians, called mathematical circles, in the former Soviet Union. It is predicated on the idea that studying mathematics can generate the same enthusiasm as playing a team sport, without necessarily being competitive.

Thus it is more like a book of mathematical recreations—except that it is more serious. Written by research mathematicians holding university appointments, it is the result of these same mathematicians' years of experience with groups of high school students. The sequences of problems are structured so that virtually any student can tackle the first few examples. Yet the same principles of problem solving developed in the early stages make possible the solution of extremely challenging problems later on. In between, there are problems for every level of interest or ability.

The mathematical circles of the former Soviet Union, and particularly of Leningrad (now St. Petersburg, where these problems were developed) are quite different from most math clubs across the globe. Typically, they were run not by teachers, but by graduate students or faculty members at a university, who considered it part of their professional duty to show younger students the joys of mathematics. Students often met far into the night, and went on weekend trips or summer retreats together, achieving a closeness and mutual support usually reserved in our country for members of athletic teams.

The development of mathematics education is an aspect of Russian culture from which we have much to learn. It is still very rare to find research mathematicians willing to devote time, energy, and thought to the development of materials for high school students.

So we must borrow from our Russian colleagues. The present book is the result of such borrowings. Some chapters, such as the one on the triangle inequality, can be used directly in our classrooms, to supplement the development in the usual textbooks. Others, such as the discussion of graph theory, stretch the curriculum with gems of mathematics which are not usually touched on in the classroom. Still others, such as the chapter on games, offer a rich source of extra-curricular materials with more structure and meaning than many.

Each chapter gives examples of mathematical methods in some of their barest forms. A game of nim, which can be enjoyed and even analyzed by a third grader, turns out to be the same as a game played with a single pawn on a chessboard. This becomes a lesson for seventh graders in restating problems, then offers an introduction to the nature of isomorphism for the high school student. The Pigeon Hole Principle, among the simplest yet most profound mathematics has to offer, becomes a tool for proof in number theory and geometry.

Yet the tone of the work remains light. The chapter on combinatorics does not require an understanding of generating functions or mathematical induction. The problems in graph theory, too, remain on the surface of this important branch of mathematics. The approach to each topic lends itself to mind play, not weighty reflection. And yet the work manages to strike some deep notes.

It is this quality of the work which the mathematicians of the former Soviet Union developed to a high art. The exposition of mathematics, and not just its development, became a part of the Russian mathematician's work. This book is thus part of a literary genre which remains largely undeveloped in the English language.

Mark Saul, Ph.D.
Bronxville Schools
Bronxville, New York

Preface to the Russian Edition

§1. Introduction

This book was originally written to help people in the former Soviet Union who dealt with extracurricular mathematical education: school teachers, university professors participating in mathematical education programs, various enthusiasts running mathematical circles, or people who just wanted to read something both mathematical and recreational. And, certainly, students can also use this book independently.

Another reason for writing this book was that we considered it necessary to record the role played by the traditions of mathematical education in Leningrad (now St. Petersburg) over the last 60 years. Though our city was, indeed, the cradle of the olympiad movement in the USSR (having seen the very first mathematical seminars for students in 1931–32, and the first city olympiad in 1934), and still remains one of the leaders in this particular area, its huge educational experience has not been adequately recorded for the interested readers.

* * *

In spite of the stylistic variety of this book's material, it is methodologically homogeneous. Here we have, we believe, all the basic topics for sessions of a mathematical circle for the first two years of extracurricular education (approximately, for students of age 12–14). Our main objective was to make the preparation of sessions and the gathering of problems easier for the teacher (or any enthusiast willing to spend time with children, teaching them non-standard mathematics). We wanted to talk about mathematical ideas which are important for students, and about how to draw the students' attention to these ideas.

We must emphasize that the work of preparing and leading a session is itself a creative process. Therefore, it would be unwise to follow our recommendations blindly. However, we hope that your work with this book will provide you with material for most of your sessions. The following use of this book seems to be natural: while working on a specific topic the teacher reads and analyzes a chapter from the book, and after that begins to construct a sketch of the session. Certainly, some adjustments will have to be made because of the level of a given group of students. As supplementary sources of problems we recommend [**13**, **16**, **24**, **31**, **33**], and [**40**].

* * *

We would like to mention two significant points of the Leningrad tradition of extracurricular mathematical educational activity:

(1) Sessions feature vivid, spontaneous communication between students and teachers, in which each student is treated individually, if possible.

(2) The process begins at a rather early age: usually during the 6th grade (age 11–12), and sometimes even earlier.

This book was written as a guide especially for secondary school students and for their teachers. The age of the students will undoubtedly influence the style of the sessions. Thus, a few suggestions:

A) We consider it wrong to hold a long session for younger students devoted to only one topic. We believe that it is helpful to change the direction of the activity even within one session.

B) It is necessary to keep going back to material already covered. One can do this by using problems from olympiads and other mathematical contests (see Appendix A).

C) In discussing a topic, try to emphasize a few of the most basic landmarks and obtain a complete understanding (not just memorization!) of these facts and ideas.

D) We recommend constant use of non-standard and "gamelike" activities in the sessions, with complete discussion of solutions and proofs. It is important also to use recreational problems and mathematical jokes. These can be found in [**5–7**, **16–18**, **26–30**].

We must mention here our predecessors—those who have tried earlier to create a sort of anthology for Leningrad mathematical circles. Their books [**32**] and [**43**], unfortunately, did not reach a large number of readers interested in mathematics education in secondary school.

In 1990–91 the original version of the first part of our book was published by the Academy of Pedagogical Sciences of USSR as a collection of articles [**21**] written by a number of authors. We would like to thank all our colleagues whose materials we used when working on the preparation of the present book: Denis G. Benua, Igor B. Zhukov, Oleg A. Ivanov, Alexey L. Kirichenko, Konstantin P. Kokhas, Nikita Yu. Netsvetaev, and Anna G. Frolova.

We also express our sincere gratitude to Igor S. Rubanov, whose paper on induction written especially for the second part of the book [**21**] (but never published, unfortunately) is included here as the chapter "Induction"

Our special thanks go to Alexey Kirichenko whose help in the early stages of writing this book cannot be overestimated. We would also like to thank Anna Nikolaeva for drawing the figures.

§2. Structure of the book

The book consists of this preface, two main parts, Appendix A "Mathematical Contests", Appendix B "Answers, Hints, Solutions", and Appendix C "References"

The first part ("The First Year of Education") begins with Chapter Zero, consisting of test questions intended mostly for students of ages 10–11. The problems of this chapter have virtually no mathematical content, and their main objective is to reveal the abilities of the students in mathematics and logic. The rest of the first part is divided into 8 chapters. The first seven of these are devoted to particular topics, and the eighth ("Problems for the first year") is simply a compilation of problems on a variety of themes.

The second part ("The Second Year of Education") consists of 9 chapters, some of which just continue the discussion in the first part (for example, the chapters "Graphs-2" and "Combinatorics-2"). Other chapters are comprised of material considered to be too complicated for the first year: "Invariants", "Induction", "Inequalities"

Appendix A tells about five main types of mathematical contests popular in the former Soviet Union. These contests can be held at sessions of mathematical circles or used to organize contests between different circles or even schools.

Advice to the teacher is usually given under the remark labelled **"For teachers"** Rare occasions of **"Methodological remarks"** contain mostly recommendations about the methodology of problem solving: they draw attention to the basic patterns of proofs or methods of recognizing and classifying problems.

§3. Technicalities and legend

(1) The most difficult problems are marked with an asterisk (*).

(2) Almost all of the problems are commented on in Appendix B: either a full solution or at least a hint and answer. If a problem is computational, then we usually provide only an answer. We do not give the solutions to problems for independent solution (this, in particular, goes for all the problems from Chapters 8 and 17).

(3) All the references can be found at the end of the book in the list of references. The books we recommend most are marked with an asterisk.

Chapter Zero

In this chapter we have gathered 25 simple problems. To solve them you do not need anything but common sense and the simplest calculational skills. These problems can be used at sessions of a mathematical circle to probe the logical and mathematical abilities of students, or as recreational questions.

* * *

Problem 1. A number of bacteria are placed in a glass. One second later each bacterium divides in two, the next second each of the resulting bacteria divides in two again, et cetera. After one minute the glass is full. When was the glass half-full?

Problem 2. Ann, John, and Alex took a bus tour of Disneyland. Each of them must pay 5 plastic chips for the ride, but they have only plastic coins of values 10, 15, and 20 chips (each has an unlimited number of each type of coin). How can they pay for the ride?

Problem 3. Jack tore out several successive pages from a book. The number of the first page he tore out was 183, and it is known that the number of the last page is written with the same digits in some order. How many pages did Jack tear out of the book?

Problem 4. There are 24 pounds of nails in a sack. Can you measure out 9 pounds of nails using only a balance with two pans? (See Figure 1.)

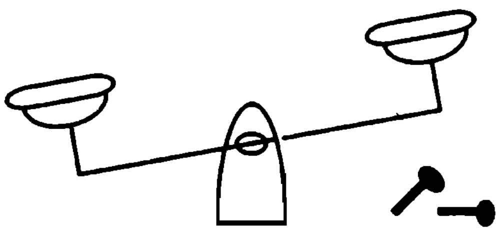

FIGURE 1

Problem 5. A caterpillar crawls up a pole 75 inches high, starting from the ground. Each day it crawls up 5 inches, and each night it slides down 4 inches. When will it first reach the top of the pole?

Problem 6. In a certain year there were exactly four Fridays and exactly four Mondays in January. On what day of the week did the 20th of January fall that year?

Problem 7. How many boxes are crossed by a diagonal in a rectangular table formed by 199×991 small squares?

Problem 8. Cross out 10 digits from the number 1234512345123451234512345 so that the remaining number is as large as possible.

* * *

Problem 9. Peter said: "The day before yesterday I was 10, but I will turn 13 in the next year." Is this possible?

Problem 10. Pete's cat always sneezes before it rains. She sneezed today. "This means it will be raining," Pete thinks. Is he right?

Problem 11. A teacher drew several circles on a sheet of paper. Then he asked a student "How many circles are there?" "Seven," was the answer. "Correct! So, how many circles are there?" the teacher asked another student. "Five," answered the student. "Absolutely right!" replied the teacher. How many circles were really drawn on the sheet?

Problem 12. The son of a professor's father is talking to the father of the professor's son, and the professor does not take part in the conversation. Is this possible?

Problem 13. Three turtles are crawling along a straight road heading in the same direction. "Two other turtles are behind me," says the first turtle. "One turtle is behind me and one other is ahead," says the second. "Two turtles are ahead of me and one other is behind," says the third turtle. How can this be possible?

Problem 14. Three scholars are riding in a railway car. The train passes through a tunnel for several minutes, and they are plunged into darkness. When they emerge, each of them sees that the faces of his colleagues are black with the soot that flew in through the open window. They start laughing at each other, but, all of a sudden, the smartest of them realizes that his face must be soiled too. How does he arrive at this conclusion?

Problem 15. Three tablespoons of milk from a glass of milk are poured into a glass of tea, and the liquid is thoroughly mixed. Then three tablespoons of this mixture are poured back into the glass of milk. Which is greater now: the percentage of milk in the tea or the percentage of tea in the milk?

* * *

Problem 16. Form a magic square with the digits 1, 2, 3, 4, 5, 6, 7, 8, and 9; that is, place them in the boxes of a 3×3 table so that all the sums of the numbers along the rows, columns, and two diagonals are equal.

Problem 17. In an arithmetic addition problem the digits were replaced with letters (equal digits by same letters, and different digits by different letters). The result is: LOVES + LIVE = THERE. How many "loves" are "there"? The answer is the maximum possible value of the word THERE.

Problem 18. The secret service of The Federation intercepted a coded message from The Dominion which read: BLASE+LBSA = BASES. It is known that equal digits are coded with equal letters, and different digits with different letters. Two giant computers came up with two different answers to the riddle. Is this possible or does one of them need repair?

Problem 19. Distribute 127 one dollar bills among 7 wallets so that any integer sum from 1 through 127 dollars can be paid without opening the wallets.

* * *

Problem 20. Cut the figure shown in Figure 2 into four figures, each similar to the original with dimensions twice as small.

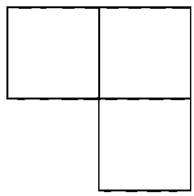

FIGURE 2

Problem 21. Matches are arranged to form the figure shown in Figure 3. Move two matches to change this figure into four squares with sides equal in length to one match.

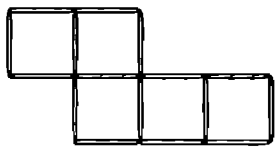

FIGURE 3

Problem 22. A river 4 meters wide makes a 90° turn (see Figure 4). Is it possible to cross the river by bridging it with only two planks, each 3.9 meters long?

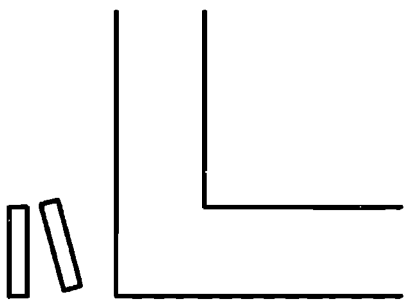

FIGURE 4

Problem 23. Is it possible to arrange six long round pencils so that each of them touches all the others?

Problem 24. Using scissors, cut a hole in a sheet of ordinary paper (say, the size of this page) through which an elephant can pass.

Problem 25. Ten coins are arranged as shown in Figure 5. What is the minimum number of coins we must remove so that no three of the remaining coins lie on the vertices of an equilateral triangle?

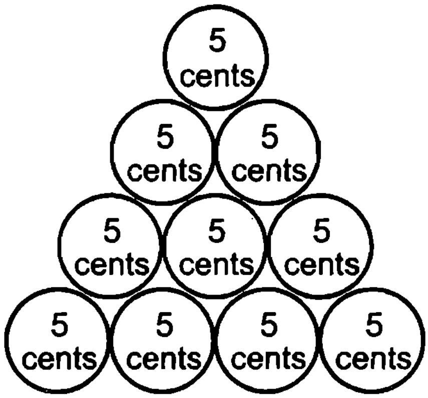

FIGURE 5

Parity

An even number is said to have even parity, and an odd number, odd parity. This concept, despite its utmost simplicity, appears in the solution of the most varied sorts of questions. It turns out to be useful in the solution of many problems, including some which are quite difficult.

The very simplicity of this theme makes it possible to pose interesting problems for students with almost no background. The same simplicity makes it even more important than usual to point out the common theme in all such problems.

§1. Alternations

Problem 1. Eleven gears are placed on a plane, arranged in a chain as shown (see Figure 6). Can all the gears rotate simultaneously?

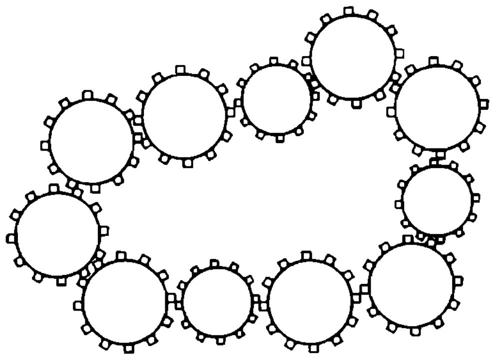

FIGURE 6

Solution. The answer is no. Suppose that the first gear rotates clockwise. Then the second gear must rotate counter-clockwise, the third clockwise again, the fourth counter-clockwise, and so on. It is clear that the "odd" gears must rotate clockwise, while the "even" gears must rotate counter-clockwise. But then the first and eleventh gears must rotate in the same direction. This is a contradiction.

The main idea in the solution to this problem is that the gears rotating clockwise and counter-clockwise alternate. Finding objects that alternate is the basic idea in the solution of the following problems as well.

Problem 2. On a chessboard, a knight starts from square $a1$, and returns there after making several moves. Show that the knight makes an even number of moves.

Problem 3. Can a knight start at square $a1$ of a chessboard, and go to square $h8$, visiting each of the remaining squares exactly once on the way?

Solution. No, he cannot. At each move, a knight jumps from a square of one color to a square of the opposite color. Since the knight must make 63 moves, the last (odd) move must bring him to a square of the opposite color from the square on which he started. However, squares $a1$ and $h8$ are of the same color.

Like Problem 3, many of the problems in this section deal with proofs that certain situations are impossible. Indeed, when a question asks whether some situation is possible, the answer in this section is invariably "no" This poses some difficulty for mathematically naive students. Their first reaction is either frustration that they cannot find the "correct" situation (fulfilling the impossible conditions) or a declaration that the situation is impossible, without a clear conception of what it might take to prove this. Here is a simple problem, related to the "odd and even" problems later in this section, which might clear up this point:

Can you find five odd numbers whose sum is 100?

A discussion can ensue, through which students are made aware that it is not just their own human failing that prevents them from finding this set of numbers, but a contradiction in the nature of the set itself. It is proof by contradiction that is at the basis of the students' confusion, as well as the notion of proof of impossibility. Problems in parity are a simple yet effective way to introduce both these concepts.

Problem 4. A closed path is made up of 11 line segments. Can one line, not containing a vertex of the path, intersect each of its segments?

Problem 5. Three hockey pucks, A, B, and C, lie on a playing field. A hockey player hits one of them in such a way that it passes between the other two. He does this 25 times. Can he return the three pucks to their starting points?

Problem 6. Katya and her friends stand in a circle. It turns out that both neighbors of each child are of the same gender. If there are five boys in the circle, how many girls are there?

Let us note an additional principle, which comes up in the solution of the previous problem: in a closed alternating chain of objects, there are as many objects of one type (boys) as there are of the other (girls).

§2. Partitioning into pairs

Problem 7. Can we draw a closed path made up of 9 line segments, each of which intersects exactly one of the other segments?

Solution. If such a closed path were possible, then all the line segments could be partitioned into pairs of intersecting segments. But then the number of segments would have to be even.

Let us single out the central point in this solution: if a set of objects can be partitioned into pairs, then there are evenly many of them. Here are some similar problems:

Problem 8. Can a 5×5 square checkerboard be covered by 1×2 dominoes?

Problem 9. Given a convex 101-gon which has an axis of symmetry, prove that the axis of symmetry passes through one of its vertices. What can you say about a 10-gon with the same properties?

Problems 10 and 11 concern a set of dominoes consisting of 2×1 rectangles with 0 to 6 spots on each square. All 28 possible pairs of numbers of spots (including doubles) are represented. The game is played by forming a chain in which squares of adjacent dominoes have equal numbers of spots.

Problem 10. All the dominoes in a set are laid out in a chain (so that the number of spots on the ends of adjacent dominoes match). If one end of the chain is a 5, what is at the other end?

Comment. A set of dominoes consists of 2×1 rectangles with 0 to 6 spots on each square. All 28 possible pairs of numbers of spots (including doubles) are represented.

Problem 11. In a set of dominoes, all those in which one square has no spots are discarded. Can the remaining dominoes be arranged in a chain?

Problem 12. Can a convex 13-gon be divided into parallelograms?

Problem 13. Twenty-five checkers are placed on a 25×25 checkerboard in such a way that their positions are symmetric with respect to one of its diagonals. Prove that at least one of the checkers is positioned on that diagonal.

Solution. If no checker occurred on the diagonal, then the checkers could be partitioned into pairs, placed symmetrically with respect to the diagonal. Therefore, there must be one (and in fact an odd number) of checkers on the diagonal.

In solving this problem, students often have trouble understanding that there may be not just one, but any odd number of checkers on the diagonal. For this problem, we may formulate our assertion about partitions into pairs thus: if we form a number of pairs from a set of oddly many objects, then at least one object will remain unpaired.

Problem 14. Let us now assume that the positions of the checkers in Problem 13 are symmetric with respect to both diagonals of the checkerboard. Prove that one of the checkers is placed in the center square.

Problem 15. In each box of a 15×15 square table one of the numbers 1, 2, 3, , 15 is written. Boxes which are symmetric to one of the main diagonals contain equal numbers, and no row or column contains two copies of the same number. Show that no two of the numbers along the main diagonal are the same.

§3. Odd and even

Problem 16. Can one make change of a 25-ruble bill, using in all ten bills each having a value of 1, 3, or 5 rubles?

Solution. It is not possible. This conclusion is based on a simple observation: the sum of evenly many odd numbers is even. A generalization of this fact is this: the parity of the sum of several numbers depends only on the parity of the number of its odd addends. If there are oddly (evenly) many odd addends, then the sum is odd (even).

Problem 17. Pete bought a notebook containing 96 pages, and numbered them from 1 through 192. Victor tore out 25 pages of Pete's notebook, and added the 50 numbers he found on the pages. Could Victor have gotten 1990 as the sum?

Problem 18. The product of 22 integers is equal to 1. Show that their sum cannot be zero.

Problem 19. Can one form a "magic square" out of the first 36 prime numbers?

A "magic square" here means a 6×6 array of boxes, with a number in each box, and such that the sum of the numbers along any row, column, or diagonal is constant.

Problem 20. The numbers 1 through 10 are written in a row. Can the signs "+" and "—" be placed between them, so that the value of the resulting expression is 0?

Note that negative numbers can also be odd or even.

Problem 21. A grasshopper jumps along a line. His first jump takes him 1 cm, his second 2 cm, and so on. Each jump can take him to the right or to the left. Show that after 1985 jumps the grasshopper cannot return to the point at which he started.

Problem 22. The numbers 1, 2, 3, ,1984, 1985 are written on a blackboard. We decide to erase from the blackboard any two numbers, and replace them with their positive difference. After this is done several times, a single number remains on the blackboard. Can this number equal 0?

§4. Assorted problems

Some more difficult problems are collected in this section. Their solutions use the ideas of parity, but also additional considerations.

Problem 23. Can an ordinary 8×8 chessboard be covered with 1×2 dominoes so that only squares $a1$ and $h8$ remain uncovered?

Problem 24. A 17-digit number is chosen, and its digits are reversed, forming a new number. These two numbers are added together. Show that their sum contains at least one even digit.

Problem 25. There are 100 soldiers in a detachment, and every evening three of them are on duty. Can it happen that after a certain period of time each soldier has shared duty with every other soldier exactly once?

Problem 26. Forty-five points are chosen along line AB, all lying outside of segment AB. Prove that the sum of the distances from these points to point A is not equal to the sum of the distances of these points to point B.

Problem 27. Nine numbers are placed around a circle: four 1's and five 0's. The following operation is performed on the numbers: between each adjacent pair of numbers is placed a 0 if the numbers are different, and a 1 if the numbers are the same. The "old" numbers are then erased. After several of these operations, can all the remaining numbers be equal?

Problem 28. Twenty-five boys and 25 girls are seated at a round table. Show that both neighbors of at least one student are boys.

Problem 29. A snail crawls along a plane with constant velocity, turning through a right angle every 15 minutes. Show that the snail can return to its starting point only after a whole number of hours.

Problem 30. Three grasshoppers play leapfrog along a line. At each turn, one grasshopper leaps over another, but not over two others. Can the grasshoppers return to their initial positions after 1991 leaps? (See Figure 7.)

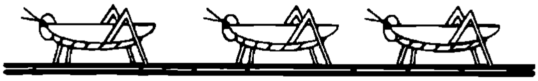

FIGURE 7

Problem 31. Of 101 coins, 50 are counterfeit, and differ from the genuine coins in weight by 1 gram. Peter has a scale in the form of a balance which shows the difference in weights between the objects placed in each pan. He chooses one coin, and wants to find out in one weighing whether it is counterfeit. Can he do this?

Problem 32. Is it possible to arrange the numbers from 1 through 9 in a sequence so that there are oddly many numbers between 1 and 2, between 2 and 3, . . , and between 8 and 9?

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Combinatorics–1

How many ways are there to drive from A to B? How many words does the Hermetian language contain? How many "lucky" six-digit numbers are there? How many   ? These and many other similar questions will be discussed in this chapter.

We will start with a few simple problems.

Problem 1. There are five different teacups and three different tea saucers in the "Tea Party" store. How many ways are there to buy a cup and a saucer?

Solution. First, let us choose a cup. Then, to complete the set, we can choose any of three saucers. Thus we have 3 different sets containing the chosen cup. Since there are five cups, we have 15 different sets ($15 = 5\ \ 3$).

Problem 2. There are also four different teaspoons in the "Tea Party" store. How many ways are there to buy a set consisting of a cup, a saucer, and a spoon?

Solution. Let us start with any of the 15 sets from the previous problem. There are four different ways to complete it by choosing a spoon. Therefore, the number of all possible sets is 60 (since $60 = 15\ \ 4 = 5 \cdot 3\ \ 4$).

In just the same way we can solve the following problem.

Problem 3. There are three towns A, B, and C, in Wonderland. Six roads go from A to B, and four roads go from B to C (see Figure 8). In how many ways can one drive from A to C?
Answer. $24 = 6 \cdot 4$.

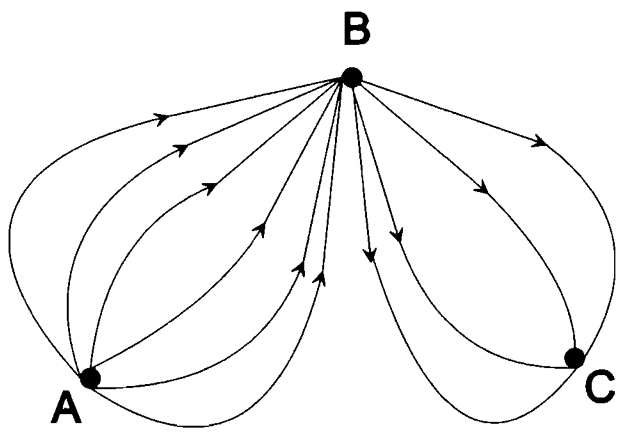

FIGURE 8

In the solution to Problem 4 we use a new idea.

Problem 4. A new town called D and several new roads were built in Wonderland (see Figure 9). How many ways are there to drive from A to C now?

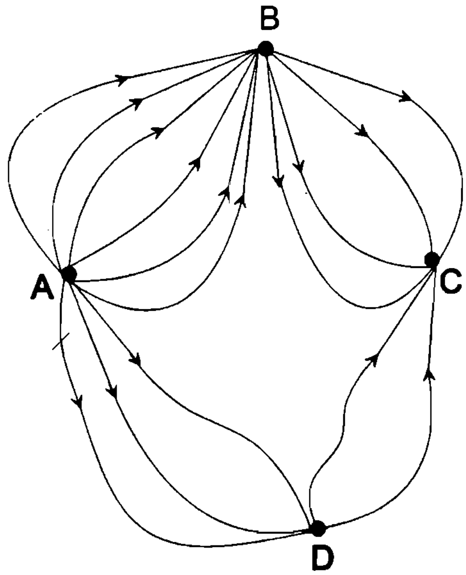

FIGURE 9

Solution. Consider two cases: our route passes either through B or through D. In each case it is quite easy to calculate the number of routes—if we drive through B then we have 24 ways to drive from A to C; otherwise we have 6 ways. To obtain the answer we must add up these two numbers. Thus we have 30 possible routes.

Dividing the problem into several cases is a very useful idea. It also helps in solving Problem 5.

Problem 5. There are five different teacups, three saucers, and four teaspoons in the "Tea Party" store. How many ways are there to buy two items with different names?

Solution. Three cases are possible: we buy a cup and a saucer, or we buy a cup and a spoon, or we buy a saucer and a spoon. It is not difficult to calculate the number of ways each of these cases can occur: 15, 20, and 12 ways respectively. Adding, we have the answer: 47.

For teachers. The main goal which the teacher must pursue during a discussion of these problems is making the students understand when we must add the numbers of ways and when we must multiply them. Of course, many problems should be presented (some can be found at the end of this chapter (Problems 28–32), in [**49**], or created by the teacher). Some possible subjects are shopping, traffic maps, arrangement of objects, etc.

Problem 6. We call a natural number "odd-looking" if all of its digits are odd. How many four-digit odd-looking numbers are there?

Solution. It is obvious that there are 5 one-digit odd-looking numbers. We can add another odd digit to the right of any odd-looking one-digit number in five ways. Thus, we have $5\ \ 5 = 25$ two-digit odd-looking numbers. Similarly, we get $5\ \ 5\ \ 5 = 125$ three-digit odd-looking numbers, and $5\ \ 5 \cdot 5 \cdot 5 = 5^4 = 625$ four-digit odd-looking numbers.

<u>For teachers.</u> In the last problem the answer has the form m^n Usually, an answer of this type results from problems where we can place an element of some given m-element set in each of n given places. In such problems the students may encounter difficulty distinguishing the two numbers m and n, therefore confusing the base and the exponent.

Here are four more similar problems.

Problem 7. We toss a coin three times. How many different sequences of heads and tails can we obtain?
Answer. 2^3

Problem 8. Each box in a 2×2 table can be colored black or white. How many different colorings of the table are there?
Answer. 2^4

Problem 9. How many ways are there to fill in a Special Sport Lotto card? In this lotto you must predict the results of 13 hockey games, indicating either a victory for one of two teams, or a draw.
Answer. 3^{13}

Problem 10. The Hermetian alphabet consists of only three letters: A, B, and C. A word in this language is an arbitrary sequence of no more than four letters. How many words does the Hermetian language contain?

Hint. Calculate separately the numbers of one-letter, two-letter, three-letter, and four-letter words.
Answer. $3 + 3^2 + 3^3 + 3^4 = 120$.

Let us continue with another set of problems.

Problem 11. A captain and a deputy captain must be elected in a soccer team with 11 players. How many ways are there to do this?

Solution. Any of 11 players can be elected as captain. After that, any of the 10 remaining players can be chosen for deputy. Therefore, we have $11\ \ 10 = 110$ different outcomes of elections.

This problem differs from the previous ones in that the choice of captain influences the set of candidates for deputy position, since the captain cannot be his or her own deputy. Thus, the choices of captain and deputy are not independent (as the choices of a cup and a saucer were in Problem 1, for example).

Below we have four more problems on the same theme.

Problem 12. How many ways are there to sew one three-colored flag with three horizontal strips of equal height if we have pieces of fabric of six colors? We can distinguish the top of the flag from the bottom.

Solution. There are six possible choices of a color for the bottom strip. After that we have only five colors to use for the middle strip, and then only four colors for the top strip. Therefore, we have $6 \cdot 5 \cdot 4 = 120$ ways to sew the flag.

Problem 13. How many ways are there to put one white and one black rook on a chessboard so that they do not attack each other?

Solution. The white rook can be placed on any of the 64 squares. No matter where it stands, it attacks exactly 15 squares (including the square it stands on). Thus we are left with 49 squares where the black rook can be placed. Hence there are $64 \cdot 49 = 3136$ different ways.

Problem 14. How many ways are there to put one white and one black king on a chessboard so that they do not attack each other?

Solution. The white king can be placed on any of the 64 squares. However, the number of squares it attacks depends on its position. Therefore, we have three cases:

a) If the white king stands in one of the corners then it attacks 4 squares (including the square it stands on). We have 60 squares left, and we can place the black king on any of them.

b) If the white king stands on the edge of the chessboard but not in the corner (there are 24 squares of this type) then it attacks 6 squares, and we have 58 squares to place the black king on.

c) If the white king does not stand on the edge of the chessboard (we have 36 squares of this type) then it attacks 9 squares, and only 55 squares are left for the black king.

Finally, we have $4 \cdot 60 + 24 \cdot 58 + 36 \cdot 55 = 3612$ ways to put both kings on the chessboard.

* * *

Let us now calculate the number of ways to arrange n objects in a row. Such arrangements are called *permutations*, and they play a significant role in combinatorics and in algebra. But before this we must digress a little bit.

If n is a natural number, then $n!$ (pronounced n *factorial*) is the product $1 \cdot 2 \cdot 3 \cdots n$. Therefore, $2! = 2$, $3! = 6$, $4! = 24$, and $5! = 120$. For convenience of calculations and for consistency, $0!$ is defined to be equal to 1.

<u>Methodological remark.</u> Before working with permutations one must know the definition of factorial and learn how to deal with this function. The following exercises may be useful.

Exercise 1. Simplify the expressions a) $10! \cdot 11$; b) $n! \cdot (n+1)$.

Exercise 2. a) Calculate $100!/98!$; b) Simplify $n!/(n-1)!$.

Exercise 3. Prove that if p is a prime number, then $(p-1)!$ is not divisible by p.

Now let us go back to the permutations.

Problem 15. How many three-digit numbers can be written using the digits 1, 2, and 3 (without repetitions) in some order?

Solution. Let us reason just the same way we did in solving Problem 12. The first digit can be any of the three given, the second can be any of the two remaining digits, and the third must be the one remaining digit. Thus we have $3 \cdot 2 \quad 1 = 3!$ numbers.

Problem 16. How many ways are there to lay four balls, colored red, black, blue, and green, in a row?

Solution. The first place in the row can be occupied by any of the given balls. The second can be occupied by any of the three remaining balls, et cetera. Finally, we have the answer (similar to that of Problem 15): $4 \quad 3 \quad 2 \cdot 1 = 4!$.

Analogously we can prove that n different objects can be laid out in a row in $n \cdot (n-1) \cdot (n-2) \cdot \qquad 2 \quad 1$ ways; that is

the number of permutations of n objects is $n!$.

For convenience of notation we introduce the following convention. Any finite sequence of English letters will be called "a word" (whether or not it can be found in a dictionary). For example, we can form six words using the letters A, B, and C each exactly once: ABC, ACB, BAC, BCA, CAB, and CBA. In the following five problems you must calculate the number of different words that can be obtained by rearranging the letters of a particular word.

Problem 17. "VECTOR"

Solution. Since all the letters in this word are different, the answer is 6! words.

Problem 18. "TRUST"

Solution. This word contains two letters T, and all the other letters are different. Let us temporarily think of these letters T as two different letters T_1 and T_2. Under this assumption we have $5! = 120$ different words. However, any two words which can be obtained from each other just by transposing the letters T_1 and T_2 are, in fact, identical. Thus, our 120 words split into pairs of identical words. This means that the answer is $120/2 = 60$.

Problem 19. "CARAVAN"

Solution. Thinking of the three letters A in this word as different letters A_1, A_2, and A_3, we get 8! different words. However, any words which can be obtained from each other just by transposing the letters A_i are identical. Since the letters A_i can be rearranged in their places in $3! = 6$ ways, all 8! words split into groups of 3! identical words. Therefore the answer is $8!/3!$.

Problem 20. "CLOSENESS"

Solution. We have three letters S and two letters E in this word. Temporarily thinking of all of them as different letters, we have 9! words. When we remember that the letters E are identical the number of different words reduces to $9!/2!$. Then, recalling that the letters S are identical, we come to the final answer: $9!/(2! \cdot 3!)$.

Problem 21. "MATHEMATICAL"
Answer. $12!/(3! \quad 2! \quad 2!)$.

This set of problems about words demonstrates one very interesting and important idea—the idea of multiple counting. That is, instead of counting the number of objects we are interested in, it may be easier to count some other objects whose number is some known multiple of the number of objects.

Here are four more problems using this method.

Problem 22. There are 20 towns in a certain country, and every pair of them is connected by an air route. How many air routes are there?

Solution. Every route connects two towns. We can choose any of the 20 towns in the country (say, town A) as the beginning of a route, and we have 19 remaining towns to choose the end of a route (say, town B) from. Multiplying, we have $20 \quad 19 = 380$. However, this calculation counted every route AB twice: when A was chosen as the beginning of the route, and when B was chosen as the beginning. Hence, the number of routes is $380/2 = 190$.

A similar problem is discussed in the chapter "Graphs–1" where we count the number of edges of a graph.

Problem 23. How many diagonals are there in a convex n-gon?

Solution. Any of the n vertices can be chosen as the first endpoint of a diagonal, and we have $n - 3$ vertices to choose from for the second end (any vertex, except the chosen one and its two neighbors). Counting the diagonals this way, we have counted every diagonal exactly twice. Hence, the answer is $n(n-3)/2$. (See Figure 10.)

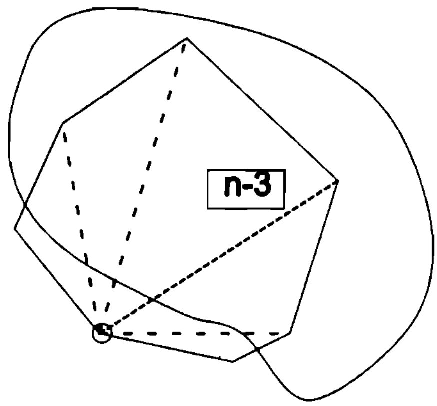

FIGURE 10

Problem 24. A "necklace" is a circular string with several beads on it. It is allowed to rotate a necklace but not to turn it over. How many different necklaces can be made using 13 different beads?

Solution. Let us first assume that it is prohibited to rotate the necklace. Then it is clear that we have 13! different necklaces. However, any arrangement of beads must be considered identical to those 12 that can be obtained from it by rotation. (See Figure 11.)
Answer: $13!/13 = 12!$.

Problem 25. Assume now that it is allowed to turn a necklace over. How many necklaces can be made using 13 different beads?

Solution. Turning the necklace over divides the number of necklaces by 2.
Answer: $12!/2$.

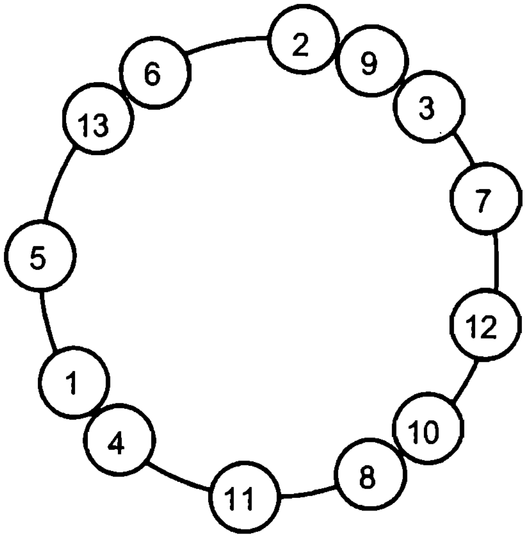

FIGURE 11

The following problem illustrates another important combinatorial idea.

Problem 26. How many six-digit numbers have at least one even digit?

Solution. Instead of counting the numbers with at least one even digit, let us find the number of six-digit numbers that do not possess this property. Since these are exactly the numbers with all their digits odd, there are $5^6 = 15625$ of them (see Problem 6). Since there are 900000 six-digit numbers in all, we conclude that the number of six-digit numbers with at least one even digit is $900000 - 15625 = 884375$.

The main idea in this solution was to use the method of complements; that is, counting (or, considering) the "unrequested" objects instead of those "requested" Here is another problem which can be solved using this method.

Problem 27. There are six letters in the Hermetian language. A word is any sequence of six letters, some pair of which are the same. How many words are there in the Hermetian language?
Answer. $6^6 - 6!$.

For teachers. In conclusion we would like to note that it is reasonable to devote a separate session to any idea which ties together the problems of each set in this chapter (and, perhaps, with other themes more distant from combinatorics). We also recommend reviewing the material already covered in previous sessions. For this reason we present here a list of problems for independent solution and for homework. In addition, you can take problems from [49] or create them yourself.

Problems for independent solution

Problem 28. There are five types of envelopes and four types of stamps in a post office. How many ways are there to buy an envelope and a stamp?

Problem 29. How many ways are there to choose a vowel and a consonant from the word "RINGER"?

Problem 30. Seven nouns, five verbs, and two adjectives are written on a blackboard. We can form a sentence by choosing one word of each type, and we do not care about how much sense the sentence makes. How many ways are there to do this?

Problem 31. Each of two novice collectors has 20 stamps and 10 postcards. We call an exchange fair if they exchange a stamp for a stamp or a postcard for a postcard. How many ways are there to carry out one fair exchange between these two collectors?

Problem 32. How many six-digit numbers have all their digits of equal parity (all odd or all even)?

Problem 33. In how many ways can we send six urgent letters if we can use three messengers and each letter can be given to any of them?

Problem 34. How many ways are there to choose four cards of different suits and different values from a deck of 52 cards?

Problem 35. There are five books on a shelf. How many ways are there to arrange some (or all) of them in a stack? The stack may consist of a single book.

Problem 36. How many ways are there to put eight rooks on a chessboard so that they do not attack each other?

Problem 37. There are N boys and N girls in a dance class. How many ways are there to arrange them in pairs for a dance?

Problem 38. The rules of a chess tournament say that each contestant must play every other contestant exactly once. How many games will be played if there are 18 participants?

Problem 39. How many ways are there to place a) two bishops; b) two knights; c) two queens on a chessboard so that they do not attack each other?

Problem 40. Mother has two apples, three pears, and four oranges. Every morning, for nine days, she gives one fruit to her son for breakfast. How many ways are there to do this?

Problem 41. There are three rooms in a dormitory: one single, one double, and one for four students. How many ways are there to house seven students in these rooms?

Problem 42. How many ways are there to place a set of chess pieces on the first row of a chessboard? The set consists of a king, a queen, two identical rooks, two identical knights, and two identical bishops.

Problem 43. How many "words" can be written using exactly five letters A and no more than three letters B (and no other letters)?

Problem 44. How many ten-digit numbers have at least two equal digits?

Problem 45.* Do seven-digit numbers with no digits 1 in their decimal representations constitute more than 50% of all seven-digit numbers?

Problem 46. We toss a die three times. Among all possible outcomes, how many have at least one occurrence of six?

Problem 47. How many ways are there to split 14 people into seven pairs?

Problem 48.* How many nine-digit numbers have an even sum of their digits?

Divisibility and Remainders

For teachers. This theme is not so recreational as some others, yet it contains large amounts of important theoretical material. Try to introduce elements of play in your sessions. Even very routine problems like the factoring of integers can be turned into a contest by asking "Who can factor this huge number first?" or "Who can find the greatest prime divisor of this number first?" Thus, sessions devoted to this topic must be prepared more carefully than others. Since divisibility also enters into the school curriculum, you can use the knowledge acquired by students there.

§1. Prime and composite numbers

Among natural numbers we can distinguish prime and composite numbers. A number is *composite* if it is equal to the product of two smaller natural numbers. For example, $6 = 2 \cdot 3$. Otherwise, and if the number is not equal to 1, it is called *prime*. The number 1 is neither prime nor composite.

Prime numbers are like "bricks", which you can use to construct all natural numbers. How can this be done? Let us consider the number 420. It is certainly composite. It can be represented, for instance, as $42 \cdot 10$. But each of the numbers 42 and 10 is composite, too. Indeed, $42 = 6 \cdot 7$, and $10 = 2 \cdot 5$. Since $6 = 2 \cdot 3$, we have $420 = 42 \cdot 10 = 6 \cdot 7 \cdot 2 \cdot 5 = 2 \cdot 3 \cdot 7 \cdot 2 \cdot 5 = 2 \cdot 2 \cdot 3 \cdot 5 \cdot 7$ (see Figure 12). This is the complete "decomposition" of our number (its representation as a product of prime numbers).

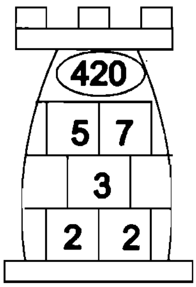

FIGURE 12

It is clear that we can factor any natural number greater than 1 in the same way. We just keep factoring the numbers we have into pairs of smaller numbers as long as we can (and if any one of the factors cannot be represented as such a product, then it is a prime factor).

But what if we try to factor the number 420 in some other way? For example, we can start with $420 = 15 \cdot 28$. It may surprise you that we will always end up with the same representation (products which differ only in the order of their factors are considered identical—we usually arrange the factors in increasing order).

This may seem evident, but it is not easy to prove. It is called the **Fundamental Theorem of Arithmetic**: any natural number different from 1 can be uniquely represented as a product of prime numbers in increasing order.

<u>**For teachers.**</u> Most of the contents of this section are connected with the Fundamental Theorem of Arithmetic.

Students should understand that the properties of divisibility are almost completely determined by the representation of a natural number as the product of prime numbers. The following exercises will help.

1. Is $2^9 \cdot 3$ divisible by 2?

Answer. Yes, since 2 is one of the factors in the decomposition of the given number.

2. Is $2^9 \cdot 3$ divisible by 5?

Answer. No, since the decomposition of this number does not contain the prime number 5.

3. Is $2^9\ \ 3$ divisible by 8?

Answer. Yes, since $8 = 2^3$, and there are nine 2's in the decomposition of the given number.

4. Is $2^9\ \ 3$ divisible by 9?

Answer. No, since $9 = 3\ \ 3$, and there is only one 3 in the decomposition of the given number (see Figure 13).

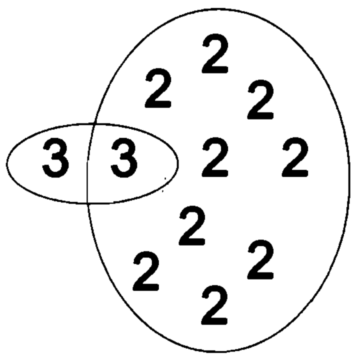

FIGURE 13

5. Is $2^9 \cdot 3$ divisible by 6?

Answer. Yes, since $6 = 2 \cdot 3$, and the decomposition of the given number contains both the prime numbers 2 and 3 (see Figure 14).

6. Is it true that if a natural number is divisible by 4 and by 3, then it must be divisible by $4\ \ 3 = 12$?