18.650 – Fundamentals of Statistics

4. Hypothesis testing

Goals

We have seen the basic notions of hypothesis testing:

- Hypotheses H_0/H_1, not symmetric, H0:status quo, H1: discovery
- Type 1/Type 2 error, level and power
- Test statistics and rejection region
- p-value

Our tests were based on CLT (and sometimes Slutsky)...

- What if data is Gaussian, σ^2 is unknown and Slutsky does not apply? T-test 需要假设样本服从正态分布
- Can we use asymptotic normality of MLE? Wald's test
  不止有CLT可以有渐进正态，MLE也可以有渐进协方差矩阵（I^-1）。 如果感兴趣的不是均值，就不能用CLT了。
- Tests about multivariate parameters $\theta = (\theta_1, \dots, \theta_d)$ (e.g.: $\theta_1 = \theta_2$)? Implicit hypotheses
- More complex tests: "Does my data follow a Gaussian distribution"? Goodness of fit.

Parametric hypothesis testing

Clinical trials

Let us go through an example to remind the main notions of hypothesis testing.

- Pharmaceutical companies use hypothesis testing to test if a new drug is efficient.
- To do so, they administer a drug to a group of patients (test group) and a placebo to another group (control group).
- We consider testing a drug that is supposed to lower LDL (low-density lipoprotein), a.k.a "bad cholesterol" among patients with a high level of LDL (above 200 mg/dL)

Notation and modelling

- Let $\Delta_{\mathrm{d}} > 0$ denote the expected decrease of LDL level (in mg/dL) for a patient that has used the drug.
- Let $\Delta_{\mathrm{c}} \geq 0$ denote the expected decrease of LDL level (in mg/dL) for a patient that has used the placebo.
- We want to know if $\Delta_d > \Delta_c$
- We observe two independent samples:
  - $X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\Delta_d, \sigma_{\mathrm{d}}^2)$ from the test group and
  - $Y_1, \ldots, Y_m \overset{iid}{\sim} \mathcal{N}(\Delta_c, \sigma_{\mathrm{c}}^2)$ from the control group.

Hypothesis testing

- Hypotheses:

$$H_0 : \Delta_c = \Delta_d \quad \text{vs.} \quad H_1 : \Delta_d > \Delta_c$$

- Since the data is Gaussian by assumption we don't need the CLT

- We have

$$\bar{X}_n \sim \mathcal{N}\left(\Delta_d, \frac{\sigma_d^2}{n}\right) \quad \text{and} \quad \bar{Y}_m \sim \mathcal{N}\left(\Delta_c, \frac{\sigma_c^2}{m}\right)$$

- Therefore

$$\frac{\bar{X}_n - \bar{Y}_m - (\Delta_\mathrm{d} - \Delta_\mathrm{c})}{\sqrt{\frac{\sigma_d^2}{n} + \frac{\sigma_c^2}{m}}} \sim \mathcal{N}(0,1)$$

方差可加

Asymptotic test

$$\frac{\bar{X}_n - \bar{Y}_m - (\Delta_c - \Delta_d)}{\sqrt{\frac{\hat\sigma_d^2}{n} + \frac{\hat\sigma_c^2}{m}}} \xrightarrow{(d)} N(0,1) \quad \text{(actually equal).}$$

$$\times \frac{\sqrt{\frac{\sigma_d^2}{n} + \frac{\sigma_c^2}{m}}}{\sqrt{\frac{\hat\sigma_d^2}{n} + \frac{\hat\sigma_c^2}{m}}} \xrightarrow[n\to\infty]{\mathbb{P}} 1 \quad \text{by CMT}$$

$\to$ Slutsky

- Assume that $m = cn$ and $n \to \infty$
- Using Slutsky's lemma, we also have

在H0下，他们相等（alpha水平是假设在H0下拒绝的概率）。

$$\frac{\bar{X}_n - \bar{Y}_m - \boxed{(\Delta_d - \Delta_c)}}{\sqrt{\frac{\boxed{\hat{\sigma}}_d^2}{n} + \frac{\boxed{\hat{\sigma}}_c^2}{m}}} \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,1)$$

m也要趋近无穷
这里assume m is a constant times n

where

scaling by 1/(n-1) leads to an unbiased estimator

$$\hat{\sigma}_{\mathrm{d}}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 \quad \text{and} \quad \hat{\sigma}_{\mathrm{c}}^2 = \frac{1}{m-1}\sum_{i=1}^{m}(Y_i - \bar{Y}_m)^2$$

- We get the the following test at asymptotic level α:

$$R_\psi = \left\{ \frac{\bar{X}_n - \bar{Y}_m}{\sqrt{\frac{\hat\sigma_d^2}{n} + \frac{\hat\sigma_c^2}{m}}} > q_\alpha \right\}$$

$= q_\alpha$ is $(1-\alpha)$ quantile of $N(0,1)$

- This is one -sided, two -sample test.

Asymptotic test

- Example $n = 70, m = 50,\ \bar{X}_n = 156.4,\ \bar{Y}_m = 132.7,$ $\hat{\sigma}_{\mathrm{d}}^2 = 5198.4,\ \hat{\sigma}_{\mathrm{c}}^2 = 3867.0,$

$$\frac{156.4 - 132.7}{\sqrt{\frac{5198.4}{70} + \frac{3867.0}{50}}} = 1.57$$

Since $q_{5\%} = 1.645$, we fail to reject H_0

- We can also compute the p-value:

$$\text{p-value} = \mathbb{P}(\mathcal{N}(0,1) > 1.57) = 0.0582$$

p-value is the probability that i would reject even more than that i'm currently doing
It's probability that the distribution of my test statistic is larger than the actual value of my test statistic


PHARMACOLOGICAL DRUG TRIAL RESULTS
OUR TRIALS SHOW THAT THE NEW DRUG PERFORMS NO BETTER THAN PLACEBO
MAYBE WE SHOULD INVEST IN PLACEBOS
CHRIS MADDEN

Small sample size

- What if $n = 20, m = 12$?
- We cannot realistically apply Slutsky's lemma

样本小，所以用不了Slutsky
虽然CLT需要用Slutsky，但是我们并不需要用CLT
我们只是在样本方差替代总体方差的时候，使用Slutsky

- We needed it to find the (asymptotic) distribution of quantities of the form

如果数据是正态分布，那么X_bar - mu也是正态分布。相当于一个均值为0的正态分布。（linear combination）

$$\frac{\boxed{\bar{X}_n - \mu}}{\sqrt{\hat{\sigma}^2}}$$

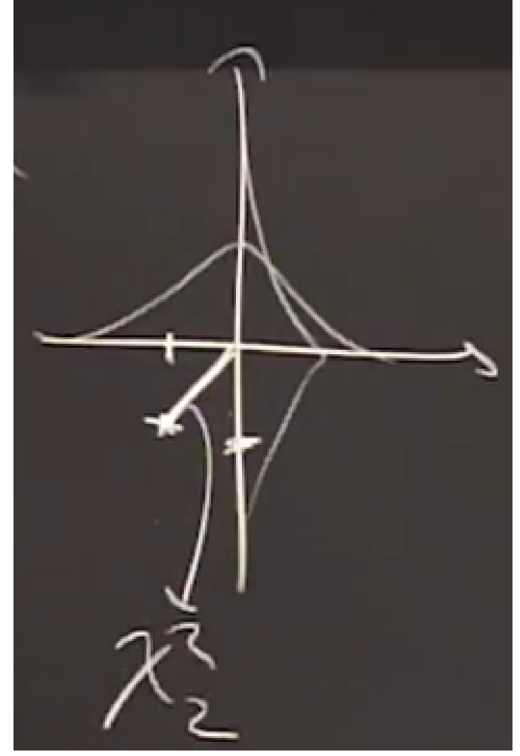
when $X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$.

- It turns out that this distribution *does not depend on μ or σ* so we can compute its quantiles

The χ^2 distribution

measuring the length of Gaussian vector(square length)

二维正态分布点离原点的距离的平方。



Definition

For a positive integer d, the χ^2 *(pronounced "Kai-squared") distribution with d degrees of freedom* is the law of the random variable $Z_1^2 + Z_2^2 + \ldots + Z_d^2$, where $Z_1, \ldots, Z_d \overset{iid}{\sim} \mathcal{N}(0, 1)$.

这里的d是指有几个data

Examples:

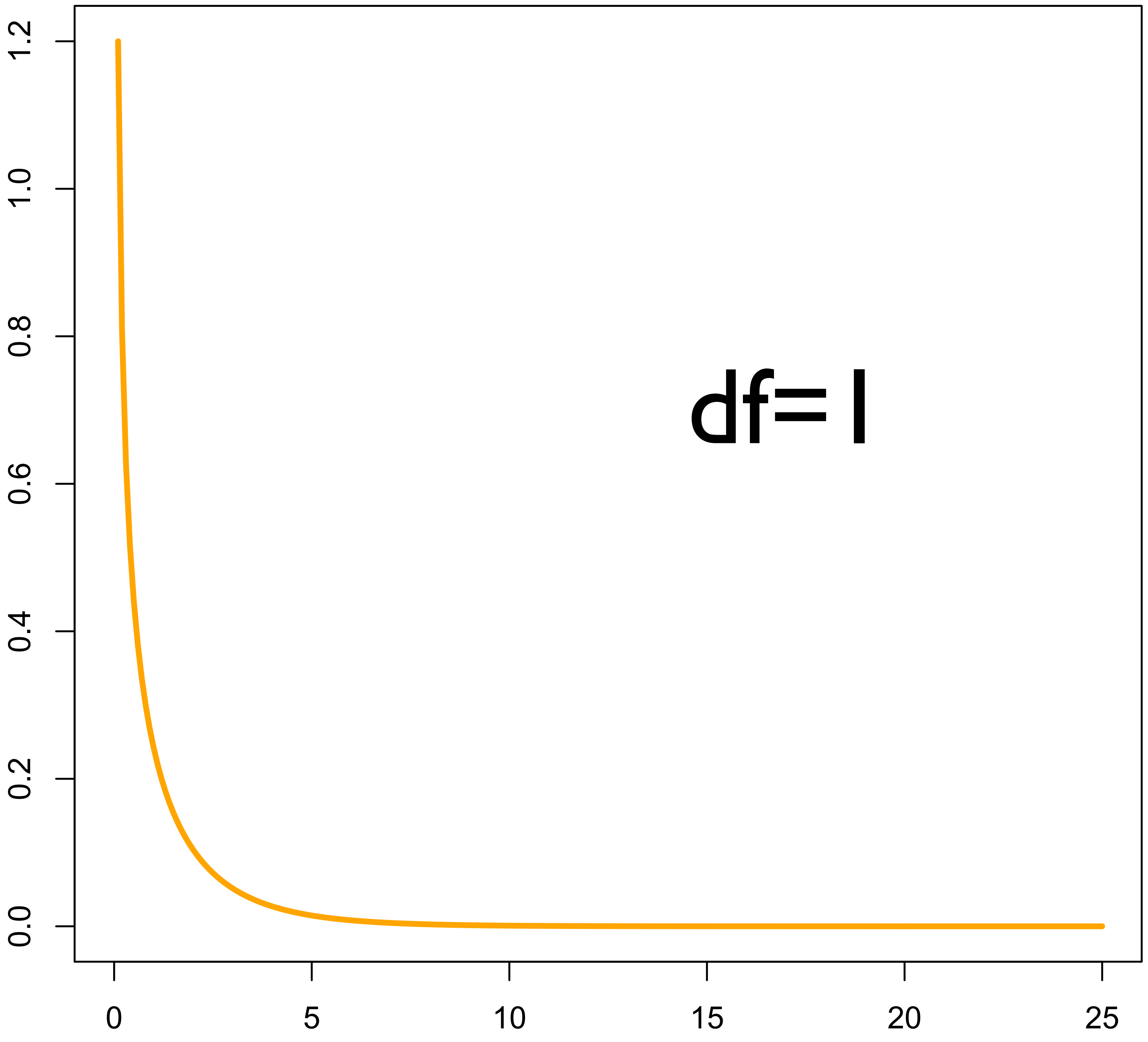
这里的d是维度

L2 norm
Euclidean norm

- If $Z \sim \mathcal{N}_d(\mathbf{0}, I_d)$, then $\|Z\|_2^2 \sim \chi_d^2$
- $\chi_2^2 = \text{Exp}(1/2)$.

pdf of $X \sim \chi_k^2 = \Gamma(\frac{k}{2}, \frac{1}{2})$

$$\frac{1}{2^{k/2}\Gamma(\frac{k}{2})} x^{k/2-1} e^{-x/2}, \quad x > 0$$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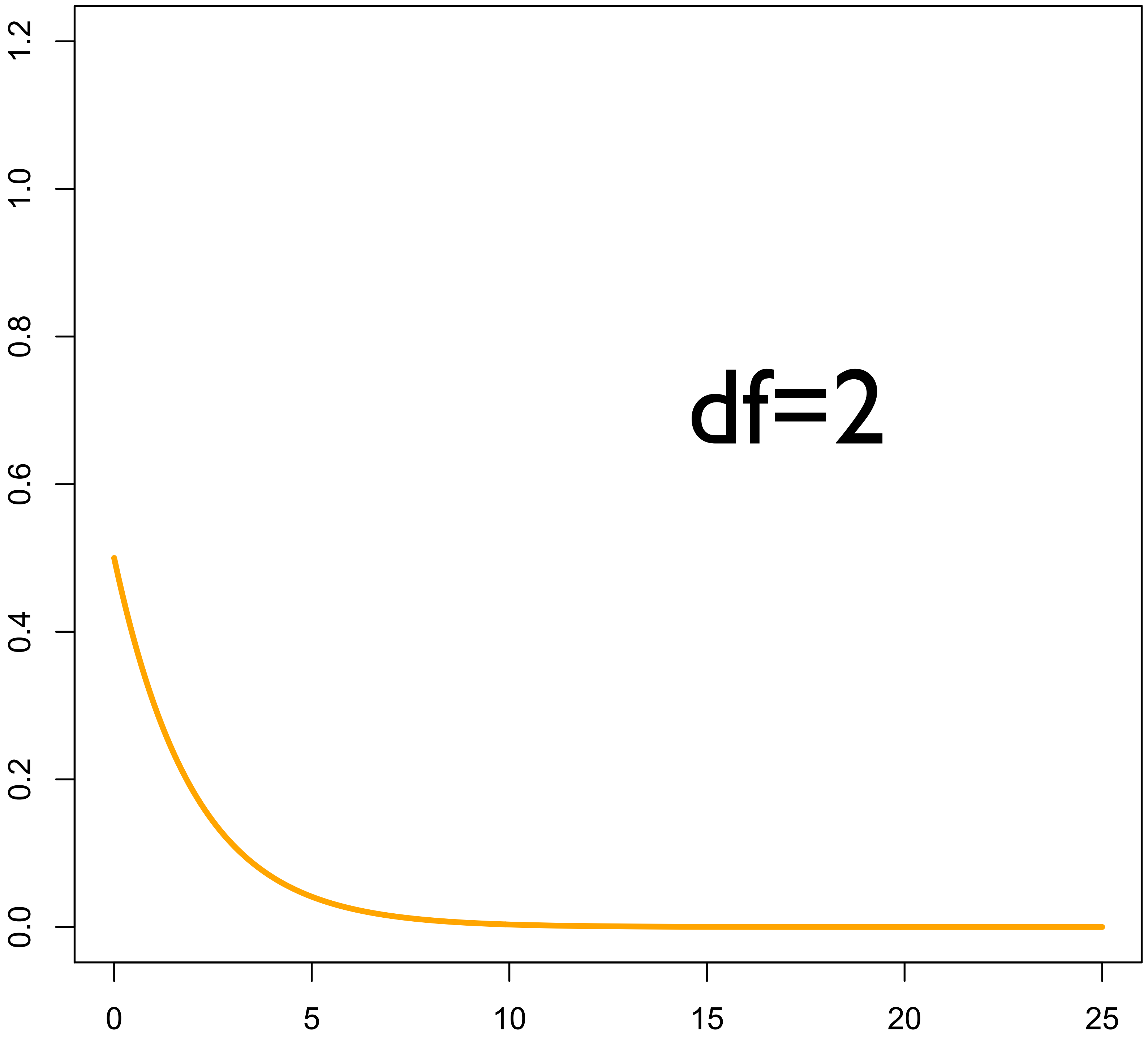
0.8
1.0
1.2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15
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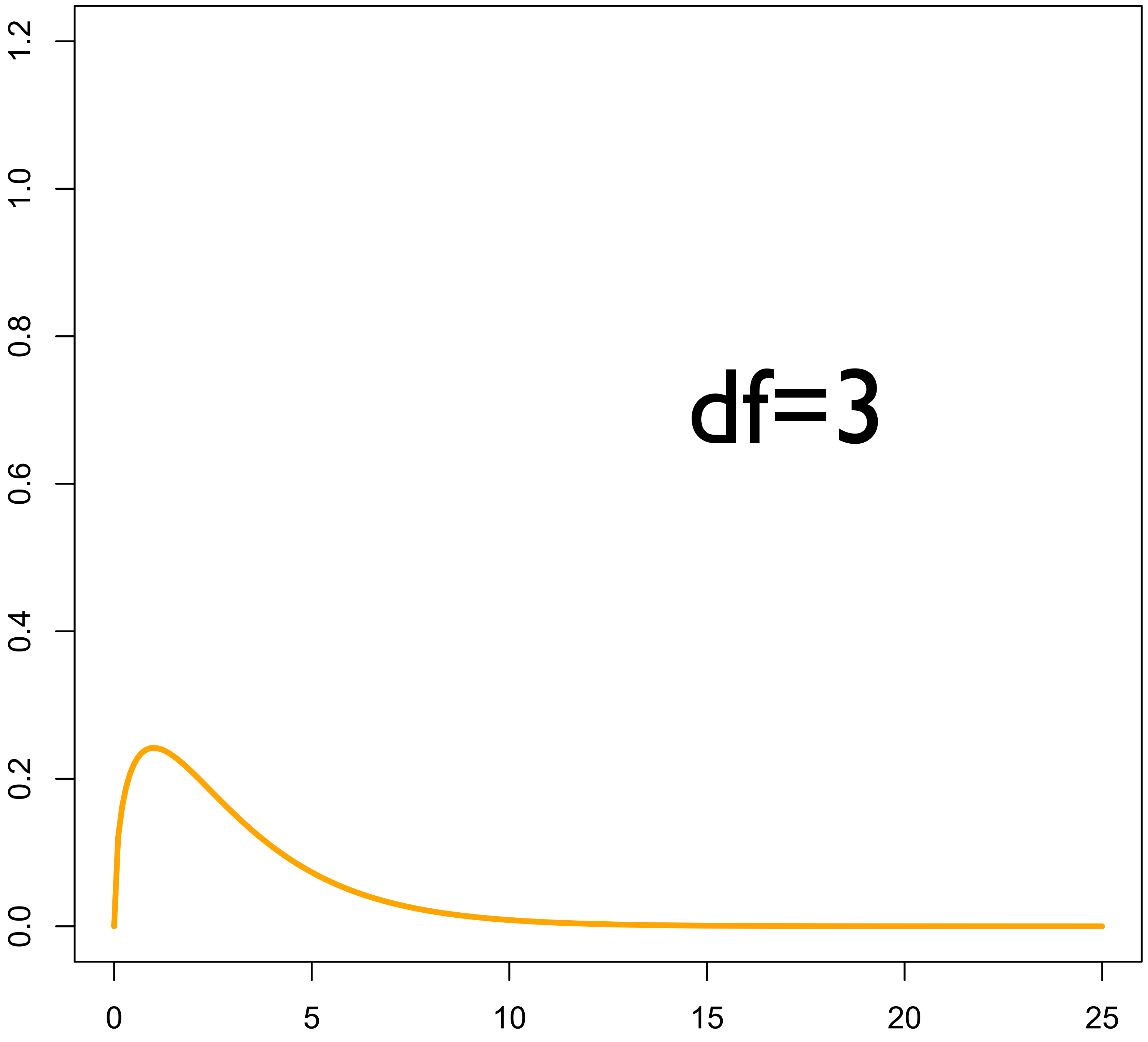
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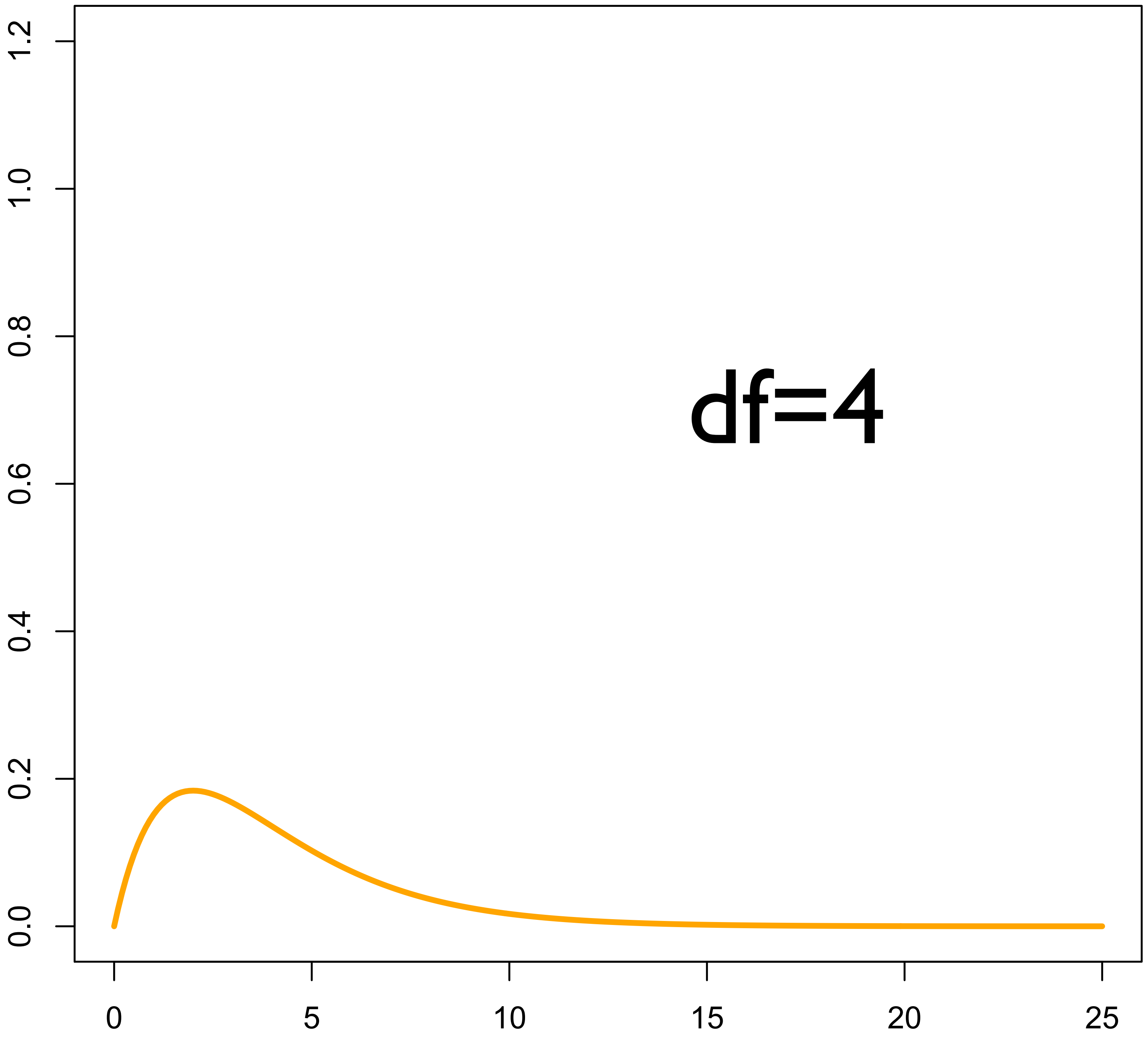

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15
20
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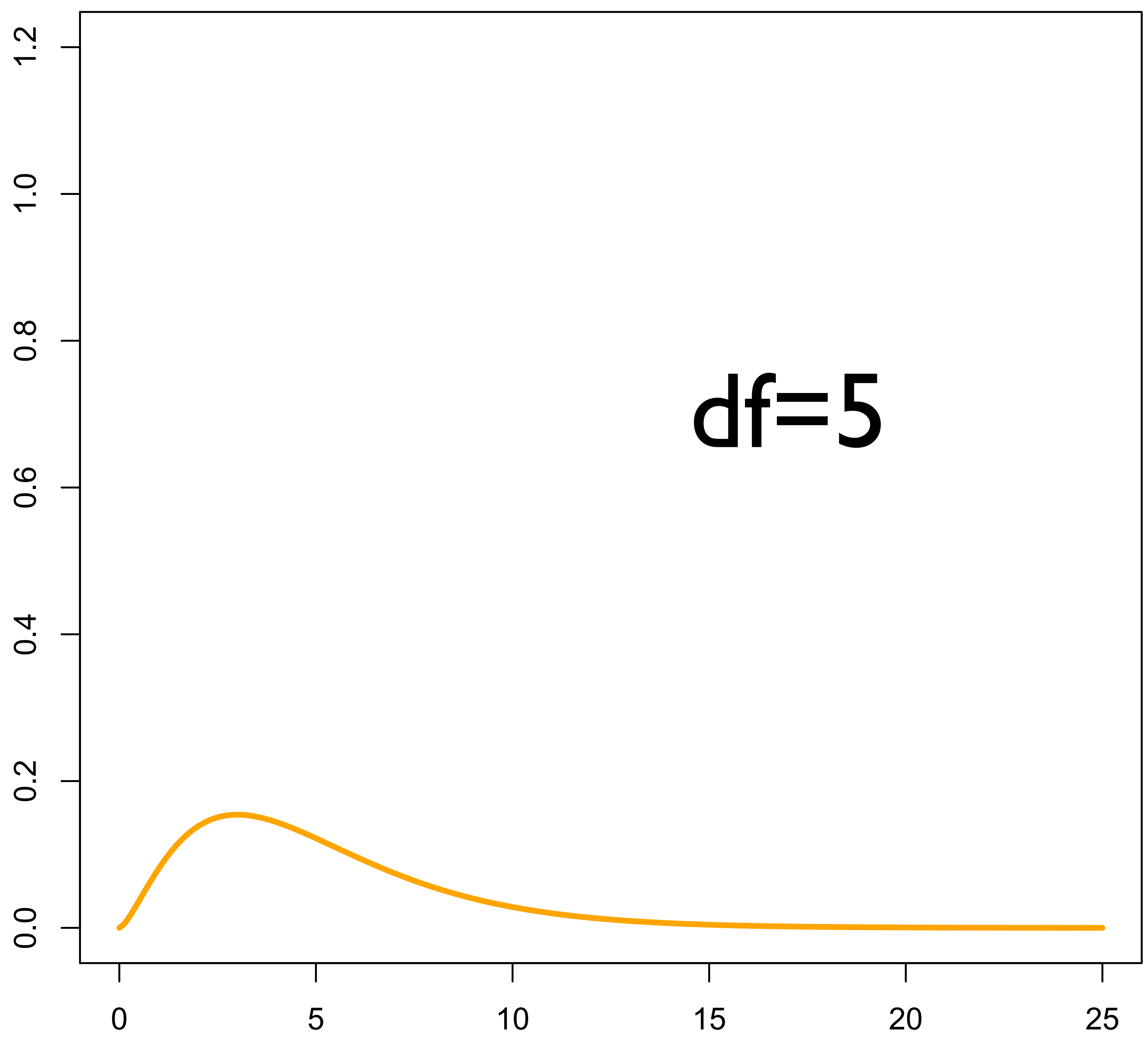
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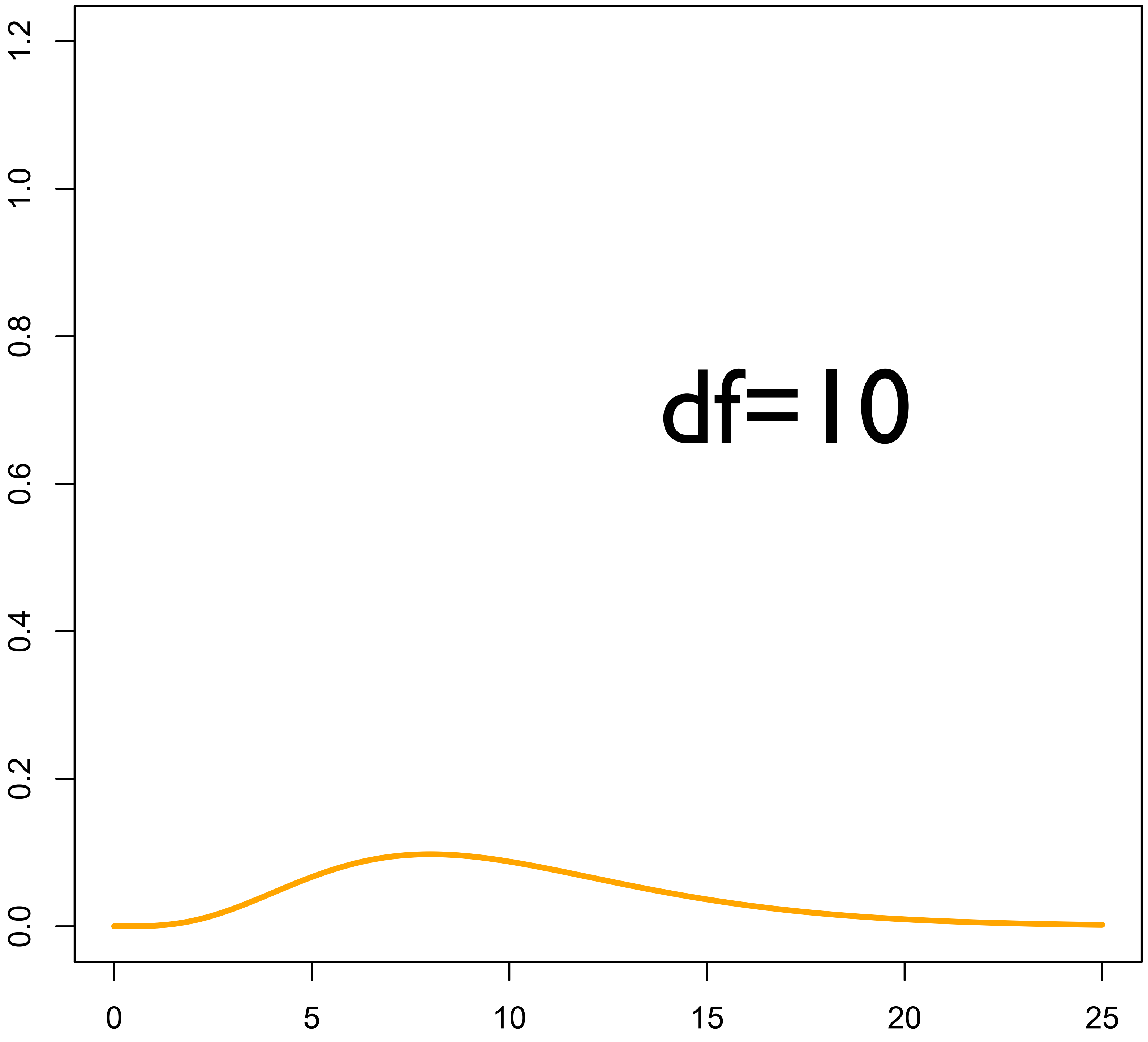

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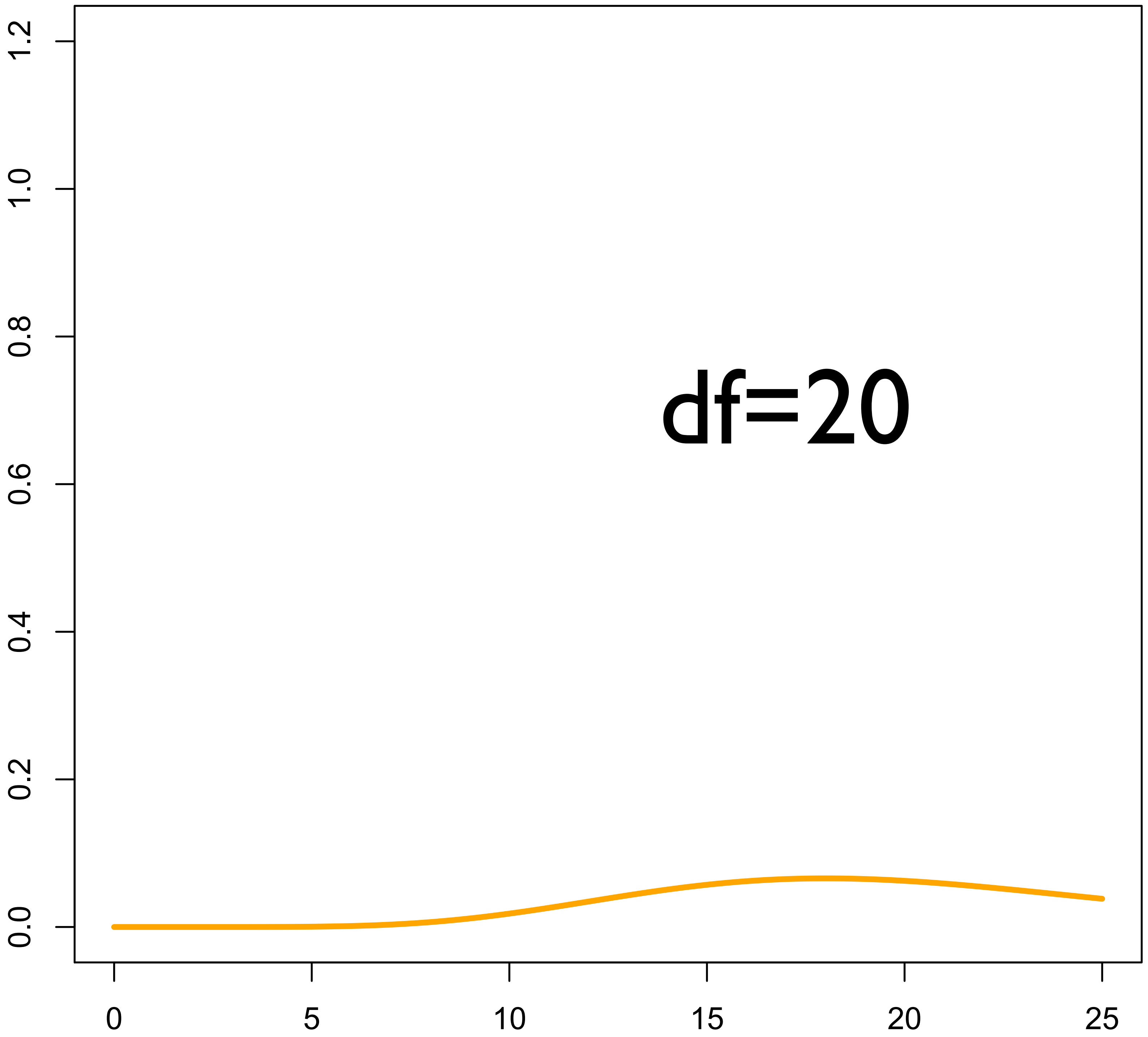
10
15
20
25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0.4
0.6
0.8
1.0
1.2
0
5
10
15
20
25

Properties χ^2 distribution (2)

Definition

For a positive integer d, the χ^2 *(pronounced "Kai-squared") distribution with d degrees of freedom* is the law of the random variable $Z_1^2 + Z_2^2 + \ldots + Z_d^2$, where $Z_1, \ldots, Z_d \overset{iid}{\sim} \mathcal{N}(0,1)$.

Properties: If $V \sim \chi_k^2$, then

- $\mathbb{E}[V] =$ $\mathbb{E}[Z_1^2] + \cdots + \mathbb{E}[Z_d^2] = d$
- $\text{var}[V] =$ $\text{Var}[Z_1^2] + \cdots + \text{Var}[Z_d^2] = 2d$

$\text{Var}[Z_1^2] = \mathbb{E}[Z_1^4] - 1 = 3 - 1 = 2$

third moment is measuring skewedness
forth moment is measuring kurtosis

如果d很大，那么chi^2 -> N(d,2d) — CLT

Important example: the sample variance

- Recall that the sample variance is given by

$$S_n = \boxed{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2} = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\bar{X}_n)^2$$

- Cochran's theorem states that for $X_1, \dots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$, if S_n is the sample variance, then
  - $\underbrace{\bar{X}_n \perp\!\!\!\perp S_n}_{\text{independent}}$; for all n. 正态分布下，样本均数和样本方差互相独立。
  - $\frac{n\boxed{S_n}}{\sigma^2} \sim \chi^2_{\boxed{n-1}}$. 这个是自由度，n个r.v.只有n-1个自由度。他不是n个独立的正态分布变量，少了一个自由度。因为你知道，这些数加起来是0，但是你当你把n个独立的正态分布变量加起来的时候，你几乎肯定得不到0。

$E[S_n] = \frac{n-1}{n}\sigma^2$

  - We often prefer the unbiased estimator of σ^2:

$$\tilde{S}_n = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 = \frac{n}{n-1} S_n$$

$$E[\tilde{S}_n] = \frac{n}{n-1} E\left[\frac{\sigma^2}{n}\chi^2_{n-1}\right] = \frac{n\sigma^2}{n-1}\,\frac{n-1}{n} = \sigma^2$$

Student's T distribution

Definition

For a positive integer d, the *Student's T distribution with d degrees of freedom* (denoted by t_d) is the law of the random variable $\frac{Z}{\sqrt{V/d}}$, 这个是定义 where $Z \sim \mathcal{N}(0,1)$, $V \sim \chi^2_d$ and $Z \perp\!\!\!\perp V$ (Z is independent of V).

这个的sample variance就是 V

VOLUME VI MARCH, 1908 No. 1

BIOMETRIKA.

THE PROBABLE ERROR OF A MEAN.

BY STUDENT.

Introduction.

ANY experiment may be regarded as forming an individual of a "population" of experiments which might be performed under the same conditions. A series of experiments is a sample drawn from this population.

Now any series of experiments is only of value in so far as it enables us to form a judgment as to the statistical constants of the population to which the experiments belong. In a great number of cases the question finally turns on the value of a mean, either directly, or as the mean difference between the two quantities.

If the number of experiments be very large, we may have precise information

Who was Student?

This distribution was introduced by **William Sealy Gosset** (1876–1937) in 1908 while he worked for the Guinness brewery in Dublin, Ireland.

$\bar{X}_n \sim N(\mu, \frac{\sigma^2}{n})$

$\frac{\bar{X}_n-\mu}{\sqrt{\sigma^2/n}} = \sqrt{n}\,\frac{\frac{\bar{X}_n-\mu}{\sigma}}{\sqrt{\frac{S_n}{\sigma^2}}}$

$\sqrt{n}\cdot\frac{\bar{X}-\mu}{\sqrt{\sigma^2}} \sim N(0,1)$

if $\bar{X}$ is Gaussian.

$\Rightarrow \sqrt{n}\cdot\frac{\bar{X}-\mu}{\sqrt{\hat{\sigma}^2}} \sim t_{n-1}$

除以标准差乘以样本标准差

Student's T test (one sample, two-sided)

两种t检验的理解方式
1，N(0,1)除以自由度为n-1的卡方分布
2，x_bar是正态分布时，如果想用样本方差估计总体方差，就要将标准正态分布换成自由度为n-1的t分布

- Let $X_1, \dots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$ where both μ and σ^2 are unknown   non-asymptotic: have to assume the distribution, .
- We want to test:

$$H_0 : \mu = 0, \quad \text{vs} \quad H_1 : \mu \neq 0$$

- Test statistic:

under H0 所以是0

$$T_n = \frac{\bar{X}_n}{\sqrt{\tilde{S}_n / n}} = \frac{\frac{\sqrt{n}\,(\bar{X}_n - \cancel{\mu})}{\sigma}}{\sqrt{\tilde{S}_n / \sigma^2}} \qquad \frac{\sqrt{n}\,(\bar{X}_n - \mu)}{\sigma} \sim \mathcal{N}(0,1)$$

- Since $\sqrt{n}\,\bar{X}_n/\sigma \sim \mathcal{N}(0,1)$ (under) and $\tilde{S}_n/\sigma^2 \sim \frac{\chi^2_{n-1}}{n-1}$ are independent by Cochran's theorem, we have:

$$T_n \sim t_{n-1}$$

- Student's test with (non asymptotic) level $\alpha \in (0, 1)$:

$$\psi_\alpha = \mathbb{1}\{|T_n| > q_{\alpha/2}\},$$

where $q_{\alpha/2}$ is the $(1 - \alpha/2)$-quantile of t_{n-1}.

Student's T test (one sample, one-sided)

- We want to test:

等于mu0和小于mu0其实是一个意思

$$H_0 : \mu \leq \mu_0, \quad \text{vs} \quad H_1 : \mu > \mu_0$$

- Test statistic:

$$T_n = \boxed{\sqrt{n}} \frac{\bar{X}_n - \mu_0}{\sqrt{\tilde{S}_n}} \sim t_{n-1} \quad \text{under } H_0$$

under H_0.
- Student's test with (non asymptotic) level $\alpha \in (0, 1)$:

$$\psi_\alpha = \mathbb{1}\{T_n > q_\alpha\},$$

where q_α is the $(1-\alpha)$-quantile of t_{n-1}

Two-sample T-test

- Back to our cholesterol example. What happens for small sample sizes?
- We want to know the distribution of

$$\frac{\bar{X}_n - \bar{Y}_m - (\Delta_{\mathrm{d}} - \Delta_{\mathrm{c}})}{\sqrt{\frac{\hat{\sigma}_{\mathrm{d}}^2}{n} + \frac{\hat{\sigma}_{\mathrm{c}}^2}{m}}}$$

- We have approximately

$$\frac{\bar{X}_n - \bar{Y}_m - (\Delta_{\mathrm{d}} - \Delta_{\mathrm{c}})}{\sqrt{\frac{\hat{\sigma}_{\mathrm{d}}^2}{n} + \frac{\hat{\sigma}_{\mathrm{c}}^2}{m}}} \sim t_N$$

where

$$N = \frac{\left(\hat{\sigma}_{\mathrm{d}}^2/n + \hat{\sigma}_{\mathrm{c}}^2/m\right)^2}{\frac{\hat{\sigma}_{\mathrm{d}}^4}{n^2(n-1)} + \frac{\hat{\sigma}_{\mathrm{c}}^4}{m^2(m-1)}} \geq \min(n, m)$$

估计自由度(越大越接
近正态，越小越保守)

(Welch-Satterthwaite formula)

Non-asymptotic test

- Example $n = 70, m = 50$, $\bar{X}_n = 156.4$, $\bar{Y}_m = 132.7$, $\hat{\sigma}^2_{\mathrm{d}} = 5198.4$, $\hat{\sigma}^2_{\mathrm{c}} = 3867.0$,

$$\frac{156.4 - 132.7}{\sqrt{\frac{5198.4}{70} + \frac{3867.0}{50}}} = \boxed{1.9248}$$

- Using the shorthand formula $N = \min(n, m) = 50$, we get $q_{5\%} = 1.68$ and

more conservative than normal
(tail has more weight)

$$\text{p-value} = \mathbb{P}[t_{50} > 1.57] = \boxed{0.029974}$$

- Using the W-S formula

$$N = \frac{\left(\frac{5198.4}{70} + \frac{3867.0}{50}\right)^2}{\frac{5198.4^2}{70^2(70-1)} + \frac{3867.0^2}{50^2(50-1)}} = 113.78$$

we round to 113 .

- We get

$$\text{p-value} = \mathbb{P}[t_{113} > 1.57] = \boxed{0.2832}$$

随着自由度增加，p值变小到和正态中的一样。(Slutsky)

Non-asymptotic test

- Example $n = 20$, $m = 12$, $\bar{X}_n = 156.4$, $\bar{Y}_m = 132.7$, $\hat{\sigma}_{\mathrm{d}}^2 = 5198.4$, $\hat{\sigma}_{\mathrm{c}}^2 = 3867.0$,

$$\frac{156.4 - 132.7}{\sqrt{\frac{5198.4}{20} + \frac{3867.0}{12}}} = 0.982$$

- Using the shorthand formula $N = \min(n, m) = 12$, we get $q_{5\%} = 1.78$ and

$$\text{p-value} = P[t_{12} > 0.982] = 17.27\,\%$$

- Using the W-S formula

$$N = \frac{\left(\frac{5198.4}{20} + \frac{3867.0}{12}\right)^2}{\frac{5198.4^2}{20^2(20-1)} + \frac{3867.0^2}{12^2(12-1)}} = 26.07$$

we round to 26.

- We get

$$\text{p-value} = P[t_{26} > 0.982] = 16.76\,\%$$

Discussion

Advantage of Student's test: Non asymptotic / Can be run on small samples + Can always use it for large sample sizes.

large data + CLT 也可以使用

Drawback of Student's test: It relies on the assumption that the sample is Gaussian (soon we will see how to test this assumption)

A test based on the MLE

- Consider an i.i.d. sample $X_1, \ldots, X_n$ with statistical model $(E, (\mathbb{P}_\theta)_{\theta \in \Theta})$, where $\Theta \subseteq \mathbb{R}^d$ $(d \geq 1)$ and let $\theta_0 \in \Theta$ be fixed and given. θ^* is the true parameter
- Consider the following hypotheses:

$$\begin{cases} H_0 : & \theta^* = \theta_0 \\ H_1 : & \theta^* \neq \theta_0. \end{cases}$$

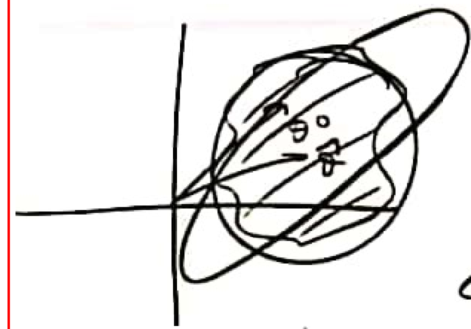
- Let $\hat{\theta}^{MLE}$ be the MLE. Assume the MLE technical conditions are satisfied.
- If H_0 is true, then MLE converges to true parameter, so we can plug in θ

sample(under H0) $$\sqrt{n}\, I(\theta_0)^{1/2} \times \left(\hat{\theta}_n^{MLE} - \theta_0\right) \xrightarrow[n \to \infty]{(d)} \mathcal{N}_d\left(0, I_d\right)$$

true $\sqrt{n}\, I(\theta^*)^{1/2}$

MLE(under H0) $\sqrt{n}\, I(\hat{\theta}_n^{MLE})^{1/2}$

Wald's test



$\|\hat{\theta}-\theta\|^2$ $\begin{pmatrix} Z_1 \\ \vdots \\ Z_d \end{pmatrix} \sim N_d(0, I_d)$

$\sum (\hat{\theta}_j - \theta_j)^2 = Z_1^2 + \dots + Z_d^2$

$n \| I(\hat{\theta})^{\frac{1}{2}} (\hat{\theta} - \theta_0) \|^2 \sim \| N_d(0, I_d) \|^2 \xrightarrow[n\to\infty]{(d)} \chi^2_d$ if H_0 is true

vector $v \in \mathbb{R}^d$, $\|v\|^2 = v^T v = \sum_{j=1}^{d} v_j^2$

$\underbrace{(\hat{\theta}-\theta_0)^T (I(\hat{\theta})^{\frac{1}{2}})^T}_{v^T} I(\hat{\theta})^{\frac{1}{2}} (\hat{\theta}-\theta_0)$

$\| I$ symmetric $I(\hat{\theta})^{1/2}$

$= (\hat{\theta}-\theta_0) I(\hat{\theta}) (\hat{\theta}-\theta_0)$

- Hence,

$$\underbrace{n\left(\hat{\theta}_n^{MLE}-\theta_0\right)^\top I(\hat{\theta}^{MLE})\left(\hat{\theta}_n^{MLE}-\theta_0\right)}_{T_n} \xrightarrow[\boxed{n\to\infty}]{(d)} \chi^2_{\boxed{d}}$$

One dimension wald's test

$\sqrt{n}\,\frac{(\hat{\theta}-\theta)}{\sigma} \xrightarrow[n\to\infty]{(d)} N(0,1)$

σ^2 known.

reject if $n\,\frac{(\hat{\theta}-\theta_0)^2}{\sigma^2} > q_\alpha(\chi^2_1)$

$\frac{n}{\sigma^2}(\hat{\theta}-\theta_0)^2 \xrightarrow[n\to\infty]{(d)} \chi^2_1$

$\Leftrightarrow \sqrt{n}\,\frac{|\hat{\theta}-\theta_0|}{\sigma} > \sqrt{q_\alpha(\chi^2_1)}$ inherently two-side

$\sqrt{n}\,\frac{|\hat{\theta}-\theta_0|}{\sigma} > q_{\frac{\alpha}{2}}(N(0,1))$

- Wald's test with asymptotic level $\alpha \in (0,1)$:

$$\psi = \mathbb{1}\{T_n > q_\alpha\},$$

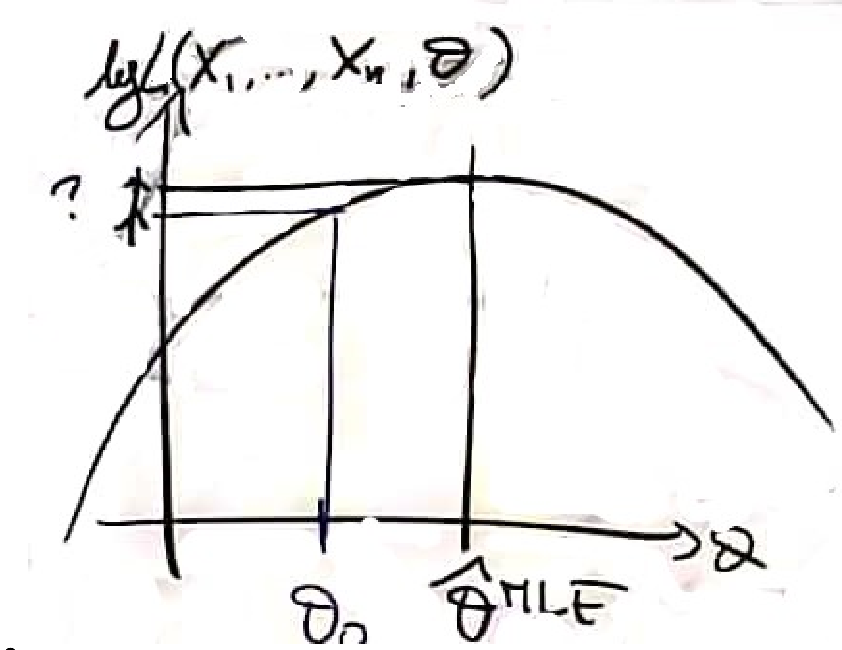
where q_α is the $(1-\alpha)$-quantile of χ^2_d (see tables).

- Remark: Wald's test is also valid if H_1 has the form "$\theta > \theta_0$" or "$\theta < \theta_0$" or "$\theta = \theta_1$"...

But less powerful

A test based on the log-likelihood

- Consider an i.i.d. sample $X_1, \dots, X_n$ with statistical model $(E, (\mathbb{P}_\theta)_{\theta \in \Theta})$, where $\Theta \subseteq \mathbb{R}^d$ $(d \geq 1)$.



- Suppose the null hypothesis has the form

$$H_0 : (\theta_{r+1}, \dots, \theta_d) = (\theta^{(0)}_{r+1}, \dots, \theta^{(0)}_d),$$

a subset of the coordinates of θ

for some fixed and given numbers $\theta^{(0)}_{r+1}, \dots, \theta^{(0)}_d$.

- Let

$$\hat{\theta}_n = \underset{\theta \in \Theta}{\operatorname{argmax}} \ \ell_n(\theta) \quad \text{(MLE)}$$

and

$$\hat{\theta}^c_n = \underset{\theta \in \Theta_0}{\operatorname{argmax}} \ \ell_n(\theta) \quad \text{("constrained MLE")}$$

where $\Theta_0 = \{ \theta \in \Theta : (\theta_{r+1}, \dots, \theta_d) = (\theta^{(0)}_{r+1}, \dots, \theta^{(0)}_d) \}$

Likelihood ratio test

Test statistic:

估计值　被限制的假设值

$$T_n = 2\left(\ell_n(\hat{\theta}_n) - \ell_n(\hat{\theta}_n^c)\right).$$

泰勒展开第二项的常数2

Wald's test:

$$\sqrt{n}(\hat{\theta}_{MLE} - \theta) \xrightarrow{(d)} N(0, I^{-1}(\theta))$$

$$\sqrt{n} \cdot \sqrt{I(\theta)} \cdot (\hat{\theta}_{MLE} - \theta) \xrightarrow{(d)} N(0, 1)$$

$$n \cdot I(\theta) \cdot (\hat{\theta}_{MLE} - \theta)^2 \xrightarrow{(d)} \chi_1^2$$

Likelihood ratio test

$$\frac{\sup_{\theta \in \Theta} L(X_1, \dots X_n | \theta)}{L(X_1, \dots X_n | \theta_0)} > c \geq 1$$

$$\Leftrightarrow \frac{L(X_1, \dots, X_n | \hat{\theta}_{MLE})}{L(X_1, \dots, X_n | \theta_0)} > c$$

$$\xrightarrow[2]{\ln()} 2(\ell(X_1, \dots, X_n | \hat{\theta}_{MLE}) - \ell(X_1 \dots X_n | \theta_0)) > \log c.$$

Wilk's thm $\xrightarrow{(d)} \chi_1^2$

Wilks' Theorem

Assume H_0 is true and the MLE technical conditions are satisfied. Then,

$$T_n \xrightarrow[\boxed{n \to \infty}]{(d)} \chi^2_{d-r}$$

d-r(constrained number of theta)
如果r是0，那么所有的theta都被限制了，就相当于问theta_hat是不是等于theta_0

Likelihood ratio test with asymptotic level $\alpha \in (0, 1)$:

$$\psi = \mathbb{1}\{T_n > q_\alpha\},$$

where q_α is the $(1 - \alpha)$-quantile of χ^2_{d-r} (see tables).

Implicit hypotheses

θ is not isolated in the equations

- Let $X_1, \ldots, X_n$ be i.i.d. random variables and let $\theta \in \mathbb{R}^d$ be a parameter associated with the distribution of X_1 (e.g. a moment, the parameter of a statistical model, etc...)

- function
  Let $g : \mathbb{R}^d \to \mathbb{R}^k$ be continuously differentiable (with $k < d$).

- Consider the following hypotheses:

$$\begin{cases} H_0 : & g(\theta) = 0 \\ H_1 : & g(\theta) \neq 0. \end{cases}$$

- E.g. $g(\theta) = (\theta_1, \theta_2)$ $(k = 2)$, or $g(\theta) = \theta_1 - \theta_2$ $(k = 1)$, or...

Delta method

- Suppose an asymptotically normal estimator $\hat{\theta}_n$ is available:

$$\sqrt{n}\left(\hat{\theta}_n - \theta\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}_d(0, \Sigma(\theta)).$$

- Delta method:

$$\sqrt{n}\left(g(\hat{\theta}_n) - g(\theta)\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}_k\left(0, \Gamma(\theta)\right),$$

where $\Gamma(\theta) = \nabla g(\theta)^\top \Sigma(\theta) \nabla g(\theta) \in \mathbb{R}^{k\times k}$.

- Assume $\Sigma(\theta)$ is invertible and $\nabla g(\theta)$ has rank k. So, $\Gamma(\theta)$ is invertible and

$$\sqrt{n}\,\Gamma(\theta)^{-1/2}\left(g(\hat{\theta}_n) - g(\theta)\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}_k\left(0, I_k\right).$$

Wald's test for implicit hypotheses

- Then, by Slutsky's theorem, if $\Gamma(\theta)$ is continuous in θ,

不知道theta是什么，under H0，我们只知道关于g(θ)的假设

$$\sqrt{n}\,\Gamma(\hat{\theta}_n)^{-1/2}\left(g(\hat{\theta}_n) - g(\theta)\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}_k\left(0, I_k\right).$$

- Hence, if H_0 is true, i.e., $g(\theta) = 0$,

$$\underbrace{ng(\hat{\theta}_n)^\top \Gamma^{-1}(\hat{\theta}_n) g(\hat{\theta}_n)}_{T_n} \xrightarrow[n\to\infty]{(d)} \chi^2_k.$$

- Test with asymptotic level α:

$$\psi = \mathbb{1}\{T_n > q_\alpha\},$$

where q_α is the $(1-\alpha)$-quantile of χ^2_k (see tables).

Goodness of fit

Goodness of fit tests

Let X be a r.v. Given i.i.d copies of X we want to answer the following types of questions:

- Does X have distribution $\mathcal{N}(0,1)$? (Cf. Student's T distribution)
- Does X have distribution $\mathcal{U}([0,1])$?
- Does X have PMF $p_1 = 0.3,\ p_2 = 0.5,\ p_3 = 0.2$

These are all *goodness of fit* (GoF) tests: we want to know if the hypothesized distribution is a good fit for the data.

Key characteristic of GoF tests: no parametric modeling.

The zodiac sign of the most powerful people is....

Can your zodiac sign predict how successful you will be later in life?
Fortune magazine collected the signs of 256 heads of the Fortune 500.

Fyi:
256/12
=21.33

Sign	Count
Aries	23
Taurus	20
Gemini	18
Cancer	23
Leo	20
Virgo	19
Libra	18
Scorpio	21
Sagittarius	19
Capricorn	22
Aquarius	24
Pisces	29

The zodiac sign of the most successful people is....

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Pisces	29

In view of this data, is there statistical evidence that successful people are more likely to be born under some sign than others?

$$p^0 = \underbrace{\left(\frac{1}{12}, \dots, \frac{1}{12}\right)}_{12 \text{ times}} \in \Delta_{12}$$

275 jurors with identified racial group.
We want to know if the jury is representative of the population of this county.

$p^0 = (0.72, 0.07, 0.12, 0.09)$

$K = 4$

Race	White	Black	Hispanic	Other	Total
# jurors	205	26	25	19	275
proportion in county	0.72	0.07	0.12	0.09	1

Discrete distribution

Let $E = \{a_1, \dots, a_K\}$ be a finite space and $(\mathbb{P}_{\mathbf{p}})_{\mathbf{p} \in \Delta_K}$ be the family of all probability distributions on E:

$p_j = \mathbb{P}[X = a_j]$

- $$\Delta_K = \left\{ \mathbf{p} = (p_1, \dots, p_K) \in (0,1)^K : \sum_{j=1}^{K} p_j = 1 \right\}.$$
- For $\mathbf{p} \in \Delta_K$ and $X \sim \mathbb{P}_{\mathbf{p}}$,

$$\mathbb{P}_{\mathbf{p}}[X = a_j] = p_j, \qquad j = 1, \dots, K.$$

Goodness of fit test

- Let $X_1, \ldots, X_n \overset{iid}{\sim} \mathbb{P}_{\mathbf{p}}$, for some unknown $\mathbf{p} \in \Delta_K$, and let $\mathbf{p}^0 \in \Delta_K$ be fixed.
- We want to test:

$$H_0\colon\ \mathbf{p} = \mathbf{p}^0 \quad \text{vs.} \quad H_1\colon\ \mathbf{p} \neq \mathbf{p}^0$$

  with asymptotic level $\alpha \in (0, 1)$.

- Example: If $\mathbf{p}^0 = (1/K, 1/K, \ldots, 1/K)$, we are testing whether $\mathbb{P}_{\mathbf{p}}$ is the uniform distribution on E.

Multinomial likelihood

K = # of modalities

- Likelihood of the model:

$X \sim \text{Multinomial}\underbrace{(p_1, \dots, p_K)}_{\vec{p}}$

$$L_n(X_1, \dots, X_n, \mathbf{p}) = p_1^{N_1} p_2^{N_2} \dots p_K^{N_K},$$

where $N_j = \#\{i = 1, \dots, n : X_i = a_j\}$.

- Let $\hat{\mathbf{p}}$ be the MLE:

$$\hat{\mathbf{p}}_j = \frac{N_j}{n}, \qquad j = 1, \dots, K.$$

⚠ $\hat{\mathbf{p}}$ maximizes $\log L_n(X_1, \dots, X_n, \mathbf{p})$ **under the constraint**

χ^2 test

$$\sqrt{n}(\hat{\mathbf{p}} - \mathbf{p}^0)^T \mathbf{1} = \sum_{i=1}^K (\hat{p}_i - p_i^0) = \sum_{i=1}^K \hat{p}_i - \sum_{i=1}^K p_i^0 = 0.$$

1，这个乘以一个全是1的列向量应该是0
2，这个一定asymptotically normal

3，但是N(0,I_k)和I_k的内积不是0 (因为Gaussian向量之间的和一定是一个Gaussian r.v.)

- If H_0 is true, then $\boxed{\sqrt{n}(\hat{\mathbf{p}} - \mathbf{p}^0)}$ is asymptotically normal, and the following holds.

$$\sqrt{n}\, I(p^0)^{1/2}(\hat{p} - p^0) \xrightarrow[n\to\infty]{} N(0, I_{\boxed{K-1}})$$

Theorem Under H_0:

4，所以这个向量一定是一个与全是1的列向量垂直的Gaussian(内积为0)

5，所以，由于限制条件，也就是所有p加起来要等于1。我们损失了一个自由度，只有K-1维度了。（在这个全是1的列向量的方向上，没有方差）

$$\underbrace{n \sum_{j=1}^K \frac{\left(\hat{\mathbf{p}}_j - \mathbf{p}_j^0\right)^2}{\mathbf{p}_j^0}}_{T_n} \xrightarrow[n\to\infty]{(d)} \chi^2_{K-1}.$$

Wald' test

自由度：support size - dimension of theta - 1

- χ^2 test with asymptotic level α: $\psi_\alpha = \mathbb{1}\{T_n > q_\alpha\}$, where q_α is the $(1-\alpha)$-quantile of χ^2_{K-1}.
- Asymptotic p-value of this test: $p - \text{value} = \mathbb{P}\left[Z > T_n | T_n\right]$, where $Z \sim \chi^2_{K-1}$ and $Z \perp\!\!\!\perp T_n$.

CDF and empirical CDF

Let $X_1, \ldots, X_n$ be i.i.d. real random variables. Recall the cdf of X_1 is defined as:

$$F(t) = \mathbb{P}[X_1 \leq t], \qquad \forall t \in \mathbb{R}.$$

It completely characterizes the distribution of X_1.

Definition

step function

The *empirical cdf* of the sample $X_1, \ldots, X_n$ is defined as:

(a.k.a sample cdf)

E[I{X<t}]

$$F_n(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{X_i \leq t\}$$

random function(depend on random observations)

count # of i which Xi < t

$$= \frac{\#\{i = 1, \ldots, n : X_i \leq t\}}{n}, \qquad \forall t \in \mathbb{R}.$$

Consistency

By the LLN, for all $t \in \mathbb{R}$,

$$F_n(t) \xrightarrow[n\to\infty]{a.s.} F(t).$$

Glivenko-Cantelli Theorem (*Fundamental theorem of statistics*)

uniform convergence(the rate of convergence is uniform, irrespective of t)

$$\sup_{\boxed{t\in\mathbb{R}}} |F_n(t) - F(t)| \xrightarrow[n\to\infty]{a.s.} 0.$$

对任意的t

If we had $F_n(t) \to F(t) \quad \forall t$

t如果depends on n或者其他情况，那么就不会uniform

$$F_n(t) = F(t) + \frac{t}{n}$$

$$\sup_{t\in\mathbb{R}} |\underbrace{F_n(t) - F(t)}_{\frac{t}{n}}| = \infty$$

Asymptotic normality

By the CLT, for all $t \in \mathbb{R}$,

average of
a bernoulli

expectation
of a bernoulli

$$\sqrt{n}\,\left(F_n(t) - F(t)\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}\left(0, F(t)(1-F(t))\right).$$

Donsker's Theorem

In the worst case, how far Fn can get to F.

Under the H0, 也就是两个相等

If F is continuous, then

pivot !

$$\sqrt{n}\,\sup_{t\in\mathbb{R}} |F_n(t) - F(t)| \xrightarrow[n\to\infty]{(d)} \sup_{0\leq t\leq 1} |\mathbb{B}(t)|,$$

F(t)

Brownian motion: random process
在0到t时间中布朗运动最大的方差
a r.v.

where $\mathbb{B}$ is a Brownian bridge on $[0, 1]$.

在0和1时，方差都是0
把在时间点0和1都在x轴上的
随机运动叫Brownian bridge

Goodness of fit for continuous distributions

- Let $X_1, \dots, X_n$ be i.i.d. real random variables with unknown cdf F and let F^0 be a **continuous** cdf. 和某个CDF比

  no single point has mass

- Consider the two hypotheses:

$$H_0 : F = F^0 \quad \text{v.s.} \quad H_1 : F \neq F^0.$$

- Let F_n be the empirical cdf of the sample $X_1, \dots, X_n$.

- If $F = F^0$, then $F_n(t) \approx F^0(t)$, for all $t \in \cancel{[0,1]}$. $\mathbb{R}$

Kolmogorov-Smirnov test

- Let $T_n = \sup_{t \in \mathbb{R}} \sqrt{n} \left| F_n(t) - F^0(t) \right|$.

- By Donsker's theorem, if H_0 is true, then $T_n \xrightarrow[n \to \infty]{(d)} Z$, where Z has a known distribution (supremum of a Brownian bridge).

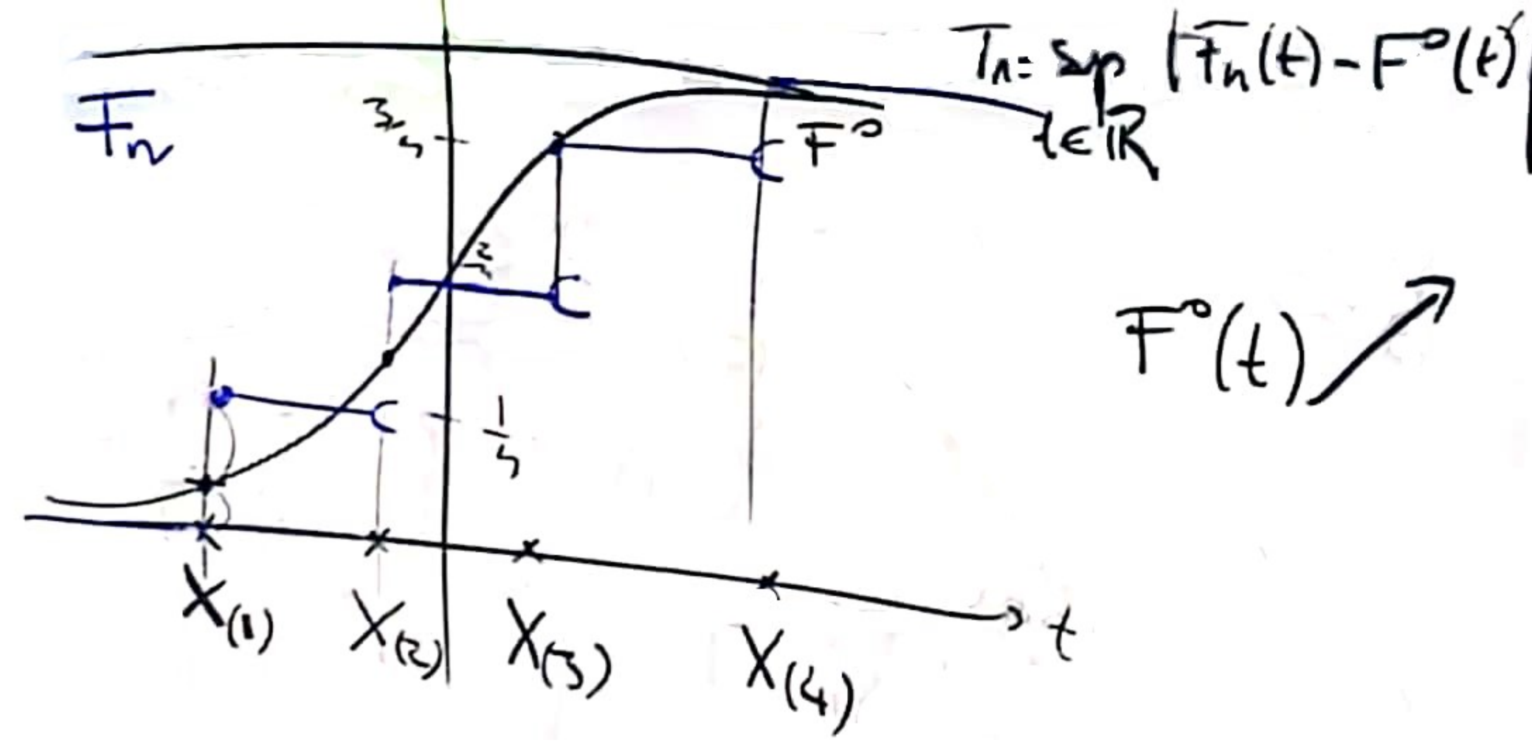
- **KS test with asymptotic level α:**

$$\delta_\alpha^{KS} = \mathbb{1}\{T_n > q_\alpha\},$$

where q_α is the $(1-\alpha)$-quantile of Z (obtained in tables).

- p-value of KS test: $\mathbb{P}[Z > T_n | T_n]$.

Computatinal issues



- In practice, how to compute T_n ?

- F^0 is non decreasing, F_n is piecewise constant, with jumps at $t_i = X_i, i = 1, \dots, n$.

- Let $X_{(1)} \le X_{(2)} \le \dots \le X_{(n)}$ be the reordered sample.

- The expression for T_n reduces to the following practical formula:

i是序数
某一段的左端点

小于第i个观测值的比率

$$T_n = \sqrt{n} \max_{i=1,\dots,n} \left\{ \max\left(\left| \frac{i-1}{n} - \boxed{F^0(X_{(i)})} \right|, \left| \frac{i}{n} - F^0(X_{(i)}) \right| \right) \right\}.$$

如果Xi的CDF是F^0的话
那么F^0(Xi)~Unif([0,1])

Pivotal distribution

- T_n is called a *pivotal statistic*: If H_0 is true, the distribution of T_n does not depend on the distribution of the X_i's and it is easy to reproduce it in simulations.

- Indeed, let $U_i = F^0(X_i), i = 1, \ldots, n$ and let G_n be the empirical cdf of $U_1, \ldots, U_n$.

- If H_0 is true, then $U_1, \ldots, U_n \overset{i.i.d.}{\sim}$ Unif ([0, 1])

  and $T_n = \sup\limits_{\boxed{0 \le x \le 1}} \sqrt{n}\,|G_n(x) - x|$.

empirical CDF of iid uniform r.v.

r.v.

$X \sim F$, $U \sim \text{Unif}[0,1]$

$X = F^{-1}(U)$ (pseudo)

$$P(X \le t) = P(F^{-1}(U) \le t)$$
$$= P(U \le F(t)) \quad \text{(F is increasing)}$$
$$= F(t)$$

$F(X) \sim \text{Unif}[0,1]$

Quantiles and p-values

- For some large integer M:
  - Simulate M i.i.d. copies $T_n^1, \ldots, T_n^M$ of T_n;
  - Estimate the $(1-\alpha)$-quantile $q_\alpha^{(n)}$ of T_n by taking the sample $(1-\alpha)$-quantile $\hat{q}_\alpha^{(n,M)}$ of $T_n^1, \ldots, T_n^M$.
- Test with approximate level α:

$$\delta_\alpha = \mathbb{1}\{T_n > \hat{q}_\alpha^{(n,M)}\}.$$

- Approximate p-value of this test:

$$\text{p-value} \approx \frac{\#\{j = 1, \ldots, M : T_n^j > T_n\}}{M}.$$

K-S table

Kolmogorov–Smirnov Tables

Critical values, $d_{alpha};(n)^{a}$, of the maximum absolute difference between sample $F_n(x)$ and population $F(x)$ cumulative distribution.

Number of trials, n	Level of significance, α			
	0.10	0.05	0.02	0.01
1	0.95000	0.97500	0.99000	0.99500
2	0.77639	0.84189	0.90000	0.92929
3	0.63604	0.70760	0.78456	0.82900
4	0.56522	0.62394	0.68887	0.73424
5	0.50945	0.56328	0.62718	0.66853
6	0.46799	0.51926	0.57741	0.61661
7	0.43607	0.48342	0.53844	0.57581
8	0.40962	0.45427	0.50654	0.54179
9	0.38746	0.43001	0.47960	0.51332
10	0.36866	0.40925	0.45662	0.48893

Other goodness of fit tests

We want to measure the distance between two functions: $F_n(t)$ and $F(t)$. There are other ways, leading to other tests:

- Kolmogorov-Smirnov: worst possible deviation

$$d(F_n, F) = \sup_{t \in \mathbb{R}} |F_n(t) - F(t)| \qquad L_\infty$$

- Cramér-Von Mises:

$$d^2(F_n, F) = \int_{\mathbb{R}} [F_n(t) - F(t)]^2 \, d\cancel{t}F(t) \qquad L_2$$

$$:= \mathbb{E}_{X \sim F}\left[|F_n(X) - F(X)|^2\right]$$

pivotal

- Anderson-Darling:

$$d^2(F_n, F) = \int_{\mathbb{R}} \frac{[F_n(t) - F(t)]^2}{F(t)(1 - F(t))} \, d\cancel{t}F(t)$$

Composite goodness of fit tests

What if I want to test: "Does X have Gaussian distribution?" but I don't know the parameters? 一般H0是deterministic，但是这里depends on estimated parameter。

Simple idea: plug-in

$$\sup_{t \in \mathbb{R}} \left| F_n(t) - \Phi_{\hat{\mu},\hat{\sigma}^2}(t) \right|$$

where 此时，我们在选择H0的过程，就让H0与数据更相似。

$$\hat{\mu} = \bar{X}_n, \qquad \hat{\sigma}^2 = S_n^2$$

and $\Phi_{\hat{\mu},\hat{\sigma}^2}(t)$ is the cdf of $\mathcal{N}(\hat{\mu}, \hat{\sigma}^2)$.

In this case Donsker's theorem is *no longer valid*. This is a common and serious mistake!

Kolmogorov-Lilliefors test (1)

Instead, we compute the quantiles for the test statistic:

$$\sup_{t \in \mathbb{R}} \left| F_n(t) - \Phi_{\hat{\mu},\hat{\sigma}^2}(t) \right|$$

把数据用我们想要对比的正态分布的参数，进行normalize

They do not depend on unknown parameters!

只是表变了，值变小了，更容易拒绝H0了

This is the Kolmogorov-Lilliefors test.

The KL quantile could be x and the KS quantile y could be $y > x$. If your test statistic is c that is $x < c < y$, then KL will reject but KS will fail to reject.

K-L table

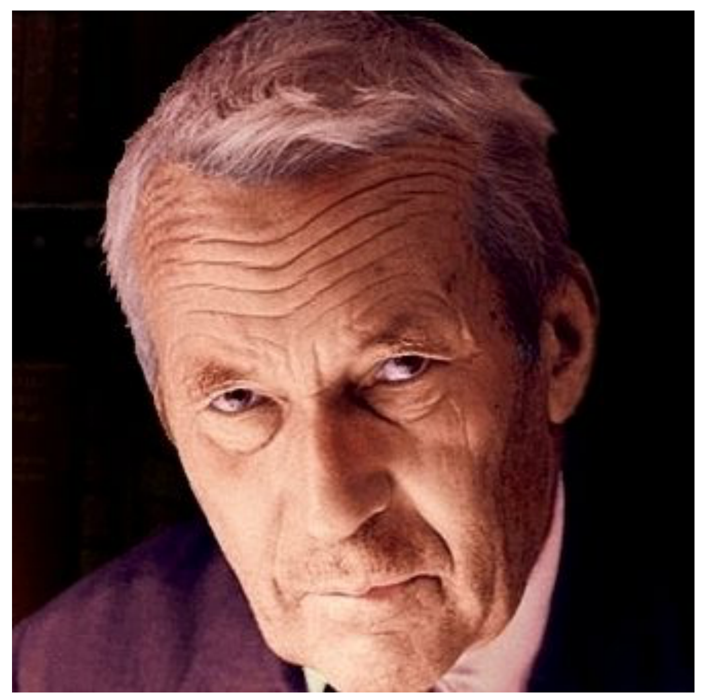
Sample Size N	Level of Significance for $D = \text{Max}\left\|F^*(X) - S_N(X)\right\|$				
	.20	.15	.10	.05	.01
4	.300	.319	.352	.381	.417
5	.285	.299	.315	.337	.405
6	.265	.277	.294	.319	.364
7	.247	.258	.276	.300	.348
8	.233	.244	.261	.285	.331
9	.223	.233	.249	.271	.311
10	.215	.224	.239	.258	.294
11	.206	.217	.230	.249	.284
12	.199	.212	.223	.242	.275
13	.190	.202	.214	.234	.268
14	.183	.194	.207	.227	.261
15	.177	.187	.201	.220	.257
16	.173	.182	.195	.213	.250
17	.169	.177	.189	.206	.245
18	.166	.173	.184	.200	.239
19	.163	.169	.179	.195	.235
20	.160	.166	.174	.190	.231

LILLIEFORS

VON MISES

WALD

KOLMOGOROV



SMIRNOV

WILKS

ANDERSON

DARLING

CRAMÈR

Quantile-Quantile (QQ) plots (1)

- Provide a visual way to perform GoF tests
- Not formal test but quick and easy check to see if a distribution is plausible.
- Main idea: we want to check visually if the plot of F_n is close to that of F or equivalently if the plot of F_n^{-1} is close to that of F^{-1}.
- More convenient to check if the points

plot的是值而不是分数，相同quantile的值，一一比较

$$\Big(F^{-1}(\frac{1}{n}), F_n^{-1}(\frac{1}{n})\Big), \Big(F^{-1}(\frac{2}{n}), F_n^{-1}(\frac{2}{n})\Big), \dots, \Big(F^{-1}(\frac{n-1}{n}), F_n^{-1}(\frac{n-1}{n})\Big)$$

X_(1) X_(2)

are near the line $y = x$.

- F_n is not technically invertible but we define

$$F_n^{-1}(i/n) = X_{(i)},$$

IF $F(\cdot)$ is CDF of $N(0,1)$

$F^{-1}(\frac{i}{n})$ is x_i such that

$$\int_{-\infty}^{x_i} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = \frac{i}{n}$$

$x_i = q_{1-\frac{i}{n}}$

the ith largest observation.

Quantile-Quantile (QQ) plots (2)

y: observation，数据！
x: cdf（一般正态分布比，就是正态的）

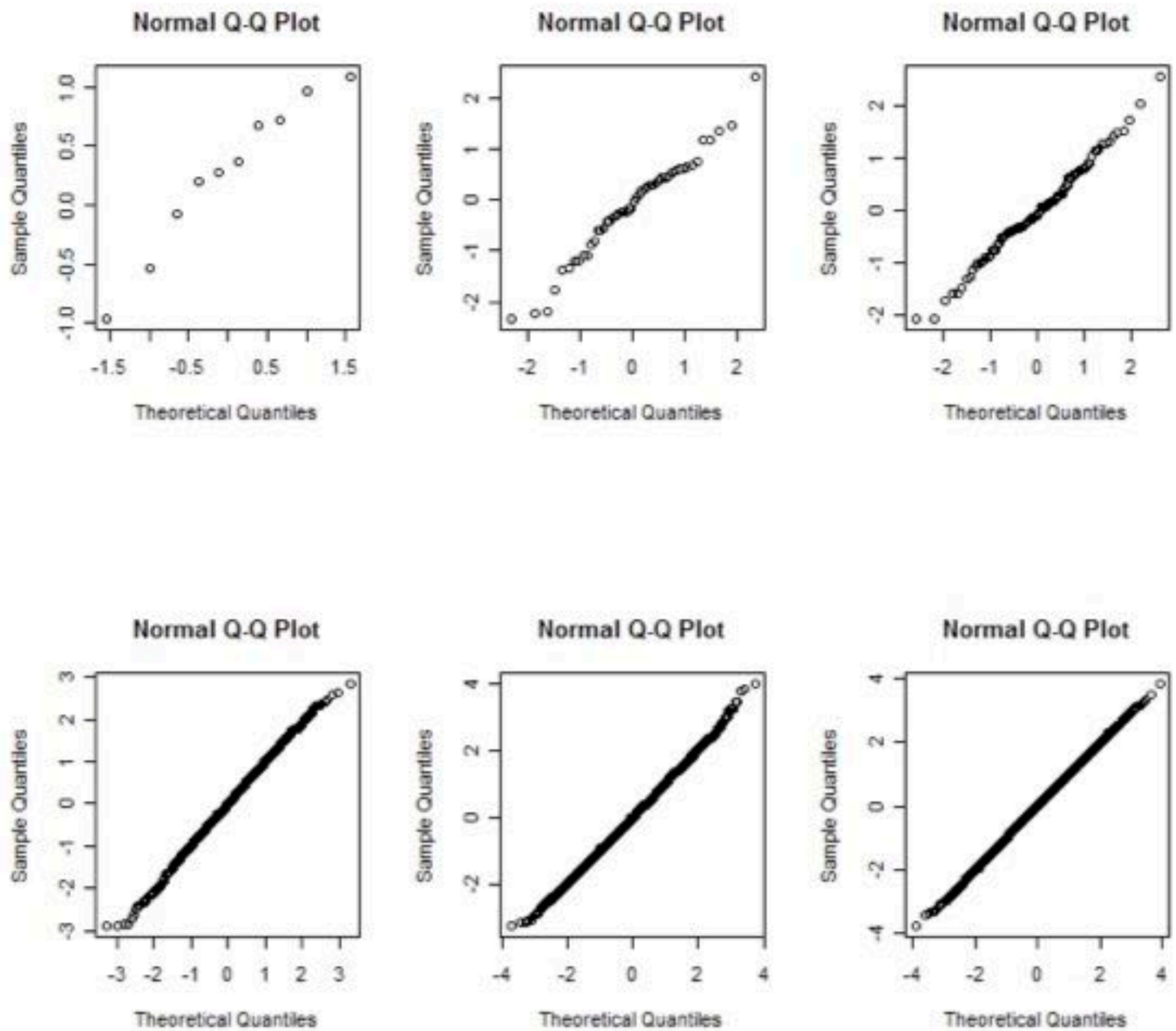
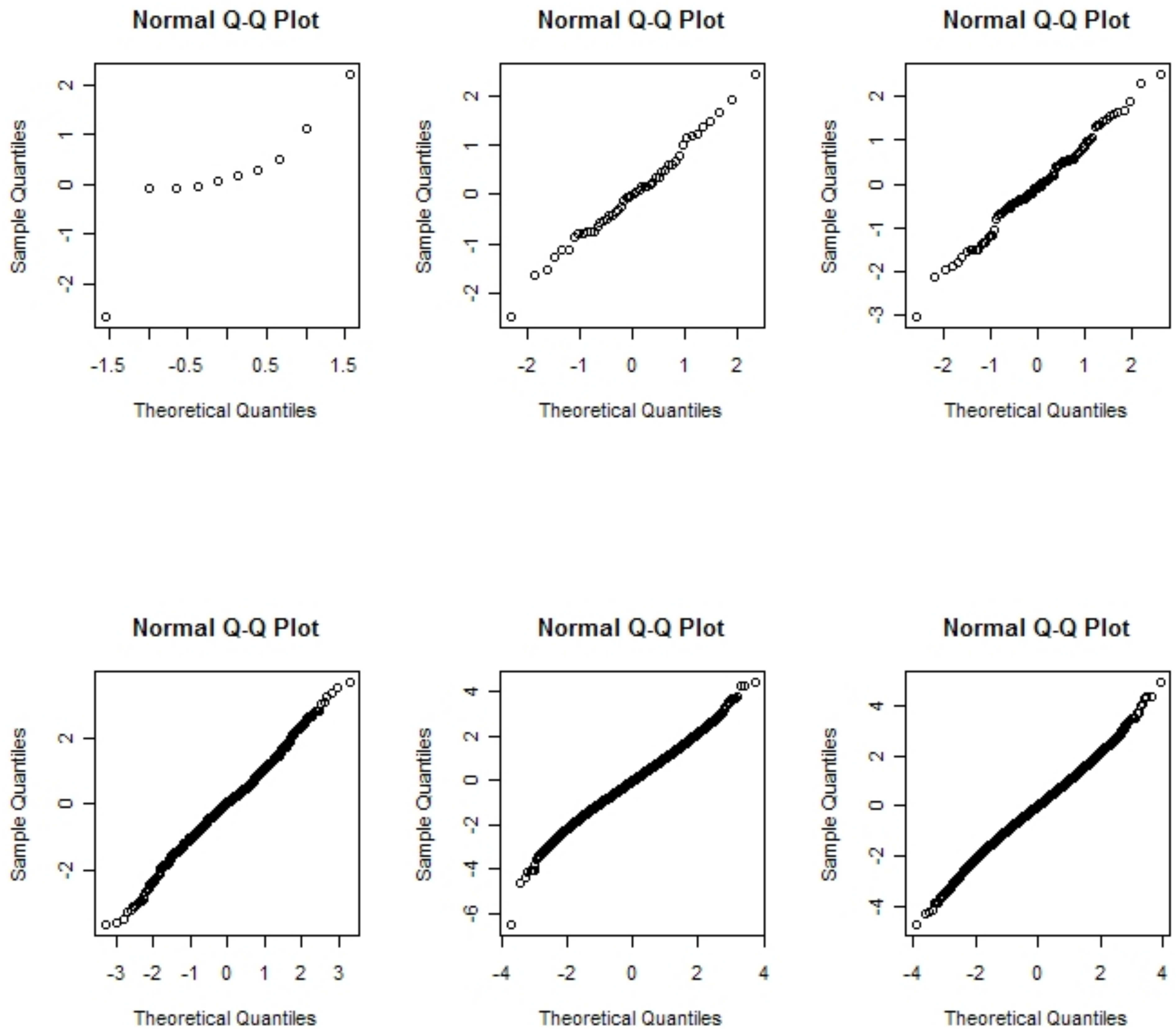
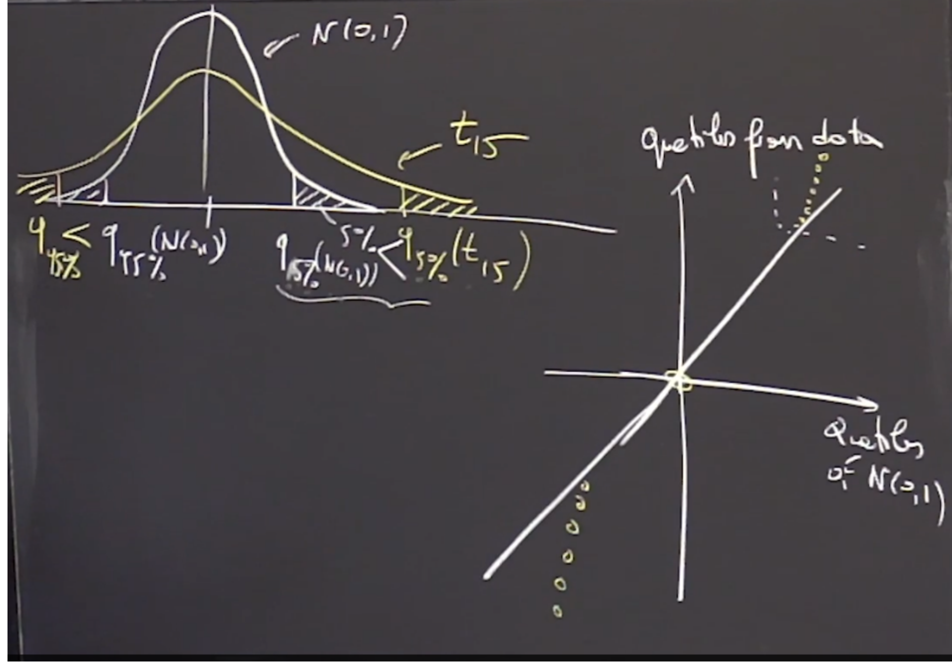


Figure 1: QQ-plots for samples of sizes 10, 50, 100, 1000, 5000, 10000 from a standard normal distribution. The upper-left figure is for sample size 10, the lower-right is for sample 10000.

Quantile-Quantile (QQ) plots (3)

重尾更多的质量在尾部，同样的quantile，Tn就会更大
轻尾更多的质量在中间，同样的quantile，Tn就会更小





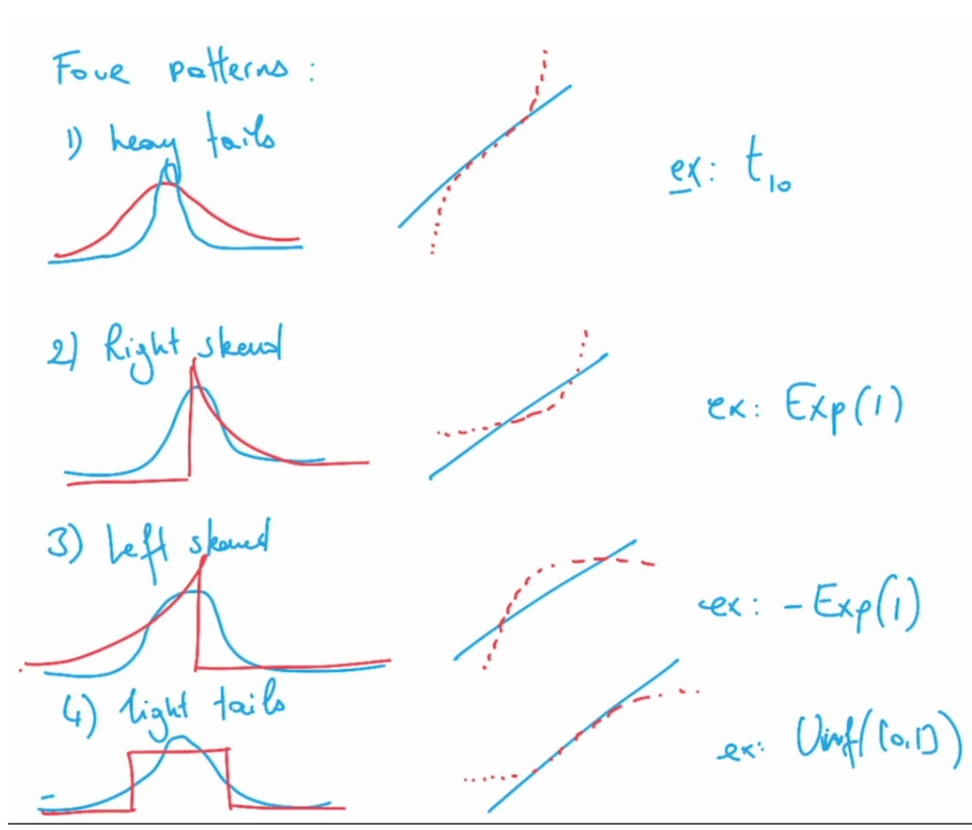


Figure 2: QQ-plots for samples of sizes 10, 50, 100, 1000, 5000, 10000 from a t_{15} distribution. The upper-left figure is for sample size 10, the lower-right is for sample 10000.