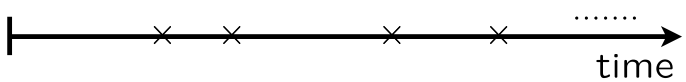
LECTURE 22: The Poisson process

- Definition of the Poisson process
  - applications
- Distribution of number of arrivals
- The time of the kth arrival
- Memorylessness
- Distribution of interarrival times

Definition of the Poisson process

Poisson



- Numbers of arrivals in disjoint time intervals are **independent**

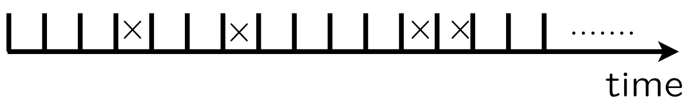
$P(k, \tau)$ = Prob. of k arrivals in interval of duration τ

- **Small interval probabilities:**
  For VERY small δ:

$$P(k,\delta) \approx \begin{cases} 1-\lambda\delta & \text{if } k=0 \\ \lambda\delta & \text{if } k=1 \\ 0 & \text{if } k>1 \end{cases} \qquad P(k,\delta) = \begin{cases} 1-\lambda\delta+O(\delta^2) & \text{if } k=0 \\ \lambda\delta+O(\delta^2) & \text{if } k=1 \\ 0+O(\delta^2) & \text{if } k>1 \end{cases}$$

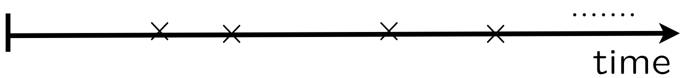
λ: "arrival rate"

Bernoulli



- Independence
- **Time homogeneity:**
  Constant p at each slot

Applications of the Poisson process



- Deaths from horse kicks in the Prussian army (1898)
- Particle emissions and radioactive decay
- Photon arrivals from a weak source
- Financial market shocks
- Placement of phone calls, service requests, etc.

Siméon Denis Poisson
(1781-1840)

The Poisson PMF for the number of arrivals

- N_τ: arrivals in $[0, \tau]$ $\qquad P(k,\tau) = \mathbf{P}(N_\tau = k)$

$n = \tau/\delta$ intervals/slots of length δ

$\mathbf{P}$(some slot contains two or more arrivals)

Bernoulli

$$p_S(k) = \frac{n!}{(n-k)!\,k!} \cdot p^k (1-p)^{n-k},$$

$$k = 0, \ldots, n$$

$\lambda = np \qquad n \to \infty \qquad p \to 0$

For fixed $k = 0, 1, \ldots,$

$$p_S(k) \to \frac{\lambda^k}{k!} e^{-\lambda},$$

$N_\tau \approx$ binomial $\qquad p = \lambda\delta + O(\delta^2)$

$np =$

$$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}, \qquad k = 0, 1, \ldots$$

Mean and variance of the number of arrivals

$$P(k,\tau) = \mathbf{P}(N_\tau = k) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}, \qquad k = 0, 1, \ldots$$

$$\boxed{\begin{array}{l} \mathbf{E}[N_\tau] = \lambda\tau \\ \\ \mathrm{var}(N_\tau) = \lambda\tau \end{array}}$$

$$\mathbf{E}[N_\tau] = \sum_{k=0}^{\infty} k \, \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!} = \cdots$$

$N_\tau \approx \text{Binomial}(n, p)$

$n = \tau/\delta, \;\; p = \lambda\delta + O(\delta^2)$

Example

- You get email according to a Poisson process, at a rate of $\lambda = 5$ messages per hour.

$$\mathbf{E}[N_\tau] = \lambda\tau$$

$$\text{var}(N_\tau) = \lambda\tau$$

- Mean and variance of mails received during a day =

- $\mathbf{P}$(one new message in the next hour) =

$$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}, \qquad k = 0, 1, \ldots$$

- $\mathbf{P}$(exactly two messages during each of the next three hours) =

The time T_1 until the first arrival

$$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}, \qquad k = 0, 1, \ldots$$

- Find the CDF: $\mathbf{P}(T_1 \leq t) =$

$$f_{T_1}(t) = \lambda e^{-\lambda t}, \quad \text{for } t \geq 0$$

Exponential(λ)

Memorylessness: conditioned on $T_1 > t$,
the PDF of $T_1 - t$ is again exponential

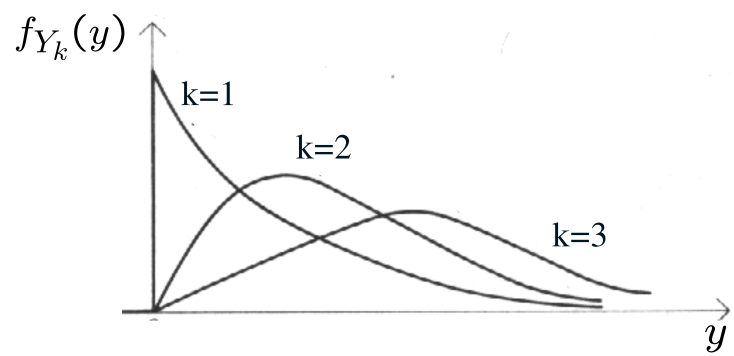
The time Y_k of the kth arrival

$$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}, \qquad k = 0, 1, \ldots$$

- Can derive its PDF by first finding the CDF
- More intuitive argument:

  $f_{Y_k}(y)\,\delta \approx \mathbf{P}(y \le Y_k \le y + \delta) =$

Erlang distribution: $f_{Y_k}(y) = \dfrac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, \quad y \ge 0$

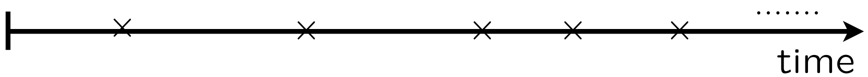


Memorylessness and the fresh-start property

- Analogous to the properties for the Bernoulli process
  - plausible, given the relation between the two processes
  - use intuitive reasoning
  - can be proved rigorously

Memorylessness and the fresh-start property

- If we start watching at time t,



  we see Poisson process, independent of the history until time t
  - time until next arrival:

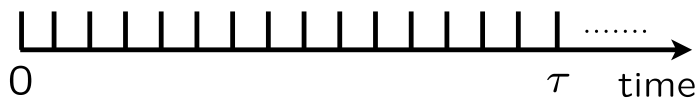
- If we start watching at time T_1,
  we see Poisson process, independent of the history until time T_1
  - hence: time between first and second arrival, $T_2 = Y_2 - Y_1$ is:
  - similarly for all $T_k = Y_k - Y_{k-1}$, $k \geq 2$

$Y_k = T_1 + \cdots + T_k$ is sum of i.i.d. exponentials

$\mathbf{E}[Y_k] = k/\lambda \qquad \text{var}(Y_k) = k/\lambda^2$

- An equivalent definition
- A simulation method

Bernoulli/Poisson relation



$n = \tau/\delta,$

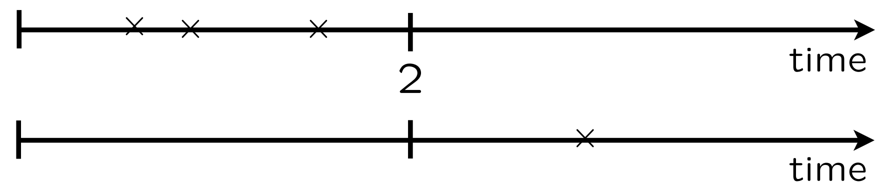
$p = \lambda\delta$

$np = \lambda\tau$

	POISSON	**BERNOULLI**
Times of Arrival	Continuous	Discrete
Arrival Rate	λ/unit time	p/per trial
PMF of # of Arrivals	Poisson	Binomial
Interarrival Time Distr.	Exponential	Geometric
Time to k-th arrival	Erlang	Pascal

Example: Poisson fishing

- Fish are caught as a Poisson process, $\lambda = 0.6$/hour
  - fish for two hours;
  - if you caught at least one fish, stop
  - else continue until first fish is caught



$\mathbf{P}$(fish for more than two hours)=

$$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}$$

$$\mathbf{E}[N_\tau] = \lambda\tau$$

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}$$

$\mathbf{P}$(fish for more than two and less than five hours)=

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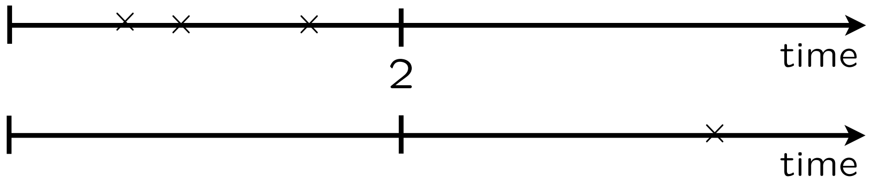

$\mathbf{P}$(catch at least two fish)=

$\mathbf{E}$[future fishing time | already fished for three hours]=

$$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}$$

$$\mathbf{E}[N_\tau] = \lambda\tau$$

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}$$

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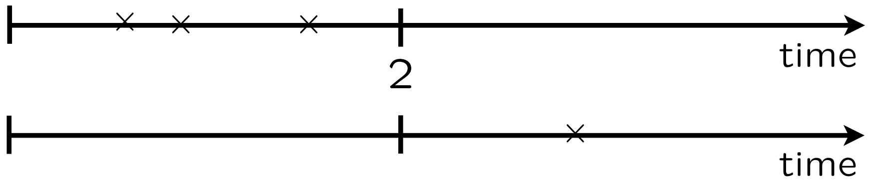

$\mathbf{E}$[total fishing time]=

$\mathbf{E}$[number of fish]=

$$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}$$

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$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}$$