

# 18.650 – Fundamentals of Statistics

## 1. Introduction and probability

# Goals

## Goals:

- ▶ To give you a solid introduction to the mathematical theory behind statistical methods;
- ▶ To provide theoretical guarantees for the statistical methods that you may use for certain applications.

At the end of this class, you will be able to

1. From a real-life situation, formulate a statistical problem in mathematical terms
2. Select appropriate statistical methods for your problem
3. Understand the implications and limitations of various methods

# Why statistics?

# In the press

# The New York Times

THE UPSHOT

## Nike Says Its \$250 Running Shoes Will Make You Run Much Faster. What if That's Actually True?

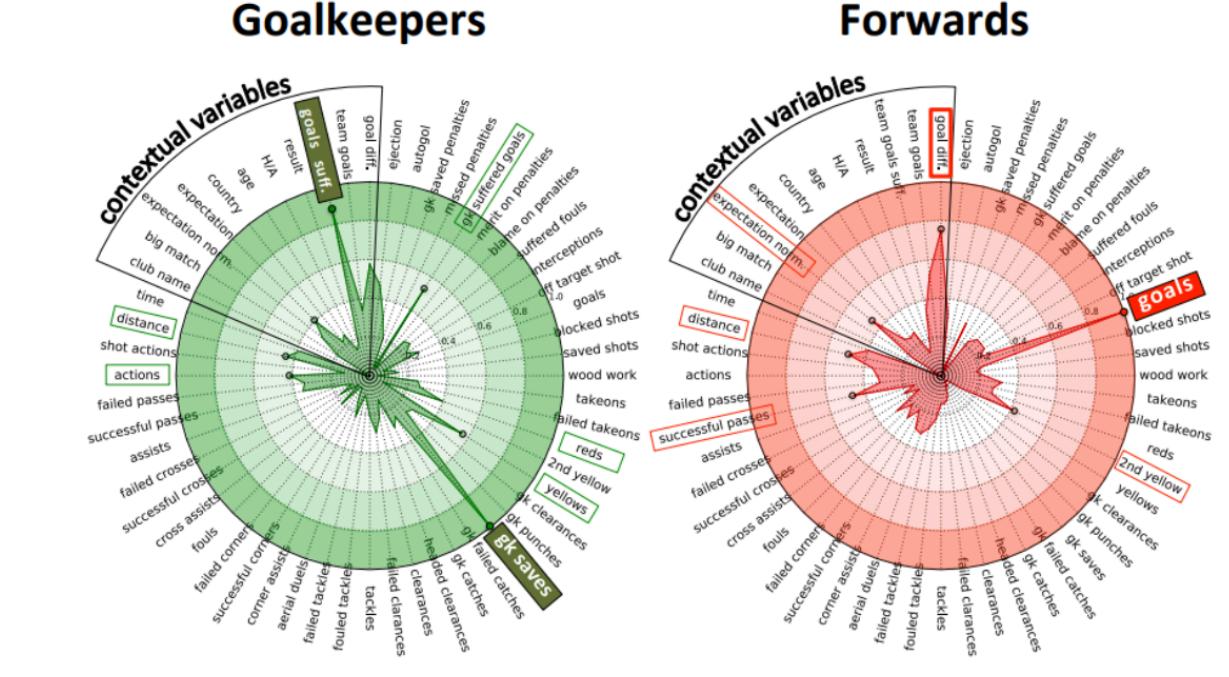
An analysis of nearly 500,000 running times estimates the effect of shoes on race performance.



# MIT Technology Review

## Data Mining Reveals the Way Humans Evaluate Each Other

Vast databases of soccer statistics expose the limited way human observers rate performance and suggest how they can do significantly better.



## In businesses

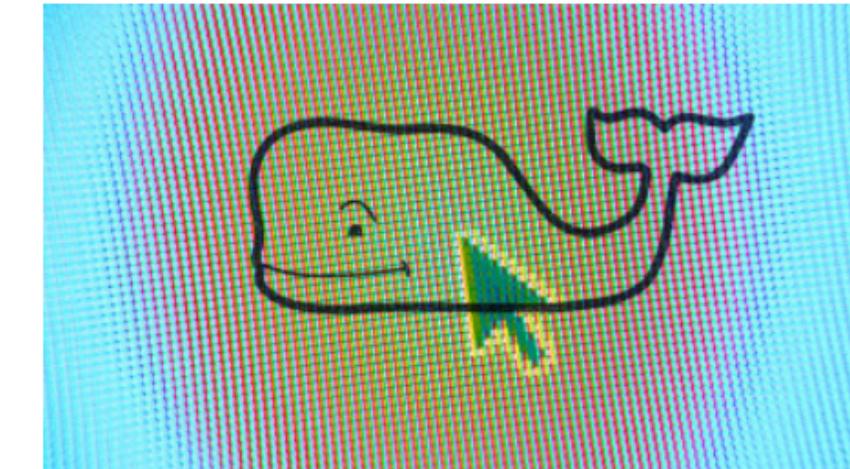
**Harvard  
Business  
Review**

### How Vineyard Vines Uses Analytics to Win Over Customers

TECHNOLOGY DIGITAL ARTICLE by Dave Sutton

A case study on how personalization is changing retail.

SAVE   SHARE   JUNE 08, 2018



**FAST COMPANY**

### AppNexus is key to AT&T's plans to use HBO for more consumer data

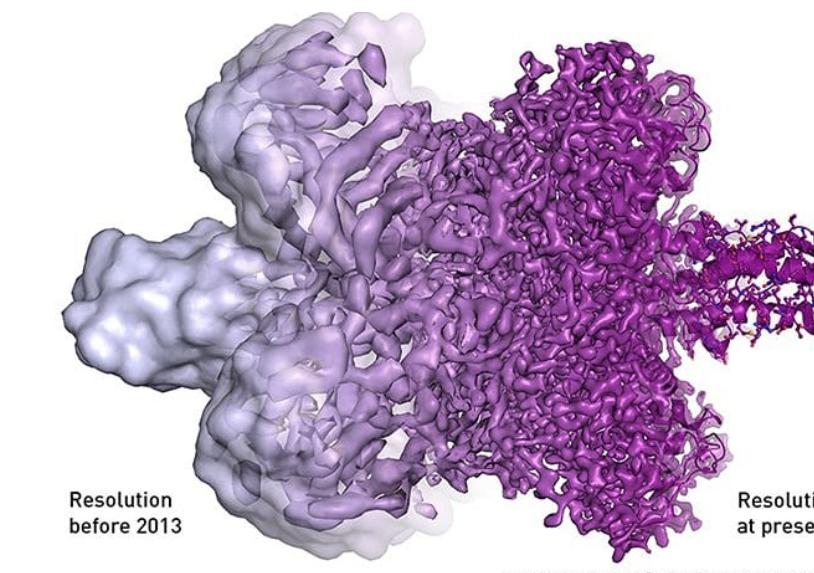
New WarnerMedia CEO John Stankey says HBO is going to "change direction a little bit," and it's all about the advertising.



In science and engineering

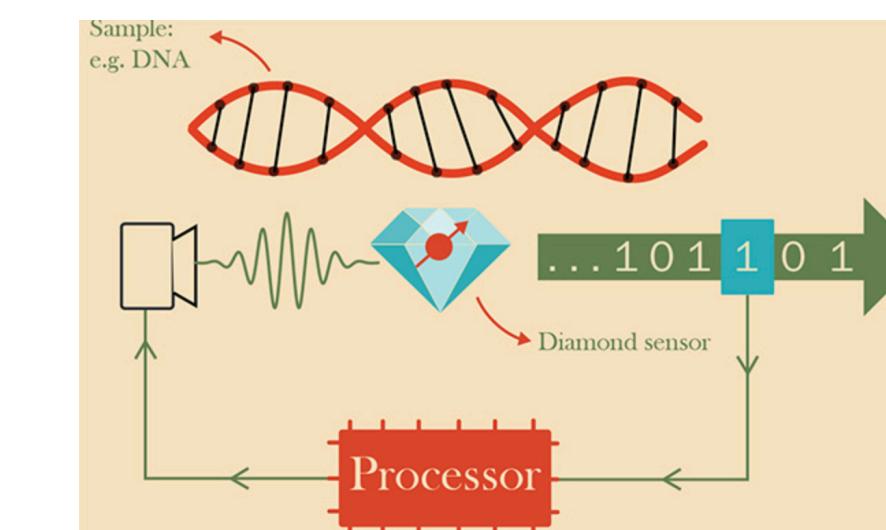
# The Guardian

What is cryo-electron microscopy, the Nobel prize-winning technique?



 IEEE  
**SPECTRUM**

**Measuring Tiny Magnetic Fields  
With an Intelligent Quantum Sensor**



# On TV

## LAST WEEK TONIGHT WITH JOHN OLIVER

"Last Week Tonight with John Oliver": Scientific Studies



**Data Science and the Art of Producing  
Entertainment at Netflix**



# Statistics, Data Science . . . and all that

Statistics, Data Science, Machine Learning, Artificial Intelligence

**What's the difference?**

# Statistics, Data Science . . . and all that

Statistics, Data Science, Machine Learning, Artificial Intelligence

## What's the difference?

- ▶ All use data to gather insight and ultimately make decisions
- ▶ Statistics is at the core of the data processing part
- ▶ Nowadays, computational aspects play an important role as data becomes larger

# Computational and statistical aspects of data science

- ▶ Computational view: data is a (large) sequence of numbers that needs to be processed by a relatively fast algorithm:  
*approximate nearest neighbors, low dimensional embeddings, spectral methods, distributed optimization*, etc.
- ▶ Statistical view: data comes from a **random process**. The goal is to learn **how this process works** in order to make predictions or to understand what plays a role in it.

To understand randomness, we need PROBABILITY.

# Probability

- ▶ Probability studies randomness (hence the prerequisite)
- ▶ Sometimes, the physical process is completely known: dice, cards, roulette, fair coins, ...

Rolling 1 die:

- ▶ Alice gets \$1 if # of dots  $\leq 3$
- ▶ Bob gets \$2 if # of dots  $\leq 2$

Who do you want to be: Alice or Bob?

Rolling 2 dice:

- ▶ Choose a number between 2 and 12
- ▶ Win \$100 if you chose the sum of the 2 dice

Which number do you choose?

$$E[A] = \frac{1}{2} \cdot \$1 = \$0.5$$

$$E[B] = \frac{1}{3} \cdot \$2 = \$0.66$$

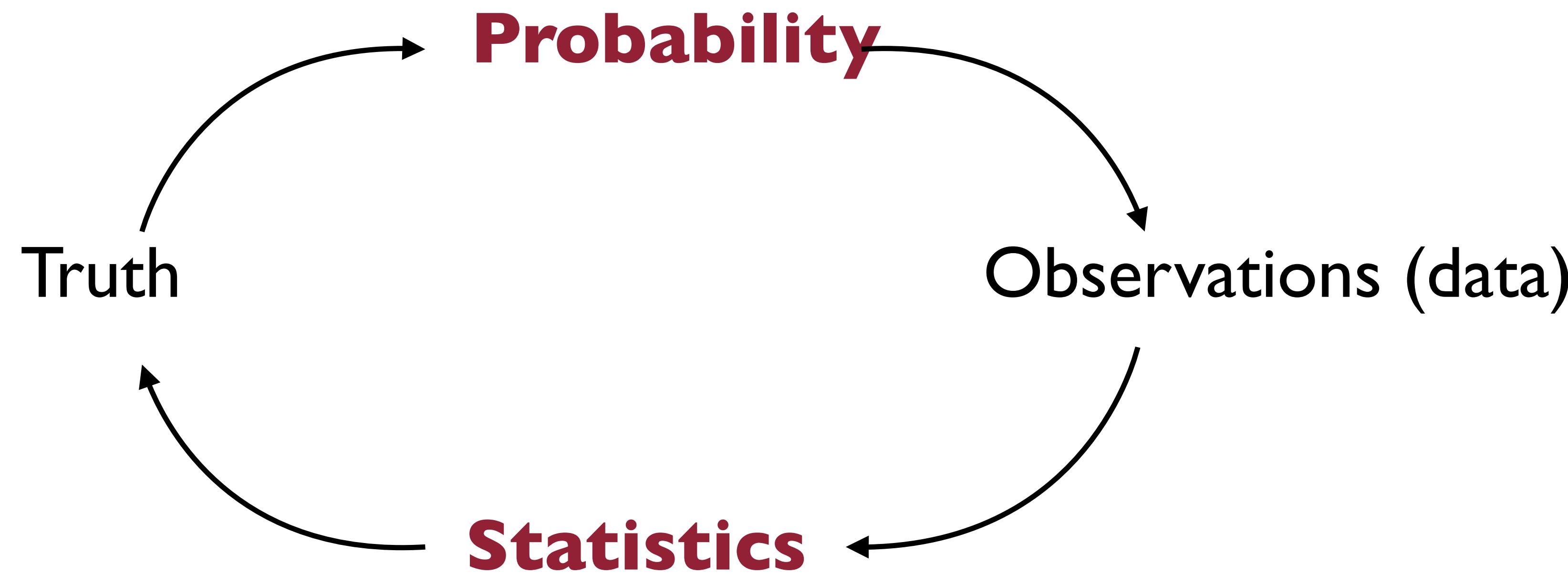
# Statistics and modeling

- ▶ Dice are well known random process from physics: 1/6 chance of each side (no need for data!), dice are independent. We can deduce the probability of outcomes, and expected \$ amounts. This is **probability**.
- ▶ How about more complicated processes? Need to estimate parameters from data. This is **statistics**
- ▶ Sometimes real randomness (random student, biased coin, measurement error, . . . )
- ▶ Sometimes deterministic but too complex phenomenon:  
**statistical modeling**

Complicated process “=” Simple process + random noise

- ▶ (good) Modeling consists in choosing (plausible) simple process **and** noise distribution.

知道model去总结model的各种属性比如mean



# Statistics vs. probability

**Probability** Previous studies showed that the drug was 80% effective. Then we can anticipate that for a study on 100 patients, in average 80 will be cured and at least 65 will be cured with 99.99% chances.

mean的置信区间

**Statistics** Observe that 78/100 patients were cured. We (will be able to) conclude that we are 95% confident that for other studies the drug will be effective on between 69.88% and 86.11% of patients

假设服从XX(正态)分布，使用XX(正态)分布的规律

# What this course is about

- ▶ Understand **mathematics** behind statistical methods
- ▶ Justify quantitative statements given modeling assumptions
- ▶ Describe interesting mathematics arising in statistics
- ▶ Provide a math toolbox to extend to other models.

## What this course is **not** about

- ▶ Statistical thinking/modeling (e.g., 15.075)
- ▶ Implementation (e.g. IDS.012)
- ▶ Laundry list of methods (e.g. AP stats)

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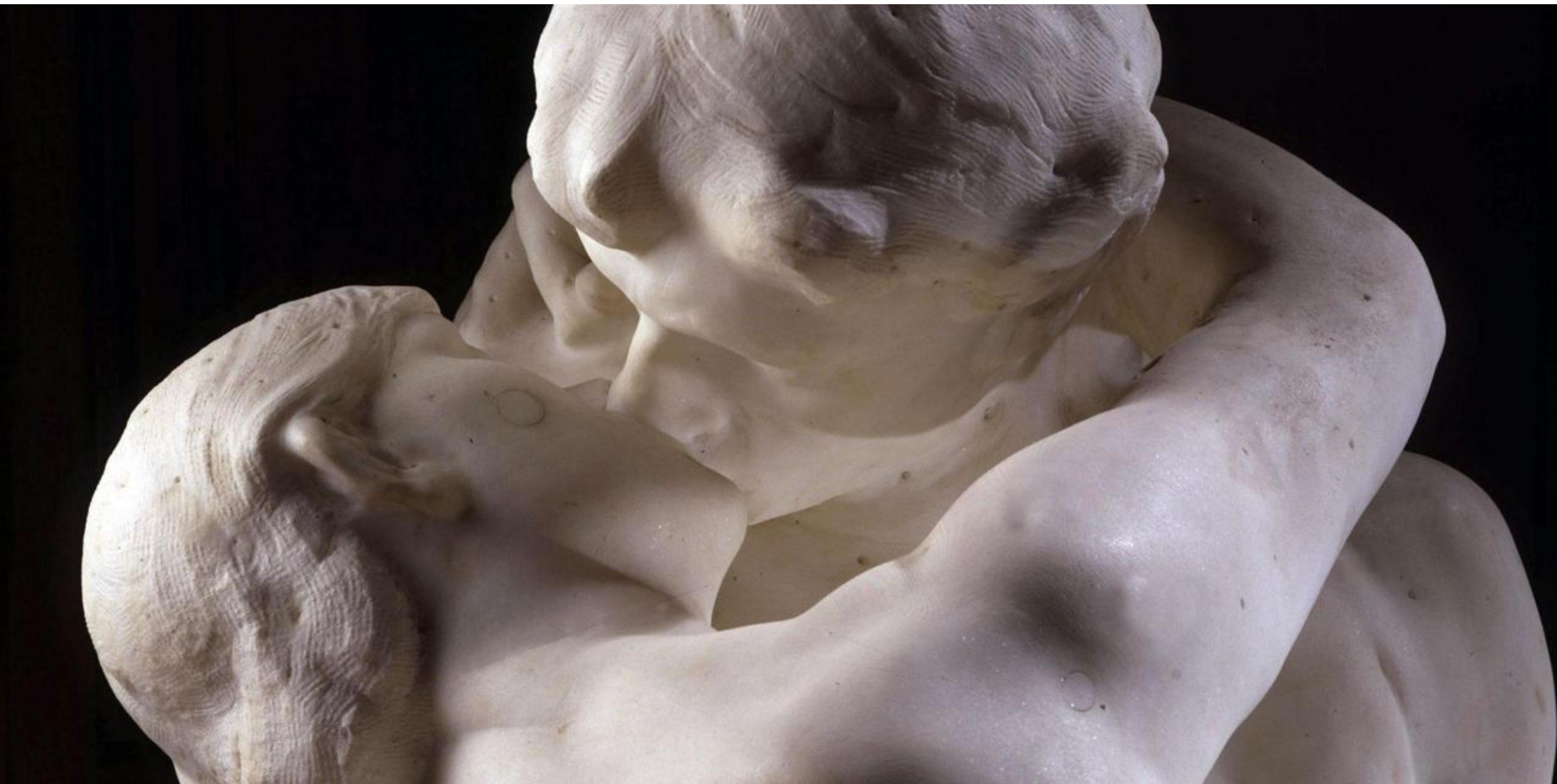
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# Let's do some statistics

# The kiss



Le baiser. Auguste Rodin. 1882.

# The kiss



Le baiser. Auguste Rodin. 1882.

# The kiss

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**Brief Communications**

*Nature* 421, 711 (13 February 2003) | doi:10.1038/421711a

**Human behaviour: Adult persistence of head-turning asymmetry**

Onur Güntürkün

**A neonatal right-side preference makes a surprising romantic reappearance later in life.**



**FULL TEXT**

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# Statistical experiment

*“A neonatal right-side preference makes a surprising romantic reappearance later in life.”*

- ▶ Let  $p$  denote the proportion of couples that turn their head to the right when kissing.
- ▶ Let us design a statistical experiment and analyze its outcome.
- ▶ Observe  $n$  kissing couples times and collect the value of each outcome (say 1 for RIGHT and 0 for LEFT);
- ▶ Estimate  $p$  with the proportion  $\hat{p}$  of RIGHT.
- ▶ Study: “Human behaviour: Adult persistence of head-turning asymmetry” (Nature, 2003):  $n = 124$  and 80 to the right so

$$\hat{p} = \frac{\underline{80}}{\underline{124}} = \underline{64.5\%}$$

# Random intuition

Back to the data:

- ▶ 64.5% is much larger than 50% so there seems to be a preference for turning right.
- ▶ What if our data was RIGHT, RIGHT, LEFT ( $n = 3$ ). That's 66.7% to the right. Even better?
- ▶ Intuitively, we need a large enough sample size  $n$  to make a call. How large?
- ▶ Another way to put the problem: for  $n = \underline{124}$ , what is the minimum number of couple "to the right" would you need to see to be convinced that  $p > 50\%$ ? 63? 72? 75? 80?

We need **mathematical modeling** to understand the accuracy of this procedure?

# A first estimator

Formally, this procedure consists of doing the following:

- ▶ For  $i = 1, \dots, n$ , define  $R_i = 1$  if the  $i$ th couple turns to the right **RIGHT**,  $R_i = 0$  otherwise.
- ▶ The estimator of  $p$  is the Sample average  $\hat{p} = \bar{R}_n = \frac{1}{n} \sum_{i=1}^n R_i$ .

$$\hat{p} = \bar{R}_n = \frac{1}{n} \sum_{i=1}^n R_i$$

What is the accuracy of this estimator ?

In order to answer this question, we propose a statistical model that describes/approximates well the experiment.

We think of the  $R_i$ 's as random variables so that  $\hat{p}$  is also a random variable. We need to understand its fluctuation.

平均数本身就是r.v.  
每一次采样不同，结果不同

sample本身不稳定

# Modelling assumptions

对观测值进行假设

Coming up with a model consists of making assumptions on the observations  $R_i, i = 1, \dots, n$  in order to draw statistical conclusions. Here are the assumptions we make:

1. Each  $R_i$  is a random variable.
2. Each of the r.v.  $R_i$  is Bernoulli with parameter  $p$ .
3.  $R_1, \dots, R_n$  are mutually independent.

$$R_i \sim \text{Ber}(p)$$

$$\begin{aligned} P(R_i=1) &= p \\ P(R_i=0) &= 1-p \end{aligned}$$

# Discussion

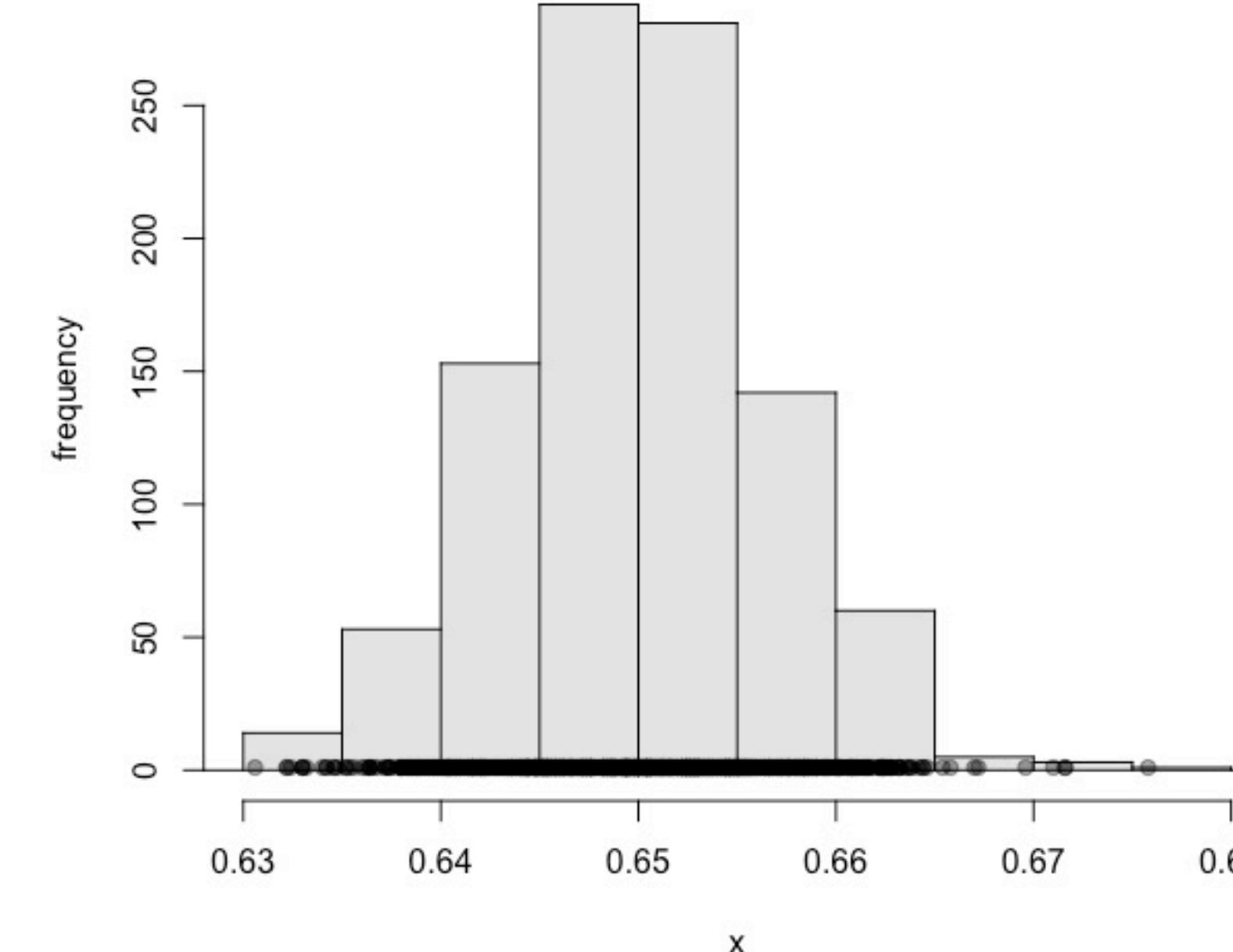
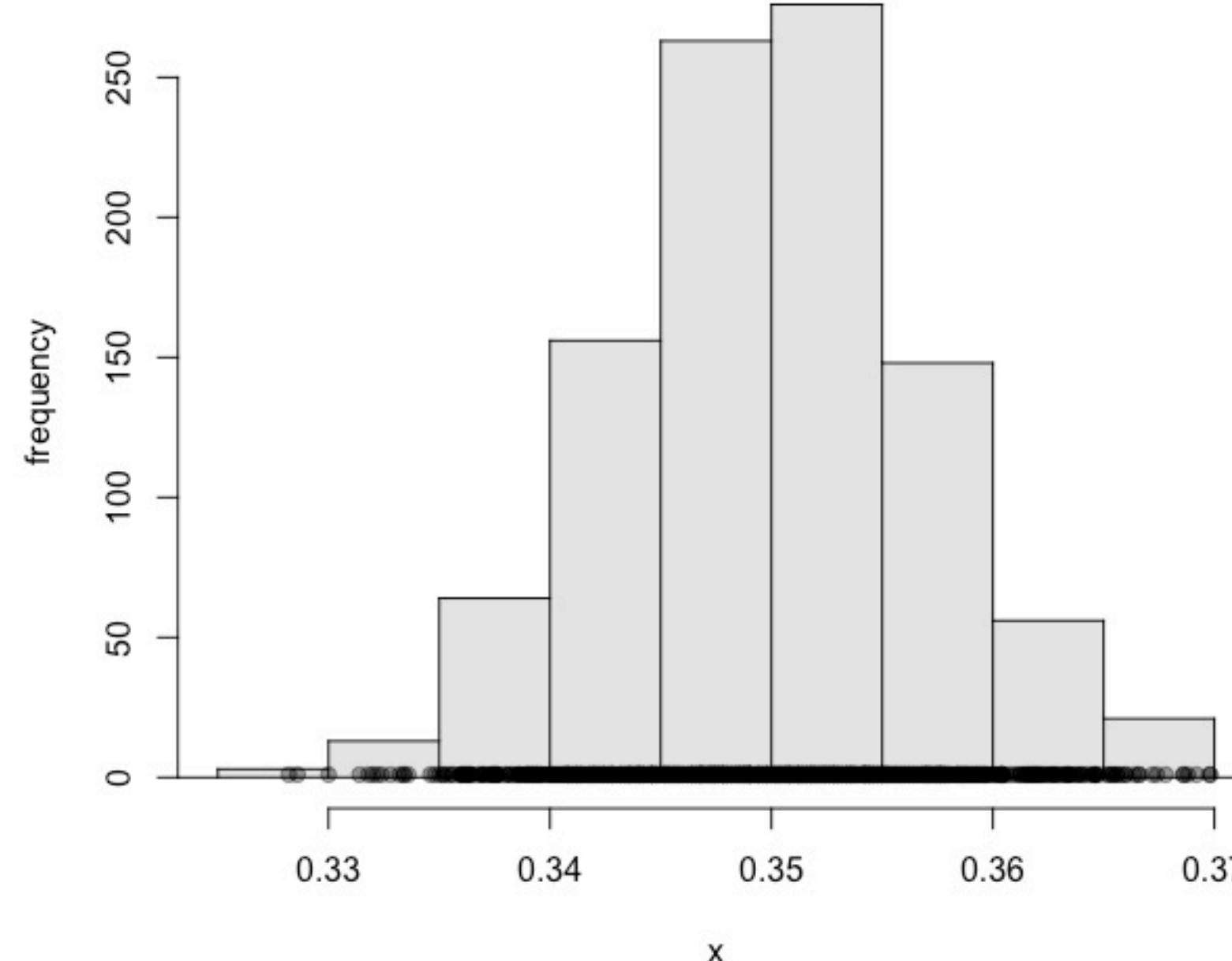
Let us discuss these assumptions.

有所有信息就不需要统计了

1. Randomness is a way of modeling lack of information; with perfect information about the conditions of kissing (including what goes on in the kissers' mind), physics or sociology would allow us to predict the outcome.
2. Hence, the  $R_i$ 's are necessarily Bernoulli r.v. since  $R_i \in \{0, 1\}$ . They could still have a different parameter  $R_i \sim \text{Ber}(p_i)$  for each couple but we don't have enough information with the data to estimate the  $p_i$ 's accurately. So we simply assume that our observations come from the same process:  $p_i = p$  for all  $i$
3. Independence is reasonable (people were observed at different locations and different times).

# Population vs. Samples

- ▶ Assume that there is a total **population** of 5,000 “airport-kissing” couples
- ▶ Assume for the sake of argument that  $p = \underline{\underline{35\%}}$  or that  $\underline{\underline{p = 65\%}}$ .
- ▶ What do **samples** of size 124 look like in each case?



# Why probability?

We need to understand probabilistic aspects of the distribution of the random variable:

$$p \rightarrow \hat{P}$$

$$\hat{p} = \bar{R}_n = \frac{1}{n} \sum_{i=1}^n R_i.$$

Specifically, we need to be able to answer questions such as:

- ▶ Is the expected value of  $\hat{p}$  close to the unknown  $p$ ?      参数estimator的期望值是不是和位置参数接近
- ▶ Does  $\hat{p}$  take values close to  $p$  with high probability?      参数estimator是不是很高概率和参数一致
- ▶ Is the variance of  $\hat{p}$  large? I.e. does  $\hat{p}$  fluctuate a lot?      参数estimator的方差大不大?

We need probabilistic tools! Most of them are about **average of independent random variables**.

$$\text{Var}(\bar{R}_n) = ?$$
$$\Pr(|\bar{R}_n - p| > 0.1) = ?$$

# Probability redux

# Averages of random variables: LLN & CLT

Let  $X, X_1, X_2, \dots, X_n$  be i.i.d. r.v.,  $\mu = \mathbb{E}[X]$  and  $\sigma^2 = \mathbb{V}[X]$ .

- ▶ Laws (weak and strong) of large numbers (LLN):

$$\bar{X}_n := \frac{1}{n} \sum_{i=1}^n X_i \xrightarrow[n \rightarrow \infty]{\mathbb{P}, \text{ a.s.}} \mu.$$

样本均数converge to 总体均数as n goes to infinity

- ▶ Central limit theorem (CLT):

样本均数converge to 正态分布as n goes to infinity

rule of thumb :

$n \geq 30$

$$\sqrt{n} \frac{\bar{X}_n - \mu}{\sigma} \xrightarrow[n \rightarrow \infty]{(d)} \mathcal{N}(0, 1).$$

(Equivalently,  $\sqrt{n} (\bar{X}_n - \mu) \xrightarrow[n \rightarrow \infty]{(d)} \mathcal{N}(0, \sigma^2)$ .)

# Another useful tool: Hoeffding's inequality

What if  $n$  is not large enough to apply CLT?

**Theorem** (Hoeffding, 1963)

Let  $n$  be a positive integer and  $X, X_1, \dots, X_n$  be i.i.d. r.v. such that  $\mu = \mathbb{E}[X]$  and

$X \in [a, b]$  almost surely

( $a < b$  are given numbers)

Then,

样本均数和总体均数的差异被bound了

$$\mathbb{P}[|\bar{X}_n - \mu| \geq \varepsilon] \leq 2e^{-\frac{2n\varepsilon^2}{(b-a)^2}}. \quad \forall \varepsilon > 0$$

This holds even for small sample sizes  $n$ .

$X_i \stackrel{\text{iid}}{\sim} \text{Ber}(p)$

$$\mathbb{P}\left(|\bar{X}_n - \mu| \geq \frac{c}{\sqrt{n}}\right) \leq 2e^{-\frac{2c^2}{(b-a)^2}}$$

# Consequences

- The LLN's tell us that

$$\bar{R}_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}, \text{ a.s.}} p.$$

$$X_i \sim \text{Ber}(\rho_i) \quad \rho_i = p \\ \forall i$$

i.i.d  
indep.

(what modeling assumptions did we use?)

- Hence, when the size  $n$  of the experiment becomes large,  $\bar{R}_n$  is a good (say "consistent") estimator of  $p$ .
- The CLT refines this by quantifying how good this estimate is: for  $n$  large enough the distribution of  $\hat{p}$  is almost:

$$\mathbb{P}(|\bar{R}_n - p| \geq \varepsilon) \simeq \mathbb{P}(|\mathcal{N}(0, \frac{p(1-p)}{n})| > \varepsilon)$$

$n \geq 30$

两种不同的测量估计好坏程度的方式  
(CLT和Hoeffding)

- In the Kiss example,  $\mathbb{P}(|\bar{R}_n - p| \geq 0.084) \simeq 5\%$
- Hoeffding's inequality tells us that 这个值是根据上面的正态分布和确定的alpha=0.05定的

$$\mathbb{P}(|\bar{R}_n - p| \geq 0.084) \leq \exp\left(-\frac{2 \cdot 124 \cdot (0.084)^2}{(1-p)^2}\right) \leq 0.35$$

同样的值用另一种方法，概率更大。  
因为这个不等式不禁，bound更大

# The Gaussian distribution

Because of the CLT, the Gaussian (a.k.a normal) distribution is ubiquitous in statistics. It is named after German Mathematician Carl Friedrich Gauss (1777–1855) in the context of the method of *least squares* (regression).

- ▶  $X \sim \mathcal{N}(\mu, \sigma^2)$
- ▶  $\mathbb{E}[X] = \mu$
- ▶  $\text{var}(X) = \sigma^2 > 0$



## Gaussian density (pdf)

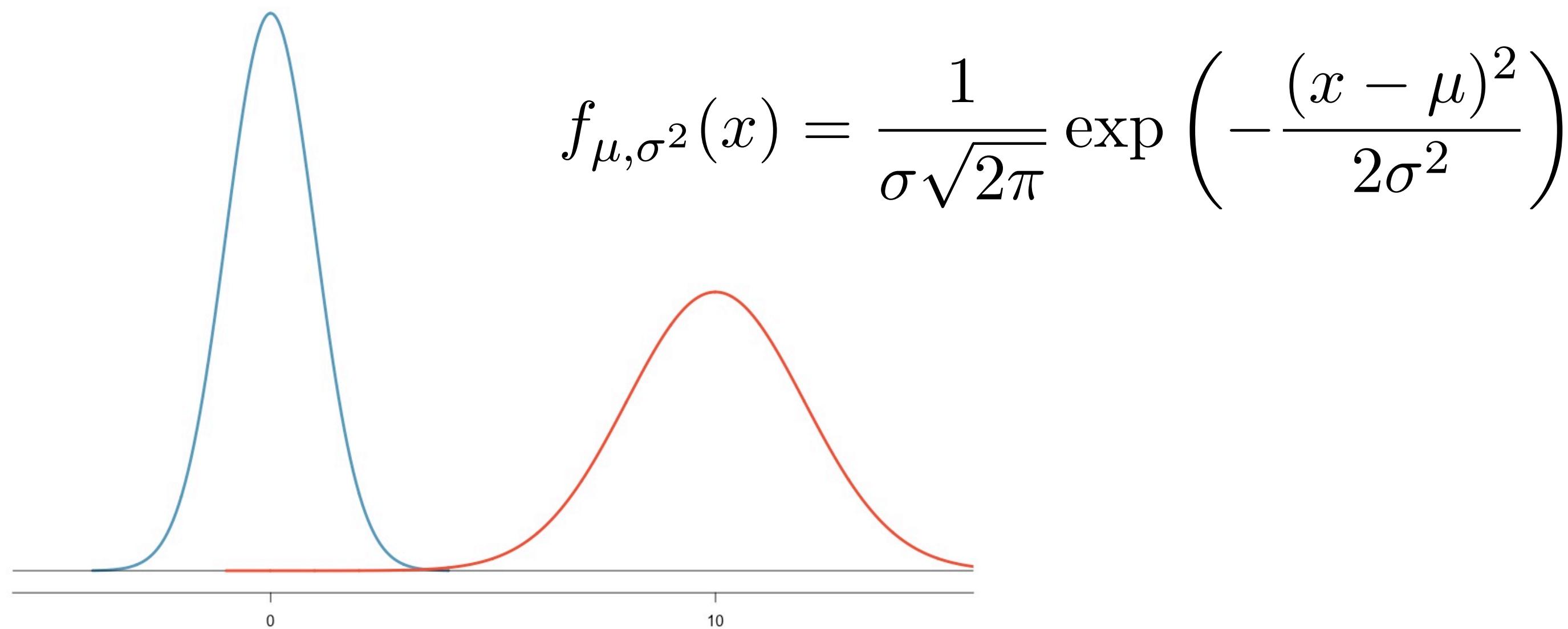


Figure 1: Two pdfs:  $\mathcal{N}(0, 1)$  and  $\mathcal{N}(10, 4)$

- ▶ Tails decay very fast (like  $e^{-\frac{x^2}{2\sigma^2}}$ ): almost in finite interval.
- ▶ There is no closed form for their cumulative distribution function (CDF). We use tables (or computers):

$$P(X \leq x) = F_{\mu, \sigma^2}(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^x \exp\left(-\frac{(t - \mu)^2}{2\sigma^2}\right) dt$$

## Some useful properties of Gaussians

Perhaps the most useful property of the Gaussian family is that it's *invariant under affine transformation*:

- $X \sim \mathcal{N}(\mu, \sigma^2)$ , then for any  $a, b \in \mathbb{R}$ ,

$$a \cdot X + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$$

- **Standardization** (a.k.a Normalization/Z-score): If  $X \sim \mathcal{N}(\mu, \sigma^2)$ , then

$$Z = \frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$$

Useful to compute probabilities from CDF of  $Z \sim \mathcal{N}(0, 1)$ :

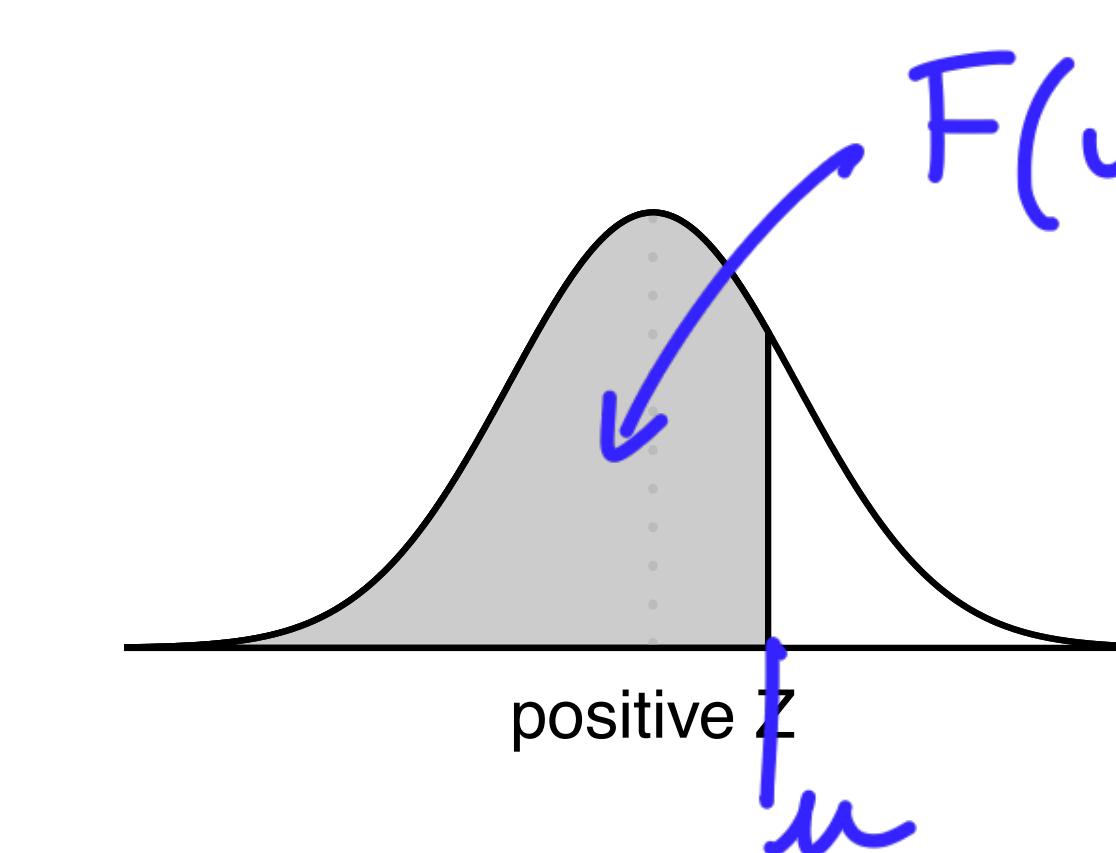
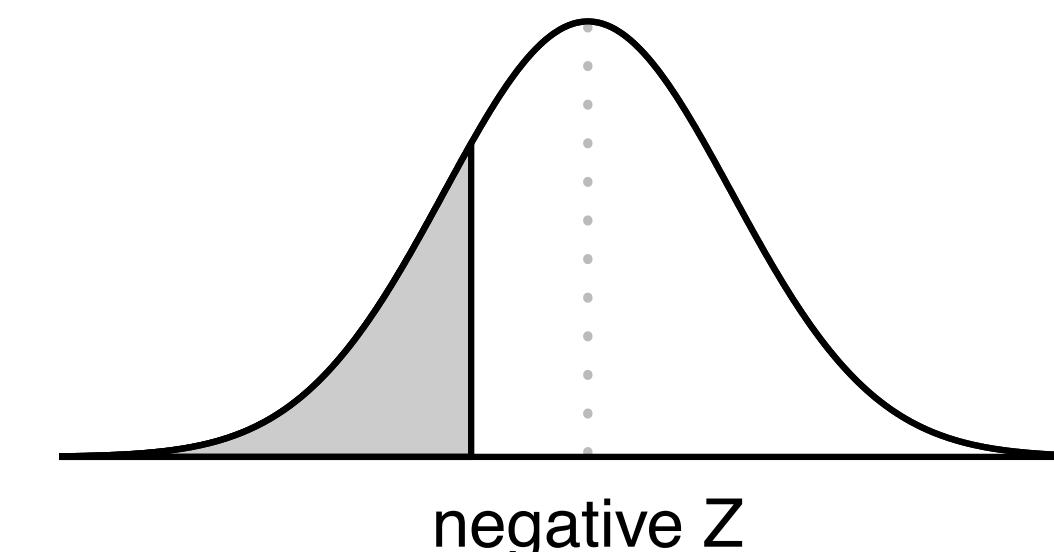
$$\mathbb{P}(u \leq X \leq v) = \mathbb{P}\left(\frac{\underline{u}-\mu}{\sigma} \leq Z \leq \frac{\underline{v}-\mu}{\sigma}\right)$$

- **symmetry**: If  $X \sim \mathcal{N}(\mu, \sigma^2)$  then  $-X \sim \mathcal{N}(-\mu, \sigma^2)$ : If  $x > 0$

$$\mathbb{P}(|X| > x) = \mathbb{P}(X > x) + \mathbb{P}(-X > -x) = 2\mathbb{P}(X > x)$$

# Gaussian probability tables

$$Z \sim N(0, 1)$$

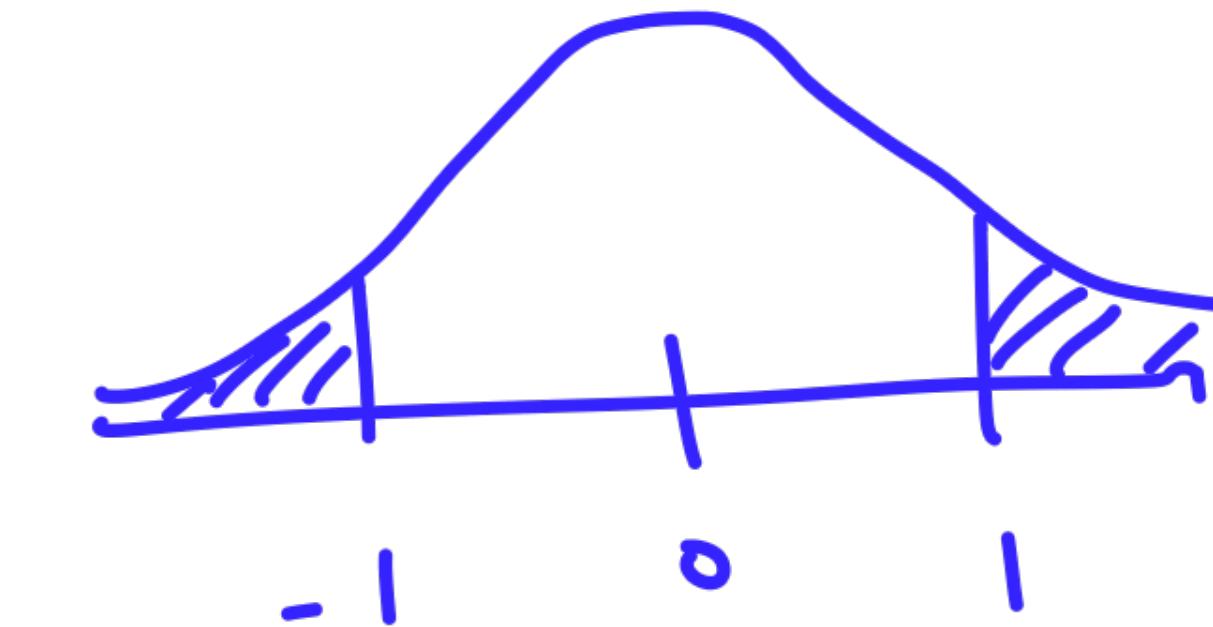


$$F(u) = P(Z \leq u)$$

$$\begin{aligned} P(Z \leq 0.76) \\ = 0.7764 \end{aligned}$$

Z	Second decimal place of Z									
	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

## Examples



Assume that  $Z \sim \mathcal{N}(0, 1)$  and compute

- $\mathbb{P}(Z \leq 1) = 0.8413$
- $\mathbb{P}(Z \geq -1) = \mathbb{P}(-z \leq 1) = \mathbb{P}(z \leq 1) = 0.8413$
- $\mathbb{P}(|Z| > 1) = 2 \mathbb{P}(z > 1) = 2 \cdot (1 - 0.8413) = 0.31$

Assume that the score distribution for a final exam is approximately  $X \sim \mathcal{N}(85, 4)$ , compute

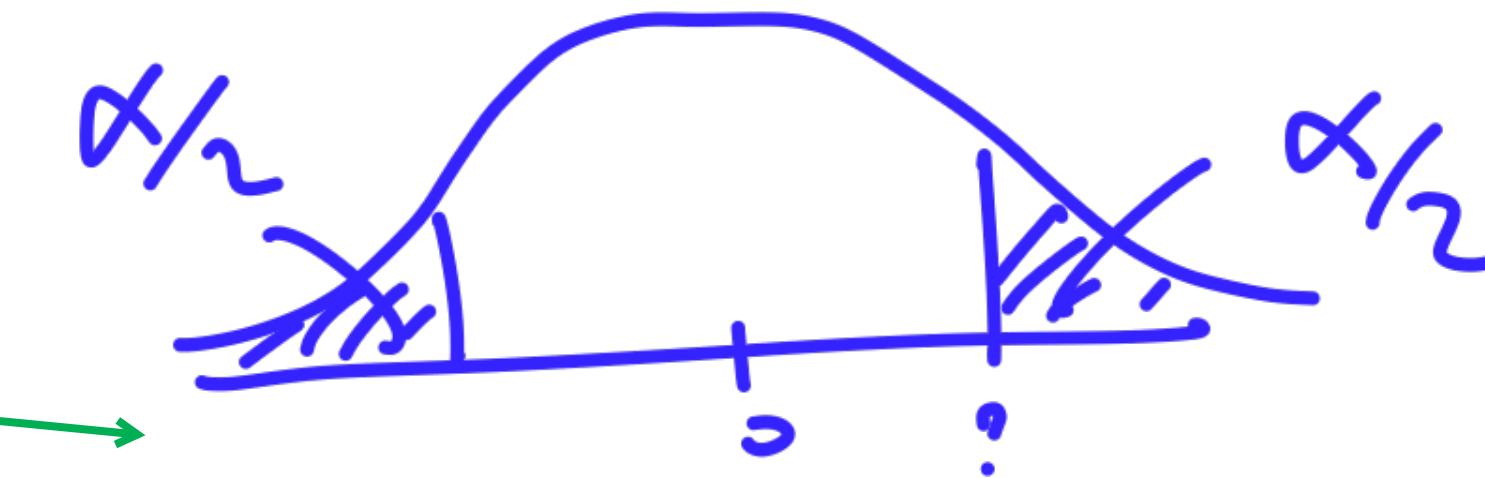
- $\mathbb{P}(X > 90) = \mathbb{P}\left(\frac{X-85}{2} > \frac{90-85}{2}\right) = \mathbb{P}(z > 2.5) = 0.62\%$
- $\mathbb{P}(80 < X < 90)$   
 $= \mathbb{P}(-2.5 < z < 2.5) = 0.9376$

More complicated: what is  $x$  such that  $\mathbb{P}(X < x) = 90\%$  ( $90^{\text{th}}$  percentile?). For that we need to read the table backwards.

## Quantiles

一个划分data的值！

X轴上的值



### Definition

Let  $\alpha$  in  $(0, 1)$ . The quantile of order  $1 - \alpha$  of a random variable  $X$  is the number  $q_\alpha$  such that

$$\mathbb{P}(X \leq q_\alpha) = 1 - \alpha$$

$\alpha = \cdot \rightarrow q_\alpha$  is  
the 90th percentile

Let  $F$  denote the CDF of  $X$ :

- ▶  $F(q_\alpha) = 1 - \alpha$
- ▶ If  $F$  is invertible, then  $q_\alpha = F^{-1}(1 - \alpha)$
- ▶  $\mathbb{P}(X > q_\alpha) = \alpha$
- ▶ If  $X = Z \sim \mathcal{N}(0, 1)$ :  $\mathbb{P}(|Z| > q_{\alpha/2}) = \alpha$

Some important quantiles of the  $Z \sim \mathcal{N}(0, 1)$  are:

$\alpha$	2.5%	5%	10%
$q_\alpha$	1.96	1.65	1.28

We get that  $\mathbb{P}(|Z| > 1.96) = 5\%$

# Three types of convergence

- $(T_n)_{n \geq 1}$  is a sequence of random variables
- $T$  is a random variable ( $T$  may be deterministic).

► Almost surely (a.s.) convergence:  $T_n$ 的r.v.值的converge , 更严格的收敛 , 存在某个 $N$  ,  $n > N$ 的时候 , 值相等。

$$T_n \xrightarrow[n \rightarrow \infty]{\text{a.s.}} T \quad \text{iff} \quad \mathbb{P} \left[ \left\{ \omega : T_n(\omega) \xrightarrow{n \rightarrow \infty} T(\omega) \right\} \right] = 1.$$

► Convergence in probability:  $T_n$ 的r.v.值的converge ,  $n$ 很大的时候 ,  $T_n$ 和 $T$ 的值差不多 , 不相等的概率越来越小。

$$T_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}} T \quad \text{iff} \quad \mathbb{P} [|T_n - T| \geq \varepsilon] \xrightarrow{n \rightarrow \infty} 0, \quad \forall \varepsilon > 0.$$

► Convergence in distribution:  $T_n$ 分布的CDF的converge , 只在乎分布 , 不在乎相互关系

$$T_n \xrightarrow[n \rightarrow \infty]{(d)} T \quad \text{iff} \quad \mathbb{E}[f(T_n)] \xrightarrow{n \rightarrow \infty} \mathbb{E}[f(T)]$$

只是期望converge吗 ?

for all continuous and bounded function  $f$ .

# Properties

- ▶ If  $(T_n)_{n \geq 1}$  converges a.s., then it also converges in probability, and the two limits are equal a.s.
- ▶ If  $(T_n)_{n \geq 1}$  converges in probability, then it also converges in distribution
- ▶ **Convergence in distribution** implies convergence of probabilities if the limit has a density (e.g. Gaussian): CDF

$$T_n \xrightarrow[n \rightarrow \infty]{(d)} T \quad \Rightarrow \quad \mathbb{P}(a \leq T_n \leq b) \xrightarrow[n \rightarrow \infty]{} \mathbb{P}(a \leq T \leq b)$$

# Exercises

a) Is the following statement correct? "If  $(T_n)_{n \geq 1}$  converges in probability, then it also converges a.s"

1. Yes
2. No

Let  $\{X_1, X_2, \dots, X_n\}$  be a sequence of r.v. such that

~~$X_n \sim \text{Ber}(\frac{1}{n})$~~ . Exercises b), c) and d) are about this sequence.

b) Let  $0 < \epsilon < 1$ ,  $n \geq 1$ . What is the value of  $P(|X_n| > \epsilon)$ ?

(answer:  $\frac{1}{n}$ )

c) Does  $\{X_n\}$  converges in probability?

1. Yes
2. No

$$\begin{aligned} P(X_n > \epsilon) &= P(X_n = 1) \\ &= \frac{1}{n} \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

## Exercises

d) Denote by  $X$  the limit of  $\{X_n\}$  (if it exists) (that is,  
 $X_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}} X$ ). What is the value of  $X$ ?

1.  $X$  does not exist
2. 0 ✓
3. 1
4. None of the above

e) Does  $\{X_n\}$  converge in distribution?

1. Yes ✓
2. No

f) What is the limit of the sequence  $\mathbb{E}[\cos(X_n)]$  as  $n$  tends to infinity?

$$\mathbb{E}[\cos(0)] = 1$$

# Addition, multiplication, division

... only for a.s. and  $\mathbb{P}$ ...

Assume

$$T_n \xrightarrow[n \rightarrow \infty]{\text{a.s./}\mathbb{P}} T \quad \text{and} \quad U_n \xrightarrow[n \rightarrow \infty]{\text{a.s./}\mathbb{P}} U$$

Then,

- ▶  $T_n + U_n \xrightarrow[n \rightarrow \infty]{\text{a.s./}\mathbb{P}} T + U,$
- ▶  $T_n U_n \xrightarrow[n \rightarrow \infty]{\text{a.s./}\mathbb{P}} TU,$
- ▶ If in addition,  $U \neq 0$  a.s., then  $\frac{T_n}{U_n} \xrightarrow[n \rightarrow \infty]{\text{a.s./}\mathbb{P}} \frac{T}{U}.$



In general, these rules **do not** apply to convergence (d).

## Slutsky's theorem

Some partial results exist for convergence in distribution on the form of *Slutsky's theorem*.

Let  $(X_n), (Y_n)$  be two sequences of r.v., such that:

$$(i) \ T_n \xrightarrow[n \rightarrow \infty]{(d)} T \quad \text{and} \quad (ii) \ U_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}} u$$

where  $T$  is a r.v. and  $u$  is a given real number (deterministic limit:  $\mathbb{P}(U = u) = 1$ ). Then,

- ▶  $T_n + U_n \xrightarrow[n \rightarrow \infty]{(d)} T + u,$
- ▶  $T_n U_n \xrightarrow[n \rightarrow \infty]{(d)} Tu,$
- ▶ If in addition,  $u \neq 0$ , then  $\frac{T_n}{U_n} \xrightarrow[n \rightarrow \infty]{(d)} \frac{T}{u}$

...

## Taking functions

Continuous functions (for all three types) . If  $f$  is a **continuous function**: 三种convergence都能用

$$T_n \xrightarrow[n \rightarrow \infty]{\text{a.s./P/(d)}} T \Rightarrow f(T_n) \xrightarrow[n \rightarrow \infty]{\text{a.s./P/(d)}} f(T).$$

Continuous mapping theorem

**Example:** Recall that by LLN,  $\bar{R}_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}, \text{ a.s.}} p$ . Therefore

$$f(\bar{R}_n) \xrightarrow[n \rightarrow \infty]{\mathbb{P}, \text{ a.s.}} f(p) \text{ for any continuous } f$$

(Only need  $f$  to be continuous around  $p$ :  $f(x)=1/x$  works if  $p > 0$ )

We also have by CLT:  $\sqrt{n} \frac{\bar{R}_n - p}{\sqrt{p(1-p)}} \xrightarrow[n \rightarrow \infty]{(d)} Z, Z \sim \mathcal{N}(0, 1)$ . So

$$f\left(\sqrt{n}(\bar{R}_n - p)\right) \xrightarrow[n \rightarrow \infty]{(d)} f(Y) \quad Y \sim \mathcal{N}(0, p(1-p))$$

⚠ not the limit of  $\sqrt{n}[f(\bar{R}_n) - f(p)]$  !!  
分布上convergence

Delta ( $\Delta$ ) method

# Recap

- ▶ Averages of random variables occur naturally in statistics
- ▶ We make modeling assumptions to apply probability results
- ▶ For large sample size they are consistent (LLN) and we know their distribution (CLT)
- ▶ CLT gives the (weakest) convergence in distribution but is enough to compute probabilities
- ▶ We use standardization and Gaussian tables to compute probabilities and quantiles
- ▶ We can make operations (addition, multiplication, continuous functions) on sequences of random variables