LECTURE 19: The Central Limit Theorem (CLT)

- WLLN: $\dfrac{X_1 + \cdots + X_n}{n} \longrightarrow \mathbf{E}[X]$

- CLT: $X_1 + \cdots + X_n \approx$ normal
  - precise statement
  - universality, usefulness
  - many examples
  - refinement for discrete r.v.s
  - application to polling

Different scalings of the sum of i.i.d. random variables

- $X_1, \ldots, X_n$ i.i.d., finite mean μ and variance σ^2

- $S_n = X_1 + \cdots + X_n$

  variance: $n\sigma^2$

- $M_n = \dfrac{S_n}{n} = \dfrac{X_1 + \cdots + X_n}{n}$

  variance: $\dfrac{\sigma^2}{n}$

- $\dfrac{S_n}{\sqrt{n}} = \dfrac{X_1 + \cdots + X_n}{\sqrt{n}}$

  variance: σ^2

The Central Limit Theorem (CLT)

- $X_1, \ldots, X_n$ i.i.d., finite mean μ and variance σ^2
- $S_n = X_1 + \cdots + X_n$ variance: $n\sigma^2$
- $\dfrac{S_n}{\sqrt{n}} = \dfrac{X_1 + \cdots + X_n}{\sqrt{n}}$ variance: σ^2

$Z_n = \dfrac{S_n - n\mu}{\sqrt{n}\,\sigma}$ $\mathbf{E}[Z_n] =$ $\text{var}(Z_n) =$

- Let Z be a standard normal r.v. (zero mean, unit variance)

Central Limit Theorem: For every z: $\lim_{n \to \infty} \mathbf{P}(Z_n \leq z) = \mathbf{P}(Z \leq z)$

- $\mathbf{P}(Z \leq z)$ is the standard normal CDF, $\Phi(z)$, available from the normal tables

Usefulness of the CLT

$S_n = X_1 + \cdots + X_n$ $\qquad$ $Z_n = \dfrac{S_n - n\mu}{\sqrt{n}\,\sigma}$ $\qquad$ $Z \sim N(0,1)$

Central Limit Theorem: For every z: $\lim_{n\to\infty} \mathbf{P}(Z_n \leq z) = \mathbf{P}(Z \leq z)$

- universal and easy to apply; only means, variances matter
- fairly accurate computational shortcut
- justification of normal models

What exactly does the CLT say? — Theory

$S_n = X_1 + \cdots + X_n$ $\qquad Z_n = \dfrac{S_n - n\mu}{\sqrt{n}\,\sigma}$ $\qquad Z \sim N(0,1)$

Central Limit Theorem: For every z: $\lim_{n\to\infty} \mathbf{P}(Z_n \le z) = \mathbf{P}(Z \le z)$

- CDF of Z_n converges to normal CDF
- results for convergence of PDFs or PMFs (with more assumptions)
- results without assuming that the X_i are identically distributed
- results under "weak dependence"
- proof: uses "transforms": $\mathbf{E}[e^{sZ_n}] \to \mathbf{E}[e^{sZ}]$, for all s

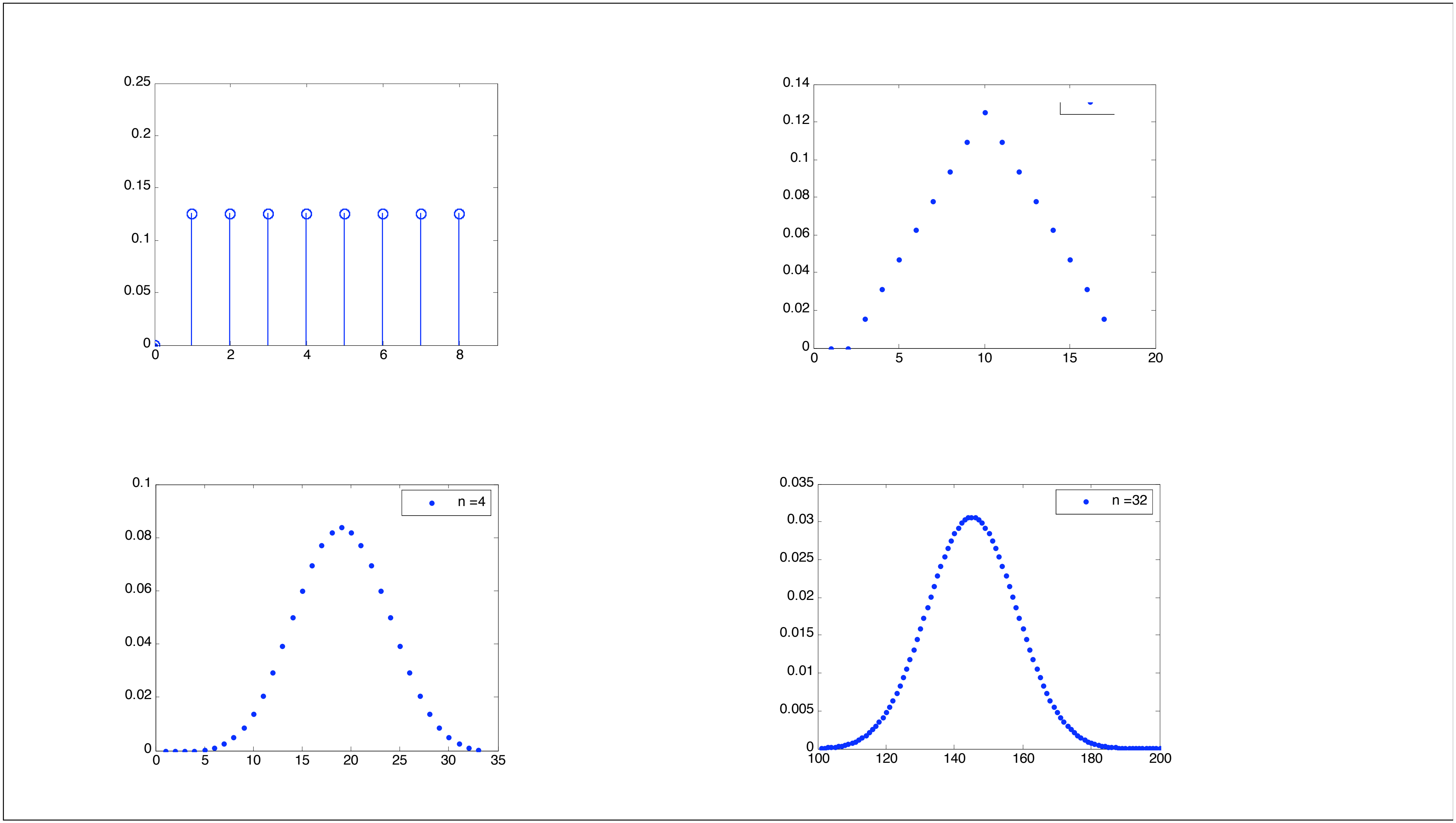
What exactly does the CLT say? — Practice

$S_n = X_1 + \cdots + X_n$ $\qquad$ $Z_n = \dfrac{S_n - n\mu}{\sqrt{n}\,\sigma}$ $\qquad$ $Z \sim N(0,1)$

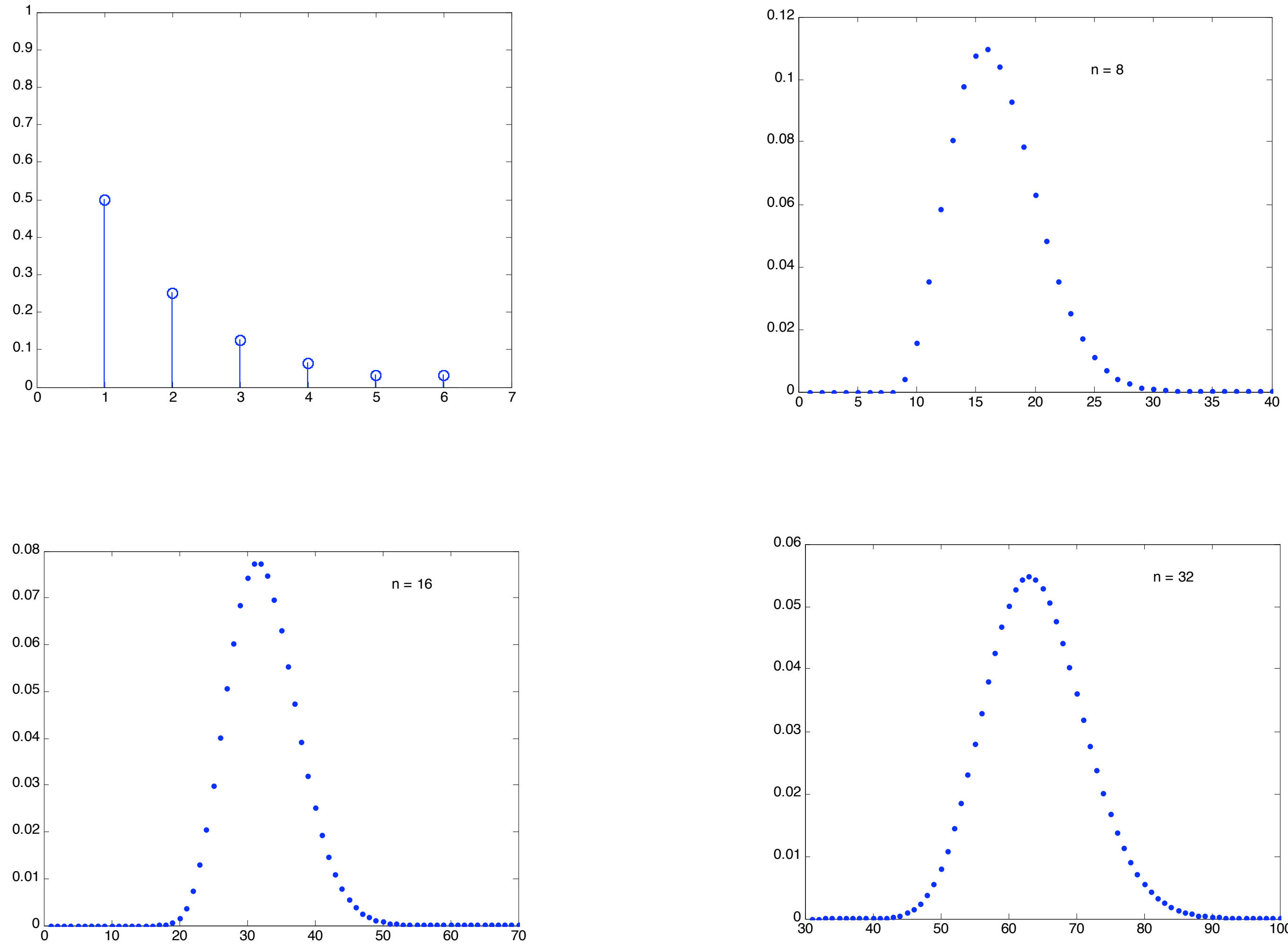
Central Limit Theorem: For every z: $\lim_{n\to\infty} \mathbf{P}(Z_n \le z) = \mathbf{P}(Z \le z)$

- The **practice** of normal approximations:
  - treat Z_n as if it were normal
  - hence treat S_n as if normal: $\mathcal{N}(n\mu, n\sigma^2)$

- Can we use the CLT when n is "moderate"?
  - usually, yes
  - symmetry and unimodality help



0.25
0.2
0.15
0.1
0.05
0
0
2
4
6
8
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0
5
10
15
20
0.1
0.08
0.06
0.04
0.02
0
0
5
10
15
20
25
30
35
n =4
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
100
120
140
160
180
200
n =32


n = 8
n = 16
n = 32

Example 1

- $\mathbf{P}(S_n \leq a) \approx b$ given two parameters, find the third

$$Z_n = \frac{S_n - n\mu}{\sqrt{n}\,\sigma}$$

- Package weights X_i, i.i.d. exponential, $\lambda = 1/2$;

$$\mu = \sigma = 2$$

- Load container with $n = 100$ packages

$\mathbf{P}(S_n \geq 210)$

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767

Example 2

- $\mathbf{P}(S_n \leq a) \approx b$ given two parameters, find the third
- Package weights X_i, i.i.d. exponential, $\lambda = 1/2$;
- Let $n = 100$. Choose the "capacity" a, so that $\mathbf{P}(S_n \geq a) \approx 0.05$.

$$Z_n = \frac{S_n - n\mu}{\sqrt{n}\,\sigma}$$

$$\mu = \sigma = 2$$

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
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1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
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1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767

Example 3

- $\mathbf{P}(S_n \leq a) \approx b$ given two parameters, find the third
- Package weights X_i, i.i.d. exponential, $\lambda = 1/2$;

$$Z_n = \frac{S_n - n\mu}{\sqrt{n}\,\sigma}$$

$$\mu = \sigma = 2$$

- How large can n be, so that $\mathbf{P}(S_n \geq 210) \approx 0.05$?

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
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1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
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1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767

Example 4

- $\mathbf{P}(S_n \leq a) \approx b$ given two parameters, find the third
- Package weights X_i, i.i.d. exponential, $\lambda = 1/2$;

$$Z_n = \frac{S_n - n\mu}{\sqrt{n}\,\sigma}$$

$$\mu = \sigma = 2$$

- Load container until weight exceeds 210
  N: number of packages loaded
- $\mathbf{P}(N > 100)$

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
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0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
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1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767

Normal approximation to the binomial

- X_i: independent, Bernoulli(p); $\quad 0 < p < 1$
- $S_n = X_1 + \cdots + X_n$: Binomial(n, p)
  - mean np, variance $np(1-p)$
- $n = 36$, $p = 0.5$; find $\mathbf{P}(S_n \le 21)$

  $np = 18 \qquad \sqrt{np(1-p)} = 3$

- CDF of $\dfrac{S_n - np}{\sqrt{np(1-p)}} \longrightarrow$ standard normal

$$\sum_{k=0}^{21} \binom{36}{k} \left(\frac{1}{2}\right)^{36} = 0.8785$$

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
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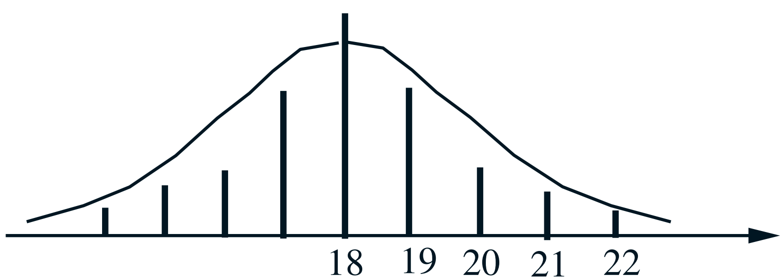
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319

The 1/2 correction for integer random variables

- $0.8413 \approx \mathbf{P}(S_n \leq 21) = \mathbf{P}(S_n < 22)$, because S_n is integer

true value 0.8785

$\mathbf{P}(S_n \leq 21.5)$



	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
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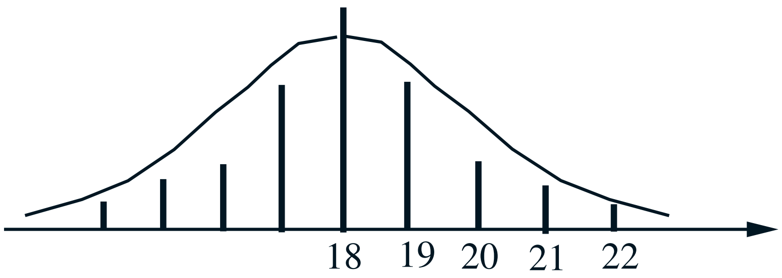
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319

De Moivre–Laplace CLT to the binomial

$\mathbf{P}(S_n = 19)$



- Exact answer:

$$\binom{36}{19}\left(\frac{1}{2}\right)^{36} = 0.1251$$

- When the 1/2 correction is used, the CLT can also approximate the binomial PMF (not just the binomial CDF)

The pollster's problem revisited

- p: fraction of population that will vote "yes" in a referendum
- ith (randomly selected) person polled: $$X_i = \begin{cases} 1, & \text{if yes,} \\ 0, & \text{if no.} \end{cases}$$
- $M_n = (X_1 + \cdots + X_n)/n$: fraction of "yes" in our sample
- Would like "small error," e.g.: $|M_n - p| < 0.01$

$$\mathbf{P}\Big(|M_n - p| \geq .01\Big)$$

$$Z_n = \frac{S_n - n\mu}{\sqrt{n}\,\sigma}$$

The pollster's problem revisited

$$\mathbf{P}\Big(|M_n - p| \geq .01\Big) \approx \mathbf{P}\left(|Z| \geq \frac{.01\sqrt{n}}{\sigma}\right) \qquad \mathbf{P}\Big(|Z| \geq .02\sqrt{n}\Big)$$

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
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0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817

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- Try $n = 10,000$

- Specs: $\mathbf{P}\Big(|M_n - p| \geq .01\Big) \leq .05$