Linear Algebra Presentation

(Vector Spaces, Transformations, Eigenvalues, SVD, and PCA)

Introduction to Linear Algebra

Linear algebra is the branch of mathematics that deals with:

- Vectors and vector spaces
- Systems of linear equations
- Matrix operations
- Applications in machine learning, computer graphics, and more

It provides the foundational tools to manipulate and understand data in high-dimensional spaces.

Vector Spaces

A vector space is a set of elements (called vectors) where:

- You can add any two vectors and get another vector.
- You can multiply any vector by a scalar and remain in the space.

Definition:

A set V with two operations $(+, \cdot)$ is a vector space over a field F if:

- Additive identity: $\exists\ 0 \in V$ such that $v + 0 = v$
- Additive inverse: $\forall v \in V$, $\exists\ -v \in V$ such that $v + (-v) = 0$
- Distributive & associative laws hold.

Examples:

- $\mathbb{R}^2$, $\mathbb{R}^3$ (2D/3D space)
- Space of all polynomials
- Space of real-valued functions

Linear Transformations

- A linear transformation $T: V \to W$ preserves vector operations:

- $T(u + v) = T(u) + T(v)$
- $T(c \cdot v) = c \cdot T(v)$

These transformations are often represented as matrices.
When applied to vectors, they stretch, rotate, or reflect them.

Matrix representation example:
If $T(x) = Ax$, then A is the matrix that defines the transformation

Eigenvalues and Eigenvectors

For a square matrix A:

$$\boxed{Av = \lambda v}$$

Where λ is an eigenvalue and v is the corresponding eigenvector. The transformation A stretches (or shrinks) the vector v without changing its direction.

Applications:

- Stability analysis
- Google's PageRank
- Quantum physics
- Principal Component Analysis (PCA)

Singular Value Decomposition (SVD)

Any matrix A can be decomposed as:

$$\boxed{A = U\Sigma V^T}$$

Where:

- U and V are orthogonal matrices
- Σ is a diagonal matrix with singular value

Why it matters:

- It’s used in image compression, noise reduction, and PCA.
- Helps uncover hidden structure in data.

PCA – Engineering Application

Principal Component Analysis (PCA) is a technique to reduce high-dimensional data into fewer dimensions while retaining important patterns.

Steps:
1. Standardize and center data
2. Compute covariance matrix
3. Perform SVD on the covariance matrix
4. Choose top k components (principal axes)
5. Project original data onto these components

Use Case: Reducing a 100-feature dataset into 2D for visualization or training.

Worked Example – Eigenvalues

Given A = [[2, 1], [1, 2]]
Find the eigenvalues and eigenvectors.

1. Solve $\det(A - \lambda I) = 0$
2. Characteristic polynomial: $(2-\lambda)^2 - 1 = 0$
3. $\lambda = 1, 3$
4. Find v such that $Av = \lambda v$

Summary

- Vector spaces help define the structure of data.
- Linear transformations are matrix-based operations on vectors.
- Eigenvalues/vectors identify key directions in transformations.
- SVD is a powerful decomposition used in data science.
- PCA, an SVD-based technique, reduces data dimensionality efficiently.