B.4 Chapter 4

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1. See the table below.

Matrix	Rows × Columns	Square	Diagonal
A	4 × 3	No	No
B	3 × 3	Yes	Yes
C	2 × 2	Yes	No
D	5 × 2	No	No
E	1 × 3	No	No
F	4 × 1	No	No
G	1 × 4	No	No
H	3 × 1	No	No

2. $$\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 13 & 4 & -8 \\ 12 & 0 & 6 \\ -3 & -1 & 5 \\ 10 & -2 & 5 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 13 & 12 & -3 & 10 \\ 4 & 0 & -1 & -2 \\ -8 & 6 & 5 & 5 \end{bmatrix}$$

$$\mathbf{B}^{\mathrm{T}} = \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_z \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_z \end{bmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{bmatrix} 15 & 8 \\ -7 & 3 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 15 & -7 \\ 8 & 3 \end{bmatrix} \qquad \mathbf{D}^{\mathrm{T}} = \begin{bmatrix} a & g \\ b & h \\ c & i \\ d & j \\ f & k \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} a & b & c & d & f \\ g & h & i & j & k \end{bmatrix}$$

$$\mathbf{E}^{\mathrm{T}} = \begin{bmatrix} 0 & 1 & 3 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \qquad \mathbf{F}^{\mathrm{T}} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} x & y & z & w \end{bmatrix}$$

$$\mathbf{G}^{\mathrm{T}} = \begin{bmatrix} 10 & 20 & 30 & 1 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 10 \\ 20 \\ 30 \\ 1 \end{bmatrix} \qquad \mathbf{H}^{\mathrm{T}} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix}$$

3. $\mathbf{AB} = (4\times3)(3\times3) = 4\times3$ $\quad$ $\mathbf{AH} = (4\times3)(3\times1) = 4\times1$
$\mathbf{BB} = (3\times3)(3\times3) = 3\times3$ $\quad$ $\mathbf{BH} = (3\times3)(3\times1) = 3\times1$
$\mathbf{CC} = (2\times2)(2\times2) = 2\times2$ $\quad$ $\mathbf{DC} = (5\times2)(2\times2) = 5\times2$
$\mathbf{EB} = (1\times3)(3\times3) = 1\times3$ $\quad$ $\mathbf{EH} = (1\times3)(3\times1) = 1\times1$
$\mathbf{FE} = (4\times1)(1\times3) = 4\times3$ $\quad$ $\mathbf{FG} = (4\times1)(1\times4) = 4\times4$
$\mathbf{GA} = (1\times4)(4\times3) = 1\times3$ $\quad$ $\mathbf{GF} = (1\times4)(4\times1) = 1\times1$
$\mathbf{HE} = (3\times1)(1\times3) = 3\times3$ $\quad$ $\mathbf{HG} = (3\times1)(1\times4) = 3\times4$

4. (a) $$\begin{bmatrix}1 & -2\\5 & 0\end{bmatrix}\begin{bmatrix}-3 & 7\\4 & 1/3\end{bmatrix} = \begin{bmatrix}(1)(-3)+(-2)(4) & (1)(7)+(-2)(1/3)\\(5)(-3)+(0)(4) & (5)(7)+(0)(1/3)\end{bmatrix}$$
$$= \begin{bmatrix}-3+(-8) & 7+(-2/3)\\-15+0 & 35+0\end{bmatrix} = \begin{bmatrix}-11 & 19/3\\-15 & 35\end{bmatrix}$$

(b) Not possible; cannot multiply a 2×2 matrix by a 1×2 vector on the right.

(c) $$\begin{bmatrix}3 & -1 & 4\end{bmatrix}\begin{bmatrix}-2 & 0 & 3\\5 & 7 & -6\\1 & -4 & 2\end{bmatrix}$$
$$= \begin{bmatrix}(3)(-2)+(-1)(5)+(4)(1) & (3)(0)+(-1)(7)+(4)(-4) & (3)(3)+(-1)(-6)+(4)(2)\end{bmatrix}$$
$$= \begin{bmatrix}-6+(-5)+4 & 0+(-7)+(-16) & 9+6+8\end{bmatrix} = \begin{bmatrix}-7 & -23 & 23\end{bmatrix}$$

(d) $$\begin{bmatrix}x & y & z & w\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}x & y & z & w\end{bmatrix}$$

(e) Not possible; cannot multiply a 1×4 vector by a 2×1 vector.

(f) $$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}m_{11} & m_{12}\\m_{21} & m_{22}\end{bmatrix} = \begin{bmatrix}m_{11} & m_{12}\\m_{21} & m_{22}\end{bmatrix}$$

(g) $$\begin{bmatrix}3 & 3\end{bmatrix}\begin{bmatrix}6 & -7\\-4 & 5\end{bmatrix} = \begin{bmatrix}(3)(6)+(3)(-4) & (3)(-7)+(3)(5)\end{bmatrix}$$
$$= \begin{bmatrix}18+(-12) & -21+15\end{bmatrix} = \begin{bmatrix}6 & -6\end{bmatrix}$$

(h) Not possible; cannot multiply a 3×3 matrix by a 2×3 matrix on the right.

5. (a) $$\begin{bmatrix}5 & -1 & 2\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$
$$= \begin{bmatrix}(5)(1)+(-1)(0)+(2)(0) & (5)(0)+(-1)(1)+(2)(0) & (5)(0)+(-1)(0)+(2)(1)\end{bmatrix}$$
$$= \begin{bmatrix}5 & -1 & 2\end{bmatrix}$$
$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}5\\-1\\2\end{bmatrix} = \begin{bmatrix}(1)(5)+(0)(-1)+(0)(2)\\(0)(5)+(1)(-1)+(0)(2)\\(0)(5)+(0)(-1)+(1)(2)\end{bmatrix} = \begin{bmatrix}5\\-1\\2\end{bmatrix}$$

(b) $\begin{bmatrix} 5 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 & -3 \\ 1 & 7 & 1 \\ -2 & -1 & 4 \end{bmatrix}$

$= \begin{bmatrix} (5)(2)+(-1)(1)+(2)(-2) & (5)(5)+(-1)(7)+(2)(-1) & (5)(-3)+(-1)(1)+(2)(4) \end{bmatrix}$

$= \begin{bmatrix} 10+(-1)+(-4) & 25+(-7)+(-2) & -15+(-1)+8 \end{bmatrix} = \begin{bmatrix} 5 & 16 & -8 \end{bmatrix}$

$$\begin{bmatrix} 2 & 5 & -3 \\ 1 & 7 & 1 \\ -2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} (2)(5)+(5)(-1)+(-3)(2) \\ (1)(5)+(7)(-1)+(1)(2) \\ (-2)(5)+(-1)(-1)+(4)(2) \end{bmatrix} = \begin{bmatrix} 10+(-5)+(-6) \\ 5+(-7)+2 \\ -10+1+8 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

(c) $\begin{bmatrix} 5 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 7 & 2 \\ 7 & 0 & -3 \\ 2 & -3 & -1 \end{bmatrix}$

$= \begin{bmatrix} (5)(1)+(-1)(7)+(2)(2) & (5)(7)+(-1)(0)+(2)(-3) & (5)(2)+(-1)(-3)+(2)(-1) \end{bmatrix}$

$= \begin{bmatrix} 5+(-7)+4 & 35+0+(-6) & 10+3+(-2) \end{bmatrix} = \begin{bmatrix} 2 & 29 & 11 \end{bmatrix}$

$$\begin{bmatrix} 1 & 7 & 2 \\ 7 & 0 & -3 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} (1)(5)+(7)(-1)+(2)(2) \\ (7)(5)+(0)(-1)+(-3)(2) \\ (2)(5)+(-3)(-1)+(-1)(2) \end{bmatrix} = \begin{bmatrix} 5+(-7)+4 \\ 35+0+(-6) \\ 10+3+(-2) \end{bmatrix} = \begin{bmatrix} 2 \\ 29 \\ 11 \end{bmatrix}$$

(d) $\begin{bmatrix} 5 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & -4 & 3 \\ 4 & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} (5)(0)+(-1)(4)+(2)(-3) & (5)(-4)+(-1)(0)+(2)(1) & (5)(3)+(-1)(-1)+(2)(0) \end{bmatrix}$

$= \begin{bmatrix} 0+(-4)+(-6) & (-20)+0+2 & 15+1+0 \end{bmatrix} = \begin{bmatrix} -10 & -18 & 16 \end{bmatrix}$

$$\begin{bmatrix} 0 & -4 & 3 \\ 4 & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} (0)(5)+(-4)(-1)+(3)(2) \\ (4)(5)+(0)(-1)+(-1)(2) \\ (-3)(5)+(1)(-1)+(0)(2) \end{bmatrix} = \begin{bmatrix} 0+4+6 \\ 20+0+(-2) \\ -15+(-1)+0 \end{bmatrix} = \begin{bmatrix} 10 \\ 18 \\ -16 \end{bmatrix}$$

6. (a) $\left((\mathbf{A}^{\mathrm{T}})^{\mathrm{T}}\right)^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}$

(b) $(\mathbf{B}\mathbf{A}^{\mathrm{T}})^{\mathrm{T}}(\mathbf{C}\mathbf{D}^{\mathrm{T}}) = \left((\mathbf{A}^{\mathrm{T}})^{\mathrm{T}}(\mathbf{B})^{\mathrm{T}}\right)(\mathbf{C}\mathbf{D}^{\mathrm{T}}) = (\mathbf{A}\mathbf{B}^{\mathrm{T}})(\mathbf{C}\mathbf{D}^{\mathrm{T}}) = \mathbf{A}\mathbf{B}^{\mathrm{T}}\mathbf{C}\mathbf{D}^{\mathrm{T}}$

(c) $\left((\mathbf{D}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}})(\mathbf{A}\mathbf{B})^{\mathrm{T}}\right)^{\mathrm{T}} = \left(\left((\mathbf{A}\mathbf{B})^{\mathrm{T}}\right)^{\mathrm{T}}(\mathbf{D}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}})^{\mathrm{T}}\right) = (\mathbf{A}\mathbf{B})\left((\mathbf{C}^{\mathrm{T}})^{\mathrm{T}}(\mathbf{D}^{\mathrm{T}})^{\mathrm{T}}\right)$

$= (\mathbf{A}\mathbf{B})(\mathbf{C}\mathbf{D}) = \mathbf{A}\mathbf{B}\mathbf{C}\mathbf{D}$

(d) $\left((\mathbf{A}\mathbf{B})^{\mathrm{T}}(\mathbf{C}\mathbf{D}\mathbf{E})^{\mathrm{T}}\right)^{\mathrm{T}} = \left(\left((\mathbf{C}\mathbf{D}\mathbf{E})^{\mathrm{T}}\right)^{\mathrm{T}}\left((\mathbf{A}\mathbf{B})^{\mathrm{T}}\right)^{\mathrm{T}}\right) = (\mathbf{C}\mathbf{D}\mathbf{E})(\mathbf{A}\mathbf{B})$

$= \mathbf{C}\mathbf{D}\mathbf{E}\mathbf{A}\mathbf{B}$

7. For each of the matrices $\mathbf{M}$, interpret the rows of $\mathbf{M}$ as basis vectors after transformation.

(a) The basis vectors $[1, 0]$ and $[0, 1]$ are transformed to $[0, -1]$ and $[1, 0]$, respectively. Thus, $\mathbf{M}$ performs a 90° clockwise rotation.

(b) The basis vectors $[1, 0]$ and $[0, 1]$ are transformed to $[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$ and $[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$, respectively. Thus, $\mathbf{M}$ performs a 45° counterclockwise rotation.

(c) The basis vectors $[1, 0]$ and $[0, 1]$ are transformed to $[2, 0]$ and $[0, 2]$, respectively. Thus, $\mathbf{M}$ performs a *uniform* scale, scaling both the x and y dimensions by 2.

(d) The basis vectors $[1, 0]$ and $[0, 1]$ are transformed to $[4, 0]$ and $[0, 7]$, respectively. Thus, $\mathbf{M}$ performs a *nonuniform* scale, scaling the x dimension by 4 and the y dimension by 7.

(e) The basis vectors $[1, 0]$ and $[0, 1]$ are transformed to $[-1, 0]$ and $[0, 1]$, respectively. Thus, $\mathbf{M}$ performs a reflection across the y axis, negating x values and leaving y values untouched.

(f) The basis vectors $[1, 0]$ and $[0, 1]$ are transformed to $[0, -2]$ and $[2, 0]$, respectively. Thus, $\mathbf{M}$ is performing a combination of transformations: it is rotating clockwise by 90° and scaling both dimensions uniformly by 2. This can be confirmed by multiplying the appropriate matrices from parts (a) and (c), which perform these transformations individually:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}.$$

8. $\mathbf{M} = \begin{bmatrix} 0 & -b_z & b_y \\ b_z & 0 & -b_x \\ -b_y & b_x & 0 \end{bmatrix}$ This matrix is skew symmetric, as desired, since $\mathbf{M}^{\mathrm{T}} = -\mathbf{M}$.

9. (a) 3 (b) 1 (c) 4 (d) 2

10. The result vector element w_i is the product of the ith row of $\mathbf{M}$ multiplied by the column vector $\mathbf{v}$. To have $w_i = v_i - v_{i-1}$, the ith row of $\mathbf{M}$ needs to capture the ith element of $\mathbf{v}$, as well as the negative of the $(i-1)$th element, but exclude all others. This means that

$$m_{ij} = \begin{cases} 1 & \text{if } j = i, \\ -1 & \text{if } j = i - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

11. The result vector element w_i is the product of the ith row of $\mathbf{N}$ multiplied by the column vector $\mathbf{v}$. To have $w_i = \sum_{j=1}^{i} v_j$, the ith row of $\mathbf{N}$ needs to capture all elements of $\mathbf{v}$ up to and including the ith element, but exclude all others. This means that

$$n_{ij} = \begin{cases} 1 & \text{if } j \leq i, \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\mathbf{N} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

12. (a) Note that the structure of $\mathbf{M}$ causes the ith row of $\mathbf{MN}$ to be equivalent to the difference between the ith and $(i-1)$th rows of $\mathbf{N}$.

    (b) Note that the structure of $\mathbf{N}$ causes the ith row of $\mathbf{NM}$ to be equivalent to the sum of the first i rows of $\mathbf{M}$.

    (c) $$\mathbf{MN} = \mathbf{NM} = \mathbf{I}_{10\times 10} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$