(b) (1) $\mathbf{v}_{ab} = [\,-3,\ 0,\ 2\,]$. $\mathbf{v}_{bc} = [\,-3,\ 0,\ -4\,]$. $x_{bc}z_{ab} - x_{ab}z_{bc} = (-3)(2) - (-3)(-4) = -18 < 0$. Thus, the NPC is turning counterclockwise.

(2) $\mathbf{v}_{ab} = [\,7,\ 0,\ 5\,]$. $\mathbf{v}_{bc} = [\,-1,\ 0,\ 3\,]$. $x_{bc}z_{ab} - x_{ab}z_{bc} = (-1)(5) - (7)(3) = -26 < 0$. Thus, the NPC is turning counterclockwise.

(3) $\mathbf{v}_{ab} = [\,6,\ 0,\ -5\,]$. $\mathbf{v}_{bc} = [\,-12,\ 0,\ -5\,]$. $x_{bc}z_{ab} - x_{ab}z_{bc} = (-12)(-5) - (6)(-5) = 90 > 0$. Thus, the NPC is turning clockwise.

(4) $\mathbf{v}_{ab} = [\,3,\ 0,\ 1\,]$. $\mathbf{v}_{bc} = [\,3,\ 0,\ 2\,]$. $x_{bc}z_{ab} - x_{ab}z_{bc} = (3)(1) - (3)(2) = -3 < 0$. Thus, the NPC is turning counterclockwise.

23.
$$\begin{aligned}
\mathbf{p}' &= \mathbf{p} + (k-1)\,(\mathbf{p}\cdot\mathbf{n})\,\mathbf{n} \\
&= \begin{bmatrix}1\\0\\0\end{bmatrix} + (k-1)\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}n_x\\n_y\\n_z\end{bmatrix}\right)\begin{bmatrix}n_x\\n_y\\n_z\end{bmatrix} \\
&= \begin{bmatrix}1\\0\\0\end{bmatrix} + (k-1)\,(n_x)\begin{bmatrix}n_x\\n_y\\n_z\end{bmatrix} \\
&= \begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}(k-1)\,{n_x}^2\\(k-1)\,n_x n_y\\(k-1)\,n_x n_z\end{bmatrix} \\
&= \begin{bmatrix}1+(k-1)\,{n_x}^2\\(k-1)\,n_x n_y\\(k-1)\,n_x n_z\end{bmatrix}
\end{aligned}$$

24.
$$\begin{aligned}
\mathbf{p}' &= \cos\theta\,(\mathbf{p} - (\mathbf{p}\cdot\mathbf{n})\,\mathbf{n}) + \sin\theta\,(\mathbf{n}\times\mathbf{p}) + (\mathbf{p}\cdot\mathbf{n})\,\mathbf{n} \\
&= \cos\theta\left(\begin{bmatrix}1\\0\\0\end{bmatrix} - \left(\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}n_x\\n_y\\n_z\end{bmatrix}\right)\begin{bmatrix}n_x\\n_y\\n_z\end{bmatrix}\right) + \sin\theta\left(\begin{bmatrix}n_x\\n_y\\n_z\end{bmatrix}\times\begin{bmatrix}1\\0\\0\end{bmatrix}\right) + \left(\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}n_x\\n_y\\n_z\end{bmatrix}\right)\begin{bmatrix}n_x\\n_y\\n_z\end{bmatrix} \\
&= \cos\theta\left(\begin{bmatrix}1\\0\\0\end{bmatrix} - n_x\begin{bmatrix}n_x\\n_y\\n_z\end{bmatrix}\right) + \sin\theta\begin{bmatrix}0\\n_z\\-n_y\end{bmatrix} + n_x\begin{bmatrix}n_x\\n_y\\n_z\end{bmatrix} \\
&= \cos\theta\begin{bmatrix}1-{n_x}^2\\-n_x n_y\\-n_x n_z\end{bmatrix} + \sin\theta\begin{bmatrix}0\\n_z\\-n_y\end{bmatrix} + \begin{bmatrix}{n_x}^2\\n_x n_y\\n_x n_z\end{bmatrix} \\
&= \begin{bmatrix}\cos\theta - {n_x}^2\cos\theta\\-n_x n_y\cos\theta\\-n_x n_z\cos\theta\end{bmatrix} + \begin{bmatrix}0\\n_z\sin\theta\\-n_y\sin\theta\end{bmatrix} + \begin{bmatrix}{n_x}^2\\n_x n_y\\n_x n_z\end{bmatrix} \\
&= \begin{bmatrix}\cos\theta - {n_x}^2\cos\theta + {n_x}^2\\-n_x n_y\cos\theta + n_z\sin\theta + n_x n_y\\-n_x n_z\cos\theta - n_y\sin\theta + n_x n_z\end{bmatrix} \\
&= \begin{bmatrix}{n_x}^2\,(1-\cos\theta) + \cos\theta\\n_x n_y\,(1-\cos\theta) + n_z\sin\theta\\n_x n_z\,(1-\cos\theta) - n_y\sin\theta\end{bmatrix}
\end{aligned}$$

(6) $\mathbf{x}$ is not visible to the NPC.

$$\cos\theta = \frac{\begin{bmatrix}5\\-2\end{bmatrix}\cdot\left(\begin{bmatrix}-3\\0\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right)}{\left\|\begin{bmatrix}5\\-2\end{bmatrix}\right\|\left\|\begin{bmatrix}-3\\0\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right\|} = \frac{8}{(\sqrt{29})(\sqrt{16})} \approx 0.371 < 0.707$$

(7) $\mathbf{x}$ is not visible to the NPC.

$$\cos\theta = \frac{\begin{bmatrix}5\\-2\end{bmatrix}\cdot\left(\begin{bmatrix}-6\\-3.5\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right)}{\left\|\begin{bmatrix}5\\-2\end{bmatrix}\right\|\left\|\begin{bmatrix}-6\\-3.5\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right\|} = \frac{0}{(\sqrt{29})(\sqrt{65.25})} = 0 < 0.707$$

(c) The NPC can see a distance of only 7 units, so only those points that are both within the FOV and within this distance will be visible.

(1) $\mathbf{x}$ is visible to the NPC.

$$\left\|\begin{bmatrix}0\\0\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right\| = \left\|\begin{bmatrix}3\\-4\end{bmatrix}\right\| = \sqrt{25} = 5 < 7$$

(2) $\mathbf{x}$ is not visible to the NPC; it is outside the FOV.

(3) $\mathbf{x}$ is not visible to the NPC; it is outside the FOV.

(4) $\mathbf{x}$ is not visible to the NPC; it is outside the FOV.

(5) $\mathbf{x}$ is not visible to the NPC.

$$\left\|\begin{bmatrix}5\\5\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right\| = \left\|\begin{bmatrix}8\\1\end{bmatrix}\right\| = \sqrt{65} \approx 8.062 > 7$$

(6) $\mathbf{x}$ is not visible to the NPC; it is outside the FOV.

(7) $\mathbf{x}$ is not visible to the NPC; it is outside the FOV.

22. (a) Let $\mathbf{v}_{ab} = \mathbf{b} - \mathbf{a}$ and $\mathbf{v}_{bc} = \mathbf{c} - \mathbf{b}$. Since the three points lie in the xz-plane, the two vectors also lie in the xz-plane and we have

$$\mathbf{v}_{ab} = \begin{bmatrix}x_{ab}\\0\\z_{ab}\end{bmatrix}, \qquad \mathbf{v}_{bc} = \begin{bmatrix}x_{bc}\\0\\z_{bc}\end{bmatrix}.$$

Taking the cross product of the vectors in the order that the points are traversed gives.

$$\mathbf{v}_{ab} \times \mathbf{v}_{bc} = \begin{bmatrix}0\\x_{bc}z_{ab} - x_{ab}z_{bc}\\0\end{bmatrix}$$

The sign of $x_{bc}z_{ab} - x_{ab}z_{bc}$ can then be used to determine the NPC's turning direction. Because we are working in a left-handed coordinate system, if the value is negative, the NPC is turning counterclockwise; if it's positive he's turning clockwise. The special case of 0 signifies that the NPC is either walking forward in a straight line or walks forward and then back along the same line.

(7) $\mathbf{x}$ can be either in front of or behind the NPC, depending on how we've decided to handle this special case.

$$\begin{bmatrix}5\\-2\end{bmatrix}\cdot\left(\begin{bmatrix}-6\\-3.5\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right)=\begin{bmatrix}5\\-2\end{bmatrix}\cdot\begin{bmatrix}-3\\-7.5\end{bmatrix}=(5)(-3)+(-2)(-7.5)=0$$

21. (a) To determine whether the point $\mathbf{x}$ is visible to the NPC, compare $\cos\theta$ to $\cos(\phi/2)$. If $\cos\theta \geq \cos(\phi/2)$, then $\mathbf{x}$ is visible to the NPC.

The value of $\cos(\phi/2)$ can be obtained from the FOV angle. To get $\cos\theta$ use the dot product

$$\cos\theta=\frac{\mathbf{v}\cdot(\mathbf{x}-\mathbf{p})}{\|\mathbf{v}\|\|\mathbf{x}-\mathbf{p}\|}.$$

(b) The NPC's FOV is 90°, so the value we are interested in is $\cos(45^\circ)\approx 0.707$.

(1) $\mathbf{x}$ is visible to the NPC.

$$\cos\theta=\frac{\begin{bmatrix}5\\-2\end{bmatrix}\cdot\left(\begin{bmatrix}0\\0\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right)}{\left\|\begin{bmatrix}5\\-2\end{bmatrix}\right\|\left\|\begin{bmatrix}0\\0\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right\|}=\frac{23}{(\sqrt{29})(\sqrt{(25)})}\approx 0.854\geq 0.707$$

(2) $\mathbf{x}$ is not visible to the NPC.

$$\cos\theta=\frac{\begin{bmatrix}5\\-2\end{bmatrix}\cdot\left(\begin{bmatrix}1\\6\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right)}{\left\|\begin{bmatrix}5\\-2\end{bmatrix}\right\|\left\|\begin{bmatrix}1\\6\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right\|}=\frac{16}{(\sqrt{29})(\sqrt{20})}\approx 0.664<0.707$$

(3) $\mathbf{x}$ is not visible to the NPC.

$$\cos\theta=\frac{\begin{bmatrix}5\\-2\end{bmatrix}\cdot\left(\begin{bmatrix}-6\\0\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right)}{\left\|\begin{bmatrix}5\\-2\end{bmatrix}\right\|\left\|\begin{bmatrix}-6\\0\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right\|}=\frac{-7}{(\sqrt{29})(\sqrt{25})}$$
$$\approx -0.260<0.707$$

(4) $\mathbf{x}$ is not visible to the NPC.

$$\cos\theta=\frac{\begin{bmatrix}5\\-2\end{bmatrix}\cdot\left(\begin{bmatrix}-4\\7\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right)}{\left\|\begin{bmatrix}5\\-2\end{bmatrix}\right\|\left\|\begin{bmatrix}-4\\7\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right\|}=\frac{-11}{(\sqrt{29})(\sqrt{10})}\approx -0.646<0.707$$

(5) $\mathbf{x}$ is visible to the NPC.

$$\cos\theta=\frac{\begin{bmatrix}5\\-2\end{bmatrix}\cdot\left(\begin{bmatrix}5\\5\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right)}{\left\|\begin{bmatrix}5\\-2\end{bmatrix}\right\|\left\|\begin{bmatrix}5\\5\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\right\|}=\frac{38}{(\sqrt{29})(\sqrt{65})}\approx 0.875\geq 0.707$$

$$\mathbf{p}_{\text{BackUpperLeft}} = \begin{bmatrix} c_x - r_x \\ c_y + r_y \\ c_z - r_z \end{bmatrix}, \qquad \mathbf{p}_{\text{BackUpperRight}} = \begin{bmatrix} c_x + r_x \\ c_y + r_y \\ c_z - r_z \end{bmatrix},$$

$$\mathbf{p}_{\text{BackLowerLeft}} = \begin{bmatrix} c_x - r_x \\ c_y - r_y \\ c_z - r_z \end{bmatrix}, \qquad \mathbf{p}_{\text{BackLowerRight}} = \begin{bmatrix} c_x + r_x \\ c_y - r_y \\ c_z - r_z \end{bmatrix}.$$

20. (a) Use the sign of the dot product between $\mathbf{v}$ and $\mathbf{x} - \mathbf{p}$ to determine whether the point $\mathbf{x}$ is in front of or behind the NPC. This follows from the geometric interpretation of the dot product,

$$\mathbf{v} \cdot (\mathbf{x} - \mathbf{p}) = \|\mathbf{v}\| \|\mathbf{x} - \mathbf{p}\| \cos\theta,$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{x} - \mathbf{p}$.

Both $\|\mathbf{v}\|$ and $\|\mathbf{x} - \mathbf{p}\|$ are always positive, leaving the sign of the dot product entirely up to the value of $\cos\theta$. If $\cos\theta > 0$ then θ is less than 90° and $\mathbf{x}$ is *in front of* the NPC. Similarly, if $\cos\theta < 0$ then θ is greater than 90° and $\mathbf{x}$ is *behind* the NPC.

The special case of $\mathbf{v} \cdot (\mathbf{x} - \mathbf{p}) = 0$ means that $\mathbf{x}$ lies either directly to the left or right of the NPC. If this case does not need to be handled explicitly, it can arbitrarily be assigned to mean either in front of or behind.

(b) (1) $\mathbf{x}$ is in front of the NPC.

$$\begin{bmatrix} 5 \\ -2 \end{bmatrix} \cdot \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \end{bmatrix} = (5)(3) + (-2)(-4) = 23$$

(2) $\mathbf{x}$ is in front of the NPC.

$$\begin{bmatrix} 5 \\ -2 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 \\ 6 \end{bmatrix} - \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} = (5)(4) + (-2)(2) = 16$$

(3) $\mathbf{x}$ is behind the NPC.

$$\begin{aligned} \begin{bmatrix} 5 \\ -2 \end{bmatrix} \cdot \left(\begin{bmatrix} -6 \\ 0 \end{bmatrix} - \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ -4 \end{bmatrix} &= (5)(-3) + (-2)(-4) \\ &= -7 \end{aligned}$$

(4) $\mathbf{x}$ is behind the NPC.

$$\begin{bmatrix} 5 \\ -2 \end{bmatrix} \cdot \left(\begin{bmatrix} -4 \\ 7 \end{bmatrix} - \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix} = (5)(-1) + (-2)(3) = -11$$

(5) $\mathbf{x}$ is in front of the NPC.

$$\begin{bmatrix} 5 \\ -2 \end{bmatrix} \cdot \left(\begin{bmatrix} 5 \\ 5 \end{bmatrix} - \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 1 \end{bmatrix} = (5)(8) + (-2)(1) = 38$$

(6) $\mathbf{x}$ is in front of the NPC.

$$\begin{bmatrix} 5 \\ -2 \end{bmatrix} \cdot \left(\begin{bmatrix} -3 \\ 0 \end{bmatrix} - \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -4 \end{bmatrix} = (5)(0) + (-2)(-4) = 8$$

(b) (1) The unit circle for the L^1 norm is a square with sides of length $\sqrt{2}$ rotated by 45°.

(2) The unit circle for the L^2 norm is the well-known unit circle we all know and love.

(3) The unit circle for the infinity norm is a square with sides of length 2.

Note that all three unit circles include the vectors $[1,0]$, $[0,1]$, $[-1,0]$, $[0,-1]$.

16. The man buys a box or has a piece of luggage that is 2 feet long, 2 feet wide, and 2 feet tall. If the object is very thin, such as a sword, then he can put the object diagonally in the box or luggage. The longest such object he could carry on is $\sqrt{2^2+2^2+2^2} \approx 3.46$ feet.

17. Let $\mathbf{s} = \mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}+\mathbf{e}+\mathbf{f}$. From inspection of Figure 2.11 we see that

$$\mathbf{s} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}.$$

We confirm this numerically using the above equation and the values of the other vectors, also obtained from inspection of Figure 2.11:

$$\begin{aligned}
\mathbf{s} &= \mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}+\mathbf{e}+\mathbf{f} \\
&= \begin{bmatrix} -1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \end{bmatrix} + \begin{bmatrix} -6 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ -3 \end{bmatrix} \\
&= \begin{bmatrix} (-1)+1+3+(-1)+(-6)+(-1) \\ 3+3+(-2)+(-2)+4+(-3) \end{bmatrix} \\
&= \begin{bmatrix} -5 \\ 3 \end{bmatrix}
\end{aligned}$$

18. Left-handed.

19. (a) Let $\mathbf{c} = \begin{bmatrix} c_x \\ c_y \end{bmatrix}$ and $\mathbf{r} = \begin{bmatrix} r_x \\ r_y \end{bmatrix}$. Then

$$\mathbf{p}_{\text{UpperLeft}} = \begin{bmatrix} c_x - r_x \\ c_y + r_y \end{bmatrix}, \qquad \mathbf{p}_{\text{UpperRight}} = \begin{bmatrix} c_x + r_x \\ c_y + r_y \end{bmatrix},$$

$$\mathbf{p}_{\text{LowerLeft}} = \begin{bmatrix} c_x - r_x \\ c_y - r_y \end{bmatrix}, \qquad \mathbf{p}_{\text{LowerRight}} = \begin{bmatrix} c_x + r_x \\ c_y - r_y \end{bmatrix}.$$

(b) Let $\mathbf{c} = \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix}$ and $\mathbf{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$. Then

$$\mathbf{p}_{\text{FrontUpperLeft}} = \begin{bmatrix} c_x - r_x \\ c_y + r_y \\ c_z + r_z \end{bmatrix}, \qquad \mathbf{p}_{\text{FrontUpperRight}} = \begin{bmatrix} c_x + r_x \\ c_y + r_y \\ c_z + r_z \end{bmatrix},$$

$$\mathbf{p}_{\text{FrontLowerLeft}} = \begin{bmatrix} c_x - r_x \\ c_y - r_y \\ c_z + r_z \end{bmatrix}, \qquad \mathbf{p}_{\text{FrontLowerRight}} = \begin{bmatrix} c_x + r_x \\ c_y - r_y \\ c_z + r_z \end{bmatrix},$$

$$= \sqrt{\left(a_x^2+a_y^2+a_z^2\right)\left(b_x^2+b_y^2+b_z^2\right)\left(1-\frac{(a_xb_x+a_yb_y+a_zb_z)^2}{\left(a_x^2+a_y^2+a_z^2\right)\left(b_x^2+b_y^2+b_z^2\right)}\right)}$$
$$= \sqrt{\left(a_x^2+a_y^2+a_z^2\right)\left(b_x^2+b_y^2+b_z^2\right)-(a_xb_x+a_yb_y+a_zb_z)^2}$$
$$= \sqrt{a_y^2b_z^2-2a_ya_zb_yb_z+a_z^2b_y^2+a_z^2b_x^2-2a_xa_zb_xb_z+a_x^2b_z^2+a_x^2b_y^2-2a_xa_yb_xb_y+a_y^2b_x^2}.$$

Starting from both ends, we have met in the middle, proving that $\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|\sin\theta$.

15. (a) (1) $\left\|\begin{bmatrix}3 & 4\end{bmatrix}\right\|_1 = |3|+|4| = 7$

$\left\|\begin{bmatrix}3 & 4\end{bmatrix}\right\|_2 = \sqrt{|3|^2+|4|^2} = 5$

$\left\|\begin{bmatrix}3 & 4\end{bmatrix}\right\|_3 = \sqrt[3]{|3|^3+|4|^3} = \sqrt[3]{91} \approx 4.498$

$\left\|\begin{bmatrix}3 & 4\end{bmatrix}\right\|_\infty = \max\left(|3|,|4|\right) = 4$

(2) $\left\|\begin{bmatrix}5 & -12\end{bmatrix}\right\|_1 = |5|+|-12| = 17$

$\left\|\begin{bmatrix}5 & -12\end{bmatrix}\right\|_2 = \sqrt{|5|^2+|-12|^2} = 13$

$\left\|\begin{bmatrix}5 & -12\end{bmatrix}\right\|_3 = \sqrt[3]{|5|^3+|-12|^3} = \sqrt[3]{1853} \approx 12.283$

$\left\|\begin{bmatrix}5 & -12\end{bmatrix}\right\|_\infty = \max\left(|5|,|-12|\right) = 12$

(3) $\left\|\begin{bmatrix}-2 & 10 & -7\end{bmatrix}\right\|_1 = |-2|+|10|+|-7| = 19$

$\left\|\begin{bmatrix}-2 & 10 & -7\end{bmatrix}\right\|_2 = \sqrt{|-2|^2+|10|^2+|-7|^2} = \sqrt{153} \approx 12.369$

$\left\|\begin{bmatrix}-2 & 10 & -7\end{bmatrix}\right\|_3 = \sqrt[3]{|-2|^3+|10|^3+|-7|^3} = \sqrt[3]{1351} \approx 11.055$

$\left\|\begin{bmatrix}-2 & 10 & -7\end{bmatrix}\right\|_\infty = \max\left(|-2|,|10|,|-7|\right) = 10$

(4) $\left\|\begin{bmatrix}6 & 1 & -9\end{bmatrix}\right\|_1 = |6|+|1|+|-9| = 16$

$\left\|\begin{bmatrix}6 & 1 & -9\end{bmatrix}\right\|_2 = \sqrt{|6|^2+|1|^2+|-9|^2} = \sqrt{118} \approx 10.863$

$\left\|\begin{bmatrix}6 & 1 & -9\end{bmatrix}\right\|_3 = \sqrt[3]{|6|^3+|1|^3+|-9|^3} = \sqrt[3]{946} \approx 9.817$

$\left\|\begin{bmatrix}6 & 1 & -9\end{bmatrix}\right\|_\infty = \max\left(|6|,|1|,|-9|\right) = 9$

(5) $\left\|\begin{bmatrix}-2 & -2 & -2 & -2\end{bmatrix}\right\|_1 = |-2|+|-2|+|-2|+|-2| = 8$

$\left\|\begin{bmatrix}-2 & -2 & -2 & -2\end{bmatrix}\right\|_2 = \sqrt{|-2|^2+|-2|^2+|-2|^2+|-2|^2} = 4$

$\left\|\begin{bmatrix}-2 & -2 & -2 & -2\end{bmatrix}\right\|_3 = \sqrt[3]{|-2|^3+|-2|^3+|-2|^3+|-2|^3} = \sqrt[3]{32} \approx 3.175$

$\left\|\begin{bmatrix}-2 & -2 & -2 & -2\end{bmatrix}\right\|_\infty = \max\left(|-2|,|-2|,|-2|,|-2|\right) = 2$

13. (a) $\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} (-1)(1) - (0)(0) \\ (0)(0) - (0)(1) \\ (0)(0) - (-1)(0) \end{bmatrix} = \begin{bmatrix} -1 - 0 \\ 0 - 0 \\ 0 - 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} (0)(0) - (1)(-1) \\ (1)(0) - (0)(0) \\ (0)(-1) - (0)(0) \end{bmatrix} = \begin{bmatrix} 0 - (-1) \\ 0 - 0 \\ 0 - 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(b) $\begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} (4)(-1) - (1)(-2) \\ (1)(1) - (-2)(-1) \\ (-2)(-2) - (4)(1) \end{bmatrix} = \begin{bmatrix} -4 - (-2) \\ 1 - 2 \\ 4 - 4 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \times \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} (-2)(1) - (-1)(4) \\ (-1)(-2) - (1)(1) \\ (1)(4) - (-2)(-2) \end{bmatrix} = \begin{bmatrix} -2 - (-4) \\ 2 - 1 \\ 4 - 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

(c) $\begin{bmatrix} 3 \\ 10 \\ 7 \end{bmatrix} \times \begin{bmatrix} 8 \\ -7 \\ 4 \end{bmatrix} = \begin{bmatrix} (10)(4) - (7)(-7) \\ (7)(8) - (3)(4) \\ (3)(-7) - (10)(8) \end{bmatrix} = \begin{bmatrix} 40 - (-49) \\ 56 - 12 \\ -21 - 80 \end{bmatrix} = \begin{bmatrix} 89 \\ 44 \\ -101 \end{bmatrix}$

$$\begin{bmatrix} 8 \\ -7 \\ 4 \end{bmatrix} \times \begin{bmatrix} 3 \\ 10 \\ 7 \end{bmatrix} = \begin{bmatrix} (-7)(7) - (4)(10) \\ (4)(3) - (8)(7) \\ (8)(10) - (-7)(3) \end{bmatrix} = \begin{bmatrix} -49 - 40 \\ 12 - 56 \\ 80 - (-21) \end{bmatrix} = \begin{bmatrix} -89 \\ -44 \\ 101 \end{bmatrix}$$

14. Let $\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$. Then $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$ and $\mathbf{a} \times \mathbf{b} =$ $\begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$. From $\|\mathbf{a} \times \mathbf{b}\|$, we have:

$$\begin{aligned} \|\mathbf{a} \times \mathbf{b}\| &= \sqrt{(a_y b_z - a_z b_y)^2 + (a_z b_x - a_x b_z)^2 + (a_x b_y - a_y b_x)^2} \\ &= \sqrt{a_y^2 b_z^2 - 2a_y a_z b_y b_z + a_z^2 b_y^2 + a_z^2 b_x^2 - 2a_x a_z b_x b_z + a_x^2 b_z^2 + a_x^2 b_y^2 - 2a_x a_y b_x b_y + a_y^2 b_x^2}. \end{aligned}$$

If we now consider $\|\mathbf{a}\|\|\mathbf{b}\| \sin\theta$, we find that:

$$\begin{aligned} \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta &= \|\mathbf{a}\|\|\mathbf{b}\| \sqrt{1 - \cos^2\theta} \\ &= \sqrt{a_x^2 + a_y^2 + a_z^2}\ \sqrt{b_x^2 + b_y^2 + b_z^2}\ \sqrt{1 - \left(\frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2}\sqrt{b_x^2 + b_y^2 + b_z^2}} \right)^2} \end{aligned}$$

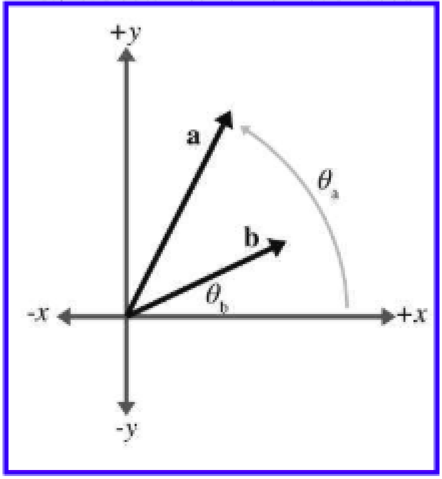
$$\begin{aligned}
\mathbf{v}_\perp &= \mathbf{v} - \mathbf{v}_\| \\
&= \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 7/2 \\ 7/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 - 7/2 \\ 3 - 7/2 \\ -1 - 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1 \end{bmatrix}
\end{aligned}$$

11. Define a triangle using the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a} - \mathbf{b}$, and let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$. Then the squared length of the edge $\mathbf{a} - \mathbf{b}$ is:

$$\begin{aligned}
\|\mathbf{a} - \mathbf{b}\|^2 &= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\
&= \mathbf{a} \cdot \mathbf{a} - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} \\
&= \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2\mathbf{a} \cdot \mathbf{b} \\
&= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\,\|\mathbf{b}\| \cos\theta
\end{aligned}$$

which is the law of cosines.

12. First, let's obtain some information about the vector components.



From the figure, we have

$$\begin{aligned}
a_x &= \|\mathbf{a}\| \cos\theta_a, & a_y &= \|\mathbf{a}\| \sin\theta_a, \\
b_x &= \|\mathbf{b}\| \cos\theta_b, & b_y &= \|\mathbf{b}\| \sin\theta_b.
\end{aligned}$$

Now we can proceed with the algebraic definition of the dot product and the cosine difference identity:

$$\begin{aligned}
\mathbf{a} \cdot \mathbf{b} &= a_x b_x + a_y b_y \\
&= \|\mathbf{a}\| \cos\theta_a \|\mathbf{b}\| \cos\theta_b + \|\mathbf{a}\| \sin\theta_a \|\mathbf{b}\| \sin\theta_b \\
&= \|\mathbf{a}\|\,\|\mathbf{b}\| \left(\cos\theta_a \cos\theta_b + \sin\theta_a \sin\theta_b\right) \\
&= \|\mathbf{a}\|\,\|\mathbf{b}\| \cos\left(\theta_b - \theta_a\right) \\
&= \|\mathbf{a}\|\,\|\mathbf{b}\| \cos\theta.
\end{aligned}$$

(d) $$\text{distance}\left(\begin{bmatrix}-2\\-4\\9\end{bmatrix},\begin{bmatrix}6\\-7\\9.5\end{bmatrix}\right)=\sqrt{(6-(-2))^2+(-7-(-4))^2+(9.5-9)^2}$$
$$=\sqrt{8^2+(-3)^2+(0.5)^2}=\sqrt{64+9+0.25}$$
$$=\sqrt{73.25}\approx 8.56$$

(e) $$\text{distance}\left(\begin{bmatrix}4\\-4\\-4\\4\end{bmatrix},\begin{bmatrix}-6\\6\\6\\-6\end{bmatrix}\right)=\sqrt{(-6-4)^2+(6-(-4))^2+(6-(-4))^2+(-6-4)^2}$$
$$=\sqrt{(-10)^2+(10)^2+(10)^2+(-10)^2}$$
$$=\sqrt{100+100+100+100}$$
$$=\sqrt{400}=20$$

9. (a) $$\begin{bmatrix}2\\6\end{bmatrix}\cdot\begin{bmatrix}-3\\8\end{bmatrix}=(2)(-3)+(6)(8)=-6+48=42$$

(b) $$-7\begin{bmatrix}1 & 2\end{bmatrix}\cdot\begin{bmatrix}11 & -4\end{bmatrix}=\begin{bmatrix}-7 & -14\end{bmatrix}\cdot\begin{bmatrix}11 & -4\end{bmatrix}$$
$$=(-7)(11)+(-14)(-4)$$
$$=-21$$

(c) $$10+\begin{bmatrix}-5\\1\\3\end{bmatrix}\cdot\begin{bmatrix}4\\-13\\9\end{bmatrix}=10+((-5)(4)+(1)(-13)+(3)(9))$$
$$=10+(-20+(-13)+27)$$
$$=10+(-6)=4$$

(d) $$3\begin{bmatrix}-2\\0\\4\end{bmatrix}\cdot\left(\begin{bmatrix}8\\-2\\3/2\end{bmatrix}+\begin{bmatrix}0\\9\\7\end{bmatrix}\right)=\begin{bmatrix}-6\\0\\12\end{bmatrix}\cdot\begin{bmatrix}8\\7\\17/2\end{bmatrix}$$
$$=(-6)(8)+(0)(7)+(12)(17/2)=54$$

10. $$\mathbf{v}_{\|}=\hat{\mathbf{n}}\frac{\mathbf{v}\cdot\hat{\mathbf{n}}}{\|\hat{\mathbf{n}}\|^2}=\hat{\mathbf{n}}\frac{\mathbf{v}\cdot\hat{\mathbf{n}}}{1}=\hat{\mathbf{n}}\,(\mathbf{v}\cdot\hat{\mathbf{n}})$$
$$=\begin{bmatrix}\sqrt{2}/2\\\sqrt{2}/2\\0\end{bmatrix}\left(\begin{bmatrix}4\\3\\-1\end{bmatrix}\cdot\begin{bmatrix}\sqrt{2}/2\\\sqrt{2}/2\\0\end{bmatrix}\right)=\begin{bmatrix}\sqrt{2}/2\\\sqrt{2}/2\\0\end{bmatrix}\left(2\sqrt{2}+\frac{3\sqrt{2}}{2}+0\right)$$
$$=\begin{bmatrix}\sqrt{2}/2\\\sqrt{2}/2\\0\end{bmatrix}\frac{7\sqrt{2}}{2}=\begin{bmatrix}7/2\\7/2\\0\end{bmatrix}$$

(d) $\begin{bmatrix}-12 & 3 & -4\end{bmatrix}_{\text{norm}} = \dfrac{\begin{bmatrix}-12 & 3 & -4\end{bmatrix}}{\left\|\begin{bmatrix}-12 & 3 & -4\end{bmatrix}\right\|} = \dfrac{\begin{bmatrix}-12 & 3 & -4\end{bmatrix}}{\sqrt{(-12)^2+3^2+(-4)^2}}$

$= \dfrac{\begin{bmatrix}-12 & 3 & -4\end{bmatrix}}{13} = \begin{bmatrix}\frac{-12}{13} & \frac{3}{13} & \frac{-4}{13}\end{bmatrix}$

(e) $\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}_{\text{norm}} = \dfrac{\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}}{\left\|\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}\right\|} = \dfrac{\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}}{\sqrt{1^2+1^2+1^2+1^2}}$

$= \dfrac{\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}}{2} = \begin{bmatrix}0.5 & 0.5 & 0.5 & 0.5\end{bmatrix}$

7. (a) $\begin{bmatrix}7 & -2 & -3\end{bmatrix} + \begin{bmatrix}6 & 6 & -4\end{bmatrix} = \begin{bmatrix}7+6 & -2+6 & -3+(-4)\end{bmatrix} = \begin{bmatrix}13 & 4 & -7\end{bmatrix}$

(b) $\begin{bmatrix}2 & 9 & -1\end{bmatrix} + \begin{bmatrix}-2 & -9 & 1\end{bmatrix} = \begin{bmatrix}2+(-2) & 9+(-9) & -1+1\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\end{bmatrix}$

(c) $\begin{bmatrix}3\\10\\7\end{bmatrix} - \begin{bmatrix}8\\-7\\4\end{bmatrix} = \begin{bmatrix}3-8\\10-(-7)\\7-4\end{bmatrix} = \begin{bmatrix}-5\\17\\3\end{bmatrix}$

(d) $\begin{bmatrix}4\\5\\-11\end{bmatrix} - \begin{bmatrix}-4\\-5\\11\end{bmatrix} = \begin{bmatrix}4-(-4)\\5-(-5)\\-11-11\end{bmatrix} = \begin{bmatrix}8\\10\\-22\end{bmatrix}$

(e) $3\begin{bmatrix}a\\b\\c\end{bmatrix} - 4\begin{bmatrix}2\\10\\-6\end{bmatrix} = \begin{bmatrix}3a\\3b\\3c\end{bmatrix} - \begin{bmatrix}8\\40\\-24\end{bmatrix} = \begin{bmatrix}3a-8\\3b-40\\3c+24\end{bmatrix}$

8. (a) $\text{distance}\left(\begin{bmatrix}10\\6\end{bmatrix}, \begin{bmatrix}-14\\30\end{bmatrix}\right) = \sqrt{(10-(-14))^2+(6-30)^2}$

$= \sqrt{24^2+(-24)^2} = \sqrt{576+576}$

$= \sqrt{1152} \approx 33.94$

(b) $\text{distance}\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}-12\\5\end{bmatrix}\right) = \sqrt{(0-(-12))^2+(0-5)^2}$

$= \sqrt{12^2+(-5)^2} = \sqrt{144+25}$

$= \sqrt{169} = 13$

(c) $\text{distance}\left(\begin{bmatrix}3\\10\\7\end{bmatrix}, \begin{bmatrix}8\\-7\\4\end{bmatrix}\right) = \sqrt{(3-8)^2+(10-(-7))^2+(7-4)^2}$

$= \sqrt{(-5)^2+17^2+3^2} = \sqrt{25+289+9}$

$= \sqrt{323} \approx 17.97$

3. $\mathbf{a} = [0, 2]$ $\quad \mathbf{b} = [0, -2]$ $\quad \mathbf{c} = [0.5, 2]$
   $\mathbf{d} = [0.5, 2]$ $\quad \mathbf{e} = [0.5, -3]$ $\quad \mathbf{f} = [-2, 0]$
   $\mathbf{g} = [-2, 1]$ $\quad \mathbf{h} = [2.5, 2]$ $\quad \mathbf{i} = [6, 1]$

4. (a) *The size of a vector in a diagram doesn't matter; we just need to draw it in the right place.* **False.** This is reversed; for vectors, size matters (meaning the length of the vector), position doesn't.

   (b) *The displacement expressed by a vector can be visualized as a sequence of axially aligned displacements.* **True.**

   (c) *These axially aligned displacements from the previous question must occur in order.* **False.** We can apply them in any order and get the same end result.

   (d) *The vector $[x, y]$ gives the displacement from the point (x, y) to the origin.* **False.** This is reversed; the vector $[x, y]$ gives the displacement from the origin to the point (x, y).

5. (a) $-\begin{bmatrix}3 & 7\end{bmatrix} = \begin{bmatrix}-3 & -7\end{bmatrix}$

   (b) $\left\|\begin{bmatrix}-12 & 5\end{bmatrix}\right\| = \sqrt{(-12)^2 + 5^2} = \sqrt{169} = 13$

   (c) $$\begin{aligned}\left\|\begin{bmatrix}8 & -3 & 1/2\end{bmatrix}\right\| &= \sqrt{8^2 + (-3)^2 + (1/2)^2} = \sqrt{64 + 9 + (1/4)} \\ &= \sqrt{293/4} \approx 8.56\end{aligned}$$

   (d) $3\begin{bmatrix}4 & -7 & 0\end{bmatrix} = \begin{bmatrix}(3)(4) & (3)(-7) & (3)(0)\end{bmatrix} = \begin{bmatrix}12 & -21 & 0\end{bmatrix}$

   (e) $\begin{bmatrix}4 & 5\end{bmatrix}/2 = \begin{bmatrix}2 & 5/2\end{bmatrix}$

6. (a) $$\begin{aligned}\begin{bmatrix}12 & 5\end{bmatrix}_{\text{norm}} &= \frac{\begin{bmatrix}12 & 5\end{bmatrix}}{\left\|\begin{bmatrix}12 & 5\end{bmatrix}\right\|} = \frac{\begin{bmatrix}12 & 5\end{bmatrix}}{13} = \begin{bmatrix}\frac{12}{13} & \frac{5}{13}\end{bmatrix} \\ &\approx \begin{bmatrix}0.923 & 0.385\end{bmatrix}\end{aligned}$$

   (b) $$\begin{aligned}\begin{bmatrix}0 & 743.632\end{bmatrix}_{\text{norm}} &= \frac{\begin{bmatrix}0 & 743.632\end{bmatrix}}{\left\|\begin{bmatrix}0 & 743.632\end{bmatrix}\right\|} = \frac{\begin{bmatrix}0 & 743.632\end{bmatrix}}{\sqrt{0^2 + 743.632^2}} \\ &= \frac{\begin{bmatrix}0 & 743.632\end{bmatrix}}{743.632} = \begin{bmatrix}0 & 1\end{bmatrix}\end{aligned}$$

   (c) $$\begin{aligned}\begin{bmatrix}8 & -3 & 1/2\end{bmatrix}_{\text{norm}} &= \frac{\begin{bmatrix}8 & -3 & 1/2\end{bmatrix}}{\left\|\begin{bmatrix}8 & -3 & 1/2\end{bmatrix}\right\|} \approx \frac{\begin{bmatrix}8 & -3 & 1/2\end{bmatrix}}{8.56} \\ &\approx \begin{bmatrix}0.935 & -0.350 & 0.058\end{bmatrix}\end{aligned}$$

B.2 Chapter 2

(Page 71.)

1. (a) **a** is a 2D row vector. **b** is a 3D column vector. **c** is a 4D column vector.

   (b) $b_y + c_w + a_x + b_z = 0 + 6 + (-3) + 5 = 8$

2. (a) *"How much do you weigh?"* Your weight is a scalar quantity. But the force of gravity, which pulls you downwards, is a vector, and so if you said that weight was a vector for that reason, you are also correct. ("My weight is 150 lbs of force in the *downward* direction.")

   (b) *"Do you have any idea how fast you were going?"* The officer is probably referring to the *speed* of your vehicle, which is a scalar quantity.

   (c) *"It's two blocks north of here."* Vector quantity.

   (d) *"We're cruising from Los Angeles to New York at 600 mph, at an altitude of 33,000 ft."* The speed "600 mph" is a scalar quantity. Since New York is east of Los Angeles, you could reasonably infer an eastward direction, so "600 mph eastward" is a velocity, which is a vector quantity. Likewise, "33,000 ft" is a scalar quantity, although if you're a stickler, you might say that a direction of "up" is implied, in which case "33,000 ft up" is a vector quantity.