4.4 Exercises

(Answers on page 759.)

Use the following matrices for questions 1–3:

$$\mathbf{A} = \begin{bmatrix} 13 & 4 & -8 \\ 12 & 0 & 6 \\ -3 & -1 & 5 \\ 10 & -2 & 5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_z \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 15 & 8 \\ -7 & 3 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} a & g \\ b & h \\ c & i \\ d & j \\ f & k \end{bmatrix} \qquad \mathbf{E} = \begin{bmatrix} 0 & 1 & 3 \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} 10 & 20 & 30 & 1 \end{bmatrix} \qquad \mathbf{H} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

1. For each matrix, give the dimensions of the matrix and identify whether it is square and/or diagonal.

2. Transpose each matrix.

3. Find all the possible pairs of matrices that can be legally multiplied, and give the dimensions of the resulting product. Include "pairs" in which a matrix is multiplied by itself. (Hint: there are 14 pairs.)

4. Compute the following matrix products. If the product is not possible, just say so.

   (a) $\begin{bmatrix} 1 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 7 \\ 4 & 1/3 \end{bmatrix}$

   (b) $\begin{bmatrix} 6 & -7 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} 3 & 3 \end{bmatrix}$

   (c) $\begin{bmatrix} 3 & -1 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 3 \\ 5 & 7 & -6 \\ 1 & -4 & 2 \end{bmatrix}$

   (d) $\begin{bmatrix} x & y & z & w \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

   (e) $\begin{bmatrix} 7 & -2 & 7 & 3 \end{bmatrix} \begin{bmatrix} -5 \\ 1 \end{bmatrix}$

   (f) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$

   (g) $\begin{bmatrix} 3 & 3 \end{bmatrix} \begin{bmatrix} 6 & -7 \\ -4 & 5 \end{bmatrix}$

   (h) $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

5. For each of the following matrices, multiply on the left by the row vector $[5, -1, 2]$. Then consider whether multiplication on the right by the column vector $[5, -1, 2]^{\mathrm{T}}$ will give the same or a different result. Finally, perform this multiplication to confirm or correct your expectation.

   (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   (b) $\begin{bmatrix} 2 & 5 & -3 \\ 1 & 7 & 1 \\ -2 & -1 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 7 & 2 \\ 7 & 0 & -3 \\ 2 & -3 & -1 \end{bmatrix}$

This is an example of a *symmetric* matrix. A square matrix is symmetric if $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$.

(d) $\begin{bmatrix} 0 & -4 & 3 \\ 4 & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix}$

This is an example of a *skew symmetric* or *antisymmetric* matrix. A square matrix is skew symmetric if $\mathbf{A}^{\mathrm{T}} = -\mathbf{A}$. This implies that the diagonal elements of a skew symmetric matrix must be 0.

6. Manipulate the following matrix expressions to remove the parentheses.

   (a) $\left(\left(\mathbf{A}^{\mathrm{T}}\right)^{\mathrm{T}}\right)^{\mathrm{T}}$

   (b) $\left(\mathbf{B}\mathbf{A}^{\mathrm{T}}\right)^{\mathrm{T}}\left(\mathbf{C}\mathbf{D}^{\mathrm{T}}\right)$

   (c) $\left(\left(\mathbf{D}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\right)\left(\mathbf{A}\mathbf{B}\right)^{\mathrm{T}}\right)^{\mathrm{T}}$

   (d) $\left(\left(\mathbf{A}\mathbf{B}\right)^{\mathrm{T}}\left(\mathbf{C}\mathbf{D}\mathbf{E}\right)^{\mathrm{T}}\right)^{\mathrm{T}}$

7. Describe the transformation $\mathbf{a}\mathbf{M} = \mathbf{b}$ represented by each of the following matrices.

   (a) $\mathbf{M} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

   (b) $\mathbf{M} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

   (c) $\mathbf{M} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

   (d) $\mathbf{M} = \begin{bmatrix} 4 & 0 \\ 0 & 7 \end{bmatrix}$

   (e) $\mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

   (f) $\mathbf{M} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$

8. For 3D row vectors $\mathbf{a}$ and $\mathbf{b}$, construct a 3×3 matrix $\mathbf{M}$ such that $\mathbf{a} \times \mathbf{b} = \mathbf{a}\mathbf{M}$. That is, show that the cross product of $\mathbf{a}$ and $\mathbf{b}$ can be represented as the matrix product $\mathbf{a}\mathbf{M}$, for some matrix $\mathbf{M}$. (Hint: the matrix will be skew-symmetric.)

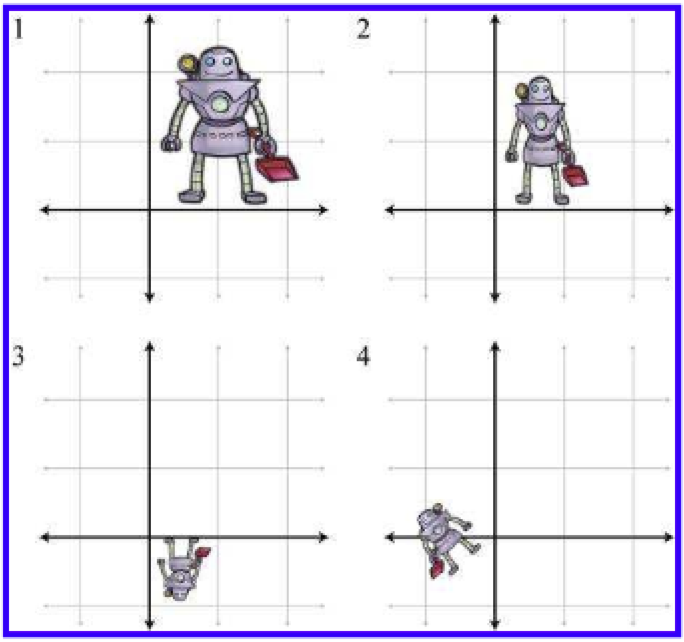
9. Match each of the following figures (1–4) with their corresponding transformations.

   (a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

   (b) $\begin{bmatrix} 2.5 & 0 \\ 0 & 2.5 \end{bmatrix}$

   (c) $\begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

   (d) $\begin{bmatrix} 1.5 & 0 \\ 0 & 2.0 \end{bmatrix}$



10. Given the 10×1 column vector $\mathbf{v}$, create a matrix $\mathbf{M}$ that, when multiplied by $\mathbf{v}$, produces a 10×1 column vector $\mathbf{w}$ such that

$$w_i = \begin{cases} v_1 & \text{if } i = 1, \\ v_i - v_{i-1} & \text{if } i > 1. \end{cases}$$

Matrices of this form arise when some continuous function is discretized. Multiplication by this *first difference* matrix is the discrete equivalent of continuous differentiation. (We'll learn about differentiation in Chapter 11 if you haven't already had calculus.)

11. Given the 10×1 column vector $\mathbf{v}$, create a matrix $\mathbf{N}$ that, when multiplied by $\mathbf{v}$, produces a 10×1 column vector $\mathbf{w}$ such that

$$w_i = \sum_{j=1}^{i} v_j.$$

In other words, each element becomes the sum of that element and all previous elements.

This matrix performs the discrete equivalent of *integration*, which as you might already know (but you certainly will know after reading Chapter 11) is the inverse operation of differentiation.

12. Consider $\mathbf{M}$ and $\mathbf{N}$, the matrices from Exercises 10 and 11.

    (a) Discuss your expectations of the product $\mathbf{MN}$.

    (b) Discuss your expectations of the product $\mathbf{NM}$.

    (c) Calculate both $\mathbf{MN}$ and $\mathbf{NM}$. Were your expectations correct?