2.14 Exercises

(Answers on page 746.)

1. Let

$$\mathbf{a} = \begin{bmatrix} -3 & 8 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 16 \\ -1 \\ 4 \\ 6 \end{bmatrix}.$$

   (a) Identify **a**, **b**, and **c**, as row or column vectors, and give the dimension of each vector.

   (b) Compute $b_y + c_w + a_x + b_z$.

2. Identify the quantities in each of the following sentences as scalar or vector. For vector quantities, give the magnitude and direction. (Note: some directions may be implicit.)

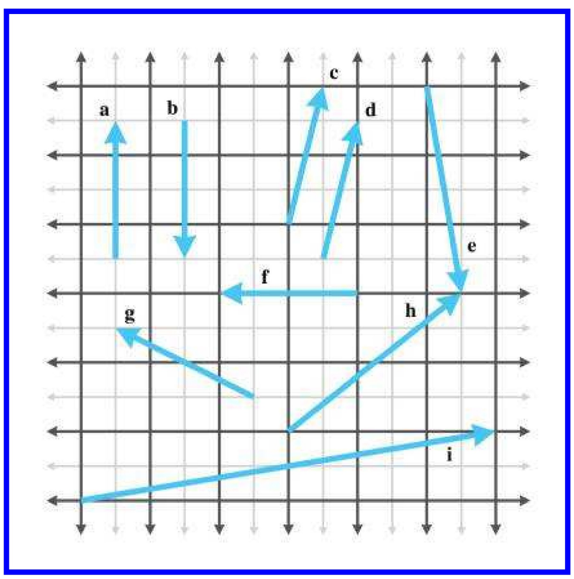
   (a) How much do you weigh?

   (b) Do you have any idea how fast you were going?

   (c) It's two blocks north of here.

   (d) We're cruising from Los Angeles to New York at 600 mph, at an altitude of 33,000 ft.

3. Give the values of the following vectors. The darker grid lines represent one unit.



4. Identify the following statements as true or false. If the statement is false, explain why.
   (a) The size of a vector in a diagram doesn't matter; we just need to draw it in the right place.
   (b) The displacement expressed by a vector can be visualized as a sequence of axially aligned displacements.
   (c) These axially aligned displacements from the previous question must occur in order.
   (d) The vector $[x, y]$ gives the displacement from the point (x, y) to the origin.
5. Evaluate the following vector expressions:
   (a) $-\begin{bmatrix}3 & 7\end{bmatrix}$
   (b) $\left\|\begin{bmatrix}-12 & 5\end{bmatrix}\right\|$
   (c) $\left\|\begin{bmatrix}8 & -3 & 1/2\end{bmatrix}\right\|$
   (d) $3\begin{bmatrix}4 & -7 & 0\end{bmatrix}$
   (e) $\begin{bmatrix}4 & 5\end{bmatrix}/2$
6. Normalize the following vectors:
   (a) $\begin{bmatrix}12 & 5\end{bmatrix}$
   (b) $\begin{bmatrix}0 & 743.632\end{bmatrix}$
   (c) $\begin{bmatrix}8 & -3 & 1/2\end{bmatrix}$
   (d) $\begin{bmatrix}-12 & 3 & -4\end{bmatrix}$
   (e) $\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}$
7. Evaluate the following vector expressions:
   (a) $\begin{bmatrix}7 & -2 & -3\end{bmatrix} + \begin{bmatrix}6 & 6 & -4\end{bmatrix}$
   (b) $\begin{bmatrix}2 & 9 & -1\end{bmatrix} + \begin{bmatrix}-2 & -9 & 1\end{bmatrix}$
   (c) $\begin{bmatrix}3 \\ 10 \\ 7\end{bmatrix} - \begin{bmatrix}8 \\ -7 \\ 4\end{bmatrix}$
   (d) $\begin{bmatrix}4 \\ 5 \\ -11\end{bmatrix} - \begin{bmatrix}-4 \\ -5 \\ 11\end{bmatrix}$
   (e) $3\begin{bmatrix}a \\ b \\ c\end{bmatrix} - 4\begin{bmatrix}2 \\ 10 \\ -6\end{bmatrix}$

8. Compute the distance between the following pairs of points:

   (a) $\begin{bmatrix}10\\6\end{bmatrix}, \begin{bmatrix}-14\\30\end{bmatrix}$

   (b) $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}-12\\5\end{bmatrix}$

   (c) $\begin{bmatrix}3\\10\\7\end{bmatrix}, \begin{bmatrix}8\\-7\\4\end{bmatrix}$

   (d) $\begin{bmatrix}-2\\-4\\9\end{bmatrix}, \begin{bmatrix}6\\-7\\9.5\end{bmatrix}$

   (e) $\begin{bmatrix}4\\-4\\-4\\4\end{bmatrix}, \begin{bmatrix}-6\\6\\6\\-6\end{bmatrix}$

9. Evaluate the following vector expressions:

   (a) $\begin{bmatrix}2\\6\end{bmatrix} \cdot \begin{bmatrix}-3\\8\end{bmatrix}$

   (b) $-7\begin{bmatrix}1 & 2\end{bmatrix} \cdot \begin{bmatrix}11 & -4\end{bmatrix}$

   (c) $10 + \begin{bmatrix}-5\\1\\3\end{bmatrix} \cdot \begin{bmatrix}4\\-13\\9\end{bmatrix}$

   (d) $3\begin{bmatrix}-2\\0\\4\end{bmatrix} \cdot \left(\begin{bmatrix}8\\-2\\3/2\end{bmatrix} + \begin{bmatrix}0\\9\\7\end{bmatrix}\right)$

10. Given the two vectors

$$\mathbf{v} = \begin{bmatrix}4\\3\\-1\end{bmatrix}, \quad \hat{\mathbf{n}} = \begin{bmatrix}\sqrt{2}/2\\\sqrt{2}/2\\0\end{bmatrix},$$

    separate $\mathbf{v}$ into components that are perpendicular and parallel to $\hat{\mathbf{n}}$. (As the notation implies, $\hat{\mathbf{n}}$ is a unit vector.)

11. Use the geometric definition of the dot product

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$$

    to prove the law of cosines.

12. Use trigonometric identities and the algebraic definition of the dot product in 2D

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y$$

    to prove the geometric interpretation of the dot product in 2D. (Hint: draw a diagram of the vectors and all angles involved.)

13. Calculate $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$ for the following vectors:
    (a) $\mathbf{a} = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$
    (b) $\mathbf{a} = \begin{bmatrix} -2 & 4 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 & -2 & -1 \end{bmatrix}$
    (c) $\mathbf{a} = \begin{bmatrix} 3 & 10 & 7 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 8 & -7 & 4 \end{bmatrix}$
14. Prove the equation for the magnitude of the cross product
$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin\theta.$$
(Hint: make use of the geometric interpretation of the dot product and try to show how the left and right sides of the equation are equivalent, rather than trying to derive one side from the other.)
15. Section 2.8 introduced the norm of a vector, namely, a scalar value associated with a given vector. However, the definition of the norm given in that section is not the only definition of a norm for a vector. In general, the *p-norm* of an n-dimensional vector is defined as
$$\|\mathbf{x}\|_p \equiv \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$
Some of the more common p-norms include:
    - The L^1 norm, a.k.a. Taxicab norm ($p = 1$):
    $$\|\mathbf{x}\|_1 \equiv \sum_{i=1}^{n} |x_i|.$$
    - The L^2 norm, a.k.a. Euclidean norm ($p = 2$). This is the most common and familiar norm, since it measures geometric length:
    $$\|\mathbf{x}\|_2 \equiv \sqrt{\sum_{i=1}^{n} x_i^2}.$$
    - The infinity norm, a.k.a. Chebyshev norm ($p = \infty$):
    $$\|\mathbf{x}\|_\infty \equiv \max\left(|x_1|, \ldots, |x_n|\right).$$

Each of these norms can be thought of as a way to assigning a length or size to a vector. The Euclidean norm was discussed in Section 2.8. The Taxicab norm gets its name from how a taxicab would measure distance driving the streets of a city laid out in a grid (e.g., Cartesia from Section 1.2.1). For example, a taxicab that drives 1 block east and 1 block north drives a total distance of 2 blocks, whereas a bird flying "as the crow flies" can fly in a straight line from start to finish and travel only $\sqrt{2}$ blocks (Euclidean norm). The Chebyshev norm is simply the absolute value of the vector component with the largest absolute value. An example of how this norm can be used is to consider the number of moves required to move a king in a game of chess from one square to another. The immediately surrounding squares require 1 move, the squares surrounding those require 2 moves, and so on.

(a) For each of the following find $\|\mathbf{x}\|_1$, $\|\mathbf{x}\|_2$, $\|\mathbf{x}\|_3$, and $\|\mathbf{x}\|_\infty$:

(1) $\begin{bmatrix} 3 & 4 \end{bmatrix}$

(2) $\begin{bmatrix} 5 & -12 \end{bmatrix}$

(3) $\begin{bmatrix} -2 & 10 & -7 \end{bmatrix}$

(4) $\begin{bmatrix} 6 & 1 & -9 \end{bmatrix}$

(5) $\begin{bmatrix} -2 & -2 & -2 & -2 \end{bmatrix}$

*(b) Draw the unit circle (i.e., the set of all vectors with $\|\mathbf{x}\|_p = 1$) centered at the origin for the L^1 norm, L^2 norm, and infinity norm.

16. A man is boarding a plane. The airline has a rule that no carry-on item may be more than two feet long, two feet wide, or two feet tall. He has a very valuable sword that is three feet long, yet he is able to carry the sword on board with him.[9] How is he able to do this? What is the longest possible item that he could carry on?

17. Verify Figure 2.11 numerically.

18. Is the coordinate system used in Figure 2.27 a left-handed or right-handed coordinate system?

19. One common way of defining a bounding box for a 2D object is to specify a center point $\mathbf{c}$ and a *radius vector* $\mathbf{r}$, where each component of $\mathbf{r}$ is half the length of the side of the bounding box along the corresponding axis.

(a) Describe the four corners $\mathbf{p}_{\text{UpperLeft}}$, $\mathbf{p}_{\text{UpperRight}}$, $\mathbf{p}_{\text{LowerLeft}}$, and $\mathbf{p}_{\text{LowerRight}}$.

(b) Describe the eight corners of a bounding cube, extending this idea into 3D.

[9]Please ignore the fact that nowadays this could never happen for security reasons. You can think of this exercise as taking place in a Quentin Tarantino movie.

20. A nonplayer character (NPC) is standing at location $\mathbf{p}$ with a forward direction of $\mathbf{v}$.

    (a) How can the dot product be used to determine whether the point $\mathbf{x}$ is in front of or behind the NPC?

    (b) Let $\mathbf{p} = \begin{bmatrix}-3 & 4\end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix}5 & -2\end{bmatrix}$. For each of the following points $\mathbf{x}$ determine whether $\mathbf{x}$ is in front of or behind the NPC:

    (1) $\mathbf{x} = \begin{bmatrix}0 & 0\end{bmatrix}$
    (2) $\mathbf{x} = \begin{bmatrix}1 & 6\end{bmatrix}$
    (3) $\mathbf{x} = \begin{bmatrix}-6 & 0\end{bmatrix}$
    (4) $\mathbf{x} = \begin{bmatrix}-4 & 7\end{bmatrix}$
    (5) $\mathbf{x} = \begin{bmatrix}5 & 5\end{bmatrix}$
    (6) $\mathbf{x} = \begin{bmatrix}-3 & 0\end{bmatrix}$
    (7) $\mathbf{x} = \begin{bmatrix}-6 & -3.5\end{bmatrix}$

21. Extending the concept from Exercise 20, consider the case where the NPC has a limited field of view (FOV). If the total FOV angle is ϕ, then the NPC can see to the left or right of its forward direction by a maximum angle of $\phi/2$.

    (a) How can the dot product be used to determine whether the point $\mathbf{x}$ is visible to the NPC?

    (b) For each of the points $\mathbf{x}$ in Exercise 20 determine whether $\mathbf{x}$ is visible to the NPC if its FOV is $90°$.

    (c) Suppose that the NPC's viewing distance is also limited to a maximum distance of 7 units. Which points are visible to the NPC then?

22. Consider three points labeled $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in the xz plane of our left-handed coordinate system, which represent waypoints on an NPC's path.

    (a) How can the cross product be used to determine whether, when moving from $\mathbf{a}$ to $\mathbf{b}$ to $\mathbf{c}$, the NPC makes a clockwise or counterclockwise turn at $\mathbf{b}$, when viewing the path from above?

    (b) For each of the following sets of three points, determine whether the NPC is turning clockwise or counterclockwise when moving from $\mathbf{a}$ to $\mathbf{b}$ to $\mathbf{c}$:

    (1) $\mathbf{a} = \begin{bmatrix}2 & 0 & 3\end{bmatrix}$, $\mathbf{b} = \begin{bmatrix}-1 & 0 & 5\end{bmatrix}$, $\mathbf{c} = \begin{bmatrix}-4 & 0 & 1\end{bmatrix}$
    (2) $\mathbf{a} = \begin{bmatrix}-3 & 0 & -5\end{bmatrix}$, $\mathbf{b} = \begin{bmatrix}4 & 0 & 0\end{bmatrix}$, $\mathbf{c} = \begin{bmatrix}3 & 0 & 3\end{bmatrix}$
    (3) $\mathbf{a} = \begin{bmatrix}1 & 0 & 4\end{bmatrix}$, $\mathbf{b} = \begin{bmatrix}7 & 0 & -1\end{bmatrix}$, $\mathbf{c} = \begin{bmatrix}-5 & 0 & -6\end{bmatrix}$
    (4) $\mathbf{a} = \begin{bmatrix}-2 & 0 & 1\end{bmatrix}$, $\mathbf{b} = \begin{bmatrix}1 & 0 & 2\end{bmatrix}$, $\mathbf{c} = \begin{bmatrix}4 & 0 & 4\end{bmatrix}$

23. In the derivation of a matrix to scale along an arbitrary axis, we reach a step where we have the vector expression

$$\mathbf{p}' = \mathbf{p} + (k-1)\,(\mathbf{p}\cdot\mathbf{n})\,\mathbf{n},$$

where $\mathbf{n}$ is an arbitrary vector $[n_x, n_y, n_z]$ and k is an arbitrary scalar, but $\mathbf{p}$ is one of the cardinal axes. Plug in the value $\mathbf{p} = [1,0,0]$ and simplify the resulting expression for $\mathbf{p}'$. The answer is not a vector expression, but a single vector, where the scalar expressions for each coordinate have been simplified.

24. A similar problem arises with the derivation of a matrix to rotate about an arbitrary axis. Given an arbitrary scalar θ and a vector $\mathbf{n}$, substitute $\mathbf{p} = [1,0,0]$ and simplify the value of $\mathbf{p}'$ in the expression

$$\mathbf{p}' = \cos\theta\,(\mathbf{p} - (\mathbf{p}\cdot\mathbf{n})\,\mathbf{n}) + \sin\theta\,(\mathbf{n}\times\mathbf{p}) + (\mathbf{p}\cdot\mathbf{n})\,\mathbf{n}.$$

What's our vector, Victor?

— Captain Oveur in *Airplane!* (1980)