Homework 4 APPM 4600

1 $g_\mu(x) = \mu \min(x, 1-x)$ on $[0,1]$, $0 \le \mu \le 2$.
want fixed points, so $x = g_\mu(x)$. Can use

$$\min(x, 1-x) = -|x - \tfrac{1}{2}| + \tfrac{1}{2}$$

a) show that for $\mu \in [0,2]$, $g_\mu(x) \in [0,1]$ $\forall x \in [0,1]$.

$$|x - \tfrac{1}{2}| \le \tfrac{1}{2} \Rightarrow -\tfrac{1}{2} \le x - \tfrac{1}{2} \le \tfrac{1}{2} \quad \forall x \in [0,1]$$

$$\Rightarrow -|x - \tfrac{1}{2}| + \tfrac{1}{2} \in [0, \tfrac{1}{2}]$$

$$\Rightarrow \mu(-|x - \tfrac{1}{2}| + \tfrac{1}{2}) \in [0, \tfrac{\mu}{2}]$$

since $0 \le \mu \le 2$, $\frac{\mu}{2} \le 1$

$$\Rightarrow g_\mu(x) = \mu(-|x - \tfrac{1}{2}| + \tfrac{1}{2}) \in [0,1] \quad \forall x \in [0,1]$$

b) $0 \le \mu < 1$.
: Prove $\exists!$ fixed point of g_μ in $[0,1]$

$$g_\mu(x) = \begin{cases} \mu x & 0 \le x \le 1/2 \\ \mu(1-x) & 1/2 \le x \le 1 \end{cases}$$

$\forall x, y \in [0,1]$. Try

- $x, y \le \frac{1}{2}$: $|\mu x - \mu y| = \mu |x - y|$
- $x, y \ge \frac{1}{2}$: $|\mu(1-x) - \mu(1-y)| = \mu |x - y|$
- $x \le \frac{1}{2} \le y$: $|\mu x - \mu + \mu y| = \mu |x + y - 1| \le \mu |x - y|$

↑ same for $y \le \frac{1}{2} \le x$

$\Rightarrow |g_\mu(x) - g_\mu(y)| \le \mu |x - y|$, and since $0 \le \mu < 1$, g_μ is a contraction map. By the Banach fixed point theorem, $\exists!$ fixed point in $[0,1]$. $\square$

ii. Given an arbitrary starting point $x_0 \in [0,1]$, would the fixed point iteration necessarily find the fixed point?
yes, the algorithm will converge by the Banach fixed point theorem.
we have $|g(x)-g(y)| \le \mu|x-y|$, so at every step, the error shrinks by a factor of μ.

iii. what is the fixed point?

$$g_\mu(x) = \begin{cases} \mu x & 0 \le x \le 1/2 \quad \text{(left side)} \\ \mu(1-x) & 1/2 \le x \le 1 \quad \text{(Right side)} \end{cases}$$

for the left side, $\mu x = x \Rightarrow x(\mu-1)=0$
if $\mu=1$ (not true, we have $0 \le \mu < 1$)
or $x=0$ ← fixed point
Right side : $\mu(1-x) = x \Rightarrow \mu = x+\mu x$
$\Rightarrow x = \frac{\mu}{1+\mu}$ for $x \in [1/2, 1]$
$\frac{\mu}{1+\mu} \ge \frac{1}{2} \Rightarrow 2\mu \ge 1+\mu \Rightarrow \mu \ge 1$ (never true for $0 \le \mu < 1$)

Thus the only fixed point for $0 \le \mu < 1$ is $x^* = 0$.

c) what are fixed points if $\mu=1$.
By the argument in B, $L = \mu$, so if $\mu=1$, $g_1(x)$ is not a contraction, so Banach doesn't apply, however,

$g_1(x) = -|x-\frac{1}{2}|+\frac{1}{2} = x$
$- x \le \frac{1}{2}$: $x - \frac{1}{2} + \frac{1}{2} = x = x \quad \forall x \in [0, 1/2]$
$- x \ge \frac{1}{2}$: $-x+1 = x \Rightarrow 2x = 1 \Rightarrow x = 1/2$

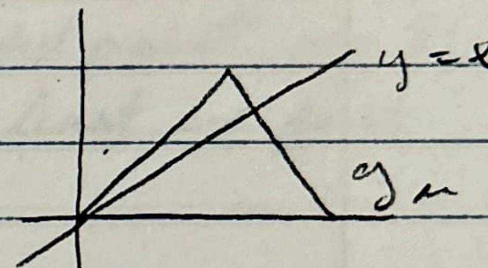
so $g_1(x)$ fixed $\forall x \in [0, 1/2]$.

d) Prove that when $1 \leq \mu \leq 2$, $\exists$ 2 fixed points in $[0,1]$, find and would contraction mapping then work? would fixed point ~~theorem~~ iteration work?

Here, $L = \mu \Rightarrow 1 < L \leq 2$, so this is not a contraction and the Banach theorem does not apply. (Recall this is $\Rightarrow$ not $\Leftarrow$)

By the argument in (6)(iii),

$$x_1^* = 0 \quad \text{and} \quad x_2^* = \frac{\mu}{1+\mu}.$$



The fixed point iteration will not converge!

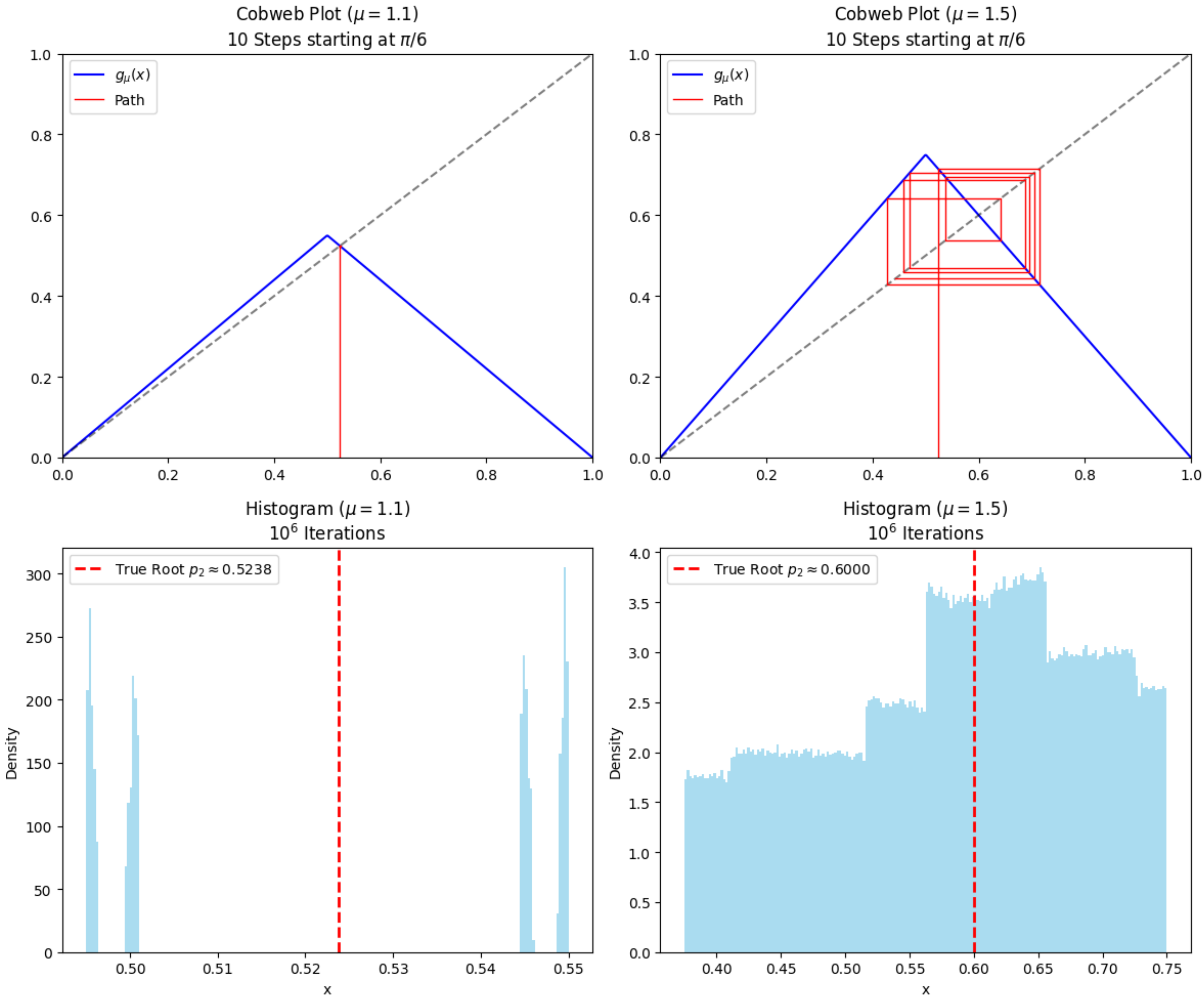
$|e_k| \leq \text{const } L^k$

but $1 < L \leq 2 \Rightarrow L^k \to \infty$!

In [10]:

```python
import numpy as np
import matplotlib.pyplot as plt

def g(x, mu):
    return mu * (-np.abs(x - 0.5) + 0.5)

def run_simulation(mu, x0, iterations):
    history = np.zeros(iterations)
    x = x0
    for i in range(iterations):
        x = g(x, mu)
        history[i] = x
    return history

# Parameters
x0 = np.pi / 6
mu_values = [1.1, 1.5]
steps_cobweb = 10
steps_hist = 10**6

fig, axes = plt.subplots(2, 2, figsize=(12, 10))

for i, mu in enumerate(mu_values):
    # Calculate true root
    p2 = mu / (1 + mu)
    # --- COBWEB PLOTS (Top Row) ---
    ax_cb = axes[0, i]
    x_vals = np.linspace(0, 1, 500)
    ax_cb.plot(x_vals, [g(v, mu) for v in x_vals], 'b', label=f'$g_\\mu(x)$')
    ax_cb.plot([0, 1], [0, 1], 'k--', alpha=0.5)

    # Generate cobweb path
    curr_x = x0
    px, py = [x0], [0]
    for _ in range(steps_cobweb):
        y_next = g(curr_x, mu)
        px.extend([curr_x, y_next])
        py.extend([y_next, y_next])
        curr_x = y_next

    ax_cb.plot(px, py, 'r', linewidth=1, label='Path')
    ax_cb.set_title(f'Cobweb Plot ($\\mu={mu}$)\n10 Steps starting at $\\pi/6$')
    ax_cb.set_xlim(0, 1)
    ax_cb.set_ylim(0, 1)
    ax_cb.legend(loc='upper left')
```

```python
    # --- HISTOGRAMS (Bottom Row) ---
    ax_hist = axes[1, i]
    data = run_simulation(mu, x0, steps_hist)

    # Plot histogram
    ax_hist.hist(data, bins=200, color='skyblue', density=True, alpha=0.7)

    # Vertical line at true root
    ax_hist.axvline(p2, color='red', linestyle='--', linewidth=2,
                    label=f'True Root $p_2 \\approx {p2:.4f}$')
    ax_hist.set_title(f'Histogram ($\\mu={mu}$)\n$10^6$ Iterations')
    ax_hist.set_xlabel('x')
    ax_hist.set_ylabel('Density')
    ax_hist.legend()

plt.tight_layout()
plt.show()
```



Cobweb Plot ($\mu = 1.1$)
10 Steps starting at $\pi/6$
$g_\mu(x)$
Path
Cobweb Plot ($\mu = 1.5$)
10 Steps starting at $\pi/6$
$g_\mu(x)$
Path
Histogram ($\mu = 1.1$)
10^6 Iterations
True Root $p_2 \approx 0.5238$
Density
x
Histogram ($\mu = 1.5$)
10^6 Iterations
True Root $p_2 \approx 0.6000$
Density
x

Here, we see that neither converges, as we would expect a spike around the true root if they converged.

$= \text{const}\, L^k$

but $1 < L \le 2 \Rightarrow L^k \to \infty$!

g_n

2 which will converge to p? Give order

a) $x_{n+1} = -16 + 6x_n + {}^{12}/_{x_n}$, p is longer fixed

$-16 + 6x_n + 12x_n^{-1} = x_n$

multiply by x_n: $-16x_n + 6x_n^2 + 12 = x_n^2$

$\Rightarrow 5x_n^2 - 16x_n + 12 = 0$

$(5x-6)(x-2) = 0$

$\Rightarrow p = 2, \frac{6}{5}$ $\quad 2 > {}^6/_5$

$g(x) = -16 + 6x + {}^{12}/_x$; $g'(x) = 6 - {}^{12}/_{x^2}$

$|g'(2)| = |6 - {}^{12}/_4| = 3 \not< 1 \Rightarrow$ No convergence

b) $x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}$, p unique $\Rightarrow \frac{2}{3}x_n + \frac{1}{x_n^2} = x_n$

$\Rightarrow \frac{1}{x_n^2} = \frac{1}{3}x_n \Rightarrow 3 = x_n^3 \Rightarrow x_n^* = \sqrt[3]{3}$

$g(x) = \frac{2}{3}x + x^{-2}$; $g'(x) = \frac{2}{3} - 2x^{-3}$

$|g'(\sqrt[3]{3})| = |\frac{2}{3} - \frac{2}{3}| = 0$

$g''(\sqrt[3]{3}) = \frac{6}{(\sqrt[3]{3})^4} = \frac{6}{3^{4/3}} \neq 0 \leftarrow$ Quadratic convergence

Homework 4

2 c) $x_{n+1} = \frac{12}{1+x_n}$; p larger fixed

$\Rightarrow 12/1+x_n = x_n \Rightarrow x_n^2 + x_n - 12 = 0$

$x = \frac{-1 \pm \sqrt{1-4(1)(-12)}}{2} = \frac{-1 \pm 7}{2} = 3, -4 \quad x^* = 3$

$g(x) = 12/1+x$; $g'(x) = \frac{0-12}{(1+x)^2} = -\frac{12}{(1+x)^2}$

$|g'(3)| = |-\frac{12}{16}| = \frac{3}{4} < 1 \Rightarrow$ Linear conv.

Rate of convergence: $\mu = \frac{3}{4}$