Continuous Systems; Waves

PHYS 301: Analytical Mechanics

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Problem 1

A string has initial conditions of

$$q[x,\ 0] = \frac{8\,A\,x^2}{L^2} - \frac{8\,A\,x^3}{L^3};$$

$$\dot{q}[x,\ 0] = 0.$$

Discuss the motion; that is, find the characteristic frequencies and calculate the amplitude of the n^{th} mode. Use Manipulate to produce an animated graph of your result.

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Clear[A]
Clear[L]
Clear[r]
```

$$\text{FullSimplify}\left[\frac{2}{L}\int_0^L\left(\frac{8\,A\,x^2}{L^2} - \frac{8\,A\,x^3}{L^3}\right)\text{Sin}\left[\frac{\pi\,r\,x}{L}\right]dx\right]$$

$$-\frac{16\,A\left(2\,\pi\,r + 4\,\pi\,r\,\text{Cos}[\pi\,r] + \left(-6 + \pi^2\,r^2\right)\text{Sin}[\pi\,r]\right)}{\pi^4\,r^4}$$

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A = 1;
L = 5;
ω = 1;
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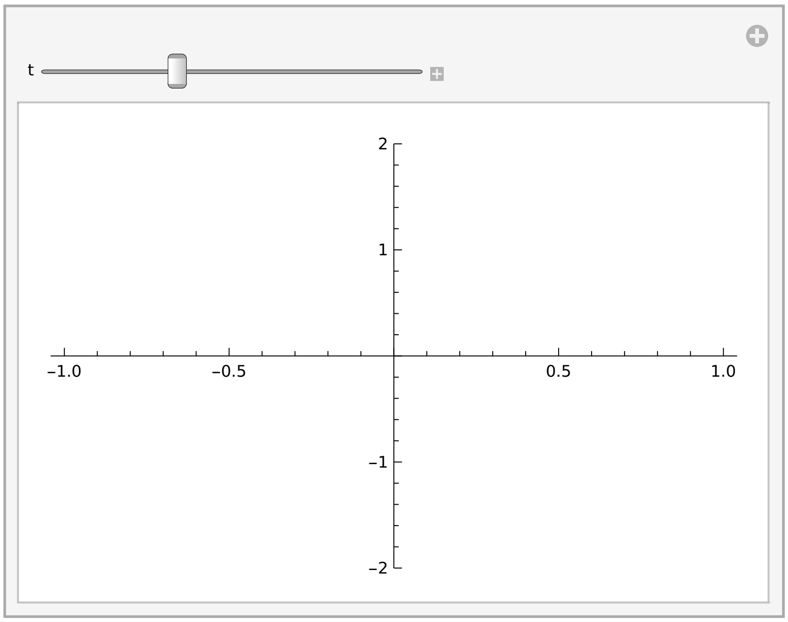
$$\text{Manipulate}\left[\text{Plot}\left[\text{Re}\left[\text{Sum}\left[-\frac{16\,A\left(2\,\pi\,r+4\,\pi\,r\,\text{Cos}[\pi\,r]+\left(-6+\pi^2\,r^2\right)\text{Sin}[\pi\,r]\right)}{\pi^4\,r^4}\,e^{i\,\omega\,t}\,\text{Sin}\left[\frac{\pi\,r\,x}{L}\right],\ \{r,\ 4\}\right]\right],\ \{x,\ 0,\ 5\},\ \text{PlotRange}\to\{-2,\ 2\}\right],\ \{t,\ 0,\ 5\}\right]$$



Problem 2

A string with no initial displacement is set into motion by being struck over a length $2\,s$ about its center. The center section is given an initial velocity v_0. Describe the subsequent motion. *Hint:*

$$q[x,\ 0] = 0;$$

$$\dot{q}[x,\ 0] = \begin{cases} v_0, & \frac{L}{2} - s < x < \frac{L}{2} + s \\ 0, & \text{otherwise} \end{cases}.$$

$$\text{FullSimplify}\left[\frac{2\,v\,L}{\pi^2}\left(\sqrt{\left(\frac{e}{t}\right)}\right)\left(\text{Cos}\left[\pi\left(\frac{1}{2}+\frac{s}{L}\right)\right]-\text{Cos}\left[\pi\left(\frac{1}{2}-\frac{s}{L}\right)\right]\right)\right]$$

$$-\frac{4\,L\,\sqrt{\frac{e}{t}}\,v\,\text{Sin}\left[\frac{\pi\,s}{L}\right]}{\pi^2}$$

$$\text{FullSimplify}\left[\frac{2\,v\,L}{4\,\pi^2}\left(\sqrt{\left(\frac{e}{t}\right)}\right)\left(\text{Cos}\left[2\,\pi\left(\frac{1}{2}+\frac{s}{L}\right)\right]-\text{Cos}\left[2\,\pi\left(\frac{1}{2}-\frac{s}{L}\right)\right]\right)\right]$$

0

$$\text{FullSimplify}\left[\frac{2\,v\,L}{9\,\pi^2}\left(\sqrt{\left(\frac{e}{t}\right)}\right)\left(\text{Cos}\left[3\,\pi\left(\frac{1}{2}+\frac{s}{L}\right)\right]-\text{Cos}\left[3\,\pi\left(\frac{1}{2}-\frac{s}{L}\right)\right]\right)\right]$$

$$\frac{4\,L\,\sqrt{\frac{e}{t}}\,v\,\text{Sin}\left[\frac{3\,\pi\,s}{L}\right]}{9\,\pi^2}$$

$$\text{FullSimplify}\left[\frac{2\,v\,L}{16\,\pi^2}\left(\sqrt{\left(\frac{e}{t}\right)}\right)\left(\text{Cos}\left[4\,\pi\left(\frac{1}{2}+\frac{s}{L}\right)\right]-\text{Cos}\left[4\,\pi\left(\frac{1}{2}-\frac{s}{L}\right)\right]\right)\right]$$

0

Problem 3

Consider a wave packet with a Gaussian amplitude distribution

$A[k] = B\,e^{-\sigma\,(k-k_0)^2}$

where $2/\sqrt{\sigma}$ is equal to the $1/e$ width of the packet. Find the initial wave function $\Psi[x, 0]$. Sketch the shape of this wave packet.

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Clear[ω]
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```
t = 0;
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$$\int_{-\infty}^{\infty} B\,e^{-\sigma\,(k-k0)^2}\,e^{i\,(\omega\,t-k\,x)}\,dk$$

$$\text{ConditionalExpression}\left[\frac{B\,e^{-\frac{x\,(x+4\,i\,k0\,\sigma)}{4\,\sigma}}\,\sqrt{\pi}}{\sqrt{\sigma}},\ \text{Re}[\sigma]>0\right]$$

```
B = 1;
k0 = 1;
σ = 1;
```

$$\text{Plot}\left[\text{Re}\left[\frac{B\, e^{-\frac{x\,(x+4\, i\, k0\, \sigma)}{4\, \sigma}}\, \sqrt{\pi}}{\sqrt{\sigma}}\right],\ \{x, -10, 10\},\ \text{PlotRange} \to \{-.5, 2\}\right]$$