Calculus of Variations

PHYS 301: Analytical Mechanics

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Problem 1

Consider the function $f = \left(\frac{dy}{dx}\right)^2$, where $y(x) = \sin x$. Add to $y(x)$ the function $\eta(x) = x^2 - \pi x$, and plot $y(\alpha, x)$. Find $J(\alpha)$ between the limits of $x = 0$ and $x = \pi$. (You probably want to use *Mathematica* to solve the integral.) For what value of α does the stationary value of $J(\alpha)$ occur?

Problem 2

Find the ratio of the radius R to the height H of a right-circular cone of fixed volume V that minimizes the surface area A. (*Answer:* $H = \sqrt{8}\, R$.)

Problem 1

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Simplify[(Cos[x] + α (2 x - π))^2]
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$(-\pi \alpha + 2 x \alpha + \mathrm{Cos}[x])^2$

$\alpha = \frac{12}{\pi^3}$;

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Plot[Sin[x] + α (x^2 - π x), {x, 0, π}]
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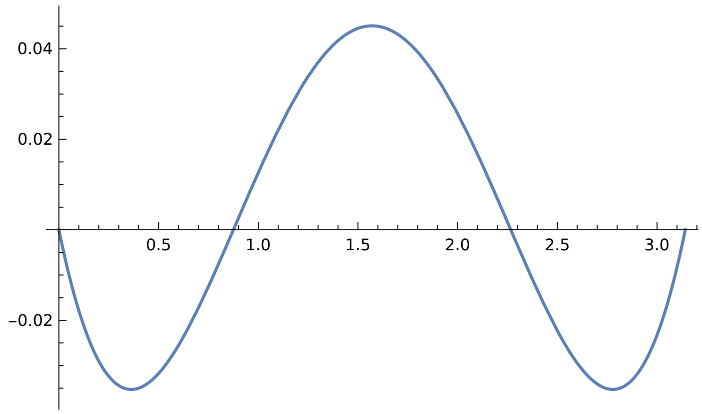

$\int_0^\pi (\text{Cos}[x])^2 \, dx$

$\frac{\pi}{2}$

$\int_0^\pi 4\,\alpha\,x\,\text{Cos}[x] \, dx$

$-8\,\alpha$

$-\int_0^\pi 2\,\alpha\,\pi\,\text{Cos}[x] \, dx$

0

$\int_0^\pi 4\,\alpha^2\,x^2 \, dx$

$\frac{4\,\pi^3\,\alpha^2}{3}$

$\int_0^\pi -4\,\alpha^2\,x\,\pi \, dx$

$-2\,\pi^3\,\alpha^2$

$\int_0^\pi \alpha^2\,\pi^2 \, dx$

$\pi^3\,\alpha^2$

Problem 2

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A = π r (r + √(h² + r²));
V = π r² h/3;
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D[A, r]
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$$\pi r \left(1 + \frac{r}{\sqrt{h^2 + r^2}}\right) + \pi \left(r + \sqrt{h^2 + r^2}\right)$$

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D[V, r]
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$$\frac{2 h \pi r}{3}$$

$$\text{Solve}\left[\pi r \left(1 + \frac{r}{\sqrt{h^2 + r^2}}\right) + \pi \left(r + \sqrt{h^2 + r^2}\right) + \lambda \frac{2 h \pi r}{3} == 0,\ \lambda\right]$$

$$\left\{\left\{\lambda \to -\frac{3 \left(r + \sqrt{h^2 + r^2}\right)^2}{2 h r \sqrt{h^2 + r^2}}\right\}\right\}$$

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D[A, h]
```

$$\frac{h \pi r}{\sqrt{h^2 + r^2}}$$

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D[V, h]
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$$\frac{\pi r^2}{3}$$

$$\text{Solve}\left[\frac{h \pi r}{\sqrt{h^2 + r^2}} + -\frac{3 \left(r + \sqrt{h^2 + r^2}\right)^2}{2 h r \sqrt{h^2 + r^2}} \left(\frac{\pi r^2}{3}\right) == 0\ ,\ h\right]$$

$$\left\{\left\{h \to -2 \sqrt{2}\ r\right\}, \left\{h \to 2 \sqrt{2}\ r\right\}\right\}$$