Pendulum Model

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I. INTRODUCTION

The pendulum has long been used as a timekeeping device such as a grandfather clock or to keep a beat like a metronome. While the clock needs to just keep track of one measurement of time, the metronome has to be able to perform at a multitude of different time-steps. This can be done by either changing the length of the pendulum or the magnitude of the angle that it is pulled away from its' equilibrium position. In this experiment, the simple pendulum model will be tested in two ways. First, the model will be tested for small oscillations (angles less than 12°) as a function of length. The model will then be tested as a function of amplitude for a variety of angles between 0° and 60°. The accuracy of the model will be assessed and the model will be updated if the pursuit of a better fit is required.

II. THEORY

To mathematically describe the period of a pendulum, you must begin with the equation for torque on a mass, as follows:

$$T = |F||r|Sin\theta \tag{1}$$

Where F is the force acting on the mass and r is the radius of the pendulum, which will just be referred to as the length of the pendulum, l. The only force acting on the pendulum is the force of gravity:

$$F = -mg \tag{2}$$

Substituting in the force and our redefined r, Equation 1 becomes:

$$T = |mg||l|Sin\theta \tag{3}$$

From here we can utilize Newton's Second Law for Torques:

$$T = I\alpha = I\ddot{\theta} \tag{4}$$

With I being the moment of inertia of the mass and $\ddot{\theta}$ being the angular acceleration of the mass. The moment of inertia of a point mass is mr^2 and continuing the redefined r, will be ml^2. Combining Equation 3 and Equation 4 produces:

$$I\ddot{\theta} = |mg||l|Sin\theta \tag{5}$$

$$ml^2\ddot{\theta} + mglSin\theta = 0 \tag{6}$$

This has turned into a differential equation that must be simplified to isolate $\ddot{\theta}$. To do so, a factor of m and a factor of l may be removed from the equation, and the entire equation may be divided by l, isolating $\ddot{\theta}$, and producing:

$$\ddot{\theta} + \frac{g}{l}Sin\theta = 0 \tag{7}$$

Equation 7 is the model for a full pendulum, one that is not limited to the small angles of the Simple Pendulum model that are desired for the experiment. In order to acquire the Simple Pendulum model, one must apply the small-angle approximation.

A small angle is any angle that is under 12°. The small angle approximation is when the sine of that angle is taken and the result is within one percent of what the original angles value in radians was. The table below illustrates the process of the small angle approximation, as well as the threshold for its application.

$\theta(Degrees)$	$\theta(Radians)$	Sinθ	Percent Difference
5	.0873	.0872	.13
10	.1745	.1736	.51
15	.2588	.2618	1.15

TABLE I. Small Angle Approximation Values

Applying this small angle approximation to Equation 7 removes the sine from the equation and creates the Simple Pendulum model that is desired for this experiment:

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \tag{8}$$

Solving the differential equation that is Equation 8, you get the following values of θ, $\dot{\theta}$, and $\ddot{\theta}$, which are then plugged back in to the Simple Pendulum Model, which will be simplified further. The values of the θ's are as follows:

$$\theta = \theta_0 Sin(\omega t) \tag{9}$$

$$\dot{\theta} = -\omega\theta_0 Cos(\omega t) \tag{10}$$

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$$\ddot{\theta} = -\omega^2\theta_0 Sin(\omega t) \tag{11}$$

With no $\dot{\theta}$ in Equation 8, it will not be used. Plugging in the values for θ and $\ddot{\theta}$ into the Simple Pendulum Model produces:

$$-\omega^2\theta_0 Sin(\omega t) + \frac{g}{l}\theta_0 Sin(\omega t) = 0 \tag{12}$$

Factoring out the like terms of $\theta_0 Sin(\omega t)$,Equation 12 is as follows:

$$\theta_0 Sin(\omega t)(-\omega + \frac{g}{l}) = 0 \tag{13}$$

The two solutions for Equation 13 are that $Sin(\omega t) = 0$ or that $(-\omega + \frac{g}{l}) = 0$. The latter solution is more fruitful in the pursuit of an equation for the period of a pendulum and rearranged is:

$$\omega = \sqrt{\frac{g}{l}} \tag{14}$$

The final step is to plug the solution for ω into the standard equation for period:

$$T = \frac{2\pi}{\omega} \tag{15}$$

Which produces the desired result for the period of a pendulum from the Simple Pendulum Model that is Equation 8:

$$T = 2\pi\sqrt{\frac{l}{g}} \tag{16}$$

For the first part of the experiment which is the changing of the length of the pendulum while maintaining a consistent small angle, it is expected that the results will follow Equation 16 and the period will increase as the square root of the length of the pendulum. For the second part of the experiment, when the length of the pendulum is kept constant, but the angle is changing and can be as great as 60 degrees, it is expected that the data does not fit the model accurately and that a new model will have to be derived in order to accurately describe the period of the pendulum that is not limited to small angles.

III. EXPERIMENT

In order to test the simple pendulum model's accuracy, the Vernier Photogate sensor is used to measure the period of the pendulum. This is done by placing the photogate beneath the pendulum at its' equilibrium position. When the mass is pulled back, it passes through the photogate and trips the sensor once, a second time on the way back, and then a third time on the way back down. The period is the time from the mass first passing through the sensor to the third time it passes through the sensor.

To set up for the experiment to be performed, the string was tied to a washer, and the mass was secured on the end of the string. Then, the washer, mass, and string were hung on the hook which allowed the string to hang at the equilibrium position.

The first part of the experiment measures the period of the pendulum with a fixed small angle but a total of seven different string lengths. The zero end of the meter stick is placed at the pivot point of the pendulum. By looking perpendicular to the meter stick and level with the center of mass the length of the pendulum is measured and recorded. Next, the mass is pulled back to $5°$ (will also be done at $10°$ and $30°$). The mass is released to measure the period using the photogate. This step is repeated 5 times at this particular length for the string. All steps and measurements are repeated for a total of 7 different string lengths.

The second part of the experiment measures the period of the pendulum at fixed lengths of $1m$, $1.5m$, and $2m$, varying the angles up to $60°$. The mass is released and the period is measured using the photogate. These steps are repeated to acquire a total of 5 measurements of the period of the pendulum at each angle, varying up to $60°$ at each length.

IV. ANALYSIS

The first step in analyzing the data was to create the two models that would serve as the fit functions. Both of these models were derived from the Simple Pendulum Model, Equation 16. The first fit function from the model was for the relationship of period vs. length with a constant angle. The model is as follows:

$$y = c(1)\sqrt{x} \tag{17}$$

From the model, y represents the period of the pendulum and x represents the lengths of the pendulum. This model was applied to the data collected from the pendulum when released from a constant angle of 5, 10, and 30 degrees. While the model applied to all 3 scenarios was the same, the values of the coefficient in the function were the same for 5 and 10 degrees but differed for the 30-degree pendulum. For the 5 and 10 degree pendulums $c(1) = \frac{2\pi}{\sqrt{g}}$. To find the value for the 30-degree pendulum, a new model must be derived. The derivation begins at Equation 7 from the Theory section of this paper. This is the differential equation representing the period of a pendulum before the application of the small-angle approximation. Solving this differential equation without the small-angle approximation produces a model for a pendulum unrestricted by the size of the angle. This nonlinear differential equation is quite difficult to solve but boils down to a Taylor expansion. The Taylor expansion produces the following equation:

$$T = 2\pi\sqrt{\frac{l}{g}}(1 + \frac{\theta^2}{16}) \tag{18}$$

This model is used as the fit function for the 30 degree pendulum as well as the period vs. angle relationship further in the analysis. The fit function is still $y = c(1)\sqrt{x}$ for the pendulum, but from this Full Pendulum Model:

$$c(1) = \frac{2\pi}{\sqrt{g}}(1 + \frac{\theta^2}{16}) \tag{19}$$

The graphs with the applied fit function for the period vs. length relationship are below. It is clear by inspection that each pendulum fits the desired function, Equation 17.

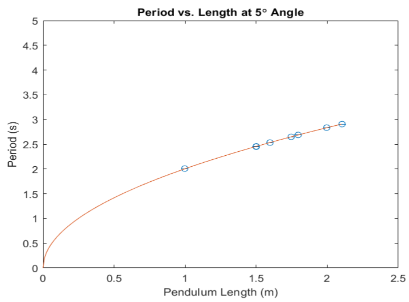

FIG. 1. This graph shows the relationship between period and length of a pendulum released from a 5 degree angle.

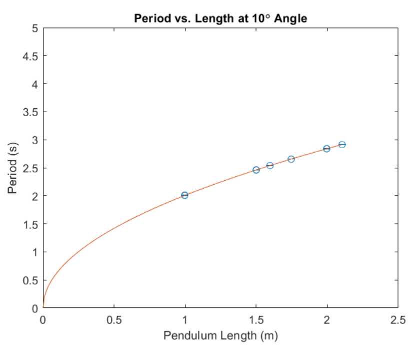

FIG. 2. This graph shows the relationship between period and length of a pendulum released from a 10 degree angle.

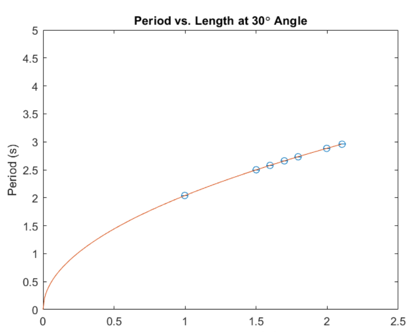

FIG. 3. This graph shows the relationship between period and length of a pendulum released from a 30 degree angle.

The values of the coefficients are given in the table below:

Degree of Pendulum	Coefficient Value
5	$2.0056 \pm .0007$
10	$2.0092 \pm .0005$
30	$2.0404 \pm .0005$

TABLE II. Part1Coeff

The second step in analyzing the data was to apply the Simple Pendulum Model, Equation 16, to the data for the period vs angle relationship. The fit function was similar to that of the first part of the analysis, but to remove the length dependence from the model. This produced a fit function and coefficient value of:

$$y = c(1) \tag{20}$$

$$c(1) = 2\pi\sqrt{\frac{l}{g}} \tag{21}$$

The graphs of period vs angle for the pendulum lengths of 1, 1.5 and 2 meters did not fit the Simple Pendulum Model fit function as can be seen below. Since the graphs did not fit the function, a table of coefficient values was not produced.

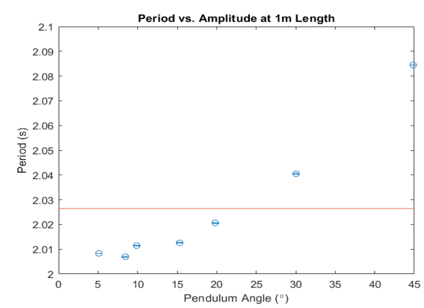

FIG. 4. This graph shows the relationship between period and angle of a 1 meter long pendulum.

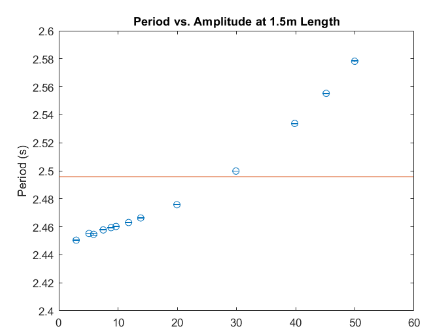

FIG. 5. This graph shows the relationship between period and angle of a 1.5 meter long pendulum.

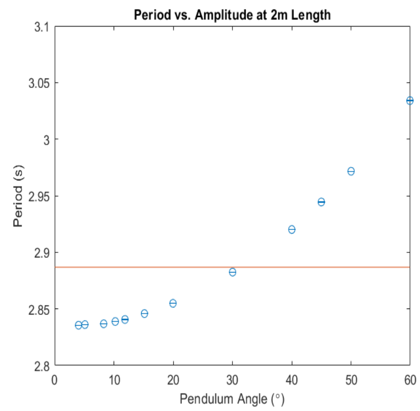

FIG. 6. This graph shows the relationship between period and angle of a 2 meter long pendulum.

With these graphs not fitting the Simple Pendulum Model, the derivation of the new model was necessary. This is the same model used for the 30 degree pendulum from the first part of the analysis, Equation 18. However, the fit function differs slightly as the variance in period is not a result of the length, but of the angle. The revised fit function is:

$$y = c(1)(1 + c(2)\theta^2) \tag{22}$$

The coefficients are defined in the table below:

Coefficient	Equation
c(1)	$2\pi\sqrt{\frac{l}{g}}$
c(2)	$\frac{1}{16}$

TABLE III. Part2Coeff

The graphs with the revised fit function for the pendulums of set length and varying release angle are as follows:

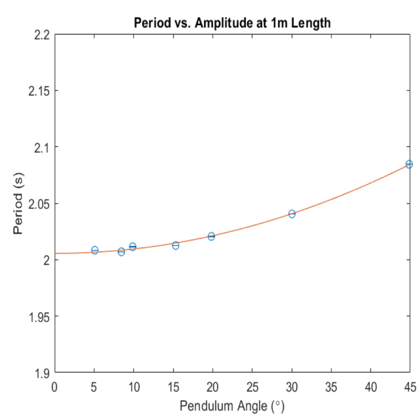

FIG. 7. This graph shows the relationship between period and angle of a 1 meter long pendulum with the updated model.

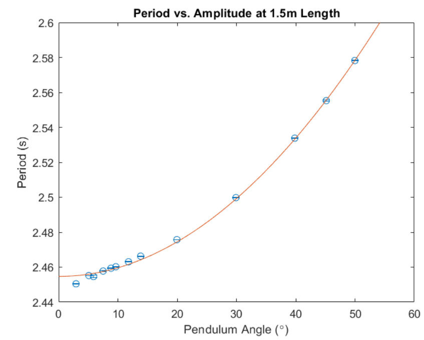

FIG. 8. This graph shows the relationship between period and angle of a 1.5 meter long pendulum with the updated model.

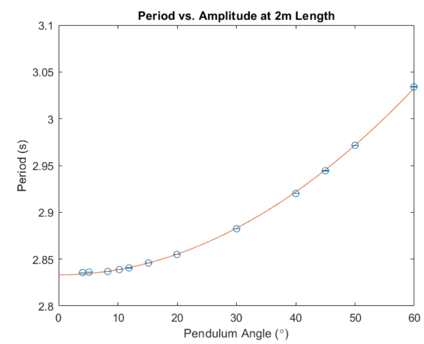

FIG. 9. This graph shows the relationship between period and angle of a 2 meter long pendulum with the updated model.

The values of the coefficients for the second part of the analysis can be seen at the top of page 5 in Table IV.

The table above shows that the value for c(2) is 0, which we know cannot be true. MatLab reported the value of c(2) to be 0 with an error of $5{*}10^{-7}$, $2{*}10^{-7}$, and $1{*}10^{-7}$ for the 1-meter, 1.5-meter, and 2-meter pendulum lengths. While the value is not the $\frac{1}{16}$ that was expected, it is also not quite zero. A zero c(2) would result in a graph that shows no angular dependence of the period, which would not fit the data.

V. CONCLUSION

For small oscillations, the simple pendulum model describes the system as a function of length correctly. The simple pendulum model suggests that the period will increase as the square root of the length of the pendulum. This was demonstrated to be correct in our model. The simple pendulum model does not accurately describe the oscillation of a pendulum for angles ranging from 0° to 60°. An adjusted model, Equation 22 that does not utilize the small-angle approximation and this model accurately described the oscillation of a pendulum for angles between 0° and 60°.

Length (m)	c(1) Value	c(2) Value
1	$2.0056 \pm .0008$	0.000 ± 0.0000
1.5	$2.4547 \pm .0006$	0.000 ± 0.0000
2	$2.8332 \pm .0004$	0.000 ± 0.0000

TABLE IV. Part2Coeff2