Coupled Oscillations

PHYS 301: Analytical Mechanics

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Problem 1

Consider a mass m_1 attached to a wall with a spring with constant $3\,\kappa$. This mass is in turn attached to another mass m_2 with a spring with constant $2\,\kappa$. (There is nothing else attached to m_2.) What are the normal frequencies of oscillation if $m_1 = m_2$?

```
Solve[0 == 6 k^2 - 7 k ω^2 m + ω^4 m^2, ω]
```

$$\left\{\left\{\omega \to -\frac{\sqrt{k}}{\sqrt{m}}\right\}, \left\{\omega \to \frac{\sqrt{k}}{\sqrt{m}}\right\}, \left\{\omega \to -\frac{\sqrt{6}\ \sqrt{k}}{\sqrt{m}}\right\}, \left\{\omega \to \frac{\sqrt{6}\ \sqrt{k}}{\sqrt{m}}\right\}\right\}$$

Problem 2

Consider the two-object, three-spring system from Thursday's class, with $\kappa_1 = 10$, $\kappa_2 = 9$, and $\kappa_{12} = 1$. Both objects have mass 1. For the initial condition where m_2 is held at its equilibrium position and m_1 is pulled a distance A to the right of its equilibrium position and released from rest, find the resulting motion. Use software such as Mathematica to plot an x-vs-t graph of the motion of each mass. Then, if using Mathematica, use the following command to visualize the motion, with the appropriate definitions for **x1[t]** and **x2[t]**.

```
Clear[w]
FullSimplify[Solve[0 == 109 - 21 r^2 + r^4, r]]
```

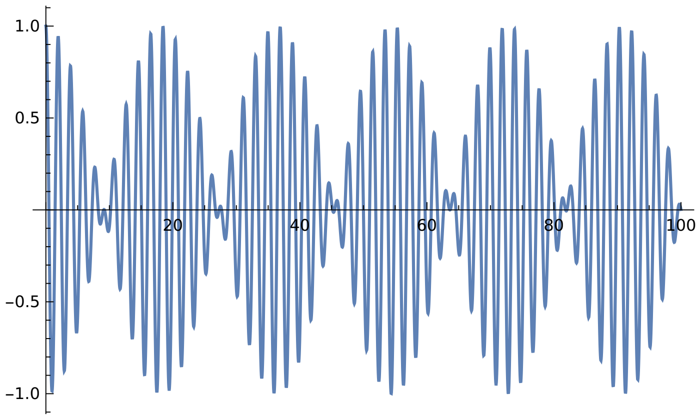
$$\left\{\left\{r \to \text{Root}\left[109 - 21\,\#1^2 + \#1^4\ \&,\ 2\right]\right\},\right.$$
$$\left.\left\{r \to \sqrt{\frac{1}{2}\left(21 - \sqrt{5}\right)}\right\}, \left\{r \to -\sqrt{\frac{1}{2}\left(21 + \sqrt{5}\right)}\right\}, \left\{r \to \sqrt{\frac{1}{2}\left(21 + \sqrt{5}\right)}\right\}\right\}$$

$$w = \sqrt{\frac{1}{2}\left(21 - \sqrt{5}\right)};$$

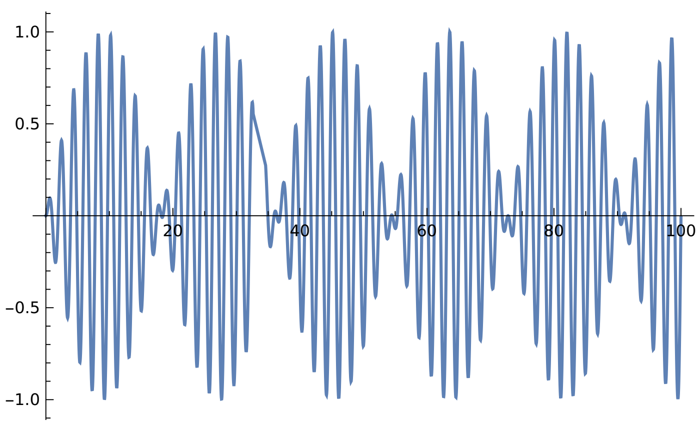
$$z = \sqrt{\frac{1}{2}\left(21 + \sqrt{5}\right)};$$

```
x[t_] := A (Cos[(z + w)/2 t] × Cos[(z - w)/2 t]);
y[t_] := A ( Sin[(z + w)/2 t] × Sin[(z - w)/2 t]);
```

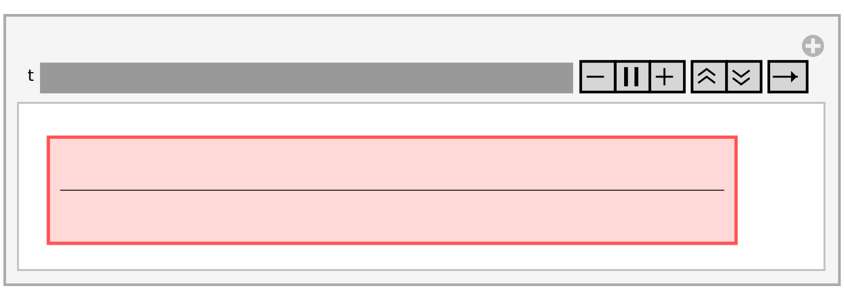
```
A = 1;
Plot[x[t], {t, 0, 100}]
```




```
Plot[y[t], {t, 0, 100}]
```




```
Manipulate[Graphics[{Line[{{-3, 0}, {13, 0}}], Disk[{x[t], 0}], Disk[{10 + y[t], 0}]}], {t, 0, ∞}]
```



Problem 3

A mass M moves horizontally along a smooth rail. A pendulum is hung from the mass with a weightless rod of length b and mass m at its end.

a) Let x_1 be the coordinate for the mass on the rail, and let x_2 and y_2 be the coordinates for the pendulum bob. Write x_2 and y_2 in terms of x_1 and θ, the vertical angle of displacement for the pendulum. (Drawing a picture helps!)

b) Find the kinetic and potential energies of the system, assuming small oscillations (e.g., $\cos\theta \approx 1 - \theta^2/2$). Keep terms of order θ^2 and $\dot{\theta}^2$, but discard terms such as $\theta^2\,\dot{\theta}$.

c) Construct the M and A matrices.

d) Find the normal frequencies of oscillation.

```
FullSimplify[-w^2 (M + m) (m g b - w^2 m b^2) - (-w^2 m b) (-w^2 m b)]
```

$$b\,m\,w^2\left(-g\,(m+M) + b\,M\,w^2\right)$$

```
Solve[0 == b m w^2 (-g (m + M) + b M w^2), w]
```

$$\left\{\{w \to 0\},\ \{w \to 0\},\ \left\{w \to -\frac{\sqrt{g}\,\sqrt{m+M}}{\sqrt{b}\,\sqrt{M}}\right\},\ \left\{w \to \frac{\sqrt{g}\,\sqrt{m+M}}{\sqrt{b}\,\sqrt{M}}\right\}\right\}$$

e) Find the normal modes of oscillation, and describe the physical conditions under which they occur.

$$\texttt{FullSimplify}\left[\frac{\left(-\left(g - \frac{g\,(m+M)}{M}\right) a\right)}{\frac{g\,(m+M)}{b\,M}}\right]$$

$$\frac{a\,b\,m}{m+M}$$

Problem 4

Find the eigenfrequencies of a 10-oscillator system fixed on either end, each particle with mass 1.0 kg and connected to its nearest neighbors by springs with the spring constant 0.4 kg/s.

To do this, you will need 10 coordinates (as defined for Mathematica in the vector below). ($x_0 = x_{11} = 0$ always; these are the walls, *i.e.* your boundary conditions). These are your q_k.

```
x1 =.; x2 =.;
q = {0, x1, x2, x3, x4, x5, x6, x7, x8, x9, x10, 0};
```

a. Find T and U for an arbitrary number of masses n.

b. Show that this reduces to the results of TM section 12.2 for $n = 2$. (*T and U for n=2 NOT result*)

```
dq = {0, dx1, dx2, dx3, dx4, dx5, dx6, dx7, dx8, dx9, dx10, 0};
```

c. Set $n = 10$ and find $\boldsymbol{A}$ and $\boldsymbol{M}$ computationally. If you are using Mathematica, you may use this $\vec{q}$.

```
U = Sum[1/2 K (q[[n]] - q[[n - 1]])^2, {n, 2, 12}];
T = Sum[1/2 m dq[[n]]^2, {n, 2, 12}] ;
MatrixForm[A = Table[D[D[U, q[[n]]], q[[p]]], {n, 2, 11}, {p, 2, 11}]]
```

$$\begin{pmatrix} 2K & -K & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -K & 2K & -K & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -K & 2K & -K & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -K & 2K & -K & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -K & 2K & -K & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -K & 2K & -K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -K & 2K & -K & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -K & 2K & -K & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -K & 2K & -K \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -K & 2K \end{pmatrix}$$

```
MatrixForm[ M = Table[ D[D[T, dq[[n]], dq[[p]]]], {n, 2, 11}, {p, 2, 11}]]
```

$$\begin{pmatrix} m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m \end{pmatrix}$$

d. *Numerically* solve the characteristic equation to find the ω's. (To help you check your answer, one of the eigenfrequencies I got was $\omega = 0.525463$.)

```
Det[(A - ω^2 M)]
```

$$11\,K^{10} - 220\,K^{9}\,m\,\omega^{2} + 1287\,K^{8}\,m^{2}\,\omega^{4} - 3432\,K^{7}\,m^{3}\,\omega^{6} + 5005\,K^{6}\,m^{4}\,\omega^{8} - 4368\,K^{5}\,m^{5}\,\omega^{10} + 2380\,K^{4}\,m^{6}\,\omega^{12} - 816\,K^{3}\,m^{7}\,\omega^{14} + 171\,K^{2}\,m^{8}\,\omega^{16} - 20\,K\,m^{9}\,\omega^{18} + m^{10}\,\omega^{20}$$

```
m = 1;
K = .4;
Simplify[
  Solve[0 == 11 K^10 - 220 K^9 m ω^2 + 1287 K^8 m^2 ω^4 - 3432 K^7 m^3 ω^6 + 5005 K^6 m^4 ω^8 - 4368 K^5 m^5 ω^10 +
     2380 K^4 m^6 ω^12 - 816 K^3 m^7 ω^14 + 171 K^2 m^8 ω^16 - 20 K m^9 ω^18 + m^10 ω^20, ω]]
```

{{ω → -1.25204}, {ω → -1.21367}, {ω → -1.1506}, {ω → -1.06411}, {ω → -0.955956},
{ω → -0.828341}, {ω → -0.683863}, {ω → -0.525463}, {ω → -0.356367}, {ω → -0.180016},
{ω → 0.180016}, {ω → 0.356367}, {ω → 0.525463}, {ω → 0.683863}, {ω → 0.828341},
{ω → 0.955956}, {ω → 1.06411}, {ω → 1.1506}, {ω → 1.21367}, {ω → 1.25204}}