Special Functions

PHYS 310 : Mathematical Methods in Physics

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Problem 1

Use the properties of Gamma functions to solve these expressions by hand. Use *Mathematica* to check your answers.

a

$$\frac{\Gamma\left[\frac{1}{2}\right]\times\Gamma[4]}{\Gamma\left[\frac{9}{2}\right]}$$

```
Gamma[1 / 2] // N
```

```
1.77245
```

```
Gamma[4] // N
```

```
6.
```

$\text{Gamma}\left[\frac{9}{2}\right]$ // N

```
11.6317
```

b

$$\Gamma\left[\frac{1}{3}\right]\times\Gamma\left[\frac{2}{3}\right]$$

$\text{Gamma}\left[\frac{1}{3}\right]\times\text{Gamma}\left[\frac{2}{3}\right]$ // FullSimplify

$$\frac{2\,\pi}{\sqrt{3}}$$

c

$\Gamma\left[-\frac{5}{4}\right] \times \Gamma\left[\frac{9}{4}\right]$

Gamma$\left[\frac{-5}{4}\right] \times$ **Gamma**$\left[\frac{9}{4}\right]$ **// N**

4.44288

Problem 2

Express these integrals as Gamma functions, showing all work by hand. Use *Mathematica* to check your answers.

a

$\int_0^\infty x^{2/3}\, e^{-x}\, dx$

$\int_0^\infty x^{2/3}\, e^{-x}\, dx$

Gamma$\left[\frac{5}{3}\right]$

Gamma$\left[\frac{5}{3}\right]$ **// N**

0.902745

$\frac{2}{3}$ **Gamma**$\left[\frac{2}{3}\right]$ **// N**

0.902745

b

$\int_0^1 x^2 \left(\ln \frac{1}{x}\right)^3 dx$

(Gamma[4]) $\frac{-1}{81}$ **// N**

-0.0740741

c

$$\int_0^\infty x^{-1/3} e^{-8x} \, dx$$

$$\text{Gamma}\left[\frac{2}{3}\right]\left(8^{\frac{-1}{3}}\right)\left(\frac{1}{8}\right) \text{ // N}$$

0.0846324

Problem 3

Express these integrals as Beta functions, then as Gamma functions, showing all work by hand. When possible, give an exact answer in terms of π, $\sqrt{2}$, etc. Use *Mathematica* to check your answers.

a

$$\int_0^2 \frac{x^2}{\sqrt{2-x}} \, dx$$

$$\left(\frac{\left(\text{Gamma}[3] \times \text{Gamma}\left[\frac{1}{2}\right]\right)}{\text{Gamma}\left[\frac{7}{2}\right]}\right) \text{ // N}$$

1.06667

$$\frac{1}{2^{2.5}}$$

0.176777

b

$$\int_0^1 \frac{1}{\sqrt{1-x^3}} \, dx$$

$$\frac{\left(\text{Gamma}[1] \times \text{Gamma}\left[\frac{1}{2}\right]\right)}{\text{Gamma}\left[\frac{3}{2}\right]} \text{ // N}$$

2.

c

$$\int_0^{\pi/2} \sqrt{\sin^3\theta\cos^5\theta}\, d\theta$$

```
(1/4 Gamma[1/4] 3/4 Gamma[3/4]) / 4 // N
```

0.20826

Problem 4

a

Find the length of the graph of $y = \sin x$ over half a period symbolically. Use *Mathematica* to find its numerical value.

$$\int_0^{\pi} \sqrt{\left(1 + \mathrm{Cos}[x]^2\right)}\, dx$$

```
Integrate[Sqrt[(1 + Cos[x]^2)], {x, 0, π}] // N
```

3.8202

b

Find the area of the region bounded by the equation $x^3 + y^3 = 8$ in the first quadrant in terms of a Beta function.

```
1/4 Beta[4/3, 1/3] // N
```

0.66249

Problem 5

Use the Taylor expansion of the error function to estimate erf(0.1). How far off is that from the actual value of erf(0.1)? Would you call this a “good” approximation for $x = 0.1$?

```
Series[Erf[x], {x, 0, 5}]
```

$$\frac{2\,x}{\sqrt{\pi}} - \frac{2\,x^3}{3\,\sqrt{\pi}} + \frac{x^5}{5\,\sqrt{\pi}} + O[x]^6$$

$$\left(\frac{2\,(.1)}{\sqrt{\pi}} - \frac{2\,(.1^3)}{3\,\sqrt{\pi}} + \frac{(.1^5)}{5\,\sqrt{\pi}}\right)$$

0.112463

Erf[.1]

0.112463