Vector Calculus II

PHYS 310 : Mathematical Methods in Physics

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Problem 1

The height of a certain hill (in feet) is given by $h(x, y) = 10\left(2\,x\,y - 3\,x^2 - 4\,y^2 - 18\,x + 28\,y + 12\right)$, where y is the distance in miles (north), x the distance east of South Hadley, Massachusetts.

a

Where is the top of the hill located?

b

How high is the hill?

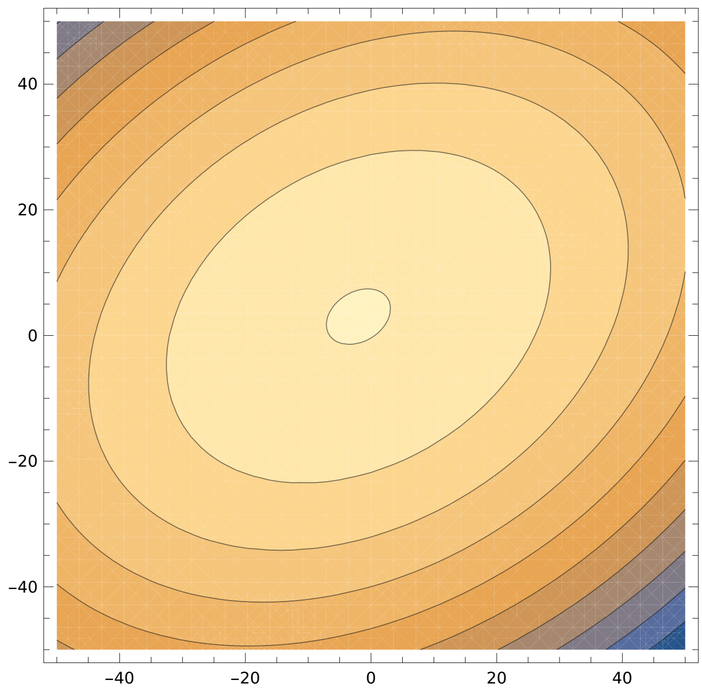
c

How steep is the slope (in feet per mile) at a point 1 mile north and one mile east of South Hadley? In what direction is the slope steepest, at that point?

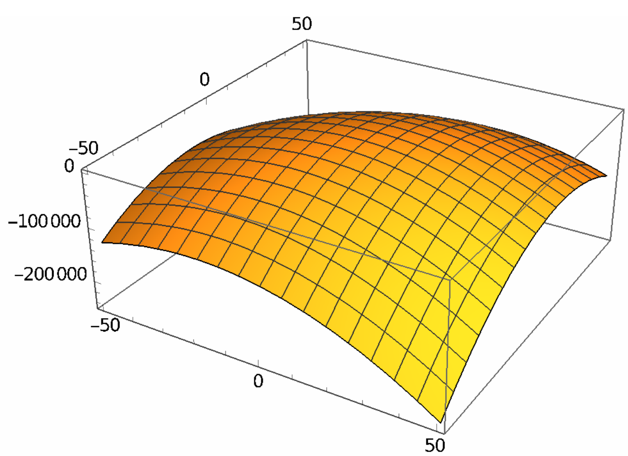
d

Make a contour map and a 3D plot of the area surrounding South Hadley, from 8 miles west to 4 miles east, and 2 miles south to 8 miles north. Plot only positive heights on the 3D graph; use **If** to help you out.

```
ContourPlot[10 (2 x y - 3 x^2 - 4 y^2 - 18 x + 28 y + 12), {x, -50, 50}, {y, -50, 50}]
```




```
Plot3D[10 (2 x y - 3 x^2 - 4 y^2 - 18 x + 28 y + 12), {x, -50, 50}, {y, -50, 50}]
```



Problem 2

Find the divergence and curl of each of the following vector functions. Check each answer with *Mathematica.*

a

$\vec{B} = x^2\,\hat{x} + 3\,x\,z^2\,\hat{y} - 2\,x\,z\,\hat{z}$

```
Div[{x^2, 3 x z^2, -2 x z}, {x, y, z}]
```

0

```
Curl[{x^2, 3 x z^2, -2 x z}, {x, y, z}]
```

$\{-6\, x\, z,\ 2\, z,\ 3\, z^2\}$

b

$\vec{S} = x\, y\, \hat{x} + 2\, y\, z\, \hat{y} + 3\, z\, x\, \hat{z}$

```
Div[{x y, 2 y z , 3 z x}, {x, y, z}]
```

$3\, x + y + 2\, z$

```
Curl[{x y, 2 y z , 3 z x}, {x, y, z}]
```

$\{-2\, y,\ -3\, z,\ -x\}$

c

$\vec{E} = y^2\, \hat{x} + \left(2\, x\, y + z^2\right) \hat{y} + 2\, y\, z\, \hat{z}$

```
Div[{y^2, (2 x y + z^2), 2 y z}, {x, y, z}]
```

$2\, x + 2\, y$

```
Curl[{y^2, (2 x y + z^2), 2 y z}, {x, y, z}]
```

$\{0,\ 0,\ 0\}$

Problem 3

Find the Laplacian of each of the following scalar functions. Check each answer with *Mathematica.*

a

$V = x^2 + 2\, x\, y + 3\, z + 4$

```
Laplacian[x^2 + 2 x y + 3 z + 4, {x, y, z}]
```

2

b

$\Phi = \sin[x] \times \sin[y] \times \sin[z]$

```
Laplacian[Sin[x] × Sin[y] × Sin[z], {x, y, z}]
```

-3 Sin[x] × Sin[y] × Sin[z]

c

$W = e^{-5x} \sin[4 y] \times \cos[3 z]$

```
Laplacian[e^(-5 x) Sin[4 y] × Cos[3 z], {x, y, z}]
```

0

Problem 4

Find the work done by the force $\vec{F} = (y + z)\,\hat{x} - (x + z)\,\hat{y} + (x + y)\,\hat{z}$ along each of the following *closed* paths.

a

The circle $x^2 + y^2 = 1$ in the $z = 0$ plane, taken counterclockwise.

$$W = \int_0^{2\pi} -1 \, d\phi$$

-2π

b

The circle $x^2 + z^2 = 1$ in the $y = 0$ plane, taken counterclockwise.

$$W = \int_0^{2\pi} \mathrm{Cos}[2\phi] \, d\phi$$

0

c

The curve starting from the origin and going successively along the x-axis to (1, 0, 0), parallel to the z-axis to (1, 0, 1), parallel to the $x = 0$ plane to (1, 1, 1), and back to the origin along $x = y = z$.

$$\text{Path1W} = \int_0^1 (0+0)\,x - (x+0)\,y + (1+0)\,z \, dx$$

$$W = \int_0^1 0\,x \, dx$$

0

$$\text{Path2W} = \int_0^1 dz$$

1

$$\text{Path3W} = \int_0^1 -2 \, dy$$

-2

$$\text{Path4W} = \int_1^0 2\,x \, dx$$

-1

TotalWork = Path1W + Path2W + Path3W + Path4W

0 + 1 + -2 + -1

-2

d

From the origin to $(0, 0, 2\pi)$ on the curve $x = 1 - \cos t$, $y = \sin t$, $z = t$, and back to the origin along the z-axis.

$$W = \int_0^{2\pi} (t\,\mathrm{Sin}[t] - t\,\mathrm{Cos}[t] + \mathrm{Sin}[t] + 2) \, dt$$

2π

$$W = \int_{2\pi}^{0} 0 \, dz$$

0

$$0 + 2\pi$$

2π