Scott Kovacs Exam 2

Problem 1

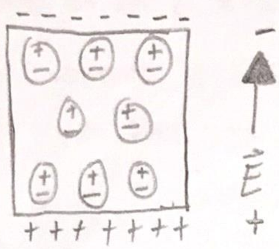
a) What does it mean for a dielectric to be polarized?

To polarize a dielectric, you put it in an electric field. Reacting to that electric field, each atom will have its own dipole moment induced by the $\vec{E}$ field. Each dipole moment will line up in the direction of the $\vec{E}$ field. If the material has polar molecules, the molecules will experience a force that lines them up with the electric field. Either way, be it atomic or molecular, the polarization of a dielectric is the lining up of the dipoles in the same direction of the magnetic field.

What a polarized dielectric looks like vs. a non-polarized dielectric:

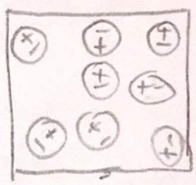
Polarized

w/ the polarized dielectric, the dipoles line up w/ the electric field: (− faces + b/c opposites attract, likes repel).



Non-Polarized:

w/o an electric field, the dielectric is not polarized, so the dipoles of the atoms ↳ or molecules have nothing making them line up



- A little further into how this happens is just a reference to the free charge, "charge we control", like the electric field in the above picture. We can introduce it, which causes polarization, or take it away which leads to a non-polarized dielectric.

b) For a dielectric to be linear and isotropic it must meet these requirements:

to be linear: Must obey the equation $\vec{P} = \varepsilon_0 \chi_e \vec{E}$, polarization is proportional to the field, $|P| \sim |E|$.
↳ (or similar configurations) *

to be isotropic: the polarization does not depend on the direction of $\vec{E}$. So applying the electric field along a different axis results in the same polarization strength.

* by similar configurations I mean it always has to be $\vec{P}$ and $\vec{E}$. Not $\vec{P}$ and $\vec{E}^2$ or $\vec{P}$ and $\vec{E}^3$, just $\vec{P}$ and $\vec{E}$

ex) $\vec{P} = \varepsilon_0 (\chi_e + \chi_e kr) \vec{E}$ is still linear b/c to $\vec{P}$ and $\vec{E}$

this example is also isotropic b/c its not dependent on any axis

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Problem 2

a) sketch vector fields ① ② + ③

b) explain how you know the field meets the requirements

Maxwell's Equations

Electrostatics: $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$ (Gauss's Law)

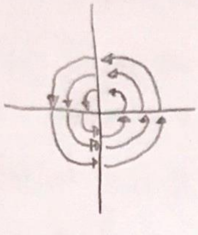
$\vec{\nabla} \times \vec{E} = 0$

Magnetostatics: $\vec{\nabla} \cdot \vec{B} = 0$

$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ (Ampere's Law)

① Could be a B-field but not $\vec{E}$-field:

- From Maxwells equations, to be a Magnetic field but not an electric field, the field must a divergence of 0 and a non-zero curl.



- This is the field of $F(x,y) = -y\hat{x} + x\hat{y}$.

$$\text{div}(-y,x) = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x)$$

$$0 = 0 + 0$$

$$\text{div} = 0$$

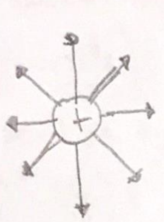
$$\text{curl}(-y,x,0) = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{z}$$

$$\text{curl} = (0-0)\hat{x} + (0-0)\hat{y} + (1 - -1)\hat{z}$$

$$\text{curl} = 2$$

② Could be an $\vec{E}$-field but not a $\vec{B}$-field:

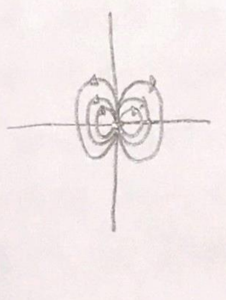
- From Maxwells equations, to be an electric field but not a magnetic field, the field must have zero curl but non-zero divergence.



- This is the field of a positive point charge
- This field clearly has a nonzero divergence, as the field lines spread out as they get farther away from the point charge.
- The curl is zero by inspection, there's no curvature of the lines at all.

③ Could be either a $\vec{B}$ or an $\vec{E}$-field:

- From Maxwells equation, this example satisfies the zero divergence of the magnetic field, while also satisfying the zero curl of the electric field. The pure monopole.



- Satisfies $\vec{\nabla} \cdot \vec{B} = 0$ b/c there are no magnetic monopoles in nature.
- taking the curl of the equation for an electric dipole gives you 0.

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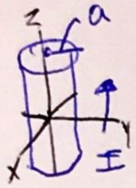
Problem 3

- ∞ long wire w/ circular cross section has radius a & is oriented w/ its axis along the z-axis. It carries a steady current that travels in the $+\hat{z}$ direction and is distributed in some way as a function of distance from the radius. You want to know the magnetic field inside and outside the wire.



a) Using only the differential form of Ampere's Law, explain how you know the direction of the magnetic field inside and outside the wire: $\hat{s}, \hat{\phi}, \hat{z}$. Do NOT say RHR.

b) What variable could the magnetic field depend on: s, ϕ, or z? (no hints)

Explain how you know, w/o citing known results for wires, for both inside and out.

c) Suppose the current is distributed such that the current density inside the wire is $\vec{J} = \frac{2I}{\pi a^4} s^2 \hat{z}$ where s is the distance to the axis. Find $\vec{B}$ inside the wire.

a) The differential form of Ampere's Law is as follows:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

- we know the current is a function of s in the $\hat{z}$ direction, $I(s)\hat{z}$ and current density is $J = \frac{I}{A} = \frac{I(s)}{\pi s^2}\hat{z}$

- so the right half of our equation is $\mu_0 \frac{I(s)}{\pi s^2}\hat{z}$, so we need the left half to produce a $+\hat{z}$ direction only.

$$\vec{\nabla} \times \vec{B} = \left(\frac{1}{s}\frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial B_s}{\partial z} - \frac{\partial B_z}{\partial s}\right)\hat{\phi} + \frac{1}{s}\left(\frac{\partial}{\partial s}(sB_\phi) - \frac{\partial B_s}{\partial \phi}\right)\hat{z}$$

- we know we have to get a $+\hat{z}$ back, so we can "eliminate" the $\hat{s}$ & $\hat{\phi}$ portions of the curl.

$\frac{1}{s}\left(\frac{\partial}{\partial s}(sB_\phi) - \frac{\partial B_s}{\partial \phi}\right)\hat{z}$ and to get a $+\hat{z}$ back, we won't use the $-\frac{\partial B_s}{\partial \phi}$ portion so we are left with just the portion w/ B_ϕ in it.

$\boxed{\text{So the magnetic field would only be in the } \hat{\phi} \text{ direction both inside and out}}$

b) As for what variable the magnetic field could depend on, it $\boxed{\text{would only be } s.}$

From part a) we widdled down $\vec{\nabla} \times \vec{B}$ to only $\frac{1}{s}\left(\frac{\partial}{\partial s}(sB_\phi)\right)$ to get the appropriate direction of the current ($\vec{B}$ has to be in $\hat{\phi}$ since I is in $\hat{z}$)

So the only variable that $\vec{B}$ would depend on is s, since it is the only variable we are taking the partial derivative of w/in the curl of $\vec{B}$

This would be the case for both inside & outside the wire

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Problem 3 (continued)

c) If $\vec{J} = \frac{2I}{\pi a^4} s^2 \hat{z}$ we can use the integral form of Ampere's Law

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} \quad , \quad w/ \ I_{enc} = \int \vec{J} \cdot d\vec{a}$$

$I_{enc} = \oint J \cdot d\vec{a}$ and $da = s\,ds\,d\phi$ in cylindrical coordinates

$$I_{enc} = \int_0^{2\pi}\int_0^{s} \frac{2I}{\pi a^4} s^2 \hat{z}\ s\,ds\,d\phi$$

- limits of integration are $0-2\pi$ for ϕ since it's a full circle and $0-s$ for s to account for the charge for any point inside the wire.

$$= \frac{2I}{\pi a^4}\int_0^{2\pi}\int_0^{s} s^3\,ds\,d\phi$$

$$= \frac{2I}{\pi a^4}\int_0^{2\pi} d\phi \ \frac{1}{4}s^4 = \frac{Is^4}{2\pi a^4}(2\pi) = \frac{Is^4}{a^4}$$

- Plug into Ampere's Law:

$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$ $\qquad dl = 2\pi s$, circumference of a circle

$$B_{in}(2\pi s) = \mu_0\left(\frac{Is^4}{a^4}\right)$$

$$B_{in} = \frac{\mu_0 I s^4}{a^4} \Big/ 2\pi s$$

$$\boxed{\vec{B}_{in} = \frac{\mu_0 I s^3}{2\pi a^4}\hat{\phi}}$$

* Amperian surface used was inside cylinder w/ radius s:

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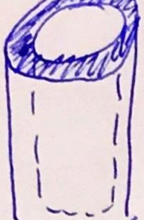
Problem 4

— Very long, hollow cylinder of inner radius a, outer radius b carries a "frozen in" polarization, $\vec{P} = P_0 s^2 \hat{s}$ which carries no free charge.

a) The condition for using the integral form of Gauss's law to determine $\vec{D}$ is that $\vec{\nabla} \times \vec{D} = 0$. <u>Without first calculating $\vec{D}$</u>, how can you determine that this condition is met? (For any scenario, not just this one). Does that condition hold true for this problem in particular? Justify



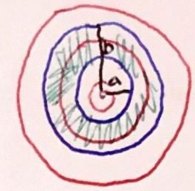
– For the given $\vec{P}$ find the following (in any order):

b) Bound volume & bound surface charge densities

c) $\vec{D}$ everywhere

d) $\vec{E}$ everywhere

Top View w/ Gaussian Surfaces



3 Gaussian surfaces:

① $s < a$ hollow part

② $a < s < b$ thickness of shell

③ $s > b$ outside shell

a) We know by definition that $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$.

We also know by Gauss's Law in electrostatics that the curl of $\vec{E}$ is 0 ($\vec{\nabla} \times \vec{E} = 0$).

So, if $\vec{D}$ were to have any curl, it would have to come from the polarization, $\vec{P}$. $\vec{\nabla} \times \vec{D} = \vec{\nabla} \times \vec{P}$

In general, if $\vec{P}$ has curl, so will $\vec{D}$.

In this case $\vec{P}$ is only in the $\hat{s}$ direction and has no curl.

$\vec{\nabla} \times \vec{P} = 0$ so $\vec{\nabla} \times \vec{D} = 0$, the condition for using the integral form of Gauss's Law is met.

b) Bound volume & Bound surface charge densities:

<u>Bound surface density</u>: $\sigma_b = \vec{P} \cdot \hat{n}$ $\hat{n}$ is the normal unit vector. @ $s=a$, $\hat{n}$ is inward, so $\hat{n} = -\hat{s}$

@ $s=b$, $\hat{n}$ is outward, so $\hat{n} = +\hat{s}$

$\sigma_b = P_0 s^2 \hat{s} \cdot \hat{n}$

$\sigma_b = P_0 s^2$

$$\boxed{\begin{array}{ll} \underline{@s=a} & \underline{@s=b} \\ \sigma_b = -P_0 a^2 \hat{s} & \sigma_b = +P_0 b^2 \hat{s} \end{array}}$$

<u>Bound Volume density</u>:

$\rho_b = -\vec{\nabla} \cdot \vec{P}$ $\vec{P} = P_0 s^2 \hat{s}$ * no P_ϕ or P_z *

$$\rho_b = -\left(\frac{1}{s}\frac{\partial}{\partial s}(s P_s)\right)$$

$$= -\frac{1}{s}\frac{\partial}{\partial s}(s P_0 s^2)$$

$$= -\frac{1}{s}\frac{\partial}{\partial s}(P_0 s^3)$$

$$\rho_b = -\frac{1}{s}(3P_0 s^2)$$

$$\boxed{\rho_b = -3P_0 s}$$

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Problem 4 (continued)

c) Find $\vec{D}$ everywhere.

Region ①: $s<a$ "hollow part"

Use integral form of Gauss's Law:

$\oint \vec{D} \cdot d\vec{a} = Q_{fenc}$

* problem states there is no free charge (even if there was some it wouldn't be in the hollow part) so $Q_{fenc}=0$

$\oint \vec{D} \cdot d\vec{a} = 0 \rightarrow \boxed{\vec{D}_{①}=0}$

Region ② $a<s<b$

integral form of Gauss's Law:

$\oint \vec{D} \cdot d\vec{a} = Q_{fenc}$ * problem states $Q_{fenc}=0$

so $\boxed{\vec{D}_{②}=0}$

Region ③:

$\oint \vec{D} \cdot d\vec{a} = Q_{fenc}$

$=0$

$\boxed{\vec{D}_{③}=0}$

d) Find $\vec{E}$ everywhere.

Region ①: $\vec{D}=0$

$\vec{P}$ also $=0$ because the polarization is in the material, not in the hollow part

$\vec{D}=\varepsilon_0\vec{E}+\vec{P}$

$0=\varepsilon_0\vec{E}+0$ $\boxed{\vec{E}_{①}=0}$

Region ② $\vec{D}=0$

$\vec{P}=P_0 s^2\hat{s}$ w/in material

$\vec{D}=\varepsilon_0\vec{E}+\vec{P}$

$0=\varepsilon_0\vec{E}+P_0 s^2\hat{s}$

$\boxed{\vec{E}=\frac{-P_0 s^2}{\varepsilon_0}\hat{s}}$

Region ③:

$\vec{D}=0$ $\vec{P}=0$

so $\boxed{\vec{E}_{③}=0}$

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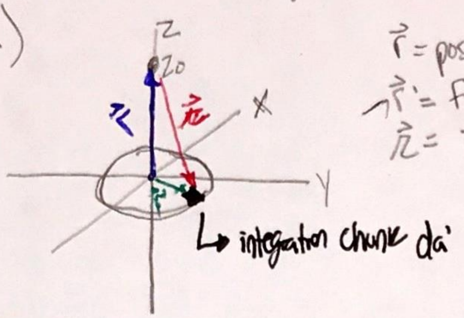
Problem 5

- Use Biot-Savart Law to find $\vec{B}$ a distance z_0 from center of disk radius R that lies in x-y plane, carries a surface charge density σ while spinning on an axis @ ω creating a steady surface current.

a) Clearly mark integration "chunk" and draw & label $\vec{r}$, $\vec{r}'$, $\mathscr{r}$, & $\hat{\mathscr{r}}$

b) Set up integral to find $\vec{B}$ at z_0 on axis & simplify. Do not solve.

a)



$\vec{r}$ = position vector from origin to field point

$\vec{r}'$ = from origin to source point

$\vec{\mathscr{r}} = \vec{r} - \vec{r}'$ field to source

$\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{|\mathscr{r}|} = \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}$

$\mathscr{r} = |\vec{r}-\vec{r}'|$

$\vec{r} = z_0\hat{z}$

$\vec{r}' = s\hat{s}$

$\vec{\mathscr{r}} = z_0\hat{z} - s\hat{s}$

$|\mathscr{r}| = \sqrt{z_0^2+s^2}$

$\hat{\mathscr{r}} = \frac{z_0\hat{z}-s\hat{s}}{\sqrt{z_0^2+s^2}}$

b) Biot-Savart Law for surface current:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}) \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, da'$$

- plug in $\hat{\mathscr{r}}$ and $\mathscr{r}^2$

* will be using cylindrical coordinates

$\vec{K}(\vec{r}) = \sigma\vec{v}$ $\quad v = \frac{w}{r}$ $\quad w = vr$

$\vec{K} = \sigma w r\,\hat{\phi}$ → $r = s$ in cylindrical

$\vec{K}(\vec{s}) = \sigma w s\,\hat{\phi}$

$$B(z_0) = \frac{\mu_0}{4\pi}\int_0^{2\pi}\int_0^{R} \frac{\sigma w s\,\hat{\phi} \times \frac{(z_0\hat{z}-s\hat{s})}{\sqrt{z_0^2+s^2}}}{\left(\sqrt{z_0^2+s^2}\right)^2}\, s\,ds'\,d\phi'$$

* $da' = s\,ds'\,d\phi'$ in cylindrical

* Limits of integration for ds' are $0 \to R$ b/c its the radius of the disk & $0 \to 2\pi$ for $d\phi'$ b/c it goes all the way around

$$B(z_0) = \frac{\mu_0}{4\pi}\int_0^{2\pi}\int_0^{R} \frac{\sigma w s\,\hat{\phi} \times (z_0\hat{z}-s\hat{s})}{(z_0^2+s^2)^{3/2}}\, s\,ds'\,d\phi'$$

$$\boxed{B(z_0) = \frac{\mu_0}{4\pi}\int_0^{2\pi}\int_0^{R} \frac{\sigma w s^2 z_0\,\hat{s} + \sigma w s^3\,\hat{z}}{(z_0^2+s^2)^{3/2}}\, ds'\,d\phi'}$$

to simplify the cross product

$\langle a_\phi b_z - a_z b_\phi,\ a_z b_s - a_s b_z,\ a_s b_\phi - b_s a_\phi \rangle$

$$\begin{array}{c|ccc} & s & \phi & z \\ a & 0 & \sigma w s & 0 \\ b & -s & 0 & z_0 \end{array}$$

$\langle \sigma w s z_0 - 0,\ (0\cdot -s)-(0\cdot z_0),\ (0\cdot 0)-(-s\sigma w s) \rangle$

$\langle \sigma w s z_0,\ 0,\ \sigma w s^2 \rangle$