# Correct Resolution of the Twin Paradox

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In the following, I explain the "Twin Paradox", which is supposed to be a paradoxical consequence of the Special Theory of Relativity (STR). I give the correct resolution of the "paradox," explaining why STR is not inconsistent as it appears at first glance. I also debunk two common, incorrect responses to the paradox. This should help the reader to understand Special Relativity and to see how the theory is coherent.

## I. Background: Time and Length Dilation

According to STR, different intertial reference frames (RF's) can disagree on the time that elapses between two events, as well as on the length of an object. These discrepancies are described by the Lorentz Transformation equations:

$$L = L_0\sqrt{1-\frac{v^2}{c^2}} \qquad\qquad T = \frac{T_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

Here, $L_0$ is the length an object has in a reference frame in which it is at rest, while L is the length the object has according to a reference frame in which it is moving at velocity *v*. Similarly, $T_0$ is the time between some pair of events in a reference frame in which the events take place at the same (spatial) place, whereas T is the time between those events according to a reference frame in which the events are separated by a path of velocity *v* (that is, the ratio of the spatial distance between the events to the temporal distance between the events is *v*). *c* is the speed of light.

For example: suppose I'm in a spaceship that's traveling at $c/2$ (half the speed of light) relative to the Earth. Suppose that in *my* reference frame, I measure my ship to be 100 feet long. So $L_0$ = 100 ft. Then observers on Earth are going to measure my spaceship to have a length of:

$$L = L_0\sqrt{1-\frac{v^2}{c^2}} = 100\sqrt{1-\frac{(c/2)^2}{c^2}} = 100\sqrt{1-\frac{1}{4}} = 50\sqrt{3} \approx 86.6\ \mathit{ft}.$$

Similarly, suppose that I have a clock in my spaceship. At the start of my journey, my clock reads 12:00. My clock measures elapsed time according to *my* reference frame (i.e., the reference frame in which the spaceship is at rest). So when the clock strikes 1:00, 1 hour has passed in my reference frame. So $T_0$ = 1 hr. However, people on Earth will see the time between these two events (my clock striking 12:00 and my clock striking 1:00) as being:

$$T = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{(c/2)^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{1}{4}}} = \frac{2}{\sqrt{3}} \approx 1.15 \ hr. \approx 69.2 \ minutes$$

So people on Earth will perceive my spaceship as having *shrunk* (since it is 87 feet long, whereas it was 100 feet long when it was at rest), and they will perceive my clock as moving *slower* than normal (because it takes it 69 minutes, instead of 60 minutes, to register the passage of an hour).

## II. The Twin "Paradox"

Here's a statement of the "Twin Paradox", designed to make it seem paradoxical:

> Suppose that two identical twins start out on Earth, the same age. One of them gets in a spaceship and flies around really fast. Call this one "Space Twin". The other twin, "Earth Twin", stays on Earth the whole time. Finally, Space Twin comes home and meets up with Earth Twin. When they meet, which one will have aged more?
>
> Well, according to *Earth* Twin, Space Twin was moving around really fast, so Space Twin aged slower (like in the example above), so Space Twin is going to look younger.
>
> But according to Space Twin, it was *Earth* Twin who was moving the whole time (in Space Twin's reference frame, Space Twin was stationary). So Earth Twin was aging slower, so Earth Twin is going to look younger.
>
> According to STR, both reference frames are equally valid. But presumably, these answers can't both be true; when they see each other, it's not going to be that each of them looks younger than the other. So it looks as if STR with its time dilation postulate is incoherent.

The above thinking is in fact wrong; time dilation is not incoherent. I'll explain why below. But first I want to clear the ground of some fallacious explanations.

## III. Two Wrong Responses to the Twin Paradox

I have heard two wrong responses (or claims about the response) to the twin paradox. First, my high school physics teacher said that the resolution of the twin paradox had to do with *General*

Relativity, which was beyond the scope of his class. That's false. General Relativity is Einstein's theory of gravitational phenomena. The Twin Paradox has nothing to do with gravitation, nor has it anything to do with the spacetime curvature postulated in GR. As we'll see in section IV, the resolution of the "paradox" can be given purely within a flat (Minkowski) spacetime.

The second false resolution I have heard is that it is the Space Twin who will have aged slower, because it was he who had to *accelerate* in order to get up to his great speed. In STR, there is an objective distinction between accelerated and uniform (non-accelerated) motion. So it's an objective fact that Space Twin underwent acceleration while Earth Twin didn't. Therefore, Space Twin is the one who was really going fast, so he'll have aged slower. Now, how many things are wrong with this response?

a. The response is profoundly un-relativistic. It supposes that Space Twin is the one who was "really" moving during his voyage, because he's the one who accelerated. (That's like what *Newton* says about absolute motion.) In relativity theory, there is no objective fact about who is moving at any given point in time; at any time, any given person or object is moving according to some RF's and stationary according to others, and all inertial RF's are equally legitimate. So the proposed response can't possibly be the resolution of the problem according to STR.

b. There is no requirement, in STR, that there be some initial conditions in which everything starts out at relative rest. Thus, we may stipulate that Space Twin does *not* undergo any acceleration to get up to his great speed—instead, suppose that Space Twin and Earth Twin just *start out* in relative motion. Space Twin is already going at near light speed (relative to Earth) at the start of our story. (Equivalently, Earth Twin is already going at near light speed relative to Space Twin at the start of the story.)

c. There is no mechanism in STR for the *past history* of some state of motion (or of anything else, for that matter) to directly exert causal power on things now. The Lorentz Transformation Equations discussed in section I above, for example, do not in any way make use of any facts about past history to determine the amount of length or time dilation. Nor does anything in General Relativity do so. Nor, in fact, does *any* modern physical theory allow the past to directly influence the present (once the current state of the world is given, how we got to that state is irrelevant to what will happen next). So even if Space Twin had to undergo acceleration in the past to get to his current speed, that would be irrelevant to what effects his current speed has on him.

d. Finally, notice that the Lorentz Equations do not take account of acceleration at all—length dilation and time dilation are purely a function of the "rest length" ($L_0$) and "rest time" ($T_0$), the velocity $v$, and the speed of light $c$. Acceleration plays no role. Nor is this different in

General Relativity or any other theory (that is, no other theory gives any modified Lorentz transformations where acceleration appears in the equations). Given that, acceleration can't have anything to do with the solution to the paradox.

## IV. Correct Resolution of the "Paradox"

There are two cases to consider: In one scenario, the two twins are in uniform motion throughout the story (they neither accelerate nor decelerate, nor change their direction of motion). In another scenario, Space Twin goes off into space for a while, then turns around (deceleration/acceleration) and comes back to Earth.

### *A. The Case of Uniform Motion*

In the first case, the twins can only have one meeting—without accelerating or decelerating, Space Twin will never return to Earth, so the meeting in which they finally see who has aged more will never happen. What does STR say about this scenario? It says that *each* twin will age slower, as measured by the other twin. How can this be? The secret lies in the relativity of simultaneity. Consider the space-time diagram in Figure 1.

In this diagram, line $t$ is the time axis, according to Earth Twin, while $s$ is the space axis (or: a simultaneity slice) according to Earth Twin. Earth Twin thinks that he's always stationary, and thus that he remains on the $t$ axis (he does not change his spatial location—his position along $s$—as time passes). He thinks that any path parallel to $t$ is the path of a stationary object (that is, any such path is stationary in his reference frame). Also, Earth Twin would say that all the spacetime points that lie on $s$ are simultaneous with each other, and all of them occur at (say) $t = 0$. Earth Twin would say that any line parallel to $s$ also marks a collection of spacetime points that are simultaneous with each other. For instance, the horizontal line labeled "$t = 4$" indicates all the spacetime points (or all the events) that occur at time 4. Earth Twin thinks that Space Twin is moving off to the right, on the path labeled $t'$.

Things look different according to Space Twin. Space Twin says that $t'$, not $t$, is the time axis. He would say that any path parallel to $t'$ is the path of a stationary object. He would say that Earth Twin, who occupies line $t$, is moving uniformly *to the left*. Now here's a slightly surprising point: Space Twin's *space* axis is $s'$, rather than $s$. (When you tilt your time axis to the right, you have to tilt your space axis *toward* your time axis. To see why this is true, consider the fact that the speed of light has to be a constant for all reference frames. The speed of light is the ratio of $\Delta$(spatial distance) to $\Delta$(temporal distance) along the path of the light ray. To keep that ratio constant, if you consider an RF whose time axis is tilted to the right, its space axis must be tilted up.) Space Twin would say that all events lying on $s'$ (not the events lying on $s$) are simultaneous

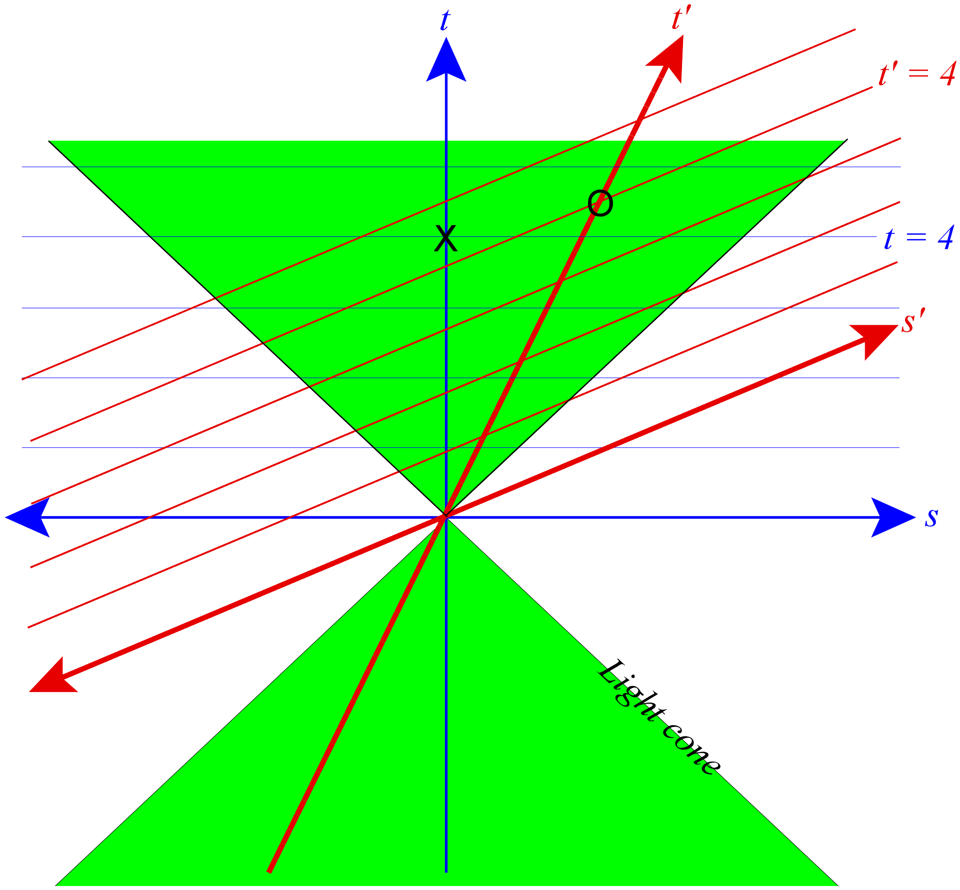


**Figure 1**

with each other. He would likewise say that any line parallel to $s'$ marks off a set of simultaneous spacetime points. For instance, the slanted line labeled "$t' = 4$" represents everything that happens at time 4, according to Space Twin.

Now, here's why both twins are aging slower, relative to each other. Consider the point where Earth Twin's clock strikes 4. I have marked this point with an "X" on the diagram. This is where Earth Twin is when 4 units of time have passed according to his reference frame. Now compare the point where Space Twin's clock strikes 4. This happens on the spaceship (so it's on path $t'$), and it happens when 4 units of time have passed in *Space Twin's* reference frame. I've marked this point with an "O". Finally, consider the question: which comes first in time: the X or the O?

Well, in Earth Twin's RF, the X comes before the O, because the O appears above the "$t =$ 4" line on the diagram. But according to Space Twin's RF, the O comes before the X, because the X appears above the "$t' = 4$" line in the diagram. Both RF's are equally good. So: *each* twin's clock strikes 4 *after* the other clock does, as measured by the other twin.

### *B. The Two-Meeting Case, from Earth's Reference Frame*

Now to the puzzling case: what happens if Space Twin comes back to Earth, and meets his brother—who will be older at that point?

To show precisely what STR predicts about this, we need to fill in some details of the scenario. Let's say that Space Twin took off away from Earth at .5*c*. Ten years later (as measured from Earth's RF), Space Twin turned around and headed back at .5*c*, until he got back to Earth. So, on Earth, 20 years elapse while Space Twin is away. Figure 2 shows how things look from Earth's frame of reference.

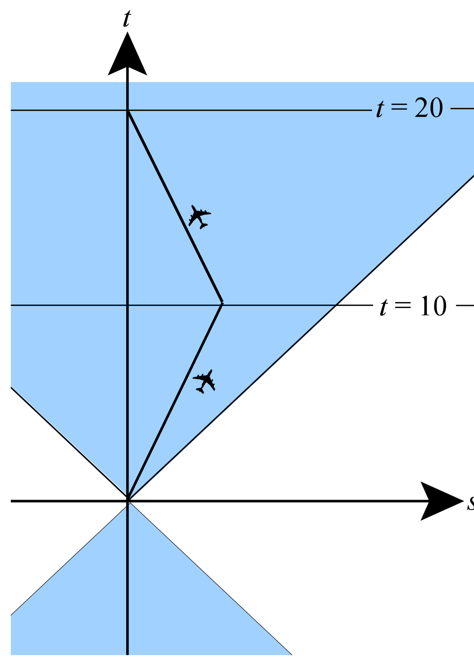

**Figure 2**

Earth Twin just sits around on the *t* axis, while Space Twin takes the dotted path labeled ✈. During his trip *away* from Earth (the first ten years), Space Twin's velocity as measured from Earth is .5*c*. The time that this half of the trip takes, as measured from Earth, is 10 years. So the Lorentz time dilation equation tells us:

$$T = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \Rightarrow \quad 10 = \frac{T_0}{\sqrt{1 - \frac{(.5c)^2}{c^2}}} \quad \Rightarrow \quad T_0 = 5\sqrt{3} \approx 8.66 \ yrs.$$

Note that I put in 10 for T, because 10 years is the time that that part of the trip takes as measured by a reference frame in which Space Twin is *not* at rest but is moving at velocity *v* (remember that that's what we said "T" stood for in the equation in section I). What we're solving for is $T_0$, the time that elapses according to a reference frame in which the path is stationary (again, that's what we said "$T_0$" stood for in section I). Now, since Space Twin ages 8.66 years during the first half of his trip (because that's how much time passes in his reference frame), he'll age twice that much in the whole trip, or **17.32 years**. So Earth Twin, having aged 20 years, winds up looking older.

### *C. The Two-Meeting Case, from Space Twin's Original Reference Frame*

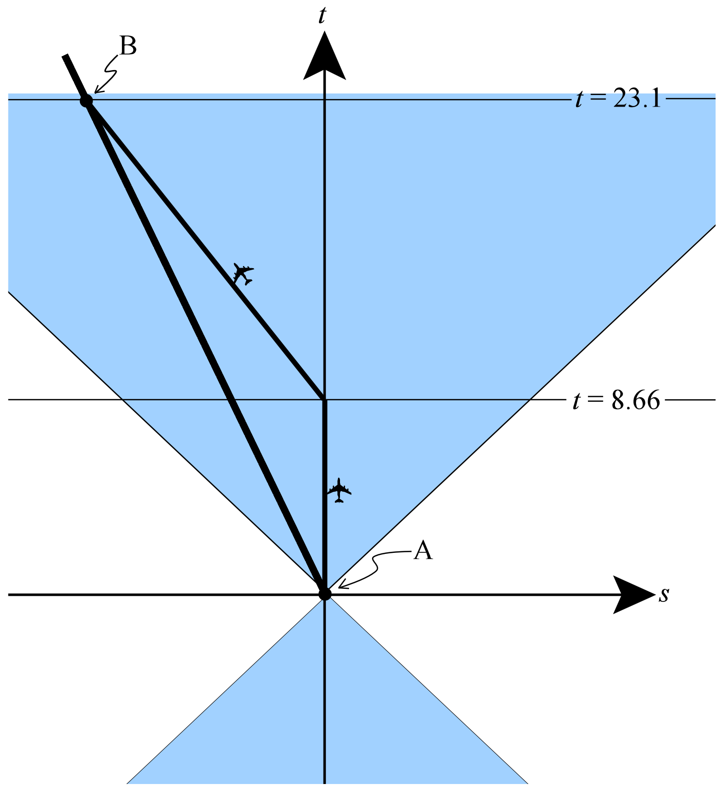

**Figure 3**

To see whether the Theory of Relativity is consistent, let's examine how things look from the reference frame in which the Earth is moving to the left at .5*c*. In this RF, Space Twin starts out stationary while Earth moves away from him at .5*c*. The Earth is in uniform motion (moving inertially), so it stays at .5*c* throughout the story, according to this RF. However, at some point Space Twin (again designated by the ✈) decides he wants to meet up with Earth Twin again, so he fires his rockets and goes extra fast to *catch up* to the Earth, which has been zooming away all this time. Figure 3, then, is how things look from the standpoint of this RF. Point A is the point where Earth Twin and Space Twin first part ways. Point B is the point where they reunite.

Note that there is *no* reference frame in which Space Twin is at rest throughout the story: he can be moving to the right for the first half of the story then moving to the left for the second half, or he can be stationary at first and then move to the left, or he can be moving to the right at first and then stop. But he cannot be stationary throughout the story, according to any legitimate (inertial) RF. That's because, in relativity, even though there's no objective fact about how fast one is moving at any time, there *is* an objective fact about whether one accelerates, and at the point at which Space Twin is firing his rockets, he is accelerating (or decelerating)—he's either doing it to *turn around*, or to move to the left to *catch up* to the Earth, or to *slow down and stop* so the Earth can catch up to him (this would be in the RF that has Earth and Space Twin both start out moving to the right, and then Space Twin stops).

Now, the following, I claim, is what happens according to the reference frame of Figure 3:

> Earth travels to the left at .5*c* for 23.1 years. Space Twin sits on his spaceship, stationary, for 8.66 years. Then he fires up his rockets and travels at .8*c* to the left for 14.44 years (= 23.1 yrs. - 8.66 yrs.), after which he finally catches up to the Earth.

To see that everything is consistent (or at least the story about who's aging more than whom), let's see how much each twin should age according to this story.

How much does Earth Twin age between point A and point B?

$$T = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \Rightarrow \quad 23.1 = \frac{T_0}{\sqrt{1 - \frac{(.5c)^2}{c^2}}} \quad \Rightarrow \quad T_0 = 23.1\sqrt{\frac{3}{4}} \approx 20 \ \mathit{yrs.}$$

Note that T is 23.1, because T is the time between A and B according to an RF in which the path between them is *not* stationary. $T_0$ is the time between A and B according to an RF in which the path is stationary. The result that $T_0$ = 20 years is what we want, because we already said (in section B above) that Earth Twin ages 20 years. (Don't get too excited about this, though—I got the 23.1 figure in the first place by solving the above equation in reverse.)

The real test is to see what happens to Space Twin—how much will *he* age between point A and point B? Well, during the first part of the story, while he's stationary, he ages 8.66 years (there's no time dilation to calculate, since according to this RF, Space Twin is stationary during this time). Now what happens is that during the second part of the story—the part where Space Twin is moving to the left at .8*c*—he's going to age extra slowly, so that he'll wind up younger than Earth Twin at the end. Before calculating that, though, you might be wondering where I got the .8*c* figure. Well, Space Twin is going to catch up to the Earth after 23.1 years. He's already wasted 8.66 years just sitting around. So he's going to have to cover that distance in the remaining 14.44 years. Now, the spatial distance he has to cover is just the distance that the Earth covers in the 23.1 years that pass between point A and point B. That spatial distance is given by the formula:

$$\mathit{distance} = \mathit{velocity} \times \mathit{time} = (.5c) \times (23.1) = 11.55c$$

So Space Twin is going to have to go a distance of 11.55*c* in 14.44 years. So his speed will have to be:

$$\mathit{velocity} = \frac{\mathit{distance}}{\mathit{time}} = \frac{11.55c}{14.44} \approx .8c$$

Now, how much will Space Twin age during this second part of our story? He's going at .8*c* for 14.44 years. So the Time Dilation equation tells us:

$$T = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \Rightarrow \quad 14.44 = \frac{T_0}{\sqrt{1 - \frac{(.8c)^2}{c^2}}} \quad \Rightarrow \quad T_0 = 14.44\sqrt{.36} \approx 8.66 \textit{ yrs.}$$

Note that once again, we're solving for $T_0$, since we're interested in how much time passes on the spaceship, according to the RF in which the spaceship is at rest; that gives us how much the people on the ship age.

Now, the total amount that Space Twin ages is:

$$\frac{\textit{aging during first}}{\textit{part of story}} + \frac{\textit{aging during second}}{\textit{part of story}} = 8.66 + 8.66 = \mathbf{17.32\ yrs.}$$

This is exactly the amount of aging that was predicted using the first reference frame, in section B above. So the whole story is consistent. Space Twin ages 17.32 years, while Earth Twin ages 20 years, no matter which reference frame you use.

### *D. The Two-Meeting Case, from Space Twin's Final Reference Frame*

I mentioned that there's a third reference frame we could use: that's the RF in which Space Twin is stationary in the *second* half of the story. This RF looks like this (see Figure 4).

Here, the calculations are pretty much the same as in section C above, so I won't bother repeating them. Here's what happens according to this third RF:

> Earth travels to the right at .5*c* for 23.1 years. At first, Space Twin is traveling to the right at .8*c*. He does this for 14.44 years, after which he fires his rockets to *stop* and let the Earth catch up, which it does after an additional 8.66 years. During the first part of the story, even though

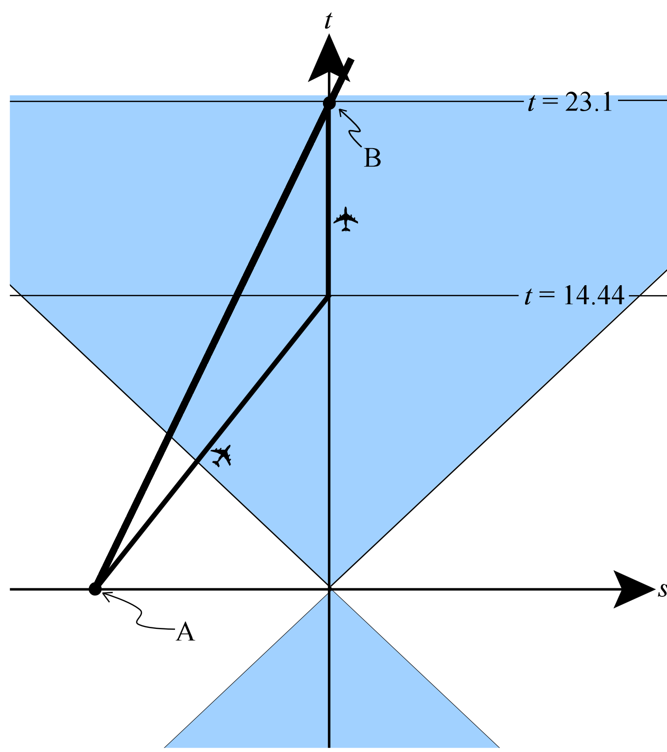


**Figure 4**

14.44 years pass, Space Twin only ages 8.66 years because he's going so fast (.8*c*). During the second part of the story, he ages another 8.66 years, for a total of 17.32 years. Also, although the whole story (between point A and point B) takes 23.1 years, people on Earth only age 20 years due to the fact that they're moving at .5*c*.

And that's it. Whichever of the 3 RF's you choose, you get the same result as to how much Earth Twin ages and how much Space Twin ages—even though the RF's disagree about almost everything else (e.g., who was moving at any given point in time, how much time passed between point A and point B, and what the spatial distance between point A and point B is). STR is designed to work this way. Anything that's clearly observable, all the RF's will agree on. How much someone has aged is observable. How much time has "really" passed, however, is not. That's why STR is designed to make the former *invariant* (it comes out the same no matter what RF you use), while the latter is not (it varies between reference frames).