Nonlinear Interaction of Nonconcentric Spherical Sound Waves in an Ideal Fluid

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Presented by Academician A.M. Dykhne on August 19, 1994

Received August 23, 1994

Quasilinear approximation is one of the simplest methods of perturbation theory used in nonlinear acoustics. Its idea is to find corrections ("secondary sound field") to the linear ("primary") field. These corrections are calculated as second-order terms of the direct asymptotic expansion of the sound field in the acoustical Mach number series. Because of its simplicity, the quasilinear approximation is widely used when the nonlinear effects *per se* are sufficiently weak. However, currently known exact solutions obtained within the framework of the quasilinear approximation are scarce; they are limited to quasi-one-dimensional cases (including the problems of interaction of plane [1], collinear cylindrical and spherical waves [2]) and the problem of interaction of plane and spherical waves [3].

In this paper, we present an exact (within the framework of the quasilinear approximation) solution to the problem of interaction of two nonconcentric harmonic spherical waves in an unbounded ideal fluid. Its importance for the theory of quasilinear approximation is due to two circumstances. First, this solution is more general than the previously known three dimensional solutions [2, 3]. Second, it can be used to write out a solution to the problem of interaction of arbitrary stationary sound fields in an ideal fluid in terms of integrals over spatially distributed sources of the primary (linear) sound field, thus being a quasilinear-approximation counterpart of the conventional Green's function.

In addition, the exact solution presented below has an unexpectedly simple form (although it may describe a rather complicated spatial structure).

Consider two spherical harmonic sound waves $p_1(\boldsymbol{r})\exp(-i\omega_1 t)$ and $p_2(\boldsymbol{r})\exp(-i\omega_2 t)$:

$$
\begin{aligned}
p_1 &= A_1 \frac{\exp(ik_1|\boldsymbol{r}-\boldsymbol{r}_{10}|)}{k_1|\boldsymbol{r}-\boldsymbol{r}_{10}|},\\
p_2 &= A_2 \frac{\exp(ik_2|\boldsymbol{r}-\boldsymbol{r}_{20}|)}{k_2|\boldsymbol{r}-\boldsymbol{r}_{20}|}.
\end{aligned}
\tag{1}
$$

Their interaction gives rise to new components $p_+(\boldsymbol{r})\exp(-i\omega_+ t)$ and $p_-(\boldsymbol{r})\exp(-i\omega_- t)$ of the sound field with sum and difference frequencies, otherwise referred to as the secondary sound field. For definiteness, we will tentatively consider the sum-frequency field p_+ only.

It is well known [2] that the secondary sound field resulting from the interaction of the two stationary components p_1 and p_2 of the primary field can be found by solving a nonuniform Helmholtz equation with the product of the primary-waves in its right-hand side (the Westervelt equation):

$$(\Delta + k_+^2)p_+ = \beta k_+^2 \frac{p_1 p_2}{\rho c^2}, \tag{2}$$

where k_+ is the sum-frequency wavenumber, β is the nonlinear parameter of the medium, ρ is the density and c is the sound velocity in the fluid. The full expression for the secondary field (formally obtained in the quasilinear approximation) differs from the solution to equation (2) only by "nonaccumulating" (i.e. virtually negligible) terms which can be written out explicitely [2]. Therefore, we can assume, for simplicity, that the required expression for p_+ is provided by a solution to equation (2), which can be written in terms of the spatial integral Green's function as the following spatial integral:

$$p_+ = -\frac{\beta A_1 A_2}{\rho c^2}\frac{k_+^2}{k_1 k_2}\iiint dV \frac{\exp(ik_1|\boldsymbol{r}' - \boldsymbol{r}_{10}|)}{|\boldsymbol{r}' - \boldsymbol{r}_{10}|}\frac{\exp(ik_2|\boldsymbol{r}' - \boldsymbol{r}_{20}|)}{|\boldsymbol{r}' - \boldsymbol{r}_{20}|}\frac{\exp(ik_+|\boldsymbol{r}' - \boldsymbol{r}|)}{4\pi|\boldsymbol{r}' - \boldsymbol{r}|}. \tag{3}$$

Thus, the problem has reduced to the calculation of the integral (3) subject to the condition

$$k(\omega) = \omega / c. \tag{4}$$

which corresponds to the case of ideal fluid. At this point, we should specify that the formally impeccable method of calculation of the integral (3) by expanding the spherical waves contained therein in terms of cylindrical functions is very tedious. Therefore, we will use a different method of finding the solution for p_+, which is more elegant, albeit not so impeccable from the formal point of view. We seek the desired solution through an analysis of the asymptotic expression for p_+, obtained in [4] assuming that $k_1 a \gg 1$ $k_2 a \gg 1$ ($a \equiv |\boldsymbol{r}_{10} - \boldsymbol{r}_{20}|$ is the distance between the sources):

$$\begin{aligned} p_+ \sim A & \frac{\exp\Big(ik_1 r_1 + ik_2 r_2 - i\frac{\theta^2}{2}\frac{k_1 r_1 k_2 r_2}{k_1 r_1 + k_2 r_2}\Big)}{k_1 r_1 + k_2 r_2} \\ & \times \left[\mathrm{Ei}\left(i\frac{\theta^2}{2}\frac{k_1 r_1 k_2 r_2}{k_1 r_1 + k_2 r_2}\right) - \mathrm{Ei}\left(i\frac{\theta^2}{2}\frac{k_1 r_1 k_2 r_2}{k_1 r_1 + k_2 r_2} + i\frac{\theta^2}{2}\frac{k_2 r_1 r_2}{r_2 - r_1}\right)\right], \quad r_2 > r_1. \end{aligned} \tag{5}$$

Here, $A \equiv \frac{\beta A_1 A_2}{\rho c^2}\frac{ik_+^2}{2k_1 k_2}$, $\mathrm{Ei}(x)$ is the exponential integral, θ is the angle subtended by the segment $[\boldsymbol{r}_{10}, \boldsymbol{r}_{20}]$ with its vertex at the observation point; $\boldsymbol{r}_1 \equiv |\boldsymbol{r} - \boldsymbol{r}_{10}|$, $\boldsymbol{r}_2 \equiv |\boldsymbol{r} - \boldsymbol{r}_{20}|$ (Fig. 1). This expression is derived under the condition

$$k_+ = k_1 + k_2, \tag{6}$$

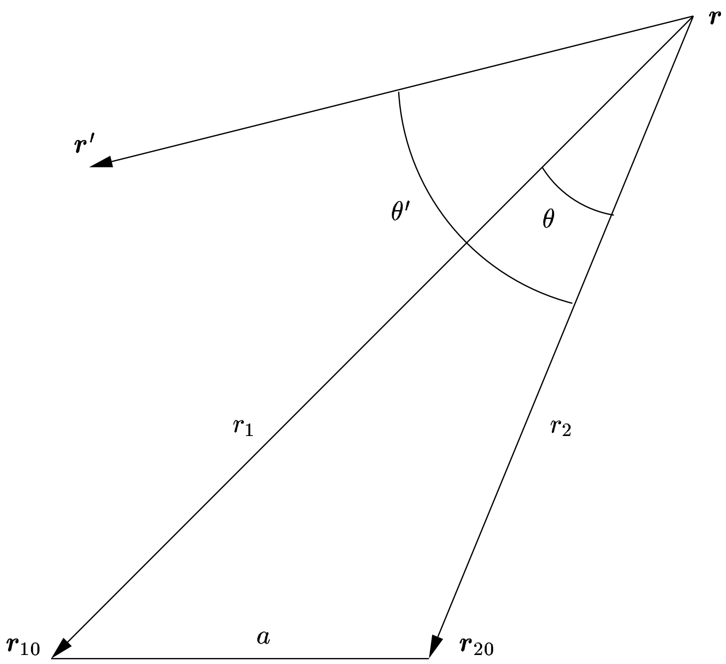

Figure 1: The geometry of the problem.

which extends somewhat beyond the framework of the ideal-fluid model implied by 4, since it admits the existence of an absorption term linearly depending on frequency. However, this extension does not have any specific physical implication.

Formula (5) carries a "trace" of the coordinate system used in [4] to perform an approximate calculation of the integral (3). It was a spherical coordinate system with its origin $r' = 0$ at the observation point $\boldsymbol{r}$ and the polar axis $\theta' = 0$ passing through the primary wave source closest to the observation point; for definiteness, we assumed it to be the source located at $\boldsymbol{r}_{10}$. We can try to rewrite (5) in a vectorial form invariant with respect to the coordinate system. To do so, we note that

$$k_1 r_1 + k_2 r_2 - |k_1 \boldsymbol{r}_1 + k_2 \boldsymbol{r}_2| = \frac{\theta^2}{2} \frac{k_1 r_1 k_2 r_2}{k_1 r_1 + k_2 r_2} + O(\theta^4),$$
$$r_1 + a - r_2 = \frac{\theta^2}{2} \frac{k_2 r_1 r_2}{r_2 - r_1} + O(\theta^4).$$

Accordingly, we can rewrite (5) as

$$p_+ \sim A \frac{\exp(iR)}{R} \{ \mathrm{Ei}[i(R_+ - R)] - \mathrm{Ei}[i(R_1 - R)] \}, \qquad r_2 > r_1, \tag{7}$$

where

$$R \equiv |k_1 \boldsymbol{r}_1 + k_2 \boldsymbol{r}_2|, \quad R_+ \equiv k_1 r_1 + k_2 r_2, \quad R_1 \equiv k_+ r_1 + k_2 a.$$

When $r_2 \to \infty$ and r_1 fixed, this expression (upon an appropriate normalization) takes the form of the already known exact solution to the problem of interaction of a plane wave and a spherical wave [3]. It retains the assymetry with respect to the simultaneous permutation $k_1 \leftrightarrow k_2$, $\boldsymbol{r}_1 \leftrightarrow \boldsymbol{r}_2$. However, it is easy to see that the second term in the braces is significant at $k_2 a \gg 1$ only in the vicinity of the semiaxis $\theta = 0$, $\quad r_2 > r_1$, and $r_2 > a$; outside this region, it is much smaller than the first term. This allows us to symmetrize expression (7) with the respect to the sources of the primary field:

$$p_+ \sim A \frac{\exp(iR)}{R} \{\mathrm{Ei}[i(R_+ - R)] - \mathrm{Ei}[i(R_1 - R)] - \mathrm{Ei}[i(R_2 - R)]\}, \tag{8}$$

where $R_2 \equiv k_+ r_2 + k_1 a$.

This expression becomes an exact solution to the problem of interaction between a plane wave and a spherical wave not only when $r_2 \to \infty$ and r_1 is fixed but also when $r_1 \to \infty$ and r_2 is fixed. Furthermore, it also has a qualitatively new property [as compared to (5) and (7)] in that it does not have a singularity as $a \to 0$. A closer scrutiny of the expression (8) shows that its value at $a \to 0$ and (for simplicity) $r \to \infty$ coincides with the corresponding solution to the problem of interaction of concentric spherical waves [2]:

$$p_+ \sim -A \frac{\exp(ik_+ r)}{k_+ r} \left[\ln(2ik_+ r) + C + O(k_+ a) + O(1/k_+ r)\right], \qquad a \to 0, \ r \to \infty, \tag{9}$$

where $C = 0.577...$ is the Euler constant.

From the formal point of view, the change from (5) to (7) and (8) is not quite substantiated due to the lack of estimates of accuracy for (5), (7) and (8). However, the form of the expression (8) and its agreement with the known solutions for $a \to 0$ and $a \to \infty$ suggest that it may be "close" to the exact solution to the problem [expression (8) *per se* does not satisfy the initial equation (2)].

To make the final step, we can use the following argument. Only the real part of the expression obtained for p_+ is physically meaningful. Consequently, sign reversal in all wavenumbers must result in a complex conjugate expression. However, as a result of the reversal, the expression in the braces in formula (8) behaves at $r \to \infty$ as r^{-1}, in contrast to the $\ln r$ behavior observed previously.

Exploration in this direction leads to a final expression for p_+:

$$\boxed{\begin{aligned} &p_+ = f(R) + f(-R), \\ &f(R) \equiv A \frac{\exp(iR)}{R} \{\mathrm{Ei}[i(R_+ - R)] - \mathrm{Ei}[i(R_1 - R)] - \mathrm{Ei}[i(R_2 - R)]\}. \end{aligned}} \tag{10}$$

Straightforward substitution shows that, under condition (6), this expression exactly satisfies equation (2). However, to recognize it as a physically valid solution, one must ensure that it comprises outgoing waves only.

The solution (10) does not contain any incoming waves; to be more precise, their amplitude falls off, as $R \to \infty$, more rapidly than any power of the distance. This behavior, which is also characteristic of other solutions obtained in the quasilinear approximation [2], signifies the necessity of refining the concepts of incoming and outgoing waves for a space with unbounded distributed sources; physically, there obviously exist incoming waves from the region of $r' > r$. Consequently, there is a hypothetical possibility of a situation such that certain groups of sources generate incoming waves, which mutually annihilate at infinity. Therefore, it appears that we can obtain a formal proof of the fact that expression (10) is equivalent to the integral (3) only by direct calculation of the integral.

The solution (10) "absorbs" the aforementioned solutions to the problems of interaction of two concentric spherical waves (in the limit $a \to \infty$) and between a plane wave and a spherical wave (in the limit $r_1 \to \infty$ or $r_2 \to \infty$, with appropriate normalization).

Writing the solution (10) in a somewhat different form,

$$
\begin{aligned}
&p_+ = g(R_+) - g(R_1) - g(R_2),\\
&g(R_i) \equiv A\frac{\exp(iR)}{R}\,\mathrm{Ei}[i(R_i - R)] - A\frac{\exp(-iR)}{R}\,\mathrm{Ei}[i(R_i + R)],
\end{aligned}
\tag{11}
$$

we note that substituting $g(R_+)$ into equation (2) yields the correct right-hand side, whereas $g(R_1)$ and $g(R_2)$ satisfy the uniform Helmholtz equation. However, $g(R_+)$ has a singularity when $R = R_+$, i.e. on the line passing through the points r_1 and r_2, except for its segment contained between these points. This singularity is canceled out by setting $g(R_1)$ (at $r_2 > r$) and $g(R_2)$ (at $r_1 > r$).

In most practical applications, we may ignore the second term on the right-hand side of (10), due to its rapid decrease (as r^{-2}) and restrict ourselves to expression (8).

An alternative simplification is often possible. In the region where "accumulation" effects are important, "fast" oscillations of the solution, due to variations in the position of the observation point, are contained in the factor $\exp(iR)$. Therefore, when expression (8) is implemented in numerical algorithms (e.g., in calculations of directivity patterns of radiating parametric arrays), far-field computations do not require calculation of the factor given by expression in the braces for each position of the observation point.

The simplest way of transition from solution (10) to the difference-frequency field is to make the formal changes $k_1 \to -k_1^*$ (or $k_2 \to -k_2^*$), $k_+ \to k_-$, $p_+ \to p_-$, appropriately modifying the time factor. As in the case of sum-frequency field, the fact that the denominator in (10) vanishes at $R = 0$ does not mean singularity of the whole expression.

We emphasize that the expression (10) may be considered as a solution to the physical problem only with certain reservations. These concern the free-space model used here (a real source always occupies a finite volume) and the assumption of weakness of the nonlinear effects. Whereas the former limitation is important only for closely spaced spaced sources of dimensions approximately equal to or greater than the wavelength, the latter one also applies to "point" sources due to the mutual modulation of their sound fields. Such modulation occurs when the distance between the sources is sufficiently short, irrespective of their type (mass or volume sources) and the choice of spatial variables (in terms of Lagrangian or Eulerian coordinates).

In conclusion, note that the solution presented above for the case of sound waves in an ideal fluid can be extended to other cases of wave interaction in media with quadratic nonlineriaty.

The author would like to thank S.A. Rybak for attention to this work.

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