Scattering of sound by a vortex filament

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Abstract

An angularly uniform asymptotic expansion of the sound field at large distances in the scattering of a plane harmonic sound wave by a straight vortex filament is derived on the basis of the Born approximation. The resulting expression describes both the scattered sound field and the entrainment of the incident wave in the velocity field of the vortex.

The scattering of sound by a straight vortex filament, or two-dimensional point vortex, poses one of the standard problems of the theory of sound scattering by hydrodynamic flows. It has been discussed many times in literature, usually within the framework of the Born approximation. Pitaevskii [1] has calculated the amplitude of scattering of a plane sound wave by a vortex filament and used the resulting expression to estimate the phonon part of the mutual friction force in liquid helium. O'Shea [2] has investigated the two-dimensional problem of the scattering of the field of a point source by a vortex filament. He showed that the scattered sound field can be represented by the sum of two components, one of which describes the entrainment of the incident wave in the velocity field of the vortex filament[1], while the second characterizes the scattered field "proper" (subsequent references to the "scattered" field will be understood to mean this component specifically). Such an interpretation of the components of the sound field is possible, based on the analysis of their directions of propagation and on the way in which the field decays with distance. A result similar to Pitaevskii's has been obtained by Ferziger [4] in a study of the problem of determining the low-frequency asymptotic behavior of the sound field scattered by an Oseen vortex[2].

[1]The author mentions the refraction of sound waves; however, it is a well-known fact [Ref. 3, p. 373 (Russ.ed)] that sound refraction does not take place in a potential velocity field

[2]It can be shown that the solution of problems in sound scattering by a point vortex can serve as the low-frequency asymptotic form of the sound field in sound scattering by an arbitrary localized two-dimensional vortex as long as a critical layer is not present

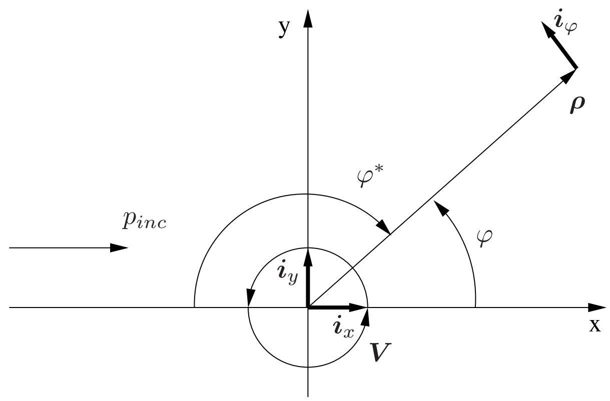

Figure 1: Geometry of the problem (in the xy plane)

Nonetheless, the asymptotic expressions obtained for the scattered field by these and other [5, 6] authors have a major shortcoming: they are singular in the forward direction. This is a consequence of using the farfield asymptotic representation of the Green's function in carrying out the integration over the secondary field sources. However, the "farfield" concept itself is meaningless in solving the given problem, owing to the long-range character of the scattered field. As a result, the dimensions of the scattering region shrink implicitly (even though the integration itself is carried out between infinite limits).

Just how substantial are the "losses" in this approach? The first consideration is forfeiture of the principal (in magnitude) term of the asymptotic expansion of the sound field at large distances, which is the term characterizing "entrainment" of the incident wave in the potential velocity field of the vortex filament (the phase distortions of its front are disregarded, because they do not alter the direction of propagation in the given case – this being possible by virtue of the anisotropy of the scattering system). Second (and because of this), the singularity obtained in the expression for the scattered field is misinterpreted as resulting from the slow decay of the velocity field of the vortex with increasing distance. Its very existence is attributed to the model character of the system. However, it will be shown that not even in the model situation does the scattered field have a singularity at infinity[3].

Let us a straight vortex filament exist and coincide with the z axis of a Cartesian coordinate system (Fig. 1). We write the velocity $\boldsymbol{V}$ of the fluid in the field of the vortex in terms of the wave number k of a plane sound wave

[3] We mention an alternative approach to the solution of problems in sound scattering by localized vortices: The incident plane wave is expanded in cylindrical (or spherical) harmonics, and the problem is solved separately for each partial model (see, e.g., [7, 8]). For the above-stated reasons, when this approach is used, not only the scattering of lower harmonics, but also of higher (in general, all) harmonics must be taken into account in order to calculate the entrainment of the incident wave in the presence of potential flow around the localized vortex core, and this is not usually done

incident on the filament:

$$\boldsymbol{V} = \frac{\varepsilon c_0}{k\rho}\boldsymbol{i}_\varphi, \tag{1}$$

$$p_{inc} = \exp(ik_x x + ik_z z - i\omega t), \tag{2}$$

where c_o is the sound velocity, ε is a small parameter, ω is the sound frequency, $\rho = \sqrt{x^2+y^2}$, $\mathbf{i}_\varphi = (-y/\rho, x/\rho)$, and p_{inc} is the pressure in the wave incident on the filament.

We adopt the Lighthill's equation [9] as the basic equation:

$$\Box p = -\rho_0 \frac{\partial^2}{\partial x_i \partial x_j}(u_i u_j), \tag{3}$$

where p is pressure, ρ_0 – density, and $\boldsymbol{u}$ – velocity.

To find the secondary sound field, the Born approximation will be used:

$$\Box p_s = -2\rho_0 \frac{\partial^2}{\partial x_i \partial x_j} v_i V_j, \tag{4}$$

where $\boldsymbol{v}$ is the velocity in the unperturbed (incident) sound field.

Questions can arise here as to the correctness of this approximation in application to the given problem in view of the long-range character of the sources in the right-hand side of the Eq. (3) (they decay with the increasing distance as ρ^{-1}) and the existence of a singularity at $\rho = 0$. The detailed analysis of these questions is not part objective of the present study. We merely note that smallness of the secondary sound field in comparison with the incident field is sufficient for the approach to be correct in the first case. The correctness of ignoring the singularity at zero can be demonstrated by analyzing behavior of the solution of the differential equation for zeroth harmonic. Physically, by using Born approximation, we neglect scattering by the pressure "hole" at the center of the vortex, as well as scattering by the field density gradient created by pressure gradient and the compressibility of the fluid.

Taking into account the the two-dimensional potential character of the fluid velocity field (1) and the form of sound velocity field given by

$$\boldsymbol{v} = \frac{\boldsymbol{k}}{\omega\rho_0} p_{inc}, \qquad \boldsymbol{k} = (k_x, 0, k_z), \tag{5}$$

the Eq. (4) may be rewritten as:

$$(\Delta + k^2)\, p_s = -Q_\perp - Q_\parallel, \tag{6}$$

$$\begin{aligned} Q_\perp &= 2\rho_0 \frac{\partial}{\partial x}\Big[V_x\big(\frac{\partial}{\partial x}v_x + \frac{\partial}{\partial z}v_z\big)\Big], \\ &= -2i\varepsilon \frac{\partial}{\partial x}\left[\frac{y}{x^2+y^2}\exp(ik_x x)\right]\exp(ik_z z - i\omega t), \end{aligned} \tag{7}$$

$$\begin{aligned} Q_\parallel &= -2\rho_0 \big(\frac{\partial}{\partial x}V_x\big)\big(\frac{\partial}{\partial z}v_z\big) \\ &= 2i\varepsilon \sin^2(\theta_{inc})\frac{\partial}{\partial y}\Big[\frac{x}{x^2+y^2}\exp(ik_x x)\Big]\exp(ik_z z - i\omega t), \end{aligned} \tag{8}$$

where $V_x = (\boldsymbol{V}, \boldsymbol{i}_x)$, $u_x = (\boldsymbol{u}, \boldsymbol{i}_x)$, $u_z = (\boldsymbol{u}, \boldsymbol{i}_z)$, and $\sin^2(\theta_{inc}) = k_z^2/k^2$. Accordingly, we obtain the secondary sound field p_s in the form

$$p_s = p_\perp + p_\parallel, \tag{9}$$

$$p_{\perp,\parallel} = \frac{i}{4} \iint dx' dy' Q(x', y')_{\perp,\parallel} H_0^{(1)}(k_x |\boldsymbol{\rho'} - \boldsymbol{\rho}|), \tag{10}$$

where $H_o^{(1)}(x)$ is a Hankel function, and $\rho = (x, y)$. We calculate $p_\perp$ and $p_\parallel$ in succession by using the following integral representation for the Hankel function:

$$H_0^{(1)}(k_x |\boldsymbol{\rho'} - \boldsymbol{\rho}|) = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \frac{\exp[i\xi(y' - y) + i\sqrt{k_x^2 - \xi^2}|x' - x|]}{\sqrt{k_x^2 - \xi^2}} d\xi \tag{11}$$

Substituting Eqs. (7) and (11) in Eq. (10) and integrating by parts with respect to x'

$$\begin{aligned} p_\perp =& (-2i\varepsilon) \frac{i}{4} \frac{1}{\pi} \iint dx' dy' \frac{\partial}{\partial x'} \left(\frac{y'}{x'^2 + y'^2} \exp(ik_x x') \right) \\ & \qquad \int d\xi \frac{\exp[i\xi(y' - y) + i\sqrt{k_x^2 - \xi^2}|x' - x|]}{\sqrt{k_x^2 - \xi^2}} \\ =& \frac{\varepsilon}{2\pi} \iint d\xi dy' \left\{ \frac{y'}{x'^2 + y'^2} \exp(ik_x x') \frac{\exp[\dots]}{\sqrt{k_x^2 - \xi^2}} \bigg|_{x=-\infty}^{\infty} \right. \\ & \qquad \left. - \int dx' \frac{y'}{x'^2 + y'^2} \exp(ik_x x') i \,\mathrm{sgn}(x' - x) \exp[\dots] \right\}, \end{aligned}$$

[on the assumption that $\mathrm{Im}(k_x) = 0$], we obtain

$$p_\perp = -\frac{i\varepsilon}{2\pi} \iiint d\xi dx' dy' \frac{y' \,\mathrm{sgn}(x' - x)}{x'^2 + y'^2} \exp[ik_x x' + i\xi(y' - y) + i\sqrt{k_x^2 - \xi^2}|x' - x|] \tag{12}$$

[we omit the factor $\exp(ik_z z - i\omega t)$ everywhere]. After integrating first with respect to y',

$$I_y \equiv \int\limits_{-\infty}^{\infty} \frac{y' \exp(i\xi y')}{y'^2 + x'^2} dy' = \frac{1}{2} \int\limits_{-\infty}^{\infty} \left(\frac{1}{y' - i|x'|} + \frac{1}{y' + i|x'|} \right) \exp(i\xi y') dy'$$

(if $\xi \gtreqless 0$ then for convergence we need $\mathrm{Im}\, y'_{res} \gtreqless 0$, where y'_{res} – pole in which the integrand residue is calculated)

$$\begin{aligned} &= \pi i \,\mathrm{sgn}(\xi) \exp(-|\xi x'|); \\ p_\perp &= \frac{\varepsilon}{2} \iint d\xi dx' \; \mathrm{sgn}(\xi) \,\mathrm{sgn}(x' - x) \exp(ik_x x - i\xi y + i\sqrt{k_x^2 - \xi^2}|x' - x| - |\xi x'|). \end{aligned}$$

and then with respect to x' [on the assumption that $\mathrm{Im}(k_x) = 0$],

$$I_x \equiv \int\limits_{-\infty}^{\infty} \mathrm{sgn}(x'-x)\exp(ik_x x' - i\xi y + i\sqrt{k_x^2-\xi^2}|x'-x| - |\xi x'|)\, dx';$$

$$
\begin{aligned}
&x<0: \\
&\quad I_x = -\int\limits_{-\infty}^{x} \exp(ik_x x' - i\sqrt{k_x^2-\xi^2}(x'-x) + |\xi|x')dx' \\
&\qquad\qquad + \int\limits_{x}^{0} \exp(ik_x x' + i\sqrt{k_x^2-\xi^2}(x'-x) + |\xi|x')dx' \\
&\qquad\qquad + \int\limits_{0}^{\infty} \exp(ik_x x' + i\sqrt{k_x^2-\xi^2}(x'-x) - |\xi|x')dx' \\
&\qquad = -\frac{\exp(ik_x x' - i\sqrt{k_x^2-\xi^2}(x'-x) + |\xi|x')}{ik_x - i\sqrt{k_x^2-\xi^2} + |\xi|}\Bigg|_{-\infty}^{x} \\
&\qquad\qquad + \frac{\exp(ik_x x' + i\sqrt{k_x^2-\xi^2}(x'-x) + |\xi|x')}{ik_x + i\sqrt{k_x^2-\xi^2} + |\xi|}\Bigg|_{x}^{0} \\
&\qquad\qquad + \frac{\exp(ik_x x' + i\sqrt{k_x^2-\xi^2}(x'-x) - |\xi|x')}{ik_x + i\sqrt{k_x^2-\xi^2} - |\xi|}\Bigg|_{0}^{\infty} \\
&\qquad = -\frac{\exp(ik_x x + |\xi|x)}{ik_x - i\sqrt{k_x^2-\xi^2} + |\xi|} - \frac{\exp(ik_x x + |\xi|x)}{ik_x + i\sqrt{k_x^2-\xi^2} + |\xi|} \\
&\qquad\qquad + \frac{\exp(-i\sqrt{k_x^2-\xi^2}x)}{ik_x + i\sqrt{k_x^2-\xi^2} + |\xi|} - \frac{\exp(-i\sqrt{k_x^2-\xi^2}x)}{ik_x + i\sqrt{k_x^2-\xi^2} - |\xi|}; \\
&\quad \frac{-1}{ik_x - i\sqrt{k_x^2-\xi^2} + |\xi|} + \frac{-1}{ik_x + i\sqrt{k_x^2-\xi^2} + |\xi|} = -\frac{1}{|\xi|} + \frac{i}{k_x}, \\
&\quad \frac{-1}{ik_x + i\sqrt{k_x^2-\xi^2} + |\xi|} + \frac{-1}{ik_x + i\sqrt{k_x^2-\xi^2} - |\xi|} = \frac{|\xi|}{k_x(k_x + \sqrt{k_x^2-\xi^2})}; \\
&\quad I_x = \left(-\frac{1}{|\xi|} + \frac{i}{k_x}\right)\exp(ik_x x + |\xi|x) + \frac{|\xi|}{k_x(k_x + \sqrt{k_x^2-\xi^2})}\exp(-i\sqrt{k_x^2-\xi^2}x)
\end{aligned}
$$

$x > 0:$

$$
\begin{aligned}
I_x = &-\int\limits_{-\infty}^{0} \exp(ik_x x' - i\sqrt{k_x^2-\xi^2}(x'-x) + |\xi|x')dx'; \\
&- \int\limits_{0}^{x} \exp(ik_x x' - i\sqrt{k_x^2-\xi^2}(x'-x) - |\xi|x')dx' \\
&+ \int\limits_{x}^{\infty} \exp(ik_x x' + i\sqrt{k_x^2-\xi^2}(x'-x) - |\xi|x')dx' \\
= &-\frac{\exp(ik_x x' - i\sqrt{k_x^2-\xi^2}(x'-x) + |\xi|x')}{ik_x - i\sqrt{k_x^2-\xi^2} + |\xi|}\Bigg|_{-\infty}^{0} \\
&+ \frac{\exp(ik_x x' - i\sqrt{k_x^2-\xi^2}(x'-x) - |\xi|x')}{ik_x - i\sqrt{k_x^2-\xi^2} - |\xi|}\Bigg|_{0}^{x} \\
&+ \frac{\exp(ik_x x' + i\sqrt{k_x^2-\xi^2}(x'-x) - |\xi|x')}{ik_x + i\sqrt{k_x^2-\xi^2} - |\xi|}\Bigg|_{x}^{\infty} \\
= &-\frac{\exp(i\sqrt{k_x^2-\xi^2}x)}{ik_x - i\sqrt{k_x^2-\xi^2} + |\xi|} + \frac{\exp(i\sqrt{k_x^2-\xi^2}x)}{ik_x - i\sqrt{k_x^2-\xi^2} - |\xi|} \\
&- \frac{\exp(ik_x x - |\xi|x)}{ik_x - i\sqrt{k_x^2-\xi^2} - |\xi|} - \frac{\exp(ik_x x - |\xi|x)}{ik_x + i\sqrt{k_x^2-\xi^2} - |\xi|};
\end{aligned}
$$

$$
\frac{-1}{ik_x - i\sqrt{k_x^2-\xi^2} + |\xi|} + \frac{1}{ik_x - i\sqrt{k_x^2-\xi^2} - |\xi|} = -\frac{|\xi|}{k_x(k_x - \sqrt{k_x^2-\xi^2})},
$$

$$
\frac{-1}{ik_x - i\sqrt{k_x^2-\xi^2}x - |\xi|x} + \frac{-1}{ik_x + i\sqrt{k_x^2-\xi^2}x - |\xi|x} = \frac{1}{|\xi|} + \frac{i}{k_x};
$$

$$
I_x = -\frac{|\xi|}{k_x(k_x - \sqrt{k_x^2-\xi^2})}\exp(i\sqrt{k_x^2-\xi^2}x) + \left(\frac{1}{|\xi|} + \frac{i}{k_x}\right)\exp(ik_x x - |\xi|x);
$$

$$
I_x = -\frac{|\xi|\operatorname{sgn}(x)}{k_x\left[k_x - \sqrt{k_x^2-\xi^2}\operatorname{sgn}(x)\right]}\exp(i\sqrt{k_x^2-\xi^2}|x|) + \left[\frac{\operatorname{sgn}(x)}{|\xi|} + \frac{i}{k_x}\right]\exp(ik_x x - |\xi||x|),
$$

$$
\begin{aligned}
p_\perp = -\frac{\varepsilon}{2}\int\limits_{-\infty}^{\infty} \exp(-i\xi y)\Bigg\{ &-\frac{\xi\operatorname{sgn}(x)}{k_x[k_x - \sqrt{k_x^2-\xi^2}\operatorname{sgn}(x)]}\exp(i\sqrt{k_x^2-\xi^2}|x| \\
&+ \left[\frac{\operatorname{sgn}(x)}{\xi} + \frac{i\operatorname{sgn}(\xi)}{k_x}\right]\exp(ik_x x - |\xi||x|)\Bigg\} d\xi
\end{aligned}
$$

Eq. (12) yields:

$$p_\perp = I_1 + I_2 + I_3, \tag{13}$$

$$I_1 = \frac{\varepsilon}{2}\,\mathrm{sgn}(x)\exp(ik_x x)\int_{-\infty}^{\infty}\frac{1}{\xi}\exp(-i\xi y - |\xi x|)\,d\xi$$

$$= \frac{\varepsilon}{2}\,\mathrm{sgn}(x)\exp(ik_x x)\ \mathrm{v.p.}\!\int_{-\infty}^{\infty}\frac{1}{\xi}\exp(-i\xi y - |\xi x|)d\xi$$

$$= \frac{\varepsilon}{2}\,\mathrm{sgn}(x)\exp(ik_x x)\int_{0}^{\infty}\frac{1}{\xi}\exp(-\xi|x|)\left[\exp(-i\xi y) - \exp(i\xi y)\right]\,d\xi$$

$$= -i\varepsilon\,\mathrm{sgn}(x)\exp(ik_x x)\int_{0}^{\infty}\frac{\sin(\xi y)}{\xi}\exp(-\xi|x|)\,d\xi$$

$$= -i\varepsilon\,\mathrm{sgn}(x)\exp(ik_x x)\,\mathrm{arctg}\left(\frac{y}{|x|}\right)$$

$$= -i\varepsilon\,\mathrm{arctg}(\frac{y}{x})\exp(ik_x x), \tag{14}$$

$$I_2 = \frac{i\varepsilon}{2k_x}\exp(ik_x x)\int_{-\infty}^{\infty}\mathrm{sgn}(\xi)\exp(-i\xi y - |\xi x|)\,d\xi$$

$$= \frac{i\varepsilon}{2k_x}\exp(ik_x x)\left[\int_{0}^{\infty}\exp(-i\xi y - \xi|x|)\,d\xi - \int_{0}^{\infty}\exp(i\xi y - \xi|x|)\,d\xi\right]$$

$$= \frac{i\varepsilon}{2k_x}\exp(ik_x x)\left(\frac{-1}{-iy-|x|} - \frac{-1}{iy-|x|}\right)$$

$$= \frac{i\varepsilon}{2k_x}\exp(ik_x x)\left(\frac{1}{|x|+iy} - \frac{1}{|x|-iy}\right)$$

$$= \frac{\varepsilon y}{k_x\rho^2}\exp(ik_x x)$$

$$= \frac{\varepsilon\sin(\varphi)}{k_x\rho}\exp(ik_x x) = O[(k_x\rho)^{-1}], \tag{15}$$

$$I_3 = -\frac{\varepsilon}{2k_x}\int_{-\infty}^{\infty}\frac{k_x\,\mathrm{sgn}(x) + \sqrt{k_x^2-\xi^2}}{\xi}\exp[-i\xi y + i\sqrt{k_x^2-\xi^2}|x|]\,d\xi, \tag{16}$$

where φ is the polar angle (see Fig. 1). An exact expression can not be obtained for the last integral, and so it is necessary to restrict the calculation to its asymptotic form in the limit $(k_x\rho) \to \infty$. Its value in this case is determined by the presence of the saddle point $\xi_s = k_x \sin(\varphi)$ and the pole $\xi_p = 0$ at $x > 0$. An asymptotic expansion of the integral (16) in powers of $(k_x\rho)^{-1/2}$ can be obtained by standard methods (see, e.g., Ref. [10]). Here we confine the analysis to the usual error limits $O[(k_x\rho)^{-1}]$ for problems with cylindrical geometry [relying heavily on the relation $\mathrm{Im}(k_x) = 0$ in this case]:

$$
\begin{aligned}
I_3 &= -\frac{\varepsilon}{2k_x}\, v.p.\!\int_{-\infty}^{\infty} \frac{k_x\,\mathrm{sgn}(x) + \sqrt{k_x^2-\xi^2}}{\xi} \exp[-i\xi y + i\sqrt{k_x^2-\xi^2}|x|]\, d\xi \\
&= \frac{\varepsilon}{2k_x}\, v.p.\!\int_{-\infty}^{\infty} \frac{k_x\,\mathrm{sgn}(x) + \sqrt{k_x^2-\xi^2}}{\xi} \exp[i\xi y + i\sqrt{k_x^2-\xi^2}|x|]\, d\xi \\
&= \frac{\varepsilon}{2k_x}\left[I_s + 2\pi i k_x \theta(x) \exp(ik_x x)\,\mathrm{sgn}(\delta_s)\right], \\
I_s &= \int_{\Gamma_s} \frac{k_x\,\mathrm{sgn}(x) + \sqrt{k_x^2-\xi^2}}{\xi} \exp[i\xi y + i\sqrt{k_x^2-\xi^2}|x|]\, d\xi;
\end{aligned}
$$

$$
\begin{aligned}
&I_s \equiv \int\limits_{\Gamma_s} f(\xi)\exp[q(\xi)]\,d\xi,\\
&q(\xi) = i\xi y + i\sqrt{k_x^2-\xi^2}|x|,\\
&f(\xi) = \frac{k_x\,\mathrm{sgn}(x) + \sqrt{k_x^2-\xi^2}}{\xi},\\
&q'(\xi) = iy - \frac{i\xi|x|}{\sqrt{k_x^2-\xi^2}},\\
&q''(\xi) = -\frac{ik_x^2|x|}{(k_x^2-\xi^2)^{3/2}},\\
&\xi_s: \quad q'(\xi_s)=0,\\
&\xi_s = k_x\frac{y}{\rho} = k_x\sin(\varphi),\\
&q(\xi_s) = ik_x\rho \quad \text{(never crosses branch points } \xi=\pm k_x),\\
&q''(\xi_s) = -\frac{ik_x^2|x|}{k_x^3|\cos^3(\varphi)|} = -\frac{i\rho}{k_x\cos^2(\varphi)},\\
&f(\xi_s) = \frac{k_x[\mathrm{sgn}(x)+|\cos(\varphi)|]}{k_x\sin(\varphi)} = \mathrm{sgn}(x)\frac{1+\cos(\varphi)}{\sin(\varphi)} = \mathrm{sgn}(x)\frac{\cos(\varphi/2)}{\sin(\varphi/2)},\\
&\xi_p: \quad \lim_{\xi\to\xi_p} f(\xi_p)(\xi-\xi_p) = a = const,\\
&\xi_p = 0,\\
&a = k_x[1+\mathrm{sgn}(x)] = 2k_x\theta(x),\\
&h \equiv \sqrt{\frac{2}{|q''(\xi_s)|}}\exp[i\tau] = \sqrt{\frac{2k_x}{\rho}}|\cos(\varphi)|\exp(-i\pi/4),\\
&(\tau = -\pi/4),\\
&s_p \equiv \sqrt{|q(\xi_s)-q(\xi_p)|}\exp[i\arg(s_p)] = \sqrt{k_x\rho - k_x|x|}\exp[i\arg(s_p)],\\
&\arg\left(\frac{\xi_s-\xi_p}{s_s-s_p}\right) = \tau,\\
&\arg(s_p) = \pi + \arg[\sin(\varphi)] - \tau,\\
&\frac{a}{s_p} = \frac{2k_x\theta(x)}{k_x^{1/2}\sqrt{\rho-|x|}}\exp[-i\pi/4 - i\pi - \arg(\sin\varphi)] = -\sqrt{\frac{2k_x}{\rho}}\frac{\theta(x)\,\mathrm{sgn}(\varphi)}{|\sin(\varphi/2)|}\exp(-i\pi/4)\\
&\quad = -\sqrt{\frac{2k_x}{\rho}}\frac{\theta(x)}{\sin(\varphi/2)}\exp(-i\pi/4);
\end{aligned}
$$

$$I_s = \exp[q(\xi_s)]\left\{\pm ia\pi\exp(-s_p^2)\,\mathrm{erfc}(\mp is_p) + \pi^{1/2}\left[hf(\xi_s) + \frac{a}{s_p}\right]\right\} + O[(k_x\rho)^{-1}], \qquad \mathrm{Im}(s_p) \gtrless 0,$$

$$\mathrm{Im}(s_p) \lessgtr 0, \quad \varphi \gtrless 0,$$
$$\pm ia\pi \to -2i\pi k_x\theta(x)\,\mathrm{sgn}(y),$$
$$\exp(-s_p^2) = \exp(-ik_x\rho + ik_x x),$$
$$\mathrm{erfc}(\mp is_p) \to \mathrm{erfc}[i\,\mathrm{sgn}(y)\sqrt{k_x\rho - k_x|x|}e^{i\pi - i\arg[\sin(\varphi)] - i\tau}]$$
$$= \mathrm{erfc}[\exp(-i\pi/4)\sqrt{k_x\rho - k_x|x|}],$$
$$I_s = -2i\pi k_x\theta(x)\,\mathrm{sgn}(y)\exp(ik_x x)\,\mathrm{erfc}\left[\exp(-i\pi/4)\sqrt{k_x\rho - k_x x}\right] + \sqrt{\frac{2\pi k_x}{\rho}}\exp(ik_x x - i\pi/4)\frac{\cos(\varphi)\cos(\varphi/2) - \theta(x)}{\sin(\varphi/2)},$$
$$I_3 = -i\varepsilon\pi\theta(x)\,\mathrm{sgn}(y) exp(ik_x x)\,\mathrm{erfc}\left[\exp(-i\pi/4)\sqrt{k_x\rho - k_x x}\right] + i\varepsilon\pi\theta(x)\,\mathrm{sgn}(y)\exp(ik_x x) + \varepsilon\sqrt{\frac{\pi}{2k_x\rho}}\exp(ik_x\rho - i\pi/4)\frac{\cos(\varphi)\cos(\varphi/2) - \theta(x)}{\sin(\varphi/2)} + O[(k_x\rho)^{-1}],$$

$$I_3 = i\varepsilon\pi\theta(x)\,\mathrm{sgn}(y)\exp(ik_x x)\,\mathrm{erf}\left[\exp(-i\pi/4)\sqrt{k_x\rho - k_x x}\right] + \varepsilon\sqrt{\frac{\pi}{2k_x\rho}}\exp(ik_x x - i\pi/4)\frac{\cos(\varphi)\cos(\varphi/2) - \theta(x)}{\sin(\varphi/2)} + O[(k_x\rho)^{-1}]. \quad (17)$$

Here $\mathrm{erf}(x)$ is the probability integral, and $\theta(x)$ is the theta function, $\theta(x) = \{0,\ x \le 0;\ 1,\ x > 0\}$.

To calculate $p_\parallel$, an expansion of $H_0^{(1)}(k_x|\boldsymbol{\rho}' - \boldsymbol{\rho}|)$ analogous to (11) is used:

$$H_0^{(1)}(k_x|\boldsymbol{\rho}' - \boldsymbol{\rho}|) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\exp[i\xi(x' - x) + i\sqrt{k_x^2 - \xi^2}|y' - y|]}{\sqrt{k_x^2 - \xi^2}}\,d\xi. \qquad (18)$$

Substituting Eqs. (18) and (8) in Eq. (10) and integrating by parts in respect to y',

$$p_\parallel = (2i\varepsilon)\frac{i}{4}\frac{1}{\pi}\sin^2(\theta_{inc})\iint dx'dy'\exp(ik_x x')\frac{\partial}{\partial y'}\left(\frac{x'}{x'^2 + y'^2}\right)\int d\xi\frac{\exp[i\xi(x' - x) + i\sqrt{k_x^2 - \xi^2}|y' - y|]}{\sqrt{k_x^2 - \xi^2}}$$
$$= -\frac{\varepsilon\sin^2(\theta_{inc})}{2\pi}\iint d\xi dx'\left\{(\dots)\Big|_{y=-\infty}^{\infty} - \int dy'\exp(ik_x x')\frac{x'}{x'^2 + y'^2}i\,\mathrm{sgn}(y' - y)\exp[\dots]\right\},$$

we obtain

$$p_{\parallel} = \frac{i\varepsilon \sin^2(\theta_{inc})}{2\pi} \iiint d\xi dy' dx' \frac{x' \operatorname{sgn}(y'-y)}{x'^2 + y'^2}$$
$$\exp[ik_x x' + i\xi(x'-x) + i\sqrt{k_x^2 - \xi^2}|y'-y|]. \qquad (19)$$

After integrating first with respect to x'

$$I_x \equiv \int_{-\infty}^{\infty} \frac{x'}{x'^2+y'^2} \exp[i(k_x+\xi)x']dx' = \pi i \operatorname{sgn}(k_x+\xi)\exp(-|k_x+\xi||y'|),$$
$$p_{\parallel} = -\frac{\varepsilon \sin^2(\theta_{inc})}{2} \iint d\xi dy' \operatorname{sgn}(y'-y)\operatorname{sgn}(\xi+k)$$
$$\exp\left(-i\xi x - |k_x+\xi||y'| + i\sqrt{k_x^2-\xi^2}|y'-y|\right)$$

nd then with respect to y',

$$I_y \equiv \int_{-\infty}^{\infty} \operatorname{sgn}(y'-y)\exp\left(-|k_x+\xi||y'| + i\sqrt{k_x^2-\xi^2}|y'-y|\right)$$

$y < 0:$

$$I_y = \ldots = \frac{\operatorname{sgn}(\xi+k_x)}{k_x}\left[-\exp(|k_x+\xi|y) + \exp(-i\sqrt{k_x^2-\xi^2}y)\right]$$

$y > 0:$

$$I_y = \ldots = \frac{\operatorname{sgn}(\xi+k_x)}{k_x}\left[\exp(-|k_x+\xi|y) - \exp(i\sqrt{k_x^2-\xi^2}y)\right]$$

$$I_y = \frac{\operatorname{sgn}(\xi+k_x)}{k_x}\operatorname{sgn}(y)\left[\exp(-|k_x+\xi||y|) - \exp(i\sqrt{k_x^2-\xi^2}|y|)\right],$$
$$p_{\parallel} = \frac{\varepsilon \sin^2(\theta_{inc})}{2k_x}\operatorname{sgn}(y)\int_{-\infty}^{\infty}\left[-\exp(-|k_x+\xi|y) + \exp(i\sqrt{k_x^2-\xi^2}y)\right]\exp(-i\xi x)\,d\xi$$

Eq. (19) yields:

$$p_{\parallel} = I_4 + I_5, \qquad (20)$$
$$I_4 = -\frac{\varepsilon \sin^2(\theta_{inc})}{2k_x}\operatorname{sgn}(y)\int_{-\infty}^{\infty}\exp(-|\xi+k_x||y| - i\xi x)\,d\xi$$

$$
\begin{aligned}
I_\xi &\equiv \int\limits_{-\infty}^{\infty} \exp\left[(\xi+k_x)|y| - i\xi x\right] d\xi + \int\limits_{-k_x}^{\infty} \exp\left[-(\xi+k_x)|y| - i\xi x\right] d\xi \\
&= \frac{\exp\left[(\xi+k_x)|y| - i\xi x\right]}{|y| - ix}\Big|_{-\infty}^{-k_x} + \frac{\exp\left[-(\xi+k_x)|y| - i\xi x\right]}{-|y| - ix}\Big|_{-k_x}^{\infty} \\
&= \exp(ik_x x)\left(\frac{1}{|y|+ix} + \frac{1}{|y|-ix}\right) = \frac{2\sin(|\varphi|)}{\rho}\exp(ik_x x); \\
I_4 &= -\frac{\varepsilon \sin^2(\theta_{inc})}{2k_x}\operatorname{sgn}(y) I_\xi \\
&= -\frac{\varepsilon \sin^2(\theta_{inc})\sin(\varphi)}{k_x\rho}\exp(ik_x x) = O[(k_x\rho)^{-1}],
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
I_5 &= -\frac{\varepsilon \sin^2(\theta_{inc})}{2k_x}\operatorname{sgn}(y)\int\limits_{-\infty}^{\infty} \exp[i\sqrt{k_x^2-\xi^2}|y| - i\xi x]\, d\xi \\
&= \frac{\varepsilon\pi \sin^2(\theta_{inc})}{2k_x}\frac{\partial}{\partial y} H_0^{(1)}(k_x\rho).
\end{aligned} \tag{22}
$$

From Eqs. (13)–(17) and (20)–(22) and taking into account

$$
\pi\theta(x)\operatorname{sgn}(y) - \operatorname{arctg}(\frac{y}{x}) = \begin{cases} \pi-\varphi, & 0 \le \varphi \le \pi \\ \pi+\varphi, & -\pi \le \varphi \le 0, \end{cases}
$$

we finally obtain an expression for the perturbation of the sound field in the velocity field of the vortex filament:

$$
\boxed{
\begin{aligned}
p_s = {} & i\varepsilon\varphi^* \exp(ik_x x) \\
& + \varepsilon\sqrt{\frac{\pi}{2k_x\rho}}\exp(ik_x\rho - i\pi/4)\frac{\cos(\varphi)\cos(\varphi/2) - \theta(x)}{\sin(\varphi/2)} \\
& - i\varepsilon\pi\theta(x)\operatorname{sgn}(y)\exp(ik_x x)\operatorname{erfc}\left[\exp(-i\pi/4)\sqrt{k_x\rho - k_x x}\right] \\
& + \frac{\varepsilon\pi\sin^2(\theta_{inc})}{2k_x}\frac{\partial}{\partial y}H_0^{(1)}(k_x\rho) + O[(k_x\rho)^{-1}],
\end{aligned}
} \tag{23}
$$

where φ^* is the polar angle measured from the backward direction (see Fig. 1), $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$, and $|\varphi^*| < \pi$.

Equation (23) is the main result of the present study. Its first term describes the geometrical entrainment of the sound field in the velocity field of the vortex; to within a constant factor, it is equal to the integral of the projection of the local velocity of the field of the vortex filament onto the x axis, evaluated along $(-\infty, x]$ interval. Because of the nature of the filament velocity field, this term does not decay with increasing distance at a fixed angle φ. Moreover, it has a discontinuity on the axis $\varphi = 0$ ($\varphi^* = \pm\pi$), since the magnitude of the effect has opposite signs in reflection about the x axis.

The second term in Eq. (23) corresponds to the scattered field. Unlike the previous results, it has no singularity in the forward direction ($\varphi = 0$). However, it has a discontinuity at $x = 0$.

The third term cancels the discontinuity of the first component in the forward direction and discontinuity of the second component at $x = 0$.

$$
\begin{aligned}
&(a) \quad k_x(\rho - x) \to 0 : \\
&\quad - i\varepsilon\pi\theta(x)\,\mathrm{sgn}(y)\exp(ik_x x)\,\mathrm{erfc}\left[\exp(-i\pi/4)\sqrt{k_x\rho - k_x x}\right] \to -i\varepsilon\pi\,\mathrm{sgn}(y)\exp(ik_x x); \\
&\quad i\varepsilon\varphi^*\exp(ik_x x) \to i\varepsilon\pi\,\mathrm{sgn}(y)\exp(ik_x x). \\
&(b) \quad x \to +0, \quad k_x(\rho - x) \to \infty : \\
&\quad - i\varepsilon\pi\theta(x)\,\mathrm{sgn}(y)\exp(ik_x x)\,\mathrm{erfc}\left[\exp(-i\pi/4)\sqrt{k_x\rho - k_x x}\right] \\
&\qquad \to -i\varepsilon\pi\,\mathrm{sgn}(y)\frac{1}{\pi^{1/2}}\frac{\exp(ik_x\rho)}{\exp(-i\pi/4)\sqrt{k_x\rho}} \\
&\qquad = \varepsilon\sqrt{\frac{\pi}{k_x\rho}}\,\mathrm{sgn}(y)\exp(ik_x\rho - i\pi/4); \\
&\quad \varepsilon\sqrt{\frac{\pi}{2k_x\rho}}\exp(ik_x\rho - i\pi/4)\frac{\cos(\varphi)\cos(\varphi/2) - \theta(x)}{\sin(\varphi/2)} \\
&\qquad \to \varepsilon\sqrt{\frac{\pi}{2k_x\rho}}\exp(ik_x\rho - i\pi/4)\frac{-1}{1/2^{1/2}}\,\mathrm{sgn}(y) \\
&\qquad = -\varepsilon\sqrt{\frac{\pi}{k_x\rho}}\,\mathrm{sgn}(y)\exp(ik_x\rho - i\pi/4).
\end{aligned}
$$

It "links" the entrained and scattered fields, and that merely reflects the fact that the phenomena of entrainment of the sound field and forward scattering *per se* are indistinguishable because the corresponding components of the secondary field with spatial phase dependences of $\exp(ik_x x)$ and $\exp(ik_x\rho)$, correspondingly, propagate in the same direction.

The fourth term exists only in oblique incidence of the sound wave and, like the second term, refers to the scattered field.

This result casts doubts on the thesis that the Born approximation is invalid for calculating the sound field at small angles ($\varphi \to 0$) [2, 4, 5, 6].

The assumption adopted in the present article, $\mathrm{Im}(k_x) = 0$, is essential to the derivation of Eq. (23). This fact is physically attributable to the long-range character of the sources in Eq. (3): An arbitrary small exponential growth of the incident wave with distance will impart infinite amplitude to the perturbation calculated in the Born approximation.

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