Supplement to "Nonlinear Interaction of Nonconcentric Spherical Sound Waves in an Ideal Fluid" – check by direct substitution that Eq. (10) is a solution of Eq. (2)

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We will check that the following expression:

$$\boxed{\begin{aligned} p_+ = {} & \frac{\exp(iR)}{R}\left\{\mathrm{Ei}[i(R_+ - R)] - \mathrm{Ei}[i(R_1 - R)] - \mathrm{Ei}[i(R_2 - R)]\right\} \\ & - \frac{\exp(-iR)}{R}\left\{\mathrm{Ei}[i(R_+ + R)] - \mathrm{Ei}[i(R_1 + R)] - \mathrm{Ei}[i(R_2 + R)]\right\}, \end{aligned}} \tag{1}$$

$$\begin{aligned} & R \equiv |k_1 \boldsymbol{r}_1 + k_2 \boldsymbol{r}_2|, \\ & R_1 \equiv k_+ r_1 + k_2 a, \\ & R_2 \equiv k_+ r_2 + k_1 a, \\ & R_+ \equiv k_1 r_1 + k_2 r_2, \\ & \boldsymbol{r}_1 \equiv \boldsymbol{r} - \boldsymbol{r}_{10}, \\ & \boldsymbol{r}_2 \equiv \boldsymbol{r} - \boldsymbol{r}_{20}, \\ & a \equiv |\boldsymbol{r}_{10} - \boldsymbol{r}_{20}|, \end{aligned}$$

is a solution for the following inhomogeneous Helmholtz equation:

$$\boxed{(\Delta + k_+^2)p_+ = -2ip_1(\boldsymbol{r})p_2(\boldsymbol{r}),} \tag{2}$$

$$\begin{aligned} & p_1 \equiv \frac{\exp(ik_1|\boldsymbol{r} - \boldsymbol{r}_{10}|)}{|\boldsymbol{r} - \boldsymbol{r}_{10}|}, \\ & p_2 \equiv \frac{\exp(ik_2|\boldsymbol{r} - \boldsymbol{r}_{20}|)}{|\boldsymbol{r} - \boldsymbol{r}_{20}|}, \end{aligned}$$

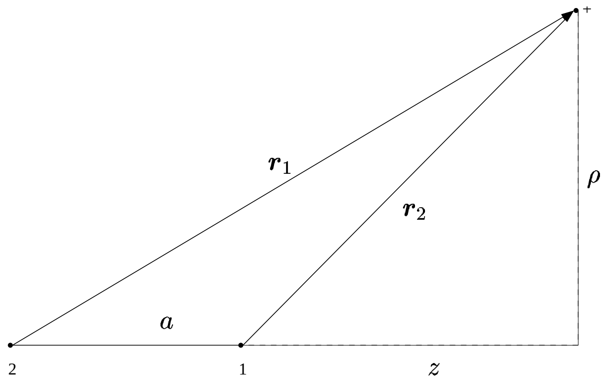

Figure 1: The geometry.

provided that

$$\boxed{k_+ = k_1 + k_2.}$$

These calculations will be made in cylindrical coordinate system shown in Fig. 1. In this system

$$
\begin{aligned}
&\boldsymbol{r} = 0, \\
&\boldsymbol{r}_1 = (z + a, \rho, 0), \\
&\boldsymbol{r}_2 = (z, \rho, 0), \\
&R = \sqrt{(k_+ z + k_2 a)^2 + (k_+ \rho)^2}, \\
&R_1 = k_+ \sqrt{z^2 + \rho^2} + k_2 a, \\
&R_2 = k_+ \sqrt{(z + a)^2 + \rho^2} + k_1 a, \\
&R = k_1 \sqrt{z^2 + \rho^2} + k_2 \sqrt{(z + a)^2 + \rho^2}.
\end{aligned}
$$

	∂_ρ	∂_z	∂_ρ^2	∂_z^2
R	$\frac{k_+^2 \rho}{R}$	$\frac{k_+ (k_z + k_2 a)}{R}$	$\frac{k_+^2 (k_+ z + k_2 a)^2}{R^3}$	$\frac{k_+^4 \rho^2}{R^3}$
R_1	$\frac{k_+ \rho}{r_1}$	$\frac{k_+ z}{r_1}$	$\frac{k_+ z^2}{r_1^3}$	$\frac{k_+ \rho^2}{r_1^3}$
R_2	$\frac{k_+ \rho}{r_2}$	$\frac{k_+ (z+a)}{r_2}$	$\frac{k_+ (z+a)^2}{r_2^3}$	$\frac{k_+ \rho^2}{r_2^3}$
R_+	$\frac{k_1 \rho}{r_1} + \frac{k_2 \rho}{r_2}$	$\frac{k_1 z}{r_1} + \frac{k_2 (z+a)}{r_2}$	$\frac{k_1 z^2}{r_1^3} + \frac{k_2 (z+a)^2}{r_2^3}$	$\frac{k_1 \rho^2}{r_1^3} + \frac{k_2 \rho^2}{r_2^3}$

$$(\partial_\rho R)^2 + (\partial_z R)^2 = k_+^2, \qquad \partial_\rho^2 R + \partial_z^2 R = \frac{k_+^2}{R},$$

$$(\partial_\rho R_1)^2 + (\partial_z R_1)^2 = k_+^2, \qquad \partial_\rho^2 R_1 + \partial_z^2 R_1 = \frac{k_+}{r_1},$$

$$(\partial_\rho R_2)^2 + (\partial_z R_2)^2 = k_+^2, \qquad \partial_\rho^2 R_2 + \partial_z^2 R_2 = \frac{k_+}{r_2},$$

$$(\partial_\rho R_+)^2 + (\partial_z R_+)^2 = k_+^2 - D, \qquad \partial_\rho^2 R_+ + \partial_z^2 R_+ = \frac{k_1}{r_1} + \frac{k_2}{r_2},$$

$$D \equiv \frac{2k_1k_2}{r_1r_2}\left[r_1r_2 - \rho^2 - z(z+a)\right] = \frac{k_1k_2}{r_1r_2}\left[a^2 - (r_1-r_2)^2\right].$$

$$p_+ \equiv (p_+^+ - p_1^+ - p_2^+) + (p_+^- - p_1^- - p_2^-)$$

$$p_i^+ \equiv \frac{\exp(iR)}{R}\,\mathrm{Ei}[i(R_i - R)], \qquad p_i^- \equiv \frac{\exp(-iR)}{-R}\,\mathrm{Ei}[i(R_i + R)], \qquad i = 1, 2, +.$$

We will proceed for p_i^+ (the sum frequency case) only; the expressions for p_i^- (difference frequency case) are very similar, one needs only to replace $R \to -R$.

$$\Delta = \frac{1}{\rho}\,\partial_\rho\,\rho\,\partial_\rho + \partial_z^2,$$

$$p_i \equiv g f_i, \qquad g = \frac{\exp(iR)}{R}, \qquad f_i = \mathrm{Ei}\left[i(r_i - R)\right];$$

$$\frac{1}{\rho}\,\partial_\rho\,\rho\,\partial_\rho\, p_i = \left(\frac{1}{\rho} g_\rho\right) + \left(\frac{1}{\rho} g + 2g_\rho\right) f_\rho + g f_{\rho\rho},$$

$$\partial_z^2\, p_i = g_z z f + 2 g_z f_z + g f_z z,$$

$$g_\rho = i\left(\frac{\partial\,\rho R}{R} - \frac{\partial\,\rho R}{R^2}\right)\exp(iR) = \left(i\,\partial_\rho - \frac{\partial\,\rho R}{R}\right)\frac{\exp(iR)}{R},$$

$$g_{\rho\rho} = \left[\frac{i\,\partial_\rho^2 R}{R} - \frac{i(\partial_\rho R)^2}{R^2} - \frac{\partial_r\, ho^2 R}{R^2} + \frac{2(\partial_\rho R)^2}{R^3}\right] + i\,\partial_\rho R\left(\frac{i\,\partial_\rho R}{R} - \frac{\partial_\rho R}{R^2}\right)\exp(iR) =$$

$$= \left[i\,\partial_\rho^2 R - (\partial_\rho R)^2 - \frac{2i(\partial_\rho R)^2}{R} - \frac{\partial_\rho^2 R}{R} + \frac{2(\partial_\rho R)^2)}{R^2}\right]\frac{\exp(iR)}{R},$$

$$f_\rho = \frac{\exp\left[i(R - R_i)\right]}{R_i - R}(\partial_\rho R_i - \partial_\rho R),$$

$$f_{\rho\rho} = i\frac{\exp\left[i(R_i - R)\right]}{R_i - R}(\partial_\rho R_i - \partial_\rho R)^2 - i\frac{\exp\left[i(R_i - R)\right]}{(R_i - R)^2}(\partial_\rho R_i - \partial_\rho R)^2$$

$$+\, i\frac{\exp\left[i(R_i - R)\right]}{R_i - R}(\partial_\rho^2 R_i - \partial_\rho^2 R)$$

$$
\begin{aligned}
\frac{1}{\rho}\,\partial_\rho\,\rho\,\partial_\rho\,p_i = {} & \left[i\frac{\overset{\textcircled{1}}{\partial_\rho} R}{\rho} - \frac{\overset{\textcircled{2}}{\partial_\rho} R}{\rho R} + i\,\overset{\textcircled{1}}{\partial^2_\rho} R - (\overset{\textcircled{2}}{\partial_\rho} R)^2 - \frac{2i(\overset{\textcircled{1}}{\partial_\rho} R)^2}{R} - \frac{\overset{\textcircled{2}}{\partial^2_\rho} R}{R} + \frac{2(\overset{\textcircled{2}}{\partial_\rho} R)^2}{R^2} \right] p_i \\
& + \left(\frac{\overset{\textcircled{4}}{1}}{\rho} + 2i\,\overset{\textcircled{3}}{\partial_\rho} R - \frac{2\,\overset{\textcircled{4}}{\partial_\rho} R}{R} \right) \frac{\partial_\rho R_i - \partial_\rho R}{R_i - R} \frac{\exp(iR)}{R} \\
& + \left[\frac{i(\partial_\rho R_i \overset{\textcircled{3}}{-} \partial_\rho R)^2}{R_i - R} - \frac{(\partial_\rho R_i \overset{\textcircled{5}}{-} \partial_\rho R)^2}{(R_i - R)^2} + \frac{\partial^2_\rho R_i \overset{\textcircled{6}}{-} \partial^2_\rho R}{R_i - R} \right] \frac{\exp(iR)}{R};
\end{aligned}
$$

$$
\begin{aligned}
\partial^2_z\,p_i = {} & \left[i\,\overset{\textcircled{1}}{\partial^2_z} R - (\overset{\textcircled{2}}{\partial_z} R)^2 - \frac{2i(\overset{\textcircled{1}}{\partial_z} R)^2}{R} - \frac{\overset{\textcircled{2}}{\partial^2_z} R}{R} + \frac{2(\overset{\textcircled{2}}{\partial_z} R)^2}{R^2} \right] p_i \\
& + \left(2i\,\overset{\textcircled{3}}{\partial_z} R - \frac{2\,\overset{\textcircled{4}}{\partial_z} R}{R} \right) \frac{\partial_z R_i - \partial_z R}{R_i - R} \frac{\exp(iR_i)}{R} \\
& + \left[\frac{i(\partial_z R_i \overset{\textcircled{3}}{-} \partial_z R)^2}{R_i - R} - \frac{(\partial_z R_i \overset{\textcircled{5}}{-} \partial_z R)^2}{(R_i - R)^2} + \frac{\partial^2_z R_i \overset{\textcircled{6}}{-} \partial^2_z R}{R_i - R} \right] \frac{\exp(iR_i)}{R}.
\end{aligned}
$$

① :

$$
i\frac{\partial_\rho R}{\rho} + i\,\partial^2_\rho R - \frac{2i(\partial_\rho R)^2}{R} + i\,\partial^2_z R - \frac{2i(\partial_z R)^2}{R} = i\frac{k_+^2}{R} + i\frac{k_+^2}{R} - 2i\frac{k_+^2}{R} = 0 \quad \text{O.K.}
$$

② :

$$
\begin{aligned}
& -\frac{\partial_\rho R}{\rho R} - (\partial_\rho R)^2 - \frac{\partial^2_\rho R}{R} + 2\frac{(\partial_\rho R)^2}{R^2} - (\partial_z R)^2 - \frac{\partial^2_z R}{R} + 2\frac{(\partial_z R)^2}{R^2} = \\
& \quad -\cancel{\frac{k_+^2}{R^2}} - \frac{k_+^4\rho^2}{R^2} - \cancel{\frac{k_+^2(k_+z + k_2a)^2}{R^4}} + 2\cancel{\frac{k_+^4\rho^2}{R^4}} - \frac{k_+^2(k_+z + k_2a)^2}{R^2} - \cancel{\frac{k_+4\rho^2}{R^4}} \\
& \quad + 2\cancel{\frac{k_+^2(k_+z + k_2a)^2}{R^2}} = -k_+^2 \quad \text{O.K.}
\end{aligned}
$$

③ :

$i = 1, 2:$

$$2\,\partial_\rho\, R(\partial_\rho\, R_i - \partial_\rho\, R) + (\partial_\rho\, R_i - \partial_\rho\, R)^2 + 2\,\partial_z(\partial_z\, R_i - \partial_z\, R) + (\partial_z\, R_i - \partial_z\, R)^2$$
$$= (\partial_\rho\, R_i)^2 + (\partial_z\, R_i)^2 - (\partial_\rho\, R)^2 - (\partial_z\, R)^2 = 0 \quad \text{O.K.}$$

$i = +:$

$$i\left[(\partial_\rho\, R_i)^2 + (\partial_z\, R_i)^2 - (\partial_\rho\, R)^2 - (\partial_z\, R)^2\right]\left(\frac{1}{R_+ - R} - \frac{1}{R_+ + R}\right)\frac{\exp(iR_+)}{R}$$
$$= i(-D)\frac{2R}{R_+^2 - R^2}\frac{\exp(iR_+)}{R} =$$
$$(\text{as} \quad R_+^2 - R^2 = Dr_1r_2)$$
$$= -2i\frac{R}{r_1r_2}\frac{\exp(iR_+)}{R} = -2i\frac{\exp(ik_1r_1)}{r_1}\frac{\exp(ik_2r_2)}{r_2} \quad \text{O.K. – see Eq. (2).}$$

④ :

$$④ = \left(\frac{1}{\rho} - 2\frac{\partial_\rho\, R}{R}\right)\left(\frac{\partial_\rho\, R_i - \partial_\rho\, R}{R_i - R} - \frac{\partial_\rho\, R_i + \partial_\rho\, R}{R_i + R}\right)$$
$$+ \left(-2\frac{\partial_z\, R}{R}\right)\left(\frac{\partial_z\, R_i - \partial_z\, R}{R_i - R} - \frac{\partial_z\, R_i + \partial_z\, R}{R_i + R}\right)$$
$$= 2\left(\frac{1}{\rho} - 2\frac{\partial_\rho\, R}{R}\right)\left(\frac{R\,\partial_\rho\, R_i - R_i\,\partial_\rho\, R}{R_i^2 - R^2}\right) - 2\frac{2\,\partial_z\, R}{R}\frac{R\,\partial_z\, R_i - R_i\,\partial_z\, R}{R_i^2 - R^2};$$
$$\frac{1}{\rho} - 2\frac{\partial_\rho\, R}{R} = \frac{1}{\rho} - 2\frac{k_+^2\rho}{R^2} = \frac{(k_+z + k_2a)^2 - (k_+\rho)^2}{\rho R^2} \qquad \left[= \frac{[k_+(z+a) - k_1a]^2 - (k_+\rho)^2}{\rho R^2}\right];$$
$$-2\frac{\partial_z\, R}{R} = -\frac{2k_+\rho(k_+z + k_2a)}{\rho R^2};$$

$i = 1:$

$$\partial_\rho R_1 \cdot R - \partial_\rho R \cdot R_1 = \frac{k_+\rho}{r_1}R - \frac{k_+^2\rho}{R}(k_+r_1 + k_2a)$$

$$= \frac{k_+\rho k_2 a}{r_1 R}(2k_+z + k_2a - k_+r_1);$$

$$\partial_z R_1 \cdot R - \partial_z R \cdot R_1 = \frac{k_+z}{r_1}R - \frac{k_+(k_+z + k_2a)}{R}(k_+r_1 + k_2a)$$

$$= \frac{k_2a}{r_1R}\left[(k_2a + k_+z)(k_+z - k_+r_1) - (k_+\rho)^2\right];$$

$$\frac{(k_+z + k_2a)^2 - (k_+\rho)^2}{\cancel{\rho}R^2}\frac{k_+\cancel{\rho}k_2a}{r_1R}(2k_+z + k_2a - k_+r_1)$$

$$- \frac{2k_+\cancel{\rho}(k_+z + k_2a)}{\cancel{\rho}R^2}\frac{k_2a}{r_1R}\left[(k_2a + k_+z)(k_+z - k_+r_1) - (k_+\rho)^2\right]$$

$$= \frac{k_+k_2a}{r_1R^3}\left[(k_+z + k_2a)^2(k_2a + \cancel{2k_+z} - \cancel{k_+r_1} - \cancel{2k_+z} + \cancel{2}k_+r_1)\right.$$

$$\left. -(k_+\rho)^2(\cancel{k_2a} + \cancel{2k_+z} - k_+r1 - \cancel{2k_+z} - \cancel{2}k_2a)\right]$$

$$= \frac{k_+k_2a}{r_1R^{\cancel{3}}}\cancel{R^2}(k_+r_1 + k_2a) = \frac{k_+k_2aR_1}{r_1R},$$

$$④ = \frac{k_+k_2aR_1}{r_1R}\frac{2}{R_1^2 - R^2}$$

$i = 2:$

$$
\begin{aligned}
&\partial_\rho R_2 \cdot R - \partial_\rho R \cdot R_2 = \frac{k_+\rho}{r_2} R - \frac{k_+^2\rho}{R}(k_+ r_2 + k_1 a) \\
&\qquad = \frac{k_+\rho}{r_2 R}\left[2k_+(z+a) + k_+ r_2 - k_1 a\right]; \\
&\partial_z R_2 \cdot R - \partial_z R \cdot R_2 = \frac{k_+(z+a)}{r_2} R - \frac{k_+[k_+(z+a) - k_1 a]}{R}(k_+ r_2 + k_1 a) \\
&\qquad = -\frac{k_1 a}{r_2 a}\left\{[k_+ z(z+a) - k_1 a][k_+ r_2 + k_+(z+a)] - (k_+\rho)^2\right\}; \\
&\frac{[k_+(z_+ a) - k_1 a]^2 - (k_+\rho)^2}{\cancel{\rho} R^2}\left(-\frac{k_+\cancel{\rho} k_1 a}{r_2 R}\right)\left[2k_+(z+a) + k_+ r_2 - k_1 a\right] \\
&\qquad - \frac{2k_+\cancel{\rho}[k_+(z+a) - k_1 a]}{\cancel{\rho} R^2}\left(-\frac{k_1 a}{r_2 R}\right)\left\{[k_+(z+a) - k_1 a][k_+ r_2 + k_+(z+a)] - (k_+\rho)^2\right\} \\
&\qquad = \frac{k_+ k_1 a}{r_2 R^3}\left\{[k_+(z+a) - k_1 a]^2[-\cancel{2k_+(z+a)} - \cancel{k_+ r_2} + k_1 a + \cancel{2} k_+ r_2 + \cancel{2k_+(z+a)}]\right. \\
&\qquad\quad \left. +(k_+\rho)^2[\cancel{2k_+(z+a)} + k_+ r_2 - k_1 a - \cancel{2k_+(z+a)} + 2k_1 a]\right\} \\
&\qquad = \frac{k_+ k_1 a R_2}{r_2 R}, \\
&④ = \frac{k_+ k_1 a R_2}{r_2 R}\frac{2}{R_2^2 - R^2}
\end{aligned}
$$

$i = + :$

$$\partial_\rho R_+ \cdot R - \partial_\rho R \cdot R_+ = \left(\frac{k_1\rho}{r_1} + \frac{k_2\rho}{r_2}\right) - \frac{k_+^2\rho}{R}(k_1r_1 + k_2r_2)$$

$$= \frac{k_1\rho}{r_1R}k_2a(2k_+z + k_2a) + \frac{k_2\rho}{r_2R}k_1a[-2k_+(z+a) + k_1a];$$

$$\partial_z R_+ \cdot R - \partial_z R \cdot R_+ = \left[\frac{k_1z}{r_1} + \frac{k_2(z+a)}{r_2}\right]R - \frac{k_+(k_+z + k_2a)}{R}(k_1r_1 + k_2r_2)$$

$$= \frac{k_2a}{r_1R}\left[k_1z(k_+z + k_2a) - k_1k_+\rho^2\right] + \frac{k_1a}{r_2R}\left\{k_1(z+a)[k_+(z+a)z - k_1a] + k_2k_+\rho^2\right\}$$

$$\frac{(k_+z + k_2a)^2 - (k_+\rho)^2}{\cancel{\rho}R^2}\frac{k_1\cancel{\rho}k_2a}{r_1R}(2k_+z + k_2a)$$

$$- \frac{2k_+\cancel{\rho}(k_+z + k_2a)}{\cancel{\rho}R^2}\frac{k_2a}{r_1R}[k_1z(k_+z + k_2a) - k_1k_+\rho^2]$$

$$= \frac{k_1k_2a}{r_1R^3}\left[(k_+z + k_2a)^2(2\cancel{k_+z} + k_2a - 2\cancel{k_+z}) + (k_+\rho)^2(-2\cancel{k_+z} - \cancel{k_2a} + 2\cancel{k_+z} + \cancel{2}k_2a)\right]$$

$$= \frac{k_1(k_2a)^2}{r_1R};$$

$$\frac{[k_+(z+a) - k_1a]^2 - (k_+\rho)^2}{\cancel{\rho}R^2}\frac{k_2\cancel{\rho}k_1a}{r_2R}[-2k_+(z+a) + k_1a)$$

$$- \frac{2k_+\cancel{\rho}[k_+(z+a) - k_1a]}{\cancel{\rho}R^2}\frac{k_1a}{r_2R}\left\{-k_2(z+a)[k_+(z+a) - k_1a] + k_1k_+\rho^2\right\}$$

$$= \frac{k_1k_2a}{r_2R^3}\left\{[k_+(z+a) - k_1a]^2[-2\cancel{k_+(z+a)} + k_1a + 2\cancel{k_+(z+a)}]\right.$$

$$\left. + (k_+\rho)^2[-2\cancel{k_+(z+a)} - \cancel{k_1a} - 2\cancel{k_+(z+a)} + \cancel{2}k_1a)\right\}$$

$$= \frac{k_2(k_1a)^2}{r_2R};$$

$$④ = \frac{2}{R_+^2 - R^2}\left[\frac{k_1(k_2a)^2}{r_1r_2R} + \frac{k_2(k_1a)^2}{r_2R}\right] = \frac{2}{R_+^2 - R^2}\frac{k_1k_2a^2R_+}{r_1r_2R}$$

⑤ :

$$⑤ = -\frac{(\partial_\rho R_i - \partial_\rho R)^2}{(R_i - R)^2} + \frac{(\partial_\rho R_i + \partial_\rho R)^2}{(R_i + R)^2} - \frac{(\partial_z R_i - \partial_z R)^2}{(R_i - R)^2} + \frac{(\partial_z R_i + \partial_z R)^2}{(R_i + R)^2}$$

$$= \frac{4}{(R_i^2 - R^2)^2}\left\{(\partial_\rho R_i\,\partial_\rho R + \partial_z R_i\,\partial_z R)(R_i^2 + R^2)\right.$$

$$\left. - \left[(\partial_\rho R_i)^2 + (\partial_z R_i)^2 + (\partial_\rho R)^2 + (\partial_z R)^2\right]R_iR\right\}$$

$i = 1:$

$$
\begin{aligned}
⑤ &= \frac{4k_+^2}{(R_1^2 - R^2)^2}\left[\frac{(k_+\rho)^2 + k_+ z(k_+ z + k_2 a)}{k_+ r_1 R}(R_1^2 + R^2) - 2R_1 R\right] \\
&= \frac{2k_+^2}{(R_1^2 - R^2)^2}\frac{(R_1^2 + R^2 - 2k_2 a R_1)(R_1^2 + R^2) - 4k_+ r_1 R_1 R^2}{k_+ r_1 R} \\
&= \frac{2k_+^2}{(R_1^2 - R^2)^2}\frac{(R_1^2 + R^2)^2 - 4R_1^2 R^2 + {}^2\cancel{4}k_2 a R_1 R^2 - 2k_2 a R_1^3 - \cancel{2k_2 a R_1 R^2}}{k_+ r_1 R} \\
&= \frac{2k_+}{r_1 R} - \frac{2}{R_1^2 - R^2}\frac{2k_+ k_2 a R_1}{r_1 R}
\end{aligned}
$$

$i = 2:$

$$
\begin{aligned}
⑤ &= \frac{4k_+^2}{(R_2^2 - R^2)^2}\left[\frac{(k_+\rho)^2 + k_+(z + a)[k_+(z + a) - k_1 a]}{k_+ r_2 R}(R_2^2 + R^2) - 2R_2 R\right] \\
&= \frac{2k_+^2}{(R_2^2 - R^2)^2}\frac{(R_2^2 + R^2 - 2k_1 a R_2)(R_2^2 + R^2) - 4k_+ r_2 R_2 R^2}{k_+ r_2 R} \\
&= \frac{2k_+^2}{(R_2^2 - R^2)^2}\frac{(R_2^2 + R^2)^2 - 4R_2^2 R^2 + {}^2\cancel{4}k_1 a R_2 R^2 - 2k_1 a R_2^3 - \cancel{2k_1 a R_2 R^2}}{k_+ r_2 R} \\
&= \frac{2k_+}{r_2 R} - \frac{2}{R_2^2 - R^2}\frac{2k_+ k_1 a R_2}{r_2 R}
\end{aligned}
$$

$i = +:$

$$
\begin{aligned}
\partial_\rho R + \partial_\rho R + \partial_z R_+ \partial_z R &= \left(\frac{k_1\rho}{r_1} + \frac{k_2\rho}{r_2}\right) + \left[\frac{k_1 z}{r_1} + \frac{k_2(z + a)}{r_2}\right]\frac{k_+(k_+ z + k_2 a)}{R} \\
&= \frac{k_+^2 R_+}{R} + \frac{k_1 k_2 k_+ a z}{r_1 R} - \frac{k_1 k_2 k_+ a(z + a)}{r_2 R};
\end{aligned}
$$

$$
\begin{aligned}
(\partial_\rho R_+)^2 + (\partial_z R_+)^2 + (\partial_\rho R)^2 + (\partial_z R)^2 &= \left(\frac{k_1\rho}{r_1} + \frac{k_2\rho}{r_2}\right)^2 + \left[\frac{k_1 z}{r_1} + \frac{k_2(z + a)^2}{r_2}\right]^2 + k_+^2 \\
&= 2k_+^2 + \frac{2k_1 k_2}{r_1 r_2}[\rho^2 + z(z + a) - r_1 r_2];
\end{aligned}
$$

$$
\begin{aligned}
⑤ = \frac{4}{(R_+^2 - R^2)^2}\Bigg\{&\left[\frac{k\,+^2 R_+}{R} + \frac{k_1 k_2 k_+ z a}{r_1 R} - \frac{k_1 k_2 k_+ (z + a) a}{r_2 R}\right](R_+^2 + R^2) \\
&- \left[2k_+^2 + 2\frac{k_1 k_2}{r_1 r_2}(\rho^2 + z(z + a) - r_1 r_2)\right] R_+ R\Bigg\}
\end{aligned}
$$

⑥ :

$$
\begin{aligned}
\textcircled{6} &= \left(\partial_\rho^2 R_i + \partial_z^2 R_i - \partial_\rho^2 R - \partial_z^2 R\right) \frac{1}{R_i - R} - \left(\partial_\rho^2 R_i + \partial_z^2 R_i + \partial_\rho^2 R + \partial_z^2 R\right) \frac{1}{R_i + R} \\
&= \frac{2}{R_i^2 - R^2} \left[R(\partial_\rho^2 R_i - \partial_z^2 R_i) - R_i(\partial_\rho^2 R + \partial_z^2 R)\right]
\end{aligned}
$$

$i = 1:$

$$
\begin{aligned}
\textcircled{6} &= \frac{2}{R_1^2 - R^2} \left[R \left(\frac{k_+ z^2}{r_1^3} + \frac{k_+ \rho^2}{r_1^3}\right) - R_1 \frac{k_1^2 (k_+ z + k_2 a)^2 + k_+^4 \rho^2}{R^3}\right] \\
&= \frac{2}{R_1^2 - R^2} \left(R \frac{k_+}{r_1} - R_1 \frac{k_+^2}{R}\right) = \frac{2}{R_1^2 - R^2} \frac{k_+ R^2 - k_+^2 r_1 R_1}{r_1 R}
\end{aligned}
$$

$i = 2:$

$$
\begin{aligned}
\textcircled{6} &= \frac{2}{R_2^2 - R^2} \left\{R \left[\frac{k_+ (z + a)^2}{r_2^3} + \frac{k_+ \rho^2}{r_2^3}\right] - R_2 \frac{k_+^2}{R}\right\} \\
&= \frac{2}{R_2^2 - R^2} \left(R \frac{k_+}{r_2} - R_2 \frac{k_+^2}{R}\right) = \frac{2}{R_2^2 - R^2} \frac{k_+ R^2 - k_+^2 r_2 R_2}{r_2 R}
\end{aligned}
$$

$i = +:$

$$
\begin{aligned}
\textcircled{6} &= \frac{2}{R_+^2 - R^2} \left\{R \left[\frac{k_1 z^2}{r_1^3} + \frac{k_2 (z + a)^2}{r_2^3} + \frac{k_1 \rho^2}{r_1^3} + \frac{k_2 \rho^2}{r_2^3}\right] - R_+ \frac{k_+^2}{R}\right\} \\
&= \frac{2}{R_+^2 - R^2} \left[R \left(\frac{k_1}{r_1} + \frac{k_2}{r_2}\right) - R_+ \frac{k_+^2}{R}\right]
\end{aligned}
$$

④ + ⑤ + ⑥ :

$i = 1:$

$$
\begin{aligned}
\textcircled{4} + \textcircled{5} + \textcircled{6} &= \left(\frac{2}{R_1^2 - R^2} \frac{k_+ k_2 a R_1}{r_1 R}\right) + \left(\frac{2k_+}{r_1 R} - \frac{2}{R_1^2 - R^2} \frac{2k_+ k_2 a R_1}{r_1 R}\right) \\
&\quad + \left(\frac{2}{R_1^2 - R^2} \frac{k_+ R^2 - k_+^2 r_1 R_1}{r_1 R}\right) \\
&= \frac{2}{R_1^2 - R^2} \frac{1}{r_1 R} \left(\cancel{k_+ k_2 a R_1} + k_+ R_1 - \cancel{k_+ R^2} - \cancel{2} k_+ k_2 a R_1 + \cancel{k_+ R^2} - k_+^2 r_1 R_1\right) \\
&= \frac{2}{R_1^2 - R^2} \frac{k_+}{r_1 R} (R_1 - k_+ r_1 - k_2 a) = 0 \qquad \text{O.K.}
\end{aligned}
$$

$i = 2:$

$$
\begin{aligned}
④ + ⑤ + ⑥ &= \left(\frac{2}{R_2^2 - R^2}\frac{k_+ k_1 a R_2}{r_2 R}\right) + \left(\frac{2k_+}{r_2 R} - \frac{2}{R_2^2 - R^2}\frac{2k_+ k_1 a R_2}{r_2 R}\right) \\
&\quad + \left(\frac{2}{R_2^2 - R^2}\frac{k_+ R^2 - k_+^2 r_2 R_2}{r_2 R}\right) \\
&= \frac{2}{R_2^2 - R^2}\frac{2}{r_2 R}(\cancel{k_+ k_1 a R_2} + k_+ R_2 - \cancel{k_+ R^2} - \cancel{2} k_+ k_1 a R_2 + \cancel{k_+ R^2} - k_+^2 r_2 R_2) \\
&= \frac{2}{R_2^2 - R^2}\frac{k_+}{r_2 R}(R_2 - k_+ r_2 - k_1 a) = 0 \qquad \text{O.K.}
\end{aligned}
$$

$i = +:$

$$
\begin{aligned}
④ + ⑤ + ⑥ &= \left(\frac{2}{R_+^2 - R^2}\times\right) \quad \frac{k_1 k_2 a^2 R_+}{r_1 r_2 R} + \left(\frac{k_1}{r_1} + \frac{k_2}{r_2}\right) R - k_+^2 \frac{R_+}{R} \\
&+ \frac{2}{R_2^2 - R^2}\left\{\left[\frac{k_+^2 R_+}{R} + \frac{k_1 k_2 k_+ z a}{r_1 R} - \frac{k_1 k_2 k_+ (z+a) a}{r_2 R}\right](R_+^2 + R^2)\right. \\
&\left. - \left[k_+^2 + \frac{2k_1 k_2}{r_1 r_2}(\rho^2 + z(z+a) - r_1 r_2)\right] R_+ R\right\}
\end{aligned}
$$

$$
\left(
\begin{array}{l}
2\rho^2 + 2z(z+a) - 2r_1 r_2 = (r_1 - r_2)^2 - a^2 \\
\dfrac{k_1 k_2 a^2 R_+}{r_1 r_2 R} - \dfrac{2k_1 k_2}{R_+^2 - R^2}\dfrac{-a^2}{r_1 r_2} R_+ R = \dfrac{1}{R_+^2 - R^2}\dfrac{k_1 k_2 a^2 R_+ (R_+^2 + R^2)}{r_1 r_2 R} \\
k_2 k_+ z a = \dfrac{1}{2}[R^2 - (k_+ r_1)^2 - (k_2 a)^2] \\
- k_1 k_+ (z+a) a = \dfrac{1}{2}[R^2 - (k_+ r_2)^2 - (k_1 a)^2]
\end{array}
\right)
$$

$$
\begin{aligned}
&= \left(\frac{k_1}{r_1} + \frac{k_2}{r_2}\right) R - k_+^2 \frac{R_+}{R} + \frac{1}{R_+^2 - R^2} \Big\{ \Big[2k_+^2 R_+ r_1 r_2 + (R^2 - [k_+ r_1]^2) k_1 r_2 \\
&\quad + (R^2 - [k_+ r_2]^2) k_2 r_1 \Big] \frac{R_+^2 + R^2}{r_1 r_2 R} - 2 \left[2k_+^2 + \frac{k_1 k_2 (r_1 - r_2)^2}{r_1 r_2} \right] R_+ R \Big\} \\
&= \left(\frac{k_1}{r_1} + \frac{k_2}{r_2}\right) R - k_+^2 \frac{R_+}{R} + \frac{1}{R_+^2 - R^2} \left\{ \left[\frac{k_+^2 R_+}{R} + \left(\frac{k_1}{r_1} + \frac{k_2}{r_2}\right) R \right] (R_+^2 + R^2) \right. \\
&\quad \left. - \left[4k_+^2 + 2 \frac{k_1 k_2 (r_1 - r_2)^2}{r_1 r_2} \right] R_+ R \right\} \\
&= \frac{1}{R_+^2 - R^2} \left\{ \left(\frac{k_1}{r_1} + \frac{k_2}{r_2}\right) R (R_+^2 - \cancel{R^2}) + \left(\frac{k_1}{r_1} + \frac{k_2}{r_2}\right) R (R_+^2 + \cancel{R^2}) \right. \\
&\quad \left. -2 \left[k_+^2 + \frac{k_1 k_2 (r_1 - r_2)^2}{r_1 r_2} \right] R_+ R \right\} \\
&= \frac{2R_+ R}{R_+^2 - R^2} \left[\left(\frac{k_1}{r_1} + \frac{k_2}{r_2}\right) R - k_+^2 - \frac{k_1 k_2 (r_1 - r_2)^2}{r_1 r_2} \right] \\
&= \frac{2}{R_+^2 - R^2} \frac{1}{r_1 r_2} \left[(k_1 r_2 + k_2 r_1)(k_1 r_1 + k_2 r_2) - k_+^2 r_1 r_2 - k_1 k_2 (r_1 - r_2)^2 \right] \\
&= \frac{2}{R_+^2 - R^2} \frac{1}{r_1 r_2} \left[k_1^2 r_1 r_2 + \cancel{k_1 k_2 r_1^2} + \cancel{k_1 k_2 r_2^2} + k_2^2 r_1 r_2 - k_+^2 r_1 r_2 - \cancel{k_1 k_2 r_1^2} \right. \\
&\quad \left. - \cancel{k_1 k_2 r_2^2} + 2 k_1 k_2 r_1 r_2 \right] = 0 \qquad \text{O.K.}
\end{aligned}
$$