Nonlinear interaction of nonconcentric spherical waves

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Abstract

A high-frequency asymptotic solution of the problem of interaction between two nonconcentric harmonic spherical waves in an unbounded homogeneous fluid medium is obtained in the quasilinear approximation. The resulting expression describes not only phenomena associated with the cumulative growth of the sound field in space, but also the transition from noncumulative to cumulative interactions. The high-frequency asymptotic behavior of the sound field far from primary sources is determined for a dispersive and (or) absorbing medium.

The problem investigated in the present article – the nonlinear interaction of nonconcentric spherical waves in an unbounded homogeneous fluid medium – is one of several canonical (standard) problems of nonlinear acoustics. Its solution in the quasilinear approximation[1,2] can be used as an analog of the Green's function for determining the secondary sound field in the interaction of harmonic sound fields from distributed sources of arbitrary configuration in an unbounded fluid[3,4,5].

In contrast with the simpler problem of the interaction of plane and spherical harmonic waves, for which an exact solution has been found in quasilinear approximation[6], the curvature of the wave fronts of both primary waves in the given situation greatly complicates the analysis of the secondary sound field. It is no longer possible to find an exact solution in this case, so that only asymptotic expressions can be obtained. We propose to obtain a high-frequency asymptotic solution, uniform over all space[1)], for the secondary sound field, taking into account all phenomena associated with its cumulative growth in space. We note that previously derived expressions for the secondary sound field (the author knows of only one paper on the topic[7]) have a more specialized character in that they describe the behavior of the sound field only in bounded spatial domains "near to" and "far from" the symmetry axis of the system, i.e. in the near and far fields.

Let two harmonic spherical waves p_1 and p_2 be given:

$$p_1 = A_1 \frac{\exp(ik_1|\mathbf{r}-\mathbf{r}_1|)}{k_1|\mathbf{r}-\mathbf{r}_1|}, \qquad p_2 = A_2 \frac{\exp(ik_2|\mathbf{r}-\mathbf{r}_2|)}{k_2|\mathbf{r}-\mathbf{r}_2|}, \tag{1}$$

where k_1 and k_2 are the wave numbers at the corresponding frequencies [the time factor of the form $\exp(i\omega t)$ is dropped everywhere]. We have the following relation for the component of the secondary field at (for definiteness) the sum frequency, correct to within noncumulative effects (see, e.g., Refs. [2] and [4]):

$$p_+(\mathbf{r}) = -\frac{\beta A_1 A_2}{\rho_0 c_0^2} \int \frac{\exp(ik_1|\mathbf{r}_1-\mathbf{r}'|)}{k_1|\mathbf{r}_1-\mathbf{r}'|} \frac{\exp(ik_2|\mathbf{r}_2-\mathbf{r}'|)}{k_2|\mathbf{r}_2-\mathbf{r}'|} \frac{\exp(ik_+|\mathbf{r}-\mathbf{r}'|)}{4\pi|\mathbf{r}-\mathbf{r}'|}\, dV', \tag{2}$$

where β is the nonlinearity parameter, ρ_0 and c_0 are the density and sound velocity in the unperturbed fluid, and k_+ is the wave number at the sum frequency. The integrand is symmetric with respect to the points $\mathbf{r}_1$, $\mathbf{r}_2$, and r; accordingly, if we were able to find an exact solution, it would also have to be symmetric with respect to $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}$. However, from the physical point of view (from the point of view of the nature of this solution) such symmetry is not complete, because the point $\mathbf{r}$ is distinguished from the other two by the relation $k_+ \approx k_1 + k_2$. This consideration sheds at least some light on why a spherical coordinate system (r, θ, φ) with center at the "distinguished" point, i.e., the observation point $\mathbf{r}$, and the axis $\theta = 0$ passing through the center of one of the spherical waves (p_1 for definiteness; Fig. 1) is useful for the ensuing analysis [2)]. We assume for definiteness that

$$r_2 > r_1. \tag{3}$$

We expand the the functions contained in the integrand of Eq. 2 in spherical harmonics in the adopted coordinate system:

$$\frac{\exp(ik_1|\mathbf{r}_1-\mathbf{r}'|)}{|\mathbf{r}_1-\mathbf{r}'|} = ik_1 \sum_{n=0}^{\infty} (2n+1) P_n(\cos\theta') j_n(k_1 r_{1<}) h_n^{(1)}(k_1 r_{1>}), \tag{4}$$

$$\begin{aligned}\frac{\exp(ik_2|\mathbf{r}_2-\mathbf{r}'|)}{|\mathbf{r}_2-\mathbf{r}'|} &= ik_2 \sum_{m=0}^{\infty} \sum_{l=m}^{\infty} \varepsilon_m \frac{(2l+1)!(l-m)!}{(l+m)!)} \cos[m(\varphi-\varphi')] \\ &\quad \times P_m^l(\cos\theta) P_m^l(\cos\theta') j_m(k_2 r_{2<}) h_m^{(1)}(k_2 r_{2>}),\end{aligned} \tag{5}$$

where $j_n(x)$ and $h_n^{(1)}(x)$ are spherical Bessel and Hankel functions; $P_m^l(x)$ denotes Legendre polynomials, $P_n(x) = P_n^0(x)$; $r_{1<} \equiv \min(r_1, r')$, $r_{1>} \equiv \max(r_1, r')$, $r_{2<} \equiv \min(r_2, r')$, $r_{2>} \equiv \max(r_2, r')$; and $\varepsilon_m \equiv \{1, m=0; 2, m = 1, 2, \ldots\}$. Substituting the representations (4) and (5) in the integral (2) and integrating with respect to the angular coordinates, we obtain

$$\begin{aligned}p_+(\mathbf{r}) =& \frac{\beta A_1 A_2}{\rho_0 c_0^2} k_+^2 \int dr' r'^2 \frac{\exp(ik_+ r')}{r'} \sum_{n=0}^{\infty} (2n+1) P_n(\cos\theta) \\ & \times j_n(k_1 r_{1<}) h_n^{(1)}(k_1 r_{1>}) j_n(k_2 r_{2<}) h_n^{(1)}(k_2 r_{2>}).\end{aligned} \tag{6}$$

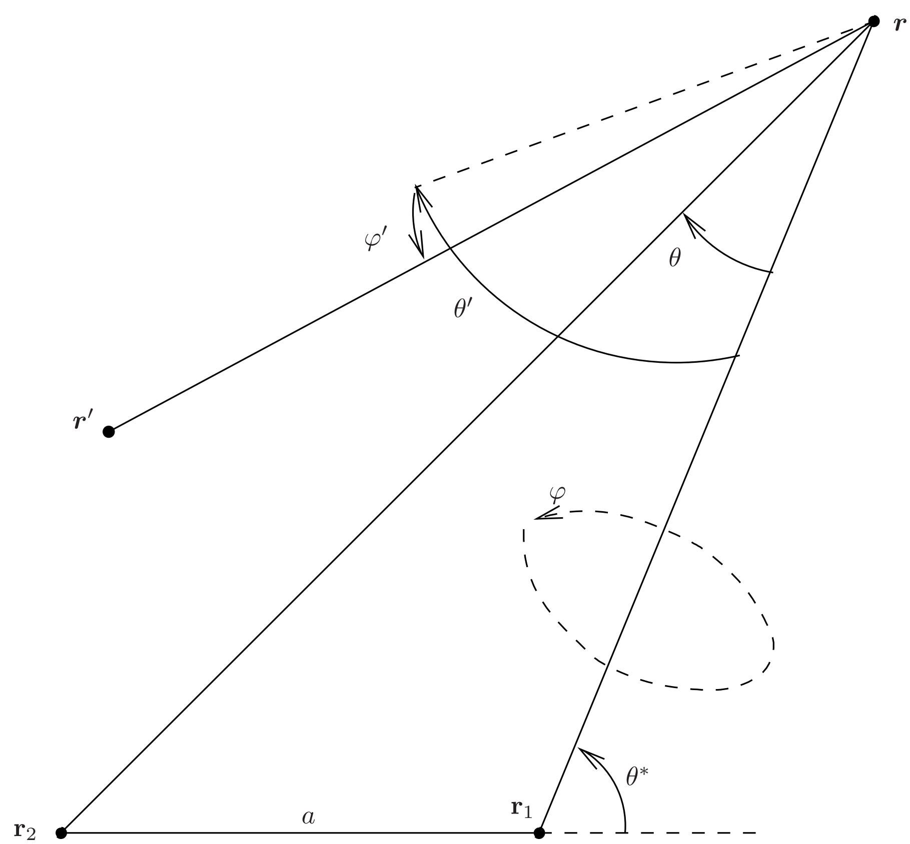

Figure 1: Interaction of two spherical waves: geometry of the problem. $\mathbf{r}_1, \mathbf{r}_2$ – primary sources, $\mathbf{r}$ – observation point.

The spatial growth of waves arriving at the observation point from the sources of the secondary field is possible only in the domain $r' < \min(r_1, r_2) = r_1$. Accordingly, Eq. (6) can be rewritten as follows within the same error limits as before:

$$p_+(\mathbf{r}) = \frac{\beta A_1 A_2}{\rho_0 c_0^2} k_+^2 \int\limits_0^{r_1} dx\, x \exp(ik_+ x) \sum_{n=0}^{\infty} (2n+1) P_n(\cos\theta) \\ \times j_n(k_1 x) h_n^{(1)}(k_1 r_1) j_n(k_2 x) h_n^{(1)}(k_2 r_2). \tag{7}$$

We now estimate the sum in the integral in Eq. (7) in that order. To do so, we note that only those terms in the sum whose phase does not vary too strongly in integration are of primary importance in estimating the value of the sum. This approximation corresponds to the inclusion on cumulative effects in the interaction of the primary waves.

The value of the sum in Eq. (7) is determined by two factors: the position of the stationary point n_s and the rapid decay of the amplitudes of the spherical Bessel functions with indices greater than the value of the argument, i.e., with $n > k_< x$, where $k_< \equiv \min(k_1, k_2)$. The position of the stationary point n_s is determined from the equation

$$\theta + \cos^{-1}\left(\frac{n_s}{k_1 x}\right) + \cos^{-1}\left(\frac{n_s}{k_2 x}\right) - \cos^{-1}\left(\frac{n_s}{k_1 r_1}\right) - \cos^{-1}\left(\frac{n_s}{k_2 r_2}\right) = 0, \tag{8}$$

which is obtained by replacing the sum in Eq. (7) by an integral, replacing Bessel functions by their Debye asymptotic forms, and then setting the derivative of the exponent function of the exponential to zero.

We can assume $\theta \ll 1$ in the adopted coordinate system within the stated error limits. The secondary field is small at observation points for which this condition does not hold, because cumulative growth does not take place. In this case, assuming that the arguments of the arc cosine are small in Eq. (8), we have

$$n_s \sim \frac{\theta}{\left(\frac{1}{k_1 x} - \frac{1}{k_1 r_1}\right) + \left(\frac{1}{k_2 x} - \frac{1}{k_2 r_2}\right)}. \tag{9}$$

We assume that $r_1 \approx r_2 \gg a$ for simplicity. It follows from Eq. (9) in this case that if the condition

$$r_1 - x \gg a\theta^*$$

is satisfied, so is the condition

$$n_s \ll k_< x. \tag{10}$$

Consequently, for the case in question, $r_1 = r_2 \gg a^{3)}$, condition (10) is satisfied everywhere except in a small domain of the variable of integration x

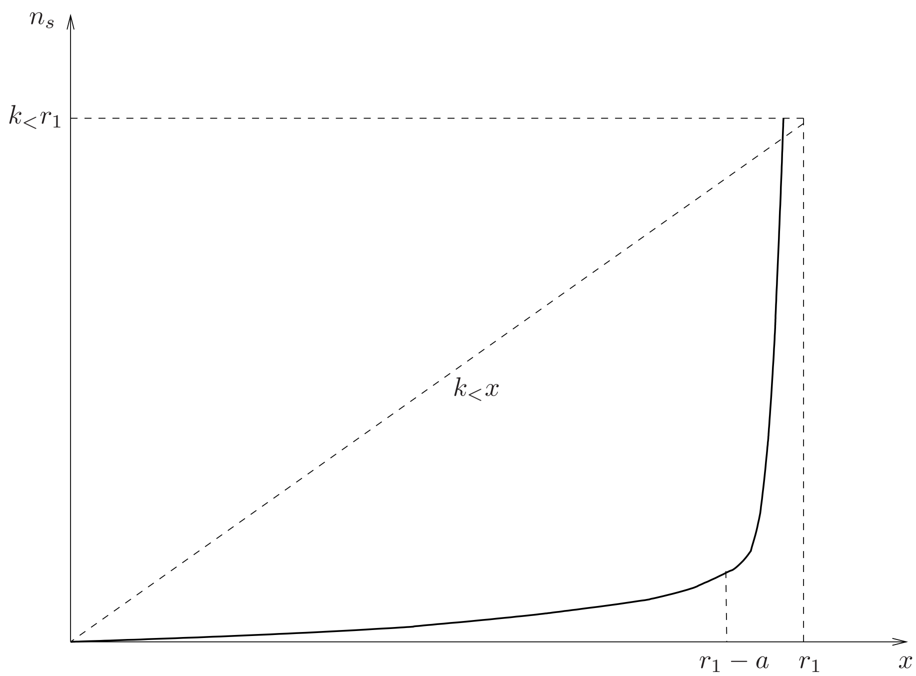

Figure 2: Typical behaviour of the function $n_s(x)$ in the case $r_1 \approx r_2 \gg a$.

(see Fig. 2)[4)]. This condition is the key to the method proposed below for the asymptotic estimation of p_+.

This method essentially involves the representation of the Bessel functions in Eq. (7) by asymptotic forms that will have the necessary accuracy in the synchronism domain and will also describe the transition from cumulative to noncumulative interactions as x varies.

The asymptotic solution

$$h_n^{(1)}(x) \sim (ix)^{-1} \exp[i(x - n\pi/2)], \quad x \gg n \tag{11}$$

is not sufficiently accurate from this point of view, because it does not describe the transition from cumulative to noncumulative interactions, whereas Debye asymptotic formula

$$\begin{aligned} h_{n-1/2}^{(1)}(x) \sim & x^{-1/2}(x^2 - n^2)^{-1/4} \\ & \times \exp\left\{ i\left[(x^2 - n^2)^{1/2} - n\cos^{-1}(n/x) - \pi/4) \right] \right\}, \quad |x - n| \gg n^{1/3} \end{aligned} \tag{12}$$

is more accurate than we need. All the necessary requirements are satisfied by the asymptotic solution

$$h_n^{(1)}(x) \sim (ix)^{-1} \exp\left[i(x - n\pi/2 + n^2/2x) \right], \quad x \gg n, \tag{13}$$

which is obtained from (12) by expanding the argument of the exponential function in n/x. The additional term in the exponent [in comparison with (11)] describes the additional phase shift attending the transition from the “far” field of the nth harmonic to its “near” field, i.e., it describes the desynchronyzation of the harmonics in this case. On the whole, using asymptotic expression (13) for analysis is ostensibly equivalent to making transition from the solution of inhomogeneous Helmholtz equation to the solution of an inhomogeneous parabolic equation involving the first derivative with respect to the radial coordinate[7].

The asymptotic solution (13) can be used in the products of Bessel functions in Eq. (7) if n is close to the stationary point n_s and if relation (8) is taken into account; the function $j_n(x)$ can then be replaced by the function $(1/2)h_n^{(2)}(x)$, because synchronism is possible only for this term in the representation $j_n(x) = (1/2)[h_n^{(1)}(x) + h_n^{(2)}(x)]$. Consequently, instead of representation (7) we now have

$$p_+(\mathbf{r}) = \frac{\beta A_1 A_2}{\rho_0 c_0^2} \frac{k_+^2}{4k_1 k_2} \frac{1}{k_1 r_1 k_2 r_2} \exp(ik_1 r_1 + ik_2 r_2) \int_0^{r_1} \frac{1}{x} \exp(i\delta_+ x) S(\mathbf{r}, x) dx, \tag{14}$$

$$S(\mathbf{r}, x) = \sum_{n=0}^{k_< x} (2n+1) P_n(\cos\theta) \exp[-in^2/2\alpha(\mathbf{r}, x)], \tag{15}$$

$$\frac{1}{\alpha} \equiv \frac{1}{k_1 x} + \frac{1}{k_2 x} - \frac{1}{k_1 r_1} - \frac{1}{k_2 r_2}, \tag{16}$$

where $\delta_+ = k_+ - k_1 - k_2$.

We replace the sum in Eq. (15) asymptotically by the integral

$$S(\mathbf{r}, x) \sim 2 \int_0^\infty \nu J_0(\nu\theta) \exp(i\nu^2/2\alpha)\, d\nu = -2i\alpha \exp(i\theta^2\alpha/2), \tag{17}$$

where $J_0(x)$ is a Bessel function. We substitute Eq. (17) in (14):

$$\begin{aligned} p_+(\mathbf{r}) \sim -\frac{\beta A_1 A_2}{\rho_0 c_0^2} \frac{ik_+^2}{2k_1k_2} \frac{1}{k_1r_1k_2r_2} \exp(ik_1r_1 + ik_2r_2) \\ \times \int_0^{r_1} \frac{\alpha(x)}{x} \exp[i\delta_+ x + i\alpha(x)\theta^2/2]\, dx. \end{aligned} \tag{18}$$

Equation (18) is fundamental to the analysis of the secondary sound field in the interaction of nonconcentric spherical waves in an unbounded dispersive fluid medium. In the case of zero dispersion

$$\delta_+ = 0, \tag{19}$$

Eq. (18) yields the expression for the secondary field

$$\begin{aligned} p_+(\mathbf{r}) \sim & \frac{\beta A_1 A_2}{\rho_0 c_0^2} \frac{ik_+^2}{2k_1k_2} \frac{\exp\left(ik_1r_1 + ik_2r_2 - i\frac{\theta^2}{2}\frac{k_1r_1k_2r_2}{k_1r_1+k_2r_2}\right)}{k_1r_1 + k_2r_2} \\ & \times \left\{ \mathrm{Ei}\left[i\frac{\theta^2}{2}\frac{k_1r_1k_2r_2}{k_1r_1+k_2r_2}\right] - \mathrm{Ei}\left[i\frac{\theta^2}{2}\left(\frac{k_1r_1k_2r_2}{k_1r_1+k_2r_2} + \frac{k_2r_1r_2}{r_2 - r_1}\right)\right]\right\}. \end{aligned} \tag{20}$$

The asymptotic expression for p_+ can be obtained in its final form on the basis of the fact that the above-derived expression (20) is valid for small θ and by writing the arguments of the exponential function and the integral exponential function in a vector form that is invariant relative to the coordinate system. Accordingly, using the relations

$$\begin{aligned} k_1r_1 + k_2r_2 - |k_1\mathbf{r}_1 + k_2\mathbf{r}_2| &= \frac{\theta^2}{2}\frac{k_1r_1k_2r_2}{k_1r_1 + k_2r_2} + O(\theta^4), \\ r_1 + a - r_2 &= \frac{\theta^2}{2}\frac{r_1r_2}{r_2 - r_1} + O(\theta^4), \end{aligned}$$

we rewrite Eq. (20) in the asymptotically equivalent form

$$p_+(\mathbf{r}) \sim \frac{\beta A_1 A_2}{\rho_0 c_0^2} \frac{ik_+^2}{2k_1k_2} \{\mathrm{Ei}\,[i(R_+ - R)] - \mathrm{Ei}\,[i(R_1 - R)]\} \frac{\exp(iR)}{R}, \tag{21}$$

where $R \equiv |k_1\mathbf{r}_1 + k_2\mathbf{r}_2|$, $R_+ \equiv k_1r_1 + k_2r_2$, $R_1 \equiv k_+r_1 + k_2a$, end $\mathrm{Ei}(x)$ is the integral exponential function.

Equations (20) and (21) are the required asymptotic solutions for the secondary field. One argument on favour of the expression (21) is the fact that it goes over to the exact solution for the secondary sound field in the interaction of a plane wave with a spherical wave[6] in the limit $r_2 \to \infty$. On the other hand, the expression (21) differs from (20) in that it implies a discontinuity of the solution p_+ upon crossing the plane $r_1 = r_2$ because of the asymmetry of the argument of the second integral exponential function in the brackets with respect to the sources of the primary sound field. Note that the difference between the values of the expression (21) for p_+ before and after crossing the plane $r_1 = r_2$ becomes insignificant at high frequencies, i.e., for

$$k_1 a \gg 1, \quad k_2 a \gg 1. \tag{22}$$

We also note that the error of Eq. (20) is probably determined mainly by the transition from the sum between finite limits (15) to the integral between infinite limits (17). However, this error is obviously smaller, the larger the wave parameters of the problem $k_1 a$ and $k_2 a$. Consequently, the expression obtained here can be regarded as a "high-frequency" asymptotic solution for p_+.

In the light of the foregoing it is reasonable to formulate an expression for p_+ that will be valid in all space and not just in the domain $r_1 < r_2$. This is achieved by combining (21) with the analogous expression obtained after interchanging the sources:

$$p_+(\mathbf{r}) \sim \frac{\beta A_1 A_2}{\rho_0 c_0^2} \frac{i k_+^2}{2 k_1 k_2} \{ \mathrm{Ei}\,[i(R_+ - R)] - \mathrm{Ei}\,[i(R_1 - R)] \\ - \mathrm{Ei}\,[i(R_2 - R)] \} \frac{\exp(iR)}{R}, \tag{23}$$

where $R_2 \equiv k_+ r_2 + k_1 a$.

An estimate of the range $r_1 = a^*$ at which synchronism sets in can be obtained from the above-derived expressions (20), (21) and (23). This is the range at which the smallest of the arguments of the integral exponential functions becomes of order of unity, .i.e., $|R_+ - R| \approx 1$:

$$a^* = \frac{k_1 k_2 a^2 \sin^2 \theta^*}{2 k_+} \qquad (\theta^* \not\approx 0), \tag{24}$$

where θ^* is the angle between $\mathbf{r}_1$ and a line drawn through the sources. With the exception of the neighbourhood $\theta^* \approx 0$, the secondary field is small at $r < a^*$ and begins to grow cumulatively at $r > a^*$; as it grows, its amplitude increases approximately as $\ln(r/a^*)$. Diffraction effects become significant in the neigborhood $\theta^* \approx 0$, so that the concept of the range at which synchronism sets in becomes meaningless in this neighbourhood. Nonetheless, an expression for the secondary sound field on the axis $\theta = 0$ is readily obtained from (20) or (21):

$$p_+(\mathbf{r}) \sim \frac{\beta A_1 A_2}{\rho_0 c_0^2} \frac{i k_+^2}{2 k_1 k_2} \ln\left(\frac{k_1 a}{k_+ r_2}\right) \frac{\exp(i k_+ r_1 + i k_2 a)}{k_+ r_1 + k_2 a}. \tag{25}$$

(to be added)

Figure 3: Typical behavior of the function $\| \mathrm{Ei}[i(R_+ - R)] - \mathrm{Ei}[i(R_1 - R)] \| / R$ with the variation of θ^* for different values of $r_1, k_1 = k_2 = 10^3, a = 1$. 1) $r_1 = 0.01$; 2) 0.1; 3) 1; 4) 10; 5) 100.

Figure 3 shows typical curves of the amplitude of the secondary sound field as a function of the angle θ^*, calculated for several values of r_1 according to expression (21).

Expression(18) is amendable to effective numerical and asymptotic analysis in the case of an absorbing and (or) dispersive medium. A change of variable of integration reduces the integral in Eq.(18) to the form

$$I = A\exp(i\delta_+ B - i\theta^2 A/2) \int\limits_{B-r_1}^{B} \frac{1}{x} \exp(-i\delta_+ x + i\theta^2 AB/2x)\, dx, \tag{26}$$

$$A \equiv \frac{k_1 r_1 k_2 r_2}{k_1 r_1 + k_2 r_2}, \quad B \equiv \frac{k_+ r_1 r_2}{k_1 r_1 + k_2 r_2}.$$

The form of the integral in Eq. (26) admits effective asymptotic analysis [8]. However we propose to use a different approach, approximating (26) by special functions.

Let the observation point be a region of "well-developed" dispersion:

$$|\delta_+| B \gg 1, \tag{27}$$

while

$$|\delta_+|(B - r_1) \ll 1. \tag{28}$$

In this case

$$\int\limits_{B-r_1}^{B} \frac{1}{x} \exp(-i\delta_+ x + i\theta^2 AB/2x)\, dx \sim \int\limits_{0}^{\infty} \frac{1}{x} \exp(-i\delta_+ x + i\theta^2 AB/2x)\, dx$$
$$- \int\limits_{0}^{B-r_1} \frac{1}{x} \exp(i\theta^2 AB/2x)\, dx = \mathrm{Ei}\left[\frac{i\theta^2 AB}{2(B - r_1)}\right] + 2K_0[(2\delta_+ \theta^2 AB)^{1/2}], \tag{29}$$

where $K_0(x)$ is a modified Bessel function (provided that all integrals in the above expression do exist). As a result, for p_+ we obtain

$$p_+(\mathbf{r}) \sim \frac{\beta A_1 A_2}{\rho_0 c_0^2} \frac{ik_+^2}{2k_1 k_2} \frac{\exp(iR)}{R} \exp(i\delta_+ B)$$
$$\times \left\{ Ei\left[\frac{i\theta^2 AB}{2(B - r_1)}\right] + 2K_0[(2\delta_+ \theta^2 AB)^{1/2}] \right\}. \tag{30}$$

This expression for the secondary sound field associated with the interaction of two spherical waves in an unbounded fluid medium with arbitrary dispersion and absorption is valid in the region of well developed dispersion $|delta_+|r \gg 1$ and encompasses a wide range of physically diverse cases. For example, the argument Z of the function K_0 in the expression (30) is of the order of $Z \sim 2(|\delta_+|a^*)^{1/2}$, so that the cases $Z \ll 1$ and $Z \gg 1$ correspond to situations in which the influence of dispersion (absorption) fir a given direction θ^* becomes appreciable long after or long before synchronism sets in [the validity of the expression (30) for $Z \ll 1$ is attributable to the fact that the asymptotic forms of the Bessel functions (13) used in the analysis provide the necessary accuracy not only in the domain where synchronism exists, but also in the transition domain; however, the parameter $\|\delta_+\|a^*$ must not be too small].

The expressions (23) and (30) for the secondary sound field at the sum frequency represent the fundamental result of the present study; the transition from them to expressions for the difference-frequency field is purely a formal exercise: it is sufficient to let $k_2 \to -k_2^*$, $k_+ \to k_-$, and $\delta_+ \to \delta_-$. However, there is still question of whether it is necessary to satisfy condition $k_- a \gg 1$. The form of the combination-frequency Green's function in the coordinate system adopted for the analysis of the integral (2) certainly seems to indicate that this condition is not necessary in order for the expressions (20), (21), (23), and (30) to be valid in the case of the difference-frequency field; as before, the satisfaction of conditions (22) is sufficient.

The author is grateful to L. M. Lyamshev for scientific supervision.

Notes

1) The term "uniform over all space" should be interpreted as follows here: Let p_+ be the exact solution, and let $\tilde{p}_+$ be the resulting asymptotic solution; then for any $\varepsilon > 0$ a frequency ω_ε exists such that the relation $|p_+ - \tilde{p}_+|k_+ r_1 r_2/(r_1+r_2) \leq \varepsilon$ will be satisfied for any frequency $\omega > \omega_\varepsilon$ and and for any distances r_1 and r_2 from the observation point to the primary sources. In this case, since the error is absolute in nature, an asymptotic solution that is uniform over all space does not necessarily describe noncumulative effects.

2) If one of the primary sources is chosen as the center of the spherical coordinate system, the final asymptotic expressions will be nonuniform in angle.

3) Indirect confirmation of the correctness of the subsequent analysis in all other cases follows from the fact that he final asymptotic expression (23) goes over to the exact solution of the problem of interaction between a plane wave and a spherical wave[6] in the limit $r_2 \to \infty$.

4) Synchronism between the interacting waves obviously does not occur in the domain $r_1 - x \leq s\theta^*$.

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