Nonlinear interaction of a plane wave and a spherical wave

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August 10, 1987

Abstract

An exact solution is obtained for the combination-frequency components of the sound field generated by interaction of plane and spherical waves of arbitrary frequency in a homogeneous fluid medium.

The problem of the interaction of plane and spherical sound waves has arisen frequently in previous studies of parametric receiving arrays (see, e.g., Ref. [1]). The object of investigation was the interaction of a spherical pump wave with a low-frequency signal wave. The ray approximation proposed by Zverev and Kalachev [2] was used to determine the combination-frequency mode. It essentially entails integration of the phase shift of the high-frequency wave in the direction of propagation as a result of nonlinear interaction with the low-frequency wave. The resulting equation corresponds to Westervelt solution for plane waves [3] if the spherical wave is interpreted as a plane wave with a slowly varying amplitude [4]. Another approach has been proposed by Westervelt [5]. It entails expansion of the spherical wave in plane waves with subsequent recourse to the solution of the problem of interaction between two plane waves in a low-dispersion medium [6]. As a result, the combination-frequency components of the secondary wave field are expressed in terms of an integral with respect to plane waves; it was shown that this integral gives the same solution as the ray approximation in the limit of a low-frequency plane wave.

In the present article we use direct integration to obtain an exact solution for the combination-frequency components of the secondary field produced in the interaction of a plane wave and a spherical wave of arbitrary frequency in a homogeneous non-dispersive fluid medium.

Following Ref. [7] (p. 871), we write the equation for the pressure p'' of the secondary sound field in the form

$$\Box p'' = -2\beta \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} U - \Box(E + W) - q, \tag{1}$$

$$W = \frac{1}{\rho_0 c_0^2} \frac{\partial}{\partial t} p' \int\limits_{-\infty}^{t} p' dt, \tag{2}$$

where β is the nonlinearity coefficient of the fluid, U and E are the potential and total energy density of the primary sound field, ρ_0 and c_0 are the equilibrium values of the density and sound velocity, and p' is the pressure of the primary sound field. The term q, which is added for completeness, accounts for the interaction of the primary sound field with the sources. We set

$$p'' = p^{(1)} + p^{(2)} + p^{(3)}, \tag{3}$$

$$\Box p^{(1)} = -2\beta \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}U \tag{4}$$

$$p^{(2)} = -E - W, \tag{5}$$

$$\Box p^{(3)} = -q. \tag{6}$$

The term $p^{(1)}$ in the representation (3) corresponds to the scattering of sound by sound, $p^{(2)}$ has the significance of the nonlinear response of the medium, and $p^{(3)}$ is attributable to interaction of the primary field with the sources.

We specify the pressure of the spherical wave p_1 and the plane wave p_2 forming the primary field:

$$p_1(\boldsymbol{r}, t) = \frac{A_1}{k_1 r}\exp(ik_1 r - i\omega_1 t), \tag{7}$$

$$p_2(\boldsymbol{r}, t) = A_2 \exp(ik_2 z - i\omega_2 t), \tag{8}$$

where (r, θ, φ) are spherical coordinates, and $z = r\cos\theta$. We assume that the spherical wave p_1 is generated by a mass source with volume density ρs, where ρ is the density of fluid and

$$s(\boldsymbol{r}, t) = -4\pi i \frac{A_1}{\rho_0 c_0}\frac{\delta(\boldsymbol{r})}{k_1^2} exp(-\omega_1 t). \tag{9}$$

We also assume that the medium has small dispersion and absorption, so that the dispersion relation

$$k(\omega) = \omega/c_0 + \delta(\omega) + i\alpha(\omega), \qquad |\delta/k| \ll 1, \qquad |\alpha/k| \ll 1 \tag{10}$$

holds; the corresponding generalization of Eqs. (4) and (6) entails the substitution

$$\Box p^{(i)}(\boldsymbol{r}, t) \to \int_{-\infty}^{\infty} (\Delta + k^2) p_\omega^{(i)}(\boldsymbol{r}) exp(-i\omega t) d\omega, \quad i = 1, 3. \tag{11}$$

We find the combination-frequency components of the fields $p^{(1)}$, $p^{(2)}$, and $p^{(3)}$ in succession.

In the investigated case of a biharmonic primary sound field,

$$p_1(\boldsymbol{r}, t) = p_1(\boldsymbol{r})\exp(-i\omega_1 t), \qquad p_2(\boldsymbol{r}, t) = p_2(\boldsymbol{r})\exp(-i\omega_2 t),$$

we find the sum-frequency component $p_+^{(1)}(\boldsymbol{r})$ of the field $p^{(1)}(\boldsymbol{r})$ according to Eqs. (4) and (11):

$$(\Delta + k_+^2)p_+^{(1)}(\boldsymbol{r}) = \beta k_+^2 \frac{p_1(\boldsymbol{r})p_2(\boldsymbol{r})}{\rho_0 c_0^2}. \tag{12}$$

To make the transition to the difference-frequency component, $p_+^{(i)} \to p_-^{(i)}$, $i = 1, 2, 3\,;$, it is sufficient to let $k_+ \to k_-$ and $p_2 \to p_2^*$.

We write the solution $p_+^{(1)}$ of Eq. (12) in the form of a volume integral over the product of the Green's function and the density of sources on the right-hand side of Eq. (12):

$$p_+(\boldsymbol{r}) = -\beta \frac{A_1 A_2}{\rho_0 c_0^2} k_+^2 \iiint \frac{\exp(ik_1 r')}{k_1 r'} \exp(ik_2 z') \frac{\exp(ik_+|\boldsymbol{r} - \boldsymbol{r}'|)}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|}\, d\boldsymbol{r}'. \tag{13}$$

We use integral representations for the functions in the integrand of Eq. (13) [see Ref. [8], p. 54, Eq. (20)]:

$$\begin{gathered}\frac{\exp(ik_1 r')}{k_1 r'} = \frac{i}{2k_1} \int\limits_{\infty \exp(i\pi)}^{\infty} \xi H_0^{(1)}(\xi\rho) \frac{\exp(i\sqrt{k_1^2 - \xi^2}|z'|)}{\sqrt{k_1^2 - \xi^2}}\, d\xi, \\ \operatorname{Im}\left(\sqrt{k_1^2 - \xi^2}\,\right) > 0;\end{gathered} \tag{14}$$

$$\begin{gathered}\frac{\exp(ik_+|\boldsymbol{r} - \boldsymbol{r}'|)}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|} = \frac{i}{8\pi} \int\limits_{\infty \exp(i\pi)}^{\infty} \zeta H_0^{(1)}(\zeta|\boldsymbol{\rho} - \boldsymbol{\rho}'|) \frac{\exp(i\sqrt{k_+^2 - \zeta^2}|z - z'|)}{\sqrt{k_+^2 - \zeta^2}}\, d\zeta, \\ \operatorname{Im}\left(\sqrt{k_+^2 - \zeta^2}\,\right) > 0,\end{gathered} \tag{15}$$

where $H_0^{(1)}(x)$ is a Hankel function, and $\boldsymbol{\rho} = (r \sin\theta \cos\varphi, r \sin\theta \sin\varphi)$. We substitute Eqs. (14) and (15) in (13). Invoking Gegenbauer's addition theorem [Ref. [9], p. 116, Eq. (29)], we have

$$\int\limits_0^{2\pi} H_0^{(1)}(\zeta|\boldsymbol{\rho} - \boldsymbol{\rho}'|)\, d\varphi' = 2\pi J_0(\zeta\rho_<) H_0^{(1)}(\zeta\rho_>), \tag{16}$$

where $J_0(x)$ is a Bessel function, $\rho_< \equiv \min\{\rho, \rho'\}$, and $\rho_> \equiv \max\{\rho, \rho'\}$. Next, for

$$\operatorname{Im}(\xi + \zeta) > 0 \tag{17}$$

it follows from Ref. [9] [p. 104, Eq. (9) and p. 91, Eq. (36)] that

$$\int\limits_0^{\infty} H_0^{(1)}(\xi\rho') J_0(\zeta\rho_<) H_0^{(1)}(\zeta\rho_>) \rho' d\rho' = \frac{2i}{\pi} \frac{H_0^{(1)}(\xi\rho) - H_0^{(1)}(\zeta\rho)}{\zeta^2 - \xi^2}. \tag{18}$$

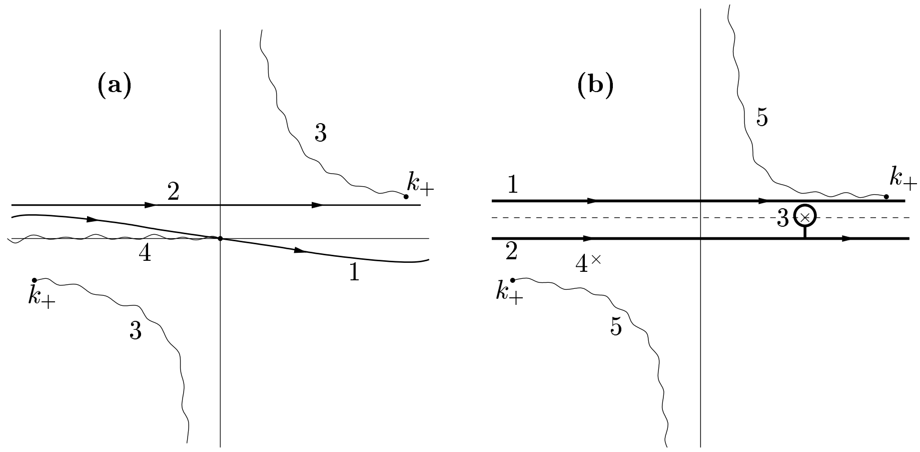

Figure 1: Contours of integration in the complex plane of ζ. (a) In the representation (15): 1 – original contour; 2 – deformed contour (Γ_2); 3 – cuts along lines $\mathrm{Im}(\sqrt{k_+^2 - \zeta^2}) = 0$; 4 – cut along half-line $\mathrm{Im}\,\zeta = 0$, $\zeta \leq 0$. (b) In the first integral in Eq. (19): 1 – original contour (Γ_2); 2 – deformed contour; 3,4 – poles at points $\zeta = \pm\xi$; 5 – cuts along lines $\mathrm{Im}(\sqrt{k_+^2 - \zeta^2}) = 0$.

We therefore obtain for $p_+^{(1)}$

$$
\begin{aligned}
p_+^{(1)}(\boldsymbol{r}) = \frac{\beta A_1 A_2}{\rho_0 c_0^2} \frac{i k_+^2}{4\pi k_1} \Big[&\int_{-\infty}^{\infty} dz' \int_{\Gamma_1} d\xi \int_{\Gamma_2} d\zeta \, \frac{\xi\zeta H_0^{(1)}(\xi\rho)}{\zeta^2 - \xi^2} \\
&\times \frac{\exp(i\sqrt{k_+^2 - \zeta^2}|z - z'| + i\sqrt{k_1^2 - \xi^2}|z'| + ik_2 z')}{\sqrt{k_+^2 - \zeta^2}\sqrt{k_1^2 - \xi^2}} \\
&+ \int_{-\infty}^{\infty} dz' \int_{\Gamma_2} d\zeta \int_{\Gamma_1} d\xi \, \frac{\xi\zeta H_0^{(1)}(\zeta\rho)}{\xi^2 - \zeta^2} \\
&\times \frac{\exp(i\sqrt{k_+^2 - \zeta^2}|z - z'| + i\sqrt{k_1^2 - \xi^2}|z'| + ik_2 z')}{\sqrt{k_+^2 - \zeta^2}\sqrt{k_1^2 - \xi^2}} \Big].
\end{aligned}
\tag{19}
$$

Since integrands in Eqs. (14) and (15) do not have singularities in the domains $0 < \mathrm{Im}\,\xi < \mathrm{Im}\,k_1$ and $0 < \mathrm{Im}\,\zeta < \mathrm{Im}\,k_+$ (Fig. 1; see also Ref. [8], pp. 27-31), we interpret the contours Γ_1 and Γ_2 as the lines $\mathrm{Im}\,\xi = \varepsilon_1$ and $\mathrm{Im}\,\xi = \varepsilon_2$, respectively, $0 < \varepsilon_1 < \varepsilon_2 < \min\{\mathrm{Im}\,k_1, \mathrm{Im}\,k_+\}$; condition (17) is satisfied in this case.

The integrand in the first integral in Eq. (19) is an odd function of the variable ζ, and the integrand in the second integral in an odd function of ξ.

Consequently, deforming the contour Γ_2 in the first integral and Γ_1 in the second integral until they coincide with the real axis, we have only to take into account the pole $\zeta = \xi$ in the first integral in Eq. (19). We obtain

$$p_+^{(1)}(\boldsymbol{r}) = \frac{\beta A_1 A_2}{\rho_0 c_0^2} \frac{k_+^2}{4k_1} \int_{-\infty}^{\infty} dz' \int_{\infty \exp(i\pi)}^{\infty} d\xi \; \frac{\xi H_0^{(1)}(\xi\rho)}{\sqrt{k_1^2 - \xi^2}\sqrt{k_+^2 - \xi^2}} \times \exp(i\sqrt{k_+^2 - \xi^2}|z - z'| + i\sqrt{k_1^2 - \xi^2}|z'| + ik_2 z'). \tag{20}$$

We require

$$\mathrm{Im}(k_1 + k_+ - k_2) > 0. \tag{21}$$

In this case

$$\int_{-\infty}^{\infty} \exp(i\sqrt{k_+^2 - \xi^2}|z - z'| + i\sqrt{k_1^2 - \xi^2}|z'| + ik_2 z')\, dz' = \frac{2i\sqrt{k_+^2 - \xi^2}\exp(ik_2 z \pm i\sqrt{k_1^2 - \xi^2} z)}{k_+^2 - k_1^2 - k_2^2 \mp 2k_2\sqrt{k_1^2 - \xi^2}} - \frac{2i\sqrt{k_1^2 - \xi^2}\exp(\pm i\sqrt{k_+^2 - \xi^2} z)}{k_+^2 - k_1^2 + k_2^2 \mp 2k_2\sqrt{k_+^2 - \xi^2}}, \qquad z \gtrless 0. \tag{22}$$

The physical significance of condition (21) is that the exponential growth of the amplitude of the plane wave p_2 in the limit $z \to \infty$ is offset by the decay of the spherical wave p_1 and the secondary waves generated in the interaction of p_1 and p_2.

The substitution of Eq. (22) into (20) yields

$$p_+^{(1)}(\boldsymbol{r}) = \frac{\beta A_1 A_2}{\rho_0 c_0^2} \frac{ik_+^2}{2k_1} \Big[\exp(ik_2 z) \int_{\infty \exp(i\pi)}^{\infty} \frac{\xi H_0^{(1)}(\xi\rho)\exp(\pm i\sqrt{k_1^2 - \xi^2} z)}{(k_+^2 - k_1^2 - k_2^2 \mp 2k_2\sqrt{k_1^2 - \xi^2})\sqrt{k_1^2 - \xi^2}}\, d\xi - \int_{\infty \exp(i\pi)}^{\infty} \frac{\xi H_0^{(1)}(\xi\rho)\exp(\pm i\sqrt{k_+^2 - \xi^2} z)}{(k_+^2 - k_1^2 + k_2^2 \mp 2k_2\sqrt{k_+^2 - \xi^2})\sqrt{k_+^2 - \xi^2}}\, d\xi, \Big] \quad z \gtrless 0. \tag{23}$$

The integrands in both integrals have simple poles at the points ξ_1 and ξ_2,

$$\xi_{1,2} = \pm \frac{i}{2k_2}\sqrt{k_1^4 + k_2^4 + k_+^4 - 2k_1^2 k_2^2 - 2k_1^2 k_+^2 - 2k_2^2 k_+^2}, \qquad \mathrm{Im}\,\xi_1 > 0.$$

The residues of the integrands in Eq. (23) at the point $\xi = \xi_1$ are equal. Consequently, making the changes of variables $\zeta = \sqrt{k_1^2 - \xi^2}$ in the first integral and $\zeta = \sqrt{k_+^2 - \xi^2}$ in the second integral, we obtain from Eq. (23) (Ref. [8], pp. 53-55)

$$p_+^{(1)}(\boldsymbol{r}) = \frac{\beta A_1 A_2}{\rho_0 c_0^2} \frac{i k_+^2}{2k_1} \Big[\exp(i k_2 z) \int_{-\infty}^{\infty} \frac{H_0^{(1)}(\sqrt{k_1^2 - \zeta^2}\rho)}{k_+^2 - k_1^2 - k_2^2 \mp 2k_2\zeta} \exp(\pm i\zeta z)\, d\zeta - \int_{-\infty}^{\infty} \frac{H_0^{(1)}(\sqrt{k_+^2 - \zeta^2}\rho)}{k_+^2 - k_1^2 + k_2^2 \mp 2k_2\zeta} \exp(\pm i\zeta z)\, d\zeta \Big], \quad z \gtrless 0. \tag{24}$$

We can now go from upper to the lower signs in both integrals by replacing ζ with $-\zeta$ and vice versa. It is possible therefore to use the upper signs exclusively, independently of the sign of z.

We write Eq. (24) for $p_+^{(1)}(\boldsymbol{r})$ in the form

$$p_+^{(1)}(\boldsymbol{r}) = \frac{\beta A_1 A_2}{\rho_0 c_0^2} \frac{i k_+^2}{2k_1 k_2} \exp\Big(\frac{k_+^2 + k_2^2 - k_1^2}{2k_2} z\Big) \big[I(k_1, \Delta_1, \boldsymbol{r}) - I(k_+, \Delta_+, \boldsymbol{r}) \big], \tag{25}$$

$$I(k, \Delta, \boldsymbol{r}) \equiv -\frac{1}{2} \exp[-i(k+\Delta)z] \int_{-\infty}^{\infty} \frac{H_0^{(1)}(\sqrt{k_1^2 - \zeta^2}\rho)}{\zeta - k - \Delta} \exp(i\zeta z)\, d\zeta, \tag{26}$$

$$\Delta_1 \equiv \frac{k_+^2 - (k_1 + k_2)^2}{2k_2}, \qquad \Delta_+ \equiv \frac{(k_+ - k_2)^2 - k_1^2}{2k_2}. \tag{27}$$

We now note that

$$\frac{\partial}{\partial z} I(k, \Delta, \boldsymbol{r}) = -\frac{i}{2} \exp[-i(k+\Delta)z] \int_{-\infty}^{\infty} H_0^{(1)}(\sqrt{k^2 - \zeta^2}\rho) \exp(i\zeta z)\, d\zeta = -\frac{1}{r} \exp[ik(r-z)] \exp(-i\Delta z).$$

From this result we obtain the alternative representation for $I(k, \Delta, \boldsymbol{r})$:

$$I(k, \Delta, \boldsymbol{r}) = -\int_{-\infty}^{z} \exp(-i\Delta t) \frac{\exp[ik(\sqrt{\rho^2 + t^2} - t)]}{\sqrt{\rho^2 + t^2}}\, dt \tag{28}$$

If dispersion is absent, we have $k_+ = k_1 + k_2, \quad \Delta_1 = \Delta_+ = 0$, and

$$I(k, 0, \boldsymbol{r}) = \mathrm{Ei}[ik(r-z)], \tag{29}$$

so that

$$\boxed{p_+^{(1)}(\boldsymbol{r}) = \frac{\beta A_1 A_2}{\rho_0 c_0^2} \frac{i k_+^2}{2k_1 k_2} \big\{ \mathrm{Ei}[ik_1(r-z)] - \mathrm{Ei}[ik_+(r-z)] \big\} \exp(ik_+ z).} \tag{30}$$

Equation (30) is our fundamental result. It shows, in particular, that the equal-amplitude surfaces of the component $p_+^{(1)}$ of the secondary sound field in the absence of the dispersion represent paraboloids of revolution with focus at point $\boldsymbol{r} = 0$.

In the limit $k_2/k_1 \to 0$

$$\mathrm{Ei}[ik_1(r-z)] - \mathrm{Ei}[ik_+(r-z)] = \int\limits_{ik_+(r-z)}^{ik_1(r-z)} \frac{\exp t}{t}\, dt$$
$$\sim \frac{k_2}{k_+} \exp[ik_+(r-z)] \exp(-iM) \frac{\sin M}{M},$$

where $M \equiv k_2(r-z)/2$. Substituting this expression in Eq. (30), we have

$$p+^{(1)} \sim \frac{\beta A_1 A_2}{\rho_0 c_0^2} \frac{i}{2} \frac{\sin M}{M} \exp(-iM) \exp(ik_+ r), \qquad \frac{k_2}{k_1} \to 0, \tag{31}$$

which agrees with the the previously obtained approximate results [1, 2].

In the general case $k_+ = k_1 + k_2 + \delta_+, \quad \delta_+ \neq 0$, making the change of variables

$$u = \sqrt{\frac{2k_1}{|\Delta_1|}} \frac{\sqrt{\rho^2 - t^2} - t}{\rho}$$

in the representation (28) for $I(k_1, \Delta_1, \boldsymbol{r})$ and

$$u = \sqrt{\frac{2k_+}{|\Delta_+|}} \frac{\sqrt{\rho^2 - t^2} - t}{\rho}$$

in the representation for $I(k_+, \Delta_+, \boldsymbol{r})$, in the case $\Delta(r-z) \ll 1$ we obtain

$$I(k_1, \Delta_1, \boldsymbol{r}) - I(k_+, \Delta_+, \boldsymbol{r}) = \int\limits_a^b \frac{\exp[i\lambda(u \pm 1/u)]}{u}\, du, \qquad \delta_+ \lessgtr 0, \tag{32}$$

where

$$a \equiv \sqrt{\frac{k_1}{k_+}} \sqrt{\frac{2k_2}{|\delta+|}} \frac{r-z}{\rho}, \qquad b \equiv \sqrt{\frac{k_+}{k_1}} \sqrt{\frac{2k_2}{|\delta+|}} \frac{r-z}{\rho}, \qquad \lambda \equiv \sqrt{\frac{k_1 k_+ |\delta_+|}{k_2}} \rho.$$

We obtain the component $p_+^{(2)}$ of the secondary sound field from the explicit equation (5):

$$p_+^{(2)} = -\frac{\beta A_1 A_2}{\rho_0 c_0^2} \frac{\exp(ik_1 r + ik_2 z)}{k_1 r} \Big(\frac{k_+^2}{2k_1 k_2} - \sin^2 \frac{\theta}{2} - i \frac{\cos\theta}{2k_1 r} \Big). \tag{33}$$

The form of the function q characterizing the sources of the components $p_+^{(3)}$ depends on the form of the sources of the sound field. When the latter are mass sources with a volume density ρs, the hydrodynamic equations give

$$q = \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\Big(s\int\limits_{-\infty}^{t} p'\,dt\Big), \tag{34}$$

so that in our case

$$q_+(\boldsymbol{r}) = \frac{\beta A_1 A_2}{\rho_0 c_0^2} 4\pi \frac{k_+^2}{2k_1 k_2}\frac{\delta(\boldsymbol{r})}{k_1}$$

and, accordingly,

$$p_+^{(3)} = \frac{\beta A_1 A_2}{\rho_0 c_0^2}\frac{k_+^2}{2k_1 k_2}\frac{\exp(ik_+ r)}{k_1 r}. \tag{35}$$

In conclusion we compare the values $p_+^{(1)}$, $p_+^{(2)}$, and $p_+^{(3)}$. It follows from Eqs. (30), (33) and (35) that when the synchronism conditions hold,

$$\left|\frac{p_+^{(1)}}{p_+^{(2)}}\right| = k_1 r \ln\frac{k_+}{k_1} \quad (k_1 z \gg 1), \qquad \left|\frac{p_+^{(1)}}{p_+^{(3)}}\right| = k_+ r \ln\frac{k_+}{k_1}, \qquad \theta = 0.$$

In the absence of synchronism, $k_1(r-z) \gg 1$, $k_+(r-z) \gg 1$, the scattering of sound by sound is not cumulative in space. The components $p_+^{(1)}$, $p_+^{(2)}$, and $p_+^{(3)}$ of the secondary field are commensurable in this case.

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