Week 9-12 Project (Project 3)

Answer problem 1 (A or B), 2 and 3. Problem 1A will be marked if both problem 1A and 1B are discussed in your report. Your report should include your code and numerical results (figures, estimates, equations), and results should be explained in words, e.g. formulation, code algorithms, numerical convergence, parameter dependence, comparison with analytic estimates, interpretations, you do not need to cover all, but you should give discussion sensibly. Brief discussion (a few paragraphs for each problem) is sufficient. Example code has been uploaded together with this project note, you should use the example code to tackle the problems below.

If you are not familiar with Octave/Matlab, you should first read through the Octave tutorial. Please see the Octave start guide to learn how to run octave. Feel free to post questions to the discussion board.

Problem 1A: 2dim Random Walk:
(a) Consider a particle which starts from the origin, and it moves in a random direction in each timestep: $(\Delta x, \Delta y) = (d\cos\theta, d\sin\theta)$ where $d = 0.01$ and θ is a random number between 0 and 2π. Plot a path of the particle up to 2000 timesteps.

Hints: run one of the example code: problem1Aa.m by typing "problem1Aa" at the Octave command prompt. You should modify this code to answer this question. To produce random angle θ, use a built-in function "rand" which produces random numbers between 0 and 1. Type "rand" at the Octave command prompt, you should get a different number between 0 and 1 each time you try. Type "help rand" at the Octave command prompt if you like to know how to use the function (help can be used to know other functions). Since the particle starts from the origin, the initial condition is $x(1) = 0$ and $y(1) = 0$. Consider how the position at timestep $i+1$ is related to the position at timestep i. By using a for-loop, you can run statements repeatedly, i.e. $x(2)$ and $y(2)$ are defined by using the previous position $x(1)$ and $y(1)$; $x(3)$ and $y(3)$ are defined by using the previous position $x(2)$ and $y(2)$,... the positions are defined recursively.

(b)Plot the positions of $N_p = 2000$ independent random particles at timestep 2000 in the x-y plane.

Hints: Nest two for-loops to obtain the final positions of $N_p = 2000$ particles. One for-loop is to evaluate the time evolution of each particle, by using another for-loop repeat the calculations N_p times.

```
for j=1:Np  % repeat Np times
    ....
    for i=1:2000   % time-evolution of a single particle
        ...
        x = x+...
        y = y+...
    end
    xfinal(j) = x; % the position of the j-th particle
    yfinal(j) = y;
    rfinal(j) = sqrt(x^2+y^2); % the distance from the origin
end
```

(c) Using the positions of the particles at timesep 2000 obtained in (b), evaluate the particle number distribution $N(r, r+\Delta r)$ for $\Delta r = 0.05$ where $r = \sqrt{x^2+y^2}$ is the distance from the origin, i.e. count the numbers of particles between 0 and Δr, between Δr and $2\Delta r$, between $2\Delta r$ and $3\Delta r$, ...

Hints: Histograms can be created by using a built-in function "histc". For vectors X and EDGES, counts=histc(X,EDGES) counts the number of the elements of X that fall in the histogram bins defined by EDGES. See one of the example code: problem1Ab.m for details. When writing a program, begin with a small number of particles (e.g $N_p = 10$). Only after completing programming and confirming it works properly, try with a larger number (e.g. $N_p = 2000$). To accurately estimate $N(r, r + \Delta r)$, a quite large number (e.g. $N_p = 10^5$ or even larger) might be needed. You can measure the time taken by a program to execute in Octave by using tic and toc function (see problem1Ab.m). First measure it for a small N_p and estimate how long it takes to complete the simulation with $N_p = 10^5$.

(d) Reevaluate the numerical results (c) for $N_p = 10^5$ and plot the numerical results with the analytic number distribution $N(r, r + \Delta r)$.

Hints: In the lecture, we discussed the probability distribution $p(x)$ for 1 dim random walk. This is readily extended to 2 dim random walk, since we can consider a walk with steps $(\pm L, \pm L)$, so that the walk is the product of walks in each dimension. The probability distribution $p(x, y)$ at timestep n satisfies

$$p(x,y) = p(x)p(y) = \frac{1}{2\pi n L^2} \exp\left(-\frac{x^2+y^2}{2nL^2}\right) = \frac{1}{\pi n d^2} \exp\left(-\frac{r^2}{nd^2}\right), \tag{1}$$

where $d = \sqrt{L^2 + L^2} = \sqrt{2}L$ is the jump size at each timestep. Since the probability distribution is normalized: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x, y)dxdy = 1$, the particle density distribution for N_p particles is given by

$$\rho(r) = \frac{N_p}{\pi n d^2} \exp\left(-\frac{r^2}{nd^2}\right). \tag{2}$$

Integrating the density between r and $r + \Delta r$, we get the number distribution,

$$N(r, r+\Delta r) \quad = \quad N_p \exp\left(-\frac{r^2}{nd^2}\right)\left\{1 - \exp\left(-\frac{\Delta r(2r+\Delta r)}{nd^2}\right)\right\}. \tag{3}$$

(e) Energy is released in the core of the Sun in the form of neutrinos and radiation. The neutrinos escape into space a few seconds after being created, due to the fact that they interact extremely weakly with matter, while the photons take a much longer time to escape. For simplicity, we assume that photons follow 2dim random walks with step size of d =1mm. Using the analytic results discussed in (d), estimate how long it takes for photons to diffuse through the Sun, e.g. When does the peak of the number distribution $N(r, r+dr) = 2\pi r\rho(r)dr$ reach the surface of the Sun? Once photons reach the surface, they freely propagate and the photon distribution near the surface might be slightly different from what we discussed in (d). For simplicity, you can ignore such effects.

Problem 1B: Heat Conduction

This problem is related to a historical work by a British physicist, Lord Kelvin of Glasgow, who published in 1862 calculations that fixed the age of the Earth at between 20 million and 400 million years. He assumed that the Earth had been created as a completely molten ball of rock, and determined the amount of time it took for the ball to cool to its present temperature distribution (he paid attention to the surface temperature gradient). His calculations did not account for the ongoing heat source in the form of radioactive decay, which was unknown at the time. That is why he significantly underestimated the age: the current estimate is about 4.5 billion years. In this problem, we also do NOT consider such internal heat sources. Another important mechanism which Kelvin neglected is convection – the transfer of heat not through heat conduction but through the movement of hot parts to the surface; this is a process familiar in home cooking. We also do not consider the convection in the problem.

Consider a ball of rock with radius a. We assume that the initial temperature distribution is homogeneous in the entire volume: $T = T_0$ at $t < 0$, and the surface of the ball is suddenly cooled at $t = 0$, and the surface temperature is kept at zero at $t \geq 0$. As discussed in the lecture, the temperature gradient at the centre should be always zero, because there are no heat sources in the rock, especially at the centre.

$$
\begin{aligned}
T(t=0, r<a) &= T_0 &(4)\\
T(t, r=a) &= 0 &(5)\\
\frac{\partial T}{\partial r}|_{r=0} &= 0 &(6)
\end{aligned}
$$

The diffusion equation for the spherical system is given by

$$
\frac{\partial T}{\partial t} = \eta \left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} \right), \qquad (7)
$$

where η is the thermal diffusivity. We rewrite the initial and boundary conditions, and the diffusion equation by using dimensionless variables: $x \equiv r/a, \tau \equiv t/(a^2/\eta)$, $u \equiv T/T_{in}$ as

$$
\begin{aligned}
u(0, x) &= 1 &(8)\\
u(\tau, 1) &= 0 &(9)\\
\frac{\partial u}{\partial x}|_{x=0} &= 0 &(10)
\end{aligned}
$$

$$
\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + \frac{2}{x}\frac{\partial u}{\partial x} \qquad (11)
$$

The surface of the sphere is at $x = 1$, and we consider the temperature distribution $u(\tau, x)$ in the range of $0 \leq x \leq 1$.

We now consider the system in the discretized grid spacetime. Grid points are specified by two positive integers: time-index n ($n = 1, 2, 3, ...$) and space-index j ($j = 1, 2, ..., N$) as $\tau_n = (n-1)\Delta_\tau$ and $x_j = (j-1)\Delta_x$. Δ_τ and $\Delta_x = 1/(N-1)$ are the time-step and grid-spacing, respectively. Approximating the derivatives by finite differences,

$$\frac{\partial u}{\partial \tau} \sim \frac{u_j^{n+1} - u_j^n}{\Delta_\tau} \tag{12}$$

$$\frac{2}{x}\frac{\partial u}{\partial x} \sim \frac{2}{(j-1)\Delta_x}\frac{u_{j+1}^n - u_j^n}{\Delta_x} \tag{13}$$

$$\frac{\partial^2 u}{\partial x^2} \sim \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta_x^2} \tag{14}$$

where $u_j^n \equiv u(\tau_n, x_j)$, the diffusion equation is given by

$$u_j^{n+1} = \left\{1 - 2\mu\left(1 + \frac{1}{j-1}\right)\right\} u_j^n + \mu u_{j-1}^n + \mu\left(1 + \frac{2}{j-1}\right) u_{j+1}^n, \tag{15}$$

where $\mu = \Delta_\tau/\Delta_x^2$. The initial and boundary conditions are

$$u_j^1 = 1 \ (1 \le j < N) \tag{16}$$

$$u_N^n = 0 \ (n \ge 1) \tag{17}$$

$$u_1^n = u_2^n \ (n \ge 1) \tag{18}$$

Note that in the lecture we used a notation in which n and j start from zero (i.e. n=0,1,2,3,..), we here used a different notation in which they start from 1 (i.e. n=1,2,3,....) because of a technical reason: the first element of a vector x is x(1) rather than x(0) in Octave.

(a) Write a numerical code to evaluate the evolution of the temperature distribution. For $\mu = 0.2$, plot the distributions at several timesteps (between $\tau = 0$ and 0.2) together.

Hints: You do not need to write code from scratch, modify problem1B.m. If the temperature distribution at timestep n is known, we can evaluate the distribution at the next timestep $n+1$ by using Equation (15) where u_j^{n+1} is given by a weighted linear sum of u_{j-1}^n, u_j^n and u_{j+1}^n (Define A, B and C in problem1B.m). However, we can not apply this equation to the boundaries $j = 1$ (or N), because it requires u_0^n (or u_{N+1}^n) to evaluate u_1^{n+1} (or u_N^{n+1}) (That's why j runs from 2 to $N-1$ in problem1B.m). The temperature at the boundaries is determined by the boundary conditions: Eqs. (17) and (18), e.g. after evaluating u_2^{n+1} by using Eq (15), the value is substituted into u_1^{n+1} (Define unew(1) and unew(N) in problem1B.m).

(b) When the rock sphere has a radius of $a = 10$ km and thermal diffusivity $\eta = 10^{-6}\mathrm{m}^2/\mathrm{s}$, using your numerical results (clearly explain how you have used them), estimate time (in units of year) taken for the temperature at the centre $r = 0$ to become the half of the initial value.

Problem 2: Circular Binary

First we consider two stars attracting each other with Newton's gravity forces (binary stars). The primary star with mass m_p is at position $\vec{x}_p$, and the secondary one with m_s is at position $\vec{x}_s$. Their equations of motion are given by

$$m_p \frac{d^2\vec{x}_p}{dt^2} = \frac{G m_p m_s}{|\vec{x}_s - \vec{x}_p|^3} \left(\vec{x}_s - \vec{x}_p\right), \tag{19}$$

$$m_s \frac{d^2\vec{x}_s}{dt^2} = \frac{G m_p m_s}{|\vec{x}_s - \vec{x}_p|^3} \left(\vec{x}_p - \vec{x}_s\right). \tag{20}$$

As discussed in the "Random Walk and Diffusion" lecture, we need to pay attention to units when writing programs. In this problem, we will use characteristic scales in the system to measure physical quantities. Using the total mass $m = m_p + m_s$, the initial binary separation a, the dynamical timescale $\sqrt{a^3/Gm}$, we define dimensionless mass, distance and time as

$$\tilde{m}_p \equiv \frac{m_p}{m}, \quad \tilde{m}_s \equiv \frac{m_s}{m}, \tag{21}$$

$$\tilde{\vec{x}}_p \equiv \frac{\vec{x}_p}{a}, \quad \tilde{\vec{x}}_s \equiv \frac{\vec{x}_s}{a}, \tag{22}$$

$$\tilde{t} \equiv \frac{t}{\sqrt{a^3/Gm}}. \tag{23}$$

Substituting these relations (e.g. $\vec{x}_p = a\tilde{\vec{x}}_p$) into the equations of motion, we get

$$\frac{d^2\tilde{\vec{x}}_p}{d\tilde{t}^2} = \frac{\tilde{m}_s}{|\tilde{\vec{x}}_s - \tilde{\vec{x}}_p|^3} \left(\tilde{\vec{x}}_s - \tilde{\vec{x}}_p\right), \tag{24}$$

$$\frac{d^2\tilde{\vec{x}}_s}{d\tilde{t}^2} = \frac{\tilde{m}_p}{|\tilde{\vec{x}}_s - \tilde{\vec{x}}_p|^3} \left(\tilde{\vec{x}}_p - \tilde{\vec{x}}_s\right), \tag{25}$$

(You should derive these equations by yourself). In the following discussion, we will drop 'tilde' for simplicity. Introducing velocities of the two stars: $\vec{v}_{p,s} = d\vec{x}_{p,s}/dt$, we rewrite the 2nd-order differential equations as a set of 1st-order differential equations,

$$\frac{d\vec{x}_p}{dt} = \vec{v}_p, \tag{26}$$

$$\frac{d\vec{v}_p}{dt} = \frac{m_s}{|\vec{x}_s - \vec{x}_p|^3} \left(\vec{x}_s - \vec{x}_p\right), \tag{27}$$

$$\frac{d\vec{x}_s}{dt} = \vec{v}_s, \tag{28}$$

$$\frac{d\vec{v}_s}{dt} = \frac{m_p}{|\vec{x}_s - \vec{x}_p|^3} \left(\vec{x}_p - \vec{x}_s\right). \tag{29}$$

or equivalently

$$\frac{d}{dt}\mathbf{X} = \mathbf{F} \tag{30}$$

where

$$\mathbf{X} \;=\; \begin{bmatrix} x_p \\ y_p \\ v_{px} \\ v_{py} \\ x_s \\ y_s \\ v_{sx} \\ v_{sy} \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} v_{px} \\ v_{py} \\ (m_s/r^3)\,(x_s - x_p) \\ (m_s/r^3)\,(y_s - y_p) \\ v_{sx} \\ v_{sy} \\ (m_p/r^3)\,(x_p - x_s) \\ (m_p/r^3)\,(y_p - y_s) \end{bmatrix}, \tag{31}$$

where $r = |\vec{x}_s - \vec{x}_p| = \sqrt{(x_s - x_p)^2 + (y_s - y_p)^2}$ and we have chosen the Cartesian coordinate as the orbits of the stars stay in the x-y plane (it is always possible for two-body problems).

As Dr Maciejewski has discussed, ordinary differential equations can be numerically integrated by using the Runge-Kutta method. I have provided such code: binary.m, RK4.m, f.m, energy.m in the problem 2 folder. The main code is binary.m in which $\mathbf{X}$ vector (x in the code) at the next timestep is recurrently evaluated until $t = tmax$. The main part of binary.m is

```
while t < tmax
     x = RK4(h,t,x,mp,ms);
     t = t+h;
end
```

where for a given timestep h, time t, vector x, and star masses m_p and m_s, the vector x at the next timestep is evaluated by using the function RK4.m. The time is also updated at each timestep: $t = t + h$. The function f.m which defines the $\mathbf{F}$ vector (f in the code) is used in the function RK4.m. Although I provided the general Runge-Kutta code RK4.m which can be used for other problems, since $\mathbf{F}$ does not explicitly depend on t in this problem, it is possible to drop the t dependence in RK4.m and f.m (you do not need to modify them, keep them in the current form).

The positions of the primary and secondary stars: $x_p = x(1), y_p = x(2), x_s = x(5), y_s = x(6)$, are saved together with time t and the binary energy E in a data file "out". If we save these quantities at each timestep, the frequent access to the hard disc drive would slow down the calculation and the data file "out" might become unnecessarily large. To avoid these problems, the date vector v is saved with a sampling period of dtp which is much larger than the timestep h for the dynamical evolution (we save only $Ns = 100$ times during the evolution).

```
if  t >= tprint
    E = energy(x,mp,ms);
    v = [t x(1) x(2) x(5) x(6) E];
    save out v -ascii -append
    tprint =  tprint + dtp;
end
```

We define the energy of the binary by using the dimensional quantities as,

$$E = \frac{1}{2}m_p v_p^2 + \frac{1}{2}m_s v_s^2 - \frac{Gm_p m_s}{r}. \tag{32}$$

The function energy.m gives this energy in units of Gm^2/a, i.e. the dimensionless energy. Since we have chosen specific mass, length, and time units, energy should be in units of Gm^2/a. Note that the energy has the dimension of (mass)*(length)^2/(time)^2.

(a) Assuming a circular binary, give the initial conditions for the dimensionless variables, i.e. the dimensionless positions and velocities of the two stars. You should be able to express these quantities by using $\tilde{m}_p$ and $\tilde{m}_s$. You can assume that the orbits of the stars in the x-y plane, and they are on x-axis at $t = 0$ (the primary is on the negative x-axis and the center of mass is at the origin). The binary rotation should be counterclockwise. Estimate the binary's rotation period in the unit of $\sqrt{a^3/Gm}$ (i.e. the dimensionless period).

(b) Evaluate the values of $\tilde{m}_p$ and $\tilde{m}_s$ for $m_p/m_s = 4$, and substitute the values into mp and ms in binary.m. The code is written with the dimensionless quantities. Using the results obtained in (a), give the values of x(1), x(2),... x(8) in binary.m. You should be able to express these quantities by using mp and ms. Set the parameter tmax in binary.m which should be long enough to allow the stars to rotate around the center of mass a few times.

(c) Run the code binary.m which will produce a data file 'out'. Using this data file and orbitplot.m, plot the numerical orbits of the two stars: the primary star (the blue solid line) and secondary star (the red dotted line).

d) Plot the evolution of the numerical energy (which should be a constant), together with the exact value $\tilde{E}_{exact} = -\tilde{m}_p\tilde{m}_s/2$. Derive this analytic estimate. Use energyplot.m to plot the energies: numerical results (the blue circles) and the exact (the red line). You need to specify mp, ms and tmax in energyplot.m before using it.

Problem 3: Hypervelocity Stars

Now we consider a binary moving around a massive black hole (BH) with mass M. The masses of the primary and secondary stars are m_p and m_s, respectively. The initial separation between the binary stars is a, and the total mass of the binary is $m = m_p + m_s$. We assume that the BH is at the origin and it does not move because of the large mass $M \gg m$, and that the interaction between the stars and the BH is described by the Newton gravity (i.e. the binary travels well outside the BH horizon. The study of relativistic hypervelocity stars is beyond the scope of this module).

The equations of motion are given by

$$\begin{aligned} m_p \frac{d^2\vec{x}_p}{dt^2} &= \frac{G m_p M}{|\vec{x}_p|^3}\,(-\vec{x}_p) + \frac{G m_p m_s}{|\vec{x}_s - \vec{x}_p|^3}\,(\vec{x}_s - \vec{x}_p)\,, &(33)\\ m_s \frac{d^2\vec{x}_s}{dt^2} &= \frac{G m_s M}{|\vec{x}_s|^3}\,(-\vec{x}_s) + \frac{G m_p m_s}{|\vec{x}_s - \vec{x}_p|^3}\,(\vec{x}_p - \vec{x}_s)\,. &(34)\end{aligned}$$

Compared to the equations of motion for an isolated binary (problem 2), there are additional BH gravity terms (i.e. the first terms in the RHS of the equations). Using the same dimensionless masses, lengths, and time discussed in the problem 2, we can rewrite the equations as

$$\begin{aligned} \frac{d^2\tilde{\vec{x}}_p}{d\tilde{t}^2} &= \frac{\tilde{M}}{|\tilde{\vec{x}}_p|^3}\left(-\tilde{\vec{x}}_p\right) + \frac{\tilde{m}_s}{|\tilde{\vec{x}}_s - \tilde{\vec{x}}_p|^3}\left(\tilde{\vec{x}}_s - \tilde{\vec{x}}_p\right), &(35)\\ \frac{d^2\tilde{\vec{x}}_s}{d\tilde{t}^2} &= \frac{\tilde{M}}{|\tilde{\vec{x}}_s|^3}\left(-\tilde{\vec{x}}_s\right) + \frac{\tilde{m}_p}{|\tilde{\vec{x}}_s - \tilde{\vec{x}}_p|^3}\left(\tilde{\vec{x}}_p - \tilde{\vec{x}}_s\right), &(36)\end{aligned}$$

where $\tilde{M} \equiv M/m$. In the following discussion, we will drop 'tilde' for simplicity. Introducing velocities of the two stars, we can further rewrite the equations as

$$\frac{d}{dt}\mathbf{X} = \mathbf{F} \tag{37}$$

where

$$\mathbf{X} = \begin{bmatrix} x_p \\ y_p \\ v_{px} \\ v_{py} \\ x_s \\ y_s \\ v_{sx} \\ v_{sy} \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} v_{px} \\ v_{py} \\ (M/r_p^3)(-x_p) + (m_s/r^3)\,(x_s - x_p) \\ (M/r_p^3)(-y_p) + (m_s/r^3)\,(y_s - y_p) \\ v_{sx} \\ v_{sy} \\ (M/r_s^3)(-x_s) + (m_p/r^3)\,(x_p - x_s) \\ (M/r_s^3)(-y_s) + (m_p/r^3)\,(y_p - y_s) \end{bmatrix}, \tag{38}$$

where $r_p = \sqrt{x_p^2 + y_p^2}$, $r_s = \sqrt{x_s^2 + y_s^2}$, and $r = \sqrt{(x_s - x_p)^2 + (y_s - y_p)^2}$. We can numerically evaluate the evolution of the system in the same way discussed for the isolated binary. The difference in the formulation between the two problems are the definition of **F** (see eqs 31 and 38) and the initial condition which will be discussed below.

We assume $M = 4 \times 10^6$ solar mass (such a massive BH is actually at the centre of our Galaxy. e.g. see `http://www.einstein-online.info/spotlights/milkyway_bh.html`), $m_p = 3.2$ solar mass and $m_s = 0.8$ solar mass. The binary is assumed to be a circular binary with separation $a = 7$ solar radius when it is located well outside the BH tidal radius $R_{tidal} = (M/m)^{1/3}a$. The angular momentum vector of the binary star rotation around its own center of mass (CM) is assumed to be parallel to that of the binary's CM rotation around the BH (under this assumption, the system becomes 2 dimensional, i.e. the stars move in x-y plane, and the calculations are simpler). We inject the binary in a parabolic orbit with periastron $R_p = 3R_{tidal}$ around the BH at $R_0 = 10R_{tidal}$ and follow the evolution (i.e. the initial distance at $t = t_0$ between the BH and the binary's CM is $R_0 = 10R_{tidal}$, and the center of mass of the binary approaches the BH in a parabolic orbit). The initial binary phase at t_0 is arbitrary, and it could take any value. For this problem, I suggest a binary phase $\phi_0 = \pi/2$ where it is the angel between $(\vec{x}_s - \vec{x}_p)$ and the positive x-axis.

Assuming the binary's CM passes through the periastron at $t = 0$, numerically evaluate the orbits of the two stars from $t = t_0 < 0$ to $t = -t_0$.

(a) Plot the evolution of the dimensionless positions (the orbits) of the two stars in the BH rest frame.

(b) Plot the evolution of the dimensionless position (the orbit) of the secondary star in the comoving frame of the primary star.

(c) Plot the energies E_p and E_s of the two stars as functions of time.

$$E_p = \frac{1}{2}m_p v_p^2 - \frac{Gm_p M}{r_p} - \frac{Gm_p m_s}{r_{ps}}, \tag{39}$$

$$E_s = \frac{1}{2}m_s v_s^2 - \frac{Gm_s M}{r_s} - \frac{Gm_p m_s}{r_{ps}}, \tag{40}$$

where r_p, r_s and r_{ps} indicate the distance between the BH and the primary, that between the BH and the secondary, and that between the two stars, respectively. When you plot the evolution, the energies and time should be in the units of Gm^2/a and $\sqrt{a^3/Gm}$, respectively (i.e. plot the dimensionless quantities).

(d) Repeat the same calculations/plots (a)-(c) for a closer encounter case with the same parameters except $R_p = 0.1R_{tidal}$.

(e) In the case of $R_p = 0.1 R_{tidal}$, the binary will be disrupted by the BH. By using your numerical results obtained in (d), estimate the velocity of the ejected star at $r = \infty$ (i.e. far away from the BH) in the units of km/s.

(f) Investigate your own problem (or one of the examples i-iii listed below) related to the tidal disruption of a binary by a massive black hole. In your report, clearly explain what you try to investigate, why you think the problem is interesting, and whether you have any astronomical implications. Since this question is challenging, the marking would be generous. Possible problems are something like (i) which binary member is more likely to be captured around the BH (i.e. negative final energy) when an unequal-mass binary is disrupted? You can investigate this question by repeating the same simulation with different binary phases ϕ_0: uniformly sample ϕ_0 between 0 and 2π. (ii) Does the orientation of the binary rotation axis (e.g. counterclockwise or clockwise) affect the outcomes (e.g. binary disruption chance, the ejection velocity)? (iii) is the binary always disrupted at a very close BH encounter $D \ll 1$?

Hints for questions (a)-(e):

(1) To take into account the differences in $\mathbf{F}$ (see Eqs 31 and 38), modify f.m (i.e. add the BH gravity terms). Since the BH mass is in the BH gravity terms, the 1st line of f.m should be

```
function dxdt = f(t,x,mb,mp,ms)
```

where a new input parameter mb (the BH mass) has been added. You should use mb as the BH mass in the BH gravity terms.

(2) You can use basically the same RK4.m, but you need to add the new input parameter mb. Use the new RK4.m in the problem 3 folder.

(3) The main code is HVS.m which is very similar to binary.m. You need to specify the values of mb, mp, and ms in HVS.m. Note that these are dimensionless masses and you should give them in units of the binary mass $m = m_p + m_s$.

(4) If the initial conditions, i.e. t and x, are properly defined in initialc.m, you can numerically evolve the system by using HVS.m, RK4.m and f.m, initialc.m and energy.m. The evaluation of the initial conditions is the most challenging part of this problem, and we use the following well-known results (see your mechanics books). A parabolic orbit of the binary's CM is given by

$$\vec{X}_{cm} = (R\cos f, R\sin f)\,, \qquad R = \frac{2R_p}{1+\cos f} \tag{41}$$

where R_p is the closest approach (periastron), f is the angle from the point of the closest approach (i.e. the angle from the positive x-axis). The angle f, known as the true anomaly, is a function of time, but analytically one has only the time as a function of f.

$$t = \frac{\sqrt{2}}{3}\sqrt{\frac{R_p^3}{GM}}\tan(f/2)\ (3 + \tan^2(f/2)) \tag{42}$$

Its differential form is

$$\frac{df}{dt} = \sqrt{\frac{GM}{R_p^3}}\frac{\sqrt{2}}{4}(1 + \cos f)^2 \tag{43}$$

For the initial radius $R_0 = 10R_{tidal}$, the initial angle is obtained from Equation (41)

$$f_0 = -\cos^{-1}\left(-1 + \frac{D}{5}\right), \tag{44}$$

where $D \equiv R_p/R_{tidal} = 3$ is the penetration factor. Using this angle, the initial dimensionless time is

$$t_0 = \frac{\sqrt{2}}{3}D^{3/2}\tan(f_0/2)\left(3 + \tan^2(f_0/2)\right). \tag{45}$$

Since the binary's CM passes through the periastron $R = R_p$ at $t = 0$ and $f = 0$, the initial time should be negative $t_0 < 0$ (and $f_0 < 0$).

Using the position of the CM $\vec{X}_{cm}$ and the positions of the two stars relative to the CM $\vec{r}_i \ \ (i = p, s)$, the positions of the two stars (relative to the BH) are given by

$$\vec{x}_p = \vec{X}_{cm} + \vec{r}_p, \qquad \vec{x}_s = \vec{X}_{cm} + \vec{r}_s \tag{46}$$

As considered in the problem 2, the initial dimensionless positions in the comoving frame of the binary's CM are given by

$$\vec{r}_p = (m_s\cos(\phi_0 + \pi), m_s\sin(\phi_0 + \pi)), \qquad \vec{r}_s = (m_p\cos\phi_0, m_p\sin\phi_0) \tag{47}$$

We assume the initial binary phase $\phi_0 = \pi/2$ in this problem. The initial velocities of the stars are given by evaluating the following equations at $t = t_0$,

$$\dot{\vec{x}}_p = \dot{\vec{X}}_{cm} + \dot{\vec{r}}_p, \qquad \dot{\vec{x}}_s = \dot{\vec{X}}_{cm} + \dot{\vec{r}}_s. \tag{48}$$

the time derivatives $\dot{\vec{r}}_p$ and $\dot{\vec{r}}_s$ should be easily evaluated as you have done in the problem 2. To estimate $\dot{\vec{X}}_{cm}$, use the following equations for the dimensionless quantities which are obtained from Eqs (41) and (43) (you should derive these by yourself),

$$\dot{R} = \frac{M^{1/3}}{\sqrt{2D}}\sin f, \qquad \dot{f} = \frac{\sqrt{2}}{4}D^{-3/2}(1 + \cos f)^2, \tag{49}$$

where we have dropped tilde (e.g. M means M/m in the equation above).

(5) Using the results above, define the initial conditions in initialc.m.

(6) E_p and E_s are evaluated in the new energy.m in the problem 3 folder.

- HVS.m: Add the values of the mb, mp and ms parameters.
- initialc.m: Evaluate the initial conditions: t and x
- f.m: use the new f.m and add BH gravity terms
- RK4.m: use the new RK4.m (no modifications needed)
- energy.m: use the new energy.m (no modifications needed)
- Modify script4.m to plot numerical results.