Linear Programming: Key Concepts and Methods

Linear programming (LP) is a mathematical technique used to find the best possible outcome—such as maximizing profit or minimizing cost—given a set of linear constraints and a linear objective function. It is widely applied in business, economics, engineering, and logistics for optimal resource allocation.

Graphical Method

- Used for LP problems with two decision variables.
- Steps:
  - Formulate the problem: Define the objective function and constraints.
  - Plot each constraint as a straight line on the XY-plane by converting inequalities to equations.
  - Identify the feasible region where all constraints overlap.
  - Find the corner points (vertices) of the feasible region.
  - Evaluate the objective function at each vertex; the optimal value occurs at one of these points.
- Best suited for small problems (two variables), as it allows visualization and easy identification of the optimal solution.

General Linear Programming Problem

A general LP problem can be stated as:

- **Objective:** Maximize or minimize $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$
- **Subject to constraints:**
  - $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$
  - $a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$
  - $\vdots$
  - $a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m$
- **Non-negativity:** $x_j \geq 0$ for all j

Canonical and Standard Form of LPP

- **Standard Form:**
  - All constraints are equations (equalities) with non-negative right-hand sides.

    - All variables are non-negative.
    - All constraints are of the form $ \leq $.
    - Slack variables are added to convert inequalities to equalities.
- **Canonical Form:**
    - The form used in the simplex method.
    - The objective function is to be maximized.
    - All constraints are equations (after adding slack, surplus, or artificial variables as needed).
    - The basic variables (including slacks and artificials) have coefficients of 1 in one constraint and 0 in others.

Simplex Method

- An iterative algebraic procedure for solving LP problems with more than two variables.
- Steps:
    - Convert the problem to standard/canonical form.
    - Set up the initial simplex tableau.
    - Identify the entering and leaving variables using the pivot rules (largest coefficient in the objective row for maximization).
    - Perform row operations to update the tableau.
    - Repeat until no further improvement is possible (all coefficients in the objective row are non-negative for maximization).
- The final tableau gives the optimal solution.

Artificial Variable Technique

- Used when constraints cannot be converted to equations with only slack or surplus variables (e.g., equations or "≥" constraints).
- Artificial variables are introduced to start the simplex method.
- These variables are penalized in the objective function to ensure they are removed from the solution.

Big M Method

- A specific artificial variable technique.
- Artificial variables are added with a very large penalty (M) in the objective function (e.g., $Z = \cdots - M \times (\text{artificial variable})$).
- The simplex method is applied; the optimal solution should have all artificial variables equal to zero.
- If any artificial variable remains non-zero in the final solution, the original problem is infeasible.

Dual Simplex Method

- Used when the initial solution is not feasible but optimality conditions are satisfied.
- The dual simplex iteratively restores feasibility while maintaining optimality.
- At each step, the most negative value in the right-hand side (RHS) identifies the leaving variable, and the entering variable is chosen to maintain optimality.

Two-Phase Simplex Method

- Another artificial variable technique.
- **Phase I:** Introduce artificial variables and minimize their sum to find a feasible solution.
- **Phase II:** Use the feasible solution as the starting point for the original objective function and apply the simplex method.
- Ensures that artificial variables are eliminated before optimizing the original objective.