Transportation Problem Overview

A transportation problem is a type of linear programming problem where the objective is to determine the most cost-effective way to distribute goods from multiple sources (e.g., factories) to multiple destinations (e.g., warehouses), subject to supply and demand constraints. The goal is to minimize the total transportation cost while meeting all supply and demand requirements.

Mathematical Form of the Transportation Problem

The transportation problem is formulated as a linear programming problem where the objective is to minimize the total transportation cost of shipping goods from multiple sources to multiple destinations, subject to supply and demand constraints.

Mathematical Formulation:

Let:

- m: Number of sources (origins)
- n: Number of destinations
- a_i: Supply at source i ($i = 1, 2, ..., m$)
- b_j: Demand at destination j ($j = 1, 2, ..., n$)
- c_{ij}: Cost of transporting one unit from source i to destination j
- x_{ij}: Quantity transported from source i to destination j

Objective Function:

$$\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to:

$$\sum_{j=1}^{n} x_{ij} = a_i \ \forall i = 1,2,\ldots,m \text{ (Supply constraints)}$$

$$\sum_{i=1}^{m} x_{ij} = b_j \ \forall j = 1,2,\ldots,n \text{ (Demand constraints)}$$

$$x_{ij} \geq 0 \ \forall i,j$$

This is the standard form for a balanced transportation problem, where total supply equals total demand ($\sum a_i = \sum b_j$).

Methods to Solve the Transportation Problem

1. North-West Corner Method

- **Purpose:** Finds an initial basic feasible solution (BFS).
- **Procedure:**
  - Start at the top-left (north-west) cell of the transportation table.
  - Allocate as much as possible to this cell (minimum of supply or demand).
  - Adjust the supply and demand for the row/column.
  - Move right if supply is exhausted, move down if demand is exhausted.
  - Repeat until all supplies and demands are allocated.
- **Advantages:** Simple and quick to apply.
- **Disadvantages:** Does not consider transportation costs, so the solution may not be close to optimal.

2. Least Cost Method

- **Purpose:** Also finds an initial BFS, but considers costs.
- **Procedure:**
  - Identify the cell with the lowest transportation cost.
  - Allocate as much as possible to this cell (minimum of supply or demand).
  - Adjust supply and demand, cross out satisfied rows/columns.
  - Repeat for the next lowest cost cell until all allocations are made.
- **Advantages:** Usually yields a better starting solution than the North-West Corner Method.
- **Disadvantages:** Still may not be optimal; only considers immediate costs, not the overall effect.

3. Vogel’s Approximation Method (VAM)

- **Purpose:** Provides a better initial BFS by considering penalties.
- **Procedure:**
  - For each row and column, calculate the penalty (difference between the two lowest costs).

   - Identify the row or column with the highest penalty.
   - Allocate as much as possible to the lowest cost cell in that row/column.
   - Adjust supply/demand, recalculate penalties, and repeat until complete.
- **Advantages:** Often produces an initial solution very close to the optimal.
- **Disadvantages:** More complex and time-consuming than the other two methods.

4. MODI Method (Modified Distribution Method/U-V Method)

- **Purpose:** Tests the optimality of a BFS and improves it if necessary.
- **Procedure:**
   - Assign dual variables (U for rows, V for columns) such that for occupied cells: $C_{ij} = U_i + V_j$.
   - For unoccupied cells, compute opportunity cost: $\Delta_{ij} = C_{ij} - (U_i + V_j)$.
   - If all $\Delta_{ij} \geq 0$, the solution is optimal.
   - If any $\Delta_{ij} < 0$, select the cell with the most negative value and adjust allocations along a closed loop to improve the solution.
   - Repeat until optimality is reached.
- **Advantages:** Guarantees the optimal solution.
- **Disadvantages:** Requires an initial BFS and can be computationally intensive for large problems.

5. Assignment Problem

- **Definition:** A special case of the transportation problem where supply and demand at each source and destination is exactly one (e.g., assigning workers to jobs).
- **Solution Methods:** Typically solved using the Hungarian Method, which is efficient for this specific structure[5].
- **Relation to Transportation Problem:** Can be modeled as a transportation problem with all supplies and demands equal to one.

6. Degeneracy in Transportation Problem

- **Definition:** Occurs when the number of allocations in a BFS is less than $m + n - 1$ (where m is the number of sources and n is the number of destinations).

- **Implications:** Degeneracy can cause issues in the MODI method because it requires $m + n - 1$ allocations to compute unique dual variables.
- **Remedy:** Introduce a very small allocation (epsilon, ε) in one or more empty cells to maintain the required number of allocations, ensuring the solution remains feasible and the MODI method can proceed.

Key Solution Concepts

Feasible Solution

A feasible solution is any set of non-negative allocations x_{ij} that satisfies all supply and demand constraints (i.e., all goods shipped from sources do not exceed their supply, and all demands at destinations are met exactly).

Basic Feasible Solution (BFS)

A basic feasible solution is a feasible solution where the number of positive allocations (non-zero x_{ij}) does not exceed $m + n - 1$. This is because, in the system of equations formed by the constraints, only $m + n - 1$ of them are linearly independent.

Optimal Solution

An optimal solution is a basic feasible solution that yields the minimum possible transportation cost, i.e., it minimizes the objective function Z while satisfying all constraints.

Non-Degenerate Solution

A non-degenerate basic feasible solution is a BFS in which the number of positive allocations is exactly $m + n - 1$ and all these allocations are strictly positive (none are zero).

Degenerate Basic Feasible Solution

A degenerate basic feasible solution is a BFS in which the number of positive allocations is less than $m + n - 1$. This can occur due to the structure of the supply and demand values, often resulting in some allocations being zero, which can complicate the application of methods like MODI for finding the optimal solution.