MP Lecture 1

Numerical Analysis

→ Finite Differences

$y = f(x)$

Entry (dependent var) ← y; x → Argument (independent variable)

If x is changing in AP with an interval of h

$x \rightarrow \quad x_0,\ x_0 + h,\ x + 2h,\ \ldots\ldots,\ x_0 + nh$

then ↓

$y = f(x) \rightarrow \quad y_0 = f(x_0),\ y_1 = f(x_0 + h), \ldots,\ y_n = f(x_0 + h)$

differences in y :

$$\left.\begin{aligned} \Delta y_0 &= y_1 - y_0 \\ \Delta y_1 &= y_2 - y_1 \\ \Delta y_{n-1} &= y_n - y_{n-1} \end{aligned}\right\} \text{First order forward diff}$$

operator 'Δ'

↓

Newton-forward difference operator.

Second order fwd diff ← $\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_2$.

general formula → $\Delta^n y_r = \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r$

* $\boxed{\Delta f(x) = f(x+h) - f(x)}$

Forward Difference Table

Value of x	*Value of y*	*1st. diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*	*5th diff.*
x_0	y_0					
		Δy_0				
$x_0 + h$	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
$x_0 + 2h$	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$,	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
$x_0 + 3h$	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$,	
		Δy_3		$\Delta^3 y_2$		
$x_0 + 4h$	y_4		$\Delta^2 y_3$			
		Δy_4				
$x_0 + 5h$	y_5					

when denoted

* Forward diff Table

→ Backward Differences (∇) → nabla

$$\nabla y_1 = y_1 - y_0$$
$$\nabla y_2 = y_2 - y_1$$
$$\nabla y_n = y_n - y_{n-1}$$

$$\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}$$

* Backward diff Table

* $$\boxed{\nabla f(x) = f(x) - f(x-h)}$$

→ Central difference operator (δ)
→ Averaging operator (μ)
→ Shift operator (E)

→ Relation between operators.

Central difference operator : (δ)

The central difference operator (δ) is defined as

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

$$\therefore \quad \delta y_{\frac{1}{2}} = y_1 - y_0$$

$$\delta y_{3/2} = y_2 - y_1 \; ; \; \delta y_{5/2} = y_3 - y_2$$

$$\vdots$$

$$\delta y_{n} = y_{n+\frac{1}{2}} - y_{n-\frac{1}{2}}$$

Ily $\delta^2 y_1 = \delta y_{3/2} - \delta y_{\frac{1}{2}}$, $\delta^2 y_2 = \delta y_{5/2} - \delta y_{3/2}$

in general form for m^{th} central difference.

$$\boxed{\delta^m y_n = \delta^{m-1} y_{n+\frac{1}{2}} - \delta^{m-1} y_{n-\frac{1}{2}}}$$

x	y	δy	$\delta^2 y$	$\delta^3 y$
x_0	y_0			
		δy_1		

$$\delta y_{3/2} = y_2 - y_1 \; ; \; \delta y_{5/2} = y_3 - y_2$$

$$\vdots$$

$$\delta y_{n} = y_{n+\frac{1}{2}} - y_{n-\frac{1}{2}}$$

Ily $\delta^2 y_1 = \delta y_{3/2} - \delta y_{\frac{1}{2}}$; $\delta^2 y_2 = \delta y_{5/2} - \delta y_{3/2}$

in general form for m^{th} central difference.

$$\boxed{\delta^m y_n = \delta^{m-1} y_{n+\frac{1}{2}} - \delta^{m-1} y_{n-\frac{1}{2}}}$$

x	y	δy	$\delta^2 y$	$\delta^3 y$
x_0	y_0			
		$\delta y_{\frac{1}{2}}$		
x_1 (or) (x_0+h)	y_1		$\delta^2 y_1$	
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$
x_2 (or) (x_0+2h)	y_2		$\delta^2 y_2$	
		$\delta y_{5/2}$		
x_3 (or) (x_0+3h)	y_3			

Averaging operator (μ)
(or)
Mean operator :-

It is defined as

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$$

$$\therefore \mu y_{\frac{1}{2}} = \frac{1}{2}(y_1 + y_0) \; ; \quad \mu y_{3/2} = \frac{1}{2}(y_2 + y_1) \; \ldots$$

In general $\mu y_n = \frac{1}{2}\left(y_{n+\frac{1}{2}} + y_{n-\frac{1}{2}}\right)$

Shift operator (E) :- (or) known as Enlargement operator (or) displacement operator (or) shifting operator.

It is defined as

$E f(x) = f(x+h)$; In general.
$$E^n f(x) = f(x+nh)$$

(The function $f(x)$ may be shifted to $f(x+h)$ by an operator E called shifting operator)

$E y_0 = y_1$; $E^2 y_0 = y_2$ ....

Relation b/n the operators:-

(i) $\Delta = E - 1$ (ii) $\nabla = 1 - E^{-1}$ (iii) $\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$

(iv) $\mu = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)$ and $\mu^2 = 1 + \frac{1}{4}\delta^2$

(v) $\Delta = E\nabla = \nabla E = \delta E^{\frac{1}{2}}$

(vi) $E = e^{hD}$ $\left(\because D = \frac{d}{dx}\right)$

(i) Relation b/n Δ and E

$$\Delta f(x) = f(x+h) - f(x)$$
$$E f(x) - f(x)$$
$$= (E-1) f(x)$$
$$\therefore \underline{\Delta = E - 1} \Rightarrow \underline{E = \Delta + 1}$$

Ily from the definitions, the remaining relations can be easily est

(ii) Prove that $\nabla = 1 - E^{-1} \Rightarrow 1 - \nabla = E^{-1}$

$$(1-\nabla) f(x) = f(x) - \nabla f(x)$$
$$= f(x) - [f(x) - f(x-h)]$$
$$= f(x-h) = E^{-1} f(x)$$
$$(1-\nabla) f(x) = E^{-1} f(x)$$

Ily from the definitions, the remaining relations can be easily est

(ii) Prove that $\nabla = 1 - E^{-1} \Rightarrow 1 - \nabla = E^{-1}$

$$(1-\nabla) f(x) = f(x) - \nabla f(x)$$
$$= f(x) - [f(x) - f(x-h)]$$
$$= f(x-h) = E^{-1} f(x)$$
$$(1-\nabla) f(x) = E^{-1} f(x)$$
$$\Rightarrow 1 - \nabla = E^{-1}$$

from these ① & ② $EE^{-1} = (1-\nabla)(1+\Delta)$

$$\cancel{1} = \cancel{1} + \Delta - \nabla - \nabla\Delta$$
$$\nabla(1+\Delta) = \Delta \Rightarrow \boxed{\nabla = \frac{\Delta}{1+\Delta}}$$

(iii) $\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$
$$= E^{\frac{1}{2}} f(x) - E^{-\frac{1}{2}} f(x)$$
$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

from this prove

$$\boxed{\delta = \frac{\nabla}{\sqrt{1-\nabla}}} \quad \& \quad \boxed{\delta = \frac{\Delta}{\sqrt{1+\Delta}}}$$

$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}} = (1+\Delta)^{\frac{1}{2}} - \frac{1}{(1+\Delta)^{\frac{1}{2}}}$$

$$= \frac{\cancel{1}+\Delta-\cancel{1}}{(1+\Delta)^{\frac{1}{2}}} = \frac{\Delta}{\sqrt{1+\Delta}} \,//$$

$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}} = \frac{1}{\sqrt{1-\nabla}} - \sqrt{1-\nabla}$$

$$= \frac{\cancel{1}-\cancel{1}+\nabla}{\sqrt{1-\nabla}} = \boxed{\frac{\nabla}{\sqrt{1-\nabla}}}$$

(iv) $\mu = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)$

$$\mu f(x) = \frac{1}{2}\left(f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$$

$$\mu f(x) = \frac{1}{2}\left[E^{\frac{1}{2}} f(x) + E^{-\frac{1}{2}} f(x)\right]$$

$$= \frac{1+\Delta-1}{(1+\Delta)^{\frac{1}{2}}} = \frac{\Delta}{\sqrt{1+\Delta}}$$

$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}} = \frac{1}{\sqrt{1-\nabla}} - \sqrt{1-\nabla}$$

$$= \frac{\cancel{1}-\cancel{1}+\nabla}{\sqrt{1-\nabla}} = \boxed{\frac{\nabla}{\sqrt{1-\nabla}}}$$

(iv) $\mu = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)$

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$$

$$\mu f(x) = \frac{1}{2}\left[E^{\frac{1}{2}} f(x) + E^{-\frac{1}{2}} f(x)\right]$$

$$= \frac{1}{2} f(x)\left[E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right]$$

$$\mu = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)$$

$$\mu^2 = \frac{1}{4}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)^2 = \frac{1}{4}\left(E + E^{-1} + 2\right)$$

$$= \frac{1}{4}\left[\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)^2 + 4\right]$$

$$= \frac{1}{4}\left(\delta^2 + 4\right) \Rightarrow 1 + \frac{1}{4}\delta^2$$

$$\boxed{\mu^2 = 1 + \frac{1}{4}\delta^2}$$

(v) $\Delta = E\nabla = \nabla E = \delta E^{\frac{1}{2}}$

$\Delta = E - 1$

$\nabla = \cancel{1 - E^{-1}}\ 1 - E^{-1}$

$= 1 - \frac{1}{E} = \frac{E-1}{E}$

$\therefore \nabla = \frac{\Delta}{E} \Rightarrow \underline{\underline{\Delta = E\nabla}}$

$\underline{\Delta = E - 1}$

(vi) $E = e^{hD}$ $\left(\because D = \frac{d}{dx} \right)$

D is differential operator defined by

$D f(x) = f'(x)$

By using Taylor's theorem, we have

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \cdots$$

$$= f(x) + h\, D f(x) + \frac{h^2}{2!} D^2 f(x)$$

$$= \left(1 + hD + \frac{h^2}{2!} D^2 + \cdots\right) f(x)$$

by definition

$E f(x) = f(x+h)$

$$E = 1 + hD + \frac{h^2}{2!} D^2 + \cdots = e^{hD}$$ //