CS754 Assignment-4

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Declaration: The work submitted is our own, and we have adhered to the principles of academic honesty while completing and submitting this work. We have not referred to any unauthorized sources, and we have not used generative AI tools for the work submitted here.

Question 2

Part (a)

We are sampling from a set of n coupons, with replacement.

So far, from the previous $j-1$ trials, we have been getting unique coupons. So, now we have $n-j+1$ coupons left, out of which we can select any one for uniqueness. Total varieties of coupons = n. So, the probability of picking up a unique coupon is: $\frac{n-j+1}{n}$. (For $j > n$, this probability is 0, because of pigeon-hole principle.)

$q_1 = \frac{n-1+1}{n} = 1$ (This also follows from the fact that the first coupon will be always unique.)

Part (b)

We get our first head in the k^{th} trial, so we get tail in the initial $k-1$ trials. the probability of getting a head is q, so that of a tail is $1-q$. Since, each trial is independent of the previous trials, the overall probability is:

$P = (1-q)^{k-1}q$

(where we have also considered the probability of getting a head in the k^{th} trial.)

Part (c)

The probability that $Y = k$ has been calculated in the previous part:

$P(Y = k) = (1-q)^{k-1}q$

To find the expectation of the random variable Y, we need to calculate the following sum:

$1 \times P(Y = 1) + 2 \times P(Y = 2) + \dots$

That is, the trial number multiplied my the probability corresponding to that trial.

Writing the above expression in the summation form, we get:

$$\mathbb{E}(Y) = \sum_{k=1}^{\infty} k(1-q)^{k-1}q$$

Let us call this expression as S, then:

$$S \times (1-q) = \sum_{k=1}^{\infty} k(1-q)^k q$$

Subtracting the lower expression from the one above, we get:

$$S - S(1-q) = (q + 2(1-q)q + 3(1-q)^2 q + \dots) - ((1-q)q + 2(1-q)^2 q + \dots)$$

We will combine the terms with common powers of $(1-q)$, so we get:

$$S \times q = q + q(1-q) + q(1-q)^2 + \dots + \infty$$

We can use infinite GP expression:

$$S(q) = \frac{q}{1-(1-q)}$$

So, we get $\mathbb{E}(Y) = \frac{1}{q}$.

Part (d)

We need to find the variance of Y, which can be written as:

$$Var(Y) = E(Y^2) - (E(Y))^2$$

The rightmost term, has already been calculated in the previous part. So, let us calculate $E(Y^2)$:

$$\mathbb{E}(Y^2) = \sum_{k=1}^{\infty} k^2(1-q)^{k-1} q$$

Let us calculate the following expression:

$$\mathbb{E}(Y^2) - 2\mathbb{E}(Y^2)(1-q) + \mathbb{E}(Y^2)(1-q)^2$$

$$= \sum_{k=1}^{\infty} k^2(1-q)^{k-1} q - 2\sum_{k=1}^{\infty} k^2(1-q)^k q + \sum_{k=1}^{\infty} k^2(1-q)^{k+1} q = \mathbb{E}(Y^2)(1-(1-q))^2 = \mathbb{E}(Y^2)(q^2)$$

Once again, we compbine terms with common powers of $1-q$. So, we get:

$$\mathbb{E}(Y^2)(q^2) = q + 2^2(1-q)q - 2(1-q)q + q\sum_{k=1}^{\infty}(1-q)^{k+1}((k+2)^2 - 2(k+1)^2 + k^2)$$

$$\mathbb{E}(Y^2)(q^2) = q + 2(1-q)q + 2q\sum_{k=1}^{\infty}(1-q)^{k+1}$$

$$\mathbb{E}(Y^2)(q^2) = q + 2(1-q)q + 2q\frac{(1-q)^2}{1-(1-q)} = q + 2q - 2q^2 + 2 - 2q + 2q^2$$

So, we get:

$$\mathbb{E}(Y^2) = \frac{2-q}{q^2}$$

From this, we get the variance as:

$$Var(Y) = \frac{2-q}{q^2} - (\frac{1}{q})^2 = \frac{1-q}{q^2}$$

Part (c)