CS754 Assignment-3

Saksham Rathi, Ekansh Ravi Shankar, Kshitij Vaidya

Declaration: The work submitted is our own, and we have adhered to the principles of academic honesty while completing and submitting this work. We have not referred to any unauthorized sources, and we have not used generative AI tools for the work submitted here.

Question 5

Solution

---

1 Relationship Between Shifted Particle Images

Consider two observed particle images Q1 and Q2 corresponding to a 3D density map, each in different 3D orientations and 2D shifts. Let Q1 be obtained by translating a zero-shift particle image P1 by $(\delta x_1, \delta y_1)$. Let Q2 be obtained by translating a zero-shift particle image P2 by $(\delta x_2, \delta y_2)$. The common line for the particle images P1, P2 passes through the origins of their respective coordinate systems at angles θ_1 and θ_2 with respect to their respective X axes.

1.1 Mathematical Derivation

1. In real space, the relationship between the shifted and unshifted images is:

$$Q_1(x, y) = P_1(x - \delta x_1, y - \delta y_1) \tag{1}$$
$$Q_2(x, y) = P_2(x - \delta x_2, y - \delta y_2) \tag{2}$$

2. We can take the Fourier Transform of the System to obtain:

$$\mathcal{F}[Q_1](s, t) = e^{-2\pi i(s\delta x_1 + t\delta y_1)} \cdot \mathcal{F}[P_1](s, t) \tag{3}$$
$$\mathcal{F}[Q_2](s, t) = e^{-2\pi i(s\delta x_2 + t\delta y_2)} \cdot \mathcal{F}[P_2](s, t) \tag{4}$$

3. Along the common line, the Fourier space coordinates can be written as:

$$\text{For Q1:} \quad (s, t) = (r\cos\theta_1, r\sin\theta_1) \tag{5}$$
$$\text{For Q2:} \quad (s, t) = (r\cos\theta_2, r\sin\theta_2) \tag{6}$$

where r is the distance from the origin in Fourier space.

4. The Fourier slice theorem implies that along this common line:

$$\mathcal{F}[P_1](r\cos\theta_1, r\sin\theta_1) = \mathcal{F}[P_2](r\cos\theta_2, r\sin\theta_2) \tag{7}$$

5. Therefore, we can write:

$$\frac{\mathcal{F}[Q_1](r\cos\theta_1, r\sin\theta_1)}{\mathcal{F}[Q_2](r\cos\theta_2, r\sin\theta_2)} = \frac{e^{-2\pi i(r\cos\theta_1\delta x_1 + r\sin\theta_1\delta y_1)}}{e^{-2\pi i(r\cos\theta_2\delta x_2 + r\sin\theta_2\delta y_2)}} \tag{8}$$

6. Simplifying:

$$\frac{\mathcal{F}[Q_1](r\cos\theta_1, r\sin\theta_1)}{\mathcal{F}[Q_2](r\cos\theta_2, r\sin\theta_2)} = e^{-2\pi ir[(\delta x_1\cos\theta_1 + \delta y_1\sin\theta_1) - (\delta x_2\cos\theta_2 + \delta y_2\sin\theta_2)]} \tag{9}$$

7. The phase difference between corresponding points on the common line is:

$$\Delta\phi(r) = -2\pi r[(\delta x_1\cos\theta_1 + \delta y_1\sin\theta_1) - (\delta x_2\cos\theta_2 + \delta y_2\sin\theta_2)] \tag{10}$$

8. This is a linear function of r with slope:

$$m = -2\pi[(\delta x_1\cos\theta_1 + \delta y_1\sin\theta_1) - (\delta x_2\cos\theta_2 + \delta y_2\sin\theta_2)] \tag{11}$$

1.2 Determining Shifts

To determine $\delta x_1, \delta y_1, \delta x_2, \delta y_2$:

1. Identify the common line and determine θ_1 and θ_2.
2. Compute the Fourier transforms of Q1 and Q2.
3. Calculate the phase difference $\Delta\phi(r)$ along the common line.
4. Measure the slope m of this phase difference.

This gives one equation with four unknowns. To solve:

1. Set one image (e.g., Q1) as reference: $(\delta x_1, \delta y_1) = (0, 0)$
2. Use multiple common lines between pairs of images for additional equations

1.3 Extension to N Images

For N images, we have 2N unknowns $(\delta x_i, \delta y_i)$ for $i = 1$ to N.

1. For each pair of images (i,j), identify their common line and angles θ_i and θ_j.
2. Compute the phase difference slope along each common line.
3. Set up a system of equations. For each pair (i,j):

$$m_{ij} = -2\pi[(\delta x_i\cos\theta_i + \delta y_i\sin\theta_i) - (\delta x_j\cos\theta_j + \delta y_j\sin\theta_j)] \tag{12}$$

4. We can form up to $\binom{N}{2}$ equations from all possible image pairs.

5. Solve the system using least squares optimization when we have more equations than unknowns (typically when $N > 4$).

1.4 Number of Knowns and Unknowns

1. Unknowns: 2N (the shifts $\delta x_i, \delta y_i$ for $i = 1$ to N)

2. Knowns: Up to $\binom{N}{2}$ equations from common line pairs

3. Additional constraint: Can set one image as reference, reducing unknowns to $2(N-1)$