

CS754 Assignment-5

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Declaration: The work submitted is our own, and we have adhered to the principles of academic honesty while completing and submitting this work. We have not referred to any unauthorized sources, and we have not used generative AI tools for the work submitted here.

Question 4

Solution

Given a learned dictionary D , any image I can be represented as a sparse linear combination of the dictionary atoms. The representation is given by:

$$I = D \cdot \alpha \quad (1)$$

where α is the sparse coefficient vector.

Part (a)

Considering the subset \mathcal{P}_1 of the class of images \mathcal{S} , the images in \mathcal{P}_1 are scaled by the affine transformation A_1 and the images in \mathcal{P}_2 are scaled by the affine transformation A_2 . Thus, any image in \mathcal{P}_1 can be represented as:

$$\tilde{I} = \begin{cases} A_1 D \alpha, & \text{if } I \in \mathcal{P}_1, \\ A_2 D \alpha, & \text{if } I \in \mathcal{S}_2. \end{cases} \quad (2)$$

Now, we consider the augmented dictionary \tilde{D} , which is obtained by augmenting the original dictionary D with the affine transformations A_1 and A_2 . The augmented dictionary can be represented as:

$$\tilde{D} = [A_1 D \quad A_2 D] \quad (3)$$

This dictionary \tilde{D} can be used to represent the images in both \mathcal{P}_1 and \mathcal{P}_2 . The representation of the images in \mathcal{P}_1 and \mathcal{P}_2 can be expressed as:

$$\tilde{I} = \tilde{D} \cdot \tilde{\alpha} \quad (4)$$

where $\tilde{\alpha}$ is the augmented coefficient vector. The augmented coefficient vector can be represented as:

$$\tilde{\alpha} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \quad (5)$$

where α is the original coefficient vector and 0 is a zero vector of appropriate size. This representation allows us to represent the images in both \mathcal{P}_1 and \mathcal{P}_2 using the augmented dictionary \tilde{D}

and the augmented coefficient vector $\tilde{\alpha}$. Note that $\tilde{\alpha}$ positions the 0 vector and α as the first or second half of the augmented coefficient vector, depending on whether the image is from \mathcal{P}_1 or \mathcal{P}_2 .

Part (b)

Class \mathcal{S}_2 is obtained by applying an intensity transformation given by :

$$I_{new}^i(x, y) = \alpha \cdot (I_{old}^i(x, y))^2 + \beta \cdot I_{old}^i(x, y) + \gamma \quad (6)$$

To find the dictionary that can represent the images in \mathcal{S}_2 , we can use the following approach:

1. This is nonlinear pointwise transformation. As the dictionary learning is linear, the transformation cannot be incorporated by simply transforming the atoms of the dictionary.
2. We assume small signal variations that the dictionary, we approximate the effect of this transformation using a Taylor expansion. The first-order approximation of the transformation can be represented as:

$$D_i^{new} \approx \alpha \cdot D_i^2 + \beta \cdot D_i + \gamma \mathbf{1} \quad (7)$$

$$D_{\mathcal{S}_2} = [\alpha D^2 + \beta D + \gamma] \quad (8)$$

where D^2 is the element-wise square of the matrix D .

Part (c)

For Class \mathcal{S}_4 , the images are obtained by downsampling the images in Class \mathcal{S}_3 by a factor of k in both the x and y directions. The image in the new class can be represented as:

$$I_{new}^i(x, y) = I_{old}^i(kx, ky) \quad (9)$$

To obtain the dictionary of the new class, we can similarly downsample the dictionary of the old class by a factor of k in both the x and y directions. The downsampled dictionary can be represented as:

$$D_{new}^i(x, y) = D_{old}^i(kx, ky) \quad (10)$$

This new dictionary is a valid dictionary for the new class of images, as it can represent the images in Class \mathcal{S}_4 using the same sparse linear combination of the dictionary atoms. The representation of the images in Class \mathcal{S}_4 can be expressed as:

$$I_{new}^i(x, y) = D_{new}^i \cdot \alpha_{new}^i \quad (11)$$

where α_{new}^i is the sparse coefficient vector for the new class of images.

Part (d)

Let \mathcal{S}_5 be obtained by convolving images in \mathcal{S} with a blur kernel $h = \sum_i \lambda_i b_i$, where $b_i \in B$ are known basis blur kernels, and λ_i are known coefficients.

Let $C_{b_i} \in \mathbb{R}^{n \times n}$ be the convolution matrix corresponding to blur kernel b_i . The blur operation can be modeled as:

$$C_h = \sum_i \lambda_i C_{b_i} \quad (12)$$

Then, we apply C_h to each dictionary atom:

$$D_{S_5} = C_h D = \left(\sum_i \lambda_i C_{b_i} \right) D = \sum_i \lambda_i C_{b_i} D \quad (13)$$

This transformed dictionary represents the effect of blur on each dictionary atom.

Part (e)

S_6 consists of 1D signals obtained by applying the Radon transform at a known angle θ to each image in S . Define $\mathcal{R}_\theta : \mathbb{R}^n \rightarrow \mathbb{R}^m$ as the Radon transform operator at angle θ .

Then, apply \mathcal{R}_θ to each atom in D to get:

$$D_{S_6} = \mathcal{R}_\theta D \quad (14)$$

This transformed dictionary can now sparsely represent the Radon projections of images in S .