CS754 Assignment-5

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Declaration: The work submitted is our own, and we have adhered to the principles of academic honesty while completing and submitting this work. We have not referred to any unauthorized sources, and we have not used generative AI tools for the work submitted here.

Question 3

Solution

Introduction

Robust principal component analysis (RPCA) decomposes a data matrix as

$$M = L + S, \qquad \text{where } L \text{ is low–rank and } S \text{ is sparse in the canonical basis.}$$

The paper "A Dictionary–Based Generalization of Robust PCA with Applications to Target Localization in Hyperspectral Imaging" extends RPCA by modelling

$$M = L + DS,$$

where $D \in \mathbb{R}^{n\times d}$ is a known dictionary and $S \in \mathbb{R}^{d\times m}$ is an unknown sparse coefficient matrix, and L is a low rank matrix. Solving this new formulation requires new techniques, which is mentioned in the theorem below.

Theorem

This is solved by solving the optimisation problem

$$\min_{L,S} \|L\|_* + \lambda_e \|S\|_1 \text{ s.t. } L + DS = M$$

for the entry wise sparsity case, where S has atmost s_e non-zero entries, and

$$\min_{L,S} \|L\|_* + \lambda_c \|S\|_1 \text{ s.t. } L + DS = M$$

for the column wise sparsity case, where S has atmost s_c non-zero columns. We will focus on the non-zero entries case.

Let $M = L + DS$ with $\text{rank}(L) = r$, S has at most $s_e \leq \frac{(1-\mu)^2}{2}\frac{m}{r}$ non-zero entries, D satisfies the *generalised frame property* with bounds $\alpha_\ell \leq \alpha_u$. Given, $\mu \in [0,1)$, $\gamma_U \in [0,1]$, $\gamma_V \in [r/m, 1]$ and ξ_e as defined in the paper, define

$$\lambda_{\min} = \frac{1+C_e}{1-C_e}\xi_e, \qquad \lambda_{\max} = \frac{\sqrt{\alpha_\ell(1-\mu)} - \sqrt{r\alpha_u}\mu}{\sqrt{s_e}},$$

where C_e depends on several model parameters. If

$$r < \left(\sqrt{\frac{\alpha_\ell}{\alpha_u}}\frac{1-\mu}{\mu} - \frac{\xi_e}{\sqrt{\alpha_u}\mu}\frac{1+C_e}{1-C_e}\sqrt{s_e}\right)^2,$$

then, $\lambda_{\min} < \lambda_{\max}$, and then for every $\lambda_e \in [\lambda_{\min}, \lambda_{\max}]$ the program

$$\min_{L,S} \|L\|_* + \lambda_e\|S\|_1 \quad \text{s.t. } M = L + DS$$

recovers the exact pair (L, S). The result holds for both *thin* ($d \leq n$) and *fat* ($d > n$), under certain assumptions.

How this advances RPCA

This generalised the outlier (that is, the sparsity matrix) model. Hence, the matrix S needs to be sparse in a known dictionary, and could be dense otherwise. The standard RPCA assumes $D = I_n$, and that $S \in \mathbb{R}^{n\times m}$ needs to be sparse.
However, with this model, D can either be *thin* ($d \leq n$) or *fat* ($d > n$), which leads to flexibility in choosing the known dictionary and checking if such a sparsity matrix exists or is useful. This essentially creates a much more useful generalisation over standard RPCA.

Application: Target Localisation in Hyperspectral Imaging

In a number of applications, the data may not be inherently low-rank, but may be decomposed as a superposition of a low-rank component, and a component which has a sparse representation in a known dictionary. This scenario is encountered in target identification applications in hyperspectral(HS) imaging where the a priori knowledge of the target signatures (dictionary), can be leveraged for localization.
The paper mainly focussed on the Indian Pines Dataset and the Pavia University Dataset, which are popular choices for collecting HS images for various remote sensing applications. Several methods were used for dictionary selection, and both thin and fat dictionaries were considered. This method gave the largest true positive rate, and lowest false positive rate, when compared to other methods which means that the method is able to detect all elements in the class while rejecting elements outside the class.