Assignment 1: CS 754, Advanced Image Processing

Due date: 4th Feb before 11:55 pm

Remember the honor code while submitting this (and every other) assignment. All members of the group should work on and understand all parts of the assignment. We will adopt a zero-tolerance policy against any violation.

Submission instructions: You should ideally type out all the answers in MS office or Openoffice (with the equation editor) or, more preferably, using Latex. In either case, prepare a pdf file. Create a single zip or rar file containing the report, code and sample outputs and name it as follows: A1-IdNumberOfFirstStudent-IdNumberOfSecondStudent-IdNumberOfThirdStudent.zip. (If you are doing the assignment alone, the name of the zip file is A1-IdNumber.zip. If it is a group of two students, the name of the file should be A1-IdNumberOfFirstStudent-IdNumberOfSecondStudent.zip). Upload the file on moodle BEFORE 11:55 pm on 4th Feb, which is the time that the submission is due. No assignments will be accepted after a cutoff deadline of 10 am on 5th Feb. Note that only one student per group should upload their work on moodle, although all group members will receive grades. Please preserve a copy of all your work until the end of the semester. If you have difficulties, please do not hesitate to seek help from me. The time period between the time the submission is due and the cutoff deadline is to accommodate for any unprecedented issues. But no assignments will accepted after the cutoff deadline.

Please write down the following declaration in your report: The work submitted is our own, and we have adhered to the principles of academic honesty while completing and submitting this work. We have not referred to any unauthorized sources, and we have not used generative AI tools for the work submitted here.

1. Consider a $m \times n$ sensing matrix $\boldsymbol{A}$ ($m < n$) with order-s restricted isometry constant (RIC) of δ_s. Let $\mathcal{S}$ be a subset of up to s elements from $\{1, 2, ..., n\}$. Let $\boldsymbol{A}_{\mathcal{S}}$ be a $m \times |S|$ sub-matrix of $\boldsymbol{A}$ with columns corresponding to indices in $\mathcal{S}$. Let λ_{max} be the maximum of the maximal eigenvalue of any matrix $\boldsymbol{A}_{\mathcal{S}}^T \boldsymbol{A}_{\mathcal{S}}$ (i.e. the maximum is taken across all possible subsets of size up to s). Let λ_{min} be the minimum of the minimal eigenvalue of any matrix $\boldsymbol{A}_{\mathcal{S}}^T \boldsymbol{A}_{\mathcal{S}}$ (i.e. the minimum is taken across all possible subsets of size up to s). Then prove that $\delta_s = \max(1 - \lambda_{min}, \lambda_{max} - 1)$. [15 points] **Answer:** By the definition of RIC, we have $(1 - \delta_s)\|\boldsymbol{x}\|_2^2 \leq \|\boldsymbol{Ax}\|_2^2 \leq (1 + \delta_s)\|\boldsymbol{x}\|_2^2$ for any s-sparse vector $\boldsymbol{x}$. Note that δ_s is the smallest number which satisfies this constraint. Now, we have $\lambda_{min} = \min_{S, \boldsymbol{z} \in \mathbb{R}^s} \dfrac{\|\boldsymbol{A}_S \boldsymbol{z}\|^2}{\|\boldsymbol{z}\|^2}$ where $\boldsymbol{z}$ is any s-sparse vector. Likewise, we have $\lambda_{max} = \max_{S, \boldsymbol{z} \in \mathbb{R}^s} \dfrac{\|\boldsymbol{A}_S \boldsymbol{z}\|^2}{\|\boldsymbol{z}\|^2}$. Hence, we have $\lambda_{min}\|\boldsymbol{x}\|_2^2 \leq \|\boldsymbol{Ax}\|_2^2 \leq \lambda_{max}\|\boldsymbol{x}\|_2^2$ for any s-sparse vector $\boldsymbol{x}$. [7 points for this step]

   Consequently, we must have $1 - \delta_s \leq \lambda_{min}$ and $1 + \delta_s \geq \lambda_{max}$. [4 points]

   Hence $\delta_s \geq 1 - \lambda_{min}$ and $\delta_s \geq \lambda_{max} - 1$. This gives us $\delta_s \geq \max(1 - \lambda_{min}, \lambda_{max} - 1)$. However δ_s is the smallest number that should satisfy this property (by the definition of RIC) and hence $\delta_s = \max(1 - \lambda_{min}, \lambda_{max} - 1)$. [4 points]

2. Consider positive integers s and t such that $s < t$. Argue which of the following statements is true: (i) $\delta_s < \delta_t$, (ii) $\delta_t < \delta_s$, (iii) $\delta_s = \delta_t$, (iv) It is not possible to establish a precise equality/inequality between δ_s and δ_t. [10 points]

   Answer: We use the notation from the previous question. We clearly see that $\lambda_{min}^{(s)} \geq \lambda_{min}^{(t)}$ and $\lambda_{max}^{(s)} \leq \lambda_{max}^{(t)}$. Consequently, we see that $\delta_s \leq \delta_t$. The correct options are both (i) and (iii). Proper reasoning must be mentioned otherwise no marks are to be awarded.

3. For a unique solution to the P1 problem, we require that $\delta_{2s} < 0.41$ as given in class. What is the corresponding upper bound for δ_{2s} in order for the P0 problem to give a unique solution? (Hint: Look at the proof of the uniqueness of the solutions to the P0 problem, and see the definition of RIC) [15 points]
Answer: The only meaningful values of δ_{2s} are those that lie within the interval $(0, 1)$, i.e. not including 1. The corresponding bound for P0 is $\delta_{2s} < 1$ which is a less stringent assumption on the RIC as compared to the P1 problem. If $\delta_{2s} = 1$, then we will have by the definition of RIC $0 \leq \|\boldsymbol{\Phi x}\|_2^2 \leq (1 + \delta_{2s})\|\boldsymbol{x}\|_2^2$ for any s-sparse vector $\boldsymbol{x}$. This implies that an s-sparse vector $\boldsymbol{x}$ could lie in the nullspace of $\boldsymbol{\Phi}$. Hence, there could be some $2s$ columns of $\boldsymbol{\Phi}$ which are linearly dependent, and hence a unique solution to P0 cannot be guaranteed.
Marking scheme: Mentioning the bound correctly carries 5 points. 10 points for the reasoning as to why $\delta_{2s} < 1$.

4. Please do a google search to find out some application of compressed sensing to efficiently sense some sort of signal. In your report, state the application and state which research paper or article you are referring to. Clearly explain how the measurements are acquired, what the underlying unknown signal is and what the measurement matrix is. Please exclude applications to compressive MRI, pooled testing or any compressive architecture which is covered in the slides on CS systems. [15 points]
The chosen paper must clearly have a component of signal sensing. Data science applications of compressed sensing are not acceptable and will receive no points. Examples of compressively sensed signals can include microscopy, ECG, EEG, seismic signals, radar (DoA estimation), radio telescopes, etc. 5 points each for stating the underlying signal, the measurement vector and the measurement matrix. If the hardware part is not clearly explained, a total of 7 points to be deducted from 15.

   For example, here is a paper dealing with compressive imaging in flourescence microscopy: `https://www.pnas.org/doi/epdf/10.1073/pnas.1119511109`. Here the measurement vector $\boldsymbol{y}$ is obtained by the dot-product of a random binary pattern and the underlying image $\boldsymbol{x}$. The binary pattern is generated by a DMD array which provides the modulation. The PMT computes the summation part of the dot-product. The image size is 1024×768 whereas the length of the measurement vector is 65536.

5. Construct a synthetic image $\boldsymbol{f}$ of size 32×32 in the form of a sparse linear combination of k randomly chosen 2D DCT basis vectors. Simulate m compressive measurements of this image in the form $\boldsymbol{y} = \boldsymbol{\Phi}\text{vec}(\boldsymbol{f})$ where $\text{vec}(\boldsymbol{f})$ stands for a vectorized form of $\boldsymbol{f}$, $\boldsymbol{y}$ contains m elements and $\boldsymbol{\Phi}$ has size $m \times 1024$. The elements of $\boldsymbol{\Phi}$ should be independently drawn from a Rademacher matrix (i.e. the values of the entries should independently be -1 and $+1$ with probability 0.5). Your job is to implement the OMP algorithm to recover $\boldsymbol{f}$ from $\boldsymbol{y}, \boldsymbol{\Phi}$ for $k \in \{5, 10, 20, 30, 50, 100, 150, 200\}$ and $m \in \{100, 200, ..., 1000\}$. In the OMP iterations, you may assume knowledge of the true value of k. Each time, you should record the value of the RMSE given by $\|\text{vec}(\boldsymbol{f}) - \text{vec}(\hat{\boldsymbol{f}})\|_2 / \|\text{vec}(\boldsymbol{f})\|_2$. For $k \in \{5, 50, 200\}$, you should plot a graph of RMSE versus m and plot the reconstructed images with appropriate captions declaring the value of k, m. Also plot the ground truth image. For $m \in \{500, 700\}$, you should plot a graph of RMSE versus k and plot the reconstructed images with appropriate captions declaring the value of k, m. Also plot the ground truth image. Comment on the behaviour of these plots. Repeat all these tasks with the CoSAMP, another greedy algorithm from equation (10) of the paper 'CoSaMP: iterative signal recovery from incomplete and inaccurate samples' which you can find at `https://dl.acm.org/doi/10.1145/1859204.1859229`. For implementing this algorithm, you should again assume knowledge of the true k. A local copy of this paper is also uploaded onto the homework folder. [15 + 15 = 30 points]
Solution: The code for OMP and COSAMP can be found in the folder. The report must contain the plots of the RMSE versus k for fixed m, RMSE versus m for fixed k and reconstructed images. If these plots are missing, you will lose 2 points for each plot missing. If the reconstructed images are not shown, you will lose another 3 points. In general, you should RMSE decreasing with m and increasing with k.

6. This homework problem is inspired from one of the questions asked to me in class. Consider a signal g with n elements where g is the Dirac comb consisting of spikes separated by $\sqrt{n}$ in time. Let F be the support set of g in the Fourier domain (i.e. the set of frequencies at which its Fourier transform is non-zero), and let Ω be the set of frequencies at which the Fourier transform of g is measured. Let us assume that Ω is chosen uniformly at random. We want to derive lower bounds on the size of Ω in order to be able to reconstruct g

exactly from these measurements with a probability of at least $1 - n^{-M}$ where $M > 0$. To this end, answer the following questions. Do not merely quote theorems or results, but answer this from first principles: [5+5+5=15 points]

(a) If the intersection of Ω with F is a null set, then we definitely have no chance of recovering g. What is the probability of this happening in terms of $|\Omega|, n, |F|$? Here $|F|$ stands for the cardinality of F.

(b) Argue that this probability is lower bounded by $(1 - |\Omega|/n)^{|F|}$.

(c) Hence derive a lower bound on $|\Omega|$. Use the assumption that $|\Omega| \ll n$ so that $\log(1 - |\Omega|/n) \approx -|\Omega|/n$.

Solutions:

(a) The probability of F and Ω having a null intersection is given by $P = \dfrac{C(n-|F|,|\Omega|)}{C(n,|\Omega|)}$. Simplication yields
$P = \dfrac{(n-|\Omega|)(n-|\Omega|-1)(n-|\Omega|-2)...(n-|\Omega|-|F|+1)}{n(n-1)(n-2)...(n-F+1)}$.

(b) Method 1, using upper bounds. We have:

$$P = \frac{(n-|\Omega|)(n-|\Omega|-1)(n-|\Omega|-2)...(n-|\Omega|-|F|+1)}{n(n-1)(n-2)...(n-F+1)} \tag{1}$$
$$= (1-\frac{|\Omega|}{n})(1-\frac{|\Omega|}{n-1})...(1-\frac{|\Omega|}{n-|F|+1}) \tag{2}$$
$$\leq (1-\frac{|\Omega|}{n})^{|F|}. \tag{3}$$

In method 2, we use lower bounds. We note the following:

$$P = \frac{(n-|\Omega|)(n-|\Omega|-1)(n-|\Omega|-2)...(n-|\Omega|-|F|+1)}{n(n-1)(n-2)...(n-F+1)} \tag{4}$$
$$= (1-\frac{|\Omega|}{n})(1-\frac{|\Omega|}{n-1})...(1-\frac{|\Omega|}{n-|F|+1}) \tag{5}$$
$$\geq (1-\frac{2|\Omega|}{n})^{|F|}. \tag{6}$$

The last inequality is because we have $1 - \frac{|\Omega|}{n-q} \geq 1 - \frac{2|\Omega|}{n}$ for any $r \leq n/2$.

(c) Continuing method 1, we want the upper bound on the probability P (of the bad event of null intersection) to itself be at most n^{-M} for $M > 0$, we have the following:

$$(1-\frac{|\Omega|}{n})^{|F|} \leq n^{-M} \tag{7}$$
$$|F| \log(1-\frac{|\Omega|}{n}) \leq -M \log n \tag{8}$$
$$\therefore |F|(\frac{|\Omega|}{n}) \geq M \log n \tag{9}$$
$$\therefore |\Omega| \geq Mn \log n/|F| \implies |\Omega| \geq M\sqrt{n} \log n \text{ as } |F| = \sqrt{n}. \tag{10}$$

You may wonder why the number of measurements, i.e. $|\Omega|$, is inversely proportional to $|F|$. The reason is that if you are doing Fourier domain measurements, then your signal should be sparse in the time domain (and thus be dense in the Fourier domain, i.e. $|F|$ should be large). If it is too sparse in the Fourier domain, i.e. $|F|$ is small, you will need many more measurements. In general note, that if T is the time domain support of a signal, then we must have $|T| + |F| \geq 2\sqrt{n}$ which is called the Fourier uncertainty principle. Equality is met for Dirac combs for which $|T| = |F| = \sqrt{n}$.
Continuing with method 2, we again upper bound P by n^{-M} for $M > 0$. This yields

$$(1-\frac{2|\Omega|}{n})^{|F|} \leq P \leq n^{-M}, \tag{11}$$
$$|F|\frac{2|\Omega|}{n} \geq M \log n \implies |\Omega| \geq Mn \log n/2|F|. \tag{12}$$

Marking Scheme: Either method 1 or method 2 will be accepted. It is interesting to see that both yield essentially the same result, apart from constant factors. Alternative proofs will be accepted as long as they are correct.