CS754 Assignment-5

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Declaration: The work submitted is our own, and we have adhered to the principles of academic honesty while completing and submitting this work. We have not referred to any unauthorized sources, and we have not used generative AI tools for the work submitted here.

Question 5

Solution

0.1 Part (a)

0.1.1 Cost Function:

$$J_1(A_r) = \|A - A_r\|_F^2 \tag{1}$$

where $A \in \mathbb{R}^{m \times n}$ is a known matrix, and A_r is a rank-r approximation of A with $r < m, r < n$.

0.1.2 Minimization Method:

Compute the rank-r approximation A_r using the **Singular Value Decomposition (SVD)** of A. Let:

$$A = U\Sigma V^T \tag{2}$$

Then the best rank-r approximation is:

$$A_r = U_r \Sigma_r V_r^T \tag{3}$$

where Σ_r retains only the top r singular values.

0.1.3 Application

Image Compression. Low-rank approximation is widely used to compress images while retaining the most significant information.

0.2 Part (b)

0.2.1 Cost Function

$$J_2(R) = \|A - RB\|_F^2 \tag{4}$$

where $A, B \in \mathbb{R}^{n \times m}$, $m > n$, and $R \in \mathbb{R}^{n \times n}$ is an orthonormal matrix ($R^T R = I$).

0.2.2 Minimization Method

We compute the SVD of AB^T:

$$AB^T = U\Sigma V^T \tag{5}$$

Then the optimal R is:

$$R = UV^T \tag{6}$$

0.2.3 Application

Image Registration. Used to align two sets of landmark points or feature descriptors under rotation/reflection.

0.3 Part (c)

0.3.1 Cost Function

$$J_3(A) = \|C - A\|_F^2 + \lambda \|A\|_1 \tag{7}$$

where $C \in \mathbb{R}^{m\times n}$ is known and $\|\cdot\|_1$ denotes element-wise ℓ_1-norm.

0.3.2 Minimization Method

This is a **soft-thresholding problem**, solved using the **Iterative Shrinkage-Thresholding Algorithm (ISTA)** or **FISTA** for faster convergence. A closed-form solution for element-wise shrinkage is:

$$A_{ij} = \text{sign}(C_{ij}) \cdot \max(|C_{ij}| - \lambda, 0) \tag{8}$$

0.3.3 Application

Image Denoising. Promotes sparsity to eliminate noise while preserving important signal components.

0.4 Part (d)

0.4.1 Cost Function

$$J_4(A) = \|C - A\|_F^2 + \lambda \|A\|_* \tag{9}$$

where $\|\cdot\|_*$ is the nuclear norm (sum of singular values), and $C \in \mathbb{R}^{m\times n}$ is known.

0.4.2 Minimization Method

Solved using the **Singular Value Thresholding (SVT)** algorithm. Let $C = U\Sigma V^T$ be the SVD of C. Then apply soft-thresholding to singular values:

$$A = U \cdot \text{diag}(\max(\sigma_i - \lambda, 0)) \cdot V^T \tag{10}$$

0.4.3 Application

Image Inpainting. Used to recover missing parts of an image by promoting low-rank structure.