CS754 Assignment-2

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Declaration: The work submitted is our own, and we have adhered to the principles of academic honesty while completing and submitting this work. We have not referred to any unauthorized sources, and we have not used generative AI tools for the work submitted here.

Question 4

Part (a)

We are given a matrix A and a vector v such that $Av = 0$. The condition $||v_S||_1 < ||v_{\bar{S}}||_1$ is true for set S that contains the indices of the s largest absolute value entries of v.

We will prove that this is true for any other set S such that $|S| \leq s$.

Let us assume to the contrary that there exists a set R such that $|R| = k \leq s$ and $||v_R||_1 \geq ||v_{\bar{R}}||_1$.

By definition, S contains the indices of the s largest absolute value entries of v, meaning that $\sum_{i \in S} |v_i|$ is maximized over all subsets of size s. Moreover, the set R has a cardinality of $k \leq s$, therefore, $\sum_{i \in R} |v_i| \leq \sum_{i \in S} |v_i|$.

Combining these three inequalities, we get:

$$||v_{\bar{S}}||_1 > ||v_S||_1 \geq ||v_R||_1 \geq ||v_{\bar{R}}||_1 \tag{1}$$

From above, we get $||v_{\bar{S}}||_1 + ||v_S||_1 > ||v_R||_1 + ||v_{\bar{R}}||_1$. However, we also have $||v_{\bar{S}}||_1 + ||v_S||_1 = ||v||_1 = ||v_R||_1 + ||v_{\bar{R}}||_1$. Therefore, we have a contradiction, and the assumption that there exists a set R such that $|R| = k \leq s$ and $||v_R||_1 \geq ||v_{\bar{R}}||_1$ is false.

Part (b)

We define $\sigma_{s,1} = inf_{||w_0|| \leq s} ||v - w||_1$. We need to minimize the l_1 norm of $v - w$ over all vectors w that have at most s non-zero entries. For this, we can choose w to be the vector v with all but the s largest absolute value entries set to zero (a greedy choice). Now, the value of $\sigma_{s,1}$ is equal to the absolute sum of $n - s$ entries of v. (This is somewhat similar to what we had taken in the previous part.)

We need to prove that MNSP implies that $||v||_1 \leq 2\sigma_{s,1}$. From MNSP, we have that if $v \in nullspace(A) - \{0\}$, then $||v_S||_1 < ||v_{\bar{S}}||_1$. We can write $||v||_1 = ||v_S||_1 + ||v_{\bar{S}}||_1$ (this is true for any set S, as the indices of v can be exhaustively partitioned into S and $\bar{S}$). So, this property also holds for the set S which contains s largest absolute value entries of v.

Now, we have $\sigma_{s,1} = ||v_{\bar{S}}||_1$ (for such a set S). Also, we have:

$$||v_S||_1 < ||v_{\bar{S}}||_1 = \sigma_{s,1}$$

The above is true for a matrix which satisfy MNSP. Therefore:

$$||v||_1 = ||v_S||_1 + ||v_{\bar{S}}||_1 < 2\sigma_{s,1}$$

Part (c)

Now we need to prove: Given a matrix $A \in \mathbb{R}^{m \times n}$, any s-sparse vector $x \in \mathbb{R}$ is a unique solution to the $P1$ problem with the constraint $y = Ax$ iff A satisfies the MNSP of order s. There are two parts of this proof (for both directions).

A satisfies MNSP $\implies$ x is a unique solution to $P1$

The P1 problem is given by:

$$\min_z ||z||_1 \quad \text{subject to} \quad Az = y$$

Assume to the contrary that there exists another s-sparse vector x' such that $Ax' = y$ and

$$||x||_1 = ||x'||_1$$

(else we can pick one of them as the unique solution).

Since, both x and x' are $s - sparse$, let S and S' be the support sets of x and x' respectively. We have $||x||_1 = ||x'||_1 \implies ||x_S||_1 = ||x'_{S'}||_1$ (as other enties are 0). Let us take $v = x - x'$. We have $Av = 0$ (as $Ax = Ax'$). Since, x and x' are s-sparse, v is 2s-sparse.

From the previous part (since A satisfies MNSP of order s), we have $||v_R||_1 < ||v_{\bar{R}}||_1$, for all subsets R of size $|R| \leq s$. Let us take R to be the indices corresponding to the s largest absolute value entries of v. Now, since v is $2s - sparse$, $||v_{\bar{R}}||_1$ is the sum of remaining $t \leq s$ non-zero entries of v (after we have already chosen the s largest absolute value entries).

Now, since we have already chosen R to contain the s largest absolute value entries of v, we have $||v_R||_1 \geq ||v_{\bar{R}}||_1$ (because $\bar{R}$ can't contain values higher than R).

This is a contradiction to the MNSP property of A because $||v_R||_1 \geq ||v_{\bar{R}}||_1$ and $||v_R||_1 < ||v_{\bar{R}}||_1$ can't hold together. Therefore, x is a unique solution to $P1$.

x is a unique solution to $P1$ $\implies$ A satisfies MNSP

Let us assume to the contrary that A does not satisfy MNSP. This means that there exists a nonzero vector $v \in null(A)$ such that $||v_S||_1 \leq ||v_{\bar{S}}||_1$ for some set S of size s.

Now, we consider the solution x to the P1 problem. It is given that this solution is unique, that is we can't find another vector x_1 such that $Ax_1 = y$ and $||x_1||_1 \leq ||x||_1$.

Consider the vector $x^\top = x + v$ (v is chosen from the above context). Since $Av = 0$ we have: $Ax^\top = A(x + v) = Ax + Av = Ax + 0 = Ax$. Thus, $x^\top$ is another feasible solution to the P1 problem. Since x is s-sparse, let S be the support set of x. We will now analyze the l_1 norm of $x^\top$:

$$||x^\top||_1 = ||x + v||_1$$

We can break $||x + v||_1$ into two sets, based on the set S:

$$||x^\top||_1 = ||x_S + v_S||_1 + ||x_{\bar{S}} + v_{\bar{S}}||_1 = ||x_S + v_S||_1 + ||v_{\bar{S}}||_1$$

(because $x_{\bar{S}} = 0$).

Now, let us use the traingle inequality:

$$||x^\top||_1 = ||x_S + v_S||_1 + ||v_{\bar{S}}||_1 \leq ||x_S||_1 + ||v_S||_1 + ||v_{\bar{S}}||_1$$

Since, $||v_S||_1 < ||v_{\bar{S}}||_1$, we have:

$$||x^\top||_1 < ||x_S||_1 + 2||v_{\bar{S}}||_1$$

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Since, $||v_S||_1 < ||v_{\bar{S}}||_1$, we have:

$$||x^\top||_1 < ||x_S||_1 + 2||v_{\bar{S}}||_1$$