Assignment 2: CS 754, Advanced Image Processing

Due date: 17th Feb before 11:55 pm

Remember the honor code while submitting this (and every other) assignment. All members of the group should work on and understand all parts of the assignment. We will adopt a zero-tolerance policy against any violation.

Submission instructions: You should ideally type out all the answers in MS office or Openoffice (with the equation editor) or, more preferably, using Latex. In either case, prepare a pdf file. Create a single zip or rar file containing the report, code and sample outputs and name it as follows: A2-IdNumberOfFirstStudent-IdNumberOfSecondStudent-IdNumberOfThirdStudent.zip. (If you are doing the assignment alone, the name of the zip file is A2-IdNumber.zip. If it is a group of two students, the name of the file should be A2-IdNumberOfFirstStudent-IdNumberOfSecondStudent.zip). Upload the file on moodle BEFORE 11:55 pm on 17th Feb, which is the time that the submission is due. No assignments will be accepted after a cutoff deadline of 10 am on 18th Feb. Note that only one student per group should upload their work on moodle, although all group members will receive grades. Please preserve a copy of all your work until the end of the semester. If you have difficulties, please do not hesitate to seek help from me. The time period between the time the submission is due and the cutoff deadline is to accommodate for any unprecedented issues. But no assignments will accepted after the cutoff deadline.

1. Refer to Theorem 1 and its proof in the paper 'The restricted isometry property and its implications for compressed sensing', a copy of which is placed in the homework folder. This theorem is same as Theorem 3 in our lecture slides. Its proof is given in this paper. Your task is to justify various steps of the proof using standard equalities and inequalities. There are 16 steps in all, and each step carries 2 points. [32 points]
   Solution:
   (a) Q1: The first inequality holds due to the triangle inequality satisfied by the vector 2-norm and the definition of measurement vector $\boldsymbol{y}$. The second inequality is because of the constraint imposed in the optimization problem.
   (b) Q2: This is true because $\|\mathbf{h}_{T_j}\|_2 = \sqrt{\sum_{i=1}^{s} h_{T_j,i}^2} \leq \sqrt{\sum_{i=1}^{s} \|\mathbf{h}_{T_j}\|_\infty^2} = s^{1/2}\|\mathbf{h}_{T_j}\|_\infty$. Also, notice that $s^{1/2}\|\mathbf{h}_{T_j}\|_\infty \leq \dfrac{s^{1/2}}{s}\sum_{i=1}^{s}\|\mathbf{h}_{T_j}\|_\infty \leq s^{-1/2}\|\mathbf{h}_{T_{j-1}}\|_1$. Also any element of $\mathbf{h}_{T_j}$ (including $\|\mathbf{h}_{T_j}\|_\infty$) is less than or equal to any element of $\mathbf{h}_{T_{j-1}}$.
   (c) Q3: $\sum_{j\geq 2}\|\mathbf{h}_{T_j}\|_2 \leq s^{-1/2}\sum_{j\geq 1}\|\mathbf{h}_{T_j}\|_1 = s^{-1/2}\|\mathbf{h}_{T_0^c}\|_1$. The last equality is because $T_0^c = T_1 \cup T_2 \cup .....$ The first inequality holds due to a simple summation starting from the previous relation $\|\boldsymbol{h}_{\boldsymbol{T_j}}\|_2 \leq s^{-1/2}\|\boldsymbol{h}_{\boldsymbol{T_{j-1}}}\|_1$.
   (d) Q4 and Q5: We have $\|\mathbf{h}_{(T_0\cup T_1)^c}\|_2 = \|\sum_{j\geq 2}\mathbf{h}_{T_j}\|_2$ as $\forall j, \mathbf{h}_{T_j}$ have disjoint support. The next inequality follows by triangle inequality and the last one is because we earlier proved that $\sum_{j\geq 2}\|\mathbf{h}_{T_j}\|_2 \leq s^{-1/2}\|\boldsymbol{h}_{\boldsymbol{T_0^c}}\|_1$.
   (e) Q6: Reverse Triangle inequality on $|x_i + h_i|$ in two different directions.
   (f) Q7: Directly uses the previous equation from the paper and re-arranges the terms. It also uses $\|\boldsymbol{x}\|_1 = \|\boldsymbol{x}_{\boldsymbol{T^0}}\|_1 + \|\boldsymbol{x}_{\boldsymbol{T^{0c}}}\|_1$.
   (g) Q8: This is almost directly given in the paper via equations 11 and 12 and the relationship $\|\boldsymbol{h}_{\boldsymbol{T^0}}\|_1 \leq s^{1/2}\|\boldsymbol{h}_{\boldsymbol{T^0}}\|_2$.

(h) Q9: The first inequality is due to the Cauchy Schwartz inequality. The second inequality is due to the fact that $\|\boldsymbol{\Phi h}\|_2 \leq 2\varepsilon$ from part 1, and by invoking the upper bound of the RIP for $\|\boldsymbol{\Phi h}_{\boldsymbol{T^0 \cup T^1}}\|_2 \leq \sqrt{1+\delta_{2s}}\|\boldsymbol{h}_{\boldsymbol{T^0 \cup T^1}}\|_2$ as $\boldsymbol{h}_{\boldsymbol{T^0 \cup T^1}}$ is $2s$-sparse.

(i) Q10: This comes from Lemma 2.1 in the paper, but extended to vectors that have magnitude greater than 1. See lecture slides for more details. Also note that the Lemma 2.1 can be invoked because $\boldsymbol{h}_{\boldsymbol{T^0}}$ and $\boldsymbol{h}_{\boldsymbol{T^j}}$ are sparse vectors with disjoint support.

(j) Q11: Consider a 2-element vector $\mathbf{w} = (\|\mathbf{h_{T_0}}\|_2, \|\mathbf{h_{T_1}}\|_2)$. Then $\|\mathbf{w}\|_1 = \|\mathbf{h_{T_0}}\|_2 + \|\mathbf{h_{T_1}}\|_2$. We know that $\|\mathbf{w}\|_1 \leq \sqrt{2}\|\mathbf{w}\|_2 = \sqrt{2}\|\mathbf{h_{T_0 \cup T_1}}\|_2$ since the support sets T_0 and T_1 are disjoint.

(k) Q12: We need to be very careful here as so many steps are involved! From the RIP of $\boldsymbol{\Phi}$ for order $2s$ with RIC δ_{2s}, we know that $(1-\delta_{2s})\|\mathbf{h}_{T_0\cup T_1}\|_2^2 \leq \|\boldsymbol{\Phi}\mathbf{h}_{T_0\cup T_1}\|_2^2 =$
$\langle \boldsymbol{\Phi}\mathbf{h_{T_0 \cup T_1}}, \boldsymbol{\Phi}\mathbf{h}\rangle - \langle \boldsymbol{\Phi}\mathbf{h_{T_0 \cup T_1}}, \sum_{j\geq 2} \boldsymbol{\Phi}\mathbf{h_{T_j}}\rangle$(as shown in the paper after equation 13)
$\leq |\langle \boldsymbol{\Phi}\mathbf{h_{T_0 \cup T_1}}, \boldsymbol{\Phi}\mathbf{h}\rangle| + |\langle \boldsymbol{\Phi}\mathbf{h_{T_0 \cup T_1}}, \sum_{j\geq 2} \boldsymbol{\Phi}\mathbf{h_{T_j}}\rangle|$
$\leq 2\epsilon\sqrt{1+\delta_{2s}}\|\mathbf{h}_{T_0\cup T_1}\|_2 + |\langle \boldsymbol{\Phi}\mathbf{h_{T_0 \cup T_1}}, \sum_{j\geq 2} \boldsymbol{\Phi}\mathbf{h_{T_j}}\rangle|$
(using Cauchy-Schwartz inequality, equation 9 from the paper, right side of RIP and Q9)
$\leq 2\epsilon\sqrt{1+\delta_{2s}}\|\mathbf{h}_{T_0\cup T_1}\|_2 + |\langle \boldsymbol{\Phi}\mathbf{h_{T_0}} + \boldsymbol{\Phi}\mathbf{h_{T_1}}, \sum_{j\geq 2} \boldsymbol{\Phi}\mathbf{h_{T_j}}\rangle|$
(as T_0 and T_1 are disjoint sets and hence $\boldsymbol{\Phi}\mathbf{h}_{T_0\cup T_1} = \boldsymbol{\Phi}\mathbf{h}_{T_0} + \boldsymbol{\Phi}\mathbf{h}_{T_1}$)
$\leq 2\epsilon\sqrt{1+\delta_{2s}}\|\mathbf{h}_{T_0\cup T_1}\|_2 + \delta_{2s}(\|\mathbf{h_{T_0}}\|_2 + \|\mathbf{h_{T_1}}\|_2)\|\sum_{j\geq 2}\|\mathbf{h_{T_j}}\|_2$
from Lemma 2.1 of the paper since $\boldsymbol{h}_{\boldsymbol{T^0}}, \boldsymbol{h}_{\boldsymbol{T^1}}, \boldsymbol{h}_{\boldsymbol{T_j}}$ are all vectors with disjoint support
$\leq 2\epsilon\sqrt{1+\delta_{2s}}\|\mathbf{h}_{T_0\cup T_1}\|_2 + \delta_{2s}\sqrt{2}(\|\mathbf{h_{T_0 \cup T_1}}\|_2)\|\sum_{j\geq 2}\|\mathbf{h_{T_j}}\|_2$ as $\|\boldsymbol{h}_{\boldsymbol{T_0}}\|_2 + \|\boldsymbol{h}_{\boldsymbol{T_1}}\|_2 = \|\boldsymbol{h}_{\boldsymbol{T_0 \cup T_1}}\|_2$.

(l) Q13: This follows straightforwardly from the previous step. Just divide the leftmost and rightmost sides by $\|\boldsymbol{h}_{T_0\cup T_1}\|_2$, and from equation 10 of the paper.

(m) Q14: Follows from straightforward algebra using equation 12 of the paper.

(n) Q15: Follows from triangle inequality.

(o) Q16: This follows in a very straightforward way using Lemma 2.2 from the paper which shows that $\|\boldsymbol{h}_{\boldsymbol{T_0}}\|_1 \leq \rho\|\boldsymbol{h}_{\boldsymbol{T_0^c}}\|_1$. The paper has already derived a bound for $\|\boldsymbol{h}_{\boldsymbol{T_0^c}}\|_1$. This produces the bound for $\|\boldsymbol{h}\|_1$. (Food for thought: can this be easily extended for the noisy case as well? Why (not?))

2. Your task here is to implement the ISTA algorithm:

(a) Consider the 'Barbara' image from the homework folder. Add iid Gaussian noise of mean 0 and variance 4 (on a [0,255] scale) to it, using the 'randn' function in MATLAB. Thus $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{\eta}$ where $\boldsymbol{\eta} \sim \mathcal{N}(0,4)$. You should obtain $\boldsymbol{x}$ from $\boldsymbol{y}$ using the fact that patches from $\boldsymbol{x}$ have a sparse or near-sparse representation in the 2D-DCT basis.

(b) Divide the image shared in the homework folder into patches of size 8×8. Let $\boldsymbol{x_i}$ be the vectorized version of the i^{th} patch. Consider the measurement $\boldsymbol{y_i} = \boldsymbol{\Phi x_i}$ where $\boldsymbol{\Phi}$ is a 32×64 matrix with entries drawn iid from $\mathcal{N}(0,1)$. Note that $\boldsymbol{x_i}$ has a near-sparse representation in the 2D-DCT basis $\boldsymbol{U}$ which is computed in MATLAB as 'kron(dctmtx(8)',dctmtx(8)')'. In other words, $\boldsymbol{x_i} = \boldsymbol{U\theta_i}$ where $\boldsymbol{\theta_i}$ is a near-sparse vector. Your job is to reconstruct each $\boldsymbol{x_i}$ given $\boldsymbol{y_i}$ and $\boldsymbol{\Phi}$ using ISTA. Then you should reconstruct the image by averaging the overlapping patches. You should choose the α parameter in the ISTA algorithm judiciously. Choose $\lambda = 1$ (for a [0,255] image). Display the reconstructed image in your report. State the RMSE given as $\|X(:) - \hat{X}(:)\|_2/\|X(:)\|_2$ where $\hat{X}$ is the reconstructed image and X is the true image. Repeat this with the 'goldhill' image (take the top-left portion of size 256 by 256 only). [12 points]

(c) Implement both the above cases using the FISTA algorithm from the research paper `https://epubs.siam.org/doi/10.1137/080716542`. [12 points]

(d) Read the research paper and explain in which precise mathematical sense the FISTA algorithm is faster than ISTA. Also, why is it faster than ISTA? [12 points]

Solution: See sample codes in the homework folder. For the last part: The ISTA algorithm has a convergence rate of $O(1/k)$ whereas the FISTA algorithm has a convergence rate of $O(1/k^2)$ where k is the number of iterations, making FISTA the faster algorithm. The mathematical reason for this is due to Lemma 4.1, 4.2, 4.3 and Theorem 4.1. FISTA remains a first order method and hence does not require significantly more computation than ISTA per iteration. In fact, FISTA requires only one gradient computation per iteration but the soft thresholding is applied on a quantity derived from the previous two estimates of $\boldsymbol{x}$. Theorem 4.1 establishes that the number of iterations required by FISTA to reach an error ϵ is proportional to $C/\sqrt{\epsilon}$.

3. Perform a google search to find out a research paper that uses group testing in data science or machine learning. Explain (i) the specific ML/DS problem targeted in the paper, (ii) the pooling matrix, measurement vector and unknown signal vector in the context of the problem being solved in this paper, (iii) the algorithm used in the paper to solve this problem. You can also refer to references within chapter 1 of the book `https://arxiv.org/abs/1902.06002`. [16 points]
Solution: An example of group testing for an ML application is multi-label classification, where each datapoint $\boldsymbol{x}_i$ can belong to at most k out of d classes where $k \ll d$. The paper under consideration is 'Multilabel Classification with Group Testing and Codes' from ICML 2017. The url is `https://proceedings.mlr.press/v70/ubaru17a.html`. Instead of individually testing whether $\boldsymbol{x}_i$ belongs to each of the d classes, the proposed method checks whether $\boldsymbol{x}_i$ belongs to each of $m = O(k \log d)$ pools where each pool is created by a union of a small number of labels (individual classes). Since $m \ll d$ for $k \ll d$, this saves on the total number of classifiers invoked. The pooling matrix is so designed that each classes participates in a fixed, non-zero number of pools, using Reed-Solomon codes. The measurement vector is the result of the classification of $\boldsymbol{x}_i$ on each of the m pools and is a binary vector with m values. Using a group testing given via Algorithm 1 and 2, we can determine the d-element binary membership vector of $\boldsymbol{x}_i$, i.e. we can determine to which of the d classes $\boldsymbol{x}_i$ belongs. The algorithm does the following: The lth coordinate of the d-element label vector of $\boldsymbol{x}_i$ is set to 1 if the number of coordinates with a value of 1 in $\boldsymbol{A}_l$ (the lth column of $\boldsymbol{A}$) but not in the predicted reduced label vector is less than $e/2$ where e is a tolerance value accounting for the fact that the classifiers may make errors. This technique returns correct label vectors even if there are errors in up to $e/2$ binary classifiers.

4. A matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ where $m < n$ is said to satisfy the modified null space property (MNSP) relative to a set $S \subset [n] := \{1, 2, ..., n\}$ if for all $\boldsymbol{v} \in \text{nullspace}(\boldsymbol{A}) - \{\boldsymbol{0}\}$, we have $\|\boldsymbol{v}_S\|_1 < \|\boldsymbol{v}_{\bar{S}}\|_1$ where $\bar{S}$ stands for the complement of the set S. The matrix $\boldsymbol{A}$ is said to satisfy MNSP of order s if it satisfies the MNSP relative to any set $S \subset [n]$ where $|S| \leq s$. Now answer the following questions:

   (a) Consider a given matrix $\boldsymbol{A}$ and $\boldsymbol{v} \in \text{nullspace}(\boldsymbol{A}) - \{\boldsymbol{0}\}$. Suppose the condition $\|\boldsymbol{v}_S\|_1 < \|\boldsymbol{v}_{\bar{S}}\|_1$ is true for set S that contains the indices of the s largest absolute value entries of $\boldsymbol{v}$. Then is this condition also true for any other set S such that $|S| \leq s$? Why (not)?
   Solution: Yes, this implies that the condition is true for any other set S such that $|S| \leq s$. Consider a set $T \neq S$ with $|T| \leq s$, where T does not contain all the indices of the s largest absolute value entries of $\boldsymbol{v}$. Notice that $\|\boldsymbol{v}_T\|_1 < \|\boldsymbol{v}_S\|_1$ and that $\|\boldsymbol{v}_{\bar{T}}\|_1 > \|\boldsymbol{v}_{\bar{S}}\|_1$. Since $\|\boldsymbol{v}_S\|_1 < \|\boldsymbol{v}_{\bar{S}}\|_1$, it is clear that $\|\boldsymbol{v}_T\|_1 < \|\boldsymbol{v}_{\bar{T}}\|_1$. Hence this property is true for the set T, i.e. any other such set T.

   (b) Show that the MNSP implies that $\|\boldsymbol{v}\|_1 < 2\sigma_{s,1}(\boldsymbol{v})$ for $\boldsymbol{v} \in \text{nullspace}(\boldsymbol{A}) - \{\boldsymbol{0}\}$ where $\sigma_{s,1}(\boldsymbol{v}) := \inf_{\|\boldsymbol{w}\|_0 \leq s} \|\boldsymbol{v} - \boldsymbol{w}\|_1$.
   Solution: Given $\|\boldsymbol{v}_S\|_1 < \|\boldsymbol{v}_{\bar{S}}\|_1$, we add $\|\boldsymbol{v}_{\bar{S}}\|_1$ to both sides of the inequality. This yields $\|\boldsymbol{v}\|_1 < 2\|\boldsymbol{v}_{\bar{S}}\|_1$. Now if S is chosen to be the set with the $s := |S|$ largest absolute value indices of $\boldsymbol{v}$, the RHS is equal to $\sigma_{s,1}(\boldsymbol{v})$. (Aside: Since $\|\boldsymbol{v}_S\|_1 < \|\boldsymbol{v}_{\bar{S}}\|_1$, we have $2\|\boldsymbol{v}_S\|_1 < \|\boldsymbol{v}\|_1$. This part is not required for the question.)

   (c) Given a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ of size $m \times n$, any s-sparse vector $\boldsymbol{x} \in \mathbb{R}^n$ is a unique solution of the P1 problem with the constraint $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ if and only if $\boldsymbol{A}$ satisfies the MNSP of order s.
   Solution: Consider a set S where $|S| = s$. Let us assume that any vector $\boldsymbol{x}$ whose support set is S, is the unique solution to P1, i.e., it is the unique minimizer of $\|\boldsymbol{w}\|_1$, provided it satisfies $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}\boldsymbol{w}$. We now want to prove MNSP of $\boldsymbol{A}$ for the set S. Let $\boldsymbol{v}$ belong to the non-trivial nullspace of $\boldsymbol{A}$. Now the vector $\boldsymbol{v}_S$ has a support set S and hence it is the unique minimizer of $\|\boldsymbol{w}\|_1$ such that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}\boldsymbol{w}$. Now $\boldsymbol{A}\boldsymbol{v} = \boldsymbol{0}$ as $\boldsymbol{v}$ lies in the nullspace of $\boldsymbol{A}$. Hence $\boldsymbol{A}\boldsymbol{v}_S = -\boldsymbol{A}\boldsymbol{v}_{\bar{S}}$. Also, $\boldsymbol{v}_S \neq -\boldsymbol{v}_{\bar{S}}$ since $\boldsymbol{v} \neq \boldsymbol{0}$. Then both the vectors $\boldsymbol{v}_S$ and $-\boldsymbol{v}_{\bar{S}}$ satisfy the constraint of problem P1. As $\boldsymbol{v}_S$ is the unique minimizer, this implies that $\|\boldsymbol{v}_S\|_1 < \|\boldsymbol{v}_{\bar{S}}\|_1$, i.e. it shows that $\boldsymbol{A}$ obeys the MNSP for set S. As this is true for any set S, we conclude that $\boldsymbol{A}$ obeys MNSP of order s.

Now, we will prove the other direction. Suppose $\boldsymbol{A}$ obeys MNSP for set S. We now have to prove uniqueness of the solution for P1. Let $\boldsymbol{x} \in \mathbb{R}^n$ be a vector with support set S. Let $\boldsymbol{w} \neq \boldsymbol{x}$ be another vector in $\mathbb{R}^n$ such that $\boldsymbol{Ax} = \boldsymbol{Aw}$. The vector $\boldsymbol{v} := \boldsymbol{x} - \boldsymbol{w}$ lies in the nontrivial nullspace of $\boldsymbol{A}$. Now, we have the following: $\|\boldsymbol{x}\|_1 = \|\boldsymbol{x} - \boldsymbol{w}_S + \boldsymbol{w}_S\|_1 \leq \|\boldsymbol{x} - \boldsymbol{w}_S\|_1 + \|\boldsymbol{w}_S\|_1 = \|\boldsymbol{x}_S - \boldsymbol{w}_S\|_1 + \|\boldsymbol{w}_S\|_1 = \|\boldsymbol{v}_S\|_1 + \|\boldsymbol{w}_S\|_1$. Here note that $\boldsymbol{x}_{\bar{S}}$ contains all zeroes. Moreover, due to MNSP, we know that $\|\boldsymbol{v}_S\|_1 < \|\boldsymbol{v}_{\bar{S}}\|_1$. Hence $\|\boldsymbol{x}\|_1 < \|\boldsymbol{v}_{\bar{S}}\|_1 + \|\boldsymbol{w}_S\|_1$. Now, as $\boldsymbol{x}_{\bar{S}}$ contains all zeroes, we know that $\boldsymbol{v}_{\bar{S}} = \boldsymbol{x}_{\bar{S}} - \boldsymbol{w}_{\bar{S}} = -\boldsymbol{w}_{\bar{S}}$. This produces $\|\boldsymbol{x}\|_1 < \|-\boldsymbol{w}_{\bar{S}}\|_1 + \|\boldsymbol{w}_S\|_1 = \|\boldsymbol{w}_{\bar{S}}\|_1 + \|\boldsymbol{w}_S\|_1 = \|\boldsymbol{w}\|_1$. This proves uniqueness of the solution for P1. These arguments are true for any set S such that $|S| = s$.

[4+4+8=16 points]