Homework 1: CS 726, Spring 2025

March 1, 2025

1. Consider the joint probability distribution $Pr(x_1, \dots, x_n)$. By the chain rule of probability, we can write:

$$Pr(x_1, \dots, x_n) = \prod_{i=1}^{n} Pr(x_i | x_1, \dots, x_{i-1}).$$

Without loss of generality (WLOG), consider a topological ordering of the nodes in the Directed Acyclic Graph (DAG). Each variable x_i is conditionally independent of its non-descendants given its parents x_{π_i}. Thus:

$$Pr(x_i | x_1, \dots, x_{i-1}) = Pr(x_i | x_{\pi_i}),$$

where x_{π_i} denotes the parents of node i.

Substituting this into the chain rule expression, we get:

$$Pr(x_1, \dots, x_n) = \prod_{i=1}^{n} Pr(x_i | x_{\pi_i}).$$

Now, it is given that:

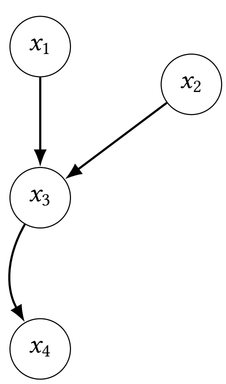
$$Pr(x_1, \dots, x_n) = \prod_{i=1}^{n} f_i(x_i, x_{\pi_i}),$$

$$\sum_{x_i} f_i(x_i, x_{\pi_i}) = 1.$$

Comparing these two factorizations of $Pr(x_1, \dots, x_n)$, we conclude that:

$$f_i(x_i, x_{\pi_i}) = Pr(x_i | x_{\pi_i}).$$

2. (a) $4(1 + e)^2$



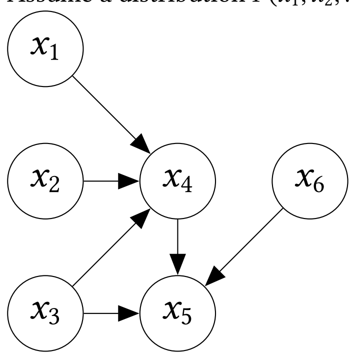
(b)

(c)

$$Pr(x_1 | Pa(x_1)) = Pr(x_1) = \frac{1}{2}$$
$$Pr(x_2 | Pa(x_2)) = Pr(x_2) = \frac{1}{2}$$
$$Pr(x_3 | Pa(x_3)) = Pr(x_3 | x_1, x_2) =$$

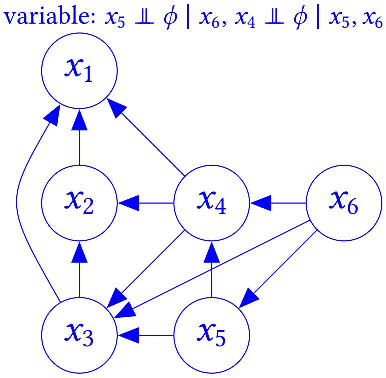
$$\begin{cases} Pr(x_3 = 0 \mid x_1 = 0, x_2 = 0) = \frac{1}{e+1} \\ Pr(x_3 = 1 \mid x_1 = 0, x_2 = 0) = \frac{e}{e+1} \\ Pr(x_3 = 0 \mid x_1 = 0, x_2 = 1) = \frac{e}{e+1} \\ Pr(x_3 = 1 \mid x_1 = 0, x_2 = 1) = \frac{1}{e+1} \\ Pr(x_3 = 0 \mid x_1 = 1, x_2 = 0) = \frac{e}{e+1} \\ Pr(x_3 = 1 \mid x_1 = 1, x_2 = 0) = \frac{1}{e+1} \\ Pr(x_3 = 0 \mid x_1 = 1, x_2 = 1) = \frac{1}{e+1} \\ Pr(x_3 = 1 \mid x_1 = 1, x_2 = 1) = \frac{e}{e+1} \end{cases}$$

3. Assume a distribution $P(x_1, x_2, \ldots, x_6)$ is represented by the Bayesian Network H below:



   (a) In the BN H, list all variables that are unconditionally independent of x_6 ..1
   All variables except x_5 are independent of x_6.

   (b) Using the D-separation rule on the above network to get answers to yes/no questions on conditional independencies, draw a fresh minimal and correct Bayesian network G using the variable order $x_6, x_5, x_4, x_3, x_2, x_1$ ..3
   Following the order of variables, we find that the following CIs hold in the above Bayesian Network H for each variable: $x_5 \perp\!\!\!\perp \phi \mid x_6$, $x_4 \perp\!\!\!\perp \phi \mid x_5, x_6$, $x_3 \perp\!\!\!\perp \phi \mid x_4, x_5, x_6$, $x_2 \perp\!\!\!\perp x_5, x_6 \mid x_3, x_4$, $x_1 \perp\!\!\!\perp x_5, x_6 \mid x_2, x_3, x_4$.



4. You are given the following statements about the conditional independence of a set of 6 variables A, B, C, D, E, F

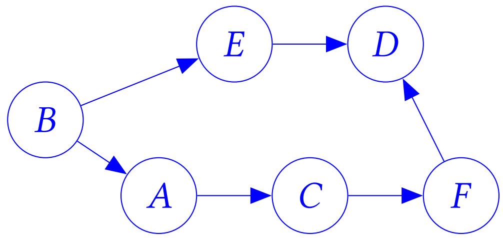
   (a) C is independent of B given A

   (b) F is independent of $\{A, B\}$ given C

   (c) E is independent of $\{A, C, F\}$ given B

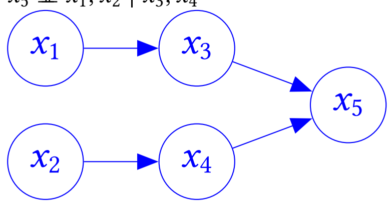
   (d) D is independent of $\{B, A, C\}$ given E, F

   Given these statements, state which of the following are true. Give a brief justification for your answer. No marks without the right justification. [You may find it useful to first draw a suitable graphical model that is possible to draw given the above statements.]
   (1) Is F independent of B given A? (2) Is C independent of E? ..4
   To answer this question, we will draw a BN to represent the given CIs. The list of given CIs hints at the order of variables as B, A, C, F, E, D. Using these, we can construct the following BN:



Using the d-separation rule, we find that the only path between B and F is blocked by A, so (1) is true. There is an unblocked path between C and E via B, so (2) is false.

5. Draw a Bayesian network over five variables $x_1, \dots, x_5$ assuming the variable order x_1, x_2, x_3, x_4, x_5. For this ordering, assume that the following set of local CIs hold in the distribution: $x_1 \perp\!\!\!\perp x_2$, $x_3 \perp\!\!\!\perp x_2 \mid x_1$, $x_4 \perp\!\!\!\perp x_1, x_3 \mid x_2$, $x_5 \perp\!\!\!\perp x_1, x_2 \mid x_3, x_4$ ..2



6. In the above Bayesian network, use only local conditional independencies (CIs) and the standard conditional probability axioms (2.7 to 2.10 from Chapter 2 of your textbook) to prove that $x_3 \perp\!\!\!\perp x_4$ ..5

(a) $x_4 \perp x_3, x_1 \mid x_2$ (given)
(b) $x_4 \perp x_3 \mid x_1, x_2$ (Weak Union applied to (a))
(c) $x_3 \perp x_4 \mid x_1, x_2$ (Symmetry on (b)
(d) $x_3 \perp x_2 \mid x_1$ (given)
(e) $x_3 \perp x_4, x_2 \mid x_1$ (Contraction on (c) and (d))
(f) $x_3 \perp x_4 \mid x_1$ (Decomposition on (e))
(g) $x_1 \perp x_4 \mid x_2$ (Decomposition on (a))
(h) $x_1 \perp x_2$ (given)
(i) $x_1 \perp x_4, x_2$ (Contraction on (g) and (h))
(j) $x_1 \perp x_4$ (Decomposition on (i)
(k) $x_4 \perp x_3, x_1$ (Contraction on (f) and (j))
(l) $x_4 \perp x_3$ (Decomposition on (k)

7. Prove that in a Bayesian Network (BN) where x_i, x_j are any two nodes with no edge between them, then $x_i \perp\!\!\!\perp x_j \mid \mathrm{Pa}(x_i), \mathrm{Pa}(x_j)$. [Use d-separation.] ..5

Let us consider two cases for any path between x_i and x_j

path type-1 :- atleast one of $\{Pa(x_i), Pa(x_j)\}$ appears in path.

let us call it is x_k, and without loss of generality, x_k is parent of x_i

$$x_i \leftarrow x_k \rightleftharpoons - - - \rightleftharpoons x_j$$

i.e., path is blocked x_k in this case

So $x_k \in Z$ (where $Z = \{Pa(x_i), Pa(x_j)\}$)

path type-2 :- None of the parents appear in path. So both edges from $x_i \& x_j$ are outgoing i.e.,

$$x_i \rightarrow \cdot \rightleftharpoons \cdot \rightleftharpoons \cdot\cdot \rightleftharpoons \cdot \leftarrow x_j$$

Claim:- There exists atleast one v-structure in any such path

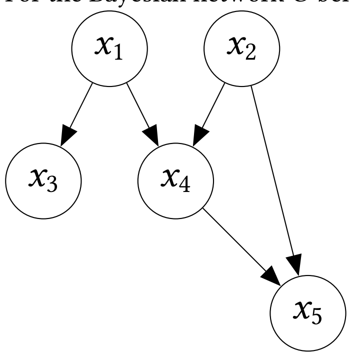
Proof:- Let us assume there is no v-structure. So node after x_i also should have right directing edge. This continues till end i.e., $\rightarrow \cdot \leftarrow x_j$ which is v-structure & contradiction.

let this v-structure node be x_v. by assumption of no parent in path, $x_v \notin Z$

i.e., path blocked by x_v in this case

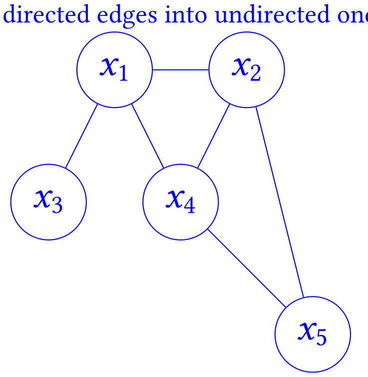
$\therefore$ by the rules of d-separation, $x_i \perp\!\!\!\perp x_j | Z$ i.e., $x_i \perp\!\!\!\perp x_j | Pa(x_i), Pa(x_j)$

8. For the Bayesian network G below, perform the following operations



(a) Convert it into an undirected graphical model H. ..2

Since x_1 and x_2 are co-parents of x_4, moralize the graph by adding an edge between them. Finally, convert all directed edges into undirected ones.



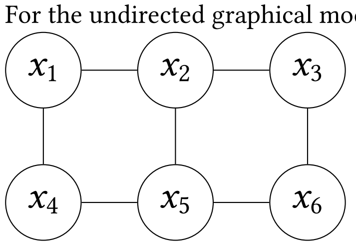
(b) List two CIs that hold in G but do not hold in H. ..2

It is easy to see that moralization introduces new dependencies that exist in H but not in G.

$x_1 \perp\!\!\!\perp x_2$ $x_2 \perp\!\!\!\perp x_3$ $x_3 \perp\!\!\!\perp x_5$

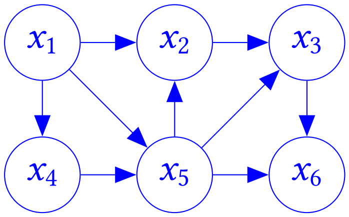
NOTE: The CI $x_1 \perp\!\!\!\perp x_5 \mid x_4$ does not hold in G, as conditioning on x_4 unblocks the V structure involving x_1, x_2 and x_4, resulting in an alternate path $x_1 \rightarrow x_4 \leftarrow x_2 \rightarrow x_5$.

9. For the undirected graphical model H below, perform the following operations



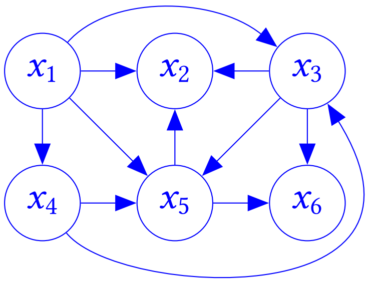
(a) Convert it into a BN G using variable order $x_1, x_4, x_5, x_2, x_3, x_6$. ..2

Total number of edges: 9



(b) Choose a different variable order that leads to adding more edges in G than in the above ordering. ..2

Many orderings are possible here. Consider the following ordering: $x_1, x_4, x_3, x_5, x_6, x_2$



(c) List two CIs that hold in H but do not hold in G. ..2

Two new edges introduced in G invalidates the following dependencies from H.

$x_1 \perp\!\!\!\perp x_5 \mid x_2, x_4 \quad x_5 \perp\!\!\!\perp x_3 \mid x_2, x_6$