CS 602 - Applied Algorithms: Assignment 2

Total Marks - 60

Instructions. Please try to be brief, clear, and technically precise. Use pseudo-codes to describe the algorithms. To solve the problems, one may assume that the instances are in general position, unless stated otherwise. Novelty in the answer carries marks.

Question 1[15 Marks] Let $(\mathcal{X}, \mathcal{R})$ and $(\mathcal{X}, \mathcal{R}')$ be two set systems with bounded VC dimension.

Show that $\mathsf{VC\ dim}(\mathcal{R} \cup \mathcal{R}') \leq \mathsf{VC\ dim}(\mathcal{R}) + \mathsf{VC\ dim}(\mathcal{R}') + 1$.

Hint: Use Sauer's lemma.

Question 2[15 Marks] *Recall the definition of a metric space. We now introduce a metric called the* ***shortest-path metric*** *of a graph. Given a simple, undirected graph G with vertex set V, the distance between any two vertices $u, v \in V$ is defined as the length of the shortest path connecting u and v in G, where the path length is measured by the number of edges it contains. We assume that G is connected. As a simple example, consider the complete graph K_n, which gives the n-point* ***equilateral space****, where the distance between every pair of vertices is 1.*

Show that any tree and its corresponding shortest-path metric on the vertices can be *isometrically embedded* into l_1 (also known as Manhattan distance).

Hint: Begin by considering the embedding of trees with unit-length edges.

Question 3[15 Marks] Consider the star graph below and its corresponding shortest-path metric on the vertices (as defined in question 2).

1. Prove that this graph cannot be isometrically embedded into l_2.
2. Show that minimum distortion for embedding this graph into l_2 is $\frac{2}{\sqrt{3}}$.

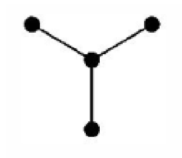

Figure 1: Star graph

Question 4[15 Marks] Let $\mathcal{U}$ be a set of size n, and let $\mathcal{X} \subseteq \mathcal{U}$ be a subset of size k. Let $\xi : \mathcal{U} \to [k]$ be a coloring of the elements of $\mathcal{U}$, chosen uniformly at random (i.e. each element of $\mathcal{U}$ is colored with one of the k colors uniformly and independently at random). Then the probability that the elements of X are colored with pairwise distinct colors is at least e^{-k}.