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Article *in* ACM Transactions on Algorithms · March 2013

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The traveling salesman problem for lines, balls and planes*

Adrian Dumitrescu† Csaba D. Tóth‡

Friday 31st October, 2014

Abstract

We revisit the traveling salesman problem with neighborhoods (TSPN) and propose several new approximation algorithms. These constitute either first approximations (for hyperplanes, lines, and balls in $\mathbb{R}^d$, for $d \geq 3$) or improvements over previously best approximations achievable in comparable times (for unit disks in the plane).

(I) Given a set of n hyperplanes in $\mathbb{R}^d$, a TSP tour whose length is at most $O(1)$ times the optimal can be computed in $O(n)$ time, when d is constant.

(II) Given a set of n lines in $\mathbb{R}^d$, a TSP tour whose length is at most $O(\log^3 n)$ times the optimal can be computed in polynomial time, when d is constant.

(III) Given a set of n lines in $\mathbb{R}^d$, a TSP tour whose length is at most $O(1)$ times the optimal can be computed in polynomial time, when d is constant. Our technique improves the previous best approximation in the plane, and generalizes to higher dimensions.

Keywords: Traveling salesman, group Steiner tree, linear programming, minimum-perimeter rectangular box, approximation algorithm, lines, planes, hyperplanes, unit disks and balls.

1 Introduction

In the Euclidean Traveling Salesman Problem (ETSP), given a set of points in the plane (or in the Euclidean space $\mathbb{R}^d$, $d \geq 3$), one seeks a shortest tour (closed curve) that visits each point. In the *TSP with neighborhoods* (TSPN), first studied by Arkin and Hassin [2], each point is replaced by a (possibly disconnected) region. The tour must visit at least one point in each of the given regions (i.e., it must intersect each region). A tour for a set of neighborhoods is also referred to as a TSP tour. Since the Euclidean TSP is known to be NP-hard in $\mathbb{R}^d$ for every $d \geq 2$ [25, 26, 44], TSPN is also NP-hard for every $d \geq 2$. TSP is recognized as one of the corner-stone problems in combinatorial optimization. See [39, 40] for a list of related problems in geometric network optimization.

*A preliminary version of this paper appeared in the Proceedings of the 24th ACM-SIAM Symposium on Discrete Algorithms, New Orleans, LA, 2013, SIAM, pp. 828–843.

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Supported in part by the NSF grant DMS-1001667.

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Related work. It is known that the Euclidean TSP admits a polynomial-time approximation scheme in $\mathbb{R}^d$, where $d = O(1)$, due to Arora [3] and Mitchell [38]. Subsequent running time improvements have been obtained by Rao and Smith [46]; specifically, the running time of their PTAS is $O(f(\varepsilon)\, n \log n)$, where $f(\varepsilon)$ grows exponentially in $1/\varepsilon$. In contrast, TSPN in general is harder to approximate. Certain instances are known to be APX-hard. Research efforts focused on approximations for families of neighborhoods with "nice" geometric properties. Typically, improved approximation methods are available when the neighborhoods are pairwise disjoint, or fat, or have comparable sizes. We briefly review previous work most closely related to our results.

Arkin and Hassin [2] gave constant-factor approximations for translates of a convex region, translates of a connected region, and more generally, for regions with diameters *parallel* to a common direction, and with the ratio between the longest and the shortest diameter bounded by a constant.

For n connected (possibly overlapping) neighborhoods in the plane, TSPN can be approximated with ratio $O(\log n)$ by the algorithm of Mata and Mitchell [34]. See also the survey by Bern and Eppstein [4] for a short outline of this algorithm. Subsequent running time improvements have been offered by Elbassioni et al. [20] and by Gudmundsson and Levcopoulos [28]. At its core, the $O(\log n)$-approximation relies on the following early result by Levcopoulos and Lingas [33]: Every (simple) rectilinear polygon P with n vertices, r of which are reflex, can be partitioned in $O(n \log n)$ time into rectangles whose total perimeter is $\log r$ times the perimeter of P.

The first PTAS for TSPN (beyond the PTAS for points) was obtained for fat regions of about the same size and of bounded depth in the plane by Dumitrescu and Mitchell [16]; see [50] for recent adjustments. In particular, this includes the case of pairwise-disjoint unit disks. For the related *minimum corridor connection* problem, Bodlaender et al. [6] present a PTAS for the case of fat rooms of comparable size (based on Arora's PTAS for Euclidean TSP), and give a constant-factor approximation algorithm for the case of fat rooms of varying sizes.

Using an approximation algorithm due to Slavik [49] for Euclidean group TSP (see below), de Berg et al. [11] obtain constant-factor approximations for disjoint fat convex regions in the plane, not necessarily of comparable size. Elbassioni et al. [19] improve the runtime of the approximation algorithm. Subsequently, Elbassioni et al. [20, 21] give constant-factor approximations for (possibly intersecting) fat convex regions of comparable size. Preliminary work by Mitchell gives (i) a PTAS [41] for bounded depth fat regions of arbitrary sizes in the plane; in particular for disjoint fat regions in the plane, and (ii) constant-factor approximations [42] for pairwise-disjoint connected neighborhoods of any size or shape. Chan and Elbassioni [8] gave a quasipolynomial-time approximation scheme for fat, weakly disjoint regions in $\mathbb{R}^d$, for $d = O(1)$ (which also works in metric spaces with bounded doubling dimension).

Finally, interesting variants are those with unbounded neighborhoods, such as lines or planes. For TSPN with n lines in the plane, an exact solution can be found in $O(n^5)$ time [7, 12, 51, 52] (see also [32]), and a 1.28-approximation can be computed in $O(n)$ time [14]. In contrast, TSPN for lines in $\mathbb{R}^3$ is NP-hard. The status of TSPN for planes in $\mathbb{R}^3$ appears to be unknown.

Regarding the degree of approximation achievable, TSPN with arbitrary neighborhoods is generally APX-hard [11, 48], and it remains so even for segments of nearly the same length [20]. For instance, approximating TSPN for connected regions in the plane within a factor smaller than 2 is intractable (NP-hard) [48]. The problem is also APX-hard for disconnected regions [48], the simplest case being point-pair regions [13]. It is conjectured that approximating TSPN for disconnected regions in the plane within a $O(\log^{1/2} n)$ factor is intractable [48]. Similarly, it is conjectured that approximating TSPN for connected regions in $\mathbb{R}^3$ within a $O(\log^{1/2} n)$ factor and for disconnected regions in $\mathbb{R}^3$ within a $O(\log^{2/3} n)$ factor [48] are intractable. Moreover, proving these conjectures seems to require advances in complexity, rather than geometry.

Region type	Old ratio	New ratio	NP-hard
Hyperplanes in $\mathbb{R}^d$	—	$(1+\varepsilon)\, 2^{d-1}/\sqrt{d}$	open
Planes in $\mathbb{R}^3$	—	2.31 in $O(n)$ time	open
Lines in $\mathbb{R}^d$	—	$O(\log^3 n)$	yes
Disjoint unit disks in the plane	3.55	—	yes
Unit disks in the plane	7.62	6.75	yes
Disjoint unit balls in $\mathbb{R}^3$	—	7.01	yes
Unit balls in $\mathbb{R}^3$	—	100.61	yes
Unit balls in $\mathbb{R}^d$	—	$O(7.73^d)$	yes

Table 1: Old and new (asymptotic) approximation ratios obtained in polynomial time. The ratios in rows 3–7 (for disks and balls) are obtained by using the black box PTAS for computing point tours. Disjoint unit disks admit a PTAS [16] but its time complexity is higher than that of the PTAS for points. The old ratios listed in column 2 are from [16].

Our results. In this paper we present several improved approximation algorithms for TSPN, for three types of neighborhoods: (i) hyperplanes in $\mathbb{R}^d$; (ii) lines in $\mathbb{R}^d$; (iii) congruent disks in the plane and congruent balls in $\mathbb{R}^d$. Our results are summarized in Table 1.

We start with hyperplanes in $\mathbb{R}^d$. No approximation algorithm was known for this type of neighborhoods. For constant d, we can compute constant-factor approximations in linear time.

Theorem 1. *Given a set of n hyperplanes in $\mathbb{R}^d$, and $\varepsilon > 0$, a TSP tour whose length is at most $(1+\varepsilon)\, 2^{d-1}/\sqrt{d}$ times the optimal can be computed in $O(d\, \varepsilon^{1-d}\, n)$ time. In particular for $d = 3$, a TSP tour whose length is at most 2.31 times the optimal can be computed in $O(n)$ time.*

We continue with lines in $\mathbb{R}^d$, a problem much harder to deal with. Note that an instance with parallel lines reduces to an instance of the Euclidean TSP for points in one dimension lower (namely the points of intersection between the given lines orthogonal to a hyperplane). Here we obtain the first approximations.

Theorem 2. *Given a set of n lines in $\mathbb{R}^d$, where d is a constant positive integer, a TSP tour whose length is at most $O(\log^3 n)$ times the optimal can be computed in polynomial time.*

For n unit disks in the plane (resp., unit balls in $\mathbb{R}^3$), we give constant-factor approximations by using a black box that computes a good tour of at most n points (the centers of a suitable subset of disks, resp., balls). While for disjoint unit disks in $\mathbb{R}^2$ the existence of a PTAS has been established a decade ago [16], it has remained open for disjoint unit balls in $\mathbb{R}^3$. The constant-factor approximations we give here reduce one instance of TSPN with disks (or balls) to one with points, and thus lead to faster and also conceptually simpler algorithms. For unit disks we obtain an improved approximation factor 6.75; the previous best ratio is 7.62, that also holds for *translates* of a convex region [16]. Let $T(n, d, \varepsilon)$ denote the running time for computing a $(1+\varepsilon)$-approximation of an optimal tour of n points in $\mathbb{R}^d$; recall that $T(n, d, \varepsilon)$ is currently exponential in $1/\varepsilon$ [46].

Theorem 3. *Given a set of n unit disks in the plane, and $\varepsilon > 0$, a TSP tour whose length is at most $\left(\frac{7}{3} + \frac{8\sqrt{3}}{\pi}\right)(1+\varepsilon)$ times the optimal, apart from an additive constant, can be computed in time $O(T(n, 2, 1.8\,\varepsilon))$. In particular, a TSP tour whose length is at most 6.75 times the optimal can be computed in time $O(T(n, 2, 0.0018))$. Alternatively, a TSP tour whose length is at most 8.52 times the optimal can be computed in time $O(n^{3/2} \log^5 n)$.*

For congruent balls in $\mathbb{R}^3$ we give the first explicit constant approximation factor, *not* in the $O(1)$ form.

Theorem 4. *Given a set of n unit balls in $\mathbb{R}^3$, and $\varepsilon > 0$, a TSP tour whose length is at most $54\sqrt{3}(1+\varepsilon)$ times the optimal, apart from an additive constant, can be computed in time $O(T(n,3,\varepsilon))$. In particular, a TSP tour whose length is at most 100.61 times the optimal can be computed in time $O(T(n,3,0.01))$. Alternatively, a TSP tour whose length is at most 104.1 times the optimal can be computed in time $O(n^3)$.*

The above result generalizes to congruent balls in $\mathbb{R}^d$ for any fixed dimension d; the proof is analogous to that of Theorem 4 for the 3-dimensional version.

Theorem 5. *Given a set of n unit balls in $\mathbb{R}^d$, and $\varepsilon > 0$, a TSP tour whose length is at most $O(7.73^d)$ times the optimal can be computed in time $O(T(n,d,\varepsilon))$.*

Preliminaries. Let $\mathcal{R}$ be a set of regions in $\mathbb{R}^d$, $d \geq 2$. A set $\Xi \subset \mathbb{R}^d$ *intersects* $\mathcal{R}$ if Ξ intersects each region in $\mathcal{R}$, that is, $\Xi \cap r \neq \emptyset$, $\forall r \in \mathcal{R}$. A shortest TSP tour for a set $\mathcal{R}$ of regions (neighborhoods), denoted by $\mathrm{OPT}(\mathcal{R})$, is a shortest closed curve in the ambient space that intersects $\mathcal{R}$.

The Euclidean length of a curve γ is denoted by $\mathrm{len}(\gamma)$, or just $|\gamma|$ when there is no danger of confusion. Similarly, the total (Euclidean) length of the edges of a geometric graph G or a polygon P is denoted by $\mathrm{len}(G)$ and $\mathrm{per}(P)$, respectively. For a hyperrectangle (rectangular box) Q in $\mathbb{R}^d$ with sides $w_1, \ldots, w_d$, the total edge length $\mathrm{per}(Q) = 2^{d-1} \sum_{i=1}^d w_i$ is called its *perimeter*.

For $\alpha \geq 1$, we say that an approximation algorithm (for TSPN) has ratio α if its output tour ALG satisfies $\mathrm{len}(\mathrm{ALG}) \leq \alpha \, \mathrm{len}(\mathrm{OPT})$, where OPT is an optimal tour, and has *asymptotic* ratio α if its output satisfies $\mathrm{len}(\mathrm{ALG}) \leq \alpha \, \mathrm{len}(\mathrm{OPT}) + \beta$ for some constant $\beta \geq 0$.

The convex hull of a set $A \subset \mathbb{R}^d$ is denoted by $\mathrm{conv}(A)$. The Cartesian coordinates of a point $p \in \mathbb{R}^d$ are denoted by $x_1(p), \ldots, x_d(p)$. For a line segment $s \in \mathbb{R}^3$, $\Delta_1(s), \ldots, \Delta_d(s)$ denote the lengths of its projections on the d coordinate axes.

2 The illusions and pitfalls of localization

Given a set $\mathcal{R}$ of n regions, it would be helpful to find a convex set that contains an optimal tour $\mathrm{OPT} = \mathrm{OPT}(\mathcal{R})$ and whose diameter is a polynomial in n and perhaps other parameters, such as an upper bound on $\mathrm{diam}(\mathrm{OPT})$. A convex set C_1 that intersects $\mathcal{R}$ is often easy to compute. It is tempting to believe (as it has been suggested by several researchers) that if C_1 is scaled up by some suitable polynomial factor, the resulting convex set C_2 might contain OPT. Finding such a set C_2 would allow using standard approximation techniques (such as discretization, convex approximation tools, etc.).

In this section, we show that this naïve approach is infeasible when the regions in $\mathcal{R}$ are lines or hyperplanes in $\mathbb{R}^d$. Let $\lambda(x,y)$ be a given polynomial of 2 variables with positive coefficients. We present constructions for a set of n lines and a set of n hyperplanes, respectively, such that the minimum intersecting ball B_1 is centered at the origin, but $\lambda \, B_1$ fails to contain OPT, where $\lambda = \lambda(n, \mathrm{diam}(B_1))$. Moreover: (i) the shortest TSP tour contained in $\lambda \, B_1$ is a $\Theta(\sqrt{n})$-approximation for lines, in contrast with the $O(\log^3 n)$-approximation in Theorem 2, and (ii) the shortest TSP tour contained in $\lambda \, B_1$ is a c-approximation for hyperplanes, where $c > 1$ is a constant, which rules out a $(1+\varepsilon)$-approximation algorithm using this approach.

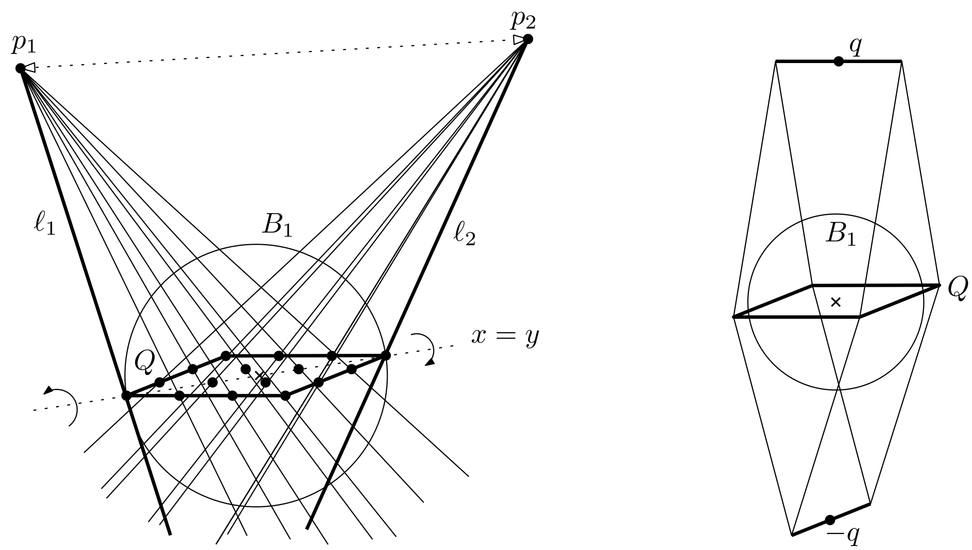

Figure 1: Left: a set $\mathcal{L}$ of nearly vertical lines that intersect a square Q in a grid-like pattern, and their minimum intersecting ball B_1. Right: a set of four nearly vertical planes containing four sides of a square $Q = [-1, 1]^2$ in the xy-plane, and their minimum intersecting ball B_1.

Lines in $\mathbb{R}^3$. For an integer n and a polynomial $\lambda(x, y)$, we construct a set $\mathcal{L}$ of n lines in $\mathbb{R}^3$. Consider the square $Q = [-1, 1]^2$ in the xy-plane (Fig. 1 (left)). Let B_1 be the ball of radius $\sqrt{2}$ centered at the origin, and note that $Q \subset B_1$. We first construct two skew lines in $\mathbb{R}^3$ whose minimum intersecting ball is B_1. Start with two vertical lines passing through $(1, 1, 0)$ and $(-1, -1, 0)$, and observe that they intersect any horizontal plane at two points at distance $2\sqrt{2}$ apart. Rotate these lines about the horizontal line $\ell_0 : y = x$ by some small angle α and $-\alpha$, respectively, to obtain two skew lines ℓ_1 and ℓ_2. As ℓ_1 and ℓ_2 remain orthogonal to ℓ_0, the minimum intersecting ball of ℓ_1 and ℓ_2 is still B_1. Choose α such that ℓ_1 and ℓ_2 intersect the horizontal plane $z = n\,\lambda(n, 4)$ at two points, p_1 and p_2, at distance 4 apart. We now define the set $\mathcal{L}$ of n lines as follows: $\mathcal{L}$ contains ℓ_1 and ℓ_2, about half of the lines in $\mathcal{L}$ pass through p_1 and the other half pass through p_2. The lines in $\mathcal{L}$ are nearly vertical and intersect Q in a square grid pattern, where any two intersection points are at distance at least $2/\sqrt{n}$ apart.

Lemma 1. *Every TSP tour γ lying in $\lambda\, B_1$ satisfies $\mathrm{len}(\gamma) \geq \frac{\sqrt{n}}{8}\,\mathrm{len}(\mathrm{OPT})$. In particular, $\lambda\, B_1$ does not contain the optimal tour $\mathrm{OPT} = \mathrm{OPT}(\mathcal{L})$ or any $o(\sqrt{n})$-approximation of it.*

Proof. Note that the tour that visits points p_1 and p_2, of length $2|p_1p_2| = 8$, intersects all lines. Consequently, $\mathrm{len}(\mathrm{OPT}) \leq 8$ and $\mathrm{diam}(\mathrm{OPT}) \leq 4$. Consider a tour γ lying in $\lambda\, B_1$ and let γ' be the orthogonal projection of γ onto the xy-plane, where $\mathrm{len}(\gamma') \leq \mathrm{len}(\gamma)$. Since the lines in $\mathcal{L}$ are nearly vertical, the orthogonal projections of the line segments in $\{\ell \cap \lambda\, B_1 : \ell \in \mathcal{L}\}$ have length at most $2/n$, and they each contain distinct grid points within Q. Since the distance between any two grid points is at least $2/\sqrt{n}$, we have $\mathrm{len}(\gamma') \geq n(2/\sqrt{n} - 4/n) = 2\sqrt{n} - 4 \geq \sqrt{n}$, and so $\mathrm{len}(\gamma) \geq \sqrt{n} \geq \frac{\sqrt{n}}{8}\,\mathrm{len}(\mathrm{OPT})$, as required. □

Planes in $\mathbb{R}^3$. For an integer n and a polynomial $\lambda(x, y)$, we construct a set $\mathcal{H}$ of n planes in $\mathbb{R}^3$. Consider the unit square $Q = [-1, 1]^2$ in the xy-plane (Fig. 1 (right)). Let the first 4 planes in $\mathcal{H}$ each contain one side of Q. The two planes containing the two sides of Q parallel to the x-axis intersect in a line parallel to the x-axis and containing the point $q = (0, 0, h)$, where h is large, specifically $h = n\,\lambda(n,3)$. The two planes containing the sides of Q parallel to the y-axis intersect in a line parallel to the y-axis and containing the point $-q = (0,0,-h)$. By symmetry, the minimum intersecting ball of these four planes is centered at the origin, and its radius is at least $1 - 1/h$ and at most 1. Arrange the remaining $n-4$ planes in $\mathcal{H}$ such that they all contain the point $q = (0,0,h)$, are tangent to the ball B_1, and the tangency points are uniformly distributed along a horizontal circle $C \subset \partial B_1$. By construction, B_1 is the minimum intersecting ball of the n planes in $\mathcal{H}$.

Lemma 2. *Every TSP tour γ lying in λB_1 satisfies $\mathrm{len}(\gamma) \geq \frac{\pi}{2}(1 - O(1/n))\,\mathrm{len}(\mathrm{OPT})$. In particular, λB_1 does not contain the optimal tour $\mathrm{OPT} = \mathrm{OPT}(\mathcal{H})$ or any $(1+\varepsilon)$-approximation of it for a sufficiently small $\varepsilon > 0$.*

Proof. Note that the triangle formed by the point q and its orthogonal projections onto the two planes containing the two sides of Q parallel to the y-axis is a tour for $\mathcal{H}$. The length of this tour is at most $4 + 4/h$. Consequently, $\mathrm{len}(\mathrm{OPT}) \leq 4 + 4/h$ and $\mathrm{diam}(\mathrm{OPT}) \leq 3$. Consider a tour γ lying in λB_1, and let γ' be the orthogonal projection of γ to the xy-plane, where $\mathrm{len}(\gamma') \leq \mathrm{len}(\gamma)$. Since the planes in $\mathcal{H}$ are nearly vertical, the orthogonal projections of the disks in $\{H \cap \lambda B_1 : H \in \mathcal{H}\}$ are ellipses of width at most $2/n$. The first four ellipses each contain a side of the square Q. The remaining ellipses form $\lfloor (n-4)/2 \rfloor$ pairs such that the major axes of any pair are on parallel lines at distance at least $2 - 2/n$ apart, and the directions of the pairs are uniformly distributed. Consequently, the width of γ' is at least $2 - O(1/n)$, and so $\mathrm{len}(\gamma') \geq 2\pi(1 - O(1/n))$ $\geq \frac{\pi}{2}(1 - O(1/n))\,\mathrm{len}(\mathrm{OPT})$, as required. □

Easy weak approximations. Finding a minimum-radius ball B_1 that intersects a set of n hyperplanes (resp., lines) in $\mathbb{R}^d$ is an LP-type problem [18]; for a fixed d, such a ball can be computed in $O(n)$ time. This immediately leads to a simple 2^{d-1}-approximation for hyperplanes and a $O(n^{1-1/(d-2)})$-approximation for lines in $\mathbb{R}^d$. Indeed, since the minimum enclosing ball B_{OPT} of an optimal tour OPT also intersects all n hyperplanes (resp., lines), it is clear that $\mathrm{diam}(B_1) \leq \mathrm{diam}(B_{\mathrm{OPT}})$. Since B_{OPT} is spanned by up to $d+1$ points, it is easy to see that $\mathrm{len}(\mathrm{OPT}) \geq 2\,\mathrm{diam}(B_{\mathrm{OPT}})$. On the other hand, a Hamiltonian cycle of the 2^d vertices of an enclosing hypercube of B_1 intersects all hyperplanes (cf., Observation 1), and has length at most $2^d \mathrm{diam}(B_1)$. For n lines in $\mathbb{R}^d$, one can compute all intersection points of the n lines with the boundary of B_1, and return an approximate tour for these $2n$ points of length $\mathrm{diam}(B_1) \cdot O(n^{1-1/(d-2)})$ by a result of Few [23].

In Section 3 we obtain a better approximation for TSPN with n hyperplanes, a ratio close to $2^{d-1}/\sqrt{d}$, by using hyperrectangles instead of balls and a careful analysis. In Section 4, we use a completely different approach to achieve a much better $O(\log^3 n)$-approximation for TSPN with n lines in $\mathbb{R}^3$.

3 TSPN for hyperplanes in $\mathbb{R}^d$

In this section we prove Theorem 1: we present a constant factor approximation algorithm for TSPN for a set $\mathcal{H}$ of n hyperplanes in $\mathbb{R}^d$ with ratio $(1+\varepsilon)\frac{2^{d-1}}{\sqrt{d}}$ and running in $O(n)$ time, for constant d and $\varepsilon > 0$. In particular, for $\varepsilon = 0.0002$, we get the approximation ratios 2.31 in $\mathbb{R}^3$, 4.001 in $\mathbb{R}^4$, and 7.16 in $\mathbb{R}^5$.

Our algorithm is based on solving low-dimensional linear programs; it combines ideas from [14, 15, 16, 32]. We show below (Lemma 4) that any closed curve $\gamma \subset \mathbb{R}^d$ is contained in a rectangular box of edge lengths $w_1, \ldots, w_d$ such that $\sum_{i=1}^{d} w_i \leq \frac{\sqrt{d}}{2}\,\mathrm{len}(\gamma)$. We apply this result to the optimal tour OPT($\mathcal{H}$). Then we use linear programming to compute a $(1+\varepsilon)$-approximation for the minimum-perimeter rectangular box intersecting $\mathcal{H}$, and use a Hamilton cycle of the 2^d vertices as an approximate tour.

Let Q be rectangular box in $\mathbb{R}^d$ such that the d extents of Q are $w_1 \le w_2 \le \ldots \le w_d$. It is not difficult to see (by induction on d) that Q admits a Hamiltonian cycle of total length

$$\tau(Q) = 2^{d-1}w_1 + 2^{d-2}w_2 + \ldots + 2w_{d-1} + 2w_d = w_d + \sum_{j=1}^{d} 2^{d-j} w_j.$$

The *orientation* of a rectangular box Q in $\mathbb{R}^d$ is given by an orthonormal basis whose vectors are parallel to the edges of Q. Select a set $A = \{\alpha_1, \ldots, \alpha_m\}$ of $m = O(\varepsilon^{1-d})$ orientations that cover all possible orientations within an error of $\varepsilon/(d-1)$. That is, for any orientation α, there is an orientation $\alpha' \in A$ and a matching between the orthogonal bases α and α' so that the angle between any two corresponding vectors is at most $\varepsilon/(d-1)$.

Algorithm A1.

Step 1: Let $m = O(\varepsilon^{1-d})$. For each $i = 1, \ldots, m$, compute a minimum-perimeter rectangular box Q_i with orientation α_i that intersects $\mathcal{H}$.

Step 2: Let Q be a box with the minimum perimeter over all m directions, found above. Return a Hamiltonian cycle of the 2^d vertices of Q, of length $\tau(Q)$, as depicted in Fig. 2 (right).

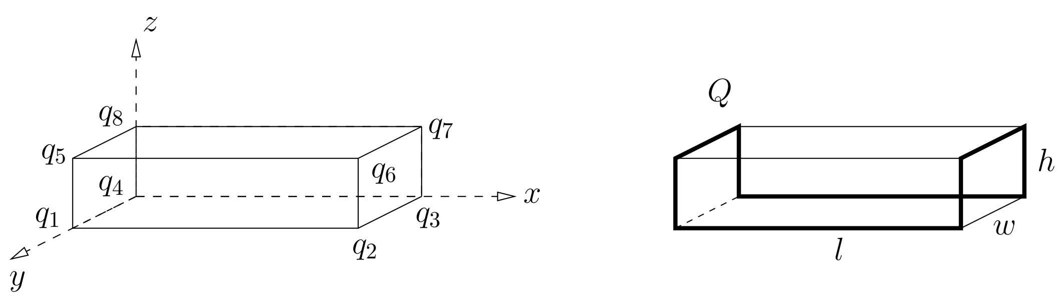

Figure 2: Left: An axis-aligned rectangular box Q. Right: a Hamiltonian cycle (in bold lines) of length $2l + 2w + 4h$ of the vertices of Q that visits all planes intersecting Q.

For each iteration $i = 1, \ldots, m$, we compute the box Q_i by linear programming. By a suitable rotation of the set $\mathcal{H}$ of hyperplanes, the box Q_i is axis-aligned. This can be obtained in $O(n)$ time per iteration. For a hyperplane σ, let $\vec{u}(\sigma)$ denote the unit vector orthogonal to σ with a positive x_d-coordinate. An axis-aligned rectangular box in $\mathbb{R}^d$ has 2^{d-1} antipodal pairs of vertices, which we denote by s_i and t_i, for $i = 1, \ldots, 2^{d-1}$, such that the vector $s_i t_i$ has a positive x_d-coordinate. Partition $\mathcal{H}$ into 2^{d-1} *types* based on the following rule (ties are broken arbitrarily):

- $\sigma \in \mathcal{H}$ is of type i, $i \in \{1, \ldots, 2^{d-1}\}$, if the $\vec{u}(\sigma)$-minimal and $\vec{u}(\sigma)$-maximal vertices of Q_i are s_i and t_i, respectively.

Let $\mathcal{H} = \bigcup_{i=1}^{2^{d-1}} \mathcal{H}_i$ be the corresponding partition of the hyperplanes given by this rule. For a hyperplane σ, that is not parallel to any coordinate axis, denote by $\sigma(p) \le 0$ (respectively, by $\sigma(p) \ge 0$) that a point $p \in \mathbb{R}^d$ lies in the closed halfspace bounded from above by σ (resp., bounded from below by σ). Observe that for $i = 1, \ldots 2^{d-1}$,

- a hyperplane $\sigma \in \mathcal{H}_i$ intersects the rectangular box Q_i if and only if $\sigma(s_i) \le 0 \le \sigma(t_i)$.

The minimum-perimeter objective is naturally expressed as a linear function. The resulting linear program has $2d$ variables $x_1, y_1, \ldots, x_d, y_d$ for the box $Q_i = [x_1, y_1] \times \ldots \times [x_d, y_d]$, and $2n + d$ constraints.

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{j=1}^{d}(y_j - x_j) \qquad \text{(LP1)} \\ \text{subject to} & \begin{cases} \sigma(s_i) \le 0 & \text{if } \sigma \in \mathcal{H}_i, \quad \forall \sigma \in \mathcal{H} \\ \sigma(t_i) \ge 0 & \text{if } \sigma \in \mathcal{H}_i, \quad \forall \sigma \in \mathcal{H} \\ x_j \le y_j & \qquad\qquad\quad \forall j \in \{1, \ldots, d\} \end{cases} \end{array}$$

Algorithm analysis. The key observation is the following.

Observation 1.

(i) *If a polygon γ intersects $\mathcal{H}$, then* $\operatorname{conv}(\gamma)$ *(and any set containing* $\operatorname{conv}(\gamma)$*) also intersects $\mathcal{H}$.*

(ii) *If a polyhedron Q intersects $\mathcal{H}$, then every Hamiltonian cycle of the vertices of Q also intersects $\mathcal{H}$.*

Let Q^* be a minimum-perimeter rectangular box intersecting $\mathcal{H}$, with side lengths denoted by $w_1, \ldots, w_d$. To account for the error made by discretization, we need the following easy fact. The planar variant was shown in [15, Lemma 2]. We include the almost identical proof for completeness.

Lemma 3. *There exists $i \in \{1, \ldots, m\}$ such that* $\operatorname{per}(Q_i) \le (1 + \varepsilon) \operatorname{per}(Q^*)$.

Proof. Consider a box Q_i, $i \in \{1, \ldots, m\}$, that minimizes the angle difference β between the orientations of Q_i and Q^*. By construction, there exists $i \in \{1, \ldots, m\}$ such that the angle β between the orientations of Q_i and Q^* is at most $\varepsilon/(d-1)$, that is, $\beta \le \varepsilon/(d-1)$.

Let Q_i' be the minimum-perimeter box with the same orientation as Q_i such that Q_i' contains Q^*. By definition, $\operatorname{per}(Q_i) \le \operatorname{per}(Q_i')$. An easy trigonometric calculation shows that the corresponding sides $w_1', \ldots, w_d'$ of Q_i' are bounded from above as follows. For $j = 1, \ldots, d$, we have

$$w_j' \le w_j \cos\beta + \left(\sum_{k \ne j} w_k\right) \sin\beta \le w_j + \left(\sum_{k \ne j} w_k\right) \frac{\varepsilon}{d-1}.$$

Consequently,

$$\sum_{j=1}^{d} w_j' \le (1 + \varepsilon) \sum_{j=1}^{d} w_j,$$

that is,

$$\operatorname{per}(Q_i') \le (1 + \varepsilon) \operatorname{per}(Q^*).$$

Since $\operatorname{per}(Q_i) \le \operatorname{per}(Q_i')$, it follows that $\operatorname{per}(Q_i) \le (1 + \varepsilon) \operatorname{per}(Q^*)$, as required. $\square$

Lemma 4. *A closed curve $\gamma \subset \mathbb{R}^d$ is contained in a rectangular box Q with side lengths $w_1, \ldots, w_d$ satisfying $\sum_{j=1}^{d} w_j \le \frac{\sqrt{d}}{2} \operatorname{len}(\gamma)$. Consequently,* $\operatorname{per}(Q) \le \sqrt{d} \cdot 2^{d-2} \operatorname{len}(\gamma)$.

Proof. Let γ be a closed curve and let $Q = Q(\gamma)$ be a minimum-perimeter enclosing rectangular box. Assume for convenience that Q is axis-aligned, so that its extents in the d coordinates are $w_1, \ldots, w_d$, respectively. Since Q has minimum perimeter, γ meets each $(d-1)$-dimensional face of Q. Arbitrarily select a point a_i of γ on each of the $2d$ faces of Q, in the order traversed by γ, to obtain a polygonal closed curve $\gamma_1 = (a_1, \ldots, a_{2d})$ still enclosed in Q (duplicate points are possible). For convenience, introduce $a_{2d+1} = a_1$.

By the triangle inequality,

$$\mathrm{len}(\gamma) \geq \mathrm{len}(\gamma_1) = \sum_{i=1}^{2d} \mathrm{len}(a_i a_{i+1}). \tag{1}$$

By the Cauchy-Schwarz inequality, for $i = 1, \ldots, 2d$, we have

$$\mathrm{len}(a_i a_{i+1}) = \left(\sum_{j=1}^{d} \Delta_j^2(a_i a_{i+1}) \right)^{1/2} \geq \frac{1}{\sqrt{d}} \sum_{j=1}^{d} \Delta_j(a_i a_{i+1}). \tag{2}$$

Since γ_1 is a closed curve that visits both faces of Q orthogonal to the jth axis for each $j = 1, \ldots, d$, we have

$$\sum_{i=1}^{2d} \Delta_j(a_i a_{i+1}) \geq 2w_j, \text{ for } j = 1, \ldots, d.$$

Combined with (1) and (2), this yields $\mathrm{len}(\gamma) \geq \frac{2}{\sqrt{d}} \sum_{j=1}^{d} w_j$, as claimed. ☐

Let $L^* = \mathrm{len}(\mathrm{OPT})$ and let Q_{OPT} be a minimum-perimeter rectangular box containing OPT. By Observation 1 and Lemmas 3 and 4, we have

$$\mathrm{per}(Q_i) \leq (1+\varepsilon)\mathrm{per}(Q^*) \leq (1+\varepsilon)\mathrm{per}(Q_{\mathrm{OPT}}) \leq (1+\varepsilon)\sqrt{d} \cdot 2^{d-2} L^*. \tag{3}$$

By Observation 1, any Hamilton cycle of Q_i is a valid tour of the hyperplanes in $\mathcal{H}$, and its length is bounded above by $\mathrm{per}(Q_i)$. From (3), this length is at most $(1+\varepsilon)\sqrt{d} \cdot 2^{d-2}$ times the optimum.

We now refine the analysis and show that the length $\tau(Q_i)$ of a *shortest* Hamilton cycle of Q_i is at most $2^{d-1}/\sqrt{d}$ times the optimum. Algorithm A1 computes a tour T of length $L = \tau(Q_i) = w_d + \sum_{j=1}^{d} 2^{d-j} w_j$, where $S = \sum_{j=1}^{d} w_j \leq (1+\varepsilon)\frac{\sqrt{d}}{2} L^*$. For $i = 1, \ldots, d$ put $S_i = \sum_{j=1}^{i} w_j$ and $S = S_d$. Since $w_1 \leq w_2 \ldots \leq w_d$, we have $S_i \leq iS/d$, for $i = 1, \ldots, d$. Consequently,

$$\begin{aligned}
L &= w_d + \sum_{j=1}^{d} 2^{d-j} w_j = 2^{d-1} w_1 + 2^{d-2} w_2 + \ldots + 2w_{d-1} + 2w_d \\
&= 2S_d + \sum_{i=1}^{d-2} 2^i S_{d-i-1} \\
&\leq \frac{S}{d} \left(2d + \sum_{i=1}^{d-2} 2^i (d-i-1) \right) \\
&= \frac{S}{d} \left(\left(2d + d \sum_{i=1}^{d-2} 2^i \right) - \sum_{i=1}^{d-2} (i+1) 2^i \right) \\
&= \frac{S}{d} \left(d\, 2^{d-1} - (d-2)\, 2^{d-1} \right) = \frac{2^d}{d}\, S.
\end{aligned} \tag{4}$$

To evaluate $\sum_{i=1}^{d-2}(i+1)2^i$ in the last line of (4), we have set $F(x) = \sum_{i=2}^{d-1} x^i$, and evaluated its derivative $F'(x)$ in two ways. On one hand

$$F'(x) = \sum_{i=2}^{d-1} ix^{i-1} = \sum_{i=1}^{d-2}(i+1)x^i, \text{ thus } \sum_{i=1}^{d-2}(i+1)2^i = F'(2).$$

On the other hand,

$$F(x) = x^2 \frac{x^{d-2}-1}{x-1}, \text{ thus}$$

$$F'(x) = 2x \frac{x^{d-2}-1}{x-1} + \frac{x^2}{(x-1)^2} \left((d-2)x^{d-3}(x-1) - x^{d-2} + 1\right).$$

It follows that

$$F'(2) = 4\left(2^{d-2}-1\right) + 4\left((d-2)\,2^{d-3} - 2^{d-2} + 1\right) = (d-2)\,2^{d-1},$$

as required by (4).

Substituting now the upper bound $S \leq (1+\varepsilon)\frac{\sqrt{d}}{2}L^*$ yields

$$L \leq \frac{2^d}{d}\, S \leq (1+\varepsilon)\frac{\sqrt{d}}{2}\,\frac{2^d}{d}L^* = (1+\varepsilon)\frac{2^{d-1}}{\sqrt{d}}L^*,$$

as required.

In particular, for $d = 3$ and $\varepsilon \leq 0.00022$, we have $L \leq 2.31L^*$, thus algorithm A1 computes a tour whose length is at most 2.31 times the optimal. The algorithm solves a (large!) constant number of 6-dimensional linear programs, each in $O(n)$ time [37]. The overall time is $O(n)$. A modest number of linear programs suffices to get a weaker approximation, say 2.5 or 3.

Remark. A standard reduction from the *sorting problem* or from the *convex hull problem* as in [45], applied to a suitable set of vertical planes, shows that a shortest TSP tour for n planes cannot be computed in $O(n)$ time; that is, in the worst-case, finding it requires $\Omega(n \log n)$ time in the algebraic decision tree model of computation.

4 TSPN for lines in $\mathbb{R}^d$

In this section we prove Theorem 2. Let $\mathcal{L} = \{\ell_1, \ldots, \ell_n\}$ be a set of n lines in $\mathbb{R}^d$, $d \geq 3$. We reduce the TSPN problem to a group Steiner tree problem on a geometric graph. Specifically, we construct a geometric graph $G_{\mathcal{L}} = (V, E)$, where V is a set of points on the lines in $\mathcal{L}$, and E consists of line segments connecting some of these points; the weight of an edge is its Euclidean length. We have $V = \bigcup_{i=1}^{n} V_i$, where $V_i \subset \ell_i$ $(i = 1, \ldots, n)$ naturally form n groups, one for each line. We then run an approximation algorithm for the group Steiner tree problem on this graph.

It is well known that an optimal TSP tour for points can be 2-approximated by a minimum spanning tree (a TSP tour is obtained by doubling the edges of the MST and by using shortcuts and the triangle inequality). Reich and Widmayer [47] introduced the following *group Steiner tree* (a.k.a., *one-of-a-set Steiner tree*) problem. Given an edge weighted graph $G = (V, E)$ and g *groups* of vertices $V_1, \ldots, V_g \subseteq V$, $|V| = n$, find a tree in G of minimum weight that includes at least one vertex from each group. The problem is known to be APX-hard [5], and it cannot be approximated better than $\Omega(\log^{2-\varepsilon} n)$ for any $\varepsilon > 0$ unless NP admits quasipolynomial-time

Las Vegas algorithms [29]. The current best approximation ratio, $O(\log^2 n \log g)$ comes from the algorithm of Garg et al. [27] as further refined by Fakcharoenphol et al. [22]. As before with the MST, by doubling the edges of such a tree, and by using shortcuts and the triangle inequality, one can obtain a Hamiltonian cycle which includes at least one vertex from each group, and the approximation ratio of this cycle is of the same asymptotic order with the approximation ratio of the group Steiner tree used.

We construct the graph $G_{\mathcal{L}}$ for a set $\mathcal{L}$ of n lines in $\mathbb{R}^d$ shortly. Lemma 7 below shows that the length of a minimum group Steiner tree in $G_{\mathcal{L}}$ is a constant-factor approximation for the minimum TSP tour for $\mathcal{L}$. In our case, the graph $G_{\mathcal{L}}$ has $O(n^3)$ vertices and the number of groups is n, so the $O(\log^2 n \log g)$-approximation [22, 27] for the group Steiner tree problem on a graph with n vertices and g groups yields an $O(\log^3 n)$-approximation for TSPN with n lines in $\mathbb{R}^d$.

Intuition. A *transversal* between two lines, ℓ_1 and ℓ_2, is a line segment p_1p_2 with $p_1 \in \ell_1$ and $p_2 \in \ell_2$. A *minimum transversal* of two lines is one of minimum length; is known to be orthogonal to both lines, and if the two lines intersect, it is a segment of zero length (i.e., $p_1 = p_2$).

The graph $G_{\mathcal{L}}$ is defined in terms of a set S of transversal segments whose endpoints are on distinct lines from $\mathcal{L}$. The vertices of $G_{\mathcal{L}}$ will be the set of endpoints of the segments in S; the edges of $G_{\mathcal{L}}$ will include all segments in S, all segments along the lines in $\mathcal{L}$ between consecutive vertices, and some other edges. Let S be the set of minimum transversals between all $\binom{n}{2}$ pairs of lines. We show (cf. Lemma 5) that any segment p_ip_j, with $p_i \in \ell_i$ and $p_j \in \ell_j$, can be approximated by the detour via the minimum transversal of ℓ_i and ℓ_j, if the angle between ℓ_i and ℓ_j is bounded from below. The minimum transversals, however, are insufficient if several consecutive lines visited by an optimal tour are nearly parallel. For nearly parallel lines in $\mathcal{L}$, we include many additional transversal segments in S, which lie in several parallel planes.

Two Technical Lemmas. The approximation relies on Lemmas 5 and 6 (below). According to Lemma 5, if the directions of two lines are far apart, then a connecting segment can be approximated by a 3-segment path that detours through the minimum transversal of the two lines. According to Lemma 6, if two lines are nearly vertical and we are given some horizontal transversal segment between the lines, then the only way to find a significantly shorter transversal is to move the endpoints closer to the endpoints of the minimum transversal.

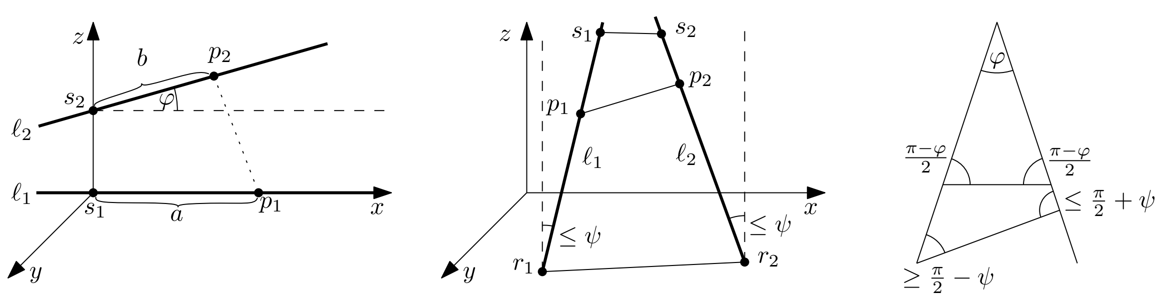

Figure 3: Left: The angle between lines ℓ_1 and ℓ_2 is φ. The distance between $p_1 \in \ell_1$ and $p_2 \in \ell_1$ is approximated by the polygonal path (p_1, s_1, s_2, p_2) that passes through the minimum transversal s_1s_2 between the two lines. Middle: If $|p_1p_2| \leq \frac{1}{3}|r_1r_2|$, then p_1 and p_2 are much closer to the minimum transversal than r_1 and r_2, respectively. Right: Two triangles with angle φ. The other two angles are equal in one triangle, and they differ by at most 2ψ in the other triangle.

Lemma 5. *Let ℓ_1 and ℓ_2 be two lines in $\mathbb{R}^d$ such that the angle between their directions is $\varphi \in (\varphi_0, \frac{\pi}{2}]$. Let s_1s_2 be their minimum transversal with $s_1 \in \ell_1$ and $s_2 \in \ell_2$. Let $p_1 \in \ell_1$ and $p_2 \in \ell_2$ be two points. Then $|p_1s_1| + |s_1s_2| + |s_2p_2| \leq \sqrt{\frac{3}{1-\cos\varphi_0}}|p_1p_2|$.*

Proof. Consider the 3-dimensional affine subspace spanned by ℓ_1 and ℓ_2. Without loss of generality, we may assume that ℓ_1 is the x-axis, $p_1 = (a, 0, 0)$, $s_1 = (0, 0, 0)$, and $s_2 = (0, 0, h)$ as in Fig. 3 (left). Let $a = |p_1s_1|$ and $b = |p_2s_2|$. The Cauchy-Schwarz inequality yields the upper bound

$$|p_1s_1| + |s_1s_2| + |s_2p_1| = a + h + b \leq \sqrt{3(a^2 + b^2 + h^2)}.$$

If $x(p_2) \geq 0$ (as in Fig. 3, left), then the law of cosines yields

$$\begin{aligned} |p_1p_2|^2 &= h^2 + a^2 + b^2 - 2ab\cos\varphi \\ &= h^2 + (a-b)^2\cos\varphi + (a^2+b^2)(1-\cos\varphi) \\ &\geq (1-\cos\varphi)(a^2+b^2+h^2) \\ &\geq (1-\cos\varphi_0)(a^2+b^2+h^2). \end{aligned}$$

If $x(p_2) \leq 0$, then $|p_1p_2|^2 = h^2 + a^2 + b^2 - 2ab\cos(\pi - \varphi) \geq h^2 + a^2 + b^2$, since $\cos(\pi - \varphi) < 0$, and we obtain $|p_1p_2|^2 \geq (1 - \cos\varphi_0)(a^2 + b^2 + h^2)$ in this case, as well. In both cases, the claimed inequality follows after taking square roots. $\square$

Lemma 6. *Let ℓ_1 and ℓ_2 be two lines in $\mathbb{R}^d$ such that the angle between their directions is $\varphi \in (0, \frac{\pi}{6}]$; and the direction of each line differs from the x_d-axis by at most $\psi \in [0, \frac{\pi}{6}]$. Let $p_1 \in \ell_1$ and $p_2 \in \ell_2$ be two arbitrary points on the two lines; let $r_1 \in \ell_1$ and $r_2 \in \ell_2$ be the intersection points of the two lines with a hyperplane orthogonal to the x_d-axis; and s_1s_2 be the minimum transversal of the two lines such that $s_1 \in \ell_1$ and $s_2 \in \ell_2$ (Fig. 3, middle). If $|p_1p_2| \leq \frac{1}{3}|r_1r_2|$, then $|p_1s_1| \leq \frac{2\sqrt{3}}{9}\,|r_1s_1|$ and $|p_2s_2| \leq \frac{2\sqrt{3}}{9}\,|r_2s_2|$.*

Proof. Let $h = |s_1s_2|$ be the distance between the two lines. Put $a = |p_1s_1|$, $b = |p_2s_2|$, $e = |r_1s_1|$, and $f = |r_2s_2|$. By the law of cosines, we have $|p_1p_2|^2 = h^2 + a^2 + b^2 - 2ab\cos\varphi$. The sum of the last three terms in this expression is $c^2 = a^2 + b^2 - 2ab\cos\varphi$, where c is the third side of a triangle with two adjacent sides of lengths a and b that meet at angle φ. Denote by β the angle of this triangle opposite to the *longer* of a and b. Then the law of sines yields $a^2 + b^2 - 2ab\cos\varphi = (\max\{a,b\})^2 \cdot \frac{\sin^2\varphi}{\sin^2\beta} \geq (\max\{a,b\})^2 \sin^2\varphi$. Consequently,

$$|p_1p_2|^2 \geq h^2 + (\max\{a,b\})^2 \sin^2\varphi. \tag{5}$$

Similarly, by the law of cosines we have $|r_1r_2|^2 = h^2 + e^2 + f^2 - 2ef\cos\varphi$. Consider a triangle where e and f are adjacent sides parallel with r_1s_1 and r_2s_2 respectively, that meet at angle φ. Since the directions of ℓ_1 and ℓ_2 differ from vertical by at most ψ, the angle opposite to the shorter of e and f is at least $\frac{\pi}{2} - \psi$. Hence the law of sines yields $e^2 + f^2 - 2ef\cos\varphi \leq (\min\{e,f\})^2 \cdot \frac{\sin^2\varphi}{\sin^2(\pi/2-\psi)}$, and consequently

$$|r_1r_2|^2 \leq h^2 + (\min\{e,f\})^2 \cdot \frac{\sin^2\varphi}{\sin^2(\pi/2-\psi)}. \tag{6}$$

The inequality $|p_1p_2| \leq \frac{1}{3}|r_1r_2|$ in combination with inequalities (5) and (6) implies

$$\begin{aligned} h^2 + (\max\{a,b\})^2 \sin^2\varphi &\leq \tfrac{1}{9}\left(h^2 + (\min\{e,f\})^2 \tfrac{\sin^2\varphi}{\sin^2(\pi/2-\psi)}\right) \\ &\leq h^2 + \tfrac{1}{9}(\min\{e,f\})^2 \tfrac{\sin^2\varphi}{\sin^2(\pi/2-\psi)}, \end{aligned} \tag{7}$$

and further (after reducing the term h^2 and taking square roots) that

$$\max\{a, b\} \leq \frac{\min\{e, f\}}{3\sin(\pi/2 - \psi)}. \tag{8}$$

If $\psi \in [0, \frac{\pi}{6}]$, then $\sin\left(\frac{\pi}{2} - \psi\right) \geq \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and so $\max\{a, b\} \leq \frac{2\sqrt{3}}{9}\min\{e, f\}$. It follows that $a \leq \frac{2\sqrt{3}}{9}e$ and $b \leq \frac{2\sqrt{3}}{9}f$, as required. $\square$

Construction of graph $G_{\mathcal{L}}$. We define $G_{\mathcal{L}}$ in terms of a set S of transversal segments among the lines: let the vertices of $G_{\mathcal{L}}$ be the set of endpoints of the segments in S; the edges of $G_{\mathcal{L}}$ include all segments in S, and all segments along the lines in $\mathcal{L}$ between consecutive vertices. We use two types of segments, S_1 and S_2, with $S = S_1 \cup S_2$. Let S_1 be the set of minimum transversals between all $\binom{n}{2}$ pairs of lines in $\mathcal{L}$. The segments in S_2 will connect families of "nearly parallel" lines in $\mathcal{L}$. The definition of these families requires some preparation.

The *direction* of a line in $\mathbb{R}^d$ is represented by a parallel line thorough the origin. The space of directions is the set of lines passing through the origin, also known as the Grassmann manifold $\mathrm{Gr}(1, d)$. It is a compact metric space, where the distance between two directions is the angle in $[0, \frac{\pi}{2}]$ between the two lines passing through the origin. Since $\mathrm{Gr}(1, d)$ is compact, it can be covered by $\tau = O(1)$ neighborhoods of radius $\pi/12$ (where τ depends on the constant $d \geq 3$). Let $B(L_j, \frac{\pi}{12})$, $j = 1, \ldots, \tau$, be a cover of all directions with $\frac{\pi}{12}$-neighborhoods for some $L_j \in \mathrm{Gr}(1, d)$. That is, $\mathrm{Gr}(1, d) = \bigcup_{j=1}^{\tau} B(j, \frac{\pi}{12})$. By increasing the radii of these neighborhoods from $\frac{\pi}{12}$ to $\frac{\pi}{6}$, it is also clear that

$$\mathrm{Gr}(1, d) = \bigcup_{j=1}^{\tau} B\left(L_j, \frac{\pi}{6}\right).$$

For $j = 1, \ldots, \tau$, let $\mathcal{L}_j \subset \mathcal{L}$ denote the set of lines whose directions are in $B(L_j, \frac{\pi}{6})$. Observe that the angle between any two lines in a set $\mathcal{L}_j$ is at most $\frac{\pi}{3}$, which is the diameter of $B(L_j, \frac{\pi}{6})$. Note also that the sets $\mathcal{L}_j$, $j = 1, \ldots, \tau$ are not necessarily disjoint. In any case, we have

$$\mathcal{L} = \bigcup_{j=1}^{\tau} \mathcal{L}_j.$$

For $t \geq 1$ and a set of points P in $\mathbb{R}^d$, a *t-spanner* is a geometric graph G on the vertex set P such that the Euclidean length of the shortest path in G between any two vertices, p and q, is at most $t|pq|$. For a set of n points in $\mathbb{R}^d$ and a constant $t > 1$, a bounded-degree t-spanner can be computed in $O(n \log^{d-1} n)$ time [43]. For $d = 2$, the Delaunay triangulation is a 2-spanner [55] with $O(n)$ edges that can be computed in $O(n \log n)$ time.

We now define the segments in S_2; see Fig. 4(left). For each endpoint p of each segment in S_1, let $\ell(p) \in \mathcal{L}$ denote the line containing p. For every such segment endpoint p, and for every $j = 1, \ldots, \tau$ such that $\ell(p) \in \mathcal{L}_j$, let $H_j(p)$ denote the hyperplane orthogonal to L_j and containing p, and let $P(p, j) = \{H_j(p) \cap \ell : \ell \in \mathcal{L}_j\}$ be the set of intersection points of $H_j(p)$ with the lines in $\mathcal{L}_j$. Construct a 2-spanner for the at most n points in $P(p, j) \subset H_j(p)$, and add all edges of the 2-spanner to the set S_2. This completes the description of the segments in S_2.

$S = S_1 \cup S_2$ determines $G_{\mathcal{L}}$. The $\binom{n}{2}$ segments in S_1 have $n(n-1)$ endpoints, and for each endpoint we compute no more than $\tau = O(1)$ 2-spanners, each with $O(n)$ vertices and edges. So S_2 contains $O(n^3)$ segments and $G_{\mathcal{L}}$ has $O(n^3)$ vertices and edges.

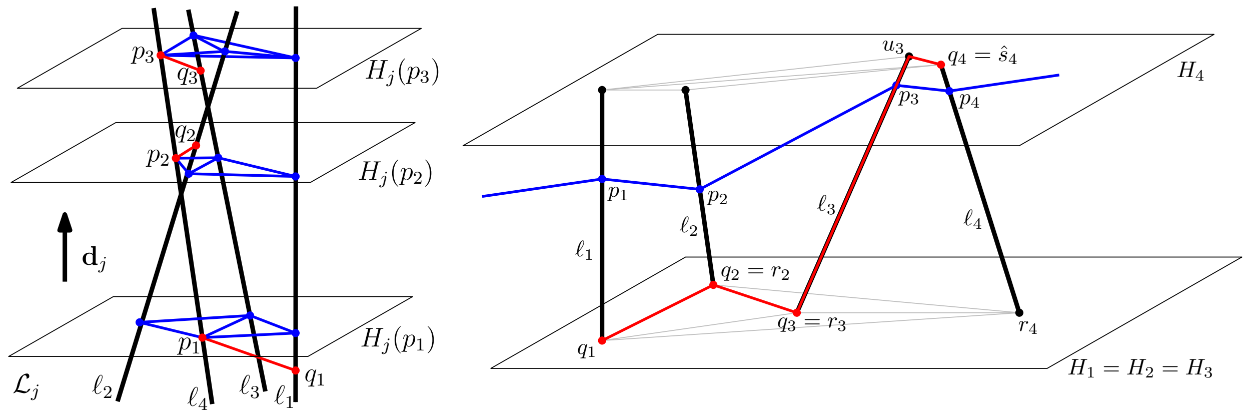

Figure 4: Left: A set of four lines $\ell_1, \ldots, \ell_4$, whose directions are within $\pi/6$ from the vertical direction $\mathbf{d}_j$. The minimum transversals between ℓ_4 and the other three lines are p_1q_1, p_2q_2 and p_3q_3. For $i = 1, 2, 3$, we insert a 2-spanner in the horizontal planes $H_1(p_i)$ containing point p_i. (2-spanners in other parallel planes are not shown in this figure.) Right: An optimal tour OPT visits the lines $\ell_1, \ell_2, \ell_3, \ell_4$ at points p_1, p_2, p_3, p_4, respectively. We construct a path $\gamma_1 = (q_1q_2q_3q_4)$ that visits these lines in the same order. Points q_1, q_2, q_3 are in a horizontal plane $H_1 = H_2 = H_3$. Point q_4 is in a hyperplane $H_4 \neq H_3$ because $|q_3r_4| > 3|p_3p_4|$.

Group Steiner tree yields a constant-factor approximation for TSP with lines. The main result of this section is the following lemma.

Lemma 7. *Let $\mathcal{L}$ be a set of n lines in $\mathbb{R}^d$. Then the length of a minimum group Steiner tree in $G_{\mathcal{L}}$ is $O(1)$ times the length of a minimum TSP tour for the lines in $\mathcal{L}$.*

Proof. Let $\mathcal{L} = \{\ell_1, \ldots, \ell_n\}$, where the lines are indexed so that an optimal TSP tour is $\mathrm{OPT}(\mathcal{L}) = (p_1, \ldots, p_n)$ with $p_i \in \ell_i$, $i = 1, \ldots, n$. We show that $G_{\mathcal{L}}$ contains a group Steiner tree T of length at most $68 \operatorname{len}(\mathrm{OPT}(\mathcal{L}))$. The proof does not use the optimality of the tour $\mathrm{OPT}(\mathcal{L})$, i.e., for any cycle $C = (p_1, \ldots, p_n)$, $p_i \in \ell_i$, we construct a group Steiner tree of length at most $68 \operatorname{len}(C)$. The tree T will consist of a main (backbone) path γ_0, and a path attached to each vertex of the backbone.

Decompose the sequence $(\ell_1, \ldots, \ell_n)$ into maximal subsequences (which we call *blocks*)

$$(\ell_{\tau(i)}, \ell_{\tau(i)+1}, \ldots, \ell_{\tau(i+1)-1}), \qquad i = 1, 2, \ldots, k$$

for some k as follows. Let $\tau(1) = 1$, and for each $i = 1, \ldots, k-1$, let $\tau(i+1)$ be the first index such that the directions of $\ell_{\tau(i)}$ and $\ell_{\tau(i+1)}$ differ by more than $\frac{\pi}{12}$. That is, the directions of the lines in the i-th block differ from the direction of $\ell_{\tau(i)}$ by at most $\frac{\pi}{12}$. This implies that if line $\ell_{\tau(i)}$ is in the neighborhood $B(L_j, \frac{\pi}{12}) \subset \mathrm{Gr}(1, d)$, then all lines in the i-th block are in $B(L_j, \frac{\pi}{6}) \subset \mathrm{Gr}(1, d)$.

Consider the sequence of the first elements of the blocks, $(\ell_{\tau(1)}, \ell_{\tau(2)}, \ldots, \ell_{\tau(k)})$. By construction, the directions of any two consecutive lines differ by more than $\frac{\pi}{12}$. For $i = 1, \ldots, k-1$, let $s_{\tau(i)}t_{\tau(i+1)}$ denote the minimum transversal between $\ell_{\tau(i)}$ and $\ell_{\tau(i+1)}$ with $s_{\tau(i)} \in \ell_{\tau(i)}$ and $t_{\tau(i+1)} \in \ell_{\tau(i+1)}$. The "backbone" of the group Steiner tree T is the polygonal path $\gamma_0 = (s_{\tau(1)}t_{\tau(2)}s_{\tau(2)}t_{\tau(3)} \ldots s_{\tau(k-1)}t_{\tau(k)})$. By Lemma 5, $\operatorname{len}(\gamma_0)$ is bounded from above by $9.4 \operatorname{len}(\mathrm{OPT}(\mathcal{L}))$:

$$\begin{aligned}
\operatorname{len}(\gamma_0) &= \operatorname{len}(s_{\tau(1)}t_{\tau(2)}s_{\tau(2)}t_{\tau(3)} \ldots s_{\tau(k-1)}t_{\tau(k)}) \\
&\leq \operatorname{len}(p_{\tau(1)}s_{\tau(1)}t_{\tau(2)}p_{\tau(2)}) + \ldots + \operatorname{len}(p_{\tau(k-1)}s_{\tau(k-1)}t_{\tau(k)}p_{\tau(k)}) \\
&\leq \sqrt{\frac{3}{1 - \cos(\pi/12)}} \operatorname{len}(p_{\tau(1)}p_{\tau(2)} \ldots p_{\tau(k)}) \\
&\leq 9.4 \operatorname{len}(p_1p_2 \ldots p_n) \\
&\leq 9.4 \ \mathrm{OPT}(\mathcal{L}). \qquad (9)
\end{aligned}$$

For each block $(\ell_{\tau(i)}, \ell_{\tau(i)+1}, \ldots, \ell_{\tau(i+1)-1})$, $i = 1, \ldots, k$, we attach a path γ_i to the backbone γ_0. We only explain this procedure for the first block; other blocks are treated analogously. Put $m = \tau(2) - 1$, so the first block is $(\ell_1, \ell_2, \ldots, \ell_m)$ (recall that $\tau(1) = 1$). The directions of the lines in a block are in a $\frac{\pi}{6}$-neighborhood $B(L_j, \frac{\pi}{6})$, for some direction $L_j \in \mathrm{Gr}(1, d)$. Without loss of generality, we may assume that $j = 1$ and L_1 is parallel to the x_d-axis, that is, the directions of $\ell_1, \ldots, \ell_m$ differ from the x_d-axis by at most $\frac{\pi}{6}$. For brevity, a segment parallel to the x_d-axis is called *vertical*, and a segment or hyperplane orthogonal to the x_d-axis is called *horizontal*. We construct a path γ_1 in $G_{\mathcal{L}}$ that starts from s_1 and includes some point $q_i \in \ell_i$ for each $i = 1, \ldots, m$, in this order. The path γ_1 will use horizontal transversal segments from S_2 between lines in $\mathcal{L}$, and possibly other edges along the lines $\ell_1, \ldots, \ell_m$.

We construct γ_1 incrementally. Refer to Fig. 4 (right). Initially, it is a single-vertex path $\gamma_1 = (q_1)$, where $q_1 = s_1 \in \ell_1$. We extend γ_i in $m - 1$ steps to visit points $q_i \in \ell_i$, $i = 2, \ldots, m$. In step i, ideally we would like to find a point $q_{i+1} \in \ell_{i+1}$ in the horizontal hyperplane containing q_i. However, if the distance from q_i to ℓ_{i+1} in this horizontal hyperplane is larger than $3\,|p_i p_{i+1}|$, then γ_1 will be continued in the horizontal hyperplane that contains one of the endpoints of the minimal transversal between ℓ_i and ℓ_{i+1}.

Suppose that we have already built the path γ_1 up to vertex $q_i \in \ell_i$, $i \in \{1, \ldots, m - 1\}$. We choose $q_{i+1} \in \ell_{i+1}$, and the portion of γ_1 from q_i to q_{i+1} as follows. Let H_i be the horizontal hyperplane containing q_i. Let $r_{i+1} = \ell_{i+1} \cap H_i$ (by construction, r_{i+1} is a vertex of $G_{\mathcal{L}}$). We distinguish two cases:

- If $|q_i r_{i+1}| \leq 3\,|p_i p_{i+1}|$, then let $q_{i+1} = r_{i+1}$, and extend the path γ_1 by the shortest path from q_i to q_{i+1} in the 2-spanner in H_i.
- Otherwise denote the minimum transversal between ℓ_i and ℓ_{i+1} by $\hat{s}_i \hat{s}_{i+1}$ with $\hat{s}_i \in \ell_i$ and $\hat{s}_i \in \ell_{i+1}$. Let H_{i+1} be the horizontal hyperplane containing $\hat{s}_{i+1} \in \ell_{i+1}$, and let $u_i = \ell_i \cap H_{i+1}$. Now let $q_{i+1} = \hat{s}_{i+1}$; and extend γ_1 by a path from q_i to $u_i \in H_{i+1}$ along ℓ_i, and by the shortest path from u_i to q_{i+1} in the 2-spanner in the hyperplane H_{i+1}.

For estimating $\mathrm{len}(\gamma_1)$, we consider the transversal segments and the edges along the lines in $\mathcal{L}$ separately. All transversal segments in γ_1 are horizontal. The total length of these transversal segments between q_i and q_{i+1} is at most $6\,|p_i p_{i+1}|$, and consequently, the total length of all transversal segments in γ_1 is at most $6\,\mathrm{len}(p_1 \ldots p_m)$. Indeed, in the first case the 2-spanner in H_i contains a path between q_i and q_{i+1} of length at most $2\,|q_i q_{i+1}| = 2\,|q_i r_{i+1}| \leq 6\,|p_i p_{i+1}|$. In the second case, the 2-spanner in H_{i+1} contains a path between $u_i = \ell_i \cap H_{i+1}$ and q_{i+1} of length at most $2\,|u_i q_{i+1}| = 2\,|u_i \hat{s}_{i+1}| \leq \frac{2}{\cos(\pi/6)}\,|\hat{s}_i \hat{s}_{i+1}| = \frac{4}{\sqrt{3}}\,|\hat{s}_i \hat{s}_{i+1}| \leq \frac{4}{\sqrt{3}}\,|p_i p_{i+1}|$, where the inequality $|u_i \hat{s}_{i+1}| \leq \frac{1}{\cos(\pi/6)}\,|\hat{s}_i \hat{s}_{i+1}|$ follows from the fact that the direction of ℓ_i differs from vertical by at most $\frac{\pi}{6}$, and so the right triangle $u_i \hat{s}_i \hat{s}_{i+1}$ has an interior angle at most $\frac{\pi}{6}$ at $\hat{s}_{i+1}$.

It remains to bound the total length of the edges in γ_1 that lie along the lines $\ell_1, \ldots, \ell_m$. Recall that the directions of the lines $\ell_1, \ldots, \ell_m$ differ by at most $\frac{\pi}{6}$ from vertical. Therefore, it is enough to estimate the vertical extents of the edges along these lines, which approximate the actual lengths up to a factor of $\frac{1}{\cos(\pi/6)} = \frac{2}{\sqrt{3}}$. Denoting the total vertical length of the segments in γ_1 by $Z = \sum_{i=1}^{m-1} |z(q_i) - z(q_{i+1})|$, the total length of the edges in γ_1 that lie along the lines $\ell_1, \ldots, \ell_m$ is at most $\frac{2}{\sqrt{3}} Z$.

If q_i and q_{i+1} are in the same horizontal plane, then obviously $z(q_i) = z(q_{i+1})$. Otherwise, we have $|p_i p_{i+1}| < \frac{1}{3} |q_i r_{i+1}|$ by construction. In this case, Lemma 6 is applicable, and it gives $|p_{i+1} \hat{s}_{i+1}| \leq \frac{2\sqrt{3}}{9} |r_{i+1} \hat{s}_{i+1}|$. The segments $p_{i+1} \hat{s}_{i+1}$ and $r_{i+1} \hat{s}_{i+1}$ are collinear, hence the inequality holds for their projections to the x_d-axis, that is, $|z(p_{i+1}) - z(\hat{s}_{i+1})| \leq \frac{2\sqrt{3}}{9} |z(r_{i+1}) - z(\hat{s}_{i+1})|$.

Substituting $\hat{s}_{i+1} = q_{i+1}$, we obtain

$$|z(p_{i+1}) - z(q_{i+1})| \leq \frac{2\sqrt{3}}{9}|z(q_i) - z(q_{i+1})|, \tag{10}$$

for every $i \in \{1, \ldots, m-1\}$ where $z(q_i) \neq z(q_{i+1})$. Inequality (10) holds only for the indices $i \in \{1, \ldots, m-1\}$ where the segment $q_i q_{i+1}$ of γ_1 connects two different horizontal hyperplanes. To mark these indices, we introduce the following notation. Let $\sigma(1) = 1$ and let $1 < \sigma(2) < \ldots < \sigma(t) \leq m$ be the sequence of indices such that $z(q_{\sigma(i)-1}) \neq z(q_{\sigma(i)})$ for $i = 2, \ldots, t$. With this notation, we have $z(q_{\sigma(i)}) = z(q_{\sigma(i+1)-1})$ for $i = 1, \ldots, t-1$, and hence $Z = \sum_{i=2}^{t} |z(q_{\sigma(i)-1}) - z(q_{\sigma(i)})| = \sum_{i=1}^{t-1} |z(q_{\sigma(i)}) - z(q_{\sigma(i+1)})|$. Then Z is bounded from above by:

$$\begin{aligned}
Z &= \sum_{i=1}^{t-1} |z(q_{\sigma(i)}) - z(q_{\sigma(i+1)})| \\
&\leq \sum_{i=1}^{t-1} \big(|z(q_{\sigma(i)}) - z(p_{\sigma(i)})| + |z(p_{\sigma(i)}) - z(p_{\sigma(i+1)})| + |z(p_{\sigma(i+1)}) - z(q_{\sigma(i+1)})|\big) \\
&< |z(q_{\sigma(1)}) - z(p_{\sigma(1)})| + \sum_{i=1}^{t-1} |z(p_{\sigma(i)}) - z(p_{\sigma(i+1)})| + 2\sum_{i=1}^{t-1} |z(p_{\sigma(i+1)}) - z(q_{\sigma(i+1)})| \\
&\leq |z(q_1) - z(p_1)| + \sum_{i=1}^{t-1} |z(p_{\sigma(i)}) - z(p_{\sigma(i+1)})| + \frac{4\sqrt{3}}{9} \sum_{i=1}^{t-1} |z(q_{\sigma(i+1)-1}) - z(q_{\sigma(i+1)})| \\
&\leq |z(s_1) - z(p_1)| + \sum_{i=1}^{m-1} |z(p_i) - z(p_{i+1})| + \frac{4\sqrt{3}}{9}\, Z,
\end{aligned}$$

where we used the triangle inequality, inequality (10), and the fact that $q_1 = s_1$. After rearranging, we obtain

$$\begin{aligned}
Z &\leq \frac{9}{9 - 4\sqrt{3}} \left(|z(s_1) - z(p_1)| + \sum_{i=1}^{m-1} |z(p_i) - z(p_{i+1})| \right) \\
&\leq \frac{9}{9 - 4\sqrt{3}} \left(|s_1 p_1| + \sum_{i=1}^{m-1} |p_i p_{i+1}| \right) \\
&\leq \frac{9}{9 - 4\sqrt{3}} \left(9.4\,|p_1 p_{m+1}| + \operatorname{len}(p_1 p_2 \ldots p_m)\right) \\
&\leq \frac{9}{9 - 4\sqrt{3}} \cdot 10.4 \operatorname{len}(p_1 p_2 \ldots p_{m+1}),
\end{aligned}$$

where we used $|s_1 p_1| \leq \operatorname{len}(p_1 s_1 t_{m+1} p_{m+1}) \leq 9.4|p_1 p_{m+1}|$ from (9). It follows that

$$\operatorname{len}(\gamma_1) \leq \left(6 + \frac{2}{\sqrt{3}} \cdot \frac{9}{9 - 4\sqrt{3}} \cdot 10.4 \right) \operatorname{len}(p_{\tau(1)} \ldots p_{\tau(2)}). \tag{11}$$

Since this bound holds for each block, we have thereby shown (according to (9) and (11)) that $G_{\mathcal{L}}$ contains a group Steiner tree for $\mathcal{L}$ whose length is

$$\operatorname{len}(\gamma_0) + \sum_{i=1}^{k} \operatorname{len}(\gamma_i) \leq \left(9.4 + 6 + \frac{2}{\sqrt{3}} \cdot \frac{9}{9 - 4\sqrt{3}} \cdot 10.4 \right) \operatorname{len}(\mathrm{OPT}(\mathcal{L})) \leq 68 \operatorname{len}(\mathrm{OPT}(\mathcal{L})),$$

as claimed. $\square$

5 TSPN for unit disks and balls

In this section we prove Theorems 3 and 4. Without a doubt, among the simplest neighborhoods are congruent disks [2, 16]. TSPN for unit disks is NP-hard, since when the disk centers are fixed and the radius tends to zero, the problem reduces to a TSP for points. Given a set S of n points in the plane, let $\mathcal{D} = \mathcal{D}(S, r)$ be the set of n disks of radius r centered at the points. It is known (and easy to argue) that the optimal tours for the points and the disks, respectively, are polygonal tours with at most n sides. The lengths of the optimal tours for the points and the disks are not too far from each other. Indeed, given any tour of the n disks, one can convert it into a tour of the n centers by adding detours of length at most $2r$ at each of the n visiting points (arbitrarily selected); see e.g., [16, 30]. Let $\mathrm{OPT}(S)$ denote a shortest TSP tour of S, and $\mathrm{OPT}(S, r)$ denote a shortest TSP tour of the disks of radius r centered at the points in S. Consequently, for each $n \geq 3$ and $r > 0$, we have:

$$\mathrm{len}(\mathrm{OPT}(S)) - \mathrm{len}(\mathrm{OPT}(S, r)) \leq 2\,nr. \tag{12}$$

As it is currently the case with TSP for points, the known approximation schemes are highly impractical; see the comments in [36] and [31, Ch. 13–14]. This is even more so for the approximation schemes for TSP with neighborhoods, including disks [16]. Designing more practical (efficient) algorithms with a constant approximation factor remains of high interest. The obvious motivation is to provide faster and conceptually simpler algorithmic solutions.

5.1 Unit disks: an improved approximation

Background. The current best approximation ratio for the TSP with n unit disks, 7.62, was obtained in [16]. The algorithm works by reducing the problem for n disks to one for at most n (representative) points (representative points could be shared). These points are selected after computing a *line cover* consisting of parallel lines. More generally, this ratio holds for translates of a convex region. An alternative approach (also from [16]) selects representative points from among the centers of the disks (i.e., a suitable subset). However, the approximation obtained in [16] in this way is weaker. For instance, starting from a $(1+\varepsilon)$-approximation for the center points yields a ratio of $(8+\pi)(1+\varepsilon) \leq 11.16$, provided that $\varepsilon \leq 0.001$. Starting from a 1.5-approximation (with a faster algorithm) for the center points yields a ratio of $(8+\pi)1.5 \leq 16.72$.

Here we improve the two asymptotic approximation ratios, from 7.62 to 6.75 (when using the PTAS for points), and from 11.43 to 8.52 (when using the faster 1.5-approximation for points). Somewhat surprisingly, we employ the latter approach with center points, which gave previously only a weaker bound. It is worth mentioning that the ratios for the special case of disjoint unit disks remain unchanged, at 3.55 and 5.32, respectively. We now proceed with the details.

A simple packing argument. Let $B(x)$ denote a ball of radius x centered at the origin. Let $G = (V, E)$ be a connected geometric graph in $\mathbb{R}^2$ and let $L = \mathrm{len}(G)$. Let C be the set of points at distance at most x from the edges and vertices of G. Equivalently, $C = G + B(x)$ is the Minkowski sum of G and $B(x)$. We need the following well known inequality; see also [21, Lemma 4].

Lemma 8. $\mathrm{Area}(C) \leq 2Lx + \pi x^2$. *This bound cannot be improved.*

Proof. Consider $B(x) + v_0$ for an arbitrary vertex v_0 of G and mark vertex v_0. The area covered so far is πx^2. Pick an edge uv of G where u is marked and v is unmarked. Place $B(x)$ with the center at u and translate $B(x)$ along uv (its center moves from u to v), and mark v. Observe that the newly covered area is at most $2|uv|x$. Continue and repeat this step as long as there are unmarked vertices. Since G is connected the procedure will terminate when all vertices of G are marked. It follows that the area of C is at most

$$\pi x^2 + \sum_{uv \in E(G)} 2|uv|x = 2Lx + \pi x^2,$$

as required.

Equality holds if and only if G is a straight-line path. Indeed, except for the first step (i.e., each step involving an edge) the newly covered area is strictly less than $2|uv|x$, unless all edges of G are collinear in a straight-line path. $\square$

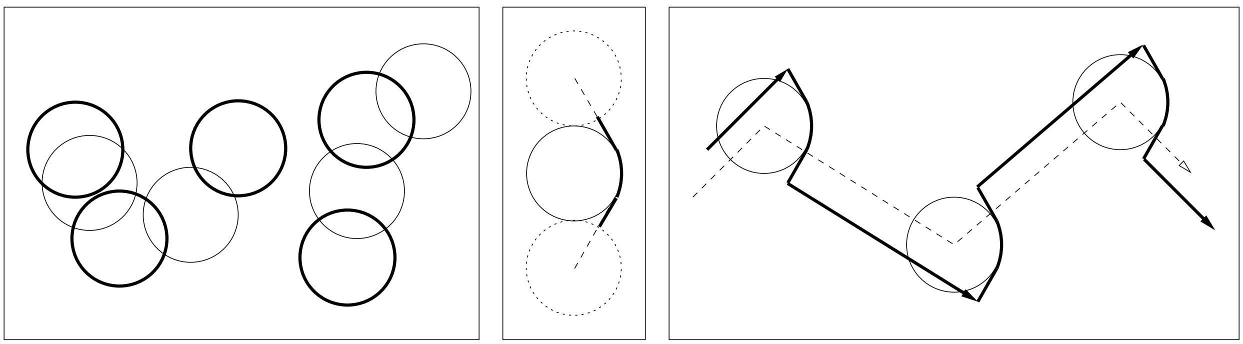

Figure 5: From left to right: (i) a line-sweep independent set (in bold lines); (ii) the curve γ; (iii) a part of the constructed disk tour.

Approximation algorithm—outline. The idea is to first compute a maximal independent set and then an approximate tour of the centers of the independent set, as in [16]. The approximate tour of the centers is then extended by detours so that it visits all the other disks (not in the independent set). However the details differ significantly in both phases of the algorithm, in order to obtain a better approximation ratio: a monotone independent set is found, and a tailored visiting procedure is employed that takes advantage of the special form of the independent set.

Let $\mathcal{D}$ be a set of unit disks. First, compute a maximal independent set of disks $\mathcal{I} \subset \mathcal{D}$ by the following line-sweep algorithm. Select a leftmost disk $\omega \in \mathcal{D}$ and include it in $\mathcal{I}$. Remove from $\mathcal{D}$ all disks intersecting ω. Repeat this selection step as long as $\mathcal{D}$ is non-empty.

We call $\mathcal{I}$ a *line-sweep independent set* or *x-monotone independent set*. Clearly, $\mathcal{I}$ is a maximal independent set in $\mathcal{D}$, that is, each disk in $\mathcal{D} \setminus \mathcal{I}$ intersects a disk in $\mathcal{I}$. Moreover, by construction, each disk in $\mathcal{D} \setminus \mathcal{I}$ intersects the right half-circle boundary of a disk in $\mathcal{I}$.

Let $L^* = \text{len}(\text{OPT}(\mathcal{D}))$ and $L^*_{\mathcal{I}} = \text{len}(\text{OPT}(\mathcal{I}))$. Obviously, $L^*_{\mathcal{I}} \le L^*$.

Algorithm. The algorithm for computing a TSP tour of the disks is as follows. Compute a (maximal) line-sweep independent set $\mathcal{I}$; write $k = |\mathcal{I}|$. Next, compute $T_{\mathcal{I}} = o_1 \dots o_k$, an α-approximate tour of the center points of disks in $\mathcal{I}$, for some constant $\alpha > 1$. If we use the PTAS for Euclidean TSP [3, 38], for a given $0 < \varepsilon < 1/2$, we have $\alpha = 1 + \varepsilon$. If we use the approximation algorithm for metric TSP due to Christofides [10], we have $\alpha = 1.5$.

Write $S_{\mathcal{I}} = \{o_1, o_2, \dots, o_k\}$. For each disk $\omega \in \mathcal{I}$, let ω^- and ω^+ be the two unit disks tangent to ω from below and from above, respectively. Let o^- and o^+ be the centers of ω^- and ω^+, respectively. See Fig. 5(ii). Let $\gamma(\omega)$ be the open curve obtained as follows: start with the tangent segment of positive slope from o^- to ω; concatenate the arc of ω subtending a center angle of $\pi/3$ and symmetric about the x-axis; concatenate the tangent segment of negative slope from ω to o^+. Now remove two unit segments, one from each endpoint of the curve obtained in the previous step. The resulting curve is $\gamma = \gamma(\omega)$. Observe that the open curve $\gamma(\omega)$ intersects any unit disk from $\mathcal{D}$ that intersects the right half-circle boundary of ω (this includes ω as well). Let v denote the vertical segment connecting the endpoints of γ. It is easy to check that

$$\begin{aligned} \operatorname{len}(\gamma) &= 2\left(\frac{\pi}{6} + 2\cos\frac{\pi}{6} - 1\right) = 2\left(\frac{\pi}{6} + \sqrt{3} - 1\right) \leq 2.512, \\ \operatorname{len}(v) &= 4 - \sqrt{3} \leq 2.268. \end{aligned} \tag{13}$$

Replace each segment $o_i o_{i+1}$ of this tour, with i odd, by a parallel segment of equal length connecting the two highest endpoints of the curves $\gamma(\omega_i)$ and $\gamma(\omega_{i+1})$. Similarly, replace each segment $o_i o_{i+1}$ of this tour, with i even, by a parallel segment of equal length connecting the two lowest endpoints of $\gamma(\omega_i)$ and $\gamma(\omega_{i+1})$. See Fig. 5(iii).

To obtain a tour (closed curve) we visit the disks in $\mathcal{I}$ in the same order as $T_{\mathcal{I}}$. After each segment, the tour traverses the corresponding curve $\gamma(\omega)$ (going up or down, as needed, in an alternating fashion). If k is even we proceed as above, while if k is odd, the curve $\gamma(\omega_1)$ is traversed in a circular way (going down along γ and up again along the vertical segment v) in order to get a closed curve. We call T the resulting tour.

Algorithm analysis. Since any disk in $\mathcal{D}$ is either in $\mathcal{I}$ or intersects the curve $\gamma(\omega)$ of some disk $\omega \in \mathcal{I}$, and since T visits all disks in $\mathcal{I}$ and contains the curves $\gamma(\omega)$ of all disks in $\mathcal{I}$, it follows that T is a valid tour for all disks in $\mathcal{D}$. Further observe that the disjoint unit disks in $\mathcal{I}$ are contained in the figure $C = T^*_{\mathcal{I}} + B(2)$. By Lemma 8,

$$\pi|\mathcal{I}| \leq \operatorname{Area}(C) \leq 4\operatorname{len}(T^*_{\mathcal{I}}) + 4\pi,$$

hence

$$k = |\mathcal{I}| \leq \frac{4}{\pi}L^*_{\mathcal{I}} + 4 \leq \frac{4}{\pi}L^* + 4. \tag{14}$$

The total length of the detours incurred by T over all disks in $\mathcal{I}$ is $k\operatorname{len}(\gamma)$ when k is even, and $k\operatorname{len}(\gamma) + \operatorname{len}(v)$ when k is odd. Hence by (13) the length of the output tour is bounded from above as follows.

$$L \leq L_{S_{\mathcal{I}}} + k\operatorname{len}(\gamma) + \operatorname{len}(v) \leq L_{S_{\mathcal{I}}} + (2.512k + 2.268). \tag{15}$$

Inequality (14) implies the following upper bound on the second term in (15).

$$2.512k + 2.268 \leq 2.512\left(\frac{4}{\pi}L^* + 4\right) + 2.268. \tag{16}$$

We next bound from above the first term in (15). The inequality (12) applied to $\mathcal{I}$ and $S_{\mathcal{I}}$ yields

$$L^*_{S_{\mathcal{I}}} \leq L^*_{\mathcal{I}} + 2k. \tag{17}$$

Since the algorithm computes a α-approximation of the optimal tour for the points in $S_{\mathcal{I}}$, by (14) we have

$$\begin{aligned} L_{S_{\mathcal{I}}} &\leq \alpha L^*_{S_{\mathcal{I}}} \leq \alpha(L^*_{\mathcal{I}} + 2k) \leq \alpha(L^* + 2k) \\ &\leq \alpha\left(L^* + 2\left(\frac{4}{\pi}L^* + 4\right)\right) \\ &\leq \alpha\left(\left(1 + \frac{8}{\pi}\right)L^* + 8\right). \end{aligned} \tag{18}$$

Substituting into (15) the upper bounds in (18) and (16) yields

$$
\begin{aligned}
L &\leq \alpha\left(\left(1+\frac{8}{\pi}\right)L^* + 8\right) + 2.512\left(\frac{4}{\pi}L^* + 4\right) + 2.268 \\
&\leq \left(\alpha\left(1+\frac{8}{\pi}\right) + 2.512\cdot\frac{4}{\pi}\right)L^* + (8\alpha + 4\cdot 2.512 + 2.268) \\
&\leq (3.5465\alpha + 3.1984)L^* + (8\alpha + 12.32). \qquad (19)
\end{aligned}
$$

For $\alpha = 1+\varepsilon$ (using the PTAS for the center points), the length of the output tour is $L \leq 6.75\,L^* + 20.4$, assuming that $\varepsilon \leq 0.001$. A more precise calculation along the lines above yields the following upper on the main term (in L^*); the constant factor appears in Theorem 3; note also that $1/0.53 > 1.8$, which explains the other parameter in Theorem 3.

$$
\left(\frac{7}{3} + \frac{8\sqrt{3}}{\pi}\right)\left(1 + \frac{\left(1+\frac{8}{\pi}\right)\varepsilon}{\left(\frac{7}{3}+\frac{8\sqrt{3}}{\pi}\right)}\right)L^* \leq \left(\frac{7}{3} + \frac{8\sqrt{3}}{\pi}\right)(1 + 0.53\,\varepsilon)L^*.
$$

The running time is dominated by that of computing a $(1+\varepsilon)$-approximation of the optimal tour of n points in $\mathbb{R}^2$.

For $\alpha = 1.5$ (using the algorithm of Christofides for the center points), the length of the output tour is $L \leq 8.52\,L^* + 24.4$. The running time is dominated by that of computing a minimum-length perfect matching on n points in the plane (n even), e.g., $O(n^{3/2}\log^5 n)$ by using the algorithm of Varadarajan [54].

Remarks. 1. If the input consists of pairwise-disjoint (unit) disks, then (18) yields improved approximations. These are not new: the case $\alpha = 1+\varepsilon$ was already analyzed in [16]; we just list them for comparison. For $\alpha = 1+\varepsilon$, (18) yields $L \leq 3.55\,L^* + 8.01$, assuming that $\varepsilon \leq 0.001$. For $\alpha = 1.5$, (18) yields $L \leq 5.32\,L^* + 12$. The approximation ratio 3.55 for disjoint unit disks is probably far from tight; the current best lower bound is 2, see [16]. The example in [30, Fig. 4] is yet another instance with a ratio (lower bound) of 2. Hence the approximation ratio 6.75 for unit disks (which uses the above) is probably also far from tight.

2. A simple example shows that one cannot extend the above approach to disks of arbitrary radii. Let $x \geq 1$. See Fig. 6(left) where $n = 3$, and Fig. 6 (right) for its analogue with arbitrarily large n. Let $x \to \infty$ and $\varepsilon \to 0$.

(i) Suppose that we first compute a maximal independent set $\mathcal{I}$ in a greedy manner, by selecting disks in increasing order of their radii. Further suppose that we start by computing T, a constant approximation for the shortest TSP tour on $\mathcal{I}$, for instance by using the algorithm of de Berg et al. [11] (recall, this algorithm works with fat, disjoint regions). In some instances, no constant factor extension (by adding suitable detours to visit the remaining disks) exists. In Fig. 6(left), $\mathrm{len}(\mathrm{OPT}(\mathcal{I})) = 2\varepsilon$, while $\mathrm{len}(\mathrm{OPT}) = 4x$. Moreover, since $x \to \infty$, no *asymptotic* constant factor can be guaranteed by this approach; indeed, for any constants α, β, there exists x large enough, such that $\alpha\, 2\varepsilon + \beta < 4x$.

(ii) Suppose that we first compute a maximal line-sweep independent set, as in our algorithm for unit disks. The same example depicted in Fig. 6(left) shows that no constant factor extension (by adding suitable detours to visit the remaining disks) exists. Moreover, as in (i), since $x \to \infty$, no *asymptotic* constant factor can be guaranteed by this approach.

3. Consider an algorithm that first computes a *maximal* independent set $\mathcal{I}$ of disks (according to some criterion), then computes a good approximate tour of the disks in $\mathcal{I}$, and then extends

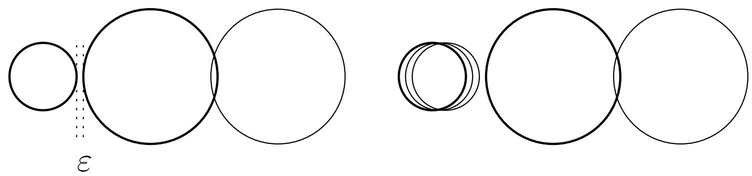

Figure 6: A set of three disks of radii 1, x, and x, centered at 0, $1+x+\varepsilon$ and $1+3x$ (left) and a set of n disks, $n \geq 3$, of radii $1,\ldots,1$, x and x (right). A maximal independent set of disks (in bold) is shown for each case.

this tour with the boundary circles of the disks in $\mathcal{I}$ (in some way). Observe that the length of the overall detour incurred in this way is proportional to $\sum_{i\in\mathcal{I}} r_i$. The following claim (and example) shows a deeper cause for which this general approach does not give a constant approximation ratio; see also [17] for refinements of this inequality and other related results.

Claim. For every $M > 0$, there exists a disk packing in the unit square $[0,1]^2$ with $\sum r_i \geq M$ and all disks tangent to the unit segment $[0,1]\times[0,0]$.

Proof. We place disks in layers of decreasing radius. Each layer consists of congruent disks placed in blocks in between consecutive tangent disks of the previous layer, or in between a disk and a vertical side, as in Fig. 7. The first layer consists of k disks of radius $1/(2k)$, for some $k \geq 1$. By

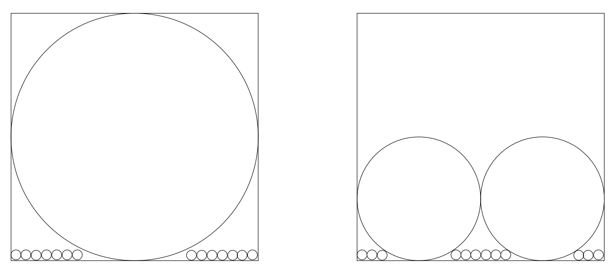

Figure 7: The first two layers of an iterative construction: $k = 1$ (left), and $k = 2$ (right).

choosing the radius of the disks in the next layer much smaller than the radius of the disks in the current layer, one can "cover" any prescribed large fraction $\rho < 1$ of the length of the bottom side of the square by disks tangent to the bottom side of the square and having the sum of radii at least $\rho/2$. Consequently, by using sufficiently many layers, one can achieve $\sum r_i \geq M$, as required. $\square$

5.2 Unit balls in $\mathbb{R}^3$: an improved approximation

We need an analogue of Lemma 8, specifically Lemma 9 below; its proof works in the same way. Let $B(x)$ denote a ball of radius x. Let $G = (V,E)$ be a connected geometric graph in $\mathbb{R}^3$ and let $L = \text{len}(G)$. Let C be the set of points at distance at most x from the edges and vertices of G. Equivalently, $C = G + B(x)$ is the Minkowski sum of G and $B(x)$.

Lemma 9. $\text{Vol}(C) \leq \pi x^2 L + \frac{4\pi}{3}x^3$. *This bound cannot be improved.*

Let $\mathcal{D}$ be a set of unit balls (as input). As in the planar case, we compute a maximal independent set of disks $\mathcal{I} \subset \mathcal{D}$ by a plane-sweep algorithm. For convenience, we sweep a horizontal plane in the positive direction of the z-axis. We call $\mathcal{I}$ a *plane-sweep independent set* or *z-monotone independent set.*

The algorithm computes a tour of $\mathcal{D}$ as follows. First, compute a maximal z-monotone independent set $\mathcal{I}$; write $k = |\mathcal{I}|$. Next, compute $T_{\mathcal{I}} = o_1 \dots o_k$, an α-approximate tour of the center points of the balls in $\mathcal{I}$, for some constant $\alpha > 1$. Write $S_{\mathcal{I}} = \{o_1, o_2, \dots, o_k\}$. For each ball $\omega \in \mathcal{I}$, let $\Gamma = \Gamma(\omega)$ be a discrete set of 28 lattice points associated with ω (relative to its center). For describing this set we will assume for convenience that the center of ω is $(0, 0, 0)$. Let $a = 1/\sqrt{3}$. Γ contains 16 points in the plane $z = a$ and 12 points in the plane $z = 3a$; see Fig. 5(ii). Specifically,

$$
\begin{aligned}
\Gamma = \{&(-3a, -3a, a), (-3a, -a, a), (-3a, a, a), (-3a, 3a, a),\\
&(-a, -3a, a), (-a, -a, a), (-a, a, a), (-a, 3a, a),\\
&(a, -3a, a), (a, -a, a), (a, a, a), (a, 3a, a),\\
&(3a, -3a, a), (3a, -a, a), (3a, a, a), (3a, 3a, a)\}\\
&\cup \{(-3a, -a, 3a), (-3a, a, 3a), (-a, -3a, 3a), (-a, -a, 3a), (-a, a, 3a), (-a, 3a, 3a),\\
&(a, -3a, 3a), (a, -a, 3a), (a, a, 3a), (a, 3a, 3a), (3a, -a, 3a), (3a, a, 3a)\}.
\end{aligned}
$$

One can check that the points in Γ admit a Hamiltonian path in which each edge has length $2a$, say $\xi(\Gamma) = \gamma_1, \gamma_2, \dots, \gamma_{28}$, starting at $\gamma_1 = (-a, -3a, a)$ and ending at $\gamma_{28} = (-a, -3a, 3a)$.

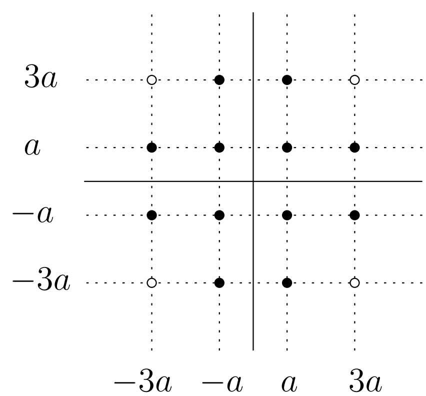

Figure 8: The set Γ has 16 points with $z = a$ and 12 points with $z = 3a$; $|\Gamma| = 28$. The hollow circles indicate the four missing points in the plane $z = 3a$.

We will prove shortly that any unit ball that intersects ω from above (i.e., the z-coordinate of its center is non-negative) contains at least one of the points in $\Gamma(\omega)$. Moreover, this also holds for ω itself.

We modify (extend) the tour $T_{\mathcal{I}} = o_1 \dots o_k$ as follows. Assume first that k is even. We replace each segment $o_i o_{i+1}$ of this tour, with i odd, by a parallel segment of equal length connecting $\gamma_1 \in \Gamma(\omega_i)$ with $\gamma_1 \in \Gamma(\omega_{i+1})$. Similarly, we replace each segment $o_i o_{i+1}$ of this tour, with i even, by a parallel segment of equal length connecting $\gamma_{28} \in \Gamma(\omega_i)$ with $\gamma_{28} \in \Gamma(\omega_{i+1})$. To obtain a tour, we visit the balls in $\mathcal{I}$ in the same order as $T_{\mathcal{I}}$. After each segment, the tour visits all the 28 points in the corresponding set $\Gamma(\omega)$ by using the Hamiltonian path $\xi(\Gamma)$ and then continues with the next segment, etc. This extension procedure can be adapted to work for odd k *without* incurring any increase in cost: specifically, the first cycle of period 2 is replaced by a cycle of period 3. For odd k, the output TSP tour has the form $T = \xi_1 \xi_2 \xi_3 \xi \xi^R \xi \xi^R \dots \xi \xi^R$, rather than the form $T = \xi \xi^R \xi \xi^R \dots \xi \xi^R$ (for k even). Here ξ^R is the path ξ traversed in the opposite direction, and ξ_1, ξ_2, ξ_3 are three suitable Hamiltonian paths on Γ (details are omitted).

Algorithm analysis. The analysis of the approximation ratio is similar to that in the planar case. The disjoint unit balls in $\mathcal{I}$ are contained in the body $C = T^*_{\mathcal{I}} + B(2)$. By Lemma 9,

$$\frac{4\pi}{3}|\mathcal{I}| \leq \mathrm{Vol}(C) \leq 4\pi \operatorname{len}(T^*_{\mathcal{I}}) + \frac{4\pi}{3}8,$$

hence

$$k = |\mathcal{I}| \leq 3\left(L^* + \frac{8}{3}\right) = 3L^* + 8. \tag{20}$$

The total length of the detours incurred by T over all balls in $\mathcal{I}$ is bounded from above by

$$(28-1)2ak = 27\frac{2}{\sqrt{3}}k = 18\sqrt{3}k. \tag{21}$$

It follows that the length of the output tour is bounded from above as follows.

$$L \leq L_{S_{\mathcal{I}}} + 18\sqrt{3}k. \tag{22}$$

The upper bound on $L_{S_{\mathcal{I}}}$ (analogue of (18)) is

$$\begin{aligned} L_{S_{\mathcal{I}}} &\leq \alpha L^*_{S_{\mathcal{I}}} \leq \alpha(L^*_{\mathcal{I}} + 2k) \leq \alpha(L^* + 2k) \leq \alpha(L^* + 2(3L^* + 8)) \\ &= 7\alpha L^* + 16\alpha. \end{aligned} \tag{23}$$

The upper bound on $18\sqrt{3}k$ (analogue of (16)) is

$$18\sqrt{3}k \leq 18\sqrt{3}(3L^* + 8) = 54\sqrt{3}L^* + 144\sqrt{3}. \tag{24}$$

Substituting into (22) the upper bounds in (23) and (24) yields

$$\begin{aligned} L &\leq (7\alpha L^* + 16\alpha) + (54\sqrt{3}L^* + 144\sqrt{3}) \\ &= (7\alpha + 54\sqrt{3})L^* + (16\alpha + 144\sqrt{3}). \end{aligned} \tag{25}$$

For $\alpha = 1 + \varepsilon$ (using the PTAS for the center points), the length of the output tour is $L \leq 100.61\, L^* + 265.6$, assuming that $\varepsilon \leq 0.01$. For $\alpha = 1.5$ (using the algorithm of Christofides for the center points), the length of the output tour is $L \leq 104.1\, L^* + 273.5$. The running time is dominated by that of computing a minimum-length perfect matching on n points in $\mathbb{R}^3$ (n even), e.g., $O(n^3)$ [24].

Lemma 10. *Let ω and ω' be two intersecting unit balls, centered at $(0,0,0)$ and (x,y,z), respectively, where $z \geq 0$. Then ω contains a point in $\Gamma(\omega)$.*

Proof. By symmetry, it suffices to prove the claim when $x, y \geq 0$. We therefore have $x, y, z \geq 0$ and $x^2 + y^2 + z^2 \leq 4$. We distinguish two cases, depending on whether $z \leq 2a$ or $z \geq 2a$. If $z \leq 2a$, we show that ω contains a point of Γ in the lower plane $\sigma_1 : z = a$; if $z \geq 2a$, we show that ω contains a point of Γ in the higher plane $\sigma_3 : z = 3a$. Write $\Gamma_1 = \Gamma \cap \sigma_1$, and $\Gamma_3 = \Gamma \cap \sigma_3$.

Case 1: $z \leq 2a$. Since $x^2 + y^2 + z^2 \leq 4$, we have $\max(x, y) \leq 2 < 4a$. The closest lattice point $\gamma = (\gamma_x, \gamma_y, \gamma_z) \in \Gamma_1$ to (x, y, z) satisfies

$$|x - \gamma_x| \leq a, \quad |y - \gamma_y| \leq a, \quad \text{and} \quad |z - \gamma_z| \leq a,$$

thus

$$(x - \gamma_x)^2 + (y - \gamma_y)^2 + (z - \gamma_z)^2 \leq 3a^2 = 1,$$

as required.

Case 2: $z \geq 2a$. Since $x^2 + y^2 + z^2 \leq 4$, we have $x^2 + y^2 \leq 4 - 4a^2 = 8/3$. Observe that the disk $x^2 + y^2 \leq 8/3$ does not intersect the interior of the square $[2a, 3a]^2$ in the plane $z = 0$. Thus the projection of (x, y, z) onto the plane $z = 0$ is contained in $[0, 3a]^2 \setminus (2a, 3a]^2$. This implies that the closest lattice point $\gamma = (\gamma_x, \gamma_y, \gamma_z) \in \Gamma_3$ to (x, y, z) satisfies

$$|x - \gamma_x| \leq a, \quad |y - \gamma_y| \leq a, \quad \text{and} \quad |z - \gamma_z| \leq a,$$

and the conclusion follows as in Case 1. $\square$

Remark. Analogous to the planar case, if the input consists of pairwise-disjoint (unit) balls, then (23) yields improved approximations. For $\alpha = 1 + \varepsilon$, (23) yields $L \leq 7.01\, L^* + 16.1$, assuming that $\varepsilon \leq 0.001$. For $\alpha = 1.5$, (23) yields $L \leq 10.5\, L^* + 24$.

Generalization to higher dimensions. The technique in this section generalizes to congruent balls in $\mathbb{R}^d$ for any fixed $d \geq 4$. First, the plane-sweep algorithm does so and yields an independent set $\mathcal{I}$. Then compute an α-approximate tour $T_{\mathcal{I}}$ of the center points of the balls in $\mathcal{I}$ for a small $\alpha \leq 1.5$.

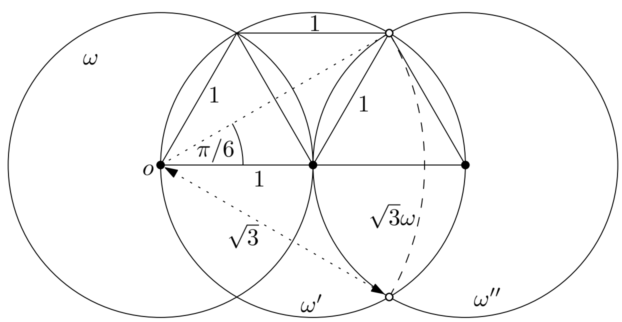

Figure 9: A unit disk ω centered at o intersects two unit disks, ω' and ω'', whose centers are at distance 1 and 2 from o. Both ω' and ω'' intersects the boundary of $\sqrt{3}\omega$ in a spherical cap of radius $\sqrt{3} \cdot \pi/6$.

For each ball $\omega \in \mathcal{I}$, we construct a finite point set $\Gamma = \Gamma(\omega)$ with the property that any unit ball that intersects ω contains at least one of the points in $\Gamma(\omega)$. Consider a unit ball ω' that intersects ω. If the distance between their centers is less than 1, then ω' contains the center of ω; otherwise ω' intersects the boundary of $\sqrt{3}\,\omega$ (i.e., the ball of radius $\sqrt{3}$ concentric with ω) in a spherical cap of radius at least $\sqrt{3}\,\frac{\pi}{6}$ in spherical distance (refer to Fig. 9). The bound $\sqrt{3}\,\frac{\pi}{6}$ is attained when the centers of ω and ω' are at distance 1 or 2 apart. Compute a maximal packing of the sphere $\partial(\sqrt{3}\omega)$ with spherical caps of radius $\sqrt{3}\,\frac{\pi}{12}$, starting with an arbitrary cap, and incrementally adding interior-disjoint caps so that each touches some previous cap.

Let $\Gamma(\omega)$ contain the centers of all caps in this maximal packing and the center of ω. Suppose a unit disk ω' intersects ω but misses $\Gamma(\omega)$. Then ω' contains a spherical cap in $\partial(\sqrt{3}\omega)$ of radius at least $\sqrt{3} \cdot \pi/6$, which contains no point in $\Gamma(\omega)$; consequently a spherical cap with the same center and radius $\sqrt{3}\,\frac{\pi}{12}$ is disjoint from all caps in the packing, contradicting maximality. Therefore $\Gamma(\omega)$ has the desired property.

We extend the tour $T_{\mathcal{I}}$ by suitable detours visiting all points in $\Gamma(\omega)$ for all $\omega \in \mathcal{I}$ and thereby obtain a tour for the input set. The analysis of the approximation ratio is similar to the 2- and 3-dimensional cases and uses volume arguments in $\mathbb{R}^d$. Let $\mathrm{Vol}_d(r)$ be the volume of a ball of radius

r in $\mathbb{R}^d$. It is well-known that

$$\mathrm{Vol}_d(r) = \begin{cases} \dfrac{\pi^{d/2}}{(d/2)!} \cdot r^d & \text{if } d \text{ is even,} \\ \dfrac{2^d \cdot \pi^{(d-1)/2}\,((d-1)/2)!}{d!} \cdot r^d & \text{if } d \text{ is odd.} \end{cases} \tag{26}$$

Combining (26) with the Stirling formula yields the following upper bound:

Lemma 11.

$$\frac{\mathrm{Vol}_{d-1}(1)}{\mathrm{Vol}_d(1)} \le (1+o(1))\sqrt{\frac{d}{2\pi}}.$$

Proof. Write $f \sim g$ whenever $\lim_{d\to\infty} f(d)/g(d) = 1$. We distinguish two cases according to the parity of d.

If d is even, then

$$\begin{aligned} \frac{\mathrm{Vol}_{d-1}(1)}{\mathrm{Vol}_d(1)} &= \frac{2^{d-1}\pi^{(d-2)/2}((d-2)/2)!}{(d-1)!}\frac{(d/2)!}{\pi^{d/2}} \\ &= \frac{2^d}{\pi}\frac{\pi^{d/2}(d/2)!}{d!}\frac{(d/2)!}{\pi^{d/2}} \sim \frac{2^d}{\pi}\frac{(2\pi d/2)\left(\frac{d}{2e}\right)^d}{\sqrt{2\pi d}\left(\frac{d}{e}\right)^d} \\ &= \frac{2^d}{\pi}\frac{\pi d}{\sqrt{2\pi d}}\frac{1}{2^d} = \sqrt{\frac{d}{2\pi}}. \end{aligned}$$

If d is odd, then

$$\begin{aligned} \frac{\mathrm{Vol}_{d-1}(1)}{\mathrm{Vol}_d(1)} &= \frac{\pi^{(d-1)/2}}{((d-1)/2)!}\frac{d!}{2^d\pi^{(d-1)/2}\,((d-1)/2)!} \\ &= \frac{d!}{2^d((d-1)/2)!((d-1)/2)!} \sim \frac{\sqrt{2\pi d}\left(\frac{d}{e}\right)^d}{2^d\,2\pi\frac{d-1}{2}\left(\frac{d-1}{2e}\right)^{d-1}} \\ &= \frac{\sqrt{2\pi d}\,d^d\,2^{d-1}\,e^{d-1}}{\pi\,e^d\,2^d\,(d-1)^d} \sim \frac{\sqrt{2\pi d}}{2e\pi}e = \sqrt{\frac{d}{2\pi}}. \end{aligned}$$

□

By Lemma 11, a volume argument analogous to (20) yields

$$k = |\mathcal{I}| \le \frac{\mathrm{Vol}_{d-1}(2)\,L^* + \mathrm{Vol}_d(2)}{\mathrm{Vol}_d(1)} \le (1+o(1))\sqrt{\frac{d}{2\pi}}\,2^{d-1}L^* + 2^d.$$

The surface area of a sphere of radius r in $\mathbb{R}^d$ is $\mathrm{Area}_{d-1}(r) = 2\pi r\mathrm{Vol}_{d-2}(r)$, and the surface area of a spherical cap of radius $r\varphi$ is bounded from below by $\mathrm{Vol}_{d-1}(r\sin\varphi)$. A volume argument yields

$$|\Gamma| \le \frac{\mathrm{Area}_{d-1}(\sqrt{3})}{\mathrm{Vol}_{d-1}(\sqrt{3}\sin(\pi/12))} + 1 \le \frac{2\pi\mathrm{Vol}_{d-2}(1)}{(\sin(\pi/12))^{d-1}\mathrm{Vol}_{d-1}(1)} + 1 \le (1+o(1))\frac{\sqrt{2\pi d}}{(\sin(\pi/12))^{d-1}}.$$

If two spherical caps of radius $\sqrt{3}\,\frac{\pi}{12}$ are in contact on the sphere $\partial(\sqrt{3}\omega)$, then the distance between their centers is $2\sqrt{3}\sin\frac{\pi}{12}$. By construction, the length of a minimum spanning tree of Γ is

$$(|\Gamma|-2)\,2\sqrt{3}\sin\frac{\pi}{12} + \sqrt{3} \le (1+o(1))\frac{2\sqrt{6\pi d}}{(\sin(\pi/12))^{d-2}},$$

and the length of a Hamiltonian cycle ξ of Γ is at most twice this length. Consequently, we obtain a tour of length

$$L \leq \alpha L^* + 2k\,\mathrm{len}(\xi) \leq \alpha L^* + 2\left((1+o(1))\sqrt{\frac{d}{2\pi}}\,2^{d-1}L^* + 2^d\right)\left((1+o(1))\frac{2\sqrt{6\pi d}}{(\sin(\pi/12))^{d-2}}\right).$$

The resulting (asymptotic) approximation ratio is

$$\alpha + (1+o(1))\frac{2\sqrt{3}\,d\,2^d}{(\sin(\pi/12))^{d-2}} = O\left(d\left(\frac{2}{\sin(\pi/12)}\right)^d\right) = O\left(7.73^d\right),$$

as claimed.

6 Conclusion

We have revisited TSP with neighborhoods and have obtained several approximation algorithms: some for neighborhoods previously less studied (lines and hyperplanes in $\mathbb{R}^d$), and some for the most previously studied (such as disks and balls). Despite the progress, one may rightfully say that the general problem of TSP with neighborhoods is far from resolved. Interesting questions remain open regarding the structure of optimal TSP tours for lines, segments, balls, and hyperplanes, and the degree of approximation achievable for these problems. We list the simplest and most natural questions (still unsolved) that we could identify.

(1) Is there a polynomial-time algorithm for planes in $\mathbb{R}^3$?

(2) Is there a constant approximation algorithm for lines in $\mathbb{R}^3$ (or in $\mathbb{R}^d$ for $d \geq 3$)? Can the current $O(\log^3 n)$ ratio be improved?

(3) Can the approximation factors for unit disks (3.55 for disjoint, 6.75 for arbitrary) and balls (7.01 for disjoint, 100.61 for arbitrary) be further reduced while maintaining or reducing the running times?

(4) Is there a constant approximation algorithm for disks (of arbitrary radii) in the plane? Is there one for balls (of arbitrary radii) in $\mathbb{R}^3$?

(5) Is there a constant approximation algorithm for parallel segments in $\mathbb{R}^3$? To start with, one can further assume that the segments are pairwise-disjoint.

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