Proof. Given a request sequence $\sigma = \sigma(1), \ldots, \sigma(m)$, we assume without loss of generality that MARKING already has a fault on the first request $\sigma(1)$.

MARKING divides the request sequence into phases. A phase starting with $\sigma(i)$ ends with $\sigma(j)$, where j, $j > i$, is the smallest integer such that the set

$$\{\sigma(i), \sigma(i+1), \ldots, \sigma(j+1)\}$$

contains $k+1$ distinct pages. Note that at the end of a phase all pages in fast memory are marked.

Consider an arbitrary phase. Call a page *stale* if it is unmarked but was marked in the previous phase. Call a page *clean* if it is neither stale nor marked.

Let c be the number of clean pages requested in the phase. We will show that

1. the amortized number of faults made by OPT during the phase is at least $\frac{c}{2}$.
2. the expected number of faults made by MARKING is at most cH_k.

These two statements imply the theorem.

We first analyze OPT's cost. Let S_{OPT} be the set of pages contained in OPT's fast memory, and let S_M be the set of pages stored in MARKING's fast memory. Furthermore, let d_L be the value of $|S_{OPT} \setminus S_M|$ at the beginning of the phase and let d_F be the value of $|S_{OPT} \setminus S_M|$ at the end of the phase. OPT has at least $c - d_L$ faults during the phase because at least $c - d_L$ of the c clean pages are not in OPT's fast memory. Also, OPT has at least d_F faults during the phase because d_F pages requested during the phase are not in OPT's fast memory at the end of the phase. We conclude that OPT incurs at least

$$\max\{c - d_L, d_F\} \geq \frac{1}{2}(c - d_L + d_F) = \frac{c}{2} - \frac{d_L}{2} + \frac{d_F}{2}$$

faults during the phase. Summing over all phases, the terms $\frac{d_L}{2}$ and $\frac{d_F}{2}$ telescope, except for the first and last terms. Thus the amortized number of page faults made by OPT during the phase is at least $\frac{c}{2}$.

Next, we analyze the cost of MARKING. Let $c(i)$ be the number of clean pages that were requested in the phase immediately before the i-th request to a stale page and let $s(i)$ denote the number of stale pages that remain before the i-th request to a stale page.

When MARKING serves the i-th request to a stale page, exactly $s(i) - c(i)$ of the $s(i)$ stale pages are in fast memory, each of them with equal probability. Thus the expected cost of the request is

$$\frac{s(i) - c(i)}{s(i)} \cdot 0 + \frac{c(i)}{s(i)} \cdot 1 \leq \frac{c}{s(i)} = \frac{c}{k - i + 1}.$$

The last equation follows because $s(i) = k - (i - 1)$. The total expected cost for serving requests to stale pages is

$$\sum_{i=1}^{s} \frac{c}{k+1-i} = c \sum_{i=2}^{k} \frac{1}{i} = c(H_k - 1).$$

We conclude that MARKING's total expected cost in the phase is bounded by cH_k. $\square$