CS 602 - Applied Algorithms: Assignment 3

Total Marks - 60

Instructions. Please try to be brief, clear, and technically precise. Use pseudo-codes to describe the algorithms. To solve the problems, one may assume that the instances are in general position, unless stated otherwise. Novelty in the answer carries marks.

- **Question 1:** [20 Marks]*Ski rental problem*

  At a ski resort, renting a ski costs 1 rupee per day, while buying skis costs B rupees. A skier arrives at the ski resort for a ski vacation and has to decide whether to rent or buy skis. However, an unknown factor is the number of remaining skiing days left before the snow melts. A randomized $\frac{e}{e-1}$ competitive algorithm exists.

  1. Construct a simple deterministic 2- competitive algorithm for the problem.
  2. Formulate the problem as a linear program.

     Hint: define an indicator variable which is set to 1 if the skier buys the skis, and for each day $i, i \in [1, k]$ (which is unknown in advance), define another indicator variable which is set to 1 if the skier decides to rent skis on day j. The constraints guarantee that on each day, we either rent skis or buy them.
  3. Formulate the dual program for this.
  4. Construct a 2-competitive algorithm using primal-dual method.

     Hint: Whenever we have a new ski day, the primal program is updated by adding a new constraint. The dual pržogram is updated by adding a new dual variable. The online requirement is that previous decisions cannot be undone. In other words, the primal variables are monotonically non-decreasing over time.

- **Question 2:** [20 Marks] Suppose we have one machine and jobs released over time: job i is released at time r_i, has size w_i, benefit b_i, and deadline

d_i. Jobs are allowed to be preempted (i.e., interrupted and later resumed) and/or partially executed (as long as it is before the deadline). Denote by $p_i \leq w_i$ the total time job i was processed before its deadline. Our goal is to maximize $\sum_i \frac{p_i}{w_i} b_i$. Assume that r_i, w_i, d_i (but not b_i) are integers and the algorithm as well as OPT are allowed to preempt jobs only at integer times. Design a 2-competitive algorithm for the problem.

- **Question 3:** [20 Marks] *MARKING Problem* : Consider a randomized paging algorithm that processes a request sequence in phases. At the beginning of each phase, all pages in the memory system are unmarked. Whenever a page is requested, it is marked. On a fault, a page is chosen uniformly at random from among the unmarked pages in fast memory, and this pages is evicted. A phase ends when all pages in fast memory are marked and a page fault occurs. Then, all marks are erased and a new phase is started.

  The competitive ratio of a randomized online algorithm A is defined with respect to an adversary. The adversary generates a request sequence σ and it also has to serve σ. When constructing σ, the adversary always knows the description of A. We hereby define the notion of oblivious adversary.

  Oblivious Adversary: The oblivious adversary has to generate a complete request sequence in advance, before any requests are served by the online algorithm. The adversary is charged the cost of the optimum online algorithm for that sequence.

  Prove that the MARKING algorithm is $2H_k$-competitive against any oblivious adversary, where $H_k = \sum_{i=1}^{k} \frac{1}{i}$ is the k-th Harmonic number.