First Order Logic: A Brief Introduction

Supratik Chakraborty
IIT Bombay

Notation

- Variables: $x, y, z, \ldots$
  - Represent elements of an underlying set
- Constants: $a, b, c, \ldots$
  - Specific elements of underlying set
- Function symbols: $f, g, h, \ldots$
  - *Arity* of function: $\#$ of arguments
  - 0-ary functions: constants
- Relation (predicate) symbols: $P, Q, R, \ldots$
  - Hence, also called "predicate calculus"
  - *Arity* of predicate: $\#$ of arguments
- Fixed symbols:
  - Carried over from prop. logic: $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$, (,)
  - New in FOL: $\exists$, $\forall$ ("quantifiers")

Equality in FOL

- A special binary predicate, used widely in maths
- Represented by special predicate symbol "$=$"
- Semantically, binary identity relation (more on this later ...)
- First-order logic with equality
  - Different expressive power vis-a-vis first-order logic
  - Most of our discussions will assume availability of "$=$"
  - Refer to as "first-order logic" unless the distinction is important

Syntax of FOL

Two classes of syntactic objects: *terms* and *formulas*

Terms

- Every variable is a term
- If f is an m-ary function, $t_1, \ldots t_m$ are terms, then $f(t_1, \ldots t_m)$ is also a term

Constants are also terms

0-ary functions

Atomic formulas

- If R is an n-ary predicate, $t_1, \ldots t_n$ are terms, then $R(t_1, \ldots t_m)$ is an atomic formula
- Special case: $t_1 = t_2$

Syntax of FOL

- *Primitive* fixed symbols: $\wedge$, $\neg$, $\exists$
  - Other choices also possible: E.g., $\vee$, $\neg$, $\forall$

Rules for formuling formulas

- Every atomic formula is a formula
- If φ is a formula, so are $\neg\varphi$ and (φ)
- If φ_1 and φ_2 are formulas, so is $\varphi_1 \wedge \varphi_2$
- If φ is a formula, so is $\exists x\, \varphi$ for any variable x

- Formulas with other fixed symbols definable in terms of formulas with primitive symbols.
  - $\varphi_1 \vee \varphi_2 \triangleq \neg(\neg\varphi_1 \wedge \neg\varphi_2)$
  - $\varphi_1 \rightarrow \varphi_2 \triangleq \neg\varphi_1 \vee \varphi_2$
  - $\varphi_1 \leftrightarrow \varphi_2 \triangleq (\varphi_1 \rightarrow \varphi_2) \wedge (\varphi_2 \rightarrow \varphi_1)$
  - $\forall x\, \varphi \triangleq \neg(\exists x\, \neg\varphi)$

FOL formulas as strings

- Alphabet (over which strings are constructed):
  - Set of variable names, e.g. $\{x_1, x_2, y_1, y_2\}$
  - Set of constants, functions, predicates, e.g. $\{a, b, f, =, P\}$
  - Fixed symbols $\{\neg, \vee, \wedge, \rightarrow, \leftrightarrow, \exists, \forall\}$
- Well-formed formula: string formed according to rules on prev. slide
  - $\forall x_1(\forall x_2\,(((x_1 = a) \vee (x_1 = b)) \wedge \neg(f(x_2) = f(x_1))))$ is well-formed
  - $\forall(\forall x_1(x_1 = ab)\neg()x_2)$ is not well-formed
- Well-formed formulas can be represented using parse trees
  - Consider the rules on prev. slide as production rules in a context-free grammar

Vocabulary

- Alphabet (over which strings are constructed):
  - Set of variable names, e.g. $\{x_1, x_2, y_1, y_2\}$
  - **Set of constants, functions, predicates, e.g.** $\{a, b, f, =\}$: **Vocabulary**
  - Fixed symbols $\{\neg, \vee, \wedge, \rightarrow, \leftrightarrow, \exists, \forall\}$

Vocabulary

- Alphabet (over which strings are constructed):
  - Set of variable names, e.g. $\{x_1, x_2, y_1, y_2\}$
  - **Set of constants, functions, predicates, e.g.** $\{a, b, f, =\}$: **Vocabulary**
  - Fixed symbols $\{\neg, \vee, \wedge, \rightarrow, \leftrightarrow, \exists, \forall\}$
- Smallest vocabulary to generate $\forall x_1 (\forall x_2 (((x_1 = a) \vee (x_1 = b)) \wedge \neg(f(x_2) = f(x_1))))$?

Vocabulary

- Alphabet (over which strings are constructed):
  - Set of variable names, e.g. $\{x_1, x_2, y_1, y_2\}$
  - **Set of constants, functions, predicates, e.g.** $\{a, b, f, =\}$: **Vocabulary**
  - Fixed symbols $\{\neg, \vee, \wedge, \rightarrow, \leftrightarrow, \exists, \forall\}$
- Smallest vocabulary to generate $\forall x_1 (\forall x_2 (((x_1 = a) \vee (x_1 = b)) \wedge \neg(f(x_2) = f(x_1))))$?
  - $\{a, b, f, =\}$

Free Variables in a Formula

Free variables are those that are not quantified in a formula.
Let free(φ) denote the set of free variables in φ

- If φ is an atomic formula, $\text{free}(\varphi) = \{x \mid \text{x occurs in } \varphi\}$
- If $\varphi = \neg\psi$ or $\varphi = (\psi)$, $\text{free}(\varphi) = \text{free}(\psi)$
- If $\varphi = \varphi_1 \wedge \varphi_2$, $\text{free}(\varphi) = \text{free}(\varphi_1) \cup \text{free}(\varphi_2)$
- if $\varphi = \exists x\, \varphi_1$, $\text{free}(\varphi) = \text{free}(\varphi_1) \setminus \{x\}$

Free Variables in a Formula

Free variables are those that are not quantified in a formula.
Let free(φ) denote the set of free variables in φ

- If φ is an atomic formula, $\text{free}(\varphi) = \{x \mid \text{x occurs in } \varphi\}$
- If $\varphi = \neg\psi$ or $\varphi = (\psi)$, $\text{free}(\varphi) = \text{free}(\psi)$
- If $\varphi = \varphi_1 \wedge \varphi_2$, $\text{free}(\varphi) = \text{free}(\varphi_1) \cup \text{free}(\varphi_2)$
- if $\varphi = \exists x\, \varphi_1$, $\text{free}(\varphi) = \text{free}(\varphi_1) \setminus \{x\}$

- What is $\text{free}(\underbrace{(\underbrace{\exists x\, \underbrace{P(x,y)}_{x,y}}_{y}) \wedge (\underbrace{\forall y\, \underbrace{Q(x,y)}_{x,y}}_{x})}_{x,y})$?

Free Variables in a Formula

Free variables are those that are not quantified in a formula.
Let free(φ) denote the set of free variables in φ

- If φ is an atomic formula, $\text{free}(\varphi) = \{x \mid \text{x occurs in } \varphi\}$
- If $\varphi = \neg\psi$ or $\varphi = (\psi)$, $\text{free}(\varphi) = \text{free}(\psi)$
- If $\varphi = \varphi_1 \wedge \varphi_2$, $\text{free}(\varphi) = \text{free}(\varphi_1) \cup \text{free}(\varphi_2)$
- if $\varphi = \exists x\, \varphi_1$, $\text{free}(\varphi) = \text{free}(\varphi_1) \setminus \{x\}$

- What is $\text{free}((\exists x\, P(x,y)) \wedge (\forall y\, Q(x,y)))$?
  - $= \text{free}((\exists x\, P(x,y))) \;\cup\; \text{free}(\forall y\, Q(x,y))$
  - $= \text{free}(P(x,y)) \setminus \{x\} \;\cup\; \text{free}(Q(x,y)) \setminus \{y\}$
  - $= \{x,y\} \setminus \{x\} \;\cup\; \{x,y\} \setminus \{y\} \;=\; \{x,y\}$

Free variables are those that are not quantified in a formula.
Let free(φ) denote the set of free variables in φ

- If φ is an atomic formula, $\text{free}(\varphi) = \{x \mid \text{x occurs in } \varphi\}$
- If $\varphi = \neg\psi$ or $\varphi = (\psi)$, $\text{free}(\varphi) = \text{free}(\psi)$
- If $\varphi = \varphi_1 \wedge \varphi_2$, $\text{free}(\varphi) = \text{free}(\varphi_1) \cup \text{free}(\varphi_2)$
- if $\varphi = \exists x\, \varphi_1$, $\text{free}(\varphi) = \text{free}(\varphi_1) \setminus \{x\}$

- What is $\text{free}((\exists x\, P(x,y)) \wedge (\forall y\, Q(x,y)))$?
  - $= \text{free}((\exists x\, P(x,y))) \;\cup\; \text{free}(\forall y\, Q(x,y))$
  - $= \text{free}(P(x,y)) \setminus \{x\} \;\cup\; \text{free}(Q(x,y)) \setminus \{y\}$
  - $= \{x,y\} \setminus \{x\} \;\cup\; \{x,y\} \setminus \{y\} \;=\; \{x,y\}$

If φ has free variables $\{x,y\}$, we write $\varphi(x,y)$

A formula with no free variables is a **sentence**, e.g. $\exists x \forall y\, f(x) = y$

Bound Variables in a Formula

Bound variables are those that are quantified in a formula.
Let $\mathrm{bnd}(\varphi)$ denote the set of bound variables in φ

- If φ is an atomic formula, $\mathrm{bnd}(\varphi) = \emptyset$
- If $\varphi = \neg\psi$ or $\varphi = (\psi)$, $\mathrm{bnd}(\varphi) = \mathrm{bnd}(\psi)$
- If $\varphi = \varphi_1 \wedge \varphi_2$, $\mathrm{bnd}(\varphi) = \mathrm{bnd}(\varphi_1) \cup \mathrm{bnd}(\varphi_2)$
- if $\varphi = \exists x\, \varphi_1$, $\mathrm{bnd}(\varphi) = \mathrm{bnd}(\varphi_1) \cup \{x\}$

Bound Variables in a Formula

Bound variables are those that are quantified in a formula.
Let $\text{bnd}(\varphi)$ denote the set of bound variables in φ

- If φ is an atomic formula, $\text{bnd}(\varphi) = \emptyset$
- If $\varphi = \neg\psi$ or $\varphi = (\psi)$, $\text{bnd}(\varphi) = \text{bnd}(\psi)$
- If $\varphi = \varphi_1 \wedge \varphi_2$, $\text{bnd}(\varphi) = \text{bnd}(\varphi_1) \cup \text{bnd}(\varphi_2)$
- if $\varphi = \exists x\, \varphi_1$, $\text{bnd}(\varphi) = \text{bnd}(\varphi_1) \cup \{x\}$

- What is $\text{bnd}(\underbrace{\underbrace{(\exists x\, \underbrace{P(x,y)}_{\phi})}_{x} \wedge \underbrace{(\forall y\, \underbrace{Q(x,y)}_{\phi}))}_{y}}_{x,y})$?

Bound Variables in a Formula

Bound variables are those that are quantified in a formula.
Let bnd(φ) denote the set of bound variables in φ

- If φ is an atomic formula, $\text{bnd}(\varphi) = \emptyset$
- If $\varphi = \neg\psi$ or $\varphi = (\psi)$, $\text{bnd}(\varphi) = \text{bnd}(\psi)$
- If $\varphi = \varphi_1 \wedge \varphi_2$, $\text{bnd}(\varphi) = \text{bnd}(\varphi_1) \cup \text{bnd}(\varphi_2)$
- if $\varphi = \exists x\, \varphi_1$, $\text{bnd}(\varphi) = \text{bnd}(\varphi_1) \cup \{x\}$

- What is $\text{bnd}((\exists x\, P(x,y)) \wedge (\forall y\, Q(x,y)))$?
  - $= \text{bnd}((\exists x\, P(x,y))) \;\cup\; \text{bnd}(\forall y\, Q(x,y))$
  - $= \text{bnd}(P(x,y)) \cup \{x\} \;\cup\; \text{bnd}(Q(x,y)) \cup \{y\}$
  - $= \emptyset \cup \{x\} \;\cup\; \emptyset \cup \{y\}$
  - $= \{x\} \;\cup\; \{y\} = \{x,y\}$!!!
- free(φ) and bnd(φ) are not complements!

Substitution in FOL

Suppose $x \in \text{free}(\varphi)$ and t is any term.
We wish to replace every free occurrence of x in φ with t, such that free variables in t stay free in the resulting formula.

Term t is free for x in φ if no free occurrence of x in φ is in the scope of $\forall y$ or $\exists y$ for any variable y occurring in t.

- $\varphi \triangleq \exists y\, R(x, y) \vee \forall x\, R(z, x)$, and t is $f(z, x)$
- $f(z, x)$ is free for x in φ, but $f(y, x)$ is not

$\varphi[t/x]$: Formula obtained by replacing each free occurrence of x in φ by t, if t is free for x in φ

- For φ defined above,
  $\varphi[f(z,x)/x] \triangleq \exists y\, R(f(\underbrace{z,x}_{\text{z,x still remain free}}), y) \vee \forall x\, \underbrace{R(z, x)}_{\text{Here x will get bounded}}$