Substitution in FOL

Suppose $x \in \text{free}(\varphi)$ and t is any term.
We wish to replace every free occurrence of x in φ with t, such that free variables in t stay free in the resulting formula.

> Term t is free for x in φ if no free occurrence of x in φ is in the scope of $\forall y$ or $\exists y$ for any variable y occurring in t.

- $\varphi \triangleq \exists y\, R(x, y) \lor \forall x\, R(z, x)$, and t is $f(z, x)$
- $f(z, x)$ is free for x in φ, but $f(y, x)$ is not → y will get bounded in first substitution
  x will get bounded in second

$\varphi[t/x]$: Formula obtained by replacing each free occurrence of x in φ by t, if t is free for x in φ

- For φ defined above,
  $\varphi[f(z, x)/x] \triangleq \exists y\, R(f(z, x), y) \lor \forall x\, R(z, x)$

$t = \underline{\underline{f}}(z, x)$

$\exists y\, \underbrace{R(x, y)}_{} \vee \forall x\, \underbrace{R(z, x)}_{}$

x z

bool R(int a, int b)
{
...
}

$x \mapsto \underline{z + x}$

$z \mapsto x * z$

```
result = false
for all int y.
    if R(z+x, y) is true then
        result = true ; exit loop

result1 = true
    for all int x
        if R(z*x, x) is false then
            result1 = false;
            exit loop
return result V result1
```

Semantics of FOL: Some Intuition

$$\varphi \triangleq \forall x \forall y \, (P(x,y) \rightarrow \exists z \, (\neg(z = x) \wedge \neg(z = y) \wedge P(x,z) \wedge P(z,y))$$

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English reading: For every x and y, if $P(x,y)$ holds, we can find z distinct from x and y such that both $P(x,z)$ and $P(z,y)$ hold.

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Case 1:

- Variables take values from **real numbers**
- $P(x,y)$ represents $\mathbf{x < y}$
- English reading simply states "real numbers are dense"
- φ is **true**

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$x \leq y \rightarrow \quad z \neq x \quad z \neq y \quad \underbrace{x \leq z \wedge z \leq y}$

Case 2:

- Variables take values from **real numbers**
- $P(x,y)$ represents $\mathbf{x \leq y}$
- English reading requires the following to be true
  - If $x = y$, there is a z such that $z \neq x$, $x \leq z$ and $z \leq x$
  - Thus, $z \neq x$ and $z = x$
- φ is **false**

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Case 3:

- Variables take values from **natural numbers**
- $P(x, y)$ represents $\mathbf{x} < \mathbf{y}$
- English reading states that "natural numbers are dense"
- φ is **false**

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Truth of φ depends on the underlying set from which variables take values, and on how constants, functions, predicates are interpreted

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Vocabulary $\mathcal{V}$: E.g. $\mathcal{V} : \underbrace{\{a, f, =, R\}}$

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   1 and 2 define a $\mathcal{V}$-**structure** $M = (U^M, (a^M, f^M, R^M))$

3. *Binding (aka environment)* $\alpha : \text{free}(\varphi) \rightarrow U$
   e.g. $\alpha(y) = 2$

$M \rightarrow (\mathbb{N}, (a, f, R))$

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Given structure M and binding α, does φ evaluate to **true**?

Notationally, does $\mathbf{M}, \alpha \models \varphi$?

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  - If t is a variable x, $\overline{\alpha}(t) = \alpha(x)$
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Didn't understand

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- $M, \alpha \models \exists x\, \varphi$ iff there is some $c \in U^M$ such that $M, \alpha[x \mapsto c] \models \varphi$, where
  - $\alpha[x \mapsto c](v) = \alpha(v)$, if variable v is different from x
  - $\alpha[x \mapsto c](x) = c$

Semantics of FOL: Illustration

$$\varphi \triangleq \underbrace{\exists x\, R(x, f(y, a))}_{\cancel{x}, y} \rightarrow \underbrace{\exists z\, (\neg(z = a) \wedge R(z, y))}_{\cancel{z}, y}$$

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- free(φ) $= \{y\}$, and $\alpha(\underline{y}) = 2$
- $M, \underline{\alpha[z \mapsto 1]} \models (\neg(z = a) \wedge R(z, y))$
- Therefore, $\underline{M, \alpha} \models \exists z\, \underline{(\neg(z = a) \wedge R(z, y))}$

$1 \neq 0$

1 2

$1 < 2$ holds

$y \mapsto 2$, $z \mapsto 1$

$z \mapsto 1$

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- Finally, $M, \alpha \models \varphi$
- Note that if $\underbrace{\alpha'(y)}_{\text{won't work}} = 1$, $M, \alpha' \not\models \varphi$