CS 208 HW3 Q1

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22B1003

Question 1

1. $L_1 = \{0^i 1^j \mid i, j \in \mathbb{N},\ i + j^2 = \text{prime}\}$

By pumping lemma, $\exists\ p$ (p = numbers of states of DFA)

such that $w \in L_1 = x \cdot y \cdot z$ $\quad |xy| \leq p \quad |y| \geq 1$

and $xy^k z \in L$ if L = regular language

Consider $w = 0^{p'} 1^{q'}$

where $p' \geq p$ (clearly, we can find such p because there are infinite numbers of primes)

(For a set of complete primes $P_1, P_2 \ldots P_n$: $(P_1 P_2 \cdots P_n + 1)$ = prime)

$w = x \cdot y \cdot z$

$|xy| \leq p$ and $|y| \geq 1$.

So y will be of the form 0^s $(s \geq 1)$

Consider $xy^{k+1}z = w'$

Now, if we assume L to be regular, then $w' \in L$ by pumping lemma.

$w' = 0^{p' + ks} 1^{q'}$

For $w' \in L$, $x = (p' + ks) + (q')^2$ = prime $\quad (p' + q'^2 = \text{prime})$

$k \in \mathbb{N}$, so consider $k = p' + (q')^2$ then

$x = (p' + q'^2)(s+1) \rightarrow$ not prime $\rightarrow w' \notin L$

Contradiction. Hence L_1 = not regular

$L_2 = \{w \mid w \text{ differs from } w^R \text{ in exactly two positions}\}$

We assume L_2 to be regular

Consider $w = 0^n 0 1^n 0^n$
$w^R = 0^n 1^n 0\, 0^n$

clearly, $w \in L_2$ as w^R and w differ at exactly two places

Let the number of states of the DFA be $p > 0$

$\left.\begin{array}{l} p \neq 1 \\ p \neq 2 \end{array}\right] \rightarrow$ no such DFA can be formed

$\Rightarrow p > 2$ Consider $n = p$

$w = x.y.z$ $|xy| \leq p$ $|y| \geq 1$

$\Rightarrow y$ will be of the form 0^s

Consider $xy^{k+1}z = w'$

$w' = 0^p 0^{ks} 0 1^p 0^p$

$(w')^R = 0^p 1^p 0\, 0^{ks} 0^p$

since $p > 2$ and if we choose $k > 2$, then
w' and $(w')^R$ differ at more than 2 positions

This implies $(w') \notin L_2$

$\Rightarrow$ Pumping lemma violated $\Rightarrow L_2$ not regular
(Proof by contradiction!)

2. L_1 = not regular $\Rightarrow$

Consider the infinite set = $\{0^i : i \geq 1\}$

Now, we need to provide a string x for every pair of words $w_1 = 0^i$ and $w_2 = 0^j$

Let us assume that such a string does not exist.

i.e. $\forall x$ $w_1 x$, $w_2 x$ either belong to L_1 simultaneously or do not do so.

Consider $x = 0^k 1$ such that

$w_1 x = 0^i 0^k 1$ and $(i+k)+(1)^2$ = prime

[since, there are infinite numbers of primes, such k will exist.]

$\Rightarrow w_1 x \in L_1$

$w_2 x = 0^j 0^k 1$ also belongs to L (else they won't be equivalent)

$\Rightarrow j+k+1$ = prime.

Consider $x' = 0^{j-i+k} 1$ (without loss of generality $j>i$)

$w_1 x' = 0^i 0^{j-i+k} 1 \in L_1$ (because $j+k+1$ = prime)

$w_2 x' = 0^j 0^{j-i+k} 1 \in L_1 \Rightarrow 2j-i+k+1$ = prime

Continuing this further, we can generate an infinite AP of primes with common difference $(j-i)$

So, $(j+k+1) + (j-i)n$ has to be prime for $\forall n$.

But, this is not true, consider $n = (j+k+1)$ for instance

[With this we have also found a suitable x distinguishing w_i and w_j]

$\Rightarrow \quad w_i \not\approx_L w_j \qquad \forall\, i, j$

We have found an infinite set of words.

L_2 = not regular $\Rightarrow$

Consider the set $= \{ 0^i 1^i \mid i \geq 2 \}$

$w_1 = 0^i 1^i \qquad w_2 = 0^j 1^j$

Now, we need to find an x such that $w_1 x \in L_2$ but $w_2 x \notin L_2$

Choose $x = 0 0^i$

$$\left.\begin{aligned} w_1 x &= 0^i 1^i 0 0^i \\ (w_1 x)^R &= 0^i 0 1^i 0^i \end{aligned}\right\} \rightarrow \text{differ at exactly two places} \Rightarrow w_1 x \in L_1$$

without loss of generality: $i > j$

$w_2 x = 0^j 1^j 0 0^i$

$(w_2 x)^R = 0^i 0 1^j 0^j$

$= 0^j 0^{i-j+1} 1^j 0^j$

$w_2 x = 0^j 1^j 0^{i-j+1} 0^j$

Case-1: $i-j+1 \geq j \rightarrow$ strings different at $j + j = 2j$ places

$\underbrace{j \geq 2}_{\text{from constraints}} \Rightarrow 2j \geq 4 \Rightarrow w_2 x \notin L_2$

Case-2: $j \geq i-j+1 \rightarrow$ strings different at $2(i-j+1)$

$i > j \Rightarrow i+1 > j+1 \Rightarrow i+1-j > 1$

$\Rightarrow 2(i+1-j) > 2 \Rightarrow w_2 x \notin L_2$

We have found a distinguishing word x for each w_1 and w_2 such that $w_1 x \in L_2$ and $w_2 x \notin L_2$

$w_1 \not\equiv_{L_2} w_2$ for an infinite set

$\Rightarrow L_2$ = not regular (By Myhill-Nerode Theorem!)