Semantic Relations in FOL

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M, α

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- **Consistency:** $\mathcal{F}$ is consistent iff there is at least one M and α such that $M, \alpha \models \varphi_i$ for all $\varphi_i \in \mathcal{F}$.
  - $\{\exists x\, R(x, y),\ \exists x\, R(f(x), y),\ \exists x\, R(f(f(x)), y), \ldots\}$ is consistent

Sematic Equivalence in FOL

$\varphi \equiv \psi$ iff $\{\varphi\} \models \psi$ and $\{\psi\} \models \varphi$.

Quantifier Equivalences

- $\forall x \forall y\, \varphi \;\equiv\; \forall y \forall x\, \varphi, \quad \exists x \exists y\, \varphi \;\equiv\; \exists y \exists x\, \varphi$
- $\forall x\, (\varphi_1 \wedge \varphi_2) \;\equiv\; (\forall x\, \varphi_1) \wedge (\forall x\, \varphi_2)$
- $\exists x\, (\varphi_1 \vee \varphi_2) \;\equiv\; (\exists x\, \varphi_1) \vee (\exists x\, \varphi_2)$
- If $x \notin \mathrm{free}(\varphi_2)$, then $Qx\, (\varphi_1 \text{ op } \varphi_2) \;\equiv\; (Qx\, \varphi_1) \text{ op } \varphi_2$, where $Q \in \{\exists, \forall\}$ and op $\in \{\vee, \wedge\}$.

Renaming Quantified Variables

Let $z \notin \mathrm{free}(\varphi) \cup \mathrm{bnd}(\varphi)$.
Then $Qx\, \varphi \;\equiv\; Qz\, \varphi[z/x]$ for $Q \in \{\exists, \forall\}$.

bounded set = $\{x\}$
free set = $\{y, w\}$

Enabler for substitution, e.g., $\exists x\, R(f(x, y), w) \;\equiv\; \exists z\, R(f(z, y), w)$
free
$f(x, y)$ not free for y in $\exists x\, R(f(x, y), w)$, but is free for y in $\exists z\, R(f(z, y), w)$.

$\exists x\, R(f(z, f(x,y)), w)$ $f(x,y)$

work

$\exists x\, R(f(x, f(x,y)), w)$

x,y should be free

x is bounded

Semantically Equivalent Transformations of FOL Formulae

Negation Normal Form

Push negations down to atomic predicates using

- DeMorgan's Laws
- $\neg\exists x\, \varphi(x) \equiv \forall x\, \neg\varphi(x)$ and $\neg\forall x\, \varphi(x) \equiv \exists x\, \neg\varphi(x)$ and

Pull quantifiers out to the left

- Rename every quantified variable to a fresh variable name
- Use rules for scoping of quantifiers in previous slide to pull all quantifiers out to the left → won't work with $\wedge$
  - $\exists x\, \varphi(x) \;\vee\; \exists x\, \psi(x) \;\equiv\; \exists x\, (\varphi(x) \vee \psi(x))$
  - $\exists x\, \varphi(x) \;\wedge\; \exists z\, \psi(z) \;\equiv\; \exists x \exists z\, (\varphi(x) \wedge \psi(z))$ } will work with $\vee$
  - $\forall x\, \varphi(x) \;\wedge\; \forall x\, \psi(x) \;\equiv\; \forall x\, (\varphi(x) \wedge \psi(x))$ } won't work with $\vee$
  - $\forall x\, \varphi(x) \;\vee\; \forall z\, \psi(z) \;\equiv\; \forall x \forall z\, (\varphi(x) \vee \psi(z))$ } work with $\wedge$

Prenex Normal Form (PNF)

First order logic formula of the form:

$$Q_1 x_1 Q_2 x_2 \dots Q_k x_k \, \varphi(x_1, x_2, \dots x_k, y_1, \dots y_n)$$

$Q_i \in \{\exists, \forall\}$ for all $i \in \{1, \dots k\}$ and $\varphi(\cdots)$ quantifier-free

- All quantifiers pulled out to the left: *quantifier prefix* of formula
  - Exact sequencing of $\forall$ and $\exists$ important
- $y_1, \dots y_n$ are free variables
- $\varphi(x_1, x_2, \dots x_k, y_1, \dots y_n)$ is quantifier free: *matrix* of formula

Every FOL formula has a semantically equivalent PNF

Special prenex normal forms

- Prenex conjunctive normal form (PCNF): matrix in CNF w.r.t. atomic predicates
- Prenex disjunctive normal form (PDNF): matrix in DNF w.r.t. atomic predicates

Every FOL formula has a sem. equivalent PCNF and PDNF.

First-order Definable Structures

- If φ is a $\mathcal{V}$-sentence (no free vars), no binding α necessary for evaluating truth of φ
  - Given $\mathcal{V}$-structure M, we can ask if $M \models \varphi$
  - Class of $\mathcal{V}$-structures defined by φ is $\{M \models \varphi\}$
- Some examples of structures: graphs, databases, number systems

Graphs as FO structures

A graph G

- U^G: set of vertices
- Vocabulary $\mathcal{V}$: $\{E, =\}$, where E is a binary (edge) relation
- Interpretation: For $a, b \in U^G$, $E^G(a, b) =$ **true** iff there is an edge from vertex a to vertex b in G

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- $\forall x \forall y\, (\neg(x = y) \rightarrow E(x, y))$
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  - (Infinite) class of all graphs with no cycles of length 3
- $\exists x \exists y\, (\neg(x = y) \wedge E(x, y) \wedge \forall z\, ((x = z) \vee (y = z)))$
  - (Finite) class of graphs with exactly two connected vertices.

Relational Databases as FO structures

A relational database D

- U^D: set of (possibly differently typed) data items
- Vocabulary $\mathcal{V}$: $\{P_1, \ldots P_k, =\}$, where P_i is a k_i-ary predicate corr. to the i^{th} table in database with k_i columns
- Interpretation: For $a_1, \ldots a_{k_i} \in U^D$, $P_i(a_1, \ldots a_{k_i}) =$ **true** iff $(a_1, \ldots a_{k_1})$ is a row of the i^{th} table

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Examples of classes of databases definable in FOL:

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  - Table StRec is the natural join of Tables Dob and Class

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Example database query:

- $\varphi(x) \triangleq \exists y \exists z\, (\mathrm{Dob}(x, y) \wedge \mathrm{After}(y, \text{“01/01/1990''}) \wedge \mathrm{Class}(x, z) \wedge \mathrm{Primary}(z))$

  Defines set of students born after “01/01/1990” and studying in a primary class.

Number systems as FO structures

Natural/real numbers with addition, multiplication, linear ordering and constants $\mathbf{0}$ and $\mathbf{1}$ (fixed interpretation)

- $\mathfrak{N} = (\mathbb{N}, \mathbf{0}, \mathbf{1}, \times, +, <, =)$
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Examples of properties expressible in FOL:

- $\mathfrak{R} \models \forall x \exists y \, (x = ((y \times y) \times y))$
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  - Not every natural number can be expressed as the sum of squares of two natural numbers. This can be done for real numbers
- $\mathfrak{N} \models \forall x \exists y\, ((x < y) \wedge (\forall z \forall w\, (y = z \times w) \rightarrow ((z = y) \vee (w = y)))$
  - There are infinitely many prime natural numbers