CS 208 HW 3 QUESTION 3

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22B1003

Question 3

1. $L = \{ w \in \Sigma^* \mid \exists\, u, v \in \Sigma^*, \; w = u.v \quad n_0(u) = n_1(v) \}$

We will claim that L is Σ^* i.e. every string of 1s and 0s belongs to L.

To prove this fact, we need to provide a possible split of $u.v$ for every string $w \in \Sigma^*$.

Consider $w = a_1 a_2 \ldots a_n$, where $a_1, a_2 \ldots a_n$ are letters of the string w; clearly $a_i = 0$ or 1.

I will try to provide an algorithmic proof for every such w.

Now, there are two extreme cases (i) $u = \epsilon$, $v = w$ or
(ii) $u = w$ $v = \epsilon$

In the first case (i) $n_0(u) = 0$
$n_1(v) = n_1(w)$

In case (ii) $n_0(u) = n_0(w)$
$n_1(v) = 0$

If w contained only 0s, then we can take case (i) as the possible split. $(n_0(u) = n_1(v) = 0)$

If w contained only 1s, then we can take case (ii) as the possible split. $(n_0(u) = n_1(v) = 0)$

Thus, we can safely assume that w contains both 1s and 0s.

Therefore $n_1(v)$ in case (i) $\neq 0$ and $n_0(u)$ in case (ii) $\neq 0$.

$w = \uparrow a_1 a_2 a_3 \dots a_{n-1} a_n \uparrow$

case (i) case (ii)

Now, if we add a_1 to u and remove it from v, there are two possible cases:

$a_1 = 1 \longrightarrow n_0(u) = 0$

$n_1(v) = n_1(w) - 1 \longrightarrow$ This decreased

$a_1 = 0 \longrightarrow n_0(u) = 1 \longrightarrow$ This increased

$n_1(v) = n_1(w)$

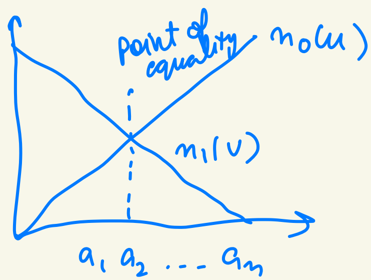
So, in both the cases, one of the sides changed by 1.

In the start (case (i)), we had $n_0(u) = 0$ and $n_1(v)$ non-zero

In the end (case (ii)), we had $n_0(u)$ non zero and $n_1(v) = 0$

And in each step, only one of them changed.

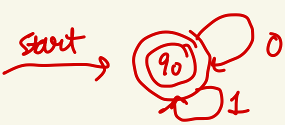
So, clearly there must exist a point where both of them are the same.



Therefore, we are successful in finding u, v for every such w.

Hence, every string $w \in \Sigma^*$ also belongs to L

Since Σ^* is regular, L is also regular. Here is the DFA:



2. Yes, there is a unique way to split w as $u.v$ for every $w \in L$.

Although the previous proof and the graph thereafter shows this fact, but we write the proof more formally here.

Let us assume, we have found a single way $w = u.v$ such that

$$n_0(u) = n_1(v) \quad \dots \quad (i)$$

(we can always do so, since $w \in L$)

We assume to the contrary that there exists another way $w = u_1 . v_1$.

Then there are two cases

(i) $len(u_1) > len(u)$ and $len(v_1) < len(v)$

(ii) $len(u_1) < len(u)$ and $len(v_1) > len(v)$

There can't be any other possibility because

$$len(u_1) + len(v_1) = len(u) + len(v) = len(w)$$

[If both correspondingly equal, then the split will be same.]

(i) In this case we can split v as $v_2 . v_1$

So $\quad w = u.v = \underbrace{(u.v_2)}_{= u_1}(v_1)$

$$n_0(u_1) = n_0(u) + n_0(v_2)$$

$$n_0(u_1) = n_1(v_1) \quad \text{(Because, this is a valid split)}$$

$$n_0(u) + n_0(v_2) = n_1(v_1) \quad \dots \quad (ii)$$

Putting (i) in (ii), we get $\quad n_1(v) + n_0(v_2) = n_1(v_1)$

$n_1(v) = n_1(v_1) + n_1(v_2)$

$n_1(v_1) + n_1(v_2) + n_0(v_2) = n_1(v_1)$

$\Rightarrow n_1(v_2) + n_0(v_2) = 0 \Rightarrow len(v_2) = 0 \Rightarrow v_2 = \varepsilon$

This implies, we have achieved the same splitting.

Hence, this case is rejected.

(ii) $u = u_1 \cdot u_2$

$w = u \cdot v = \underbrace{u_1} \cdot \underbrace{u_2 \cdot v}_{= v_1}$

$n_0(u_1) = n_1(v_1)$ [valid splitting]

$n_0(u_1) = n_1(u_2) + \underbrace{n_1(v)}_{= n_0(u) \text{ [from (i)]}}$

$n_0(u_1) = n_1(u_2) + n_0(u_1) + n_0(u_2)$

$\Rightarrow n_0(u_2) + n_1(u_2) = 0 \Rightarrow len(u_2) = 0 \Rightarrow u_2 = \varepsilon$

$\Rightarrow$ Same splitting

$\Rightarrow$ This case also rejected.

$\Rightarrow$ Both cases rejected $\rightarrow$ Contradiction!

Therefore, we have proved that for every $w \in L$, the splitting will be unique.