4. Let ϕ be $\exists x\,(P(y,z) \wedge (\forall y\,(\neg Q(y,x) \vee P(y,z))))$, where P and Q are predicate symbols with two arguments. y,z y,z x,y,z x,z x,y,z

* (a) Draw the parse tree of ϕ.

* (b) Identify all bound and free variable leaves in ϕ. (y,z) = free; bound= (x,y)

(c) Is there a variable in ϕ which has free and bound occurrences? y

* (d) Consider the terms w (w is a variable), $f(x)$ and $g(y,z)$, where f and g are function symbols with arity 1 and 2, respectively.

i. Compute $\phi[w/x]$, $\phi[w/y]$, $\phi[f(x)/y]$ and $\phi[g(y,z)/z]$. Just replace all free occurrences

ii. Which of w, $f(x)$ and $g(y,z)$ are free for x in ϕ?

iii. Which of w, $f(x)$ and $g(y,z)$ are free for y in ϕ? free variables in these should not change

(e) What is the scope of $\exists x$ in ϕ?

* (f) Suppose that we change ϕ to $\exists x\,(P(y,z) \wedge (\forall x\,(\neg Q(x,x) \vee P(x,z))))$. What is the scope of $\exists x$ now? scope because of this

5. (a) Let P be a predicate symbol with arity 3. Draw the parse tree of $\psi \stackrel{\text{def}}{=} \neg(\forall x\,((\exists y\,P(x,y,z)) \wedge (\forall z\,P(x,y,z))))$.

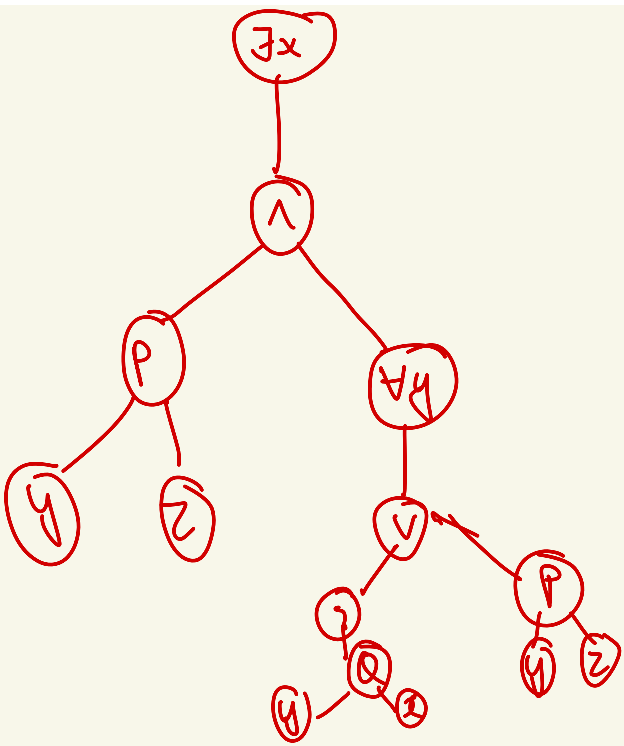
(b) Indicate the free and bound variables in that parse tree.

(c) List all variables which occur free and bound therein.

(d) Compute $\psi[t/x]$, $\psi[t/y]$ and $\psi[t/z]$, where $t \stackrel{\text{def}}{=} g(f(g(y,y)),y)$. Is t free for x in ψ; free for y in ψ; free for z in ψ?

6. Rename the variables for ϕ in Example 2.9 (page 106) such that the resulting formula ψ has the same meaning as ϕ, but $f(y,y)$ is free for x in ψ.

4) (a)



2. Recall that we use $=$ to express the equality of elements in our models. Consider the formula $\exists x \, \exists y \, (\neg(x = y) \wedge (\forall z \, ((z = x) \vee (z = y))))$. Can you say, in plain English, what this formula specifies?
3. Try to write down a sentence of predicate logic which intuitively holds in a model iff the model has (respectively)
   * (a) exactly three distinct elements
   (b) at most three distinct elements
   * (c) only finitely many distinct elements.

What 'limitation' of predicate logic causes problems in finding such a sentence for the last item?

9. Prove the validity of the following sequents in predicate logic, where F, G, P, and Q have arity 1, and S has arity 0 (a 'propositional atom'):

* (a) $\exists x\,(S \to Q(x)) \vdash S \to \exists x\,Q(x)$
  (b) $S \to \exists x\,Q(x) \vdash \exists x\,(S \to Q(x))$
  (c) $\exists x\,P(x) \to S \vdash \forall x\,(P(x) \to S)$
* (d) $\forall x\,P(x) \to S \vdash \exists x\,(P(x) \to S)$
  (e) $\forall x\,(P(x) \vee Q(x)) \vdash \forall x\,P(x) \vee \exists x\,Q(x)$
  (f) $\forall x\,\exists y\,(P(x) \vee Q(y)) \vdash \exists y\,\forall x\,(P(x) \vee Q(y))$
  (g) $\forall x\,(\neg P(x) \wedge Q(x)) \vdash \forall x\,(P(x) \to Q(x))$
  (h) $\forall x\,(P(x) \wedge Q(x)) \vdash \forall x\,(P(x) \to Q(x))$
  (i) $\exists x\,(\neg P(x) \wedge \neg Q(x)) \vdash \exists x\,(\neg(P(x) \wedge Q(x)))$
  (j) $\exists x\,(\neg P(x) \vee Q(x)) \vdash \exists x\,(\neg(P(x) \wedge \neg Q(x)))$
* (k) $\forall x\,(P(x) \wedge Q(x)) \vdash \forall x\,P(x) \wedge \forall x\,Q(x)$.
* (l) $\forall x\,P(x) \vee \forall x\,Q(x) \vdash \forall x\,(P(x) \vee Q(x))$.
*(m) $\exists x\,(P(x) \wedge Q(x)) \vdash \exists x\,P(x) \wedge \exists x\,Q(x)$.
* (n) $\exists x\,F(x) \vee \exists x\,G(x) \vdash \exists x\,(F(x) \vee G(x))$.
  (o) $\forall x\,\forall y\,(S(y) \to F(x)) \vdash \exists y S(y) \to \forall x\,F(x)$.

* (p) $\neg\forall x\,\neg P(x) \vdash \exists x\,P(x)$.
* (q) $\forall x\,\neg P(x) \vdash \neg\exists x\,P(x)$.
* (r) $\neg\exists x\,P(x) \vdash \forall x\,\neg P(x)$.

11. The proofs of the sequents below combine the proof rules for equality and quantifiers. We write $\phi \leftrightarrow \psi$ as an abbreviation for $(\phi \to \psi) \wedge (\psi \to \phi)$. Find proofs for

* (a) $P(b) \vdash \forall x\,(x = b \to P(x))$
  (b) $P(b), \forall x \forall y\,(P(x) \wedge P(y) \to x = y) \vdash \forall x\,(P(x) \leftrightarrow x = b)$
* (c) $\exists x\,\exists y\,(H(x, y) \vee H(y, x)), \neg\exists x\,H(x, x) \vdash \exists x \exists y\,\neg(x = y)$
  (d) $\forall x\,(P(x) \leftrightarrow x = b) \vdash P(b) \wedge \forall x \forall y\,(P(x) \wedge P(y) \to x = y)$.

* 12. Prove the validity of $S \to \forall x\,Q(x) \vdash \forall x\,(S \to Q(x))$, where S has arity 0 (a 'propositional atom').

* 1. Consider the formula $\phi \stackrel{\text{def}}{=} \forall x \, \forall y \, Q(g(x,y), g(y,y), z)$, where Q and g have arity 3 and 2, respectively. Find two models $\mathcal{M}$ and $\mathcal{M}'$ with respective environments l and l' such that $\mathcal{M} \vDash_l \phi$ but $\mathcal{M}' \nvDash_{l'} \phi$.

2. Consider the sentence $\phi \stackrel{\text{def}}{=} \forall x \, \exists y \, \exists z \, (P(x,y) \wedge P(z,y) \wedge (P(x,z) \to P(z,x)))$. Which of the following models satisfies ϕ?
   (a) The model $\mathcal{M}$ consists of the set of natural numbers with $P^{\mathcal{M}} \stackrel{\text{def}}{=} \{(m,n) \mid m < n\}$.
   (b) The model $\mathcal{M}'$ consists of the set of natural numbers with $P^{\mathcal{M}'} \stackrel{\text{def}}{=} \{(m, 2 * m) \mid m \text{ natural number}\}$.
   (c) The model $\mathcal{M}''$ consists of the set of natural numbers with $P^{\mathcal{M}''} \stackrel{\text{def}}{=} \{(m,n) \mid m < n + 1\}$.

5. Let ϕ be the sentence $\forall x\, \forall y\, \exists z\, (R(x, y) \to R(y, z))$, where R is a predicate symbol of two arguments.
   * (a) Let $A \stackrel{\text{def}}{=} \{a, b, c, d\}$ and $R^{\mathcal{M}} \stackrel{\text{def}}{=} \{(b, c), (b, b), (b, a)\}$. Do we have $\mathcal{M} \models \phi$? Justify your answer, whatever it is.
   * (b) Let $A' \stackrel{\text{def}}{=} \{a, b, c\}$ and $R^{\mathcal{M}'} \stackrel{\text{def}}{=} \{(b, c), (a, b), (c, b)\}$. Do we have $\mathcal{M}' \models \phi$? Justify your answer, whatever it is.
6. Consider the three sentences

$$\begin{aligned}
\phi_1 &\stackrel{\text{def}}{=} \forall x\, P(x, x) \\
\phi_2 &\stackrel{\text{def}}{=} \forall x\, \forall y\, (P(x, y) \to P(y, x)) \\
\phi_3 &\stackrel{\text{def}}{=} \forall x\, \forall y\, \forall z\, ((P(x, y) \wedge P(y, z) \to P(x, z)))
\end{aligned}$$

which express that the binary predicate P is reflexive, symmetric and transitive, respectively. Show that none of these sentences is semantically entailed by the other ones by choosing for each pair of sentences above a model which satisfies these two, but not the third sentence – essentially, you are asked to find three binary relations, each satisfying just two of these properties.

$\phi \not\models \psi \quad \neg\phi \models \psi$?

9. Let ϕ and ψ and η be sentences of predicate logic.
   (a) If ψ is semantically entailed by ϕ, is it necessarily the case that ψ is not semantically entailed by $\neg\phi$? No
   * (b) If ψ is semantically entailed by $\phi \wedge \eta$, is it necessarily the case that ψ is semantically entailed by ϕ and semantically entailed by η? No
   (c) If ψ is semantically entailed by ϕ or by η, is it necessarily the case that ψ is semantically entailed by $\phi \vee \eta$? Yes
   (d) Explain why ψ is semantically entailed by ϕ iff $\phi \rightarrow \psi$ is valid.
10. Is $\forall x\,(P(x) \vee Q(x)) \models \forall x\,P(x) \vee \forall x\,Q(x)$ a semantic entailment? Justify your answer. No
11. For each set of formulas below show that they are consistent:
   (a) $\forall x\,\neg S(x,x),\ \exists x\,P(x),\ \forall x\,\exists y\,S(x,y),\ \forall x\,(P(x) \rightarrow \exists y\,S(y,x))$
   * (b) $\forall x\,\neg S(x,x),\ \forall x\,\exists y\,S(x,y),$
     $\forall x\,\forall y\,\forall z\,((S(x,y) \wedge S(y,z)) \rightarrow S(x,z))$
   (c) $(\forall x\,(P(x) \vee Q(x))) \rightarrow \exists y\,R(y),\ \forall x\,(R(x) \rightarrow Q(x)),\ \exists y\,(\neg Q(y) \wedge P(y))$
   * (d) $\exists x\,S(x,x),\ \forall x\,\forall y\,(S(x,y) \rightarrow (x = y))$.

12. For each of the formulas of predicate logic below, either find a model which does not satisfy it, or prove it is valid:

(a) $(\forall x\, \forall y\, (S(x,y) \to S(y,x))) \to (\forall x\, \neg S(x,x))$

* (b) $\exists y\, ((\forall x\, P(x)) \to P(y))$

(c) $(\forall x\, (P(x) \to \exists y\, Q(y))) \to (\forall x\, \exists y\, (P(x) \to Q(y)))$

(d) $(\forall x\, \exists y\, (P(x) \to Q(y))) \to (\forall x\, (P(x) \to \exists y\, Q(y)))$

(e) $\forall x\, \forall y\, (S(x,y) \to (\exists z\, (S(x,z) \wedge S(z,y))))$

(f) $(\forall x\, \forall y\, (S(x,y) \to (x = y))) \to (\forall z\, \neg S(z,z))$

* (g) $(\forall x\, \exists y\, (S(x,y) \wedge ((S(x,y) \wedge S(y,x)) \to (x = y)))) \to$ $(\neg \exists z\, \forall w\, (S(z,w)))$.

(h) $\forall x\, \forall y\, ((P(x) \to P(y)) \wedge (P(y) \to P(x)))$

(i) $(\forall x\, ((P(x) \to Q(x)) \wedge (Q(x) \to P(x)))) \to ((\forall x\, P(x)) \to (\forall x\, Q(x)))$

(j) $((\forall x\, P(x)) \to (\forall x\, Q(x))) \to (\forall x\, ((P(x) \to Q(x)) \wedge (Q(x) \to P(x))))$

(k) Difficult: $(\forall x\, \exists y\, (P(x) \to Q(y))) \to (\exists y\, \forall x\, (P(x) \to Q(y)))$.
