CS 208

HW4 - Q2

SAKSHAM RATHI

22B1003

2) Given two lists of strings $u_1, u_2 \dots u_n$ and $v_1, v_2 \dots v_n$ over an alphabet Σ, does there exist a sequence of indices $i_1, i_2 \dots i_k$ such that $u_{i_1} u_{i_2} \dots u_{i_k} = v_{i_1} \dots v_{i_k}$ (This is the PCP problem)

We will take this instance of PCP and try to construct a grammar G out of this. Now, the PCP will have a solution iff w and w^R are in $L(G)$.

start symbol = S

$$S \to A \mid B$$

$$A \to u_1 A a_1 \mid u_2 A a_2 \mid \dots \mid u_n A a_n \mid \varepsilon$$

$$B \to a_1 B v_1^R \mid a_2 B v_2^R \mid \dots \mid a_n B v_n^R \mid \varepsilon$$

where $a_1 \dots a_n$ are extra symbols (different from the elements of Σ) (single letter symbols and distinct from each other, as long as n is finite, we can find such a_i's)

A will have strings of the form:

$$u_{i_1} u_{i_2} \dots u_{i_k} a_{i_k} a_{i_{k-1}} \dots a_1 \doteq w_A$$

B will have strings of the form:

$$a_i a_2 \dots a_{i_k} v_{i_k}^R v_{i_{k-1}}^R \dots v_{i_1}^R = w_B$$

Consider

$$w_B^R = v_{i_1} v_{i_2} \dots v_{i_k} a_{i_k} \dots a_{i_1}$$

Now if w_B^R has to belong to $G(S)$ then it should be part of $G(A)$.

[Because B has a_i in the start and a_is are different from Σ alphabet letters.]

$$w_B^R = w_A' \quad \text{for some } w_A' \in G(A)$$

$$v_{i_1} v_{i_2} \dots v_{i_k} \underbrace{a_{i_k} \dots a_{i_1}} = v_{j_1} \dots v_{j_l} \underbrace{a_{j_l} \dots a_{j_1}}$$

These parts must match because they are different from Σ.

$\Rightarrow$ $k = l$

and $i_t = j_t$ $\forall_{1 \le t \le k}$ $\rightarrow$ Because a_i distinct from a_j $i \neq j$.

From this we get that :

$$v_{i_1} \dots v_{i_k} = v_{i_1} \dots v_{i_k} \quad \text{for some set } (i_1 \dots i_k)$$

Similarly, we can take w_A^R and prove it to be equal to w_B

Therefore, we have deduced the following:

If PCP has a solution $i_1 \ldots i_k$ then we can find w and w^R belonging to G.

Similarly, if we can find w and $w^R \in G$, we can have a solution to the PCP problem instance.

Since we have proved that PCP reduces to our grammar G, proving the existence of a terminal string $w \in L(G)$ such that $w^R \in L(G)$ is undecidable.

* One might think that for the set $i_1 \ldots i_k$, i_{l_1} can be equal to i_{l_2} for $l_1 \neq l_2$. But it can be shown that for such cases we can remove all the repetitions and our solution will still be valid.