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22B1003

HOMEWORK 2

QUESTION -1

1. Consider a formula $F \in \Sigma$ of length x.
By length $= x$, we mean the number of characters present in the string representation of that formula.

For example : $P_1 \wedge P_2$ = formula of length 3

$((P_1 \wedge \neg P_2) \vee P_3)$ = formula of length 10 (Include brackets, negation, and, or operators)

Now, we will claim that every formula can be converted to a vector of integers through a bijective mapping. Let's create one such possible bijection.

For all the operators, (including $\wedge$, $\vee$, $\neg$, ')', '(' ...) we will assign them negative integers.

For all propositional variables ($P_1, P_2 \ldots$), we will assign them positive integers.

Mapping for operators:

$\wedge \to -1$

$\vee \to -2$

$) \to -3$

$(\to -4$

$\neg \to -5$

$\vdots$ and so on (there can be countably infinite list of operators)

Mapping for propositional variables:

$P_1 \to 1$

$P_2 \to 2$

$\vdots$

Now through this mapping lets convert the earlier two examples to vectors:

$P_1 \wedge P_2$ maps to $(1, -1, 2)^T$

$((P_1 \wedge \neg P_2) \vee P_3)$ maps to $(-4, -4, 1, -1, -5, 2, -3, -2, 3,$

One can easily see, that this mapping is bijective.

[for negative numbers correspond to operators and positive numbers correspond to variables].

Claim: For any finite t, there exists a bijection from $\mathbb{N}^t$ to $\mathbb{N}$.

Use induction $t=1 \rightarrow$ base case

use identity function

$t=2 \rightarrow$ use $g((x,y))$

$$= \frac{(x+y) \times (x+y+1)}{2} + y$$

Assume bijection from $\mathbb{N}^k$ to $\mathbb{N}$ → let this be h

$(x_1 \dots x_{k+1}) \in \mathbb{N}^{k+1}$

Define f to be $f(x_1 \dots x_{k+1}) = g(h(x_1, \dots x_k), x_{k+1})$

Clearly this is a bijection

Hence, our claim is proved.

Our formula of length x can be bijectively mapped to $\mathbb{N}^x$ vector which is bijective to $\mathbb{N}$.

Thus, number of such formulas are countably infinite.

Now, we also know that countable union of countable sets is countable

Thus, considering all formulas of ~~various~~ finite lengths and taking their union as Σ, we find that Σ must always be countable.

2. (a) To prove: Γ with $|\Gamma| \geq 2$ is irredundant iff $(\Gamma \setminus \{\gamma\}) \cup \{\neg\gamma\}$ is satisfiable for every $\gamma \in \Gamma$.

Let us assume Γ is irredundant
This implies for all $\gamma \in \Gamma$, $\exists$ one assignment α that makes all formulas $\Gamma \setminus \{\gamma\}$ true, but γ to be false ~~γ may or may not be true~~

[If no such assignment α would have existed, then $(\Sigma \setminus \{\gamma\}) \models \gamma$, but Σ = irredundant].

For all other assignments making $(\Gamma \setminus \{\gamma\})$ true, γ may or may not be true

This implies $(\Gamma \setminus \{\gamma\}) \cup \{\neg\gamma\}$ is satisfiable.

[~~the case makes~~ α will always exist because Γ is a satisfiable set. Now, this will guarantee one α which will make $\Gamma \setminus \{y\}$ = true but $\neg\gamma$ false. So, the union of these two sets will become satisfiable and we are done!]

Let us now try to prove the other way round.

So, we assume that $(\Gamma \setminus \{\gamma\}) \cup \{\neg\gamma\}$ is satisfiable for every $\gamma \in \Gamma$.

This means, for every $\gamma \in \Gamma$, there exists an assignment α of propositional variables which makes all formulas of $\Gamma \setminus \{\gamma\}$ and $\neg\gamma$ true.

Thus, for such assignment α, $\Gamma \setminus \{\gamma\}$ is true but γ is false.

By the definition of irredundancy, existence of such α for every γ makes the set of formula Γ irredundant.

Hence, we have proved both directions.

2. (b) $\Sigma' = \{ \sigma_1,$

$$(\sigma_1) \rightarrow \sigma_2$$

$$(\sigma_1 \wedge \sigma_2) \rightarrow \sigma_3$$

$$(\sigma_1 \wedge \sigma_2 \wedge \sigma_3) \rightarrow \sigma_4$$

$$\vdots$$

$$\}$$

Let us ~~...~~ assume that the i^{th} formula is a tautology. This implies that we can remove σ_i from further formulas.

For example if $(\sigma_1) \rightarrow \sigma_2$ = tautology, then it implies $\neg(\sigma_1) \vee \sigma_2$ = true $\forall$ assignments of propositional variables

if σ_1 is false, then all the other formulas will be tautologies ($\sigma_1 \wedge \sigma_2 \wedge \ldots$ = false)

If σ_2 is true, then taking its union with others is of no use, hence we can remove σ_2 from all further formulas.

~~...~~ Now, we will do this till all tautogies are removed.

We get a format of Σ'' :

$$\left(\begin{array}{l} \sigma_{l_1} \\ \sigma_{l_1} \to \sigma_{l_2} \\ \sigma_{l_1} \wedge \sigma_{l_2} \to \sigma_{l_3} \\ \vdots \end{array} \right)$$

Now, we will prove it to be irredundant.

Consider a formula γ of the form

$$\sigma_{l_1} \wedge \sigma_{l_2} \wedge \dots \wedge \sigma_{l_k} \to \sigma_{l_{k+1}}$$

If we show $\exists$ an assignment which makes this formula false, then we are done!

This is possible when $\sigma_{l_1} \wedge \sigma_{l_2} \wedge \dots \wedge \sigma_{l_k}$ is true but $\sigma_{l_{k+1}}$ is false.

~~The first~~ $\sigma_{l_1} \wedge \sigma_{l_2} \wedge \dots \wedge \sigma_{l_k} = \text{true}$ makes all the previous formulas satisfiable. $\sigma_{l_{k+1}} = \text{false}$ means the left hand side of all further formulas will be false, therefore ~~this~~ they will also get satisfied. Hence for every γ

$\Sigma \setminus \{\gamma\}$ is satisfiable but γ is not for a particular assignment α, and we are done.

[If $\circ$ there does not exist any assignment α which makes $\sigma_{l_{k+1}}$ as false, then $\sigma_{l_{k+1}}$ = tautology but this gives a contradiction, as we have already removed all such tautologies from Σ' to Σ''.

Now, we need to prove that Σ' is equisemantic to Σ''.

Consider an assignment that makes all formulas of Σ to be true. ($\sigma_1, \sigma_2 \ldots$ = true) then clearly all formulas of Σ'' are also true (both LHS and RHS of every implication true).

Similarly, for every assignment α that makes Σ'' formulas to be ~~true~~ true, will also make Σ' formulas to be true (because we have removed tautologies) to be true (because we have removed tautologies)

This in turn implies that all σ_i from Σ are true.

<u>Proof:</u> Let us assume σ_t true from $t = 1$ to n but σ_{n+1} = false this will ~~contradict~~ make

$$(\sigma_1 \wedge \sigma_2 \wedge \ldots \sigma_n) \to \sigma_{n+1} = \text{false}$$

Hence contradiction!.

Assume $\Sigma \models \varphi$ for some formula φ.

Now, consider an assignment α that makes all Σ formulas true. From assumption, φ will also become true. But from above we have that all formulas of Σ'' will also be true. This will imply for every φ if $\Sigma \models \varphi$ then $\Sigma'' \models \varphi$

With similar logic, this also holds vice versa.

$\Rightarrow \Sigma''$ is equisemantic to Σ.