CS 208 HOMEWORK 2

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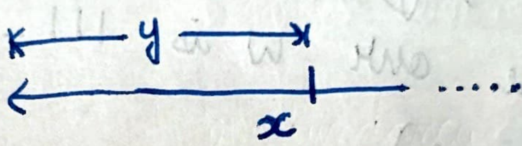
QUESTION 2

(a) Lemma proof :

For a bitstring $x \notin L$, we need to prove the existence of a finite prefix $y$ of $x$ such that for every infinite bitstring $w$, $yw \notin L$.




This means, an index $m$ exists such that any infinite bitstring concatenated after that $y$ will not belong to $L$.

Since $x \notin L$, $\exists$ a formula $F \in \Sigma$ such that $x$ does not satisfy $F$.

Since $F$ is finite (it is a formula). let $F$ contain propositional variables $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$. Take the required index to be $\max(i_1, i_2, \ldots, i_k) + 1 = m$ (say)

Now, since our formula $F$ does not contain any indices after this $m$, every bitstring $w$ for these variables having index $\geq m$, the formula will not be satisfied.

Therefore $y$ = bitstring till index $m-1$.

hence, the lemma is proved!

Now consider $x = 000\ldots$

every finite prefix of $x = y$ will have only $0$s (of length $n$, say).

Now, suppose our $w$ is $111\ldots$

(we can choose any $w$)

clearly, $yw$ belongs to the language set $L$.

But, this contradicts our lemma proof.

Therefore our assumption that language $L$ is PL definable is wrong.

(b) To show that PL-definable languages are not closed under countable union, we will take a counter example.

Take sets $\Sigma_0, \Sigma_1, \Sigma_2, \ldots, \Sigma_n, \ldots$ (countably infinite)

where set $\Sigma_i$ contains only one formula $p_i$.

Thus, our sets are:

$\{p_0\}, \{p_1\}, \{p_2\}, \ldots, \ldots$

The set $\{p_i\}$ will be satisfied by all strings with 1 at the $i^{th}$ position (rest all indices can be both 0 or 1).

Let us take the union of all of these sets.

Clearly every infinite bitstring except 000... lies in the union of the languages $L_0, L_1 \ldots$ corresponding to $\Sigma_0, \Sigma_1, \ldots$

(000... does not lie in any set, thus it won't be present in the union)

But according to previous part, this union language is not PL-definable. Thus, we have provided a counter example

Complementation:

Consider the set $\Sigma$ defined as

$\{ \neg p_0, \neg p_1, \neg p_2 \ \ldots \}$
(countably infinite elements)

Only the infinite bitstring $000\ldots$ will satisfy all of these formula $F \in \Sigma$.

[To satisfy $F_i \in \Sigma$, we need to have 0 at $i^{th}$ index, and this is true for all indices.]

Therefore the PL definable language in this case is
$L = \{000\ldots\}$ (only one element).

Let us take the complement of this:
$L' = \{$ all infinite bitstrings except $000\ldots$

But according to part (a), our language $L'$ is not PL definable.

Thus, using a counter example, we showed that PL definable languages are not closed under complementation

(c) Let us assume that our PL language does not contain every bitstring.

Then, in this case, we can find an infinite bitstring $x \notin L$. (we can find at least one such $x$)

Now, according to the lemma proved in (a) part, there exists a finite prefix $y$ of $x$ such that for any infinite bitstring $w$, $yw \notin L$.

Therefore with a given $x$, we are able to find many bitstrings $yw$ which do not belong to $L$.

Now, since $w$ is an infinite bitstring it is clear that there are infinitely uncountable such $w$, and so the number of $yw$ are also infinitely uncountable.

Therefore, if our PL language does not contain every bitstring then it will not contain uncountably many bitstrings.

Although, the proof is over, but we can give an example of $\Sigma$ such that its $L$ contains all bitstrings.

Consider $\Sigma = \{p_0 \to p_0\}$ (only single formula)

As, we can see, every bitstring satisfies $\Sigma$ and hence, we are done!