CS 208

SAKSHAM RATHI

22B1003

QUESTION 1

1. $\varphi_1 = \perp$

Consider equation ia :

$$(x_i \wedge \varphi_1) \rightarrow \varphi_2$$

If $x_i = \perp$ then $x_i \wedge \varphi_1 = \perp$
φ_2 can be $\perp$ or $\top$

If $x_i = \top$ then $x_i \wedge \varphi_1 = \perp$
φ_2 can be $\top$ or $\perp$

Now Consider equation ib :

$$(x_i \wedge \varphi_2) \rightarrow \varphi_1$$

This will be true only when $(x_i \wedge \varphi_2) = \perp$

If $x_i = \top$ then $\varphi_2 = \perp$

If $x_i = \perp$ then $\varphi_2 = \perp$ or $\top$

If any of the x_i ($i \in \{1,2 \ldots n\}$) is T,
then $\varphi_2 = \perp$

But if all of x_i are $\perp$ then
φ_2 can be $\perp$ or T.

The truth table of φ_2 will have 2^n rows.
For 1 row (with all $x_i = \perp$) we will have two possibilities of φ_2

$\therefore$ 2 semantically distinct formulas φ_2 exist.

Q3

2. $\varphi_1 = \top$

Consider equation ia

$(x_i \wedge \varphi_1) \rightarrow \varphi_2$

If $x_i = \perp$ then $x_i \wedge \varphi_1 = \perp$

φ_2 can be $\perp$ or $\top$

If $x_i = \top$ then $x_i \wedge \varphi_1 = \top$

φ_2 can be $\top$

Consider equation ib:

$(x_i \wedge \varphi_2) \rightarrow \varphi_1$

If $\varphi_1 = \top$ $\Rightarrow$ $(x_i \wedge \varphi_2)$ can be $\perp$ or $\top$

Hence, we won't get any constraint from these equations.

If all x_i are $\perp$ then φ_2 can take values $\perp$ or $\top$, else it will be $\top$.

$\therefore$ 2 semantically distinct formulas φ_2

3. We have $2n$ equations with us, all of them must be true

$\Rightarrow$ their 'and' should also be true

Consider (ia) $(x_i \wedge \varphi_1) \rightarrow \varphi_2$

$$\Leftrightarrow \sim(x_i \wedge \varphi_1) \vee \varphi_2$$

(ib) $(x_i \wedge \varphi_2) \rightarrow \varphi_1$

$$\Leftrightarrow \sim(x_i \wedge \varphi_2) \vee \varphi_1$$

Taking and of these two

(ia) $\wedge$ (ib)

$$\Rightarrow \left(\sim(x_i \wedge \varphi_1) \vee \varphi_2\right) \wedge \left(\sim(x_i \wedge \varphi_2) \vee \varphi_1\right)$$

$$\Rightarrow (\sim x_i \vee \sim\varphi_1 \vee \varphi_2) \wedge (\sim x_i \vee \sim\varphi_2 \vee \varphi_1)$$

$$\Rightarrow (\sim x_i) \vee \left((\sim\varphi_1 \vee \varphi_2) \wedge (\sim\varphi_2 \vee \varphi_1)\right)$$

$$\Rightarrow (\sim x_i) \vee \left((\sim\varphi_1 \wedge \sim\varphi_2) \vee (\sim\varphi_1 \wedge \varphi_1) \vee (\varphi_2 \wedge \varphi_1) \vee (\varphi_2 \wedge \sim\varphi_2)\right)$$

$$\Rightarrow (\sim x_i) \vee (\sim\varphi_1 \wedge \sim\varphi_2) \vee (\varphi_1 \wedge \varphi_2)$$

If x_i is false, this is always true

If x_i is true then

$(\sim Q_1 \wedge \sim Q_2) \vee (Q_1 \wedge Q_2)$

should be true.

Truth Table for this:

Q_1	Q_2	$Q_1 \wedge Q_2$	$\sim Q_1 \wedge \sim Q_2$	$(Q_1 \wedge Q_2) \vee (\sim Q_1 \wedge \sim Q_2)$
0	0	0	1	1
0	1	0	0	0
1	0	0	0	0
1	1	1	0	1

$\Rightarrow \; Q_1 = Q_2 = 0$ or $Q_1 = Q_2 = 1$

If all x_i are false then Q_1 and Q_2 can take any value $(2 \times 2 = 4)$

Else, in any case $(Q_1 = Q_2) \Rightarrow$ only 2 values

There are total 2^n rows of variables

For one row there are 4 options

For other $(2^n - 1)$ rows there are 2 options

Total number of pairs =

$$2^{2^n - 1} \times 4 = 2^{2^n - 1 + 1} = \left(2^{2^n + 1}\right)$$

4) We can take hints from the previous part in this question.

If one of the x_i is T then

$\varphi_1 = \varphi_2 \Rightarrow$ only one formula φ_2 will exist for φ_1.

But, if all of the x_i's are $\perp$ then φ_1, φ_2 can take any values.

$$\varphi_1 = \perp \longrightarrow \varphi_2 = \perp \text{ or } T$$

$$\varphi_1 = T \longrightarrow \varphi_2 = \perp \text{ or } T$$

Therefore, there does not exist any formula φ_1 such that there is exactly one formula φ_2.