CS663 Assignment-4

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Question 2

Solution

We need to find the direction f which is perpendicular to e and for which f^tCf is maximized. It is also given that all the non-zero eigen values of C are distinct and rank$(C) > 2$.

Let $C = UDU^t$ be the eigen value decomposition of C. Since C is a symmetric matrix, U is an orthogonal matrix. Let $D = diag(\lambda_1, \lambda_2, \dots, \lambda_n)$ be the diagonal matrix of eigen values of C.

Since, the eigen vectors of C form an orthonormal basis, we can write f as follows:

$$f = Ua = \sum_{i=1}^{n} a_i u_i \tag{1}$$

where a is a vector in $\mathbb{R}^n$ and u_i are the eigen vectors.

Without loss of generality, let us assume that $\lambda_1 > \lambda_2 > \lambda_3 > \dots$. The fact that rank$(C) > 2$ ensures that there are atleast three non-zero eigen values (the values being distinct follows from the question statement itself).

Since, u_1 corresponds to λ_1, we have $u_1 = e$. Also, the dot product of f and e is 0:

$$f^te = 0 \implies \sum_{i=1}^{n} a_i u_i^t e = 0 \implies a_1 = 0 \tag{2}$$

because, $u_1 = e$ and $u_i^te = 0$ for $i \neq 1$ (orthonormal eigen vectors).

We need to maximize f^tCf which is given by:

$$f^tCf = \left(\sum_{i=2}^{n} a_i u_i\right)^t C \left(\sum_{i=2}^{n} a_i u_i\right) \tag{3}$$

Since $Cu_i = \lambda_i u_i$, we can write the above equation as:

$$f^tCf = \left(\sum_{i=2}^{n} a_i u_i\right)^t \left(\sum_{i=2}^{n} a_i \lambda_i u_i\right) = \sum_{i=2}^{n} a_i^2 \lambda_i \tag{4}$$

(The last equation follows from the fact that the eigen vectors are orthonormal.)

Thus, we need to maximize $\sum_{i=2}^{n} a_i^2 \lambda_i$ subject to the constraint that $f^te = 0$ and $||f|| = 1$. (This is a weighted inequality and we need to maximizr it, so we can set the coefficient of the largest value to 1 and others to zero.) Since λ_2 is the second largest eigen value, we can maximize the above expression by setting $a_2 = 1$ and $a_i = 0$ for $i \neq 2$. Thus, the direction f is given by u_2. Also, the constraint $||f|| = 1$ is satisfied by setting $a_2 = 1$ instead of any other scalar multiple.

Solution

We need to find direction g which is perpendicular to both e and f and for which g^tCg is maximized. We have already found that $f = u_2$. We can write g as follows:

$$g = Ua = \sum_{i=1}^{n} a_i u_i \tag{5}$$

g is perpendicular to f and e. Thus, $g^t f = 0$ and $g^t e = 0$. We can write these as:

$$g^t f = 0 \implies \sum_{i=1}^{n} a_i u_i^t f = 0 \implies a_2 = 0 \tag{6}$$

$$g^t e = 0 \implies \sum_{i=1}^{n} a_i u_i^t e = 0 \implies a_1 = 0 \tag{7}$$

This again uses the fact that the eigen vectors are orthonormal. We need to maximize g^tCg which is given by:

$$g^tCg = \left(\sum_{i=3}^{n} a_i u_i\right)^t C \left(\sum_{i=3}^{n} a_i u_i\right) \tag{8}$$

Similar to the previous part, we can write the above equation as:

$$g^tCg = \left(\sum_{i=3}^{n} a_i u_i\right)^t \left(\sum_{i=3}^{n} a_i \lambda_i u_i\right) = \sum_{i=3}^{n} a_i^2 \lambda_i \tag{9}$$

Thus, we need to maximize $\sum_{i=3}^{n} a_i^2 \lambda_i$ subject to the constraint that $g^t f = 0$, $g^t e = 0$ and $||g|| = 1$. Since λ_3 is the third largest eigen value, we can maximize the above expression by setting $a_3 = 1$ and $a_i = 0$ for $i \neq 3$. Thus, the direction g is given by u_3. Also, the constraint $||g|| = 1$ is satisfied by setting $a_3 = 1$ instead of any other scalar multiple.

Therefore, we have proved that the direction of f is corresponding to the eigen vector with the second largest eigen value and the direction of g is corresponding to the eigen vector with the third largest eigen value.