CS663 Assignment 2

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Question 6

Proving $I_{xx} + I_{yy} = I_{uu} + I_{vv}$

Given,

$$(u, v) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta) \tag{1}$$

Also, for any function f:-

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial f}{\partial u}(\cos\theta) + \frac{\partial f}{\partial v}(\sin\theta) \tag{2a}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial f}{\partial u}(-\sin\theta) + \frac{\partial f}{\partial v}(\cos\theta) \tag{2b}$$

Now, let's see second partial derivatives :-

$$I_{xx} = \frac{\partial^2 I}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial I}{\partial x}\right) = \frac{\partial}{\partial u}\left(\frac{\partial I}{\partial x}\right)(\cos\theta) + \frac{\partial}{\partial v}\left(\frac{\partial I}{\partial x}\right)(\sin\theta) \qquad \text{(From Eqn. 2a)}$$

$$I_{yy} = \frac{\partial^2 I}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial I}{\partial y}\right) = \frac{\partial}{\partial u}\left(\frac{\partial I}{\partial y}\right)(-\sin\theta) + \frac{\partial}{\partial v}\left(\frac{\partial I}{\partial y}\right)(\cos\theta) \qquad \text{(From Eqn. 2b)}$$

From Eqn. 2a, (Note: I_{uv} is $\frac{\partial}{\partial v}\left(\frac{\partial I}{\partial u}\right)$ and similarly I_{vu}. Also $I_{uv} = I_{vu}$.)

$$\frac{\partial}{\partial u}\left(\frac{\partial I}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial I}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial I}{\partial u}\right)(\cos\theta) + \frac{\partial}{\partial v}\left(\frac{\partial I}{\partial u}\right)(\sin\theta) = I_{uu}(\cos\theta) + I_{uv}(\sin\theta)$$

$$\frac{\partial}{\partial v}\left(\frac{\partial I}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial I}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial I}{\partial v}\right)(\cos\theta) + \frac{\partial}{\partial v}\left(\frac{\partial I}{\partial v}\right)(\sin\theta) = I_{vu}(\cos\theta) + I_{vv}(\sin\theta)$$

From Eqn. 2b,

$$\frac{\partial}{\partial u}\left(\frac{\partial I}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial I}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial I}{\partial u}\right)(-\sin\theta) + \frac{\partial}{\partial v}\left(\frac{\partial I}{\partial u}\right)(\cos\theta) = I_{uu}(-\sin\theta) + I_{uv}(\cos\theta)$$

$$\frac{\partial}{\partial v}\left(\frac{\partial I}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial I}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial I}{\partial v}\right)(-\sin\theta) + \frac{\partial}{\partial v}\left(\frac{\partial I}{\partial v}\right)(\cos\theta) = I_{vu}(-\sin\theta) + I_{vv}(\cos\theta)$$

Substitute $\frac{\partial}{\partial u}\left(\frac{\partial I}{\partial x}\right)$ and $\frac{\partial}{\partial v}\left(\frac{\partial I}{\partial x}\right)$ in I_{xx}:-

$$\begin{aligned}
I_{xx} &= (I_{uu}(\cos\theta) + I_{uv}(\sin\theta))\cos\theta + (I_{vu}(\cos\theta) + I_{vv}(\sin\theta))\sin\theta \\
&= I_{uu}\cos^2\theta + I_{uv}\sin\theta\cos\theta + I_{vu}\cos\theta\sin\theta + I_{vv}\sin^2\theta \\
&= I_{uu}\cos^2\theta + 2I_{uv}\sin\theta\cos\theta + I_{vv}\sin^2\theta
\end{aligned}$$

Substitute $\frac{\partial}{\partial u}\left(\frac{\partial I}{\partial y}\right)$ and $\frac{\partial}{\partial v}\left(\frac{\partial I}{\partial y}\right)$ in I_{yy}:-

$$\begin{aligned}
I_{yy} &= (I_{uu}(-\sin\theta) + I_{uv}(\cos\theta))(-\sin\theta) + (I_{vu}(-\sin\theta) + I_{vv}(\cos\theta))\cos\theta \\
&= I_{uu}\sin^2\theta - I_{uv}\cos\theta\sin\theta - I_{vu}\sin\theta\cos\theta + I_{vv}\sin^2\theta \\
&= I_{uu}\sin^2\theta - 2I_{uv}\sin\theta\cos\theta + I_{vv}\cos^2\theta
\end{aligned}$$

Adding I_{xx} and I_{yy}, you can see:-

$$\begin{aligned}
I_{xx} + I_{yy} &= \{I_{uu}\cos^2\theta + 2I_{uv}\sin\theta\cos\theta + I_{vv}\sin^2\theta\} + \{I_{uu}\sin^2\theta - 2I_{uv}\sin\theta\cos\theta + I_{vv}\cos^2\theta\} \\
&= I_{uu}(\cos^2\theta + \sin^2\theta) + I_{vv}(\sin^2\theta + \cos^2\theta) \\
&= I_{uu} + I_{vv}
\end{aligned}$$

Hence proved.

Finding Second Directional Derivative in Gradient's Direction

First Directional Derivative is $\phi(f, v) = \nabla f(x, y) \cdot v = (f_x \hat{i} + f_y \hat{j}) \cdot (v_x \hat{i} + v_y \hat{j}) = v_x f_x + v_y f_y$ for any (2-variable) function f and unit vector v $(= (v_x, v_y))$.
Second Directional Derivative for f and v $(\psi(f, v))$ is First Directional Derivative of the First Directional Derivative or $\phi(\phi(f, v), v)$, which simplies to:-

$$\begin{aligned}
\phi(\phi(f, v), v) &= v_x \frac{\partial(\phi(f, v))}{\partial x} + v_y \frac{\partial(\phi(f, v))}{\partial y} \\
&= v_x \frac{\partial(v_x f_x + v_y f_y)}{\partial x} + v_y \frac{\partial(v_x f_x + v_y f_y)}{\partial y} \\
&= v_x \left(v_x \frac{\partial f_x}{\partial x} + v_y \frac{\partial f_y}{\partial x} \right) + v_y \left(v_x \frac{\partial f_x}{\partial y} + v_y \frac{\partial f_y}{\partial y} \right) \\
&= v_x^2 f_{xx} + v_x v_y f_{yx} + v_y v_x f_{xy} + v_y^2 f_{yy} \\
&= v_x^2 f_{xx} + 2 v_x v_y f_{xy} + v_y^2 f_{yy}
\end{aligned}$$

Now, $f = I$ and $v = \left(\frac{I_x}{\sqrt{I_x^2 + I_y^2}}, \frac{I_y}{\sqrt{I_x^2 + I_y^2}} \right)$, the Second Directional Derivative becomes:-

$$\begin{aligned}
&= \left(\frac{I_x}{\sqrt{I_x^2 + I_y^2}} \right)^2 I_{xx} + 2 \left(\frac{I_x}{\sqrt{I_x^2 + I_y^2}} \right) \left(\frac{I_y}{\sqrt{I_x^2 + I_y^2}} \right) I_{xy} + \left(\frac{I_y}{\sqrt{I_x^2 + I_y^2}} \right)^2 I_{yy} \\
&= \frac{I_x^2 I_{xx} + 2 I_x I_y I_{xy} + I_y^2 I_{yy}}{I_x^2 + I_y^2}
\end{aligned}$$

Hence proved.

Finding Second Directional Derivative $\perp$ to Gradient's Direction

Unit vector perpendicular to $v = (v_x, v_y)$ is $(-v_y, v_x)$ (call it u).
See that $\psi(f, v) + \psi(f, v') = f_{xx} + f_{yy}$ as shown below:-

$$\begin{aligned}
\psi(f, v) + \psi(f, v') &= \{v_x^2 f_{xx} + 2 v_x v_y f_{xy} + v_y^2 f_{yy}\} + \{u_x^2 f_{xx} + 2 u_x u_y f_{xy} + u_y^2 f_{yy}\} \\
&= \{v_x^2 f_{xx} + 2 v_x v_y f_{xy} + v_y^2 f_{yy}\} + \{(-v_y)^2 f_{xx} + 2(-v_y)(v_x) f_{xy} + (v_x)^2 f_{yy}\} \\
&= f_{xx}(v_x^2 + v_y^2) + f_{yy}(v_y^2 + v_x^2) \\
&= f_{xx} + f_{yy}
\end{aligned}$$

Hence, Second Directional Derivative perpendicular to Gradient's Direction $(\psi(I, v'))$ is:-

$$\begin{aligned}
\psi(I, v') &= I_{xx} + I_{yy} - \psi(I, v) \\
&= I_{xx} + I_{yy} - \frac{I_x^2 I_{xx} + 2 I_x I_y I_{xy} + I_y^2 I_{yy}}{I_x^2 + I_y^2} \\
&= \frac{(I_{xx} + I_{yy})(I_x^2 + I_y^2) - (I_x^2 I_{xx} + 2 I_x I_y I_{xy} + I_y^2 I_{yy})}{I_x^2 + I_y^2} \\
&= \frac{I_{xx} I_x^2 + I_{xx} I_y^2 + I_{yy} I_x^2 + I_{yy} I_y^2 - I_x^2 I_{xx} - 2 I_x I_y I_{xy} - I_y^2 I_{yy}}{I_x^2 + I_y^2} \\
&= \frac{I_{xx} I_y^2 + I_{yy} I_x^2 - 2 I_x I_y I_{xy}}{I_x^2 + I_y^2} \text{ (Ans.)}
\end{aligned}$$