CS663 Assignment-3

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Question 5

Solution

The Discrete Fourier Transform (DFT) of a signal $f(x, y)$ of size $M \times N$ and the Inverse Discrete Fourier Transform (IDFT) are respectively given by

$$F(u, v) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}$$
$$f(x, y) = \frac{1}{\sqrt{MN}} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}$$

Note that formulae given above assume that the function that is being sampled is periodic with period (M, N). Otherwise an infinite number of samples (spanning across all pairs of integers) has to be taken and the Discrete Fourier Transform and it's inverse will be an infinite double summation. Since periodicity is assumed, the sequence (ie samples) $f(x, y)$ wraps around:

$$f(x + k_1 M, y) = f(x, y + l_1 N) = f(x + k_2 M, y + l_2 N) = f(x, y)\, \forall x, y, k_i, l_i \in \mathbb{Z}$$

Let $f(x, y)$ be a real function and $F(u, v)$ be it's DFT. By linearity of the conjugate operator and the fact that $f(t)$ is real, we can show that $F^*(u, v) = F(-u, -v)$

$$\begin{aligned}
F^*(u, v) &= \left(\frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}\right)^* \\
&= \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left(f(x, y) e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}\right)^* \\
&= \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f^*(x, y) \left(e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}\right)^* \\
&= \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)} \\
&= \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{j2\pi\left(\frac{-(-u)x}{M} + \frac{-(-v)y}{N}\right)} \\
&= \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi\left(\frac{(-u)x}{M} + \frac{(-v)y}{N}\right)} \\
&= F(-u, -v) \\
\Rightarrow F^*(u, v) &= F(-u, -v)
\end{aligned}$$

Let $f(x, y)$ be real and even. Since $f(x, y)$ is even we have $f(x, y) = f(-x, -y)$. Also, f is periodic ie, $f(x + M, y +$

$N) = f(x, y)$. Combining these two equations we have,

$$\begin{aligned} f(x,y) &= f(-x,-y) \\ f(-x,-y) &= f(M-x, N-y) \\ \Rightarrow f(x,y) &= f(M-x, N-y) \end{aligned} \tag{1}$$

Also, since f and $e^{-j2\pi((\frac{ux}{M}+\frac{vy}{N}))}$ are both periodic with the same periods, the Discrete Fourier Transform of f can also be written as

$$F(u,v) = \frac{1}{\sqrt{MN}} \sum_{x=1}^{M} \sum_{y=1}^{N} f(x,y) e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)} \tag{2}$$

this is obtained by simply replacing any term with $x = 0$ by $x = M$ and any term with $y = 0$ by $y = N$ in the expression for F.

Using equations1 and2 we will show that $F(-u, -v) = F(u, v)$

$$\begin{aligned} F(-u,-v) &= \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi\left(\frac{(-u)x}{M}+\frac{(-v)y}{N}\right)} \\ &= \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(M-x,N-y) e^{-j2\pi\left(\frac{(-u)x}{M}+\frac{(-v)y}{N}\right)} \text{ (using equation1)} \\ &= \frac{1}{\sqrt{MN}} \sum_{x=1}^{M} \sum_{y=1}^{N} f(x,y) e^{-j2\pi\left(\frac{(-u)(M-x)}{M}+\frac{(-v)(N-y)}{N}\right)} \text{ (replacing } x \text{ by } M-x \text{ and } y \text{ by } N-y) \\ &= \frac{1}{\sqrt{MN}} \sum_{x=1}^{M} \sum_{y=1}^{N} f(x,y) e^{-j2\pi\left(\frac{(-u)(-x)}{M}+\frac{(-v)(-y)}{N}\right)} \text{ (using periodicity of } e^{-j2\pi\left(\frac{(-u)x}{M}+\frac{(-v)y}{N}\right)}) \\ &= \frac{1}{\sqrt{MN}} \sum_{x=1}^{M} \sum_{y=1}^{N} f(x,y) e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)} \text{ (using periodicity of } e^{-j2\pi\left(\frac{(-u)x}{M}+\frac{(-v)y}{N}\right)}) \\ &= F(u,v) \text{ (from equation2)} \\ \Rightarrow F(-u,-v) &= F(u,v) \end{aligned} \tag{3}$$

We already showed that if f is real then it's DFT F satisfies $F^*(u, v) = F(-u, -v)$. So we have $F^*(u, v) = F(-u, -v) = F(u, v)$ therefore $F^*(u, v) = F(u, v)$ ie F is real. Therefore F is both real and even and we are done.