CS663 Assignment-4

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Question 3

Solution

Let A be a real $m \times n$ matrix. A can always be expressed as $A = U\Sigma V^T$ where $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$ and $\Sigma \in \mathbb{R}^{m\times n}$, U, V being orthogonal matrices and Σ being a diagonal matrix with non negative values (called singular values) on the diagonal. Let $\Sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_{\min(m,n)})$. We will show that the squares of the non-zero singular values of A are the eigenvalues of either AA^T or A^TA. This is equivalent to showing that the the non-zero singular values of A are the squareroots of either AA^T or A^TA because, the singular values are non-negative by definition.

Part a

We will show that the non zero singular values of A are equal to the square roots of the eigen values of either AA^T or A^TA. We define $\Sigma_m^2 = \Sigma\Sigma^T$ and $\Sigma_n^2 = \Sigma^T\Sigma$.

$$\begin{aligned}
AA^T &= (U\Sigma V^T)\cdot(U\Sigma V^T)^T \\
&= U\Sigma V^T \cdot (V\Sigma^T U^T) \\
&= U\Sigma V^T V\Sigma^T U^T \\
&= U\Sigma(V^T V)\Sigma^T U^T \\
&= U\Sigma I_n \Sigma^T U^T \\
&= U\Sigma\Sigma^T U^T \\
&= U\Sigma_m^2 U^T
\end{aligned}$$

Similarly, $A^TA = V\Sigma^T\Sigma V^T = V\Sigma_n^2 V^T$.

Clearly, $\Sigma_m^2 \in R^{m\times m}$ and equals $diag(\sigma_1^2, \sigma_2^2, \ldots, \sigma_m^2) = D$. Now $m = \min(m,n)$ or $n = \min(m,n)$.

Let $m = \min(m,n)$. In this case, Σ_m^2 is a diagonal matrix (D) whose entries are squares of the singular values of A. Since AA^T is a real symmetric matrix, by **spectral theorem** it has an orthogonal decomposition given by $AA^T = \mathcal{U}\mathcal{D}\mathcal{U}^T$ where $\mathcal{D}$ is a diagonal matrix whose entries are the eigenvalues of AA^T and $\mathcal{U}$ is an orthogonal matrix. Therefore the non zero entries of D are the eigenvalues of AA^T. Since $D = \Sigma_m^2$, the non zero entries of D are the squares of the non zero singular values of A and the squares of the any non-zero singular value of A is an eigenvalue of AA^T, we are done.

Let $n = \min(m, n)$. In this case, we deal with the diagonal matrix $D = \Sigma_n^2$ whose entries are the squares of all the singular values of A (this is because $n = \min(m, n)$). The proof is very similar to the case above. Since $A^T A$ is a real symmetric matrix, it has an orthogonal decomposition $\mathcal{V}\mathcal{D}\mathcal{V}^T$ and therefore we have the non-zero entries of D to be the eigenvalues of $A^T A$ and following a similar argument we arrive at the conclusion that the square of any non-zero singular value of A is an eigen value of $A^T A$ and that any eigenvalue of $A^T A$ is the square of some non-zero singular value of A. This completes the proof.

Part b

The Frobenius norm of a matrix $A \in \mathbb{R}^{m \times n}$ is defined as

$$||A||_F = \sqrt{\left(\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}^2\right)}$$

For any matrix $A \in \mathbb{R}^{m \times n}$ we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}^2 = \text{Tr}(AA^T) = \text{Tr}(A^T A)$$

where $\text{Tr}(M) = \sum_{i=1}^{n} M_{ii}$ for any $n \times n$ square matrix M. The trace of a matrix also has the following property,

$$\text{Tr}(AB) = \text{Tr}(BA)$$

whenever AB and BA are both square matrices for two matrices A and B.

WLOG we take $m = \min(n, m)$ (in the other case we simply deal with $A^T A$ and Σ_n^2)

$$\begin{aligned}
||A||_F^2 &= \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}^2 \\
&= \text{Tr}(AA^T) \\
&= \text{Tr}((U\Sigma V^T V \Sigma^T U^T)) \\
&= \text{Tr}(U\Sigma_m^2 U^T) \\
&= \text{Tr}((U\Sigma_m^2)(U^T)) \\
&= \text{Tr}((U^T)(U\Sigma_m^2)) \\
&= (Tr)(U^T U \Sigma_m^2) \\
&= (Tr)(I_m \Sigma_m^2) \\
&= (Tr)(\Sigma_m^2) \\
&= \sum_{i=1}^{k} \sigma_i^2 \text{ (where k is the number of non-zero singular values of A)} \\
\Rightarrow ||A||_F^2 &= \sum_{i=1}^{k} \sigma_i^2
\end{aligned}$$