CS663 Assignment-3

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Question 5

Solution

The Discrete Fourier Transform (DFT) of a signal $f(x, y)$ of size $M \times N$ and the Inverse Discrete Fourier Transform (IDFT) are respectively given by

$$F(u, v) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}$$
$$f(x, y) = \frac{1}{\sqrt{MN}} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}$$

Let $f(x, y)$ be a real function and $F(u, v)$ be it's DFT. By linearity of the conjugate operator and the fact that $f(t)$ is real, we can show that $F^*(u, v) = F(-u, -v)$

$$\begin{aligned}
F^*(u, v) &= \left(\frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}\right)^* \\
&= \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left(f(x, y) e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}\right)^* \\
&= \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f^*(x, y) \left(e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}\right)^* \\
&= \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)} \\
&= \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{j2\pi\left(\frac{-(-u)x}{M} + \frac{-(-v)y}{N}\right)} \\
&= \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi\left(\frac{(-u)x}{M} + \frac{(-v)y}{N}\right)} \\
&= F(-u, -v) \\
\Rightarrow F^*(u, v) &= F(-u, -v)
\end{aligned}$$

Let $f(x, y)$ be real and even. Since $f(x, y)$ is even we have $f(x, y) = f(-x, -y)$. Also, f is periodic ie, $f(x + M, y + N) = f(x, y)$. Combining these two equations we have,

$$\begin{aligned}
f(x, y) &= f(-x, -y) \\
f(-x, -y) &= f(M - x, N - y) \\
\Rightarrow f(x, y) &= f(M - x, N - y) \qquad (1)
\end{aligned}$$

Also, since f and $e^{-j2\pi\left(\left(\frac{ux}{M} + \frac{vy}{N}\right)\right)}$ are both periodic with the same periods, the Discrete Fourier Transform of f

can also be written as

$$F(u,v) = \frac{1}{\sqrt{MN}} \sum_{x=1}^{M} \sum_{y=1}^{N} f(x,y) e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)} \tag{2}$$

this is obtained by simply replacing any term with $x = 0$ by $x = M$ and any term with $y = 0$ by $y = N$ in the expression for F.

Using equations1 and2 we will show that $F(-u,-v) = F(u,v)$

$$\begin{aligned}
F(-u,-v) &= \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi\left(\frac{(-u)x}{M} + \frac{(-v)y}{N}\right)} \\
&= \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(M-x, N-y) e^{-j2\pi\left(\frac{(-u)x}{M} + \frac{(-v)y}{N}\right)} \text{ (using equation1)} \\
&= \frac{1}{\sqrt{MN}} \sum_{x=1}^{M} \sum_{y=1}^{N} f(x,y) e^{-j2\pi\left(\frac{(-u)(M-x)}{M} + \frac{(-v)(N-y)}{N}\right)} \text{ (replacing } x \text{ by } M-x \text{ and } y \text{ by } N-y) \\
&= \frac{1}{\sqrt{MN}} \sum_{x=1}^{M} \sum_{y=1}^{N} f(x,y) e^{-j2\pi\left(\frac{(-u)(-x)}{M} + \frac{(-v)(-y)}{N}\right)} \text{ (using periodicity of } e^{-j2\pi\left(\frac{(-u)x}{M} + \frac{(-v)y}{N}\right)}) \\
&= \frac{1}{\sqrt{MN}} \sum_{x=1}^{M} \sum_{y=1}^{N} f(x,y) e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)} \text{ (using periodicity of } e^{-j2\pi\left(\frac{(-u)x}{M} + \frac{(-v)y}{N}\right)}) \\
&= F(u,v) \text{ (from equation2)} \\
\Rightarrow F(-u,-v) &= F(u,v) \qquad (3)
\end{aligned}$$

We already showed that if f is real then it's DFT F satisfies $F^*(u,v) = F(-u,-v)$. So we have $F^*(u,v) = F(-u,-v) = F(u,v)$ therefore $F^*(u,v) = F(u,v)$ ie F is real. Therefore F is both real and even and we are done.