CS663 Assignment-5

Saksham Rathi, Kavya Gupta, Shravan Srinivasa Raghavan
Department of Computer Science,
Indian Institute of Technology Bombay

Question 5

Part (a)

We prove this via a contradiction. Consider an example where $N = 2$ and both P_1 and P_2 are square matrices.
Let $P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $P_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.
The least square solution gives

$$\begin{aligned} R &= P_1 P_2^T (P_2 P_2^T)^{-1} \\ &= P_1 I^T (II^T)^{-1} \\ &= P_1 \end{aligned}$$

It was given that R is an orthonormal matrix. Clearly P_1 is not orthonormal and hence $R = P_1$ is not a valid solution. Therefore using the least squares method does not give the right answer as it does not always give a solution which is orthonormal.

Part (b)

The objective is to find the orthonormal matrix R that minimises the sqaure of the Frobenius norm of the error matrix E which equals $P_1 - RP_2$.

$$\begin{aligned} & E(R) = ||P_1 - RP_2||_F^2 \\ \Rightarrow\ & E(R) = \text{Trace}((P_1 - RP_2)^T (P_1 - RP_2)) \\ &= \text{Trace}(P_1^T P_1 - (RP_2)^T P_1 - P_1^T (RP_2) + (RP_2)^T (RP_2)) \\ &= \text{Trace}(P_1^T P_1 - P_2^T R^T P_1 - P_1^T RP_2 + P_2^T R^T (RP_2)) \\ &= \text{Trace}(P_1^T P_1 + P_2^T R^T RP_2 - P_2^T R^T P_1 - P_1^T RP_2) \\ &= \text{Trace}(P_1^T P_1 + P_2^T P_2 - P_2^T R_T P_1 - P_1^T RP_2) \ (R^T R = RR^T = I \text{ because } R \text{ is orthonormal}) \\ &= \text{Trace}(P_1^T P_1 + P_2^T P_2) - \text{Trace}(P_2^T R^T P_1) - \text{Trace}(P_1^T RP_2) \ (\text{Because Trace is a linear operator}) \\ &= \text{Trace}(P_1^T P_1 + P_2^T P_2) - \text{Trace}((P_2^T R^T P_1)^T) - \text{Trace}(P_1^T RP_2) \ (\text{Trace}(A) = \text{Trace}(A^T) \text{ for any square matrix } A) \\ &= \text{Trace}(P_1^T P_1 + P_2^T P_2) - \text{Trace}(P_1^T RP_2) - \text{Trace}(P_1^T RP_2) \\ &= \text{Trace}(P_1^T P_1 + P_2^T P_2) - 2\text{Trace}(P_1^T RP_2) \end{aligned}$$

Part (c)

As shown in the previous part $E(R) = \text{Trace}(P_1^T P_1 + P_2^T P_2) - 2\text{Trace}(P_1^T RP_2)$. Clearly for given P_1 and P_2, the first term is fixed. Therefore to minimise $E(R)$ wrt R, the term $-2\text{Trace}(P_1^T RP_2)$ should be minimised ie, $\text{Trace}(P_1^T RP_2)$ should be maximised wrt R.

Part (d)

Now the goal is to maximise Trace$(P_1^T R P_2)$ wrt R. We will first show that for any two matrices $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times m}$, Trace$(AB) =$ Trace(BA).

$$\text{Trace}(AB) = \sum_{i=1}^{m} C_{ii} \;\; (\text{where } C = AB)$$

$$\Rightarrow \text{Trace}(AB) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} A_{ij} B_{ji} \right) \tag{1}$$

Similarly,

$$\text{Trace}(BA) = \sum_{j=1}^{n} D_{jj} \;\; (\text{where } D = BA)$$

$$\Rightarrow \text{Trace}(BA) = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} B_{ji} A_{ij} \right) \tag{2}$$

Clearly, the two double summations in the RHS of equation 1 and equaion 2 are equal and can be obtained by simply reordering the summations. Therefore we have

$$\text{Trace}(AB) = \text{Trace}(BA) \tag{3}$$

We will also use the fact that the SVD of a square matrix $M = USV^T$ exists with U, V being orthonormal matrices and S being a diagonal matrix.

$$\begin{aligned}
\text{Trace}(P_1^T R P_2) &= \text{Trace}(R P_2 P_1^T) \;\; (\text{from equation 3}) \\
&= \text{Trace}(R U' S' V'^T) \;\; (\text{where } P_2 P_1^T = U' S' V'^T \text{ is the SVD of } P_2 P_1^T) \\
&= \text{Trace}(S' V'^T R U') \;\; (\text{again using the result from equation 3}) \\
&= \text{Trace}(S' X) \;\; (\text{where } X = V'^T R U' \text{ and } S' \text{ is a diagonal matrix})
\end{aligned}$$

Part (e)

Since maximising Trace$(P_1^T R P_2)$ is the same as maximising Trace$(S'X)$ where S' is a diagonal matrix and $X = V'^T R U'$, we only need to determine the matrix X such that $S'_{11} X_{11} + S'_{22} X_{22}$ is maximised where S'_{11} and S'_{22} are the singular values of $P_2 P_1^T$.
We will now show that X is also orthonormal.

$$\begin{aligned}
XX^T &= (V'^T R U') \cdot (V'^T R U')^T \\
&= (V'^T R U') \cdot (U'^T R^T V') \\
&= V'^T R (U' \cdot U'^T) R^T V' \\
&= V'^T R I R^T V', \;\; (\text{since } U, V \text{ are orthonormal}) \\
&= V'^T R R^T V' \\
&= V'^T I V' \;\; (\text{since } R \text{ is orthonormal}) \\
&= V'^T V' \\
&= I \\
\Rightarrow XX^T &= X^T X = I \\
\Rightarrow X_{11}X_{11} + X_{12}X_{12} &= 1 \,\&\, X_{11}X_{21} + X_{12}X_{22} = 0 \\
\Rightarrow X_{21}X_{11} + X_{22}X_{12} &= 0 \,\&\, X_{21}X_{21} + X_{22}X_{22} = 1 \\
\Rightarrow X_{11}^2 + X_{12}^2 &= 1 \\
\Rightarrow X_{21}^2 + X_{22}^2 &= 1 \\
\Rightarrow X_{11}X_{21} + X_{12}X_{22} &= 0
\end{aligned}$$

Since all entries of X are real numbers, $0 \leqslant X_{ij} \leqslant 1 \; \forall i, j \in [1,2]$. We need to maximise $S'_{11}X_{11} + S'_{22}X_{22}$ and since S'_{11} and S'_{22} are positive values, we only need to maximise the values of X_{11} and X_{22}. Since $X_{11} \leqslant 1$ and $X_{22} \leqslant 1$ the maximum values X_{11} and X_{22} can take are 1. Hence,

$$\text{Trace}(S'X) = X_{11}S'_{11} + X_{22}S'_{22} \leqslant ((1)S'_{11} + (1)S'_{22}) = S'_{11} + S'_{22}$$

Therefore the matrix X is,

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

since $x_{11}^2 + X_{12}^2 = 1$ and $X_{11} = 1 \Rightarrow X_{12} = 0$, similarly $X_{21} = 0$. Hence we get $X = I_2$ for maximum value of Trace$S'X$

Part (f)

Since $X = I_2$ and $X = V'^T R U'$, we get

$$\begin{aligned} V'^T R U' &= I \\ \Rightarrow V' V'^T R U' U'^T &= V' I U'^T \\ \Rightarrow IRI &= V' U'^T \text{ (since } V' \text{ and } U' \text{ are orthogonal matrices)} \\ \Rightarrow R &= V' U'^T \end{aligned}$$

If X is the identity matrix $R = V'U'^T$.

Part (g)

If R is also a rotation matrix then $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for some $\theta \in \mathbb{R}$. This gives $\det(R) = \cos^2\theta + \sin^2\theta = 1$. However if R is not a rotation matrix then $\det(R) = 1, -1$. So the additional constraint imposed is $\det(R) = 1$.