CS663 Assignment-5

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Question 1

Part (a)

Procedure:-

Let the translation operation for a (x_0, y_0) displacement on an image f_1 be defined as $f_2(x, y) = f_1(x - x_0, y - y_0)$. Given images f_1, f_2 have size $N \times N$, their (Discrete) Fourier Transforms are related as $F_2(\mu, \nu) = F_1(\mu, \nu)e^{-j2\pi(\mu x_0 + \nu y_0)/N}$ (due to 2D-DFT Translation property).
As per Equation (3) of Section I of the paper, cross spectrum of f_1 and f_2 is:-

$$C(\mu, \nu) = \frac{F_1(\mu, \nu)\, F_2^*(\mu, \nu)}{|F_1(\mu, \nu)||F_2(\mu, \nu)|} = e^{j2\pi(\mu x_0 + \nu y_0)/N} \tag{1}$$

Proof for Equation 1 above:-
Given $F_2(\mu, \nu) = F_1(\mu, \nu)e^{-j2\pi(\mu x_0 + \nu y_0)/N}$, we can write $F_2^*(\mu, \nu) = F_1^*(\mu, \nu)e^{j2\pi(\mu x_0 + \nu y_0)/N}$.

$$\begin{aligned}
\text{Now } |F_2(\mu, \nu)|^2 &= F_2^*(\mu, \nu).F_2(\mu, \nu) \\
&= F_1(\mu, \nu)e^{-j2\pi(\mu x_0 + \nu y_0)/N}.F_1^*(\mu, \nu)e^{j2\pi(\mu x_0 + \nu y_0)/N} \\
&= F_1(\mu, \nu).F_1^*(\mu, \nu) = |F_1(\mu, \nu)|^2
\end{aligned}$$

Hence $|F_2(\mu, \nu)| = |F_1(\mu, \nu)|$. Substituting these values in Equation 1, we get:-

$$C(\mu, \nu) = \frac{F_1(\mu, \nu)\, F_1^*(\mu, \nu)e^{j2\pi(\mu x_0 + \nu y_0)/N}}{|F_1(\mu, \nu)|^2} = e^{j2\pi(\mu x_0 + \nu y_0)/N}$$

Let's take the Inverse (Discrete) Fourier Transform of $C(\mu, \nu)$ to get the autocorrelation function $c(x, y)$:-

$$\begin{aligned}
c(x, y) &= \mathcal{F}^{-1}\{C(\mu, \nu)\} = \mathcal{F}^{-1}\{e^{j2\pi(\mu x_0 + \nu y_0)/N}\} \\
&= \frac{1}{N}\sum_{\mu=0}^{N-1}\sum_{\nu=0}^{N-1} e^{j2\pi(\mu x_0 + \nu y_0)/N} e^{j2\pi(\mu x + \nu y)/N} = \frac{1}{N}\sum_{\mu=0}^{N-1}\sum_{\nu=0}^{N-1} e^{j2\pi(\mu(x_0 + x) + \nu(y_0 + y))/N} \\
&= \delta(x_0 + x, y_0 + y)
\end{aligned}$$

See that $c(x, y)$ attains a non-zero value only at $(-x_0, -y_0)$, which is the displacement between the two images. Hence, the autocorrelation function of the cross spectrum is a delta function at the displacement $(-x_0, -y_0)$, hence helping us find value of (x_0, y_0).
Hence the steps are:-

1. Obtain 2D-DFT of f_1 and f_2. f_2's can be found using the Fourier Shift Theorem.
2. Compute the cross spectrum $C(\mu, \nu)$ using Equation 1.
3. Determine Inverse DFT of $C(\mu, \nu)$ and find its non-zero point. If it is at (a, b), then the displacement vector is $(-a, -b)$.

Time Complexity:-

2D-DFT (IDFT) of an image of size $M \times N$ can be found using FFT efficiently in $O(MN \log MN)$ time. Hence for a $N \times N$ image, 2D-DFT (IDFT) takes $O(NN \log NN) = O(N^2 \log N)$ time.

- Step 1 is 2D-DFT hence takes $O(N^2 \log N)$ time.
- Step 2 is a pointwise multiplication and division, hence takes $O(N^2)$ time.
- Step 3 is 2D-IDFT, hence takes $O(N^2 \log N)$ time.

Hence the total time complexity is $O(N^2 \log N)$.

Comparison with Pixel-wise image comparison procedure:-

Say there are W_x values of x-axis translation shift to be checked and so is W_y for y-axis. For each translation shift vector, we do this:-

1. Create a new image by translating f_1 by (x_0, y_0). This takes $O(N^2)$ time.
2. Compare each pixel of the new image with f_2's pixels (or take min squares difference). This takes $O(N^2)$ time.

Hence the total time complexity is $O(W_x W_y N^2)$. This is much larger than $O(N^2 \log N)$ for $W_x, W_y \gg \log N$.

Part (b)

As said in Section II of the paper, $f_2(x, y) = f_1(x \cos\theta_0 + y \sin\theta_0 - x_0,\ -x \sin\theta_0 + y \cos\theta_0 - y_0)$ (f_2 is rotated-then-translated version of f_1). We have to find rotation θ_0 and shift (x_0, y_0) values. If $F_1(\mu, \nu)$ is the 2D-DFT of f_1 and so is $F_2(\mu, \nu)$ of f_2, then:-

$$\begin{aligned}
F_2(\mu, \nu) &= \mathbf{F}[f_1(x \cos\theta_0 + y \sin\theta_0 - x_0,\ -x \sin\theta_0 + y \cos\theta_0 - y_0)] \\
&= e^{-j2\pi(\mu x_0 + \nu y_0)/N} \mathbf{F}[f_1(x \cos\theta_0 + y \sin\theta_0,\ -x \sin\theta_0 + y \cos\theta_0)] \text{ (Fourier Shift Theorem)} \\
&= e^{-j2\pi(\mu x_0 + \nu y_0)/N} F_1(\mu \cos\theta_0 + \nu \sin\theta_0,\ -\mu \sin\theta_0 + \nu \cos\theta_0) \text{ (Rotation property)}
\end{aligned}$$

Note that the Rotation property is mentioned in the lecture slides and also in Gonzales book (pg. 241). Hence not proved here.
Observe that magnitudes of the FTs are equal (Say M_1, M_2 are magnitudes of F_1, F_2):-

$$\begin{aligned}
M_2(\mu, \nu) = |F_2(\mu, \nu)| &= |e^{-j2\pi(\mu x_0 + \nu y_0)/N} F_1(\mu \cos\theta_0 + \nu \sin\theta_0,\ -\mu \sin\theta_0 + \nu \cos\theta_0)| \\
&= |F_1(\mu \cos\theta_0 + \nu \sin\theta_0,\ -\mu \sin\theta_0 + \nu \cos\theta_0)| \\
&= M_1(\mu \cos\theta_0 + \nu \sin\theta_0,\ -\mu \sin\theta_0 + \nu \cos\theta_0)
\end{aligned}$$

Convert to polar coordinate system: $(\mu, \nu) \to (\rho, \theta)$, $\rho = \sqrt{\mu^2 + \nu^2}$, $\theta = \tan^{-1}(\nu/\mu)$, $\mu =$

$\rho \cos\theta$, $\nu = \rho \sin\theta$, so:-

$$\mu \cos\theta_0 + \nu \sin\theta_0 = \rho \cos\theta \cos\theta_0 + \rho \sin\theta \sin\theta_0 = \rho(\cos\theta \cos\theta_0 + \sin\theta \sin\theta_0) = \rho \cos(\theta - \theta_0)$$
$$\nu \cos\theta_0 - \mu \sin\theta_0 = \rho \sin\theta \cos\theta_0 - \rho \cos\theta \sin\theta_0 = \rho(\sin\theta \cos\theta_0 - \cos\theta \sin\theta_0) = \rho \sin(\theta - \theta_0)$$

Hence, $M_1(\mu \cos\theta_0 + \nu \sin\theta_0, -\mu \sin\theta_0 + \nu \cos\theta_0) = M_1(\rho, \theta - \theta_0)$.
Then $M_2(\rho, \theta) = M_1(\rho, \theta - \theta_0)$. Hence this becomes like the problem in (a) part (but in 1D). Using the method in (a) part, we can find θ_0.
After the rotation angle θ_0 is found, rotate f_1 by θ_0 to get f_3. Now f_3 is related to f_2 by pure translation. Use the method in (a) part to find the translation vector (x_0, y_0).
Hence the steps are:-

1. Find 2D-DFT of f_1 and f_2.
2. Find the magnitudes of the DFTs and convert to polar coordinates.
3. Find the rotation angle θ_0 using the method in (a) part (just in 1D).
4. Rotate f_1 by θ_0 to get f_3.
5. Find the translation vector (x_0, y_0) using the method in (a) part on f_3 and f_2.