CS663 Assignment-3

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Question 7

Solution

Lemma: $\mathbf{F}\left[\frac{\partial I_{(x,y)}}{\partial t}\right]_{(\mu,\nu)} = \frac{\partial}{\partial t}(\mathbf{F}[I_{(x,y)}]_{(\mu,\nu)})$

Proof: Let's see LHS...

$$\mathbf{F}\left[\frac{\partial I_{(x,y)}}{\partial t}\right]_{(\mu,\nu)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial I_{(x,y)}}{\partial t} e^{-2\pi j(\mu x+\nu y)} dx\, dy$$

Let's see RHS...

$$\begin{aligned}
\frac{\partial}{\partial t}(\mathbf{F}[I_{(x,y)}]_{(\mu,\nu)}) &= \frac{\partial}{\partial t}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_{(x,y)} e^{-2\pi j(\mu x+\nu y)} dx\, dy\right) \\
&= \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left(\int_{-\infty}^{\infty} I_{(x,y)} e^{-2\pi j(\mu x+\nu y)} dx\right) dy && \text{(Using Newton-Leibniz Formula)} \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left(I_{(x,y)} e^{-2\pi j(\mu x+\nu y)}\right) dx\right) dy && \text{(Using Newton-Leibniz Formula)} \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \frac{\partial I_{(x,y)}}{\partial t} e^{-2\pi j(\mu x+\nu y)} dx\right) dy && \text{(As the exp term is independent from t)}
\end{aligned}$$

LHS = RHS. Hence proved.

Lemma: $\mathbf{F}\left[\frac{\partial^2 I}{\partial x^2}\right]_{(\mu,\nu)} = -4\pi^2\mu^2\mathbf{F}[I]_{(\mu,\nu)}$ and $\mathbf{F}\left[\frac{\partial^2 I}{\partial y^2}\right]_{(\mu,\nu)} = -4\pi^2\nu^2\mathbf{F}[I]_{(\mu,\nu)}$

Proof: Let's see the first one. See LHS...

$$\begin{aligned}
\mathbf{F}\left[\frac{\partial^2 I}{\partial x^2}\right]_{(\mu,\nu)} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial^2 I}{\partial x^2} e^{-2\pi j(\mu x+\nu y)} dx\, dy \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \frac{\partial^2 I}{\partial x^2} e^{-2\pi j\mu x} dx\right) e^{-2\pi j\nu y} dy \\
&= \int_{-\infty}^{\infty}\left(\frac{\partial I}{\partial x} e^{-2\pi j\mu x}\Big|_{-\infty}^{\infty} + 2\pi j\mu \int_{-\infty}^{\infty} \frac{\partial I}{\partial x} e^{-2\pi j\mu x} dx\right) e^{-2\pi j\nu y} dy && \text{(Integration by parts)} \\
&= \int_{-\infty}^{\infty}\left(0 + 2\pi j\mu\left(I e^{-2\pi j\mu x}\big|_{-\infty}^{\infty} + 2\pi j\mu \int_{-\infty}^{\infty} I e^{-2\pi j\mu x}\right) dx\right) e^{-2\pi j\nu y} dy && \text{(Integration by parts)} \\
&= \int_{-\infty}^{\infty}\left(0 + 2\pi j\mu\left(0 + 2\pi j\mu \int_{-\infty}^{\infty} I e^{-2\pi j\mu x}\right) dx\right) e^{-2\pi j\nu y} dy \\
&= (2\pi j\mu)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I e^{-2\pi j(\mu x+\nu y)} dx\, dy \\
&= -4\pi^2\mu^2\mathbf{F}[I]_{(\mu,\nu)}
\end{aligned}$$

LHS = RHS. Hence proved. Similarly the second one can be proved.

Apply Fourier Transform on the given PDE on both sides:-

$$\mathbf{F}\left[\frac{\partial I}{\partial t}\right]_{(\mu,\nu)} = \mathbf{F}\left[c\left(\frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2}\right)\right]_{(\mu,\nu)}$$
$$\implies \mathbf{F}\left[\frac{\partial I}{\partial t}\right]_{(\mu,\nu)} = c\left(\mathbf{F}\left[\frac{\partial^2 I}{\partial x^2}\right]_{(\mu,\nu)} + \mathbf{F}\left[\frac{\partial^2 I}{\partial y^2}\right]_{(\mu,\nu)}\right)$$

Apply the above lemmas:-

$$\begin{aligned}\frac{\partial}{\partial t}(\mathbf{F}[I]_{(\mu,\nu)}) &= c\left(\mathbf{F}\left[\frac{\partial^2 I}{\partial x^2}\right]_{(\mu,\nu)} + \mathbf{F}\left[\frac{\partial^2 I}{\partial y^2}\right]_{(\mu,\nu)}\right)\\ &= c\left(-4\pi^2\mu^2\mathbf{F}[I]_{(\mu,\nu)} - 4\pi^2\nu^2\mathbf{F}[I]_{(\mu,\nu)}\right)\\ &= -c4\pi^2(\mu^2+\nu^2)\mathbf{F}[I]_{(\mu,\nu)}\end{aligned}$$

Let's call $\mathbf{F}[I]_{(\mu,\nu)}$ as $F_{(\mu,\nu)}$, so our partial differential equation becomes:-

$$\frac{\partial F_{(\mu,\nu)}}{\partial t} = -c4\pi^2(\mu^2+\nu^2)F_{(\mu,\nu)}$$

Let's solve it...

$$\begin{aligned}\frac{\partial F_{(\mu,\nu)}}{F_{(\mu,\nu)}} &= -c4\pi^2(\mu^2+\nu^2)\,\partial t\\ \implies \int_0^{t_0}\frac{\partial F_{(\mu,\nu)}}{F_{(\mu,\nu)}} &= -c4\pi^2(\mu^2+\nu^2)\int_0^{t_0}\partial t\\ \implies \ln\left(\frac{F_{(\mu,\nu)(t_0)}}{F_{(\mu,\nu)(0)}}\right) &= -c4\pi^2(\mu^2+\nu^2)t_0\end{aligned}$$

Where $F_{(\mu,\nu)(t_0)}$ is the Fourier Transform of Image I at time $t = t_0$, hence taking exponent both sides...

$$F_{(\mu,\nu)(t)} = e^{-c4\pi^2(\mu^2+\nu^2)t}F_{(\mu,\nu)(0)}$$

Now take Inverse Fourier both sides...

$$\mathbf{F}^{-1}[F_{(\mu,\nu)(t)}]_{(x,y)} = \mathbf{F}^{-1}[e^{-c4\pi^2(\mu^2+\nu^2)t}F_{(\mu,\nu)(0)}]_{(x,y)}$$

Point-wise multiplication becomes convolution of the inverse fourier tranforms...

$$\begin{aligned}\mathbf{F}^{-1}[F_{(\mu,\nu)(t)}]_{(x,y)} &= \mathbf{F}^{-1}[e^{-c4\pi^2(\mu^2+\nu^2)t}]_{(x,y)} * \mathbf{F}^{-1}[F_{(\mu,\nu)(0)}]_{(x,y)}\\ \implies I_{(x,y)(t)} &= \mathbf{F}^{-1}[e^{-c4\pi^2(\mu^2+\nu^2)t}]_{(x,y)} * I_{(x,y)(0)}\end{aligned}$$

Where $I_{(x,y)(t)}$ shows the intensity at point (x, y) in Image I at time t. Now the first term of RHS is inverse fourier transform of a Gaussian which is a Gaussian! Hence the intensity obtained by running the PDE till time t can be achieved by convolution of the original intensity with a Gaussian.

Lemma: $\mathbf{F}^{-1}[e^{-c(\mu^2+\nu^2)}]_{(x,y)} = \frac{\pi}{c}e^{-\frac{\pi^2(x^2+y^2)}{c}}$
Proof: See LHS...

$$\begin{aligned}\mathbf{F}^{-1}[e^{-c(\mu^2+\nu^2)}]_{(x,y)} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-c(\mu^2+\nu^2)}e^{2\pi j(\mu x+\nu y)}d\mu\, d\nu\\ &= \left(\int_{-\infty}^{\infty} e^{-c\mu^2}e^{2j\pi\mu x}d\mu\right)\left(\int_{-\infty}^{\infty} e^{-c\nu^2}e^{2j\pi\nu y}d\nu\right)\end{aligned}$$

Consider $\int_{-\infty}^{\infty} e^{-c\mu^2}e^{2j\pi\mu x}d\mu$. It is $= \int_{-\infty}^{\infty} e^{-c\mu^2}\cos(2\pi\mu x)d\mu + j\int_{-\infty}^{\infty} e^{-c\mu^2}\sin(2\pi\mu x)d\mu$. The right integral is odd hence will become 0. The left one equals $\sqrt{\frac{\pi}{c}}e^{-\frac{\pi^2x^2}{c}}$. You can check Abramowitz and Stegun (1972, p. 302, equation 7.4.6) for this definite integral solution. Hence our above equation becomes:-

$$\mathbf{F}^{-1}[e^{-c(\mu^2+\nu^2)}]_{(x,y)} = \left(\sqrt{\frac{\pi}{c}}e^{-\frac{\pi^2x^2}{c}}\right)\left(\sqrt{\frac{\pi}{c}}e^{-\frac{\pi^2y^2}{c}}\right) = \frac{\pi}{c}e^{-\frac{\pi^2(x^2+y^2)}{c}}$$

Hence proved.

Using above lemma:-

$$\mathbf{F}^{-1}[e^{-c4\pi^2(\mu^2+\nu^2)t}]_{(x,y)} = \frac{\pi}{c4\pi^2 t}e^{-\frac{\pi^2(x^2+y^2)}{c4\pi^2 t}} = \frac{1}{4\pi ct}e^{-\frac{(x^2+y^2)}{4ct}}$$

Comparing with $e^{-\frac{x^2+y^2}{2\sigma^2}}$, we can see that $\sigma^2 = 2ct$. Hence standard deviation is $\sqrt{2ct}$.
Also from the above lemma, we see that the Gaussian will have zero mean as well.