CS663 Assignment-5

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Question 1

Part (a)

Procedure:-

Let the translation operation for a (x_0, y_0) displacement on an image f_1 be defined as $f_2(x, y) = f_1(x - x_0, y - y_0)$. Given images f_1, f_2 have size $N \times N$, their (Discrete) Fourier Transforms are related as $F_2(\mu, \nu) = F_1(\mu, \nu) e^{-j2\pi(\mu x_0 + \nu y_0)/N}$ (due to 2D-DFT Translation property).
As per Equation (3) of Section I of the paper, cross spectrum of f_1 and f_2 is:-

$$C(\mu, \nu) = \frac{F_1(\mu, \nu)\, F_2^*(\mu, \nu)}{|F_1(\mu, \nu)||F_2(\mu, \nu)|} = e^{j2\pi(\mu x_0 + \nu y_0)/N} \tag{1}$$

Proof for Equation 1 above:-
Given $F_2(\mu, \nu) = F_1(\mu, \nu) e^{-j2\pi(\mu x_0 + \nu y_0)/N}$, we can write $F_2^*(\mu, \nu) = F_1^*(\mu, \nu) e^{j2\pi(\mu x_0 + \nu y_0)/N}$.

$$\begin{aligned}
\text{Now } |F_2(\mu, \nu)|^2 &= F_2^*(\mu, \nu).F_2(\mu, \nu) \\
&= F_1(\mu, \nu) e^{-j2\pi(\mu x_0 + \nu y_0)/N}.F_1^*(\mu, \nu) e^{j2\pi(\mu x_0 + \nu y_0)/N} \\
&= F_1(\mu, \nu).F_1^*(\mu, \nu) = |F_1(\mu, \nu)|^2
\end{aligned}$$

Hence $|F_2(\mu, \nu)| = |F_1(\mu, \nu)|$. Substituting these values in Equation 1, we get:-

$$C(\mu, \nu) = \frac{F_1(\mu, \nu)\, F_1^*(\mu, \nu) e^{j2\pi(\mu x_0 + \nu y_0)/N}}{|F_1(\mu, \nu)|^2} = e^{j2\pi(\mu x_0 + \nu y_0)/N}$$

Let's take the Inverse (Discrete) Fourier Transform of $C(\mu, \nu)$ to get the autocorrelation function $c(x, y)$:-

$$\begin{aligned}
c(x, y) &= \mathcal{F}^{-1}\{C(\mu, \nu)\} = \mathcal{F}^{-1}\{e^{j2\pi(\mu x_0 + \nu y_0)/N}\} \\
&= \frac{1}{N} \sum_{\mu=0}^{N-1} \sum_{\nu=0}^{N-1} e^{j2\pi(\mu x_0 + \nu y_0)/N} e^{j2\pi(\mu x + \nu y)/N} = \frac{1}{N} \sum_{\mu=0}^{N-1} \sum_{\nu=0}^{N-1} e^{j2\pi(\mu(x_0 + x) + \nu(y_0 + y))/N} \\
&= \delta(x_0 + x, y_0 + y)
\end{aligned}$$

See that $c(x, y)$ attains a non-zero value only at $(-x_0, -y_0)$, which is the displacement between the two images. Hence, the autocorrelation function of the cross spectrum is a delta function at the displacement $(-x_0, -y_0)$, hence helping us find value of (x_0, y_0).
Hence the steps are:-

1. Obtain 2D-DFT of f_1 and f_2. f_2's can be found using the Fourier Shift Theorem.
2. Compute the cross spectrum $C(\mu, \nu)$ using Equation 1.
3. Determine Inverse DFT of $C(\mu, \nu)$ and find its non-zero point. If it is at (a, b), then the displacement vector is $(-a, -b)$.

Time Complexity:-

2D-DFT (IDFT) of an image of size $M \times N$ can be found using FFT efficiently in $O(MN \log MN)$ time. Hence for a $N \times N$ image, 2D-DFT (IDFT) takes $O(NN \log NN) = O(N^2 \log N)$ time.

- Step 1 is 2D-DFT hence takes $O(N^2 \log N)$ time.
- Step 2 is a pointwise multiplication and division, hence takes $O(N^2)$ time.
- Step 3 is 2D-IDFT, hence takes $O(N^2 \log N)$ time.

Hence the total time complexity is $O(N^2 \log N)$.

Comparison with Pixel-wise image comparison procedure:-

Say there are W_x values of x-axis translation shift to be checked and so is W_y for y-axis. For each translation shift vector, we do this:-

1. Create a new image by translating f_1 by (x_0, y_0). This takes $O(N^2)$ time.
2. Compare each pixel of the new image with f_2's pixels (or take min squares difference). This takes $O(N^2)$ time.

Hence the total time complexity is $O(W_x W_y N^2)$. This is much larger than $O(N^2 \log N)$ for $W_x, W_y \gg \log N$.

Part (b)

As said in Section II of the paper, $f_2(x, y) = f_1(x \cos\theta_0 + y \sin\theta_0 - x_0, -x \sin\theta_0 + y \cos\theta_0 - y_0)$ (f_2 is rotated-then-translated version of f_1). We have to find rotation θ_0 and shift (x_0, y_0) values.

Lemma: If 2D-FT of f is $F(\mu, \nu)$, then 2D-FT of $f(x \cos\theta_0 + y \sin\theta_0, -x \sin\theta_0 + y \cos\theta_0)$ is $F(\mu \cos\theta_0 + \nu \sin\theta_0, -\mu \sin\theta_0 + \nu \cos\theta_0)$.
Proof:

$$F'(\mu, \nu) = \int_{\infty}^{\infty} \int_{\infty}^{\infty} f(x \cos\theta_0 + y \sin\theta_0, -x \sin\theta_0 + y \cos\theta_0) e^{-j2\pi(\mu x + \nu y)} \, dx \, dy$$

Substitute $x' = x \cos\theta_0 + y \sin\theta_0$ and $y' = -x \sin\theta_0 + y \cos\theta_0$. We will get $x = x' \cos\theta_0 - y' \sin\theta_0$ and $y = x' \sin\theta_0 + y' \cos\theta_0$. Also,

$$dx \, dy = |J| \, dx' \, dy' = \begin{vmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} \\ \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} \end{vmatrix} dx' \, dy' = \begin{vmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{vmatrix} dx' \, dy' = dx' \, dy'$$

Thus,

$$\begin{aligned} F'(\mu, \nu) &= \int_{\infty}^{\infty} \int_{\infty}^{\infty} f(x', y') e^{-j2\pi(\mu(x' \cos\theta_0 - y' \sin\theta_0) + \nu(x' \sin\theta_0 + y' \cos\theta_0))} \, dx' \, dy' \\ &= F(\mu \cos\theta_0 + \nu \sin\theta_0, -\mu \sin\theta_0 + \nu \cos\theta_0) \end{aligned}$$

Hence proved.

If $F_1(\mu, \nu)$ is the 2D-FT of f_1 and so is $F_2(\mu, \nu)$ of f_2, then:-

$$\begin{aligned} F_2(\mu,\nu) &= \mathbf{F}[f_1(x\cos\theta_0 + y\sin\theta_0 - x_0,\ -x\sin\theta_0 + y\cos\theta_0 - y_0)] \\ &= e^{-j2\pi(\mu x_0+\nu y_0)}\mathbf{F}[f_1(x\cos\theta_0 + y\sin\theta_0,\ -x\sin\theta_0 + y\cos\theta_0)] \quad \text{(Fourier Shift Theorem)} \\ &= e^{-j2\pi(\mu x_0+\nu y_0)}F_1(\mu\cos\theta_0 + \nu\sin\theta_0,\ -\mu\sin\theta_0 + \nu\cos\theta_0) \quad \text{(Lemma)} \end{aligned}$$

Observe that magnitudes of the FTs are equal (Say M_1, M_2 are magnitudes of F_1, F_2):-

$$\begin{aligned} M_2(\mu,\nu) = |F_2(\mu,\nu)| &= |e^{-j2\pi(\mu x_0+\nu y_0)}F_1(\mu\cos\theta_0 + \nu\sin\theta_0,\ -\mu\sin\theta_0 + \nu\cos\theta_0)| \\ &= |F_1(\mu\cos\theta_0 + \nu\sin\theta_0,\ -\mu\sin\theta_0 + \nu\cos\theta_0)| \\ &= M_1(\mu\cos\theta_0 + \nu\sin\theta_0,\ -\mu\sin\theta_0 + \nu\cos\theta_0) \end{aligned}$$

Convert to polar coordinate system: $(\mu, \nu) \to (\rho, \theta)$, $\rho = \sqrt{\mu^2 + \nu^2}$, $\theta = \tan^{-1}(\nu/\mu)$. Then $M_2(\rho, \theta) = M_1(\rho, \theta - \theta_0)$. Hence this becomes like the problem in (a) part (but in 1D). Using the method in (a) part, we can find θ_0.
After the rotation angle θ_0 is found, rotate f_1 by θ_0 to get f_3. Now f_3 is related to f_2 by pure translation. Use the method in (a) part to find the translation vector (x_0, y_0).
Hence the steps are:-

1. Find 2D-FT of f_1 and f_2.
2. Find the magnitudes of the FTs and convert to polar coordinates.
3. Find the rotation angle θ_0 using the method in (a) part (just in 1D).
4. Rotate f_1 by θ_0 to get f_3.
5. Find the translation vector (x_0, y_0) using the method in (a) part on f_3 and f_2.