CS663 Assignment-3

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Question 2

Solution

The correlation of two continuous 2D signals in the continuous domain is represented by the equation:

$$h(x, y) = (f \otimes g)(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r, s) g(x + r, y + s) dr ds \tag{1}$$

We need to derive the 2D fourier transform of $h(x, y)$. We know that the 2D fourier transform of a function $p(x, y)$ is given by:

$$P(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y) e^{-j2\pi(ux+vy)} dx dy \tag{2}$$

Taking the fourier transform of $h(x, y)$, we get:

$$H(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y) e^{-j2\pi(ux+vy)} dx dy \tag{3}$$

Substituting the value of $h(x, y)$ from equation (1) into equation (3), we get:

$$H(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r, s) g(x + r, y + s) dr ds \right) e^{-j2\pi(ux+vy)} dx dy \tag{4}$$

Rearranging the terms, we get:

$$H(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r, s) \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x + r, y + s) e^{-j2\pi(ux+vy)} dx dy \right) dr ds \tag{5}$$

Consider $x' = x + r$ and $y' = y + v$. Thus, $x = x' - r$ and $y = y' - s$.

$$H(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r, s) \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x', y') e^{-j2\pi(u(x'-r)+v(y'-s))} dx' dy' \right) dr ds \tag{6}$$

$$H(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r, s) \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x', y') e^{-j2\pi(ux'+vy')} e^{j2\pi(ur+vs)} dx' dy' \right) dr ds \tag{7}$$

$$H(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r, s) \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x', y') e^{-j2\pi(ux'+vy')} dx' dy' \right) e^{j2\pi(ur+vs)} dr ds \tag{8}$$

$$H(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r, s) G(u, v) e^{j2\pi(ur+vs)} dr ds \tag{9}$$

This result can also be obtained by the translation theorem of fourier transformation of functions.
Let us take the complex conjugate of the equation (10):

$$H^*(u, v) = G^*(u, v) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f^*(r, s) e^{-j2\pi(ur+vs)} dr ds \tag{10}$$

$$F^*(u, v) = [\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r, s) e^{-j2\pi(ur+vs)} dr ds]^* \tag{11}$$

$$F^*(-u,-v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^*(r,s)e^{-j2\pi(ur+vs)}drds \tag{12}$$

Substituting the value of $F^*(-u,-v)$ from equation (12) into equation (10), we get:

$$H^*(u,v) = G^*(u,v)F^*(-u,-v) \tag{13}$$

$$H(u,v) = G(u,v)F(-u,-v) \tag{14}$$

Thus, the 2D fourier transform of the correlation of two continuous 2D signals is the product of the fourier transform of the two signals, with the other signal frequency being negated.

The correlation of two discrete 2D signals in the discrete domain is represented by the equation:

$$h[m,n] = (f \otimes g)[m,n] = \sum_{r=0}^{M-1}\sum_{s=0}^{N-1} f[r,s]g[m+r,n+s] \tag{15}$$

f and g are periodic functions (because they are discrete signals and can be overlapped). Therefore, we have used the above correlation formula.

We need to derive the 2D DFT of $h(x,y)$. We know that the 2D DFT of a function $p(x,y)$ is given by:

$$P(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} p(x,y)e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})} \tag{16}$$

Taking the DFT of $h(x,y)$, we get:

$$H(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} h(x,y)e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})} \tag{17}$$

Substituting the value of $h(x,y)$ from equation (1) into equation (3), we get:

$$H(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\left(\sum_{0}^{M-1}\sum_{0}^{N-1} f(r,s)g(x+r,y+s)\right)e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})} \tag{18}$$

Rearranging the terms, we get:

$$H(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(r,s)\left(\sum_{0}^{M-1}\sum_{0}^{N-1} g(x+r,y+s)e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})}\right) \tag{19}$$

Consider $x' = x+r$ and $y' = y+v$. Thus, $x = x'-r$ and $y = y'-s$. The limits of the sum on g won't change because g is periodic. (It is a discrete signal of finite size and can be overlapped.) (Translation theroem of DFT can also be used directly.)

$$H(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(r,s)\left(\sum_{0}^{M-1}\sum_{0}^{N-1} g(x',y')e^{-j2\pi(u(x'-r)+v(y'-s))}\right) \tag{20}$$

$$H(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(r,s)\left(\sum_{0}^{M-1}\sum_{0}^{N-1} g(x',y')e^{-j2\pi(ux'+vy')}e^{j2\pi(ur+vs)}\right) \tag{21}$$

$$H(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(r,s)\left(\sum_{0}^{M-1}\sum_{0}^{N-1} g(x',y')e^{-j2\pi(ux'+vy')}\right)e^{j2\pi(ur+vs)} \tag{22}$$

$$H(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(r,s)G(u,v)e^{j2\pi(ur+vs)} \tag{23}$$

Let us take the complex conjugate of the equation (23):

$$H^*(u,v) = G^*(u,v)\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f^*(r,s)e^{-j2\pi(ur+vs)} \tag{24}$$

$$F^*(u,v) = [\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(r,s)e^{-j2\pi(ur+vs)}]^* \tag{25}$$

$$F^*(-u,-v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f^*(r,s)e^{-j2\pi(ur+vs)} \tag{26}$$

Substituting the value of $F^*(-u,-v)$ from equation (26) into equation (23), we get:

$$H^*(u,v) = G^*(u,v)F^*(-u,-v) \tag{27}$$

$$H(u,v) = G(u,v)F(-u,-v) \tag{28}$$

Thus, the 2D DFT of the correlation of two discrete 2D signals is the product of the fourier transform of the two signals, with the other signal frequency being negated.