Option Pricing Models

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Mid-Term Report

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Contents

A Brief Overview of the Work

This report explains the basics of Options Trading, Black-Scholes Model, Binomial Model and Monte Carlo Simulations. It is written in LaTeX. The submission also contains a video and a code implementation. The code is written in jupyter notebook but can also be accessed through google colab. The link for the google colab is GOOGLE COLAB. The video explains the report and the code implemented. All this work is part of the *Finsearch Project* conducted by *Finance Club, IIT Bombay.*

There are a lot of references used to form this report. All of them are cited: [2], [1], [3], [6], [5] and [4]

1 Introduction

Options trading is how investors can speculate on the future direction of the overall stock market or individual securities, like stocks or bonds. Options contracts give us the choice - but not the obligation - to buy or sell an underlying asset at a specified price by a specified date.

Options are tradable contracts that investors use to speculate about whether an asset's price will be higher or lower at a certain date in the future, without any requirement to actually buy the asset in question.

1.1 Key Terminology

To understand options, we just need to know a few key terms:

- **Call Option:** A call option gives us the opportunity to buy a security at a predetermined price by a specified date.
- **Put Option:** A put option allows us to sell a security at a future date and price.
- **Strike Price:** The predetermined price is known as the strike price.
- **Premium:** The price to purchase an option is called a premium and it is calculated based on the underlying security's price and values.
- **Intrinsic Value:** Intrinsic value is the difference between an option contract's strike price and current price of the underlying asset.
- **In the money and out of the money:** Depending on the underlying security's price and the time remaining until expiration, an option is said to be in-the-money (profitable) or out-of-the-money (unprofitable).
- **European Option:** In European option, contract can only be executed at the time of expiration date.
- **American Option:** An American option allows owners to exercise their contract at any time before and including the expiration date.

1.2 Option Greeks

The option greeks are:

1. **Delta:** It measures the rate of change of options premium based on the directional movement of the underlying asset.
2. **Gamma:** Rate of change of delta itself.
3. **Vega:** Rate of change of premium based on change in volatility.
4. **Theta:** Measures the impact of premium based on time left for expiry.
5. **Rho:** Rho measures an option's sensitivity to changes in the risk-free rate of interest.

2 Black-Scholes Model

The Black-Scholes model, also known as the Black-Scholes-Merton (BSM) model, is one of the most important concepts in modern financial theory.

2.1 History

Developed in 1973 by Fischer Black, Robert Merton, and Myron Scholes, the Black-Scholes model was the first widely used mathematical method to calculate the theoretical value of an option contract, using current stock prices, expected dividends, the option's strike price, expected interest rates, time to expiration, and expected volatility.

2.2 Working of the Model

The model assumes that the stock shares or futures contracts will have a lognormal distribution of prices following a random walk with constant drift and volatility. Using this assumption, the equation derives the price of a European-style call option.

The Black-Scholes equation requires five variables. These inputs are volatility, the price of the underlying asset, the strike price of the option, the time until expiration of the option, and the risk-free interest rate.

The Black-Scholes Model makes certain assumptions:

- No dividends are paid out during the life of the option.
- Markets are random (i.e., market movements cannot be predicted).
- There are no transaction costs in buying the option.
- The risk-free rate and volatility of the underlying asset are known and constant.
- The returns of the underlying asset are normally distributed.
- The option is European and can only be exercised at expiration.

Formula

The Black-Scholes call option formula is calculated by multiplying the stock price by the cumulative standard normal probability distribution function. Therefore, the net present value (NPV) of the strike price multiplied by the cumulative standard normal distribution is subtracted from the resulting value of the previous calculation.

In mathematical notation:

$$C = SN(d_1) - Ke^{-rt}N(d_2) \tag{1}$$

where:

$$d_1 = \frac{ln(\frac{S}{K}) + (r + \frac{\sigma_v^2}{2})t}{\sigma_s\sqrt{t}}$$

$$d_2 = d_1 - \sigma_s\sqrt{t}$$

and where:
C = Call option price
S = Current Stock
K = Strike price
r = Risk-free interest rate
t = Time to maturity
N = A normal distribution

2.3 Benefits

- **Provides a Framework:** The Black-Scholes model provides a theoretical framework for pricing options. This allows investors and traders to determine the fair price of an option using a structured, defined methodology that has been tried and tested.
- **Allows for Portfolio Optimization:** The Black-Scholes model can be used to optimize portfolios by providing a measure of the expected returns and risks associated with different options. This allows investors to make smarter choices better aligned with their risk tolerance and pursuit of profit.
- **Allows for Risk Management:** By knowing the theoretical value of an option, investors can use the Black-Scholes model to manage their risk exposure to different assets.

2.4 Limitations

- **Limits Usefulness** The Black-Scholes model is only used to price European options and does not take into account that U.S. options could be exercised before the expiration date.
- **Assumes Constant Volatility** The model also assumes volatility remains constant over the option's life. In reality, this is often not the case because volatility fluctuates with the level of supply and demand.

3 Binomial Model

The binomial option pricing model is a risk-free method for estimating the value of path-dependent alternatives. With this model, investors can determine how likely they are to buy or sell at a given price in the future. According to this model, the current option value is equal to the present value of the probability-weighted future payoffs of the investment.

The model uses an iterative process for each period to determine how likely the movement will be up or down. The model effectively creates a binomial distribution of stock prices.

3.1 Assumptions

- At every point in time, the price can go to only two possible new prices, one up and one down (this is in the name, binomial).
- The underlying asset pays no dividends.
- The interest rate (discount factor) is a constant throughout the period.
- The market is frictionless, and there are no transaction costs and no taxes.
- Investors are risk-neutral, indifferent to risk.
- The risk-free rate remains constant.

3.2 Advantages

- The model is mathematically simple to calculate.
- Binomial Option Pricing is useful for American options, where the holder has the right to exercise at any time up until expiration.

3.3 Disadvantages

- A significant advantage is a multi-period view the model provides for the underlying asset's price and the transparency of the option's value over time.
- A notable disadvantage is that the computational complexity rises a lot in multi-period models.
- The most significant limitation of the model is the inherent necessity to predict future prices.

3.4 Calculations

If we set the current (spot) price of an option as S, then we can have two price movements at any given moment. The price can either go up to S+ or down to S-.

On this basis, we calculate the up(u) and down(d) factors.

$$u = \frac{s+}{s}$$

$$d = \frac{s-}{s}$$

When we have an up movement, the payoff of the call option is the maximum between zero and the spot price multiplied by the up factor and reduced with the exercise price.

$$C^{+} = max(0, uS - P_x)$$

A downward movement gives a payoff of:

$$C^{-} = max(0, dS - P_x)$$

The Binomial model effectively weighs the different payoffs with their respective probabilities and discounts them to the present value.

A put option entitles the holder to sell at the exercise price PX. When the price goes down or up, we calculate a put option as follows:

$$P^{+} = max(P_x - uS, 0)$$

$$P^{-} = max(P_x - dS, 0)$$

We can represent a general one-period call option like the following binomial tree 3.4:

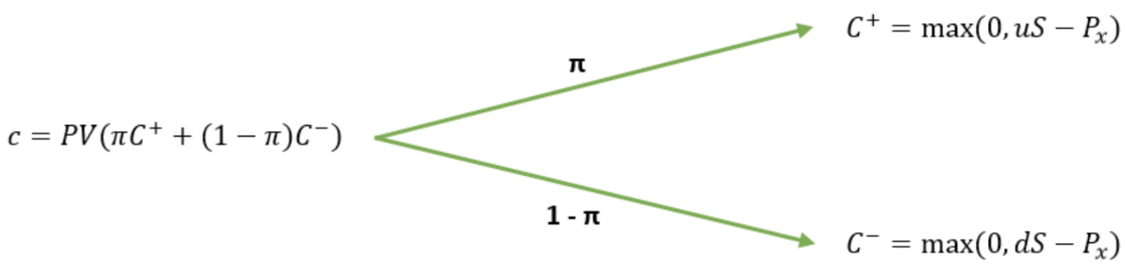

Figure 1: Binomial Tree

We can also present it as a formula:

$$c = \frac{\pi C^{+} + (1-\pi)C^{-}}{1+r} \tag{2}$$

Where π is the probability of an up move and r is the discount rate.

$$\pi = \frac{(1+tr)-d}{u-d} \tag{3}$$

where t is the period multiplier.

4 Monte Carlo Simulation

4.1 Introduction

Monte Carlo Option Pricing is a method often used in Mathematical Finance to calculate the value of an option with multiple sources of uncertainties and random features, such as changing interest rates, stock prices or exchange rates, etc. This method is named after the city of Monte Carlo, which is noted for its casinos.

4.2 Working

Monte Carlo Simulation generates a series of random variables that have similar properties to the risk factors which the simulation is trying to simulate.

A fundamental implication of asset pricing theory is that under certain circumstances, the price of a derivative security can be usefully represented as an expected value. Valuing derivatives thus reduces to computing expectations.

Valuing a derivative security by Monte Carlo typically involves simulating paths of stochastic processes used to describe the evolution of underlying asset prices, interest rates, model parameters, and other factors relevant to the security in question.

4.3 First Example

Let S(t) denote the price of the stock at time t. Consider a call option granting the holder the right to buy the stock at a fixed price K at a fixed time T in the future; the current time is t = 0. If at a time T the stock price S(T) exceeds the strike price K, the holder exercises the option for a profit of S(T)-K; if, on the other hand, S(T) $\leq$ K, the option expires worthless. The payoff to the option holder at time T is thus:

$$(S(T)-K)^+ = max\{0, S(T)-K\} \tag{4}$$

To get the present value of this payoff we multiply by a discount factor e^{-rT}, with r a continuously compounded interest rate. We denote the expected present value by $E[e^{-rT}(S(T)-K)^+]$. The Black-Scholes model describes the evolution of the stock price through the stochastic differential equation (SDE):

$$\frac{dS(t)}{S(t)} = rdt + \sigma dW(t) \tag{5}$$

with W a standard Brownian motion. The parameter σ is the volatility of the stock price and the coefficient of dt in equation 5 is the mean rate of return. In taking the rate of return to be the same as the interest rate r, we are implicitly describing the risk-neutral dynamics of the stock price.

The solution of the stochastic differential equation 5 is:

$$S(T) = S(0)exp([r - \frac{1}{2}\sigma^2]T + \sigma W(T)) \tag{6}$$

Let's apply the logarithm function to equation 6 which will allow its faster implementation when coded.

$$logS_t = logS_0 + (r - \frac{1}{2}\sigma^2)t + \sigma W(t) \tag{7}$$

5 Differences between Black-Scholes and Binomial Models

1. **Time** The Black-Scholes model assumes that the price of the underlying asset follows a continuous geometric Brownian motion. The Binomial model, on the other hand, employs discrete time, dividing the time of expiration into small periods or steps.

2. **Complexity** The Black-Scholes model uses complex mathematical formulas whereas the Binomial model is more straightforward and easier to understand.

3. **Asset Movement** The Black-Scholes model assumes that the underlying asset's price follows a log-normal distribution, meaning it can take any positive value with a higher probability of staying close to the current price. The Binomial model assumes that the underlying asset's price can only move up or down by a fixed percentage at each time step, resulting in a binomial distribution of possible prices.

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