MA105: Quiz 1 (8:15 - 9:05 a.m. on 25/08/2023)

D _ _ / T _ _ Roll Number: _ _ _ _ _ _ _ _ _ _ _ _

Name: _ A= B=

- Fill in the numbers "A" and "B" above as follows:
  If the last digit a of your roll number satisfies $0 \leq a \leq 4$, let $A = a + 5$. If $5 \leq a \leq 9$, let $A = a$.
  If the second-last digit b of your roll number satisfies $0 \leq b \leq 4$, let $B = b + 5$. If $5 \leq b \leq 9$, let $B = b$. Thus $5 \leq A, B \leq 9$.
  Example: Your Roll number is 23B0092. Then $A = 7$ and $B = 9$.
  You must use these values of A and B below. Using the wrong value of A or B even in one question may lead to the loss of all marks in this quiz.

Question 1. Write the answers **only** in the box provided below. You will get marks in questions (1)-(3) below only if you identify all the true statements and only the true statements.

(1) (1 mark) Let f be continuous on $[A, A+1]$ and differentiable on $(A, A+1)$, and suppose that $f(A)$ and $f(A+1)$ are of opposite signs and $f'(x) \neq 0$ for all $x \in (A, A+1)$. Which of the following are true?
   (a) There is a unique $x \in (A, A+1)$ such that $f(x) = 0$.
   (b) There is no $x \in (A, A+1)$ such that $f(x) = 0$.
   (c) The function $f(x)$ is always increasing in $(A, A+1)$.
   (d) There are at least two distinct points x_1, x_2 in $(A, A+1)$ such that $f(x_1) = 0 = f(x_2)$.

(a)

(2) (1 mark) Consider the function $f(x)$ defined on $\mathbb{R}$ as $f(x) = x^3 + B^2$ if $x \leq 0$ and $f(x) = (x - B)^2$ if $x > 0$. Which of the following are true?
   (a) The function $f(x)$ is strictly convex in the interval $(1, A)$.
   (b) The function $f(x)$ is differentiable at all points in $\mathbb{R}$.
   (c) The point $x = 0$ is a point of inflection for the function $f(x)$.
   (d) The function $f(x)$ has a local maximum at $x = 0$.

(a),(c),(d)

(3) (1 mark) Consider the function defined on $\mathbb{R}$ as $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ if $x \neq 0$ and $f(0) = 0$. Which of the following are true?
   (a) The function $f'(x)$ satisfies the Intermediate Value Property.
   (b) The function $f'(x)$ is continuous for all $x \in \mathbb{R}$.
   (c) The function $f(x)$ is continuous for all $x \in \mathbb{R}$.
   (d) The function $f(x)$ is twice differentiable at $x = 0$.

(a),(c)

(4) (2 marks) Find the smallest natural number N_0 such that for all $n > N_0$,

$$\left| \frac{Bn+2}{Bn+1} - 1 \right| < 10^{-2}.$$

N_0=[99/B]

$B = 5 \implies N_0 = 19$; $B = 6 \implies N_0 = 16$; $B = 7 \implies N_0 = 14$; $B = 8 \implies N_0 = 12$; $B = 9 \implies N_0 = 11$.

Note that those who have left the answer as $[\frac{99}{B}]$ have been awarded marks (in spite of not following instructions! Next time we may not be so generous).

Some students have argued that they thought that n was real. In that case, the answer becomes $[\frac{99}{B}] + 1$ (so the correct values of N_0 will become $20, 17, 15, 13, 11$ respectively). This is quite an unnatural interpretation. Nevertheless, we have decided to give you the marks if you wrote $[\frac{99}{B}] + 1$ or if you gave correct numerical value of N_0 according to this formula. If you have made this mistake but have not been given 2 marks, you may approach your TA to get your marks changed.

A few of you have written $\frac{99}{B}$ instead of $[\frac{99}{B}]$. In this case you will be given 1 mark out of 2. In case you have not been given 1 mark, contact your TA.

Question 2. (2 marks) Let $f(x)$ be defined as follows

$$f(x) = \begin{cases} A & \text{if} \quad x \leq 0 \\ x & \text{otherwise.} \end{cases}$$

Using the $\epsilon - \delta$ definition of the limit, determine whether $f(x)$ is continuous at 0.

Solution. I will give the marking scheme when $A = 5$. We will show that $f(x)$ is discontinuous at 0. We need to find an $\epsilon > 0$ such that for every $\delta > 0$ there exists at least one x with $0 < |x - 0| < \delta$ and $|f(x) - 5| \geq \epsilon$.

(1/2 mark)

(Above, you can give a 1/2 mark if the student writes $f(0)$ instead of 5. You can also give a 1/2 mark if the students negates the definition of sequential continuity.)

Let $\epsilon = 1$ (anything less than 5 will work, but the interval in which we take x below will change)

(1/2 mark)

For all $x \in (0, 4)$, $f(x) < 4$. Hence $|f(x) - 5| \geq 1$. It follows that in every interval $(-\delta, \delta)$ there is a point $x \neq 0$ such that $|f(x) - 5| \geq 1$. This shows that f is discontinuous at 0.

(1 mark)

Alternate Solution. Some of you may have shown that the left-hand and right-hand limits are different. If you have correctly used the $\epsilon - \delta$ definition of the left- and right-hand limits to do this, you will get credit for this. Those of you who have just asserted that the limits are different (i.e., you have guessed the values of the limits but have not shown that these values are actually the limits usng the $\epsilon - \delta$ definition) will not be given credit, since the question asked you explicitly to use the $\epsilon - \delta$ definition.

Common mistakes made:

1. Some students have defined continuity by saying "there exists $\delta > 0$ for all $\epsilon > 0$ ...". This is not technically correct. One should start with an $\epsilon > 0$ and then say that there exists a $\delta > 0$ (for that particular ϵ) such that the relevant inequality holds. **Nonetheless, we have given students a 1/2 mark for this.**
2. Some students have said that a function is continuous at x_0 if **there exists** $\epsilon > 0$ **etc.**. This is completely wrong and has been given no marks.
3. Some students have said "for all $\epsilon > 0$ and for all $\delta > 0$ ...". Again this is completely wrong in general. It happens to be true for the left hand limit in this problem but not for the right hand limit. This is, again, totally incorrect and has been given no marks.
4. Some students have shown correclty that the right hand limit is $0 \neq A$. They have been given full marks since $f(0) = A$ in this case.

Question 3. (3 marks) Suppose $\lim_{n\to\infty} a_n = L$. Show using the ε-N definition of the limit that the sequence $Ba_n^2 - a_n$ converges (you may not use the rules for limits or the Sandwich theorems).

Solution. I will give the marking scheme for option B=9 below:

To show that a real number l is the limit of a sequence b_n, we must show that given (any) $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$n > N \implies |b_n - l| < \epsilon.$$

In our case the sequence is $b_n = 9a_n^2 - a_n$ and we take $l = 9L^2 - L$. So we need to show that for any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$\left|9a_n^2 - a_n - 9L^2 + L\right| < \epsilon.$$

(1 mark)

(You can give one mark above, if either of the above statements is correctly written down)

We have

$$\left|9a_n^2 - a_n - 9L^2 + L\right| = \left|9a_n^2 - 9L^2 - a_n + L\right|$$
$$\leq 9|a_n^2 - L^2| + |a_n - L| = 9|a_n - L||a_n + L| + |a_n - L|$$

(1 mark)

Since a_n is a convergent sequence, it is bounded by some $M > 0$.

(1/2 mark)

Because $\lim_{n\to\infty} a_n = L$, we can find N_1 and N_2 such that

$$|a_n - L| < \epsilon/[2 \cdot 9(M + |L|)]$$

for $n > N_1$, and

$$|a_n - L| < \epsilon/2$$

for $n > N_2$.

If we take $N = \max\{N_1, N_2\}$, we get the desired result for all $n > N$.

(1/2 mark)

Alternative method (after 2.5 marks): Take $N \in \mathbb{N}$ such that

$$|a_n - L| < \frac{\epsilon}{9(M + |L|) + 1}$$

for $n > N$.

Then,

$$9|a_n - L||a_n + L| + |a_n - L| = |a_n - L|(9|a_n + L| + 1) \leq |a_n - L|(9(|a_n| + |L|) + 1) \leq |a_n - L|(9(M + |L|) + 1) < \epsilon.$$

for $n > N$. **(1/2 mark)**

Common Mistakes:

1. Some students have written "for all $\epsilon > 0$ and for all $n \in \mathbb{N}$ ..." when defining the limit. They have received no marks.
2. There are all sorts of different kinds of mistakes in the definition of the limit of a sequence - too many to list. Many answers have received 0 marks. Some incorrect answers have been given 1/2 a mark.
3. Many students have written

$$|a_n - L| < \frac{\epsilon}{9(a_n - L) - 1}.$$

This is not correct. One should say that a_n is bounded by M and proceed as in the model solution. Nonetheless, we have given partial credit of 1 mark in this case (thus, if everything else is correct, the student is likely to have got 2 marks on this question).

4. Some students have obtained complicated inequalities like

$$\left|9a_n^2 - a_n - 9L^2 + L\right| < 9\epsilon^2 + \epsilon - 1.$$

These are not helpful and usually just wrong. For example, the right hand side will be negative if ϵ is small enough. In this case no marks have been awarded.