

Tutorial sheet 1: Sequences, limits, continuity, differentiability

1. Using the (ϵ - N) definition of a limit, prove the following:

$$(i) \lim_{n \rightarrow \infty} \frac{10}{n} = 0$$

$$(ii) \lim_{n \rightarrow \infty} \frac{5}{3n+1} = 0$$

$$(iii) \lim_{n \rightarrow \infty} \frac{n^{2/3} \sin(n!)}{n+1} = 0$$

$$(iv) \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} - \frac{n+1}{n} \right) = 0$$

2. Show that the following limits exist and find them:

$$(i) \lim_{n \rightarrow \infty} \left(\frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n} \right)$$

$$(ii) \lim_{n \rightarrow \infty} \left(\frac{n!}{n^n} \right)$$

$$(iii) \lim_{n \rightarrow \infty} \left(\frac{n^3 + 3n^2 + 1}{n^4 + 8n^2 + 2} \right) \stackrel{L}{\leftarrow} \frac{n^3 + 3n^2 + 1}{n^4} = \frac{1}{n} + \frac{3}{n^2} + \frac{1}{n^4}$$

$$(iv) \lim_{n \rightarrow \infty} (n)^{1/n}$$

$$(v) \lim_{n \rightarrow \infty} \left(\frac{\cos \pi \sqrt{n}}{n^2} \right) \stackrel{L}{\leftarrow} \frac{1}{n^2}$$

$$(vi) \lim_{n \rightarrow \infty} (\sqrt{n} (\sqrt{n+1} - \sqrt{n}))$$

(sum rule)
(sandwich)

3. Show that the following sequences are not convergent:

$$\text{← } (i) \left\{ \frac{n^2}{n+1} \right\}_{n \geq 1} \quad (ii) \left\{ (-1)^n \left(\frac{1}{2} - \frac{1}{n} \right) \right\}_{n \geq 1}$$

4. Determine whether the sequences are increasing or decreasing:

$$(i) \left\{ \frac{n}{n^2+1} \right\}_{n \geq 1}$$

$$(ii) \left\{ \frac{2^n 3^n}{5^{n+1}} \right\}_{n \geq 1}$$

$$(iii) \left\{ \frac{1-n}{n^2} \right\}_{n \geq 2} \quad a_{n+1} - a_n$$

ratio test

5. Prove that the following sequences are convergent by showing that they are monotone and bounded. Also find their limits:

$$(i) a_1 = \frac{3}{2}, a_{n+1} = \frac{1}{2} \left(a_n + \frac{2}{a_n} \right) \quad \forall n \geq 1$$

$$(ii) a_1 = \sqrt{2}, a_{n+1} = \sqrt{2+a_n} \quad \forall n \geq 1$$

$$(iii) a_1 = 2, a_{n+1} = 3 + \frac{a_n}{2} \quad \forall n \geq 1$$

6. If $\lim_{n \rightarrow \infty} a_n = L$, find the following: $\lim_{n \rightarrow \infty} a_{n+1}$, $\lim_{n \rightarrow \infty} |a_n|$.

$$\text{1) (iii)} \lim_{n \rightarrow \infty} \frac{n^{2/3} \sin(n!)}{n+1} = 0$$

We need to find $n_0 \in \mathbb{N}$ such that for a given $\epsilon > 0$ we have $|a_n - l| < \epsilon$

$$\forall n \geq n_0 \quad [l=0]$$

$$a_n = \frac{n^{2/3} \sin(n!)}{n+1}$$

$$|a_n - 0| < \epsilon$$

$$|a_n| \leq \left| \frac{n^{2/3}}{n+1} \right| \leq \left| \frac{n^{2/3}}{n} \right| = \left| \frac{1}{n^{1/3}} \right|$$

(decreasing denominators)

if $\left| \frac{1}{n^{1/3}} \right| < \epsilon$
 then $|a_n| < \epsilon$

so choose such n_0

$$\left| \frac{1}{n^{1/3}} \right| < \epsilon \quad \forall n \geq n_0$$

$$\frac{1}{\epsilon} < n^{1/3} \Rightarrow n > \frac{1}{\epsilon^3}$$

$$\text{Take } n_0 = \left\lfloor \frac{1}{\epsilon^3} \right\rfloor + 1$$

since, we are able to find n_0 for every ϵ , we have proved the limit to be zero.

$$2) (i) \lim_{n \rightarrow \infty} \left(\frac{n}{n^2+1} + \frac{n}{n^2+2} + \dots + \frac{n}{n^2+n} \right)$$

The limit can be easily found using the JEE Technique (sandwich theorem)

$$t_r = \frac{n}{n^2+r} \quad (r=1 \text{ to } n)$$

$$a_n = \sum_{r=1}^n \frac{n}{n^2+r}$$

$$l = \lim_{n \rightarrow \infty} a_n$$

$$\underbrace{\frac{n}{n^2+n}}_{+} \leq t_r \leq$$

$$\underbrace{\frac{n}{n^2}}_{\downarrow} \text{decreasing denominators}$$

Increasing denominators

$$\frac{n}{n^2+n} \times n \leq \sum b_n \leq \frac{1}{n^2} \times n$$

$$\frac{n}{n+1} \leq a_n \leq 1$$

$$\lim_{n \rightarrow \infty} 1 = 1$$

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

$$\left[\left(\frac{n}{n+1} - 1 \right) < \varepsilon \Rightarrow \left(\frac{1}{n+1} \right) < \varepsilon \right] \Rightarrow n_0 = \left[\frac{1}{\varepsilon} - 1 \right] + 1$$

By sandwich theorem, limit $l = 1$

$$a_n \leq b_n \leq c_n \quad \forall n > n_0$$

and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = l$

then $\lim_{n \rightarrow \infty} b_n = l$

$$2\lambda \text{(ii)} \quad \lim_{n \rightarrow \infty} \frac{n!}{n^n}$$

$$0 < \frac{n!}{n^n} = \left(\frac{1}{n}\right)\left(\frac{2}{n}\right) \cdots \left(\frac{n}{n}\right)$$

$$= \left(\frac{1}{n}\right)\left(\frac{2}{n}\right) \cdots \left(\frac{n-1}{n}\right) \left(\frac{n}{n}\right) \leq_1 \leq_1 = 1$$

$\leq \frac{1}{n}$ (since multiplying by a quantity ≤ 1 will decrease the product)

Now, apply sandwich theorem

$$2\lambda \text{(vi)} \quad \sqrt{n} (\sqrt{n+1} - \sqrt{n}) = \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{1}{\sqrt{1 + \frac{1}{n}} + 1}$$

$$\rightarrow \frac{1}{2} \quad (\text{as } n \rightarrow \infty)$$

$$\underline{\underline{2 \text{P}(\text{IV})}} \quad \lim_{n \rightarrow \infty} n^{1/n} = ?$$

By guessing, we can get the limit to be 1 $\left[n^{1/n} = e^{\frac{1}{n} \log n} \xrightarrow[0]{\downarrow} 1 \right]$

So let us assume,

$$n^{1/n} = 1 + \underbrace{h_n}_{\downarrow}$$

variable dependent on n

so, we have,

$$n = (1 + h_n)^n$$

$$\geq 1 + nh_n + \underbrace{\binom{n}{2} h_n^2}_{\text{from series expansion}}$$

$$\geq \binom{n}{2} h_n^2 = \frac{n(n-1)}{2} h_n^2$$

$$h_n \leq \left(\frac{2}{n-1} \right)^{1/2}$$

$$\underline{\underline{\text{Claim:}}} \quad \lim_{n \rightarrow \infty} h_n = 0$$

$$\text{Proof: } \left(\frac{2}{n-1} \right)^{1/2} < \epsilon$$

$$\Rightarrow \frac{2}{\varepsilon^2} + 1 < n$$

$$n_0 = \left\lceil \frac{2}{\varepsilon^2} + 1 \right\rceil + 1$$

$$\Rightarrow \lim = 0$$

$$\lim (t+h_n) = (t+0) = t$$

$$\lim_{n \rightarrow \infty} h_n = t = 1$$

$$\text{3b) (ii)} \left\{ (-1)^n \left(\frac{1}{2} - \frac{1}{n} \right) \right\}_{n \geq 1}$$

$$= \frac{(-1)^n}{2} - \frac{(-1)^n}{n}$$

one idea $\Rightarrow n = \text{odd}$ and even \Rightarrow different limits
 \Rightarrow not convergent

$$\text{Other} \Rightarrow \lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$$

$$\left| \frac{(-1)^n}{n} \right| < \varepsilon \Rightarrow n > \frac{1}{\varepsilon}$$

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{2} = \text{not convergent}$$

hence, the difference will also not converge

~~5)(ii)~~ $a_1 = \sqrt{2}$ $a_2 > \sqrt{2} = a_1$

$$a_{n+1} = \sqrt{2+a_n}$$

Claim: $a_{n+1} \geq a_n$ [monotone]

$$\Leftrightarrow \sqrt{2+a_n} \geq a_n$$

$$\Leftrightarrow 2+a_n \geq a_n^2$$

$$\Leftrightarrow a_n^2 - a_n - 2 \leq 0$$

$$\Leftrightarrow (a_n - 2)(a_n + 1) \leq 0$$

$$\Leftrightarrow a_n \in [-1, 2]$$

Claim: $a_n < 2$

Induction $\Rightarrow a_1 = \sqrt{2} < 2$

Let $a_n < 2$

$$\text{then } a_{n+1} = \sqrt{2+a_n} < \sqrt{2+2} = 2$$

Hence sequence is monotonically increasing and bounded above, thereby convergent

$$5) \lim_{n \rightarrow \infty} a_n = l$$

$$\text{obvious : } \lim_{n \rightarrow \infty} a_{n+1} = l$$

$$\underline{\text{claim}} : \lim_{n \rightarrow \infty} |a_n| = |l|$$

$$| |a_n| - |l| | \leq |a_n - l| < \epsilon \quad \forall n > n_0$$

$$\begin{aligned} \text{Hence Proved,} \\ \Rightarrow \lim_{n \rightarrow \infty} |a_n| = |l| \end{aligned}$$

$$6) \text{ Given: } \lim_{n \rightarrow \infty} a_n = l$$

$$|a_n - l| < \epsilon \quad \forall n > n_0$$

[from definition]

$$\begin{aligned} |a_{n+1} - l| &= |a_{n+1} - a_n + a_n - l| \\ &\leq |a_{n+1} - a_n| + |a_n - l| \end{aligned}$$

$$\text{Now, we have } \lim_{n \rightarrow \infty} |a_{n+1} - a_n| = 0$$

[from a result
to be discussed
in class]

$$\Rightarrow |a_{n+1} - a_n| < \frac{\epsilon}{2} \quad \forall n > n'_0$$

[from definition]

Adding the two equations obtained

$$|a_{n+1} - l| = |a_{n+1} - a_n + a_n - l|$$

$$\leq |a_{n+1} - a_n| + |a_n - l|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \begin{array}{l} \forall n > n_0 \\ \forall n > n'_0 \end{array}$$

$$\Rightarrow |a_{n+1} - l| < \epsilon \quad \begin{array}{l} \forall n > \max \\ (n_0, n'_0) \end{array}$$

\Rightarrow The limit of a_{n+1} is l
by definition.

5b (i) $a_1 = \frac{3}{2}$ $a_{n+1} = \frac{1}{2} \left(a_n + \frac{2}{a_n} \right)$

$$a_{n+1} \geq \sqrt{a_n + \frac{2}{a_n}} = \sqrt{2} \quad (\text{AM-GM})$$

$$a_{n+1} - a_n = \frac{1}{a_n} - \frac{a_n}{2}$$

$$= \frac{2 - a_n^2}{2a_n} > 0$$

[because $a_n > \sqrt{2}$]

\Rightarrow decreasing, bounded below \rightarrow convergent $\forall n$

$$\lim_{n \rightarrow \infty} a_n = l \Rightarrow \lim_{n \rightarrow \infty} a_{n+1} = l$$

$$\Rightarrow l = \frac{1}{2} \left(l + \frac{2}{l} \right)$$

$$\Rightarrow l = 5$$