MA105 : Midsem Exam (4:00 - 6:00 p.m. on 16/09/2023)

Total Marks: 40

Division _ _ / Tutorial Batch _ _ Roll Number: _ _ _ _ _ _ _ _ _ _ _ _

Name: _ A= B=

- Fill in the numbers "A" and "B" above as follows:
  If the last digit a of your roll number satisfies $0 \leq a \leq 4$, let $A = a + 5$. If $5 \leq a \leq 9$, let $A = a$.
  If the second-last digit b of your roll number satisfies $0 \leq b \leq 4$, let $B = b + 5$. If $5 \leq b \leq 9$, let $B = b$. Thus $5 \leq A, B \leq 9$.
  Example: Your Roll number is 23B0092. Then $A = 7$ and $B = 9$.
  You must use these values of A and B below. Using the wrong value of A or B in even one question may lead to the loss of 10 or more marks in this exam.

Write the answers of Questions (1)-(14) **only** in the box provided below the questions, and that of Questions (15) and (16) in the space provided below the questions.

You will get full (respectively, partial) marks in Questions (9)-(14) below only if you select all (respectively, some but not all) of the TRUE statements and only the TRUE statements (that is, if you select a FALSE statement, you will get ZERO mark in that question).

(1) (2 marks) The directional derivative of the function $f(x, y) = Axe^y + B\cos(xy)$ at the point $(1, 0)$ and along the unit vector $v = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is

$A\sqrt{2}$

(2) (2 marks) Let $f(x, y, z) = x^2 + Axy - y^2 + z^2$, for $(x, y, z) \in \mathbb{R}^3$, and let S be the surface in $\mathbb{R}^3$ defined by $f(x, y, z) = 2$. The tangent plane to the surface S at the point $(1, 0, 1)$ on the surface is give by the equation

Equation: $2x + Ay + 2z = 4$ or $x + (A/2)y + z = 2$ or $(2, A, 2) \cdot (x - 1, y, z - 1) = 0$

(3) (2 marks) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by $f(x, y) = xy + 2y$, for $(x, y) \in \mathbb{R}^2$. Then the derivative matrix of f at (B, B) is

$[B \ \ B + 2]$

(4) (2 marks) For $n \in \mathbb{N}$, let P_n denote the partition $\{0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < 1\}$ of the interval $[0, 1]$. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = Ax^A$. Let N be the least natural number such that $U(f, P_N) - L(f, P_N) < 0.01$. Let $I \in \{150, 300, 450, 1000, 1500, 3000, 4500, 5000, 6000, 7000\}$ be the integer closest to N. Then the value of I is

$N > 100A$. For $A = 5, 6, 7, 8, 9$, $I = 450, 450, 450, 1000, 1000$, resp.

(5) (3 marks) Consider the function $f(x) = \ln x$. Let $P_2(x) = b_0 + b_1(x - B) + b_2(x - B)^2$ be the Taylor polynomial of $f(x)$ of degree (or order) 2 about the point B. Then the values of b_0, b_1 and b_2 are (your answers should be actual numbers and not functions of B)

$b_0 = \ln B$ | $b_1 = 1/B$ | $b_2 = -1/(2B^2)$

(1+1+1 marks)

(6) (3 marks) Let $\sum_{n=0}^{\infty} c_n x^n$ be the Taylor series of the function $\arctan x$ about the point 0. Then the values of c_3, c_5 and c_6 are

$c_3 = -1/3$	$c_5 = 1/5$	$c_6 = 0$

(1+1+1 marks)

(7) (3 marks) For $n \in \mathbb{N}$, define $S_n = \frac{1}{n}\sum_{i=1}^{n} \sin\left(\frac{(2i-1)A\pi}{2n}\right)$. Identify S_n as a Riemann sum $R(f, P, t)$ of a certain function $f(x)$ over the interval $[0, A\pi]$, where (P, t) is a tagged partition of the interval $[0, A\pi]$. If $t = \{t_1 < t_2 < \cdots < t_n\}$ is the tagging, what is the value of t_i? What is the function $f(x)$? What is the value of $L = \lim_{n\to\infty} S_n$?

$t_i = \frac{2i-1}{2n}A\pi$	$f(x) = \frac{1}{A\pi}\sin x$	$L = \frac{-\cos A\pi + 1}{A\pi}$, which is $\frac{2}{A\pi}$ for odd A, and 0 for even A

(1/2+1/2+2 marks)

(8) (3 marks) Let $T : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function and let $x(t), y(t)$ be differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. State the Chain Rule for the derivative of $g(t) = T(x(t), y(t))$.

$$\boxed{\frac{dg}{dt}(t_0) = \frac{\partial T}{\partial x}\Big|_{(x(t_0),y(t_0))}\frac{dx}{dt}(t_0) + \frac{\partial T}{\partial y}\Big|_{(x(t_0),y(t_0))}\frac{dy}{dt}(t_0)}$$

(2 marks)

For $(x, y) \in \mathbb{R}^2$, define $T(x, y) = x^2y + Ax$; and for $t \in \mathbb{R}$, define $x(t) = \cos t$, $y(t) = \sin t$. Using the Chain Rule as stated above, calculate the value of $g'\left(\frac{\pi}{2}\right)$, the derivative of $g(t) = T(x(t), y(t))$ at $\frac{\pi}{2}$.

$$\boxed{g'\left(\frac{\pi}{2}\right) = -A}$$

(1 mark)

(9) (2 marks) Let a_n and b_n be sequences of real numbers. Which of the following statements is/are TRUE?

(a) The sequence a_nb_n is convergent, if a_n is convergent and b_n is bounded

(b) The sequence a_nb_n is bounded, if a_n is convergent and b_n is bounded

(c) The sequence a_nb_n is convergent, if a_n is convergent and b_n is monotonically decreasing and bounded below

(d) The sequence a_nb_n is convergent, if a_n is bounded and b_n is monotonically increasing and bounded above

(b), (c)	If only (b) or only (c) is selected, 1 mark. If (a) or (d) is selected, 0 mark in total.

(10) (2 marks) Let $g(x) = \int_0^x \frac{Bt^2}{t^2+t+1}dt$, for $x \in \mathbb{R}$. Which of the following statements is/are TRUE?

(a) The function $g(x)$ has a point of inflection at $x = -2$

(b) The function $g(x)$ is increasing in the interval $(0, B)$

(c) The function $g(x)$ is concave on $(-A, 0)$

(d) The function $g(x)$ is discontinuous at $x = 0$

(a), (b)	If only (a) or only (b) is selected, 1 mark. If (c) or (d) is selected, 0 mark in total.

(11) (2 marks) Consider the real valued function $f(x,y) = \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$ defined on some subset of $\mathbb{R}^2$. Which of the following statements is/are TRUE?

(a) The natural domain of f is the set $\{(x,y) \in \mathbb{R}^2 \mid y \neq x\}$

(b) The natural domain of f is $\{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0 \text{ and } y \neq x\}$

(c) $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

(d) $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

(b), (d)	If only (b) or only (d) is selected, 1 mark. If (a) or (c) is selected, 0 mark in total.

(12) (2 marks) Consider the function $f(x,y)$ defined by $f(x,y) = \frac{x^2y-xy^2}{|x|+|y|}$ for $(x,y) \neq (0,0)$, and $f(0,0) = 0$. Which of the following statements is/are TRUE?

(a) $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

(b) f is continuous at $(0,0)$

(c) $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $(0,0)$ are 0

(d) f is differentiable at $(0,0)$

(a),(b),(c),(d)	For (a), 1/2 mark. For (b), 1/2 mark. For (c), 1/2 mark, For (d), 1/2 mark.

(13) (2 marks) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined as $f(x,y) = \sqrt[3]{xy}$, for all $(x,y) \in \mathbb{R}^2$. Which of the following statements is/are TRUE?

(a) f is discontinuous at $(0,0)$

(b) f has the directional derivatives along all unit vector vectors at $(0,0)$

(c) The gradient of f at $(0,0)$ is $(0,0)$

(d) f is not differentiable

(c), (d)	If only (c) or only (d) is selected, 1 mark. If (a) or (b) is selected, 0 mark in total.

(14) (2 marks) Let $f : \left[0, \frac{\pi}{2}\right] \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ \cos x & \text{if } x \in \mathbb{Q} \end{cases}$$

for all $x \in \left[0, \frac{\pi}{2}\right]$. Which of the following statements is/are TRUE?

(a) $L(f) = 0$

(b) $U(f) = 1$

(c) f is Riemann integrable

(d) f is continuous at all the irrational numbers

(a), (b)	If only (a) or only (b) is selected, 1 mark. If (c) or (d) is selected, 0 mark in total.

(15) (4 marks) Let $f(x)$ be a $\mathcal{C}^\infty$ (or smooth) function on $\mathbb{R}$ and let $P_n(x)$ be its Taylor polynomial of degree (or order) n about the point 1. Write down the precise formula for the remainder $R_n(x)$. Now take $f(x) = e^x$. Using this formula, determine the smallest positive integer N such that $|e^2 - P_N(2)| < 0.1$ (you must show that this does not hold for any positive integer $n < N$).

Solution: $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-1)^{n+1} = \frac{e^c(x-1)^{n+1}}{(n+1)!}$ for some c lying between 1 and x. (1 mark)

Note: Students who have written $R_n(x)$ in the forms

$$\frac{f^{(n+1)}(x)}{(n+1)\,!}(x-1)^{n+1} \text{ or } \frac{f^{(n+1)}(1)}{(n+1)\,!}(x-1)^{n+1}$$

have not received any marks in the step above because these are major errors. The whole point is that c is some point in $(1, x)$ (which one cannot usually determine). It cannot be taken to be either 1 or x.

We know that $e^2 - P_n(2) = R_n(2) = e^c \frac{1}{(n+1)!}$ for some $c \in [1, 2]$. (1 mark)

Note: Students who have taken $c = 1$ in the expression above have got no marks in the step above

Hence $|e^2 - P_n(2)| < 0.1 \Leftrightarrow e^c \frac{1}{(n+1)!} < 0.1 \Leftrightarrow (n+1)\,! > 10e^c$ where $c \in [1, 2]$. (1 mark)

Since $27 = 10 \cdot 2.7 < 10 \cdot e \leq 10e^c \leq 10e^2 < 10 \cdot 9 = 90$ and $(n+1)\,! = 24$ for $n = 3$, and $(n+1)\,! = 120$ for $n = 4$, we get that $N = 4$ is the smallest positive integer such that $(N+1)\,! > 10e^c$ for $c \in [1, 2]$, that is, $|e^2 - P_{_N}(2)| < 0.1$ for $N = 4$. (1 mark)

Note: Students have to say that $10e > 24 = (3+1)!$ in order to say that $N = 4$ is the smallest integer for which the inequality holds (i.e., it does not hold for $n = 3$). Many students have written said that $10e^2 > 24$, which is true, but does not prove what one wants, since $R_n(2) < e^2/24$.

(16) (4 marks) Show, using the definition of differentiability of a function $f : \mathbb{R}^2 \to \mathbb{R}$ at a point $(x_0, y_0) \in \mathbb{R}^2$, that the function $f(x, y) = x + Bxy$ is differentiable at $(0, 0) \in \mathbb{R}^2$.

Solution: We know that a function $f(x, y)$ is differentiable at (x_0, y_0) if

$$\lim_{(h,k)\to(0,0)} \frac{|f((x_0, y_0) + (h, k)) - f(x_0, y_0) - \left(\frac{\partial f}{\partial x}(x_0, y_0)h + \frac{\partial f}{\partial y}(x_0, y_0)k\right)|}{||(h, k)||} = 0.$$

(1 mark)

For the function $f(x, y) = x + Bxy$, $\frac{\partial f}{\partial x}(0, 0) = 1$ and $\frac{\partial f}{\partial y}(0, 0) = 0$. (1 mark)

Now, the function $f(x, y)$ is differentiable at $(0, 0)$ with the derivative (matrix) $Df(0, 0) = [1 \;\; 0]$ if and only if

$$\lim_{(h,k)\to(0,0)} \frac{|f(h, k) - h|}{||(h, k)||} = 0$$

that is,

$$\lim_{(h,k)\to(0,0)} \frac{|h + Bhk - h|}{||(h, k)||} = \lim_{(h,k)\to(0,0)} \frac{B|hk|}{||(h, k)||} = 0.$$

(1 mark)

Sine $|hk| \leq ||(h, k)||^2$, we get

$$\lim_{(h,k)\to(0,0)} \frac{B|hk|}{||(h, k)||} \leq \lim_{(h,k)\to(0,0)} \frac{B||(h, k)||^2}{||(h, k)||} \leq \lim_{(h,k)\to(0,0)} B||(h, k)|| = 0,$$

which shows that

$$\lim_{(h,k)\to(0,0)} \frac{B|hk|}{||(h, k)||} = 0$$

and hence the function $f(x, y) = x + Bxy$ is differentiable at $(0, 0)$ with the derivative $[1 \;\; 0]$.

(1 mark)