MA 105 : Calculus

D4 - Lecture 9

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Properties of the Riemann integral

From the definition of the Riemann integral we can easily prove the following properties. We assume that f and g are Riemann integrable. Then

$$\int_a^b [f(t) + g(t)]dt = \int_a^b f(t)dt + \int_a^b g(t)dt,$$

$$\int_a^b cf(t)dt = c\int_a^b f(t)dt,$$

for any constant $c \in \mathbb{R}$, and finally if $f(t) \leq g(t)$ for all $t \in [a, b]$, then

$$\int_a^b f(t)dt \leq \int_a^b g(t)dt.$$

Implicit in the properties above is the fact that if f and g are Riemann integrable, then so are $f + g$ and cf.

It is not hard to prove either of the properties. One needs only to use the corresponding properties for inf and sup.

Proving the properties of the integral

Observe that on any interval $[x_{j-1}, x_j]$,

$$\inf_{[x_{j-1},x_j]} f + \inf_{[x_{j-1},x_j]} g \leq (f+g)(x) = f(x)+g(x) \leq \sup_{[x_{j-1},x_j]} f + \sup_{[x_{j-1},x_j]} g.$$

Hence

$$\inf_{[x_{j-1},x_j]} f + \inf_{[x_{j-1},x_j]} g \leq \inf_{[x_{j-1},x_j]} (f+g) \leq \sup_{[x_{j-1},x_j]} (f+g) \leq \sup_{[x_{j-1},x_j]} f + \sup_{[x_{j-1},x_j]} g.$$

It follows that for any partitions P_1, P_2 and $P = P_1 \cup P_2$ of $[a, b]$,

$$L(f, P_1) + L(g, P_2) \leq L(f, P) + L(g, P) \leq L(f + g, P) \leq L(f + g)$$

and hence

$$\sup_{P_1} L(f, P_1) + \sup_{P_2} L(g, P_2) \leq L(f + g).$$

That is,

$$L(f) + L(g) \leq L(f + g).$$

Proof continues...

Similarly, for any partitions P_1, P_2 and $P = P_1 \cup P_2$ of $[a, b]$,

$$U(f+g) \leq U(f+g,P) \leq U(f,P)+U(g,P) \leq U(f,P_1)+U(g,P_2)$$

and hence

$$U(f+g) \leq \inf_{P_1} U(f,P_1) + \inf_{P_2} U(g,P_2).$$

That is,

$$U(f+g) \leq U(f) + U(g).$$

It follows from the above two inequalities (for the lower and upper Darboux integrals) that

$$L(f) + L(g) \leq L(f+g) \leq U(f+g) \leq U(f) + U(g).$$

Since f, g are integrable, we get

$$\int_a^b f(t)dt + \int_a^b g(t)dt \leq L(f+g) \leq U(f+g) \leq \int_a^b f(t)dt + \int_a^b g(t)dt$$

Proof continues...

and hence

$$L(f+g) = U(f+g) = \int_a^b f(t)dt + \int_a^b g(t)dt$$

that is, $\int_a^b [f(t)+g(t)]dt$ exits and

$$\int_a^b [f(t)+g(t)]dt = \int_a^b f(t)dt + \int_a^b g(t)dt.$$

Also, observe that

$$\sup_{[x_{j-1},x_j]} (cf) = c \sup_{[x_{j-1},x_j]} f \text{ and } \inf_{[x_{j-1},x_j]} (cf) = c \inf_{[x_{j-1},x_j]} f, \text{ for } c \geq 0.$$

Therefore, for $c \geq 0$ (when $c < 0$, we apply these observations to the function $(-c)f$ and use the fact that $\int_a^b (f+g) = \int_a^b f + \int_a^b g$),

$$\inf_P U(cf, P) = c \inf_P U(f, P), \quad \sup_P L(cf, P) = c \sup_P L(f, P).$$

Proof continues...

It follows that

$$U(cf) = c\int_a^b f(t)dt \quad \text{and} \quad L(cf) = c\int_a^b f(t)dt$$

and hence $L(cf) = U(cf)$, which shows that cf is integrable on $[a, b]$ and

$$\int_a^b cf(t)dt = c\int_a^b f(t)dt.$$

In case when $f(t) \geq g(t)$ for all $t \in [a, b]$, we get $f(t) - g(t) \geq 0$ for all $t \in [a, b]$ and hence $L(f - g) \geq 0$ and it follows that

$$\int_a^b [f(t) - g(t)]dt \geq 0$$

that is,

$$\int_a^b f(t)dt \geq \int_a^b g(t)dt. \qquad \square$$

Products of Riemann Integrable Functions

Theorem 6: Let $f : [a, b] \to [m, M]$ be a Riemann integrable function and let $\phi : [m, M] \to \mathbb{R}$ be a continuous function. Then the function $\phi \circ f$ (defined as $\phi \circ f(x) = \phi(f(x))$) is Riemann integrable on $[a, b]$.

The above theorem has the following interesting corollaries.

Corollary 1: Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions which are Riemann integrable on $[a, b]$. Then $f \cdot g$, $|f|$ and f^n (for any positive integer n) are Riemann integrable.

Proof: Exercise (Hint: $f \cdot g = \frac{1}{4}[(f+g)^2 - (f-g)^2]$).

Corollary 2: If $f : [a, b] \to \mathbb{R}$ is a Riemann integrable function and $[c, d] \subseteq [a, b]$. Then the function $g : [c, d] \to \mathbb{R}$ defined as $g(x) = f(x)$ for all $x \in [c, d]$ is Riemann integrable.

Proof: Exercise (Hint: First show that the characteristic function $\chi_{[c,d]}$ is integrable on $[a, b]$ and then show that $f \cdot \chi_{[c,d]}$ is integrable on $[a, b]$ by using Corollary 1, and $\int_a^b f(t)\chi_{[c,d]}(t)dt = \int_c^d g(t)dt$.)

Another property of the Riemann Integral

Theorem 7: Suppose f is Riemann integrable on $[a, b]$ and $c \in [a, b]$. Then

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt.$$

Proof: First we note that if $c = a$ or $c = b$, there is nothing to prove.

Next, if $c \in (a, b)$ we proceed as follows. If P_1 is a partition of $[a, c]$ and P_2 is a partition of $[c, b]$, then $P_1 \cup P_2 = P'$ is obviously a partition of $[a, b]$. Thus, partitions of the form $P_1 \cup P_2$ constitute a subset of the set of all partitions of $[a, b]$. For such partitions P', we have

$$L(f, P') = L(f, P_1) + L(f, P_2).$$

Let us denote by $L(f)_{[a,c]}$ (resp. $L(f)_{[c,b]}$) the Darboux lower integral of f on the interval $[a, c]$ (resp. $[c, b]$).

If we take the supremum over all partitions P_1 of $[a, c]$ and P_2 of $[c, b]$ we get

$$\sup_{P'} L(f, P') = L(f)_{[a,c]} + L(f)_{[c,b]}.$$

Now the supremum on the left hand side is taken only over partitions P' having the special form $P_1 \cup P_2$. Hence it is less than or equal to $\sup_P L(f, P)$ where this supremum is taken over all partitions P. We thus obtain

$$L(f)_{[a,c]} + L(f)_{[c,b]} \leq L(f).$$

On the other hand, for any partition $P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$, we can consider the partition $P' = P \cup \{c\}$. This will be a refinement of the partition P and can be written as a union of two partitions P_1 of $[a, c]$ and P_2 of $[c, b]$.

By the property for refinements for Darboux sums we know that $L(f, P) \leq L(f, P')$.

Thus, given any partition P of $[a, b]$, there is a refinement P' which can be written as the union of two partitions P_1 and P_2 of $[a, c]$ and $[c, b]$ respectively, and by the above inequality,

$$\sup_{P} L(f, P) \leq \sup_{P'} L(f, P'),$$

where the first supremum is taken over all partitions of $[a, b]$ and the second only over those partitions P' which can be written as a union of two partitions as above. This shows that

$$L(f) \leq L(f)_{[a,c]} + L(f)_{[c,b]},$$

so, together with the previous inequality, we get

$$L(f) = L(f)_{[a,c]} + L(f)_{[c,b]}.$$

By Corollary 2 of Theorem 6, $\int_a^c f(t)dt$ and $\int_c^b f(t)dt$ exist, and hence $L(f)_{[a,c]} = \int_a^c f(t)dt$ and $L(f)_{[c,b]} = \int_c^b f(t)dt$.

Since $L(f) = \int_a^b f(t)dt$, we obtain that

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt. \quad \square$$

The fundamental theorem of calculus: Motivation

The Fundamental Theorem of calculus allows us to relate the process of Riemann integration to the process of differentiation. Essentially, it tells us that integrating and differentiating are inverse processes. This is a tremendously useful theorem for several reasons.

It turns out that (Riemann) integrating even simple functions is much harder than differentiating them (if you don't believe me, try integrating $(\tan x)^3$ via Riemann sums!).

In practice, however, integration is what we need to do to solve physical problems. Usually, when we are studying the motion of a particle or a planet what we find is that the position of a particle, which is a function of time, satisfies some differential equation.

Solving the differential equation involves performing the inverse operation of taking some combination of derivatives. The simplest such inverse operation is taking the inverse of the first derivative, which the Fundamental Theorem says, is the same as integrating.

Calculating Integrals

Thus, calculating integrals is one of the basic things one needs to do for solving even the simplest physics and engineering problems.

The problem is that this is quite difficult to do.

Once we know the derivatives of some basic functions (polynomials, trigonometric functions, exponentials, logarithms) we can differentiate a wide class of functions using the rules for differentiation, especially the product and chain rules.

By contrast, the only rule for Riemann integration that can be proved from the basic definitions is the sum rule.

The Fundamental Theorem solves this problem (partially) because it allows us to deduce formalæ for the integrals of the products and the composition of functions from the corresponding rules for derivatives.

The Fundamental Theorem - Part I

Theorem 8: Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let

$$F(x) = \int_a^x f(t)dt$$

for any $x \in [a, b]$. Then $F(x)$ is continuous on $[a, b]$, differentiable on (a, b) and

$$F'(x) = f(x),$$

for all $x \in (a, b)$.

Proof: We know that $f(t)$ is Riemann integrable on $[a, x]$ for any $x \in [a, b]$ because of Theorem 5 (every continuous function is Riemann integrable). We show that

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x)$$

for $h > 0$. The same way one can prove the case $h < 0$.

By Theorem 7, we know that

$$\int_a^{x+h} f(t)dt = \int_a^x f(t)dt + \int_x^{x+h} f(t)dt,$$

for $x + h \in [a, b]$. Hence

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \cdot \int_x^{x+h} f(t)dt.$$

Let $m(h)$ and $M(h)$ be the constant functions given, respectively, by the infimum and supremum of the function f on $[x, x+h]$.

Then, $m(h) \leq f(t) \leq M(h)$, for all $t \in [x, x+h]$, and hence

$$m(h) \cdot h \leq \int_x^{x+h} f(t)dt \leq M(h) \cdot h.$$

Dividing by h and taking the limit gives

$$\lim_{h \to 0} m(h) \leq \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \leq \lim_{h \to 0} M(h).$$

We now show that $\lim_{h\to 0} m(h)$ exists and the value of this limit is $f(x)$. By a similar argument one can show that $\lim_{h\to 0} M(h)$ also exists and the value of this limit is also $f(x)$.

For, since $m(h) = \inf_{t\in[x,x+h]} f(t)$, for a given $\epsilon > 0$, there exists $s \in [0, h]$ such that

$$f(x+s) < m(h) + \epsilon/2 \Rightarrow f(x+s) - m(h) < \epsilon/2.$$

Now, we write

$$f(x) - m(h) = f(x) - f(x+s) + f(x+s) - m(h)$$

$$\Rightarrow |f(x) - m(h)| \leq |f(x) - f(x+s)| + |f(x+s) - m(h)|.$$

Since the function f is continuous, there exists $\delta > 0$ such that

$$|f(x+h) - f(x)| < \epsilon/2$$

whenever $|h| < \delta$.

Thus, we obtain that

$$|f(x)-m(h)| \leq |f(x)-f(x+s)|+|f(x+s)-m(h)| < \epsilon/2+\epsilon/2 = \epsilon$$

whenever $|h| < \delta$ (since $s \in [0,h]$) and hence

$$\lim_{h\to 0} m(h) = f(x).$$

Similarly, $\lim_{h\to 0} M(h) = f(x)$. Now, we go back to our inequality at the end of the last to the last slide.

By the Sandwich theorem for limits (use version 2), we see that the limit in the middle exists and is equal to $f(x)$, that is,

$$F'(x) = f(x).$$

This proves that $F(x)$ is differentiable on (a,b) and $F'(x) = f(x)$.

How to show that $F(x)$ is continuous on $[a,b]$? Can you show that $F(x)$ is continuous at the end points a, b? This I leave as an exercise. (Hint: $|\int_c^d f(t)dt| \leq \int_c^d |f(t)|dt$) $\square$

Keeping the notation as in the Theorem, we obtain

Corollary: $\int_c^d f(t)dt = \int_a^d f(t)dt - \int_a^c f(t)dt = F(d) - F(c)$ for any two points $c, d \in [a,b]$.

The Fundamental Theorem of Calculus Part II

Theorem 9: Let $f : [a, b] \to \mathbb{R}$ be given and suppose there exists a continuous function $g : [a, b] \to \mathbb{R}$ which is differentiable on (a, b) and which satisfies $g'(t) = f(t)$. Then, if f is Riemann integrable on $[a, b]$,

$$\int_a^b f(t)dt = g(b) - g(a).$$

Note that this statement does not assume that the function $f(t)$ is continuous, and hence is stronger than the corollary we have just stated.

Proof: We can write:

$$g(b) - g(a) = \sum_{i=1}^{n} [g(x_i) - g(x_{i-1})],$$

where $\{a = x_0 < x_1 < \cdots < x_n = b\}$ is an arbitrary partition of $[a, b]$.

Using the mean value theorem for each of the intervals $I_i = [x_{i-1}, x_i]$, we can write

$$g(x_i) - g(x_{i-1}) = g'(c_i)(x_i - x_{i-1}).$$

for some $c_i \in (x_{i-1}, x_i)$.

Substituting this in the previous expression and using the fact that $g'(c_i) = f(c_i)$, we get

$$g(b) - g(a) = \sum_{i=1}^{n} [f(c_i)(x_i - x_{i-1})]. \qquad (*)$$

The calculation above is valid for any partition.

Since f is Riemann integrable on $[a, b]$, for a given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left| \sum_{i=1}^{n} [f(t_i)(x_i - x_{i-1})] - \int_a^b f(t)dt \right| < \epsilon$$

for any tagged partition (P, t) of $[a, b]$ having $\|P\| < \delta$ (using the first definition of Riemann integration).

Now, we construct a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$ having $\|P\| < \delta$ and consider the tagging $t = \{c_i : x_{i-1} \leq c_i \leq x_i, 1 \leq i \leq n\}$ (of P) that we get from $(*)$ which we consider for this partition P having $\|P\| < \delta$.

Now, it follows from $(*)$ that

$$\left|g(b) - g(a) - \int_a^b f(t)dt\right| = \left|\sum_{i=1}^{n} [f(c_i)(x_i - x_{i-1})] - \int_a^b f(t)dt\right| < \epsilon.$$

Since the above inequality holds for all positive real number ϵ, we get

$$\left|g(b) - g(a) - \int_a^b f(t)dt\right| = 0,$$

that is,

$$\int_a^b f(t)dt = g(b) - g(a). \quad \square$$

FTC: Applications

Exercise 4.5. Let p be a real number and let f be a continuous function on $\mathbb{R}$ that satisfies the equation $f(x+p) = f(x)$ for all $x \in \mathbb{R}$. Show that the integral $\int_a^{a+p} f(t)dt$ has the same value for every real number a.

Solution: Consider $F(x) = \int_a^x f(t)dt,\ x \in \mathbb{R}$. Then $F'(x) = f(x)$. Note that

$$\int_{u(x)}^{v(x)} f(t)dt = \int_a^{v(x)} f(t)dt - \int_a^{u(x)} f(t)dt = F(v(x)) - F(u(x)).$$

Use the Chain rule to see that

$$\begin{aligned} \frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt &= F'(v(x))v'(x) - F'(u(x))u'(x) \\ &= f(v(x))v'(x) - f(u(x))u'(x). \end{aligned}$$

Using this formula for $G(x) = \int_x^{x+p} f(t)dt$, we see that $G'(x) = 0$ for all $x \in \mathbb{R}$, and hence $G(x)$ is constant. Thus $\int_a^{a+p} f(t)dt$ has the same value for every real number a. $\square$

Mean Value Theorem for Integrals

Theorem 10 (Mean Value Theorem for Integrals): If f is continuous on $[a, b]$, then there exists $c \in [a, b]$ such that

$$\int_a^b f(x)dx = f(c)(b-a).$$

Proof: Since the function

$$F(x) = \int_a^x f(t)dt$$

is continuous on $[a, b]$ and differentiable on (a, b) with $F'(x) = f(x)$ for all $x \in (a, b)$ (by the Fundamental Theorem of Calculus), there is $c \in (a, b)$ such that

$$f(c) = F'(c) = \frac{F(b) - F(a)}{b-a} = \frac{1}{b-a}\int_a^b f(x)dx$$

(by the Mean Value Theorem).

Thus

$$f(c)(b-a) = \int_a^b f(x)dx. \quad \square$$

The logarithmic function

Definition: The natural logarithmic function is defined on $(0, \infty)$ by

$$\ln x = \int_1^x \frac{1}{t} dt.$$

It is clear that $\ln 1 = 0$, $\ln x > 0$ for $x \in (1, \infty)$, and $\ln x < 0$ for $x \in (0, 1)$.

Theorem 11:

1. $\ln(xy) = \ln x + \ln y$
2. $\ln(\frac{x}{y}) = \ln x - \ln y$
3. $\ln(x^r) = r \ln x$, if r is a rational number.

Proof: (1). Let $f(t) = \ln(ty)$. Then, $f'(t) = \frac{1}{t}$. Therefore, by FTC - Part II, $\ln x = f(x) - f(1)$, that is, $\ln(xy) = \ln x + \ln y$. For (2), put $x = 1/y$ in (1) and get $\ln(1/y) = -\ln y$, and then use (1) again, for the product of x and $1/y$. (3) is clear if $r \in \mathbb{Z}$. Observe that $\ln x = \ln[(x^{1/q})^q] = q \ln(x^{1/q})$ for $q \neq 0 \in \mathbb{Z}$, which shows that $\ln(x^{1/q}) = (1/q) \ln x$. Now, if $r = p/q$ for $p, q \in \mathbb{Z}$, using the first case (for $p \in \mathbb{Z}$) we get $\ln(x^{p/q}) = (p/q) \ln x$. $\square$

The exponential function

Remark: $\ln x$ is increasing and concave (why?). Moreover, by IVT, there exists a number $e > 1$ such that $\ln e = 1$ (as $\exists x \in (1, \infty)$ such that $\ln x > 0$ (why?) and $\ln x^n = n \ln x \to \infty$ as $n \to \infty$).

It follows that $\ln x$ is a strictly increasing function whose range is full of $\mathbb{R}$. Therefore, it is invertible and has an inverse. We denote this by $\exp(x)$.

That is,

$$\exp(x) = y \iff \ln y = x$$

In particular, $\exp(0) = 1, \exp(1) = e$.

Since, $\ln(e^r) = r \ln e = r$, we get $e^r = \exp(r)$, when r is a rational number. Therefore, we define $e^x := \exp(x)$ for any $x \in \mathbb{R}$.

Laws of Exponents:

$e^{x+y} = e^x e^y,\ e^{x-y} = \frac{e^x}{e^y},\ (e^x)^r = e^{rx},$ if r is rational.

Proof: Use the laws of exponents for $\ln x$ (Theorem 11). $\square$

The exponential function

Theorem 12: $\frac{d}{dx}(e^x) = e^x$.

Proof: If f is differentiable with nonzero derivative, then f^{-1} is also differentiable and in this case, using the chain rule we get

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Thus, $\frac{d}{dx}(e^x) = \frac{1}{(\ln)'(e^x)} = \frac{1}{1/e^x} = e^x$. $\square$

Remark: Now, we can define a^x whenever $a > 0$ and $x \in \mathbb{R}$ as

$$a^x = e^{x \ln a}.$$

Exercise: Show that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.

Solution: Let $f(x) = \ln x$. Then $f'(x) = 1/x$. Thus, $f'(1) = 1$. But,

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{f(1+h)}{h}.$$

Thus, by using the sequential criterion for limits, if we consider the sequence $\{1/n\}$ converging to 0, then

$$1 = f'(1) = \lim_{n\to\infty} \frac{f(1+(1/n))}{(1/n)} = \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n.$$

Since the logarithmic function is continuous, we obtain that

$$\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = \ln\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)$$

and hence $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$. $\square$

Taylor series: Recall that if f is a $\mathcal{C}^\infty$ function on $\mathbb{R}$, then the Taylor series expansion of f about a is

$$f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots$$

Taylor Series for e^x

Consider the series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ for a fixed x. If we choose $N > 2x > 0$, then for all $n > N$,

$$\frac{x^{n+1}}{(n+1)!} = \frac{x^n}{n!}\frac{x}{(n+1)} \leq \frac{x^n}{n!}\frac{x}{N} \leq \frac{x^n}{n!}\frac{1}{2}.$$

Thus, for $m \geq n > N$,

$$s_m - s_n = \frac{x^{n+1}}{(n+1)!} + \cdots + \frac{x^m}{m!} \leq \frac{x^{n+1}}{(n+1)!}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-n}}\right)$$

and hence

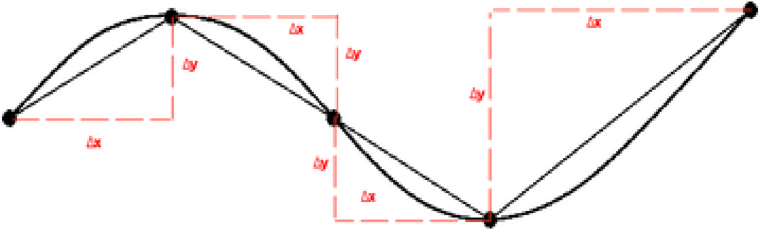
$$s_m - s_n \leq \frac{2x^{n+1}}{(n+1)!} \leq \frac{x^n}{n!}.$$

This shows that the sequence of partial sums of the Taylor series for e^x is Cauchy. Hence the series is convergent.

Does this Taylor series converge to e^x? Yes, as the Taylor's theorem insures that the remainder $R_n(x)$ associated to the function e^x converges to Zero. Therefore $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

Arc length

The picture below and the discussion on the next slide are from Wikipedia (http://en.wikipedia.org/wiki/Arc_length).



See: http://en.wikipedia.org/wiki/File:Arclength-2.png

In the picture above, the curve $y = f(x)$ is being approximated by straight line segments which form the hypotenuses of the right angled triangles shown in the picture.

The formula for arc length

Let us denote the arc length of the curve $y = f(x)$ by S.

The length of any given hypotenuse in the previous slide is given by the Pythagorean Theorem: $\sqrt{\Delta x^2 + \Delta y^2}$.

Intuitively, the sum of the lengths of the n hypotenuses appears to approximate S:

$$S \sim \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2} = \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\, \Delta x_i,$$

where "$\sim$" means approximately equal, $y_i = f(x_i)$, $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = f(x_i) - f(x_{i-1})$ corresponding to a partition $P = \{a = x_0 < \cdots < x_n = b\}$ of $[a, b]$.

The formula for arc length

Now, using the MVT for $y = f(x)$ on $[a, b]$ we get

$$S \sim \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\, \Delta x_i = \sum_{i=1}^{n} \sqrt{1 + (y'(t_i))^2}\, \Delta x_i$$

for some $t_i \in [x_{i-1}, x_i]$.

It follows that the arc length of the curve $y = f(x)$ (defined on $[a, b]$) is

$$S := \lim_{||P|| \to 0} \sum_{i=1}^{n} \sqrt{1 + (y'(t_i))^2}\, \Delta x_i = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

provided this limit exists which is equivalent of saying that the function $f'(x)$ is Riemann integrable on $[a, b]$.

Exercise 4.10. (ii) Find the length of the curve

$$y = \int_0^x \sqrt{\cos 2t}\, dt, \; 0 \le x \le \pi/4.$$

Solution: The formula for the arc length of a curve $y = f(x)$ between the points $x = a$ and $x = b$ is given by

$$\int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

For the problem at hand this gives

$$\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 2x}\, dx = \sqrt{2} \int_0^{\pi/4} \cos x \; dx = 1.$$

$\square$

Rectifiable curves

Not all curves have finite arc length! Here is an example of a curve with infinite arc length.

Example: Let $\gamma : [0,1] \to \mathbb{R}^2$ be the curve given by $\gamma(t) = (t, f(t))$, where

$$f(t) = \begin{cases} t\cos\left(\frac{\pi}{2t}\right), & \text{if } t \neq 0 \\ 0 \text{ if } t = 0 \end{cases}$$

If
http://math.stackexchange.com/questions/296397/nonrectifiable-curve
is correct, you should be able to check that this curve has infinite arc length. Try it as an exercise.

Notice that the curve above is given by a continuous function. Curves for which the arc length S is finite are called rectifiable curves.

Exercise: Show that the graphs of piecewise $\mathcal{C}^1$ functions are rectifiable.

Convergence of Power series

We have already seen the convergence of a specific power series (namely, the Taylor series for e^x). There is a general test we can use to determine if a power series converges.

Theorem 13: Let $\sum_{n=0}^{\infty} a_n(x-b)^n$ be a power series about the point b. If

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{R}$$

for some $R \in \mathbb{R}$, the series converges in the interval $(b-R, b+R)$ to a smooth function. (if the limit is 0, the series converges on the whole real line).

Roughly speaking $|a_n|$ behaves like $1/R^n$ for large n. Hence, the terms in the power series can be bounded by $|(x-b)|^n/R^n$, and this latter (geometric) series converges in $(b-R, b+R)$. This argument can be made precise. Proving that the series is smooth is trickier and we will not get into it.

A convergent Taylor series (or more generally a convergent "power series") can be differentiated and integrated "term by term". That is, if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad \text{then} \quad f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

And similarly,

$$\int_a^b \sum_{n=0}^{\infty} a_n x^n dx = \sum_{n=0}^{\infty} a_n \int_a^b x^n dx.$$

We will not be proving these facts but you can use them below.

Exercise 5: Using Taylor series write down a series for the integral

$$\int \frac{e^x}{x} dx.$$

Solution: We simply integrate term by term to get

$$\log x + x + \frac{x^2}{2 \cdot 2!} + \ldots = \log x + \sum_{k=1}^{\infty} \frac{x^k}{k \cdot k!}.$$

We can obtain Taylor series for the inverse trigonometric functions in this way. Indeed we could define the function $\arcsin x$ in this way:

$$\arcsin x = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt.$$

Now we can use the binomial theorem for the integrand. Note that the binomial theorem for arbitrary real exponents is an example of Taylor series:

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

It is not too hard to prove that the series on the right hand side above converges for $|x| < 1$. Applying the binomial theorem for $\alpha = -1/2$ to the integrand, we get

$$\arcsin x = \int_0^x \left(1 + \frac{1}{2}t^2 - \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}t^4 + \cdots\right) dt.$$

Integrating this term by term, you should verify that you get the series for $\arcsin x$ that you can derive directly from Taylor series.