MA 105 Calculus II

Lecture 7

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1. Parametrisation of curves
2. Orientation of Curves
3. Characterization of conservative fields
4. Characterization of conservative fields

Recall

- Let $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, for $n = 1, 2$, be a continuous vector field and $\mathbf{c} : [a, b] \to D$ be a C^1 curve.
  Then the line integral of $\mathbf{F}$ over $\mathbf{c}$ is

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} := \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt.$$

- $$\int_{\mathbf{c}} \mathbf{F} \cdot ds := \int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} + \ldots + \int_{\mathbf{c}_n} \mathbf{F} \cdot d\mathbf{s}.$$

- Let $\mathbf{c}$ be a curve on $[a, b]$ and $\widetilde{\mathbf{c}}(t) = \mathbf{c}(a + b - t)$, that is the curve $\widetilde{\mathbf{c}}$ traversed in the reverse direction and is denoted by $-\mathbf{c}$. Then $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = -\int_{-\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$.

Different parametrisations of the same path

Example 1: Let $\mathbf{c}_1(t) = (\cos t, \sin t)$ for $0 \leq t \leq 2\pi$. Then $c_2(t) = (\cos 2t, \sin 2t)$ for $0 \leq t \leq \pi$, the paths are different as a function but the curves traversed are the same.

Example 2: Here is an example of a simple path with three different parametrisations with the same domain.
Take the straight line segment between $(0, 0, 0)$ and $(1, 0, 0)$.

Here are three different ways of parametrising it:

$$\{t, 0, 0)\}, \quad \{(t^2, 0, 0)\} \quad \text{and} \quad \{(t^3, 0, 0)\},$$

where $0 \leq t \leq 1$.

Reparametrisation

Let $\mathbf{c} : [a, b] \to \mathbb{R}^n$ be a path which is non-singular, that is, $\mathbf{c}'(t) \neq 0$ for all $t \in [a, b]$.

- Suppose we now make change of variables $t = h(u)$, where h is $\mathcal{C}^1$ diffeomorphism (this means that h is bijective, $\mathcal{C}^1$ and so is its inverse) from $[\alpha, \beta]$ to $[a, b]$. We let $\gamma(u) = \mathbf{c}(h(u))$.
- We will assume that $h(\alpha) = a$ and $h(\beta) = b$.
- Then γ is called a reparametrisation of $\mathbf{c}$.
- Because h is a C^1 diffeomorphism, γ is also a C^1 path.

The line integral of a vector field $\mathbf{F}$ along γ is given by

$$\int_\gamma \mathbf{F} \cdot d\mathbf{s} = \int_\alpha^\beta \mathbf{F}(\gamma(u)) \cdot \gamma'(u) du = \int_\alpha^\beta \mathbf{F}(\mathbf{c}(h(u))) \cdot \mathbf{c}'(h(u)) h'(u) du,$$

where the last equality follows from the chain rule. Using the fact that $h'(u)du = dt$, we can change variables from u to t to get

$$\int_\gamma \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt = \int_\mathbf{c} \mathbf{F} \cdot d\mathbf{s}.$$

Orientation of Curves

For given two points P and Q on $\mathbb{R}^n$ for $n = 2, 3$, and a path connecting them, we can ask whether the path is traversed from P to Q or from Q to P?

Since a path from P to Q is a mapping $\mathbf{c} : [a, b] \to \mathbb{R}^n$ with $\mathbf{c}(a) = P$ and $\mathbf{c}(b) = Q$, (or vice-versa), it allows us to determine the direction in which the path is traversed. This direction of the path is called its Orientation.

If the reparametrisation $\gamma(\cdot) = \mathbf{c}(h(\cdot))$ preserves the orientation of $\mathbf{c}$, then

$$\int_\gamma \mathbf{F}.d\mathbf{s} = \int_\mathbf{c} \mathbf{F}.d\mathbf{s}.$$

If the reparamtrisation reverses the orientation, then

$$\int_\gamma \mathbf{F}.d\mathbf{s} = -\int_\mathbf{c} \mathbf{F}.d\mathbf{s}.$$

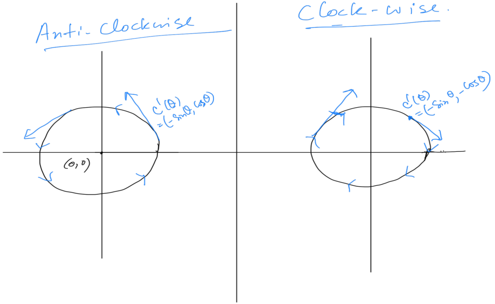
Curves on plane

Let us consider the paths lying in $\mathbb{R}^2$, namely, Planar curves.

For a simple closed planar curve, we get a choice of direction- clockwise or anti-clockwise.

Ex. $\gamma(\theta) = (\cos(\theta), \sin(\theta))$, $\theta \in [0, 2\pi]$. This is a circle with direction anti-clockwise.

Set $\gamma_1(\theta) = (\cos(\theta), -\sin(\theta))$, $\theta \in [0, 2\pi]$. It is circle with clockwise direction.



The argument just made shows that the line integral is independent of the choice of parametrisation - it depends only on the image of the non-singular parametrised path.

- A geometric curve C is a set of points in the plane or in the space that can be traversed by a parametrised path in the given direction. Often the line integral of a vector field $\mathbf{F}$ along a 'geometric curve' C is represented by $\int_C \mathbf{F}.d\mathbf{s}$ or by $\int_C F_1 dx + F_2 dy + F_3 dz$.
- To evaluate $\int_C \mathbf{F}.d\mathbf{s}$, choose a convenient parametrisation $\mathbf{c}$ of C traversing C in the given direction and then

$$\int_C \mathbf{F}.d\mathbf{s} := \int_{\mathbf{c}} \mathbf{F}.d\mathbf{s}.$$

- '$\oint_C$' means the line integral over a closed curve C.

The arc length parametrisation

There is a natural choice of parametrisation we can make which is useful in many situations. This is the parametrisation by arc length.

Recall the length of a curve $\mathbf{c}$ for a path $\mathbf{c} : [a, b] \to \mathbb{R}^3$, called its arc length, is given by

$$\ell(\mathbf{c}) = \int_a^b \|\mathbf{c}'(t)\| \; dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \; dt.$$

We now set

$$s(t) = \int_a^t \|\mathbf{c}'(u)\| du,$$

where $\mathbf{c} : [a, b] \to \mathbb{R}^3$ is a non-singular curve, from which it follows that $s'(t) = \|\mathbf{c}'(t)\|$. Why? Fundamental theorem of Calculus.

It is easy to see that s is a strictly increasing differentiable function. Let $h : [0, \ell(\mathbf{c})] \to [a, b]$ be its inverse. Then it is differentiable and its derivative is not vanishing. Define $\tilde{\mathbf{c}}(u) := \mathbf{c}(h(u))$ for $u \in [0, \ell(\mathbf{c})]$. This is called the arc length parametrisation.

Let $h(u) = t \in [a, b]$ or $s(t) = u$.
Note that

$$\begin{aligned} \frac{d\tilde{\mathbf{c}}(u)}{du} &= \mathbf{c}'(h(u))h'(u) \\ &= \mathbf{c}'(h(u))\frac{1}{s'(h(u))} \\ &= \mathbf{c}'(t)\frac{1}{\|\mathbf{c}'(t)\|} \end{aligned}$$

Using the reparametrisation theorem we get that

$$\int_{\mathbf{c}} \mathbf{F}.d\mathbf{s} = \int_{\tilde{\mathbf{c}}} \mathbf{F}.d\mathbf{s}.$$

Note,

$$\begin{aligned}\int_{\tilde{\mathbf{c}}} \mathbf{F} \cdot d\mathbf{s} &= \int_0^{\ell(\mathbf{c})} \mathbf{F}(\tilde{\mathbf{c}}(u)) \cdot \tilde{\mathbf{c}}'(u) du \\ &= \int_0^{\ell(\mathbf{c})} \mathbf{F}(\mathbf{c}(h(u))) \cdot \frac{\mathbf{c}'(h(u))}{\|\mathbf{c}'(h(u))\|} du \\ &= \int_0^{\ell(\mathbf{c})} \mathbf{F}(\mathbf{c}(h(u)) \cdot \mathbf{T}(h(u))\ du\end{aligned}$$

where $\mathbf{T}(t) = \frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t\|}$ is the unit tangent vector along the curve.

In other words, the line integral is nothing but the (Riemann) integral of the tangential component of **F** with respect to arc length.

To this end, as before we set

$$s(t) = \int_a^t \|\mathbf{c}'(u)\| du,$$

where $\mathbf{c} : [a, b] \to \mathbb{R}^3$ is a non-singular curve, from which it follows that $ds = \|\mathbf{c}'(t)\| dt$.
Integrals of scalar functions along path: Let $f : D \to \mathbb{R}$ be a continuous scalar function and $\mathbf{c} : [a, b] \longrightarrow D$ be a non-singular path. Then the path integral of f along $\mathbf{c}$ is defined by

$$\int_{\mathbf{c}} f\, ds := \int_a^b f(\mathbf{c}(t))\|\mathbf{c}'(t)\|\, dt.$$

Example. Find the circumference of the circle in $\mathbb{R}^2$ whose center is at origin and radius is r, for some $r > 0$.
Ans. Check $\int_{\mathbf{c}} f\, ds$ for $f = 1$ and $\mathbf{c}(t) = (r \cos t, r \sin t)$, for $t \in [0, 2\pi]$.

Characterization of gradient fields

Theorem (Variant of fundamental theorem of calculus)

Let $n = 2, 3$ and let $D \subset \mathbb{R}^n$.

1. *Let $\mathbf{c} : [a, b] \to D \subset \mathbb{R}^n$ be a smooth path.*
2. *Let $f : D \to \mathbb{R}$ be a differentiable function and let ∇f be continuous on $\mathbf{c}$.*

Then $\displaystyle\int_{\mathbf{c}} \nabla f . d\mathbf{s} = f(\mathbf{c}(b)) - f(\mathbf{c}(a))$.

Proof. From definition, it follows that

$$\int_{\mathbf{c}} \nabla f . d\mathbf{s} = \int_a^b \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt$$

Now the integrand on the right hand side is nothing but the directional derivative of f at $\mathbf{c}(t)$ in the direction of $\mathbf{c}'(t)$. Hence, we obtain

$$\int_a^b \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt = \int_a^b \frac{d}{dt} f(\mathbf{c}(t)) dt = f(\mathbf{c}(b)) - f(\mathbf{c}(a)).$$

- Suppose the vector field $\mathbf{F}$ is a continuous conservative field, i.e., $\mathbf{F} = \nabla f$, for some C^1 scalar function f. Then for any smooth path $\mathbf{c}$, we have
$$\int_{\mathbf{c}} \mathbf{F}.d\mathbf{s} = f(\mathbf{c}(b)) - f(\mathbf{c}(a)).$$
- This shows that the value of the line integral of a conservative field depends only on the value of the function at the end points of the curve, not on the curve itself.

Definition

The line integral of a vector field $\mathbf{F}$ is independent of path in a domain if for any $\mathbf{c}_1$ and $\mathbf{c}_2$ paths in D with the same initial and terminal points,

$$\int_{\mathbf{c}_1} \mathbf{F}.d\mathbf{s} = \int_{\mathbf{c}_2} \mathbf{F}.d\mathbf{s}.$$

Equivalently, the line integral of $\mathbf{F}$ is independent of path in D if for any closed curve $\mathbf{c}$ (why?)

$$\int_{\mathbf{c}} \mathbf{F}.d\mathbf{s} = 0.$$

Examples

Example Find the work done by the gravitational field $\mathbf{F}(x,y,z) = -\frac{mMG}{|\mathbf{r}(x,y,z)|^3}\mathbf{r}(x,y.z)$, in moving a particle with mass m and position vector $\mathbf{r}(x,y,z) = (x,y,z)$ from $(3,4,12)$ to the point $(2,2,0)$ along a piecewise-smooth curve C.

Ans Since the gravitational field is a conservative field and $\mathbf{F}(x,y,z) = \nabla f(x,y,z)$, where

$$f(x,y,z) = \frac{mMG}{|\mathbf{r}(x,y,z)|} = \frac{mMG}{\sqrt{x^2+y^2+z^2}}.$$

Using the Fundamental theorem for line integrals, the work done is

$$W = \int_{\mathbf{c}} \mathbf{F}.\mathbf{ds} = f(\mathbf{c}(b) - f(\mathbf{c}(a)) = f(2,2,0) - f(3,4,12) = mMG(\frac{1}{2\sqrt{2}} - \frac{1}{13}),$$

where $\mathbf{c} : [a,b] \to \mathbb{R}$, a parametrisation of curve C with $\mathbf{c}(a) = (3,4,12)$ and $\mathbf{c}(b) = (2,2,0)$.

Example Evaluate $\int_C y^2 dx + x dy$, where

1. $C = C_1$ is the line segment from $(-5,-3)$ to $(0,2)$,
2. $C = C_2$ is the part of parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.

Are the line integrals along C_1 and C_2 same?

Ans 1.) Consider parametrisation for C_1,
$\mathbf{c}_1(t) = (5t-5, 5t-3), \quad t \in [0,1]$. Thus $\mathbf{c}_1'(t) = (5,5)$ for all $t \in [0,1]$.
So, $\mathbf{F}(\mathbf{c}_1(t)) = ((5t-3)^2, 5t-5)$ and

$$\int_{C_1} y^2 dx + x dy = \int_0^1 [(5t-3)^2.5 + (5t-5).5]\, dt = -\frac{5}{6}.$$

2. Consider parametrisation for C_2, $\mathbf{c}_2(t) = (4-t^2, t), \quad t \in [-3,2]$. Thus $\mathbf{c}_2'(t) = (-2t, 1)$ for all $t \in [-3,2]$. So, $\mathbf{F}(\mathbf{c}_2(t)) = (t^2, 4-t^2)$ and

$$\int_{C_2} y^2 dx + x dy = \int_{-3}^2 [t^2(-2t) + (4-t^2)]\, dt = 40\frac{5}{6}.$$

Line integrals along C_1 and C_2 are Not same! Though the endpoints of C_1 and C_2 are same!