MA-105 Calculus II

Lecture 6

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October 16 - 20, 2023

1. Vector analysis

2. Curve and path

3. Line integrals of vector fields

Let $n \in \mathbb{N}$ and $\mathbb{R}^n$ be the Euclidean space defined by

$$\mathbb{R}^n := \{(x_1, x_2, \cdots, x_n) \mid x_j \in \mathbb{R}; \quad \forall\, j = 1, 2, \cdots, n\},$$

equipped with the norm

$$\|x\| := \left(\sum_{j=1}^{n} |x_j|^2\right)^{\frac{1}{2}}.$$

- Any real number is called a scalar.
- For $n \in \mathbb{N}$, any element from $\mathbb{R}^n$ is called vector. Note this means elements of $\mathbb{R}$ can be thought of both as a scalar and vector. To avoid confusion we will talk about vectors in $\mathbb{R}^n$ for $n > 1$.

Basic structure:

For any $x := (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and $y := (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ and any $a \in \mathbb{R}$:

$$x + y := (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n) \in \mathbb{R}^n, \quad \text{sum of two elements in} \quad \mathbb{R}^n$$

$$ax := (ax_1, ax_2, \cdots, ax_n) \in \mathbb{R}^n, \quad \text{Scalar multiplication.}$$

Scalar fields and Vector fields

Let D be a subset of $\mathbb{R}^n$.

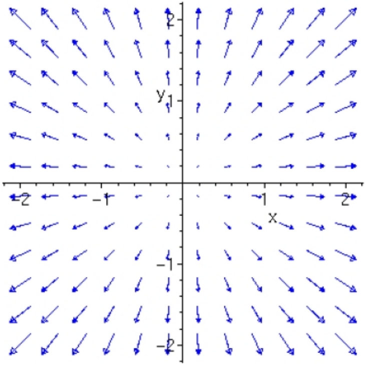
Definition: A scalar field on D is a map $f : D \to \mathbb{R}$.

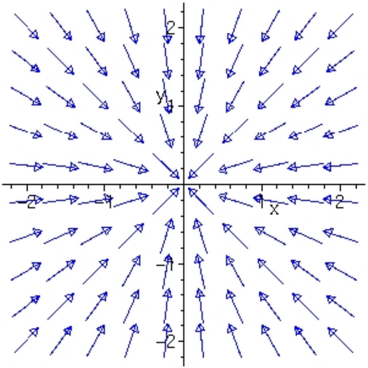
Definition A vector field on D is a map $\mathbf{F} : D \to \mathbb{R}^n$. We choose $n \geq 2$.

- A scalar field associates a number to each point of D, whereas a vector field associates a vector (of the same space) to each point of D.
- The temperature at a point on the earth is a scalar field.
- The velocity field of a moving fluid, a field describing heat flow, the gravitational field, the magnetic field etc are examples of various vector fields.

Vector fields: Examples

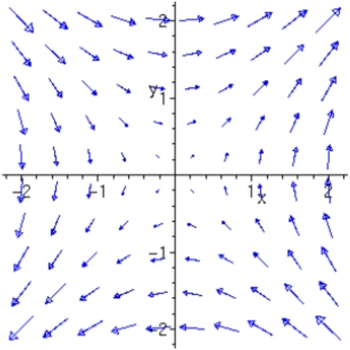
$$F_1(x, y) = (2x, 2y) \qquad F_2(x, y) = \left(\frac{-x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$$

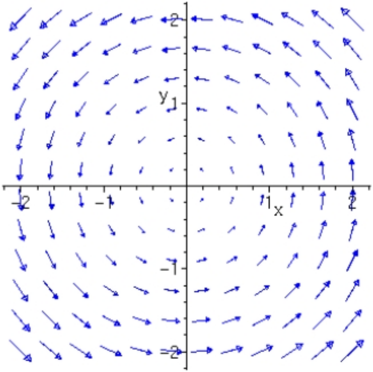




Vector fields: Examples

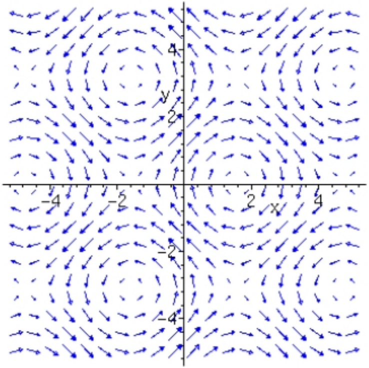
$F_3(x, y) = (y, x)$ $\qquad\qquad$ $F_4(x, y) = (-y, x)$



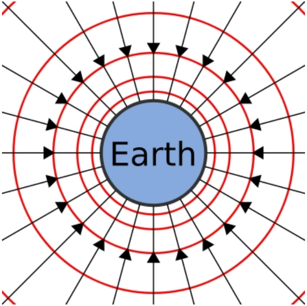


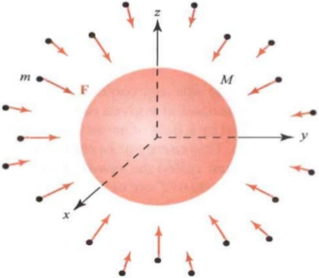
Vector fields: Examples

$$F_5(x, y) = (\sin y, \cos x)$$



Gravitation fields





The first figure describes the gravitational field of the earth whereas the second one describes that of a body with mass M. The red lines denote the direction of the force exerted on the small particles around the body.

We will assume from now on that our vector fields are smooth wherever they are defined.

One important class of vector fields are those that are given by the gradient of a scalar function. We will study these in some detail later.

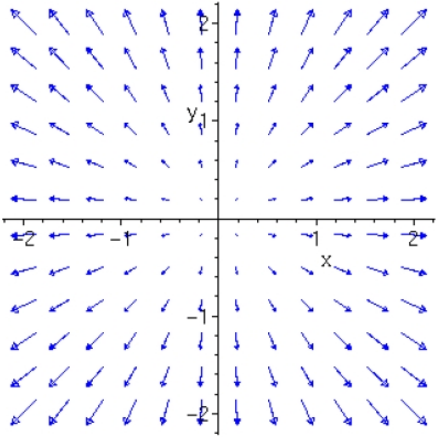
The del operator on functions: We define the del operator restricting ourselves to the case $n = 3$:

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

The del operator acts on functions $f : \mathbb{R}^3 \to \mathbb{R}$ to give a gradient vector field :

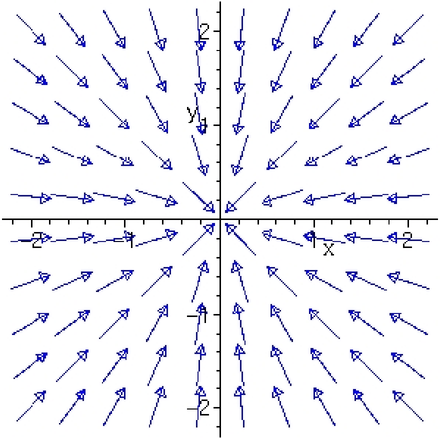
$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Thus the del operator takes scalar functions to vector fields.



$$F_1(x, y) = (2x, 2y) = \nabla(x^2 + y^2)$$

Gradient fields



$$F_2(x, y) = \left(\frac{-x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right) = \nabla \left(-\sqrt{x^2 + y^2} \right)$$

Gradient Vector fields

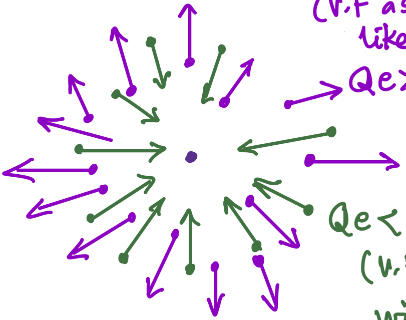
Coulomb's law says that the force acting on a charge e at a point r due to a charge Q at the origin is

$$F = -\nabla V$$

where $V = \epsilon Qe/r$ is the potential. For like charges $Qe > 0$ force is repulsive and for unlike charges $Qe < 0$ the force is attractive.

(V.F associated with like charges)

$Qe>0$



$Qe<0$

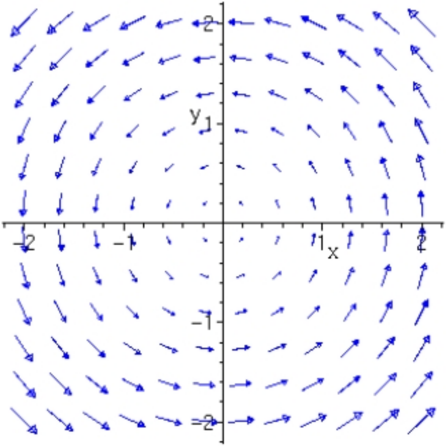
(V.F associated with unlike

Such a force is called conservative. Conservative forces are important as work done along a path will be only dependent on the end points.

Several of the examples we have seen turn out to be gradient vector fields. The natural question to ask is which vector field is a gradient field.

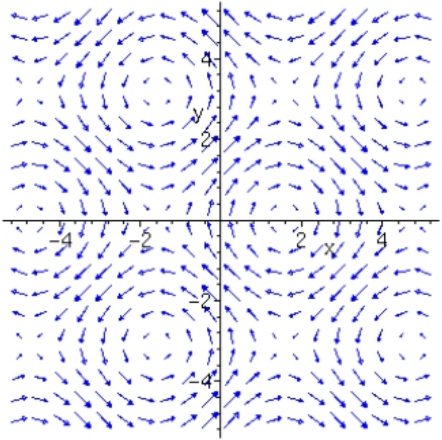
There is a neat answer to the above question, which we will see later. Not all vector fields will turn out to be gradient vector field.

Not gradient fields



$F_4(x, y) = (-y, x)$, this vector field is not ∇f for any f.

Not gradient fields



$F_5(x, y) = (\sin y, \cos x)$, this vector field is not ∇f for any f.

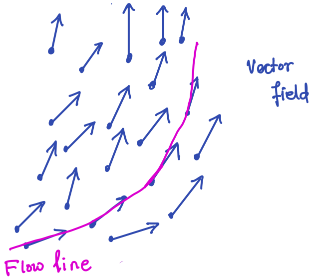
Flow lines for vector field

Vector fields also arise as the tangent vectors to the fluid flow.
Or conversely, given a vector field we can talk about its flow lines.

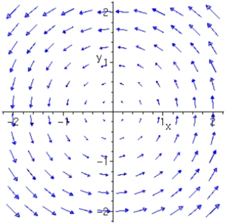
Definition If $\mathbf{F}$ is a vector field defined from $D \subset \mathbb{R}^n$ to $\mathbb{R}^n$, a flow line or integral curve is a path i.e., a map $\mathbf{c} : [a, b] \to D$ such that

$$\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t)), \quad \forall t \in [a, b].$$

In particular, $\mathbf{F}$ yields the velocity field of the path $\mathbf{c}$.



Example: Show that $c(t) = (\cos t, \sin t)$ is a flow line for the vector field $F(x, y) = -y\mathbf{i} + x\mathbf{j}$. Does it have other flow lines? Can you guess by looking at the vector field?



Finding the flow line for a given vector field involves solving a system of differential equations, if $c(t) = (x(t), y(t), z(t))$ then

$$\begin{aligned} x'(t) &= P(x(t), y(t), z(t)) \\ y'(t) &= Q(x(t), y(t), z(t)) \\ z'(t) &= R(x(t), y(t), z(t)), \end{aligned}$$

where the vector field is given by $F = (P, Q, R)$.

Such questions are dealt with in MA108.

Curve and path

Recall a path in $\mathbb{R}^n$ is a continuous map $\mathbf{c} : [a, b] \to \mathbb{R}^n$.

A curve in $\mathbb{R}^n$ is the image of a path $\mathbf{c}$ in $\mathbb{R}^n$.

Both the curve and path are denoted by the same symbol $\mathbf{c}$.

- Let $n = 3$ and $\mathbf{c}(t) = (x(t), y(t), z(t))$, for all $t \in [a, b]$. The path $\mathbf{c}$ is continuous iff each component x, y, z is continuous. Similarly, $\mathbf{c}$ is a C^1 path, i.e., continuously differentiable if and only if each component is C^1.

- A path $\mathbf{c}$ is called closed if $\mathbf{c}(a) = \mathbf{c}(b)$.

- A path $\mathbf{c}$ is called simple if $\mathbf{c}(t_1) \neq \mathbf{c}(t_2)$ for any $t_1 \neq t_2$ in $[a, b]$ other than $t_1 = a$ and $t_2 = b$ endpoints.

- If we write $\mathbf{c}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ in vector notation, the tangent vector to $\mathbf{c}(t)$ is $\mathbf{c}'(t)$, i.e.,

$$\mathbf{c}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.$$

- If a C^1 curve $\mathbf{c}$ is such that $\mathbf{c}'(t) \neq 0$ for all $t \in [a, b]$, the curve is called a regular or non-singular parametrised curve.

Examples of curves

• Let $\mathbf{c}(t) = (\cos 2\pi t, \sin 2\pi t)$ where $0 \leq t \leq 1$. This is a simple closed C^1 (actually smooth) curve.

• Let $\mathbf{c}(t) = (t, t^2)$ where $-1 \leq t \leq 5$ is a simple curve but not closed.

• Let $\mathbf{c}(t) = (\sin(2t), \sin t)$ where $-\pi \leq t \leq \pi$. It traces out a figure 8. It is not a simple but a closed C^1 curve.

• Let $\mathbf{c}(t) = (t^3, t)$ where $-1 \leq t \leq 1$. Then $\mathbf{c}$ is a part of the graph of the function $y = x^{1/3}$. This is simple but not a closed curve. Though the function $y = x^{\frac{1}{3}}$ is not a smooth function at origin, but this parametrization is regular.

Work done along a curve

• Recall from Physics, that work done by a particle on which force $\mathbf{F}$ is applied is given by the $\mathbf{F}.ds$ where ds is the displacement.

• If this is in one variable it is just the product and given by dot-product when it is in 2D or 3D space. This idea works when the displacement is straight line.

• If the particle is moving along a curve $\mathbf{c}$ then locally the curve can be approximated by a straight line.

• For a path $\mathbf{c} : [0, 1] \to \mathbb{R}^n$ for $n = 2$ or 3 if $\Delta t = t_2 - t_1$ is *very very* small then

$$\Delta s = \mathbf{c}(t_2) - \mathbf{c}(t_1) = \mathbf{c}'(\widehat{t})(t_2 - t_1)$$

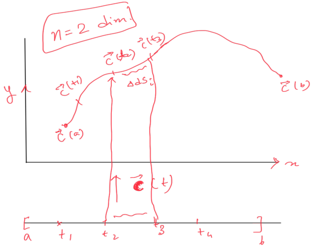
for some $\widehat{t} \in [t_1, t_2]$ by mean value theorem.

• Then work done will have to be computed over these small intervals $[t_i, t_{i+1}]$ for $i = 1, \ldots, n$.

Work done along a curve contd.

- Total work done $= \sum_{i=1}^{n} \mathbf{F}(\mathbf{c}(\widehat{t_i})) \cdot (\mathbf{c}(t_{i+1}) - \mathbf{c}(t_i))$
$= \sum_{i=1}^{n} (\mathbf{F}(\mathbf{c}(\widehat{t_i})) \cdot \mathbf{c}'(\widehat{t_i}))(t_{i+1} - t_i)$.
Does this remind you of something? Riemann sum: The limit of these Riemann sums as the length of the subintervals tends to zero, if it exists, is defined to be the line integral of the vector field $\mathbf{F}$ over the curve $\mathbf{c}$ and is denoted by

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$$



Line integrals of vector fields

Assume that the vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, for $n = 1, 2$, is continuous and the curve $\mathbf{c} : [a, b] \to D$ is C^1.

Then we define the line integral of $\mathbf{F}$ over $\mathbf{c}$ as:

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} := \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt.$$

If $\mathbf{F} = (F_1, F_2, F_3)$ and $\mathbf{c}(t) = (x(t), y(t), z(t))$, we see that

$$\begin{aligned} &\int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt \\ &= \int_a^b \left(F_1(\mathbf{c}(t))\frac{dx(t)}{dt} + F_2(\mathbf{c}(t))\frac{dy(t)}{dt} + F_3(\mathbf{c}(t))\frac{dz(t)}{dt} \right) dt. \end{aligned}$$

Because of the above form, the line integral is sometimes written as

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b F_1 dx + F_2 dy + F_3 dz.$$

The expression on the right hand side is just alternate notation for the line integral. It does not have any independent meaning.

Examples

Example 1: Evaluate

$$\int_{\mathbf{c}} x^2 dx + xydy + dz,$$

where $\mathbf{c} : [0,1] \to \mathbb{R}^3$ is given by $\mathbf{c}(t) = (t, t^2, 1)$.

Solution: Let $\mathbf{c}(t) = (t, t^2, 1)$.
Let $\mathbf{F}(x,y,z) = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)) = (x^2, xy, 1)$.

Thus $F_1(t, t^2, 1) = t^2$, $F_2(t, t^2, 1) = t^3$ and $F_3(t, t^2, 1) = 1$.

We have $\mathbf{c}'(t) = (1, 2t, 0)$, hence

$$(F_1(t,t^2,1), F_2(t,t^2,1), F_3(t,t^2,1)) \cdot \mathbf{c}'(t) = t^2 + 2t^4 + 0.$$

$$\int_{\mathbf{c}} x^2 dx + xydy + dz = \int_0^1 (t^2 + 2t^4) dt = 11/15.$$

Examples

Example 2 (Marsden, Tromba, Weinstein): Find the work done by the force field $\mathbf{F} = (x^2 + y^2)(\mathbf{i} + \mathbf{j})$ around the loop $\mathbf{c}(t) = (\cos t, \sin t)$, $0 \leq t \leq 2\pi$.
Solution: The work done is given by

$$\begin{aligned} W &= \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} \\ &= \int_0^{2\pi} \mathbf{F}(\mathbf{c}(t)).\mathbf{c}'(t)\ dt \\ &= \int_0^{2\pi} (\cos t - \sin t) dt \\ &= (\sin t + \cos t)\Big|_0^{2\pi} = 0 \end{aligned}$$

Integrating along successive paths

It is easy to see that if $\mathbf{c}_1$ is a path joining two points P_0 and P_1 and $\mathbf{c}_2$ is a path joining P_1 and P_2 and $\mathbf{c}$ is the union of these paths (that is, it is a path from P_0 to P_2 passing through P_1), which is C^1 then

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}.$$

This property follows directly from the corresponding property for Riemann integrals:

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt,$$

where c is a point between a and b.

This allows us to define integration over peicewise differentiable curves for example the perimeter of a square.

Let the curve $\mathbf{c}$ be a union of curves $\mathbf{c}_1, \ldots, \mathbf{c}_n$. We often write this as $\mathbf{c} = \mathbf{c}_1 + \mathbf{c}_2 + \ldots \mathbf{c}_n$, where end point of $\mathbf{c}_i$ is the starting point of $\mathbf{c}_{i+1}$ for all $i = 1, \ldots, n-1$.

Then we can define

$$\int_{\mathbf{c}} \mathbf{F} \cdot ds := \int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} + \ldots + \int_{\mathbf{c}_n} \mathbf{F} \cdot d\mathbf{s}.$$

Divide the curve $\mathbf{c}$ at a point p into two curves $\mathbf{c}_1$ and c_2. Then there it is easy to verify that $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}$.

Let $\mathbf{c}$ be a curve on $[a, b]$ and $\widetilde{\mathbf{c}}(t) = \mathbf{c}(a + b - t)$, that is the curve $\widetilde{\mathbf{c}}$ traversed in the reverse direction and is denoted by $-\mathbf{c}$. What is $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} + \int_{-\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$?

$\int_{c} \mathbf{F} \cdot d\mathbf{s} = -\int_{-c} \mathbf{F} \cdot d\mathbf{s}$ (use change of variables formula).