MA105 : Quiz II Solutions

- Fill in the numbers "A" and "B" above as follows:
  If the last digit a of your roll number satisfies $0 \leq a \leq 4$, let $A = a + 5$. If $5 \leq a \leq 9$, let $A = a$.
  If the second-last digit b of your roll number satisfies $0 \leq b \leq 4$, let $B = b + 5$. If $5 \leq b \leq 9$, let $B = b$. Thus $5 \leq A, B \leq 9$.
  Example: Your Roll number is 23B0092. Then $A = 7$ and $B = 9$.
  You must use these values of A and B below. Using the wrong value of A or B in even one question may lead to the loss of full marks in this exam.

Write the answers of Questions (1)-(5) **only** in the box provided below the questions.
You will get full (respectively, partial) marks in Question (5) below only if you select all (respectively, some but not all) of the TRUE statements and only the TRUE statements (that is, if you select a FALSE statement, you will get ZERO mark in that question).

Let $C = \min\{A, B\}$.

(1) (2 marks) Let P_0 be a point inside a disc of radius B such that $\min\{A, B\} - 1$ is the distance of P_0 from the centre of the disc. Let d denote the distance from arbitrary point P inside the disc to P_0. Find the average value of d^2 over the disc.

$$\boxed{\frac{B^2}{2} + (C-1)^2}$$

(2) (2 marks) What region R in xy-plane minimizes the value of

$$\int\int_R (x^2 + y^2 - AB)\, dxdy?$$

$$\boxed{R = D(0, \sqrt{AB}),\ \text{where } D(0, r) \text{ denotes the disc with centre at } 0 \text{ and radius } r}$$

(3) (2 marks) Let R be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = A, xy = \min\{A, B\} - 1$ and the lines $y = x, y = 4x$. Use the transformation $x = u/v$, $y = uv$ to evaluate the integral

$$\int\int_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right)\, dxdy.$$

$$\boxed{A - C + 1 + \tfrac{2}{3}\left(A^{3/2} - (C-1)^{3/2}\right)\log 2}$$

(4) (2 marks) Find the volume of the solid in the first octant bounded by the coordinate planes, the cylinder $x^2 + y^2 = 1$ and the plane $y + z = A$.

$$\boxed{\frac{\pi A}{4} - \frac{1}{3}}$$

(5) (2 marks) Let $f(x, y)$ be a function on $[0, 1] \times [0, 1]$ defined by

$$f(x, y) = \begin{cases} Ay & x \text{ is rational} \\ 1 + By & x \text{ is irrational} \end{cases}.$$

Which of the following statements is/are TRUE?

(a) f is not integrable on $[0, 1] \times [0, 1]$.

(b) None of the iterated integrals of f exists.

(c) One of the iterated integrals of f exists.

(d) For only one $y_0 \in [0, 1]$, the integral $\int_0^1 f(x, y_0)\, dx$ exists.

(e) For all $y_0 \in [0, 1]$, the integral $\int_0^1 f(x, y_0)\, dx$ does not exist.

Solution:

(a) is always true.

(b) is true iff $A - B \neq 2$.

(c) is true iff $A - B = 2$.

(d) is true iff $A > B$.

(e) is true iff $A \leq B$.