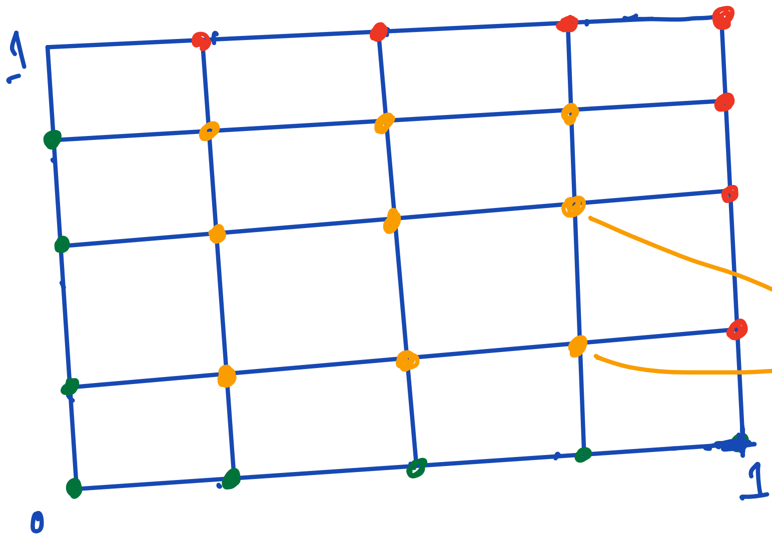
Monotone function is integrable

Regular Partitions of the unit square



These points does not contribute in $U(f,P_n) - L(f,P_n)$

$$U(f,P_n) - L(f,P_n)$$

$$\leq \frac{1}{n^2} \sum_{i=1}^{2n-1} \left(f(1,1) - f(0,0) \right)$$

$$= \frac{2n-1}{n^2} \left[f(1,1) - f(0,0) \right]$$

$$\longrightarrow 0 \quad \text{as} \quad n \to \infty .$$

Bivariate Thomae function

$$f(x,y) = \begin{cases} 1, & \text{if } x=0 \text{ and } y \in \mathbb{Q}\cap[0,1] \\ 1/q, & \text{if } x,y \in \mathbb{Q}\cap[0,1], \text{ and } x = p/q \\ 0, & \text{otherwise} \end{cases}$$

- To show f is Riemann integrable complete the following steps:

- Show that given $\varepsilon>0$, the set
$$B = \{x \in [0,1] : f(x,y) \geq \varepsilon/2 \text{ for some } y \in [0,1]\}$$
is a finite set.

- Suppose $\# B = m$ and
$$B = \{x_1, \ldots, x_m\}.$$

- Take the regular partition P of $[0,1]\times[0,1]$ of order N with $\frac{1}{N} < \frac{\varepsilon}{4m}$

- Show that $U(f,P) < \varepsilon$.

This implies f is integrable!! and $\iint f = 0$

Solve the following problems on Bivariate Thomae function.

I. Also show that the Bivariate Thomae function is continuous except the set $\{(x,y) : x \in \mathbb{Q} \cap [0,1]\}$.

$\Rightarrow$ The Bivariate Thomae function is integrable despite being discontinuous on a "large" set.

II. Using the integrability of one-variate Thomae function show that the following iterated integral

$$\int_0^1 \left(\int_0^1 f(x,y)\, dx \right) dy$$ exists.

III: However, show that for each $x \in \mathbb{Q} \cap [0,1]$, the integral $\int_0^1 f(x,y)\, dy$ does not exist.

$$f \equiv 1 \quad \text{on } [0,1] \times [0,1]$$

$$D = \{(x,y) : y, x \in \mathbb{Q} \cap [0,1]\}$$

$$= (\mathbb{Q} \cap [0,1]) \times (\mathbb{Q} \cap [0,1])$$

$$g := f\big|_D .$$

Claim: g is un integrable on D.

Consider the unit square $[0,1] \times [0,1] \supset D$.

Then the extension of g is given

by $g^*(x,y) = \begin{cases} 1 & : x, y \in \mathbb{Q} \cap [0,1] \\ 0, & \text{otherwise} \end{cases}$

We know that g^* is un integrable

over $[0,1] \times [0,1]$.