MA 105 Calculus II

Lecture 8

B. K. Das

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

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1. Characterization of conservative fields Contd.

2. Green's theorem

3. Various examples

Recall

For a given vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$ for $n = 2, 3$:

1. If $\mathbf{F}$ is a continuous, conservative vector field, i.e., $\mathbf{F} = \nabla f$, for some C^1 scalar function, then the line integral of $\mathbf{F}$ along any path C from P to Q in D given by

$$\int_C \mathbf{F}.\mathbf{ds} = f(Q) - f(P),$$

   and it only depends on the value of f, the potential function, at the initial and terminal points of the path.

2. Line integral of a conservative vector field is independent of path.

Examples

Example Find the work done by the gravitational field $\mathbf{F}(x,y,z) = -\frac{mMG}{|\mathbf{r}(x,y,z)|^3}\mathbf{r}(x,y.z)$, in moving a particle with mass m and position vector $\mathbf{r}(x,y,z) = (x,y,z)$ from $(3,4,12)$ to the point $(2,2,0)$ along a piecewise-smooth curve C.

Ans Since the gravitational field is a conservative field and $\mathbf{F}(x,y,z) = \nabla f(x,y,z)$, where

$$f(x,y,z) = \frac{mMG}{|\mathbf{r}(x,y,z)|} = \frac{mMG}{\sqrt{x^2+y^2+z^2}}.$$

Using the Fundamental theorem for line integrals, the work done is

$$W = \int_{\mathbf{c}} \mathbf{F}.d\mathbf{s} = f(\mathbf{c}(b) - f(\mathbf{c}(a)) = f(2,2,0) - f(3,4,12) = mMG(\frac{1}{2\sqrt{2}} - \frac{1}{13}),$$

where $\mathbf{c} : [a,b] \to \mathbb{R}$, a parametrisation of curve C with $\mathbf{c}(a) = (3,4,12)$ and $\mathbf{c}(b) = (2,2,0)$.

Example Evaluate $\int_C y^2dx + xdy$, where

1. $C = C_1$ is the line segment from $(-5,-3)$ to $(0,2)$,
2. $C = C_2$ is the part of parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.

Are the line integrals along C_1 and C_2 same?

Ans 1.) Consider parametrisation for C_1,
$\mathbf{c}_1(t) = (5t-5, 5t-3), \quad t \in [0,1]$. Thus $\mathbf{c}_1'(t) = (5,5)$ for all $t \in [0,1]$.
So, $\mathbf{F}(\mathbf{c}_1(t)) = ((5t-3)^2, 5t-5)$ and

$$\int_{C_1} y^2dx + xdy = \int_0^1 [(5t-3)^2.5 + (5t-5).5]\, dt = -\frac{5}{6}.$$

2. Consider parametrisation for C_2, $\mathbf{c}_2(t) = (4-t^2, t), \quad t \in [-3,2]$. Thus $\mathbf{c}_2'(t) = (-2t, 1)$ for all $t \in [-3,2]$. So, $\mathbf{F}(\mathbf{c}_2(t)) = (t^2, 4-t^2)$ and

$$\int_{C_2} y^2dx + xdy = \int_{-3}^2 [t^2(-2t) + (4-t^2)]\, dt = 40\frac{5}{6}.$$

Line integrals along C_1 and C_2 are Not same! Though the endpoints of C_1 and C_2 are same!

Conservative vector fields

In general, the line integral of a vector field depends on the path.

Fundemental theorem of calculus for line integrals yields that the line integral of a conservative field is independent of path in D.

What about the converse?

We will now prove the converse to our previous assertion under some assumption on D.

Definition: A subset D of $\mathbb{R}^n$ is called connected if it cannot be written as a disjoint union of two non-empty subsets $D_1 \cup D_2$, with $D_1 = D \cap U_1$ and $D_2 = D \cap U_2$, where U_1 and U_2 are open sets.

Definition: A subset of D of $\mathbb{R}^n$ is said to be path connected if any two points in the subset can be joined by a path (that is the image of a continuous curve) inside D.

In $\mathbb{R}^n$ we can show that an open subset is connected if and only if it is path connected. So it is sufficient to assume region of the vector field is open and connected.

Examples

Example. $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < a^2\}$ is path-connected.

Ans. If $P = (x_0, y_0)$ and $Q = (x_1, y_1)$ are in D, then which path lying in D can be defined connecting P and Q?

Path connected implies connected.

Example. $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\} \cup \{(2, 2)\}$ is connected in $\mathbb{R}^2$?
Ans No. (Why?)

Example. $D = \{(x, \sin(\frac{1}{x})) \in \mathbb{R}^2 \mid x \in (0, 1]\} \cup \{(0, 0)\}$ is connected in $\mathbb{R}^2$ but not path-connected.

Theorem

Let $\mathbf{F} : D \to \mathbb{R}^3$ *be a continuous vector field on a connected open region* D *in* $\mathbb{R}^3$. *If the line integral of* $\mathbf{F}$ *is independent of path in* D, *then* $\mathbf{F}$ *is a conservative vector field in* D.

Proof: Observe that the goal is to find a differentiable function $V : D \to \mathbb{R}$ such that

$$\mathbf{F}(x, y, z) = \nabla V(x, y, z), \quad \text{for all} \quad (x, y, z) \in D.$$

We construct such V in the following way.
Step 1 Let $P_0 = (x_0, y_0, z_0)$ be a fixed point in D. Let $P = (x, y, z)$ be an arbitrary point in D. We define

$$V(x, y, z) = \int_{\mathbf{c}_P} \mathbf{F} \cdot d\mathbf{s}, \quad \text{for all} \quad (x, y, z) \in D,$$

where $\mathbf{c}_P : [a, b] \to D$ is any path from P_0 to P.
Since D is path connected, there always exists a path from P_0 to any point $P \in D$. Hence V is defined on the whole of D.

Since the line integral of $\mathbf{F}$ is path-independent in D, $V(x, y, z)$ does not depend on which path we took from P_0 to P and hence is well-defined.

The proof of theorem contd.

Step 2 It remains to show that $\mathbf{F} = \nabla V$.

Let $\mathbf{F} = (F_1, F_2, F_3)$. Then we have to show

$$\frac{\partial V}{\partial x} = F_1, \quad \frac{\partial V}{\partial y} = F_2, \quad \frac{\partial V}{\partial z} = F_3, \quad \text{on} \quad D.$$

Evaluate $\frac{\partial V}{\partial x}(x, y, z) = \lim_{h \to 0} \frac{V(x+h,y,z) - V(x,y,z)}{h}$ for all $(x, y, z) \in D$.

From definition of V,

$$V(x + h, y, z) = \int_{\mathbf{c}_{P_h}} \mathbf{F}.d\mathbf{s},$$

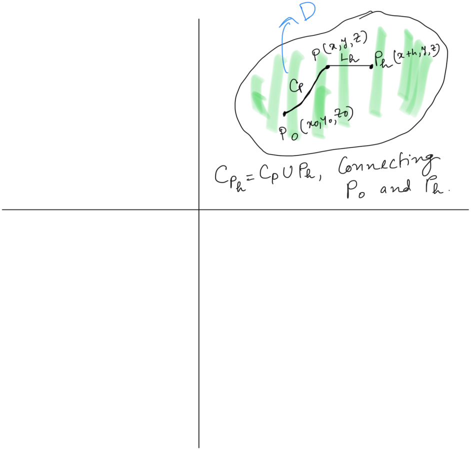
where $P_h = (x + h, y, z)$ and $\mathbf{c}_{P_h}$ is any path joining P_0 and P_h in D.

Choose $\mathbf{c}_{Ph}$ conveniently: Since D is open, for a given $P = (x, y, z) \in D$, there exists a disk contained in D with center P containing points $P_h = (x + h, y, z)$ for all h such that h is small enough. Thus for all h with $|h|$ suitable small, the straight line $\mathbf{L}_h$ joining P and P_h lies in D, where

$$\mathbf{L}_h(t) = (x + th, y, z) \quad \forall\, 0 \le t \le 1.$$

The proof of theorem contd.

We choose the path $\mathbf{c}_{P_h}$ from P_0 to P_h as the union of the two paths $\mathbf{c}_P$ from P_0 to P and the straight line $\mathbf{L}_h$ from P to P_h.



The proof of theorem contd.

From the property of line integrals we mentioned earlier

$$\int_{\mathbf{c}_{P_h}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}_P} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathbf{L}_h} \mathbf{F} \cdot d\mathbf{s}.$$

Hence it yields

$$V(x+h, y, z) = V(x, y, z)$$

$$+ \int_0^1 (F_1(x+th, y, z), F_2(x+th, y, z), F_3(x+th, y, z)) \cdot (h, 0, 0)dt$$

Thus

$$\frac{\partial V}{\partial x}(x, y, z) = \lim_{h \to 0} \frac{V(x+h, y, z) - V(x, y, z)}{h} = \lim_{h \to 0} \int_0^1 F_1(x+th, y, z)dt.$$

Due to the continuity of F_1,

$$\lim_{h \to 0} \int_0^1 F_1(x+th, y, z)dt = F_1(x, y, z).$$

The proof of theorem contd.

Hence we get

$$\frac{\partial V}{\partial x}(x,y,z) = F_1(x,y,z), \quad \forall\,(x,y,z) \in D.$$

We can similarly show that

$$\frac{\partial V}{\partial y} = F_2 \quad \text{and} \quad \frac{\partial V}{\partial z} = F_3, \quad \text{on} \quad D.$$

This proves our theorem.

In summary, for a given continuous vector field $\mathbf{F}$ in $\mathbb{R}^n$ defined on D, an open, path connected subset of $\mathbb{R}^n$, the vector field $\mathbf{F}$ is a conservative field if and only if the line integral of $\mathbf{F}$ in D is independent of path in D.

Examples Contd.

Example Determine whether or not the vector field

$$\mathbf{F}(x, y) = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}, \quad \text{in} \quad \mathbb{R}^2 \backslash \{(0,0)\},$$

is conservative.

Ans Check for the closed curve $\mathbf{c} = (\cos t, \sin t)$, $t \in [0, 2\pi]$, the line integral of $\mathbf{F}$ along $\mathbf{c}$?

$$\int_{\mathbf{c}} \mathbf{F}.d\mathbf{s} = \int_0^{2\pi} (\sin t)(\sin t) + (\cos t)(\cos t)\, dt = 2\pi.$$

so, $\int_{\mathbf{c}} \mathbf{F}.d\mathbf{s} \neq 0$, though $\mathbf{c}$ is a closed curve, and hence $\mathbf{F}$ cannot be conservative field.

However, the equivalent formulation of conservative field and the path independency of the line integral of the vector field may not be always useful to determine if a vector field conservative.

Necessary condition for conservative fields

Theorem

- *For $n = 2$, if $\mathbf{F}(x, y) = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}$ is a conservative vector field, where F_1 and F_2 have continuous first-order partial derivatives on an open region D in $\mathbb{R}^2$, then*

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \text{on} \quad D.$$

- *For $n = 3$, if $\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}$ is a conservative vector field, where F_1, F_2, F_3 have continuous first-order partial derivatives on an open region D in $\mathbb{R}^3$, then*

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} \quad \text{on} \quad D.$$

The theorem follows from a direct calculation using the fact that $\mathbf{F} = \nabla V$ and using the properties of the mixed partial derivatives of V.

Example Determine whether or not the vector field

$$\mathbf{F}(x, y) = (x - y)\mathbf{i} + (x - 2)\mathbf{j}, \quad \text{in} \quad \mathbb{R}^2$$

is conservative.

Ans Here $F_1(x, y) = x - y$ and $F_2(x, y) = x - 2$. Then

$$\frac{\partial F_1}{\partial y} = -1, \quad \text{and} \quad \frac{\partial F_2}{\partial x} = 1.$$

So by previous theorem, **F** cannot be a conservative field.

What about the converse of the theorem?

The converse is partially true under some additional hypothesis on D.

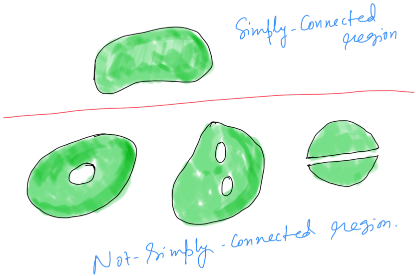
However, it is often a convenient method verifying if a vector field is conservative.

Simply connected domain

Definition

A subset D of $\mathbb{R}^n$ for $n = 2, 3$, is simply connected, if D is a connected region such that any simple closed curve lying in D encloses a region that is in D.

Basically, a simply-connected region contains no hole and cannot consist of two separate pieces.



Sufficient condition for conservative field

Theorem

Let $n = 2, 3$ and let D be an open, simply connected region in $\mathbb{R}^n$.

1. *For $n = 2$, if $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ is such that F_1 and F_2 have continuous first order partial derivatives on D satisfying*

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \text{on} \quad D,$$

   Then $\mathbf{F}$ is a conservative field.

2. *For $n = 3$, if $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is such that F_1, F_2 and F_3 have continuous first order partial derivatives on D satisfying*

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} \quad \text{on} \quad D,$$

   Then $\mathbf{F}$ is a conservative field.

We postpone the proof (Green's theorem!)

Examples

Example. Determine whether or not the vector field

$$\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}, \quad \text{in} \quad \mathbb{R}^2$$

is conservative.

Ans Note that the region $\mathbb{R}^2$ is open and simply-connected and $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable.

Let $F_1(x, y) = (3 + 2xy)$ and $F_2(x, y) = x^2 - 3y^2$. Then

$$\frac{\partial F_1}{\partial y}(x, y) = 2x = \frac{\partial F_2}{\partial x}.$$

Thus using the previous theorem, we conclude that $\mathbf{F}$ is a conservative field.

How to find a potential function f such that $\mathbf{F} = \nabla f$, for above example?

Example contd.

Let $\mathbf{F} = \nabla f$, then $\frac{\partial f}{\partial x}(x, y) = F_1(x, y)$ and $\frac{\partial f}{\partial y}(x, y) = F_2(x, y)$.
Step 1 Fixing y, solve the ODE with respect to x-variable:

$$\frac{\partial f}{\partial x}(x, y) = F_1(x, y).$$

Integrating with respect to x in both side, we get

$$f(x, y) = \int_0^x F_1(s, y)\, dx + c(y) = 3x + x^2 y + c(y).$$

Step 2 Determine the $c(y)$ using $\frac{\partial f}{\partial y}(x, y) = F_2(x, y)$. Differentiating $f(x, y)$ with respect to y,

$$\frac{\partial f}{\partial y}(x, y) = x^2 + c'(y),$$

and it has to be equal to $F_2(x, y)$.
so, $x^2 + c'(y) = x^2 - 3y^2$ and thus $c'(y) = -3y^2$. Now solving this ODE with respect to y variable:

$$c(y) = -y^3 + K,$$

In summary, for a given vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$ for $n = 2, 3$:

1. If $\mathbf{F}$ is a continuous, conservative vector field, i.e., $\mathbf{F} = \nabla f$, for some C^1 scalar function, then the line integral of $\mathbf{F}$ along any path C from P to Q in D given by
$$\int_C \mathbf{F}.\mathbf{ds} = f(Q) - f(P),$$
and it only depends on the value of f, the potential function, at the initial and terminal points of the path.
2. Let $\mathbf{F}$ be a continuous field and let D be an open connected set in $\mathbb{R}^n$. Then $\mathbf{F}$ is a conservative field if and only if the line integral of $\mathbf{F}$ is path-independent in D.
3. If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ is a C^1 conservative vector field on an open region D, then $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ on D. Similar result holds in $\mathbb{R}^3$.
4. Let D be an open, simply connected region in $\mathbb{R}^2$ and let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ be C^1 on D. Then $\mathbf{F}$ is conservative in D if and only if $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ on D. Similar result holds in $\mathbb{R}^3$.

Green's Theorem

Theorem (Green's theorem:)

1. *Let D be a bounded region in $\mathbb{R}^2$ with a positively oriented boundary ∂D consisting of a finite number of non-intersecting simple closed piecewise continuously differentiable curves.*
2. *Let Ω be an open set in $\mathbb{R}^2$ such that $\left(D \cup \partial D\right) \subset \Omega$ and let $F_1 : \Omega \to \mathbb{R}$ and $F_2 : \Omega \to \mathbb{R}$ be $\mathcal{C}^1$ functions.*

Then

$$\int_{\partial D} F_1 dx + F_2 dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy.$$

The importance of Green's theorem is that it converts a double integral into a line integral. Depending on the situation, one may be easier to evaluate than the other.

A proof of Green's theorem for regions of special type

We give a proof of Green's theorem when the region D is both of type 1 and type 2 .

Examples: Rectangles, Discs are examples of such region.

Assume that D is of Type 1

$$D = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, \quad \phi_1(x) \leq y \leq \phi_2(x)\},$$

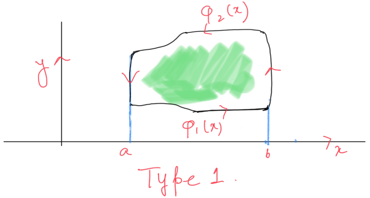
for some continuous functions ϕ_1 and ϕ_2.

Also assume there exist two continuous functions ψ_1 and ψ_2 such that D can be written as Type 2:

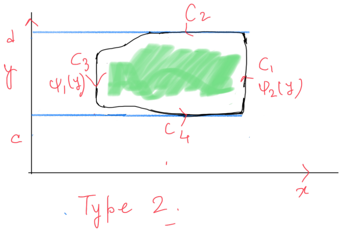
$$D = \{(x, y) \in \mathbb{R}^2 \mid c \leq y \leq d, \quad \psi_1(y) \leq x \leq \psi_2(y)\}.$$

The proof follows two main steps:

- Double integrals can be reduced to iterated integrals.
- Then the fundamental theorem of calculus can be applied to the resulting one-variable integrals.



φ2(x)
y
φ1(x)
a
b
x
Type 1.



C2
d
C3
y
ψ1(y)
C1
ψ2(y)
C4
c
x
Type 2.

The proof of Green's theorem, contd.

Step 1: Using the fact that D is a region of Type 2,

$$\iint_D \frac{\partial F_2}{\partial x} = \int_{\partial D} F_2 dy.$$

Step 2: Using the fact that D is a region of Type 1,

$$-\iint_D \frac{\partial F_1}{\partial y} = \int_{\partial D} F_1 dx.$$

Then combining the both equalities, we get our result.
Since D is a region of Type 2, it gives

$$\iint_D \frac{\partial F_2}{\partial x} dxdy = \int_c^d \int_{x=\psi_1(y)}^{\psi_2(y)} \frac{\partial F_2}{\partial x}(x,y) dxdy.$$

Using the Fundamental Theorem of Calculus we get

$$\int_c^d \int_{x=\psi_1(y)}^{\psi_2(y)} \frac{\partial F_2}{\partial x}(x,y) dxdy = \int_c^d F_2(\psi_2(y),y) - F_2(\psi_1(y),y) dy$$

The proof of Green's theorem contd.

Now let us calculate $\int_{\partial D} F_2 dy$. Note that ∂D can be written as union of four curves C_1, C_2, C_3 and C_4 such that

On C_1: $C_1 = \{(\psi_2(y), y) \in \mathbb{R}^2 \mid c \leq y \leq d\}$ with direction upwards. So,

$$\int_{C_1} F_2\, dy = \int_c^d F_2(\psi_2(y), y)\, dy.$$

On C_3: $C_3 = \{(\psi_1(y), y) \in \mathbb{R}^2 \mid c \leq y \leq d\}$ with direction downwards. So,

$$\int_{C_3} F_2\, dy = -\int_{-C_3} F_2\, dy = -\int_c^d F_2(\psi_1(y), y)\, dy.$$

On C_2 and C_4: $C_2 = \{(x, d) \mid \psi_1(d) \leq x \leq \psi_2(d)\}$ going from right to left and $C_4 = \{(x, c) \mid \psi_1(c) \leq x \leq \psi_2(c)\}$ going from left to right. In particular, they are vertical lines and y is constant along these lines. Thus, for any parametrisation of C_2 and C_4, $\frac{dy}{dt} = 0$, and

$$\int_{C_2} F_2\, dy = 0 = \int_{C_4} F_2\, dy.$$

The proof of Green's theorem contd.

$$\int_{\partial D} F_2 dy = \int_{C_1} F_2\, dy + \int_{C_2} F_2\, dy + \int_{C_3} F_2\, dy + \int_{C_4} F_2\, dy,$$

and using previous results, we obtain

$$\int_{\partial D} F_2 dy = \int_c^d F_2(\psi_2(y), y)\, dy - \int_c^d F_2(\psi_1(y), y)\, dy,$$

and thus

$$\iint_D \frac{\partial F_2}{\partial x} dxdy = \int_{\partial D} F_2 dy.$$

Similarly, using that D can be written as a region of Type 1, we get

$$\iint_D \frac{\partial F_1}{\partial y} dxdy = -\int_{\partial D} F_1 dx.$$

Subtracting the two equations above, we get

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \int_{\partial D} F_1 dx + \int_{\partial D} F_2 dy.$$

Example: Let C be the circle of radius r oriented in the counterclockwise direction, and let $F_1(x, y) = -y$ and $F_2(x, y) = x$. Evaluate

$$\int_C F_1(x, y)dx + F_2(x, y)dy.$$

Solution: Let D denote the disc of radius r. Then $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 2$. Hence, by Green's theorem

$$\int_C F_1(x, y)dx + F_2(x, y)dy = \iint_D 2dxdy = 2\pi r^2.$$

Also by the direct calculation, denoting $\mathbf{F} = (F_1, F_2)$, check
$\displaystyle\int_C \mathbf{F}.d\mathbf{s} = \int_C F_1(x, y)dx + F_2(x, y)dy = ?$.

Examples.

Example. Compute the line integral $\int_C ye^{-x}\,dx + (\frac{1}{2}x^2 - e^{-x})\,dy$, where C is the positively oriented circle in $\mathbb{R}^2$ with center at $(2,0)$ and radius 1.

Can compute directly using definition of line integral! But is there any better way?

Use Green's theorem: Set $F_1(x,y) = ye^{-x}$ and $F_2(x,y) = (\frac{1}{2}x^2 - e^{-x})$, for all $(x,y) \in D$, where $D = \{(x,y) \in \mathbb{R}^2 \mid (x-2)^2 + y^2 \leq 1\}$. Using Green's theorem,

$$\int_C ye^{-x}\,dx + (\frac{1}{2}x^2 - e^{-x})\,dy = \int\int_D \Big[\frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y)\Big]\,dxdy.$$

Now see

$$\int\int_D \Big[\frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y)\Big]\,dxdy = \int\int_D xdxdy,$$

and derive the double integral using polar coordinates: Check!

$$\int\int_D xdxdy = 2\pi.$$

Area of a region

Can the area of a region enclosed be expressed as a line integral?

If C is a positively oriented curve that bounds a region D , then the area $A(D)$ is given by (Why?)

$$A(D) = \frac{1}{2}\int_C xdy - ydx.$$

Note if $F_1(x,y) = -\frac{y}{2}$ and $F_2(x,y) = \frac{x}{2}$, for all $(x,y) \in D$, then $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$, and hence $A(D) := \int\int_D 1\, dxdy = \int\int_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dxdy$.

By Green's theorem,

$$\int_D\int_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \int_C F_1\, dx + F_2\, dy = \frac{1}{2}\int_C xdy - ydx,$$

Thus $A(D) = \frac{1}{2}\int_C xdy - ydx$.

Also note for $F_1 \equiv 0$ and $F_2(x,y) = x$, on D, $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$, thus $A(D) = \int_C x\, dy$.

Further for $F_1(x,y) = -y$ and $F_2 \equiv 0$, on D, $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$, thus $A(D) = -\int_C y\, dx$.

Example: Let us use the formula above to find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution: We parametrise the curve C by $\mathbf{c}(t) = (a\cos t, b\sin t)$, $0 \leq t \leq 2\pi$. By the formula above, we get

$$\begin{aligned}
\text{Area} &= \frac{1}{2}\int_C xdy - ydx \\
&= \frac{1}{2}\int_0^{2\pi}(a\cos t)(b\cos t)dt - (b\sin t)(-a\sin t)dt \\
&= \frac{1}{2}\int_0^{2\pi} abdt = \pi ab.
\end{aligned}$$

Polar coordinates

Suppose we are given a simple positively oriented closed curve $C : (r(t), \theta(t))$ in polar coordinates. Thus for $t \in [a, b]$ $x(t) = r(t)\cos(\theta(t))$ and $y(t) = r(t)\sin(\theta(t))$ and using chain rule formula:

$$\frac{dx}{dt} = \cos(\theta(t))\frac{dr}{dt} - r(t)\sin\theta(t)\frac{d\theta}{dt},$$
$$\frac{dy}{dt} = \sin(\theta(t))\frac{dr}{dt} + r(t)\cos\theta(t)\frac{d\theta}{dt}.$$

Then, by the area formula above, the area enclosed by C is given by

$$\begin{aligned}
&\frac{1}{2}\int_C xdy - ydx := \frac{1}{2}\int_a^b \left(x(t)\frac{dy}{dt}(t) - y(t)\frac{dx}{dt}(t)\right) dt \\
&= \frac{1}{2}\int_a^b r(t)\cos\theta(t)\sin\theta(t)\frac{dr}{dt}dt + \frac{1}{2}\int_a^b r^2(t)\cos^2\theta(t)\frac{d\theta}{dt}dt \\
&-\frac{1}{2}\int_a^b r(t)\sin\theta(t)\cos\theta(t)\frac{dr}{dt}dt + \frac{1}{2}\int_a^b r(t)^2\sin^2\theta(t)\frac{d\theta}{dt}dt \\
&= \frac{1}{2}\int_a^b r(t)^2\frac{d\theta}{dt}dt.
\end{aligned}$$

Exercise: Find the area of the cardioid $r = a(1 - \cos\theta)$, $0 \le \theta \le 2\pi$.

Solution: Parametrize the boundary curve by $t \mapsto (r(t), \theta(t))$, $t \in [0, 2\pi]$, where $r(t) = a(1 - \cos t)$ and $\theta(t) = t$. Using the formula we have just derived, the desired area is simply

$$\frac{1}{2}\int_0^{2\pi} a^2(1-\cos t)^2 dt = \frac{a^2}{2}\int_0^{2\pi}(-2\cos t + \frac{\cos 2t}{2} + \frac{3}{2})dt$$
$$= \frac{3a^2\pi}{2}.$$