$$0 \le \left| \frac{f(x+h)-f(x)}{h} \right| \le C|h|^{\alpha-1}$$

Use sandwich theorem

16. Let $f : (a,b) \to \mathbb{R}$ be a function such that

$$|f(x+h) - f(x)| \le C|h|^{\alpha}$$

for all $x, x+h \in (a,b)$, where C is a constant and $\alpha > 1$. Show that f is differentiable on (a,b) and compute $f'(x)$ for $x \in (a,b)$.

17. If $f : (a,b) \to \mathbb{R}$ is differentiable at $c \in (a,b)$, then show that

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c-h)}{2h}$$

exists and equals $f'(c)$. Is the converse true ? [Hint: Consider $f(x) = |x|$.]

18. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy

$$f(x+y) = f(x)f(y) \text{ for all } x, y \in \mathbb{R}.$$

If f is differentiable at 0, then show that f is differentiable at every $c \in \mathbb{R}$ and $f'(c) = f'(0)f(c)$.

19. Using the theorem on derivative of inverse function, compute the derivative of (i) $\cos^{-1} x, \quad -1 < x < 1$. (ii) $\quad \operatorname{cosec}^{-1} x, \ |x| > 1$.

20. Compute $\dfrac{dy}{dx}$, given

$$y = f\left(\frac{2x-1}{x+1}\right) \text{ and } f'(x) = \sin(x^2).$$

Supplement

1. A sequence $\{a_n\}_{n \ge 1}$ is said to be Cauchy if for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon, \ \forall \ m, n \ge n_0$. In other words, if we choose n_0 large enough, we can make sure that the elements of a Cauchy sequence are close to each other as we want beyond n_0. One can show that a sequence in $\mathbb{R}$ is convergent if and only if it is Cauchy. To show that a convergent sequence is Cauchy is easy. To show that every Cauchy sequence converges is harder and, moreover, involves making a precise definition of the set of real numbers. Sets in which every Cauchy sequence converges are called *complete*. Thus the set of real numbers is complete.

2. To prove that a sequence $\{a_n\}_{n \ge 1}$ is convergent to a limit L, one needs to first guess what this limit L might be and then verify the required property. However the concept of 'Cauchyness' of a sequence is an intrinsic property, that is, we can decide whether a sequence is Cauchy by examining the sequence itself. There is no need to guess what the limit might be.

3. In problem 5(i), we defined

$$a_1 = \frac{3}{2}, \quad a_{n+1} = \frac{1}{2}(a_n + \frac{2}{a_n}) \ \forall \ n \ge 1.$$

The sequence $\{a_n\}_{n \ge 1}$ is a monotonically decreasing sequence of rational numbers which is bounded below. However, it cannot converge to a rational (why?). This exhibits the need to enlarge the concept of numbers beyond rational numbers. The sequence $\{a_n\}_{n \ge 1}$ converges to $\sqrt{2}$ and its elements a_n's are used to find a rational approximation (in computing machines) of $\sqrt{2}$.

17) $f:(a,b)\to\mathbb{R}$ is differentiable at $c\in(a,b)$

$\hookrightarrow$ $$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=f'(a)$$

Consider $h>0$ $(h\to 0^+)$

$\hookrightarrow$ $$\lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h}=f'(a)$$

Consider $h<0$ $(h\to 0^-)$

Let $t=-h$

$$\to \lim_{h\to 0^-}\frac{f(a+h)-f(a)}{h}=\lim_{t\to 0^+}\frac{f(a-t)-f(a)}{-t}$$

$$=\lim_{t\to 0^+}\frac{f(a)-f(a-t)}{t}$$

$$=f'(a)$$

$$\lim_{h\to 0^+}\frac{f(c+h)-f(c-h)}{2h}$$

$$=\frac{1}{2}\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}+\frac{f(c)-f(c-h)}{h}$$

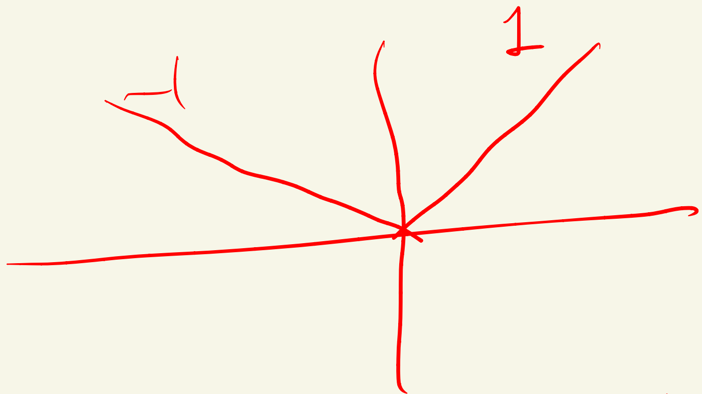
$$= \frac{1}{2}\left(f'(c) + f'(c)\right)$$

(from equations above)

$$= f'(c)$$

$\Rightarrow$ Hence the limit exists and is equal to $f'(c)$

For the converse, construct a counter example, $f(x) = |x|$



$\Rightarrow$ not differentiable at 0

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0-h)}{2h} = \lim_{h \to 0^+} \frac{|h| - |-h|}{2h}$$

$$= 0$$

Limit exists but $f'(0)$ not defined

18) $f(x+y) = f(x)\, f(y)$

$f(0) = f(0)^2$

$f(0) = 0$

$\Rightarrow f(x+0) = f(x)\, f(0) \Rightarrow f(x) = 0 \quad \forall x$

Defferentiable

$f(0) = 1$

$$\Rightarrow f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

$$= \lim_{h \to 0} \frac{f(c)\, f(h) - f(c)}{h}$$

$$= \lim_{h \to 0} f(c) \left(\frac{f(h) - f(0)}{h} \right)$$

$$f'(c) = f(c)\, f'(0)$$

Optional Exercises:

1. Show that the function f in Question 14 satisfies $f(kx) = kf(x)$, for all $k \in \mathbb{R}$.
2. Show that in Question 18, f has a derivative of every order on $\mathbb{R}$.
3. Construct an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous everywhere and is differentiable everywhere except at 2 points. $|x| + |1-x|$
4. Let $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$ Show that f is discontinuous at every $c \in \mathbb{R}$.
5. Let $g(x) = \begin{cases} x, & \text{if } x \text{ is rational,} \\ 1-x, & \text{if } x \text{ is irrational.} \end{cases}$ Show that g is continuous only at $c = 1/2$.
6. Let $f : (a, b) \to \mathbb{R}$ and $c \in (a, b)$ be such that $\lim_{x \to c} f(x) > \alpha$. Prove that there exists some $\delta > 0$ such that
$$f(c+h) > \alpha \text{ for all } 0 < |h| < \delta.$$
7. Let $f : (a, b) \to \mathbb{R}$ and $c \in (a, b)$. Show that the following are equivalent:
   (i) f is differentiable at c.
   (ii) There exist $\delta > 0$ and a function $\epsilon_1 : (-\delta, \delta) \to \mathbb{R}$ such that $\lim_{h \to 0} \epsilon_1(h) = 0$ and
   $$f(c+h) = f(c) + \alpha h + h\epsilon_1(h) \text{ for all } h \in (-\delta, \delta).$$
   (iii) There exists $\alpha \in \mathbb{R}$ such that
   $$\lim_{h \to 0} \left(\frac{|f(c+h) - f(c) - \alpha h|}{|h|} \right) = 0.$$
8. Suppose f is a function that satisfies the equation $f(x+y) = f(x) + f(y) + x^2y + xy^2$ for all real numbers x and y. Suppose also that
$$\lim_{x \to 0} \frac{f(x)}{x} = 1.$$
Find $f(0), f'(0), f'(x)$.
9. Suppose f is a function with the property that $|f(x)| \leq x^2$ for all $x \in \mathbb{R}$. Show that $f(0) = 0$ and $f'(0) = 0$.
10. Show that any continuous function $f : [0, 1] \to [0, 1]$ has a fixed point.

1) $f(x+y) = f(x) + f(y)$

$$f((x+y)+z) = f(x+y) + f(z)$$
$$= f(x) + f(y) + f(z)$$

$$\Rightarrow f(\underbrace{x+x+\dots}_{n \text{ times}}) = n f(x)$$

$$\Rightarrow f(nx) = n f(x) \quad \forall\, n \in \mathbb{N}$$

Result proved for natural numbers.

$$f(x) + f(-x) = f(x + -x) = f(0) = 0 \quad \forall\, x \in \mathbb{N}$$
$$\Rightarrow f(x) = -f(-x)$$

Rational $\mathbb{Q} \Rightarrow \frac{p}{q}$ = rational ($q \neq 0$, $p, q \in \mathbb{Z}$ integers)

$$q f\left(\frac{p}{q}\right) = \underbrace{f\left(\frac{p}{q}\right) + f\left(\frac{p}{q}\right) + \dots}_{q \text{ times}}$$

$$= f\left(\frac{p}{q} + \frac{p}{q} + \dots\right)$$

$$= f\left(\frac{p}{q} \times q\right) = f(p) = p f(1)$$

[because p is an integer

$$\Rightarrow f\left(\frac{p}{q}\right) = \frac{p}{q} f(1)$$
$$\Rightarrow f(r) = r f(1) \quad \forall\, r \in \mathbb{Q}$$

statement proved for rational numbers

Now, consider $x \in \mathbb{R}$

We can consider a sequence of rational numbers $\left(\frac{p_i}{q_i}\right)$ converging to x

$$f(x) = f\left(\lim_{i \to \infty} \frac{p_i}{q_i}\right)$$

$$= \lim_{i \to \infty} \left(f\left(\frac{p_i}{q_i}\right)\right) \quad (f = \text{continuous})$$

$$= \left(\lim_{i \to \infty} \frac{p_i}{q_i}\right) f(1)$$

$$= x f(1)$$

$\Rightarrow$ Hence proved for $\forall x \in \mathbb{R}$

2b $\quad f'(x) = f'(0) f(x)$

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

$$= \frac{f'(0) f(x+h) - f'(0) f(x)}{h}$$

$$f''(x) = f'(0) f'(x)$$

since $f'(x)$ is continuous $f''(x)$ will be.

now use induction.

Tutorial sheet 2: Rolle's theorem, MVT, maxima/minima

1. Show that all the roots of the cubic $x^3 - 6x + 3$ are real.

2. Let p and q be two real numbers with $p > 0$. Show that the cubic $x^3 + px + q$ has exactly one real root.

3. Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a)$ and $f(b)$ are of different signs and $f'(x) \neq 0$ for all $x \in (a, b)$, show that there is a unique $x_0 \in (a, b)$ such that $f(x_0) = 0$.

4. Consider the cubic $f(x) = x^3 + px + q$, where p and q are real numbers. If $f(x)$ has three distinct real roots, show that $4p^3 + 27q^2 < 0$ by proving the following:

   (i) $p < 0$.

   (ii) f has a local maximum/minimum at $\pm\sqrt{-p/3}$.

   (iii) The maximum/minimum values are of opposite signs.

5. Use the MVT to prove that $|\sin a - \sin b| \leq |a - b|$, for all $a, b \in \mathbb{R}$.

6. Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = a$ and $f(b) = b$, show that there exist distinct c_1, c_2 in (a, b) such that $f'(c_1) + f'(c_2) = 2$.

7. Let $a > 0$ and f be continuous on $[-a, a]$. Suppose that $f'(x)$ exists and $f'(x) \leq 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, show that $f(0) = 0$. Is it true that $f(x) = x$ for every x?

8. In each case, find a function f which satisfies all the given conditions, or else show that no such function exists.

   (i) $f''(x) > 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f'(1) = 1$ No, apply Rolle's Theorem

   (ii) $f''(x) > 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f'(1) = 2$

   (iii) $f''(x) \geq 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f(x) \leq 100$ for all $x > 0$

   (iv) $f''(x) > 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f(x) \leq 1$ for all $x < 0$ $g(x) = \begin{cases} \frac{1}{1-x} & x \leq 0 \\ 1+x+x^2 & x > 0 \end{cases}$

9. Let $f(x) = 1 + 12|x| - 3x^2$. Find the global maximum and the global minimum of f on $[-2, 5]$. Verify it from the sketch of the curve $y = f(x)$ on $[-2, 5]$.

10. Sketch the following curves after locating intervals of increase/decrease, intervals of concavity upward/downward, points of local maxima/minima, points of inflection and asymptotes. How many times and approximately where does the curve cross the x-axis?

    (i) $y = 2x^3 + 2x^2 - 2x - 1$

    (ii) $y = 1 + 12|x| - 3x^2$, $x \in [-2, 5]$

11. Sketch a continuous curve $y = f(x)$ having all the following properties:

    $f(-2) = 8$, $f(0) = 4$, $f(2) = 0$; $f'(2) = f'(-2) = 0$;

    $f'(x) > 0$ for $|x| > 2$, $f'(x) < 0$ for $|x| < 2$;

    $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$.

12. Give an example of $f : (0, 1) \to \mathbb{R}$ such that f is

    (i) strictly increasing and convex.

4b $f(x) = x^3 + px + q$

↳ Three distinct real roots

$r_1 < r_2 < r_3$

Rolle's Theorem ⇒ If the function is continuous on $[a,b]$ and differentiable (a,b) and $f(a) = f(b)$ then $f'(x) = 0$ for some $a < x < b$

$f(x) \to$ polynomial function →

continuous ✓

differentiable ✓

Between $(r_1, r_2) \longrightarrow f'(x)$ has one root α_1

Between $(r_2, r_3) \to f'(x)$ has one root α_2

$$f'(x) = 3x^2 + p = 0 \quad (p<0)$$

$$\Rightarrow x = \pm\sqrt{\frac{-p}{3}}$$

Let $\alpha_1 = -\sqrt{\frac{p}{3}}$ and $\alpha_2 = +\sqrt{\frac{p}{3}}$

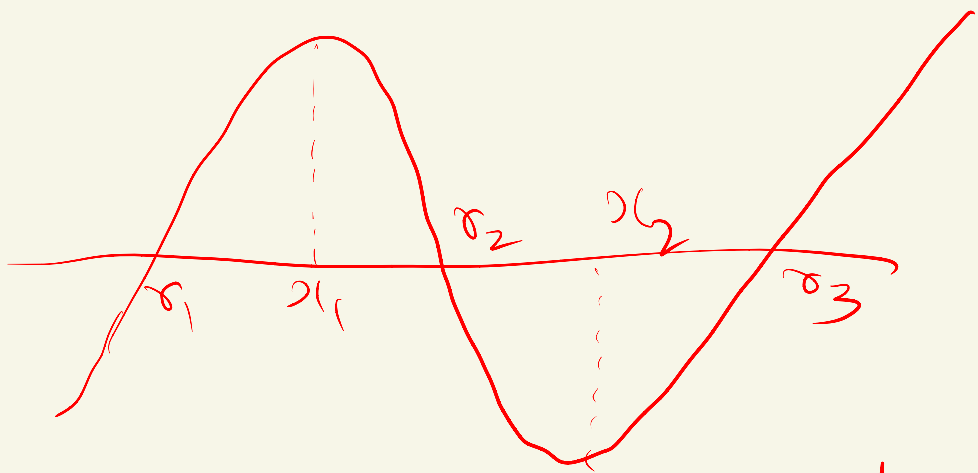
(i) For real roots to exist,

$p < 0$

$$f''(x) = 6x$$

$f''(x_1) < 0 \hookrightarrow f'(x_1) = 0$ local max.

$f''(x_2) > 0$ $f'(x_2) = 0$ $\rightarrow$ local min.



since $f(r_2) = 0$ and $f'(x)$ is negative between x_1 and x_2, $f(x_1)$ and $f(x_2)$ must be of opposite signs.

$\Rightarrow f(x_1)\, f(x_2) < 0$

$$\Rightarrow \left(f\left(\sqrt{\frac{-p}{3}}\right) f\left(+\sqrt{\frac{-p}{3}}\right)\right) < 0$$

$$f(x_1) = q + \sqrt{\frac{-4p^3}{27}}$$

$$f(x_2) = q - \sqrt{\frac{-4p^3}{27}}$$

$$f(x_1)\, f(x_2) = q^2 - \left(\frac{-4p^3}{27}\right) < 0$$

$$\Rightarrow 27q^2 + 4p^3 < 0$$

10) (ii) $1 + 12|x| - 3x^2$

$x \geq 0$: $1 + 12x - 3x^2 \rightarrow f_1$

$x < 0$: $1 - 12x - 3x^2 \rightarrow f_2$

Not differentiable at $x = 0$

$f_1' = 12 - 6x = 0 \Rightarrow x = 2$

$f_1'(2) = 0$ $\quad f_1'(0) \geq 0$

$f_1' < 0 \; \forall x > 2$ $\quad \Rightarrow f_1' > 0 \; \forall x \in [0, 2)$

$f_2' = -12 - 6x = 0 \Rightarrow x = -2$

$f_2'(2) = 0$

$f_2'(0) < 0$

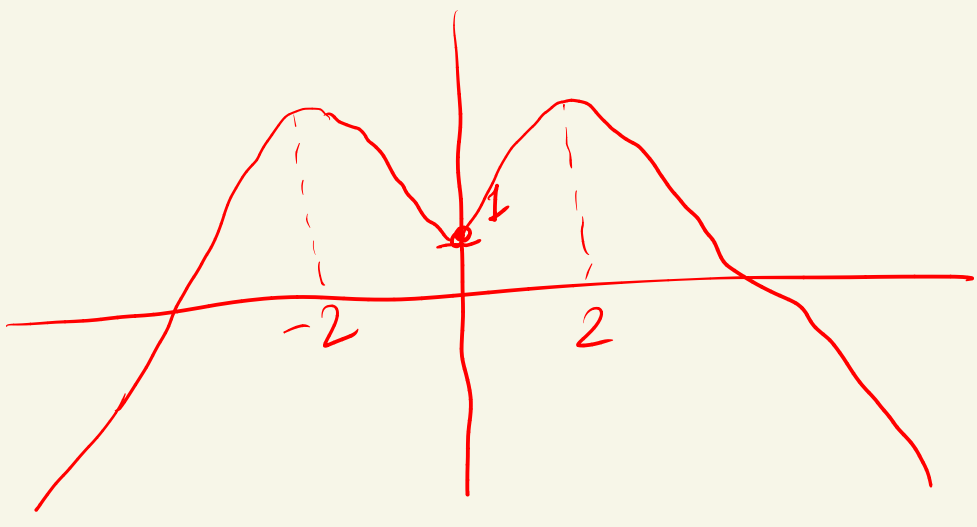
$f_2' < 0 \quad \forall x \in (-2, 0)$

$f_2' > 0 \quad \forall x > -2$

$f_1'' = -6 < 0 \quad \forall x > 0$

$f_2'' = -6 < 0 \quad \forall x < 0$

No points of inflection



Absolute maximum at $x = \pm 2$

2b $f(x) = x^3 + px + q$

$f'(x) = 3x^2 + p > 0$

$\Rightarrow$ Strictly increasing (can't have more than one root).

$$\frac{f(x)}{x^3} = 1 + \frac{p}{x^2} + \frac{q}{x^3}$$

$$\lim_{x \to \infty} \frac{f(x)}{x^3} = 1$$

$\Rightarrow f(x) > 0$ as $x \to \infty$
and $f(x) < 0$ as $x \to -\infty$

$\Rightarrow$ By IMVT (Intermediate Value Theorem)

there exists c such that $f(c) = 0$

$\Rightarrow$ One real root

5) $\dfrac{|\sin a - \sin b|}{|a-b|} = |\sin'(c)|$

for some $c \in (a,b)$

(by LMVT)

$|\sin'(c)| = |\cos(c)| \le 1$

$\Rightarrow |\sin a - \sin b| \le |a-b|$

6) Apply LMVT on $\left(a, \dfrac{a+b}{2}\right)$

$$f'(c_1) = \frac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{a+b}{2} - a}$$

Similarly $$f'(c_2) = \frac{f(b) - f\left(\frac{a+b}{2}\right)}{b - \frac{a+b}{2}}$$

Add and substitute

(ii) strictly increasing and concave.

(iii) strictly decreasing and convex.

(iv) strictly decreasing and concave.

13. Let $f, g : \mathbb{R} \to \mathbb{R}$ satisfy $f(x) \geq 0$ and $g(x) \geq 0$ for all $x \in \mathbb{R}$. Define $h(x) = f(x)g(x)$ for $x \in \mathbb{R}$. Which of the following statements are true? Why?

(i) If f and g have a local maximum at $x = c$, then so does h.

(ii) If f and g have a point of inflection at $x = c$, then so does h.

Additional Exercise

(14) Sketch the curve following the template of exercise 10 $y = \dfrac{x^2}{x^2 + 1}$

13b(i) Local maxima at c

$f(c+t) < f(c) \quad \forall\, t \in (-\delta_1, +\delta_1)$

$g(c+t) < g(c) \quad \forall\, t \in (-\delta_2, \delta_2)$

Take $\delta = \min(\delta_1, \delta_2)$

Then $f(c+t)g(c+t) < f(c)g(c)$

because $f \geq 0$, $g \geq 0$
we can use this inequality

$\Rightarrow h(c+t) \leq h(c)$

$\forall\, t \in (-\delta, \delta)$

$\Rightarrow$ Local maxima at $x = c$

NOTE = WE CAN'T ASSUME f and g to be differentiable. Use the most basic definition available to us

13b) False

Counter example:

$f(x) = 10 + \tan^{-1}(x)$

$g(x) = 10 - \tan^{-1}(x)$

$f'(x) = \frac{1}{1+x^2}$ $f''(x) = \frac{-2x}{(1+x^2)^2}$

$\downarrow$

inflection at $x=0$

similarly g has inflection at $x=0$

for inflection point, f'' should change sign

$h(x) = 100 - (\tan^{-1}(x))^2$

$h'(x) = -2\tan^{-1}(x) \frac{1}{1+x^2}$

$h''(x) = \frac{-2}{(1+x^2)^2} - 2\tan^{-1}(x) \times \frac{-2x}{1+x^2}$

No inflection point.

Tutorial sheet 3: Supplement on Taylor series

In this tutorial sheet, we will intersperse the exercises with the text, so you will have to read through the sheet somewhat carefully.

The Kerala School of Mathematics

In the fourteenth century CE, mathematicians in Kerala made a number of mathematical discoveries. Sangamagrāma Mādhavan (1350-1425 CE) appears to have been one of the founders of what is now known as the Kerala School of Mathematics, anticipating many of the later European discoveries. The following is an extract from Wikipedia (http://en.wikipedia.org/wiki/Madhava_of_Sangamagrama): Among his many contributions, he discovered the infinite series for the trigonometric functions of sine, cosine, tangent and arctangent, and many methods for calculating the circumference of a circle. One of Madhava's series is known from the text Yuktibhāṣā, which contains the derivation and proof of the power series for inverse tangent, discovered by Madhava. In the text, Jyeṣṭhadeva describes the series in the following manner:
"The first term is the product of the given sine and radius of the desired arc divided by the cosine of the arc. The succeeding terms are obtained by a process of iteration when the first term is repeatedly multiplied by the square of the sine and divided by the square of the cosine. All the terms are then divided by the odd numbers $1, 3, 5, \ldots$. The arc is obtained by adding and subtracting respectively the terms of odd rank and those of even rank. It is laid down that the sine of the arc or that of its complement whichever is the smaller should be taken here as the given sine. Otherwise the terms obtained by this above iteration will not tend to the vanishing magnitude."

Exercise 1. *Write down the Taylor series for (i)* $\cos x$*, (ii)* $\arctan x$ *about the point* 0*. Write down a precise remainder term* $R_n(x)$ *in each case.*

Exercise 2. *Our examples of Taylor's series have usually been series about the point* 0*. Write down the Taylor series of the polynomial* $x^3 - 3x^2 + 3x - 1$ *about the point* 1*.*

Exercise 3. *What is the Taylor series of the function* $1729x^{1729} + 1728x^{1728} + 1000x^{1000} + 729x^{729} + 1$ *about the point* 0*?*

Power series

Exercise 4. *Consider the series* $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ *for a fixed* x*. Prove that it converges as follows. Choose* $N > 2x$*. We see that for all* $n > N$*,*

$$\frac{x^{n+1}}{(n+1)!} < \frac{1}{2} \cdot \frac{x^n}{n!}.$$

It should now be relatively easy to show that the given series is Cauchy, and hence (by the completeness of $\mathbb{R}$*), convergent.*

Taylor series (or more generally "power series") can be differentiated and integrated "term by term". That is if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad \text{then} \quad f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

1) Taylor Series

$$\sum_{i=0}^{\infty} \frac{f^{(i)}(\alpha)\,(x-\alpha)^{i}}{i!}$$

(i) $f(x) = \cos x = 1$

$f'(x) = -\sin x = 0$

$f''(x) = -\cos x = -1$

$f'''(x) = \sin x = 0$

$f''''(x) = \cos x = 1$

continue

$$\frac{(1)(x)^0}{0!} + \frac{0\,(x-0)^1}{1!} + \frac{(-1)(x-0)^2}{2!} + \dots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

(ii) $f(x) = \arctan x$

$f'(x) = \dfrac{1}{1+x^2}$

$= 1 - x^2 + x^4 - x^6 + \dots$

(infinite GP)

$f(x) = \int 1 - x^2 + x^4 - x^6 + \dots$

$= C + x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dots$

$C = 0$ as $f(0) = 0$

$\Rightarrow \arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7}$

$= \displaystyle\sum_{k=0}^{\infty} \frac{x^{2k+1}(-1)^k}{2k+1}$

Remainder term $R_n(x)$

$$R_n(x) = f(x) - P_n(x)$$

$$R_0(x) = f(x) - P_0(x)$$

$$= f(x) - f(\alpha)$$

$$= f'(c)(x-\alpha)$$ [mean value theorem]

similarly, $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

$\cos x \Rightarrow$

$$= \frac{(\cos x)^{(n+1)}(c)\,(x)^{n+1}}{(n+1)!}$$

arc $\tan x \Rightarrow$ $\dfrac{(\tan^{-1}(x))^{n+1}(c)(x)^{n+1}}{(n+1)!}$