MA 105 : Calculus

D4 - Lecture 8

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Useful properties of the Darboux sums

One of the most useful properties of the Darboux sums is the following. If P' is a refinement of P then obviously

$$L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P).$$

This is easy to see - the lower sum computes the sum of the areas of rectangles that lie entirely below the curve while the upper sum computes the sum of the areas of rectangles whose "tops" lie above the curve.

Therefore, for any two partitions P_1 and P_2, we consider the common refinement $P = P_1 \cup P_2$ of P_1 and P_2, and get

$$L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2).$$

It follows that

$$L(f) \leq U(f).$$

Useful properties of the Darboux sums

Thus, for any partition P, we get

$$L(f, P) \leq L(f) \leq U(f) \leq U(f, P).$$

By using the above inequalities, one can easily prove the following theorems.

Theorem 1: (a). If $U(f, P) - L(f, P) < \epsilon$ for some partition P and some $\epsilon > 0$, then $U(f, P') - L(f, P') < \epsilon$ for any refinement P' of P and the same $\epsilon > 0$.

(b). If $U(f, P) - L(f, P) < \epsilon$ for the partition $P = \{x_0 < x_1 < \cdots < x_n\}$ and if $s_j, t_j \in [x_{j-1}, x_j]$ are arbitrary points, then $\sum_{j=1}^{n} |f(s_j) - f(t_j)|(x_j - x_{j-1}) < \epsilon$.

Theorem 2: A bounded function $f : [a, b] \to \mathbb{R}$ is Darboux integrable if and only if for every $\epsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon.$$

Exercise: Try proving the above theorems.

Riemann Sums

There is another way of getting at the integral due to Riemann which may be a little more intuitive and is better for calculation.

This is done via Riemann sums.

To define the notion of a Riemann sum we need one more piece of data. Suppose that for each of the intervals I_j we are given a point $t_j \in I_j$. We will denote the collection of points t_j by t. The pair (P, t) is sometimes called a tagged partition.

Definition: We define the Riemann sum associated to the function f, and the tagged partition (P, t) by

$$R(f, P, t) = \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}).$$

The norm of a partition

As must be clear, the Lower sum, Upper sum and Riemann sum all give approximations to the area between the lines $x = a$ and $x = b$ and between the curve $y = f(x)$ and the x-axis and

$$L(f, P) \leq R(f, P, t) \leq U(f, P).$$

The point is to make this statement quantitatively precise.

We define the norm of a partition P (denoted $\|P\|$) by

$$\|P\| = \max_{1 \leq j \leq n} \{|x_j - x_{j-1}|\}.$$

The norm gives some measure of the "size" of a partition, in particular, it allows us to say whether a partition is big or small.

When the size of the partition is small, it means that every interval in the partition is small.

The Riemann integral

Definition 1: A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if for some $R \in \mathbb{R}$ and every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|R(f, P, t) - R| < \epsilon,$$

for any tagged partition (P, t) of $[a, b]$ having $\|P\| < \delta$.

In other words, for all sufficiently "small" or "fine" partitions, the Riemann sums must be within ϵ of R.

Intuitively, we can see that the smaller or finer the partition, the better the area under the curve is represented by the Riemann sum.

In this case, R is called the Riemann integral of the function f on the interval $[a, b]$. It is easy to see that, for a given function f on the interval $[a, b]$, the number R is uniquely determined (how?).

Remark: Note that for a given $\epsilon > 0$ and partition P having $\|P\| < \delta$, the above condition should hold for any tagging t of P.

The Riemann integral continued...

Definition 2: A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if for some $R \in \mathbb{R}$ and every $\epsilon > 0$ there exists a partition P of $[a, b]$ such that for every tagged refinement (P', t') of P,

$$|R(f, P', t') - R| < \epsilon.$$

Remark: The nice thing about the above definition is that to check the Riemann integrability of a given function one only has to check that $|R(f, P', t') - R|$ is small for refinements of a fixed partition (for any choice of the tagging t' of P'), and not all partitions.

Theorem 3: Definition 1 and Definition 2 of the Riemann integral are equivalent.

Theorem 4: The Riemann integral exists if and only if the Darboux integral exists and in this case the two integrals are equal.

Exercise: Try proving the above theorems.

Back to our example

We now show that the function $f(x) = x$ on $[0, 1]$ is Riemann integrable using Definition 2.

Let $\epsilon > 0$ be arbitrary. For our fixed partition, we take $P = P_n$ where $n > \frac{1}{2\epsilon}$ is some fixed number.

If (P', t') is any (tagged) refinement of P_n, we have

$$L(f, P_n) \leq L(f, P') \leq R(f, P', t') \leq U(f, P') \leq U(f, P_n).$$

Now, recall that we have already computed

$$L(f, P_n) = \frac{1}{2} - \frac{1}{2n} \quad \text{and} \quad U(f, P_n) = \frac{1}{2} + \frac{1}{2n}.$$

By putting these values in above inequality we get

$$\left| R(f, P', t') - \frac{1}{2} \right| < \frac{1}{2n} < \epsilon$$

and hence the function $f(x) = x$ on $[0, 1]$ is Riemann integrable (by using Definition 2), and its integral is $\frac{1}{2}$. $\square$

Some Remarks

Remark 1: With Theorem 4 in hand, we see that the function $f(x) = x$ is also Darboux integrable $[0, 1]$. In fact, following the above proof one can easily show that the function $f(x) = x$ is Riemann/Darboux integrable on any interval $[a, b]$, for $a, b \in \mathbb{R}$. Does it also follow from Theorem 2?

Remark 2: From now onwards, we will use any of the three definitions (one of the Darboux integral and Definitions 1&2 of the Riemann integral) for computing the Riemann/Darboux integrals as essentially all of them are the same (thanks to Theorems 3&4).

Remark 3: Remember that the partition P (in Theorem 2 and Definition 2) and δ (in Definition 1) we get for a given $\epsilon > 0$ depend on ϵ, that is, P and δ in general get changed when ϵ is changed (you can correlate it with N and δ getting changed with ϵ for limits of sequences and functions).

We have already seen in the proof of the integrability of $f(x) = x$ on $[0, 1]$ (given in the last slide) that for a given $\epsilon > 0$, P is taken as the P_n, for which, $\frac{1}{n} < \epsilon$.

An example

Let us look at the following tutorial problem that is already discussed in the last tutorial class.

Exercise 4.1. Show from the first principle that the function $f : [0, 2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 \text{ if } x \in [0, 1] \\ 2 \text{ if } x \in (1, 2] \end{cases}$$

is Riemann integrable.

Remark: The solution given below is using the first principle but can you solve this exercise by considering a partition like

$$P_n = \left\{ 0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{2n-1}{n} < 2 \right\}$$

of $[0, 2]$ for $n \in \mathbb{N}$, and using Theorem 2?

The solution

Note that the Riemann integrability is same as the Darboux integrability.

Let $P = \{0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 2\}$ be an arbitrary partition of $[0, 2]$.

The point 1 lies in one of the partitions, say $[x_{i-1}, x_i]$, for some i.

We assume that $1 \neq x_i$ and treat this case first. In this case,

$$\begin{aligned} L(f, P) &= \sum_{j=1}^{i} (x_j - x_{j-1}) + \sum_{j=i+1}^{n} 2(x_j - x_{j-1}) \\ &= (x_i - x_0) + 2(x_n - x_i) \\ &= (x_i - 0) + 2(2 - x_i) \\ &= 4 - x_i, \end{aligned} \tag{1}$$

where x_i is a point in $(1, 2]$.

If $x_i = 1$ for some i, then

$$\begin{aligned} L(f,P) &= \sum_{j=1}^{i+1}(x_j - x_{j-1}) + \sum_{j=i+2}^{n} 2(x_j - x_{j-1}) \\ &= x_{i+1} - x_0 + 2(x_n - x_{i+1}) \\ &= x_{i+1} - 0 + 2(2 - x_{i+1}) \\ &= 4 - x_{i+1}, \end{aligned} \tag{2}$$

where x_{i+1} is a point in $(1,2]$.

In either case, $\sup_P L(f,P) = L(f) = 3$.

The Upper sums $U(f,P)$ can be treated in exactly the same way. In either of the cases we have treated above we get

$$U(f,P) = 4 - x_{i-1},$$

for a point $x_{i-1} \in [0,1)$. It follows that $U(f) = \inf_P U(f,P) = 3$.

It follows that $L(f) = U(f) = 3$, which shows that the function is Darboux and hence Riemann integrable, and $\int_0^2 f(x)dx = 3$. $\square$

The main theorem for Riemann integration

The main theorem of Riemann integration is the following:

Theorem 5: Let $f : [a, b] \to \mathbb{R}$ be a function that is bounded, and continuous at all but finitely many points of $[a, b]$. Then f is Riemann integrable on $[a, b]$.

In particular, continuous functions on closed and bounded intervals are Riemann integrable.

In fact, one can allow even countably many points of discontinuities and the Theorem will remain true.

Exercise 1: Those of you who have an extra interest in the course should think about trying to prove both Theorem 5 and the extension to countably many discontinuities (Warning: there is one crucial fact about continuous functions that we have not covered that you will have to discover for yourself).