MA 105 Tutorial Sheet 4 :
Line integrals and conservative fields October 24, 2023

1. Determine whether or not the given set is a) open, b) path-connected, and c) simply-connected.

   (a) $D = \{(x, y) \in \mathbb{R}^2 \mid 0 < y < 3\}$,
   (b) $D = \{(x, y) \in \mathbb{R}^2 \mid 1 < |x| < 2\}$,
   (c) $D = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 4, \quad y \geq 0\}$,
   (d) $D = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (1, 4)\}$.

2. Determine whether or not the vector field $\mathbf{F}(x, y) = 3xy\mathbf{i} + x^3y\mathbf{j}$ is a gradient on any open subset of $\mathbb{R}^2$.

3. Show that the line integral is path-independent and evaluate the integral:

$$\int_C 2xe^{-y}\, dx + (2y - x^2e^{-y})\, dy$$

   where C is any path from $(1, 0)$ to $(2, 1)$.

4. Is the line integral $\int_C ydx + xdy + xyzdz$ path-independent in $\mathbb{R}^3$?

5. Let $\mathbf{F} = \nabla f$, where $f(x, y) = \sin(x - 2y)$. Find curves C_1 and C_2 that are not closed and satisfy

$$\int_{C_1} \mathbf{F}.\mathbf{ds} = 0, \quad \int_{C_2} \mathbf{F}.\mathbf{ds} = 1.$$

6. Determine whether or not $\mathbf{F}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

   (a) $\mathbf{F}(x, y) = y^2e^{xy}\mathbf{i} + (1 + xy)e^{xy}\mathbf{j}$, for all $(x, y) \in \mathbb{R}^2$.
   (b) $\mathbf{F}(x, y) = (ye^x + \sin y)\mathbf{i} + (e^x + x\cos y)\mathbf{j}$, for all $(x, y) \in \mathbb{R}^2$.
   (c) $\mathbf{F}(x, y) = (2xy + y^{-2})\mathbf{i} + (x^2 - 2xy^{-3})\mathbf{j}$, for all $(x, y) \in \mathbb{R}^2$ and $y > 0$.

7. Let $\mathbf{F}$ be a vector field on $\mathbb{R}^2$. Find a function f such that $\mathbf{F} = \operatorname{grad} f$ and using it evaluate $\int_{\mathbf{c}} \mathbf{F} \cdot \mathbf{ds}$, where $\mathbf{F}$ and $\mathbf{c}$ are given below:

   (a) $\mathbf{F}(x, y, z) = (2xyz + \sin x)\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$ and $\mathbf{c}(t) = (\cos^5 t, \sin^3 t, t^4)$, $0 \leq t \leq \pi$.
   (b) $\mathbf{F}(x, y) = (1 + xy)e^{xy}\mathbf{i} + x^2e^{xy}\mathbf{j}$ and $\mathbf{c}(t) = (\cos t, 2\sin t)$, $0 \leq t \leq \frac{\pi}{2}$.
   (c) $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + (xy + 2z)\mathbf{k}$ and $\mathbf{c}$ is the line segment from $(1, 0, -2)$ to $(4, 6, 3)$.

8. For $\mathbf{v} = (2xy + z^3)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$, show that $\nabla\phi = \mathbf{v}$ for some ϕ and hence calculate $\oint_C \mathbf{v} \cdot d\underline{s}$ where C is any arbitrary smooth closed curve.

9. Let $S = \mathbb{R}^2 \setminus \{(0, 0)\}$. Let

$$\mathbf{F}(x, y) = -\frac{y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j} := F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}.$$

   (a) Show that $\dfrac{\partial}{\partial y}F_1(x, y) = \dfrac{\partial}{\partial x}F_2(x, y)$ on S.
   (b) Compute $\int_C \mathbf{F}.\mathbf{ds}$ where C is the circle: $x^2 + y^2 = 1$.

(c) Is $\mathbf{F}$ a conservative field on S?

10. A radial force field is one which can be expressed as $\mathbf{F}(x, y, z) = f(r)\mathbf{r}$ where $\mathbf{r} = (\mathbf{x}, \mathbf{y}, \mathbf{z})$ is the position vector and $r = \|\mathbf{r}\|$. Show that, if f is continuous, $\mathbf{F}$ is conservative in $\mathbb{R}^3$.

    (Hint. Try to guess what the potential function could be, provided f is continuous.)