MA 105 : Calculus (Autumn 2014)

Solutions to Tutorial Sheets

August 20, 2017

Solutions to Tutorial Sheet 1

(1) For a given $\epsilon > 0$, we have to find $n_0 \in \mathbb{N}$ such that $|a_n| < \epsilon$ for all $n \geq n_0$. Select $n_0 \in \mathbb{N}$ (This is possible by the *archimedean property* of $\mathbb{R}$ - but you should not probably not mention this to your students for whom this fact is surely self evident. If someone brings it up in class, you can acknowledge the comment and leave it at that.) such that

(i) $n_0 > \dfrac{10}{\epsilon}$,

(ii) $n_0 > \dfrac{5-\epsilon}{3\epsilon}$,

(iii) $n_0 > \dfrac{1}{\epsilon^3}$ as $\dfrac{1}{n^{\frac{1}{3}}} > \dfrac{n^{\frac{2}{3}}}{n+1} > |a_n|$,

(iv) $n_0 > \dfrac{2}{\epsilon}$ as $\dfrac{2}{n} > \dfrac{1}{n}\left(2 - \dfrac{1}{n+1}\right) = |a_n|$.

(2) (i) $\dfrac{n^2}{n^2+n} \leq a_n \leq \dfrac{n^2}{n^2+1} \Rightarrow \lim\limits_{n\to\infty} a_n = 1$.

(ii) $0 < \dfrac{n!}{n^n} = \left(\dfrac{n-1}{n}\right)\left(\dfrac{n-2}{n}\right)\cdots\left(\dfrac{2}{n}\right)\left(\dfrac{1}{n}\right) \leq \dfrac{1}{n} \Rightarrow \lim\limits_{n\to\infty} \dfrac{n!}{n^n} = 0$.

(iii) $0 < \dfrac{n^3+3n^2+1}{n^4+8n^2+2} = \dfrac{(1/n)+(3/n^2)+(1/n^4)}{1+(8/n^2)+(2/n^4)} < \dfrac{4}{n} \Rightarrow \lim\limits_{n\to\infty} a_n = 0$.

(iv) Let $n^{\frac{1}{n}} = 1 + h_n$. Then, for $n \geq 2$, one has

$$n = (1+h_n)^n \geq 1 + nh_n + \binom{n}{2}h_n^2 > \binom{n}{2}h_n^2.$$

Thus $0 < h_n^2 < \frac{2}{n-1}$ $(n \geq 2)$ giving $\lim_{n\to\infty} h_n = 0$. So $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

(v) Since

$$0 < \left|\frac{\cos(\pi\sqrt{n})}{n^2}\right| \leq \frac{1}{n^2},$$

it follows that $\lim_{n\to\infty} a_n = 0$.

(vi) $\sqrt{n}(\sqrt{n+1}-\sqrt{n}) = \dfrac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \dfrac{1}{\sqrt{1+\frac{1}{n}}+1} \to \dfrac{1}{2}$ as $n \to \infty$.

(3) (i) $\{\dfrac{n^2}{n+1} = (n-1) + \dfrac{1}{n+1}\}_{n\geq 1}$ is not convergent since $\dfrac{1}{n+1} \to 0$ as $n \to \infty$.

(ii) $\{(-1)^n\left(\frac{1}{2}-\frac{1}{n}\right) = \frac{(-1)^n}{2} - \frac{(-1)^n}{n}\}_{n\geq 1}$ is not convergent since $\frac{(-1)^n}{n} \to 0$ as $n \to \infty$.

(4) (i) Decreasing as $a_n = 1/\left(n+\frac{1}{n}\right)$ and $\{n+\frac{1}{n}\}_{n\geq 1}$ is increasing.

(ii) Increasing as $\frac{a_{n+1}}{a_n} = \frac{6}{5} > 1$.

(iii) Increasing as $a_{n+1} - a_n = \frac{n(n-1)-1}{n^2(1+n)^2} > 0$ for $n \geq 2$.

(5) (i) By the AM-GM inequality, we see that $a_n \geq \sqrt{2}$ for all $n \geq 2$. Consequently, $a_{n+1} - a_n = \frac{2-a_n^2}{2a_n} \leq 0$ for $n \geq 2$. Thus $\{a_n\}_{n\geq 2}$ is monotonically decreasing and bounded below by $\sqrt{2}$. So $\lim_{n\to\infty} a_n = a$ (say) exists, and $a \geq \sqrt{2}$. Also $a = \frac{1}{2}\left(a + \frac{2}{a}\right)$, i.e., $a^2 = 2$. It follows that $a = \sqrt{2}$.

(ii) By induction, $\sqrt{2} \leq a_n < 2\ \forall n$. Hence $a_{n+1} - a_n = \frac{(2-a_n)(1+a_n)}{\sqrt{2+a_n}+a_n} > 0\ \forall n$. Thus $\lim_{n\to\infty} a_n = a$ (say) exists and arguing as in (i), we find $a = 2$.

(iii) By induction, $2 \leq a_n < 6\ \forall n$. Hence $a_{n+1} - a_n = \frac{6-a_n}{2} > 0\ \forall n$. Thus $\lim_{n\to\infty} a_n = a$ (say) exists and arguing as in (i), we find $a = 6$.

(6) It is clear that $\lim_{n\to\infty} a_{n+1} = L$. The inequality $||a_n| - |L|| \leq |a_n - L|$ implies that $\lim_{n\to\infty} |a_n| = |L|$.

(7) Take $\epsilon = |L|/2$. Then $\epsilon > 0$ and since $a_n \to L$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ $\forall n \geq n_0$. Now $||a_n| - |L|| \leq |a_n - L|$ and hence $|a_n| > |L| - \epsilon = |L|/2\ \forall n \geq n_0$.

(8) Given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|a_n| < \epsilon^2\ \forall n \geq n_0$. Hence $\left|\sqrt{a_n}\right| < \epsilon$ $\forall n \geq n_0$. [Note: For a corresponding result when $a_n \to L$, see, e.g., part (ii) of Propositions 1.9 and 2.4 of [GL-1].]

(9) Both the statements are false. Consider, for example, $a_n = 1$ and $b_n = (-1)^n$.

(10) The implication "$\Rightarrow$" is obvious. For the converse, suppose both $\{a_{2n}\}_{n\geq 1}$ and $\{a_{2n+1}\}_{n\geq 1}$ converge to ℓ. Let $\epsilon > 0$ be given. Choose $n_1, n_2 \in \mathbb{N}$ such that $|a_{2n} - \ell| < \epsilon$ for all

$n \geq n_1$ and $|a_{2n+1} - \ell| < \epsilon$ for all $n \geq n_2$. Let $n_0 = \max\{n_1, n_2\}$. Then

$$|a_n - \ell| < \epsilon \text{ for all } n \geq 2n_0 + 1.$$

(11) (i) The statement is **false**. For example, consider $a = -1,\ b = 1,\ c = 0$ and define $f, g : (-1, 1) \to \mathbb{R}$ by

$$f(x) = x \quad \text{and} \quad g(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/x^2 & \text{if } x \neq 0. \end{cases}$$

(ii) The statement is **true** since $|g(x)| \leq M$ for all $x \in (a, b)$ implies that $0 \leq |f(x)g(x)| \leq M|f(x)|$ for all $x \in (a, b)$.

(iii) The statement is **true** since $\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x) \lim_{x\to c} g(x)$.

(12) Suppose $\lim_{x\to\alpha} f(x) = L$. Then $\lim_{h\to 0} f(\alpha + h) = L$. and since

$$|f(\alpha + h) - f(\alpha - h)| \leq |f(\alpha + h) - L| + |f(\alpha - h) - L|$$

it follows that

$$\lim_{h\to 0} |f(\alpha + h) - f(\alpha - h)| = 0.$$

The converse is **false**; e.g. consider $\alpha = 0$ and

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{|x|} & \text{if } x \neq 0. \end{cases}$$

(13) (i) Continuous everywhere except at $x = 0$. To see that f is not continuous at 0, consider the sequences $\{x_n\}_{n\geq 1},\ \{y_n\}_{n\geq 1}$ where

$$x_n := \frac{1}{n\pi} \quad \text{and} \quad y_n := \frac{1}{2n\pi + \frac{\pi}{2}}.$$

Note that both $x_n,\ y_n \to 0$, but $f(x_n) \to 0$ and $f(y_n) \to 1$.

(ii) Continuous everywhere. For ascertaining the continuity of f at $x = 0$, note that $|f(x)| \leq |x|$ and $f(0) = 0$.

(iii) Continuous everywhere on $[1, 3]$ except at $x = 2$.

(14) Taking $x = 0 = y$, we get $f(0+0) = 2f(0)$ so that $f(0) = 0$. By the assumption of the continuity of f at 0, $\lim_{x\to 0} f(x) = 0$. Thus,

$$\lim_{h\to 0} f(c+h) = \lim_{h\to 0}[f(c) + f(h)] = f(c)$$

showing that f is continuous at $x = c$.

Optional: First verify the equality for all $k \in \mathbb{Q}$ and then use the continuity of f to establish it for all $k \in \mathbb{R}$.

(15) Clearly, f is differentiable for all $x \neq 0$ and the derivative is

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), \ x \neq 0.$$

Also,

$$f'(0) = \lim_{h\to 0} \frac{h^2 \sin(\frac{1}{h}) - 0}{h} = 0.$$

Clearly, f' is continuous at any $x \neq 0$. However, $\lim\limits_{x\to 0} f'(x)$ does not exist. Indeed, for any $\delta > 0$, we can choose $n \in \mathbb{N}$ such that $x := 1/n\pi, \ y := 1/(n+1)\pi$ are in $(-\delta, \delta)$, but $|f'(x) - f'(y)| = 2$.

(16) The inequality

$$0 \leq \left|\frac{f(x+h) - f(x)}{h}\right| \leq c|h|^{\alpha-1}$$

implies, by the Sandwich Theorem, that

$$\lim_{h\to 0} \left|\frac{f(x+h) - f(x)}{h}\right| = 0 \ \ \forall x \in (a, b).$$

(17) For the first part, observe that

$$\begin{aligned}\lim_{h\to 0+}\frac{f(c+h)-f(c-h)}{2h} &= \lim_{h\to 0+}\frac{1}{2}\left[\frac{f(c+h)-f(c)}{h}+\frac{f(c-h)-f(c)}{-h}\right]\\ &= \frac{1}{2}[f'(c)+f'(c)] = f'(c).\end{aligned}$$

The converse is **false**; consider, for example, $f(x) = |x|$ and $c = 0$.

(18) Since $f(x+y) = f(x)f(y)$, we obtain, in particular, $f(0) = f(0)^2$ and therefore $f(0) =$ 0 or 1. If $f(0) = 0$, then

$$f(x+0) = f(x)f(0) \Rightarrow f(x) = 0 \quad \forall x.$$

Thus, trivially, f is differentiable. If $f(0) = 1$, then

$$f'(c) = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h} = f(c)\left(\lim_{h\to 0}\frac{f(h)-f(0)}{h}\right) = f'(0)f(c).$$

(19) (i) Let $f(x) = \cos(x)$. Then $f'(x) = -\sin(x) \neq 0$ for $x \in (0,\pi)$.

Thus $g(y) = f^{-1}(y) = \cos^{-1}(y), \quad -1 < y < 1$ is differentiable

and

$$g'(y) = \frac{1}{f'(x)}, \quad \text{where } x \text{ is such that } f(x) = y.$$

Therefore,

$$g'(y) = \frac{-1}{\sin(x)} = \frac{-1}{\sqrt{1-\cos^2(x)}} = \frac{-1}{\sqrt{1-y^2}}.$$

(ii) Note that

$$\operatorname{cosec}^{-1}(x) = \sin^{-1}\frac{1}{x} \quad \text{for } |x| > 1.$$

Since

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \quad \text{for } |x| < 1,$$

one has, by the Chain rule,

$$\frac{d}{dx}\,\text{cosec}^{-1}(x) = \frac{1}{\sqrt{(1-\frac{1}{x^2})}}\left(\frac{-1}{x^2}\right), \ |x| > 1.$$

(20) By the Chain rule,

$$\begin{aligned}\frac{dy}{dx} &= f'\left(\frac{2x-1}{x+1}\right)\frac{d}{dx}\left(\frac{2x-1}{x+1}\right)\\ &= \sin\left(\frac{2x-1}{x+1}\right)^2\left[\frac{3}{(x+1)^2}\right] = \frac{3}{(x+1)^2}\sin\left(\frac{2x-1}{x+1}\right)^2.\end{aligned}$$

Optional exercises

(1) Note that $f(n) = nf(1)$ for all $n \in \mathbb{N}$. Show that $f(r) = rf(1)$, for every rational number r. By continuity $f(\lambda) = \lambda f(1)$ for all real λ. If we set $f(1) = k$, we are done.

(2) This can be easily done inductively, since every derivative of f satisfies the same property as f.

(3) Consider $f(x) := |x| + |1-x|$ for $x \in \mathbb{R}$.

(4) For $c \in \mathbb{R}$, select a sequence $\{a_n\}_{n\geq 1}$ of rational numbers and a sequence $\{b_n\}_{n\geq 1}$ of irrational numbers, both converging to c. Then $\{f(a_n)\}_{n\geq 1}$ converges to 1 while $\{f(b_n)\}_{n\geq 1}$ converges to 0, showing that limit of f at c does not exist.

(5) Suppose $c \neq 1/2$. If $\{a_n\}_{n\geq 1}$ is a sequence of rational numbers and $\{b_n\}_{n\geq 1}$ a sequence of irrational numbers, both converging to c, then $g(a_n) = a_n \to c$, while $g(b_n) = 1 - b_n \to 1 - c$, and $c \neq 1 - c$. Thus g is not continuous at any $c \neq 1/2$. Further, if $\{a_n\}_{n\geq 1}$ is any sequence converging to $c = 1/2$, then $g(a_n) \to 1/2 = g(1/2)$. Hence, g is continuous at $c = 1/2$.

(6) Let $L = \lim_{x\to c} f(x)$. Take $\epsilon = L - \alpha$. Then $\epsilon > 0$ and so there exists a $\delta > 0$ such that

$$|f(c+h) - L| < \epsilon \text{ for } \quad 0 < |h| < \delta.$$

Consequently, $f(c+h) > L - \epsilon = \alpha$ for $0 < |h| < \delta$.

(7) (i) $\Rightarrow$ (ii): Choose $\delta > 0$ such that $(c-\delta, c+\delta) \subseteq (a,b)$. Take $\alpha = f'(c)$ and

$$\epsilon_1(h) = \begin{cases} \dfrac{f(c+h) - f(c) - \alpha h}{h}, & \text{if } h \neq 0 \\ 0, & \text{if } h = 0. \end{cases}$$

(ii) $\Rightarrow$ (iii): $\displaystyle\lim_{h\to 0} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} = \lim_{h\to 0} |\epsilon_1(h)| = 0$

(iii) $\Rightarrow$ (i): $\displaystyle\lim_{h\to 0} \left| \frac{f(c+h) - f(c)}{h} - \alpha \right| = 0 \Rightarrow \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$ exists and is equal to α.

(8) $f(0) = 0$, $f^{'}(0) = 1$ and $f'(x) = 1 + x^2$

(9) Follows easily from the definitions.

(10) Apply the Intermediate Value Theorem to the function $f(x) - x$.

Solutions to Tutorial Sheet 2

(1) $f(x) = x^3 - 6x + 3$ has stationary points at $x = \pm\sqrt{2}$.

Note that $f(-\sqrt{2}) = 4\sqrt{2} + 3 > 0, f(+\sqrt{2}) = -4\sqrt{2} + 3 < 0$. Therefore f has a root in $(-\sqrt{2}, \sqrt{2})$. Also, $f \to -\infty$ as $x \to -\infty$ implying that f has a root in $(-\infty, -\sqrt{2})$. Similarly, $f \to +\infty$ as $x \to +\infty$ implying that f has a root in $(\sqrt{2}, \infty)$. Since f has at most three roots, all its root are real.

(2) For $f(x) = x^3 + px + q,\ \ p > 0,\ \ f'(x) = 3x^2 + p > 0$. Therefore f is strictly increasing and can have **at most one** real root. Since

$$\lim_{x\to\pm\infty} \left(\frac{p}{x^2} + \frac{q}{x^3}\right) = 0,$$

$$\frac{f(x)}{x^3} = 1 + \frac{p}{x^2} + \frac{q}{x^3} > 0$$

for $|x|$ very large. Thus $f(x) > 0$ if x is large positive and $f(x) < 0$ if x is large negative. By the Intermediate Value Property (IVP) f must have **at least one** real root.

(3) By the IVP, there exists **at least one** $x_0 \in (a, b)$ such that $f(x_0) = 0$. If there were another $y_0 \in (a, b)$ such that $f(y_0) = 0$, then by Rolle's theorem there would exist some c between x_0 and y_0 (and hence between a and b) with $f'(c) = 0$, leading to a contradiction.

(4) Since f has 3 distinct roots, say, $r_1 < r_2 < r_3$, by Rolle's theorem $f'(x)$ has **at least two** real roots, say, x_1 and x_2 such that $r_1 < x_1 < r_2$ and $r_2 < x_2 < r_3$. Since $f'(x) = 3x^2 + p$, this implies that $p < 0$, and $x_1 = -\sqrt{-p/3}$, $x_2 = \sqrt{-p/3}$. Now, $f''(x_1) = 6x_1 < 0 \implies f$ has a local maximum at $x = x_1$. Similarly, f has a local minimum at $x = x_2$. Since the quadratic $f'(x)$ is

negative between its roots x_1 and x_2 (so that f is decreasing over $[x_1, x_2]$) and f has a root r_2 in (x_1, x_2), we must have $f(x_1) > 0$ and $f(x_2) < 0$. Further,

$$f(x_1) = q + \sqrt{\frac{-4p^3}{27}}, \; f(x_2) = q - \sqrt{\frac{-4p^3}{27}}$$

so that

$$\frac{4p^3 + 27q^2}{27} = f(x_1)f(x_2) < 0.$$

(5) For some c between a and b, one has

$$\left|\frac{\sin(a) - \sin(b)}{a - b}\right| = |\cos(c)| \leq 1.$$

(6) By Lagrange's Mean Value Theorem (MVT) there exists $c_1 \in \left(a, \frac{(a+b)}{2}\right)$ such that

$$\frac{f\left(\frac{a+b}{2}\right) - f(a)}{\left(\frac{b-a}{2}\right)} = f'(c_1)$$

and there exists $c_2 \in \left(\frac{a+b}{2}, b\right)$ such that

$$\frac{f(b) - f\left(\frac{a+b}{2}\right)}{\left(\frac{b-a}{2}\right)} = f'(c_2).$$

Clearly one has $c_1 < c_2$, and adding the above equations one obtains

$$f'(c_1) + f'(c_2) = \frac{f(b) - f(a)}{\left(\frac{b-a}{2}\right)} = 2 \text{ (as } f(b) = b, f(a) = a).$$

(7) By Lagrange's MVT, there exists $c_1 \in (-a, 0)$ and there exists $c_2 \in (0, a)$ such that

$$f(0) - f(-a) = f'(c_1)a \text{ and } f(a) - f(0) = f'(c_2)a.$$

Using the given conditions, we obtain

$$f(0) + a \leq a \text{ and } a - f(0) \leq a$$

which implies $f(0) = 0$.

Optional: Consider $g(x) = f(x) - x$, $x \in [-a, a]$. Since $g'(x) = f'(x) - 1 \leq 0$, g is decreasing over $[-a, a]$. As $g(-a) = g(a) = 0$, we have $g \equiv 0$.

(8) (i) No such function exists in view of Rolle's theorem.

(ii) $f(x) = \frac{x^2}{2} + x$

(iii) $f'' \geq 0 \Rightarrow f'$ increasing. As $f'(0) = 1$, by Lagrange's MVT we have $f(x) - f(0) \geq x$ for $x > 0$. Hence f with the required properties cannot exist.

(iv)

$$f(x) = \begin{cases} \frac{1}{1-x} & \text{if} \quad x \leq 0 \\ 1 + x + x^2 & \text{if} \quad x > 0. \end{cases}$$

(9) The points to check are the end points $x = -2$ and $x = 5$, the point of non-differentiability $x = 0$, and the stationary point $x = 2$. The values of f at these points are given by

$$f(-2) = f(2) = 13, f(0) = 1, f(5) = -14.$$

Thus, global max $= 13$ at $x = \pm 2$, and global min $= -14$ at $x = 5$.

(10) To be done in the tutorial. Asymptotes have not been discussed in class, but this curve doesn't have any, so you can skip that part.

(i) $f(x) = 2x^3 + 2x^2 - 2x - 1 \Rightarrow f'(x) = 6x^2 + 4x - 2 = 2(x+1)(3x-1)$.

Thus, $f'(x) > 0$ in $(-\infty, -1) \cup (1/3, \infty)$ so that $f(x)$ is strictly increasing in those intervals, and $f'(x) < 0$ in $(-1, 1/3)$ so that $f(x)$ is strictly decreasing in that in

Thus, $f(x)$ has a local maximum at $x = -1$, and a local minimum at $x = \frac{1}{3}$. As $f''(x) = 12x + 4$ we have that $f(x)$ is convex in $\left(-\frac{1}{3}, \infty\right)$ and concave in $\left(-\infty, -\frac{1}{3}\right.$ with a point of inflection at $x = -\dfrac{1}{3}$.

(ii) $f(x) = 1 + 12|x| - 3x^2$; f is not differentiable at $x = 0$; $f(0) = 1$. Further, $f'(x) = 0$ at $x = \pm 2, f'(x) < 0$ in $(-2, 0) \cup (2, 5]$, $f'(x) > 0$ in $(0, 2)$, and

$f''(x) = -6$ in $(-2,0) \cup (0,5)$. Thus f is concave in$(-2,0) \cup (0,5)$, decreasing in $(-2,0) \cup (2,5)$, and increasing in $(0,2)$; further, f has an absolute maximum at $x = \pm 2$.

(11) To be done in the tutorial.

(12) (i) x^2, (ii) $\sqrt{x}$, (iii) $\frac{1}{x}$, (iv) $\sin x$

(13) (i) True (easy).

(ii) False. Take $f(x) = g(x) = x^3$ at the point $x = 0$.

(14) This exercise involves asymptotes so you will have to introduce this idea. $y = \dfrac{x^2}{x^2+1} \quad \Rightarrow \lim\limits_{x \to \pm\infty} y = 1$
$y' = \dfrac{2x}{(x^2+1)^2} \Rightarrow y$ is increasing in $(0, \infty)$ and decreasing in $(-\infty, 0)$.
Further, $y'' = -\frac{2(3x^2-1)}{(x^2+1)^3}$ implies that $y'' > 0$ if $|x| < \frac{1}{\sqrt{3}}$, and $y'' < 0$ if $|x| > \frac{1}{\sqrt{3}}$.

Therefore,

$$y \text{ is convex in } \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \text{ and concave in } \mathbb{R} \setminus \left[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$$

with the points $x = \pm\dfrac{1}{\sqrt{3}}$ being the points of inflection.

Solutions to Tutorial Sheet 3

(1) The remainder terms are easy in these cases. The series are given by

(i)

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k!} \cdot x^{2k}.$$

(ii)

$$\arctan x = x - \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \qquad \text{for } |x| < 1.$$

For the series for $\arctan x$ we can proceed in two ways. The easier way is to observe that $f^{(1)}(x) = \frac{1}{1+x^2}$ and integrate this term by term. If you are using this solution, it is best to read the paragraph before 3.5 first and then do this problem.

I recall seeing a (somewhat complicated) direct solution where one can evaluate the derivatives directly at 0, but I am not able to reproduce it now. Its probably floating around on the internet somewhere.

(2) The Taylor series is just $(x-1)^3$

(3) The Taylor series is simply

$$1729x^{1729} + 1728x^{1728} + 28x^{28} + 6x^6 + 1729.$$

Indeed, for any polynomial, the Taylor series about the point 0 just gives you the polynomial back.

(4) Let us denote the partial sums of of the given series by $s_m(x)$. We would like to show that $|s_m(x) - s_n(x)|$ can be made arbitrarily small whenever m and n are sufficiently large. Let us assume that $m > n$. We see that

$$|s_m(x) - s_n(x)| = \left| \sum_{k=n+1}^{m} \frac{x^k}{k!} \right| \leq \left| \frac{x^n}{n!} \right| \left(\frac{1}{2} + \frac{1}{4} + \ldots \frac{1}{2^{m-n}} \right) \leq 2 \cdot \frac{|x^n|}{n!}.$$

If N is made sufficiently large and $n > N$, the last expression can be made as small as we please.

(5) We simply integrate term by term to get

$$\log x + x + \frac{x^2}{2\cdot 2!} + \ldots = \log x + \sum_{k=1}^{\infty} \frac{x^k}{k \cdot k!}.$$

(6) The solution will be provided later.

(7) Follow the hints given in the question.

(8)

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots = \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1} \qquad \text{for } |x| < \frac{\pi}{2}.$$

Here B_{2n} are the *Bernoulli numbers* defined by

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!},$$

that is, the Bernoulli numbers B_m are the numbers that appear in the Taylor series expansion for $\frac{t}{e^t-1}$.

(9) The key to the solution is the function

$$h(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

Recall that this is a smooth function for which $h^{(n)}(0) = 0$ for all n. Let us construct a smooth function that is 0 outside of $[c, d]$ and 1 on $[a, b] \subset [c, d]$, for any such pairs of intervals. Let $f_1(x) = h(x - c)$ and $f_2(x) = h(a - x)$. Then

$$f(x) = \frac{f_1}{f_1 + f_2}$$

has the property that f is identically 0 to the left of c and identically 1 to the right of a. Similarly, let $g_1(x) = h(d - x)$ and $g_2(x) = h(x - b)$. Then $g = g_1/g_1 + g_2$ is identically 0 to the right of d and identically 1 to the left of b. Then $k = fg$ is the desired function.

(10) Follow the hints given in the question.

Solutions to Tutorial Sheet 4

(1) The given fumction is integrable as it is monotone. Let P_n be the partition of $[0,2]$ into 2×2^n equal parts. Then $U(P_n, f) = 3$ and

$$L(P_n, f) = 1 + 1 \times \frac{1}{2^n} + 2 \times \frac{(2^n - 1)}{2^n} \to 3$$

as $n \to \infty$. Thus, $\int_0^2 f(x)dx = 3$.

(2) (a) $f(x) \geq 0 \Rightarrow U(P, f) \geq 0,\ L(P, f) \geq 0 \Rightarrow \int_a^b f(x)dx \geq 0$.

Suppose, moreover, f is continuous and $\int_a^b f(x)dx = 0$. Assume $f(c) > 0$ for some c in $[a, b]$. Then $f(x) > \frac{f(c)}{2}$ in a δ-nbhd of c for some $\delta > 0$. This implies that

$$U(P, f) > \delta \times \frac{f(c)}{2}$$

for any partition P, and hence, $\int_a^b f(x)dx \geq \delta f(c)/2 > 0$, a contradiction.

(b) on $[0, 1]$ take $f(x) = 0$ for all $x \neq 0$ and $f(0) = 0$.

(3) (i) $S_n = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{\frac{3}{2}} \longrightarrow \int_0^1 (x)^{3/2}dx = \frac{2}{5}$

(ii) $S_n = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\left(\frac{i}{n}\right)^2 + 1} \longrightarrow \int_0^1 \frac{dx}{x^2+1} = \frac{\pi}{4}$

(iii) $S_n = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{\frac{i}{n}+1}} \longrightarrow \int_0^1 \frac{dx}{\sqrt{x+1}} = 2(\sqrt{2}-1)$

(iv) $S_n = \frac{1}{n}\sum_{i=1}^{n}\cos\frac{i\pi}{n} \longrightarrow \int_0^1 \cos \pi x dx = 0$

(v) $S_n \longrightarrow \int_0^1 x dx + \int_1^2 x^{3/2}dx + \int_2^3 x^2 dx = \frac{1}{2} + \frac{2}{5}(4\sqrt{2}-1) + \frac{19}{3}$

(4) Let $F(x) = \int_a^x f(t)dt$. Then $F'(x) = f(x)$. Note that

$$\int_{u(x)}^{v(x)} f(t)dt = \int_a^{v(x)} f(t)dt - \int_a^{u(x)} f(t)dt = F(v(x)) - F(u(x)).$$

By the Chain Rule one has

$$\begin{aligned}\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt &= F'(v(x))v'(x) - F'(u(x))u'(x) \\ &= f(v(x))v'(x) - f(u(x))u'(x).\end{aligned}$$

(a) $\frac{dy}{dx} = \frac{1}{dx/dy} = \sqrt{1+y^2}$, $\frac{d^2y}{dx^2} = \frac{y}{\sqrt{1+y^2}}\frac{dy}{dx} = y$.

(b) (i) $F'(x) = \cos((2x)^2)2 = 2\cos(4x^2)$.

(ii) $F'(x) = \cos(x^2)2x = 2x\cos(x^2)$

(5) Define

$$F(x) = \int_x^{x+p} f(t)dt, \; x \in \mathbb{R}.$$

Then $F'(x) = 0$ for every x.

(6) Write $\sin\lambda(x-t)$ as $\sin(\lambda x)\cos(\lambda t) - \cos(\lambda x)\sin(\lambda t)$ in the integrand, take trems in x outside the integral, evaluate $g'(x), g''(x)$, and simplify to show LHS=RHS; from the expressions for $g(x)$ and $g'(x)$ it should be clear that $g(0) = g'(0) = 0$.

The problem could also be solved by appealing to the following theorem:

Theorem A. Let $h(t,x)$ and $\frac{\partial h}{\partial x}(t,x)$ be continuous functions of t and x on the rectangle $[a,b]\times[c,d]$. Let $u(x)$ and $v(x)$ be differentiable functions of x on $[c,d]$ such that, for each x in $[c,d]$, the points $(u(x),x)$ and $(v(x),x)$ belong to $[a,b]\times[c,d]$. Then

$$\frac{d}{dx}\int_{u(x)}^{v(x)} h(t,x)dt = \int_{u(x)}^{v(x)} \frac{\partial h}{\partial x}(t,x)dt - u'(x)h(u(x),x) + v'(x)h(v(x),x).$$

Consider now

$$g(x) = \frac{1}{\lambda}\int_0^x f(t)\sin\lambda(x-t)dt.$$

Let $h(t,x) = \frac{1}{\lambda} f(t) \sin \lambda(x-t)$, $u(x) = 0$, and $v(x) = x$. Then it follows from Theorem A that

$$g'(x) = \int_0^x f(t) \cos \lambda(x-t).$$

Again applying Theorem A, we have

$$g''(x) = -\lambda \int_0^x f(t) \sin \lambda(x-t) + f(x).$$

Thus

$$g''(x) + \lambda^2 g(x) = f(x).$$

That $g(0) = g'(0) = 0$ is obvious from the expressions for $g(x)$ and $g'(x)$.

(7) (i) $\int_0^1 y \; dx = \int_0^1 (1 + x - 2\sqrt{x}) \; dx = \frac{1}{6}$

(ii) $2 \int_0^2 (2x^2 - (x^4 - 2x^2)) \; dx = 2 \int_0^2 (4x^2 - x^4) \; dx = \frac{128}{15}$

(iii) $\int_1^3 (3y - y^2 - (3 - y)) \; dy = \int_1^3 (4y - y^2 - 3) \; dy = \frac{4}{3}$

(8) $\int_0^{1-a} (x - x^2 - ax) \; dx = \int_0^{1-a} ((1-a)x - x^2) \; dx = 4.5$ gives $\frac{(1-a)^3}{6} = 4.5$ so that $a = -2$.

(9) Required area $= 2 \times \int_0^{\pi/3} \frac{1}{2}(r_2^2 - r_1^2) d\theta = 4a^2 \int_0^{\pi/3} (8\cos^2\theta - 2\cos\theta - 1) d\theta = 4\pi a^2$.

(10) (i) Length $= \displaystyle\int_0^{2\pi} \sqrt{(1-\cos(t))^2 + \sin^2(t)} dt = \int_0^{2\pi} 2|\sin(t/2)| dt = 4\int_0^{\pi} |\sin(u)| du = 8$.

(ii) Length $= \displaystyle\int_0^{\pi/4} \sqrt{1 + y'^2} dx = \int_0^{\pi/4} \sqrt{1 + \cos(2x)} dx = \sqrt{2}\int_0^{\pi/4} |\cos(x)| dx = 1$.

(11) $\displaystyle\frac{dy}{dx} = x^2 + \left(-\frac{1}{4x^2}\right)$.

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + x^4 + \frac{1}{16x^4} - \frac{1}{2}} = x^2 + \frac{1}{4x^2}.$$

Therefore,

$$\text{Length} = \int_1^3 \left(x^2 + \frac{1}{4x^2}\right) dx = \left[\frac{x^3}{3} - \frac{1}{4x}\right]_1^3 = \frac{53}{6}.$$

The surface area is

$$S = \int_1^3 2\pi(y+1)\frac{ds}{dx}dx = \int_1^3 2\pi\left(\frac{x^3}{3}+\frac{1}{4x}+1\right)\left(x^2+\frac{1}{4x^2}\right)dx$$

$$= 2\pi\left[\frac{x^6}{18}+\frac{x^3}{3}+\frac{x^2}{6}-\frac{1}{32x^2}-\frac{1}{4x}\right]_1^3.$$

(12) The diameter of the circle at a point x is given by

$$(8-x^2)-x^2, \ \ -2 \le x \le 2.$$

So the area of the cross-section at x is $A(x) = \pi(4-x^2)^2$. Thus

$$\text{Volume } = \int_{-2}^2 \pi(4-x^2)^2 dx = 2\pi\int_0^2 (4-x^2)^2 dx = \frac{512\pi}{15}.$$

(13) In the first octant, the sections perpendicular to the y-axis are squares with

$$0 \le x \le \sqrt{a^2-y^2}, \ \ 0 \le z \le \sqrt{a^2-y^2}, \ \ 0 \le y \le a.$$

Since the squares have sides of length $\sqrt{a^2-y^2}$, the area of the cross-section at y is $A(y) = 4(a^2-y^2)$. Thus the required volume is

$$\int_{-a}^a A(y)dy = 8\int_0^a (a^2-y^2)dy = \frac{16a^3}{3}.$$

(14) Let the line be along z-axis, $0 \le z \le h$. For any fixed z, the section is a square of area r^2. Hence the required volume is $\int_0^h r^2 dz = r^2 h$.

(15) **Washer Method**

Area of washer $= \pi(1+y)^2 = \pi(1+(3-x^2))^2 = \pi(4-x^2)^2$ so that

Volume $= \int_{-2}^2 \pi(4-x^2)^2 dx = 512\pi/15$.

(This is the same integral as in (6) above).

Shell method

Area of shell $= 2\pi(y-(-1))2x = 4\pi(1+y)\sqrt{3-y}$ so that

$$\text{Volume} = \int_{-1}^{3} 4\pi(1+y)\sqrt{3-y}dy = 512\pi/15.$$

(14) **Washer Method**

Required volume = Volume of the sphere -Volume generated by revolving the shaded region around the y-axis $= 32\pi/3 - [\int_{-1}^{1} \pi x^2 dy - \pi(\sqrt{3})^2 2] = 32\pi/3 - 2\pi[\int_0^1 (4-y^2)dy - 3] = 32\pi/3 - 2\pi[11/3 - 3] = 28\pi/3$

Shell Method

Required volume = Volume of the sphere -Volume generated by revolving the shaded region around the y-axis $= 32\pi/3 - \int_{\sqrt{3}}^{2} 2\pi x(2y)dx = 32\pi/3 - 4\pi\int_{\sqrt{3}}^{2} x\sqrt{4-x^2}dx = 32\pi/3 - 4\pi(1/3) = 28\pi/3$

Solutions to Tutorial Sheet 5

(1) (i) $\{(x,y) \in \mathbb{R}^2 \mid x \neq \pm y\}$

(ii) $\mathbb{R}^2 - \{(0,0)\}$

(2) (i) A level curve corresponding to any of the given values of c is the straight line $x - y = c$ in the xy-plane. A contour line corresponding to any of the given values of c is the same line shifted to the plane $z = c$ in $\mathbb{R}^3$.

(ii) Level curves do not exist for $c = -3, -2, -1$. The level curve corresponding to $c = 0$ is the point $(0,0)$. The level curves corresponding to $c = 1, 2, 3, 4$ are concentric circles centered at the origin in the xy-plane. Contour lines corresponding to $c = 1, 2, 3, 4$ are the cross-sections in $\mathbb{R}^3$ of the paraboloid $z = x^2 + y^2$ by the plane $z = c$, i.e., circles in the plane $z = c$ centered at $(0, 0, c)$.

(iii) For $c = -3, -2, -1$, level curves are rectangular hyperbolas $xy = c$ in the xy-plane with branches in the second and fourth quadrant. For $c = 1, 2, 3, 4$, level curves are rectangular hyperbolas $xy = c$ in the xy-plane with branches in the first and third quadrant. For $c = 0$, the corresponding level curve (resp. the contour line) is the union of the x-axis and the y-axis in the xy-plane (resp. in the xyz-space). A contour line corresponding to a non-zero c is the cross-section of the hyperboloid $z = xy$ by the plane $z = c$, i.e., a rectangular hyperbola in the plane $z = c$.

(3) (i) Discontinuous at $(0,0)$. (Check $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ using $y = mx^3$).

(ii) Continuous at $(0,0)$:

$$\left|xy\frac{x^2-y^2}{x^2+y^2}\right| \leq |xy|\frac{x^2+y^2}{x^2+y^2} = |xy|.$$

(iii) Continuous at $(0,0)$:

$$|f(x,y)| \leq 2(|x|+|y|) \leq 4\sqrt{x^2+y^2}.$$

(4) (i) Use the sequential definition of limit: $(x_n, y_n) \to (a,b) \implies x_n \to a$ and $y_n \to b \implies f(x_n) \to f(a)$ and $g(y_n) \to g(b) \implies f(x_n) \pm g(y_n) \to f(a) \pm g(b)$ by the continuity of f, g and limit theorems for sequences.

(ii) $(x_n, y_n) \to (a,b) \implies x_n \to a$ and $y_n \to b \implies f(x_n) \to f(a)$ and $g(y_n) \to g(b) \implies f(x_n)g(y_n) \to f(a)g(b)$ by the continuity of f, g and limit theorems for sequences.

(iii) Follows from (i) above and the following:

$$\min\{f(x), g(y)\} = \frac{f(x)+g(y)}{2} - \frac{|f(x)-g(y)|}{2},$$
$$\max\{f(x), g(y)\} = \frac{f(x)+g(y)}{2} + \frac{|f(x)-g(y)|}{2}.$$

(5) Note that limits are different along different paths: $f(x,x) = 1$ for every x and $f(x,0) = 0$.

(6) (i) $f_x(0,0) = 0 = f_y(0,0)$.

(ii)

$$f_x(0,0) = \lim_{h\to 0} \frac{\sin^2(h)/|h|}{h} = \lim_{h\to 0} \frac{\sin^2(h)}{h|h|}$$

does not exist (Left Limit $\neq$ Right Limit). Similarly, $f_y(0,0)$ does not exist.

(7) $|f(x,y)| \leq x^2 + y^2 \Rightarrow f$ is continuous at$(0,0)$.

It is easily checked that $f_x(0,0) = f_y(0,0) = 0$.

Now,
$$f_x = 2x\left(\sin\left(\frac{1}{x^2+y^2}\right) - \frac{1}{x^2+y^2}\cos\left(\frac{1}{x^2+y^2}\right)\right).$$
The function $2x\sin\left(\frac{1}{x^2+y^2}\right)$ is bounded in any disc centered at $(0,0)$, while $\dfrac{2x}{x^2+y^2}\cos\left(\dfrac{1}{x^2+y^2}\right)$ is unbounded in any such disc.
(To see this, consider $(x,y) = \left(\frac{1}{\sqrt{n\pi}}, 0\right)$ for n a large positive integer.)

Thus f_x is unbounded in any disc around $(0,0)$.

(8) $f_x(0,0) = \lim\limits_{h\to 0}\dfrac{f(h,0)-f(0,0)}{h} = \lim\limits_{h\to 0}\sin\dfrac{1}{h}$ does not exist. Similarly $f_y(0,0)$ does not exist. Clearly, f is continuous at $(0,0)$.

(9) (i) Let $\vec{v} = (a,b)$ be any unit vector in $\mathbb{R}^2$. We have
$$\left(D_{\vec{v}}f\right)(0,0) = \lim_{h\to 0}\frac{f(h\vec{v})}{h} = \lim_{h\to 0}\frac{f(ha,hb)}{h} = \lim_{h\to 0}\frac{h^2ab\left(\frac{a^2-b^2}{a^2+b^2}\right)}{h} = 0.$$
Therefore $\left(D_{\vec{v}}f\right)(0,0)$ exists and equals 0 for every unity vector $\vec{v}\in\mathbb{R}^2$.

For considering differentiability, note that $f_x(0,0) = \left(D_{\hat{i}}f\right)(0,0) = 0 = f_y(0,0) = \left(D_{\hat{j}}f\right)(0,0)$. We have then
$$\lim_{(h,k)\to(0,0)}\frac{|f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)}\frac{|hk(h^2-k^2)|}{(h^2+k^2)^{3/2}} = 0$$
since
$$0 \le \frac{|hk(h^2-k^2)|}{(h^2+k^2)^{3/2}} \le \frac{|hk|}{\sqrt{h^2+k^2}}\frac{h^2+k^2}{h^2+k^2} \le \frac{\sqrt{h^2+k^2}\sqrt{h^2+k^2}}{\sqrt{h^2+k^2}} = \sqrt{h^2+k^2}.$$
Thus f is differentiable at $(0,0)$.

(ii) Note that, for any unit vector $\vec{v} = (a,b)$ in $\mathbb{R}^2$, we have
$$D_{\vec{v}}f(0,0) = \lim_{h\to 0}\frac{h^3a^3}{h(h^2(a^2+b^2))} = \lim_{h\to 0}\frac{a^3}{(a^2+b^2)} = \frac{a^3}{(a^2+b^2)}.$$
To consider differentiability, note that $f_x(0,0) = 1,\ f_y(0,0) = 0$ and
$$\lim_{(h,k)\to(0,0)}\frac{|f(h,k)-h\times 1-k\times 0|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)}\frac{|h^3/(h^2+k^2)-h|}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{|hk^2|}{(h^2+k^2)^{3/2}}$$

does not exist (consider, for example, $k = mh$). Hence f is not differentiable at $(0,0)$.

(iii) For any unit vector $\vec{v} \in \mathbb{R}^2$, one has

$$(D_{\vec{v}}f)(0,0) = \lim_{h\to 0} \frac{h^2(a^2+b^2)\sin\left[\frac{1}{h^2(a^2+b^2)}\right]}{h} = 0.$$

Also,

$$\lim_{(h,k)\to(0,0)} \frac{\left|(h^2+k^2)\sin\left[\frac{1}{(h^2+k^2)}\right]\right|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \sqrt{h^2+k^2}\sin\left(\frac{1}{h^2+k^2}\right) = 0;$$

therefore f is differentiable at $(0,0)$.

(10) $f(0,0) = 0,\ \ |f(x,y)| \le \sqrt{x^2+y^2} \implies f$ is continuous at $(0,0)$.

Let $\vec{v}$ be a unit vector in $\mathbb{R}^2$.

For $\vec{v} = (a,b)$, with $b \neq 0$, one has

$$(D_{\vec{v}})\, f(0,0) = \lim_{h\to 0} \frac{1}{h}\frac{hb}{|hb|}\sqrt{h^2a^2+h^2b^2} = \frac{(\sqrt{a^2+b^2})b}{|b|}.$$

If $\vec{v} = (a,0)$, then $(D_{\vec{v}}f)(0,0) = 0$. Hence $(D_{\vec{v}}f)(0,0)$ exists for every unit vector $\vec{v} \in \mathbb{R}^2$. Further,

$$f_x(0,0) = 0, \ \ f_y(0,0) = 1,$$

and

$$\lim_{(h,k)\to(0,0)} \frac{|f(h,k) - 0 - h\times 0 - k\times 1|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{\left|\frac{k}{|k|}\sqrt{h^2+k^2} - k\right|}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \left|\frac{k}{|k|} - \frac{k}{\sqrt{h^2+k^2}}\right|$$

does not exist (consider, for example, $k = mh$) so that f is not differentiable at $(0,0)$.

(11) Continuity follows from the fact that $|xy| \le x^2+y^2$.

Solutions to Tutorial Sheet 6

.(1) Solution provided in the class slides.

(2) Solution provided in the class slides.

(3) $(\nabla F)(1,-1,3) = (\dfrac{\partial F}{\partial x}(1,-1,3), \dfrac{\partial F}{\partial y}(1,-1,3), \dfrac{\partial F}{\partial z}(1,-1,3)) = 4\mathbf{j} + 6\mathbf{k}$.

The **tangent plane** to the surface $F(x,y,z) = 7$ at the point $(1,-1,3)$ is given by

$0 \times (x-1) + 4 \times (y+1) + 6 \times (z-3) = 0,$ i.e., $2y + 3z = 7$.

The **normal line** to the surface $F(x,y,z) = 7$ at the point $(1,-1,3)$ is given by

$x = 1,\ 3y - 2z + 9 = 0$.

(4) $\mathbf{u} = \dfrac{(2,2,1)}{\sqrt{2^2+2^2+1^2}} = \left(\dfrac{2}{3}, \dfrac{2}{3}, \dfrac{1}{3}\right) = \dfrac{2}{3}(\mathbf{i}+\mathbf{j}) + \dfrac{1}{3}\mathbf{k}$

and

$(\nabla F)((2,2,1)) = 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$.

Therefore,

$(D_{\mathbf{u}}F)(2,2,1) = (\nabla F)(2,2,1) \cdot \mathbf{u} = \dfrac{6}{3} - \dfrac{10}{3} + \dfrac{2}{3} = -\dfrac{2}{3}$.

(5) Given that $\sin(x+y) + \sin(y+z) = 1$ (with $\cos(y+z) \neq 0$).

(The student may assume that z is a sufficiently smooth function of x and y).

Differentiating w.r.t. x while keeping y fixed, we get

$$\cos(x+y) + \cos(y+z)\tfrac{\partial z}{\partial x} = 0. \qquad (*)$$

Similarly, differentiating w.r.t.y while keping x fixed, we get

$$\cos(x+y) + \cos(y+z)\left(1 + \tfrac{\partial z}{\partial y}\right) = 0. \qquad (**)$$

Differentiating (*) w.r.t y we have

$$-\sin(x+y) - \sin(y+z)\left(1 + \tfrac{\partial z}{\partial y}\right)\tfrac{\partial z}{\partial x} + \cos(y+z)\tfrac{\partial^2 z}{\partial x \partial y} = 0.$$

Thus, using (*) and (**), we have

$$\begin{aligned}
&\frac{\partial^2 z}{\partial x \partial y} \\
&= \frac{1}{\cos(y+z)}\left[\sin(x+y) + \sin(y+z).\left(1 + \frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x}\right] \\
&= \frac{1}{\cos(y+z)}\left[\sin(x+y) + \sin(y+z)\left(-\frac{\cos(x+y)}{\cos(y+z)}\right)\left(-\frac{\cos(x+y)}{\cos(y+z)}\right)\right] \\
&= \frac{\sin(x+y)}{\cos(y+z)} + \tan(y+z)\frac{\cos^2(x+y)}{\cos^2(y+z)}.
\end{aligned}$$

(6) We have

$$f_{xy}(0,0) = \lim_{k\to 0}\frac{f_x(0,k) - f_x(0,0)}{k},$$

where (noting that $k \neq 0$)

$$f_x(0,k) = \lim_{h\to 0}\frac{f(h,k) - f(0,k)}{h} = -k \text{ and } f_x(0,0) = \lim_{h\to 0}\frac{f(h,0) - f(0,0)}{h} = 0.$$

Therefore,

$$f_{xy}(0,0) = \lim_{k\to 0}\frac{-k-0}{k} = -1 \text{ ; similarly } f_{yx}(0,0) = 1.$$

Thus

$$f_{xy}(0,0) \neq f_{yx}(0,0).$$

By directly computing f_{xy}, f_{yx} for $(x,y) \neq (0,0)$, one observes that these are not continuous at $(0,0)$.

(In the following $H_f(a,b)$ denotes the **Hessian matrix** of a sufficiently smooth function f at the point (a,b)).

(5) (i) We have

$$f_x(-1,2) = 0 \quad = \quad f_y(-1,2); \;\; H_f(-1,2) = \begin{bmatrix} 12 & 0 \\ \\ 0 & 48 \end{bmatrix}.$$

$$D = 12 \times 48 \quad > \quad 0, \;\; f_{xx}(-1,2) = 12 > 0 \;\; \Rightarrow (-1,2) \text{ is a point of local minimum of } f.$$

(ii) We have

$$f_x(0,0) = 0 \quad = \quad f_y(0,0); \;\; H_f(0,0) = \begin{bmatrix} 6 & -2 \\ \\ -2 & 10 \end{bmatrix}.$$

$$D = 60 - 4 \quad > \quad 0, \;\; f_{xx}(0,0) = 6 > 0 \;\; \Rightarrow \;\; (0,0) \text{ is a point of local minimum of } f.$$

(8) (i) $f_x = e^{-\frac{(x^2+y^2)}{2}}\left(2x - x^3 + xy^2\right), \;\; f_y = e^{-\frac{(x^2+y^2)}{2}}\left(-2y + y^3 - x^2 y\right).$

Critical points are $(0,0), \; (\pm\sqrt{2},0), \;\; (0,\pm\sqrt{2})$.

$$H_f(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \quad \Rightarrow \;\; (0,0) \;\text{ is a saddle point of } f.$$

$$H_f(\pm\sqrt{2},0) = \begin{bmatrix} -\frac{4}{e} & 0 \\ 0 & -\frac{4}{e} \end{bmatrix} \quad \Rightarrow \;\; (\pm\sqrt{2},0) \;\text{ is a point of local maximum of } f.$$

$$H_f(0,\pm\sqrt{2}) = \begin{bmatrix} \frac{4}{e} & 0 \\ 0 & \frac{4}{e} \end{bmatrix} \quad \Rightarrow \;\; (0,\pm\sqrt{2}) \;\text{ is a point of local minimum of } f.$$

(ii) $f_x = 3x^2 - 3y^2$ and $f_y = -6xy$ imply that $(0,0)$ is the only critical point of f.

Now,

$$H_f(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, the standard derivative test fails.

However, $f(\pm\epsilon, 0) = \pm\epsilon^3$ for any ϵ so that $(0,0)$ is a saddle point of f.

(9) From $f(x,y) = (x^2 - 4x)\cos y$ $(1 \le x \le 3, -\pi/4 \le y \le \pi/4)$, we have

$f_x = (2x-4)\cos y$ and $f_y = -(x^2-4x)\sin y$.

Thus the only critical point of f is $P = (2,0)$; note that $f(P) = -4$.

Next, $g_\pm(x) \equiv f(x, \pm\frac{\pi}{4}) = \frac{(x^2-4x)}{\sqrt{2}}$ $(1 \le x \le 3)$ has $x = 2$ as the only critical point so that we consider $P_\pm = (2, \pm\frac{\pi}{4})$; note that $f(P_\pm) = \frac{-4}{\sqrt{2}}$.

We also need to check $g_\pm(1) = f(1, \pm\frac{\pi}{4})$ $(\equiv f(Q_\pm))$ and $g_\pm(3) = f(3, \pm\frac{\pi}{4})$ $(\equiv f(S_\pm))$; note that $f(Q_\pm) = \frac{-3}{\sqrt{2}}$, $f(S_\pm) = -\frac{-3}{\sqrt{2}}$.

Next, consider $h(y) \equiv f(1,y) = -3\cos y$ $(-\pi/4 \le y \le \pi/4)$. The only critical point of h is $y = 0$; note that $h(0) = f(1,0)$ $(\equiv f(M)) = -3$. ($h(\pm\pi/4)$ is just $f(Q_\pm)$).

Finally, consider $k(y) \equiv f(3,y) = -3\cos y$ $(-\pi/4 \le y \le \pi/4)$. The only critical point of k is $y = 0$; note that $k(0) = f(3,0)$ $(\equiv f(T)) = -3$. ($k(\pm\pi/4)$ is just $f(S_\pm)$).

Summarizing, we have the following table:

Points	P_+	P_-	Q_+	Q_-	S_+	S_-	T	P	M
Values	$-\frac{4}{\sqrt{2}}$	$-\frac{4}{\sqrt{2}}$	$-\frac{3}{\sqrt{2}}$	$-\frac{3}{\sqrt{2}}$	$-\frac{3}{\sqrt{2}}$	$-\frac{3}{\sqrt{2}}$	-3	-4	-3

By inspection one finds that

$f_{\min} = -4$ is attained at $P = (2,0)$ and

$f_{\max} = -\frac{3}{\sqrt{2}}$ at $Q_\pm = (1, \pm\pi/4)$ and at $S_\pm = (3, \pm\pi/4)$.

(10) Consider $\bigtriangledown(T+\lambda g)=0,\ g=0$, where

$T(x,y,z)=400xyz$ and $g(x,y,z)=x^2+y^2+z^2-1$.

Thus one has

$400yz+2\lambda x=0,\ 400xz+2\lambda y=0,\ 400xy+2\lambda z=0,$ so that

$400xyz=-2\lambda x^2=-2\lambda y^2=-2\lambda z^2$.

If $\lambda\neq 0$, then $x=\pm y, y=\pm z$ and $z=\pm x$; combining this with $g=0$ one obtains

$(x,y,z)=\left(\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}}\right)$ (with the corresponding λ being either $-200/\sqrt{3}$ or $200/\sqrt{3}$).

If $\lambda=0$, then $yz=xz=xy=0$; combining this with $g=0$ one obtains

$(x,y,z)=(\pm1,0,0)$ or $(0,\pm1,0)$ or $(0,0,\pm1)$.

Thus we need to check T at $8+6=14$ points. The first set of eight points gives $T=\frac{400}{3\sqrt{3}}$ or $-\frac{400}{3\sqrt{3}}$, and the second set of six points gives $T=0$. Hence the highest temperature on the unit sphere is $\frac{400}{3\sqrt{3}}$.

(11) Consider

$\bigtriangledown f+\lambda\bigtriangledown g_1+\mu\bigtriangledown g_2=0,\ \ g_1=0=g_2,$ where

$f(x,y,z)=xyz,\ g_1(x,y,z)=x+y+z-40,\ g_2(x,y,z)=x+y-z.$

Thus one has

$$yz+\lambda+\mu=0,\ zx+\lambda+\mu=0,\ xy+\lambda-\mu=0. \qquad (*)$$

Now, $g_1=0=g_2$ gives $z=20$ and the first two equalities in $(*)$ then give $x=y$; since one has $x+y=20$ in view of $g_2=0$, one is led to $x=y=10,\ z=20$. Then $f(10,10,20)=2000$ is the maximum value of f subject to the given constraints. (That 2000 is the *maximum* value (and not the minimum value) of f under the given constraints can be deduced by checking the value of f at some other point satisfying the constraints such as $(x,y,z)=(5,15,20)$).

(12) Consider

$\bigtriangledown f + \lambda \bigtriangledown g_1 + \mu \bigtriangledown g_2 = 0, \;\; g_1 = 0 = g_2,$ where

$f(x,y,z) = x^2+y^2+z^2, \; g_1(x,y,z) = x+2y+3z-6, \; g_2(x,y,z) = x+3y+4z-9.$

Thus one has

$$2x + \lambda + \mu = 0 \tag{A}$$

$$2y + 2\lambda + 3\mu = 0 \tag{B}$$

$$2z + 3\lambda + 4\mu = 0 \tag{C}$$

$$x + 2y + 3z - 6 = 0 \tag{D}$$

$$x + 3y + 4z - 9 = 0 \tag{E}$$

Considering $(E)-(D)$, one obtains $y+z=3$; using this and considering $(B)+(C)$ one next obtains

$$5\lambda + 7\mu = -6 \tag{F}$$

Considering $2\times(E) - 3\times(D)$, one obtains $x+z=0$; using this and considering $(A)+(C)$ one next obtains

$$4\lambda + 5\mu = 0 \tag{G}$$

Solving (F) and (G) for λ and μ, one has

$\lambda = 10$ and $\mu = -8$.

It follows now from (A), (B) and (C) that

$x = -1$, $y = 2$ and $z = 1$.

Then $f(-1,2,1) = 6$ is the minimum value of f subject to the given constraints. (That 6 is the *minimum* value (and not the maximum value) of f under the given constraints can be deduced by checking the value of f at some other point satisfying the constraints such as $(x,y,z) = (-3,0,3)$).

Solutions to Tutorial Sheet 7

(1) (i) $\int_1^e \left(\int_{\ln y}^1 dx \right) dy$

(ii) $\int_{-1}^1 \left(\int_{x^2}^1 f(x,y)dy \right) dx.$

(2) (i) $\int_0^\pi \left(\int_0^y \frac{\sin(y)}{y} dx \right) dy = \int_0^\pi \sin(y)dy = 2.$

(ii)
$$\int_0^1 \left(\int_0^x x^2 e^{xy} dy \right) dx = \int_0^1 x(e^{x^2} - 1)dx$$
$$= \frac{1}{2}(e-1) - \frac{1}{2} = \frac{1}{2}(e-2).$$

(iii)
$$\int_0^2 (\tan^{-1}(\pi x) - \tan^{-1}(x))dx = \int_0^2 \left(\int_x^{\pi x} \frac{dy}{1+y^2} \right) dx$$
$$= \int\int_{R_1+R_2} \frac{d(x,y)}{1+y^2} = \int_0^2 \left(\int_{y/\pi}^y \frac{dx}{1+y^2} \right) dy + \int_2^{2\pi} \left(\int_{y/\pi}^2 \frac{dx}{1+y^2} \right) dy$$
$$= \int_0^2 (1 - \frac{1}{\pi}) \frac{ydy}{1+y^2} + \int_2^{2\pi} (2 - \frac{y}{\pi}) \frac{dy}{1+y^2}$$
$$= \frac{\pi - 1}{2\pi} \ln(1+y^2)|_0^2 + 2\tan^{-1} y|_2^{2\pi} - \frac{1}{2\pi} \ln(1+y^2)|_2^{2\pi}$$
$$= \frac{\pi - 1}{2\pi} \ln 5 + 2(\tan^{-1} 2\pi - \tan^{-1} 2) - \frac{1}{2\pi} \left[\ln \frac{(4\pi^2+1)}{5} \right].$$

(3) $\int\int_D e^{x^2} d(x,y) = \int_0^1 \left(\int_0^{2x} e^{x^2} dy \right) dx = \int_0^1 2xe^{x^2} dx = e - 1.$

(4) Put
$$x = \frac{u-v}{2}, y = \frac{u+v}{2}.$$

Then the rectangle $R = \{\pi \leq u \leq 3\pi, \ \ -\pi \leq v \leq \pi\}$

in the uv-plane gets mapped to D, a parallelogram in the xy-plane.

Further,
$$J = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} = \frac{1}{2}$$

and then
$$\int\int_D (x-y)^2 \sin^2(x+y)dxdy = \int\int_R v^2 \sin^2(u) \frac{1}{2} dudv$$

$$= \frac{1}{2}\left(\int_{-\pi}^{\pi} v^2 dv\right)\left(\int_{\pi}^{3\pi} \sin^2(u)du\right) = \frac{1}{2}\left(2 \times \frac{\pi^3}{3}\right)(\pi) = \frac{\pi^4}{3}.$$

(5) Put

$$x = \frac{u}{v}, \quad y = uv.$$

Then the rectangle $R = \{1 \le u \le 3, 1 \le v \le 2\}$ in the uv-plane gets mapped to D in the xy-plane.

Further,

$$J = \begin{vmatrix} 1/v & -u/v^2 \\ v & u \end{vmatrix} = \frac{2u}{v}$$

and then

$$\int\int_D d(x,y) = \text{Area}(D) = \int\int_R \frac{2u}{v} dudv = \left(\int_1^3 2udu\right)\left(\int_1^2 \frac{dv}{v}\right) = 8\ln 2.$$

(6) (i) Setting

$$x = \rho\cos(\theta), y = \rho\sin(\theta), \quad 0 \le \rho \le r, 0 \le \theta \le 2\pi,$$

and using $J = \rho$, we get

$$\int\int_{D(r)} e^{-(x^2+y^2)} d(x,y) = \int_0^{2\pi}\int_0^r e^{-\rho^2}\rho d\rho d\theta = \pi(1 - e^{-r^2}).$$

Therefore, letting $r \to \infty$, we obtain the limit to be π.

(ii) By symmetry, the required limit is $\lim_{r\to\infty} \frac{\pi}{4}\left(1 - e^{-r^2}\right) = \frac{\pi}{4}.$

(iii) Let

$$I(r) = \{|x| \le r, |y| \le r\}.$$

Then

$$\int\int_{D(r)} e^{-(x^2+y^2)} d(x,y) \le \int\int_{I(r)} e^{-(x^2+y^2)} d(x,y)$$
$$\le \int\int_{D(r\sqrt{2})} e^{-(x^2+y^2)d(x,y)}.$$

Therefore, letting $r \to \infty$, we obtain the limit to be π using the Sandwich theorem.

(iv) The required integral, being one-fourth of the integral in (iii), is $\dfrac{\pi}{4}$.

(7) By symmetry, the given volume is 8 times the volume in the positive octant.

In that octant the volume lies above the region $Q = \{x \geq 0,\ \ y \geq 0,\ \ x^2 + y^2 \leq a^2\}$

and underneath the cylinder $x^2 + z^2 = a^2$.

Therefore,

$$V = 8\int_0^a \left(\int_0^{\sqrt{a^2-x^2}} \left(\int_0^{\sqrt{a^2-x^2}} 1dz\right) dy\right) dx = \frac{16a^3}{3}.$$

(8)

$$D = \{(x,y,z)| -1 \leq x \leq 1,\ -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2},\ \sqrt{x^2+y^2} \leq z \leq 1\}.$$

(9)

$$I = \int_0^{\sqrt{2}} \left(\int_0^{\sqrt{2-x^2}} \left(\int_{x^2+y^2}^{2} xdz\right) dy\right) dx.$$

We can also write the region of integration D as

$$D = \{(x,y,z)|0 \leq z \leq 2,\ 0 \leq y \leq \sqrt{z},\ 0 \leq x \leq \sqrt{z-y^2}\}.$$

Thus

$$I = \int_0^2 \left(\int_0^{\sqrt{z}} \left(\int_0^{\sqrt{z-y^2}} xdx\right) dy\right) dz = \frac{8\sqrt{2}}{15}.$$

(10) (i) Using cylindrical coordinates, one has

$$I = \int_{-1}^{1}\int_0^{2\pi}\int_0^1 (z^2r^2)rdrd\theta dz = \pi/3.$$

(ii) Using spherical coordinates, one has

$$I = \int_0^{2\pi}\int_0^{\pi}\int_0^1 (\exp(r^3)r^2 \sin\phi drd\phi d\theta = \frac{4\pi(e-1)}{3}.$$

Solutions to Tutorial Sheet 8

(1) (i) $\nabla(fg) = \sum \mathbf{i}\frac{\partial(fg)}{\partial x} = \sum \mathbf{i}\frac{\partial f}{\partial x}g + \sum \mathbf{i}f\frac{\partial g}{\partial x} = g\nabla f + f\nabla g.$

(ii) Since $\frac{\partial f^n}{\partial x} = nf^{n-1}\frac{\partial f}{\partial x}$, hence $\nabla f^n = nf^{n-1}\nabla f$.

(iii) Since,

$$\frac{\partial}{\partial x}\left(\frac{f}{g}\right) = g^{-2}\left(g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}\right),$$

$$\nabla\left(\frac{f}{g}\right) = \sum \mathbf{i}\frac{\partial}{\partial x}\left(\frac{f}{g}\right) = g^{-2}\left(g\sum \mathbf{i}\frac{\partial f}{\partial x} - f\sum \mathbf{i}\frac{\partial g}{\partial x}\right),$$

which is the desired result.

(2) (i) Note that

$$\frac{\partial r}{\partial x} = \frac{\partial(x^2+y^2+z^2)^{1/2}}{\partial x} = x/r.$$

Similarly, $\frac{\partial r}{\partial y} = y/r$ and $\frac{\partial r}{\partial z} = z/r$. Now,

$$\frac{\partial r^n}{\partial x} = nr^{n-1}\frac{\partial r}{\partial x} = nr^{n-2}x, \text{ etc.}$$

Hence, $\nabla(r^n) = nr^{n-2}\mathbf{r}$.

(ii) Letting $n = -1$ in (i) we have $\nabla(r^{-1}) = -r^{-3}\mathbf{r}$. Hence, $\mathbf{a}\cdot\nabla(r^{-1}) = -r^{-3}(\mathbf{a}\cdot\mathbf{r})$.

(iii) First we compute $\nabla(\mathbf{a}\cdot\nabla(r^{-1}))$:

$$\frac{\partial}{\partial x}\Big(\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}\Big) = \frac{\partial}{\partial x}\Big(\frac{a_1x+a_2y+a_3z}{r^3}\Big) = \frac{a_1}{r^3} + \mathbf{a}\cdot\mathbf{r}\frac{\partial r^{-3}}{\partial x} = \frac{a_1}{r^3} - 3xr^{-5}(\mathbf{a}\cdot\mathbf{r}).$$

Hence,

$$\mathbf{b}\cdot\nabla(\mathbf{a}\cdot\nabla(r^{-1})) = -r^{-3}(\mathbf{a}\cdot\mathbf{b}) + 3r^{-5}(\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r}).$$

(3) (i) $\nabla\cdot(f\mathbf{v}) = \sum \frac{\partial}{\partial x}(fv_1) = \sum \frac{\partial f}{\partial x}v_1 + f\sum \frac{\partial v_1}{\partial x} = \nabla f\cdot\mathbf{v} + f\nabla\cdot\mathbf{v}.$

(ii) $\nabla\times(f\mathbf{v}) = \sum \mathbf{i}\times\frac{\partial}{\partial x}(f\mathbf{v}) = \sum \mathbf{i}\times(\frac{\partial f}{\partial x}\mathbf{v}) + \sum \mathbf{i}\times(f\frac{\partial \mathbf{v}}{\partial x}) = \Big(\sum \frac{\partial f}{\partial x}\mathbf{i}\Big)\times\mathbf{v} + f\sum \mathbf{i}\times\frac{\partial}{\partial x}\mathbf{v}$
$= \nabla f\times\mathbf{v} + f(\nabla\times\mathbf{v}).$

(iii) $\nabla \times (\nabla \times \mathbf{v}) = \sum \mathbf{i} \times \frac{\partial}{\partial x}\Big(\sum \mathbf{j} \times \frac{\partial}{\partial y}\mathbf{v}\Big) = \sum\sum \Big(\mathbf{i} \times (\mathbf{j} \times \frac{\partial^2 \mathbf{v}}{\partial x \partial y})\Big)$

$= \sum_{\mathbf{i}}\sum_{\mathbf{j}} \mathbf{j}\Big(\mathbf{i}\cdot \frac{\partial^2 \mathbf{v}}{\partial x\partial y}\Big) - \sum \frac{\partial^2 \mathbf{v}}{\partial x^2} = \sum_{\mathbf{j}} \mathbf{j}\frac{\partial}{\partial y}\sum_{\mathbf{i}}\Big(\mathbf{i}\cdot\frac{\partial}{\partial x}\mathbf{v}\Big) - \Delta\mathbf{v} = \nabla(\nabla\cdot\mathbf{v}) - \Delta\mathbf{v}.$

(iv) Since

$\nabla \cdot (f\nabla g) = \sum \mathbf{i}\cdot\frac{\partial}{\partial x}\Big(\sum f\frac{\partial g}{\partial y}\mathbf{j}\Big) = \sum \frac{\partial}{\partial x}\Big(f\frac{\partial g}{\partial x}\Big) = \nabla f\cdot\nabla g + f\Delta g,$

we have

$$\nabla\cdot(f\nabla g) - \nabla\cdot(g\nabla f) = f\Delta g - g\Delta f.$$

(v) $\nabla\cdot(\nabla\times\mathbf{v}) = \sum \mathbf{i}\cdot\frac{\partial}{\partial x}\Big(\sum \mathbf{j}\times\frac{\partial}{\partial y}\mathbf{v}\Big) = \sum \mathbf{i}\cdot\Big(\sum\mathbf{j}\times\frac{\partial^2\mathbf{v}}{\partial x\partial y}\Big)$

$= \sum\sum(\mathbf{i}\times\mathbf{j})\cdot\frac{\partial^2\mathbf{v}}{\partial x\partial y} = \sum\mathbf{k}\cdot\Big(\frac{\partial^2\mathbf{v}}{\partial x\partial y} - \frac{\partial^2\mathbf{v}}{\partial y\partial x}\Big) = 0,$

by the equality of mixed partials.

(vi) $\nabla\times(\nabla f) = \sum\mathbf{i}\times\frac{\partial}{\partial x}\Big(\sum\mathbf{j}\frac{\partial f}{\partial y}\Big) = \sum\sum(\mathbf{i}\times\mathbf{j})\frac{\partial^2 f}{\partial x\partial y} = \sum\mathbf{k}\Big(\frac{\partial^2 f}{\partial x\partial y} - \frac{\partial^2 f}{\partial y\partial x}\Big) = 0.$

(vii) Note that

$$g\nabla f\times f\nabla g = g\sum\mathbf{i}\frac{\partial f}{\partial x}\times f\sum\mathbf{j}\frac{\partial g}{\partial y} = \sum\sum fg(\mathbf{i}\times\mathbf{j})\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}$$

$= \sum fg\Big(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\Big)\mathbf{k}$

$= fg\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{vmatrix} = \sum fg\Big(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial g}{\partial y}\frac{\partial f}{\partial z}\Big)\mathbf{i}.$

Hence,

$\nabla\cdot(g\nabla f\times f\nabla g) = \sum\mathbf{i}\cdot\frac{\partial}{\partial x}\ \sum\mathbf{i}fg\Big(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial g}{\partial y}\frac{\partial f}{\partial z}\Big)$

$= \sum\frac{\partial}{\partial x}\Big(fg\Big(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial g}{\partial y}\frac{\partial f}{\partial z}\Big)\Big) = \sum f\frac{\partial g}{\partial x}\Big(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial g}{\partial y}\frac{\partial f}{\partial z}\Big) + \sum g\frac{\partial f}{\partial x}\Big(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial g}{\partial y}\frac{\partial f}{\partial z}\Big)$

$+\sum fg\Big(\frac{\partial^2 f}{\partial x\partial y}\frac{\partial g}{\partial z} - \frac{\partial^2 f}{\partial x\partial z}\frac{\partial g}{\partial y}\Big) + \sum fg\Big(\frac{\partial^2 g}{\partial x\partial z}\frac{\partial f}{\partial y} - \frac{\partial^2 g}{\partial y\partial x}\frac{\partial f}{\partial z}\Big).$

Each of the four sums vanishes individually.

(4) (i) Since $\frac{\partial f(r)}{\partial x} = f'(r)\frac{\partial r}{\partial x} = f'(r)x/r$, we have

$$\mathrm{div}(\nabla f(r)) = \sum\frac{\partial}{\partial x}\Big(f'(r)\frac{x}{r}\Big)$$

$$= \sum f''(r)\frac{x^2}{r^2} + \sum \frac{f'(r)}{r} - \sum \frac{x^2}{r^3} f'(r) = f''(r) + \frac{2}{r} f'(r).$$

(ii) $\operatorname{div}(r^n \mathbf{r}) = \sum \frac{\partial}{\partial x}(r^n x) = \sum (r^n + n r^{n-1} \frac{x^2}{r}) = 3r^n + nr^n = (3+n)r^n$.

(iii) Note that $\nabla\left(\frac{r^{n+2}}{n+2}\right) = r^n \mathbf{r}$, by exercise 1(i). So,

$$\operatorname{curl}(r^n \mathbf{r}) = \operatorname{curl}(\operatorname{grad}\left(\frac{r^{n+2}}{n+2}\right)) = 0,$$

by exercise 3(vi). If $n = -2$, then

$$\nabla(\ln r) = r^{-2}\mathbf{r}$$

and hence,

$$\operatorname{curl}(r^{-2}\mathbf{r}) = \operatorname{curl}(\nabla \ln r) = 0.$$

(iv) Using part (i) it follows that

$$\operatorname{div}(\nabla r^{-1}) = \frac{d^2}{dr^2}\left(\frac{1}{r}\right) + \frac{2}{r}\left(\frac{d}{dr}(r^{-1})\right) = 0.$$

(5) (i) $\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \sum \mathbf{i} \cdot \big(\frac{\partial}{\partial x}(\mathbf{u} \times \mathbf{v})\big) = \sum \mathbf{i} \cdot (\frac{\partial \mathbf{u}}{\partial x} \times \mathbf{v}) + \sum \mathbf{i} \cdot (\mathbf{u} \times \frac{\partial \mathbf{v}}{\partial x})$
$= \sum (\mathbf{i} \times \frac{\partial \mathbf{u}}{\partial x}) \cdot \mathbf{v} - \sum (\mathbf{i} \times \frac{\partial \mathbf{v}}{\partial x}) \cdot \mathbf{u} = \mathbf{v} \cdot \operatorname{curl} \mathbf{u} - \mathbf{u} \cdot \operatorname{curl} \mathbf{v}$.

(**Def:** A vector-field $\mathbf{u}$ is said to be irrotational if $\nabla \times \mathbf{u} = 0$. A vector-field $\mathbf{u}$ is said to be solenoidal if $\nabla \cdot \mathbf{u} = 0$. We have now proved that if $\mathbf{u}$ and $\mathbf{v}$ are irrotational then $\mathbf{u} \times \mathbf{v}$ is solenoidal.)

(ii) $\nabla \times (\mathbf{u} \times \mathbf{v}) = \sum \mathbf{i} \times \frac{\partial}{\partial x}(\mathbf{u} \times \mathbf{v}) = \sum \mathbf{i} \times \big(\frac{\partial \mathbf{u}}{\partial x} \times \mathbf{v}\big) + \sum \mathbf{i} \times \big(\mathbf{u} \times \frac{\partial \mathbf{v}}{\partial x}\big)$
$= \sum (\mathbf{i} \cdot \mathbf{v}) \frac{\partial \mathbf{u}}{\partial x} - \sum \big(\mathbf{i} \cdot \frac{\partial \mathbf{u}}{\partial x}\big)\mathbf{v} + \sum \big(\mathbf{i} \cdot \frac{\partial \mathbf{v}}{\partial x}\big)\mathbf{u} - \sum (\mathbf{i} \cdot \mathbf{u}) \frac{\partial \mathbf{v}}{\partial x}$
$= (\mathbf{v} \cdot \nabla)\mathbf{u} - (\nabla \cdot \mathbf{u})\mathbf{v} + (\nabla \cdot \mathbf{v})\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v}$.

(iii) $\nabla(\mathbf{u} \cdot \mathbf{v}) = \sum \mathbf{i} \frac{\partial}{\partial x}(\mathbf{u} \cdot \mathbf{v}) = \sum \mathbf{i}\big(\frac{\partial \mathbf{u}}{\partial x} \cdot \mathbf{v}\big) + \sum \mathbf{i}\big(\frac{\partial \mathbf{v}}{\partial x} \cdot \mathbf{u}\big)$
$= \sum \Big[\big(\mathbf{v} \cdot \frac{\partial \mathbf{u}}{\partial x}\big)\mathbf{i} - (\mathbf{v} \cdot \mathbf{i})\frac{\partial \mathbf{u}}{\partial x}\Big] + \sum \Big[\big(\mathbf{u} \cdot \frac{\partial \mathbf{v}}{\partial x}\big)\mathbf{i} - (\mathbf{u} \cdot \mathbf{i})\frac{\partial \mathbf{v}}{\partial x}\Big] + (\mathbf{v} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{v}$
$= \mathbf{v} \times \Big(\sum \mathbf{i} \times \frac{\partial}{\partial x}\mathbf{u}\Big) + \mathbf{u} \times \Big(\sum \mathbf{i} \times \frac{\partial}{\partial x}\mathbf{v}\Big) + (\mathbf{v} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{v}$
$= \mathbf{v} \times (\nabla \times \mathbf{u}) + \mathbf{u} \times (\nabla \times \mathbf{v}) + (\mathbf{v} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{v}$.

(6) (i) Let $\mathbf{w} = f\mathbf{u}$, where f is a scalar field and $\mathbf{u}$ is a constant vector. Then,

$\mathbf{w} \cdot (\nabla \times \mathbf{w}) = f\mathbf{u} \cdot (\nabla \times f\mathbf{u}) = f\mathbf{u} \cdot (f\nabla \times \mathbf{u} + \nabla f \times \mathbf{u})$ (using 3(ii)) $=$

$f^2(\mathbf{u} \cdot (\nabla \times \mathbf{u})) + f\mathbf{u} \cdot \nabla f \times \mathbf{u} = 0$ (using 3(v)).

(ii) Here $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Thus, $\nabla \times \mathbf{v} = \nabla \times (\mathbf{w} \times \mathbf{r}) = (\mathbf{r} \cdot \nabla)\mathbf{w} - (\mathbf{w} \cdot \nabla)\mathbf{r} +$

$(\nabla \cdot \mathbf{r})\mathbf{w} - (\nabla \cdot \mathbf{w})\mathbf{r}$ (using 5(ii)) $= (\nabla \cdot \mathbf{r})\mathbf{w} - (\mathbf{w} \cdot \nabla)\mathbf{r} = 3\mathbf{w} - \mathbf{w} = 2\mathbf{w}$.

(iii) Let $f = \rho^{-1}$. Then, using problem 3(ii) and 3(vi),

$$\nabla \times \mathbf{v} = \nabla \times (f\nabla p) = f(\nabla \times \nabla p) + \nabla f \times \nabla p = -f^2(\nabla \rho \times \nabla p).$$

Hence

$$\mathbf{v} \cdot (\nabla \times \mathbf{v}) = -f^3 \, \nabla p \cdot (\nabla \rho \times \nabla p) = 0.$$

(7) Parameterize the curve C as $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$. Then, $\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$. Thus

$$\begin{aligned} \int_C f \cdot d\mathbf{r} &= \int_{-1}^{1} \left[(t^2 - 2t^3)\mathbf{i} + (t^4 - 2t^3)\mathbf{j}\right] \cdot (\mathbf{i} + 2t\mathbf{j})dt \\ &= \int_{-1}^{1} (t^2 - 2t^3) + 2t(t^4 - 2t^3)dt = -\frac{14}{15}. \end{aligned}$$

(8) A parametrization of the ellipse is given by

$$\mathbf{r}(\theta) = a\cos\theta\mathbf{i} + b\sin\theta\mathbf{j}, \quad 0 \le \theta \le 2\pi,$$

and

$$\mathbf{r}'(\theta) = -a\sin\theta\mathbf{i} + b\cos\theta\mathbf{j}.$$

Thus,

$$f(a\cos\theta, b\sin\theta)\cdot\mathbf{r}'(\theta) = [(a^2\cos^2\theta + b^2\sin^2\theta)\mathbf{i} + (a\cos\theta - b\sin\theta)\mathbf{j}]\cdot[-a\sin\theta\mathbf{i} + b\cos\theta\mathbf{j}]$$

and

$$\begin{aligned} \int_C f(x,y) \cdot d\mathbf{r} &= \int_0^{2\pi} \left[(-a^3\cos^2\theta\sin\theta - ab^2\sin^3\theta) + (ab\cos^2\theta - b^2\sin\theta\cos\theta)\right] d\theta \\ &= \pi ab. \end{aligned}$$

(9) A parametrization of the curve is given by

$$\mathbf{r}(\theta) = a\cos\theta\mathbf{i} + a\sin\theta\mathbf{j}, \quad 0 \leq \theta \leq 2\pi.$$

Thus, $x(\theta) = a\cos\theta, y = a\sin\theta$, and the required integral is

$$\int_0^{2\pi} \frac{a^2(\cos\theta + \sin\theta)(-\sin\theta) + a^2(\sin\theta - \cos\theta)\cos\theta}{a^2} d\theta = \int_0^{2\pi} \frac{-a^2}{a^2} d\theta = -2\pi.$$

(10) For the curve $z = xy, x^2 + y^2 = 1$, we can use the parametrization

$$x = \cos\theta, y = \sin\theta, z = \sin\theta\cos\theta, \quad 0 \leq \theta \leq 2\pi.$$

Thus,

$$\int ydx + zdy + xdz = \int ydx + (xy)dy + x(xdy + ydx)$$

$$= \int_0^{2\pi} [\sin\theta(-\sin\theta) + \sin\theta\cos\theta\cos\theta + \cos^2\theta\cos\theta + \sin\theta\cos\theta(-\sin\theta)]d\theta$$

$$= \int_0^{2\pi} [-\sin^2\theta + \sin\theta\cos^2\theta + \cos^3\theta - \sin^2\theta\cos\theta]d\theta$$

$$= -\int_0^{2\pi} \sin^2\theta d\theta + \int_0^{2\pi} \sin\theta\cos^2\theta d\theta$$

$$+ \int_0^{2\pi} \cos^3\theta d\theta - \int_0^{2\pi} \sin^2\theta\cos\theta d\theta$$

$$= -\pi.$$

Solutions to Tutorial Sheet 9

(1) Let

$$\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j} + ct\mathbf{k}.$$

Then $\mathbf{r}'(t) = -a\sin t\mathbf{i} + a\cos t\mathbf{j} + c\mathbf{k}$ and

$\|\mathbf{r}^{'}(t)\| = \sqrt{a^2 + c^2}$.

Hence

$$s = \int_0^t \sqrt{a^2+c^2}du = (\sqrt{a^2+c^2})t.$$

Thus the arc length parametrization is

$$\mathbf{r}(s) = a\cos\left(\frac{s}{\sqrt{a^2+c^2}}\right)\mathbf{i} + a\sin\left(\frac{s}{\sqrt{a^2+c^2}}\right)\mathbf{j} + \frac{cs}{\sqrt{a^2+c^2}}\mathbf{k}.$$

(2)

$$\int_{C_1} = -\int_{-1}^{+1} \frac{x^2dx}{(1+x^2)^2} = -\int_{-\pi/4}^{+\pi/4} \sin^2\theta = -\pi/4 + \frac{1}{2},$$

$$\int_{C_2} = -\int_{-1}^{+1} \frac{dy}{(1+y^2)^2} = -\int_{-\pi/4}^{+\pi/4} \cos^2\theta d\theta = -\pi/4 - 1/2,$$

$$\int_{C_3} = \int_{-1}^{+1} \frac{-x^2dx}{(1+x^2)}^{2} = -\pi/4 + \frac{1}{2},$$

$$\int_{C_4} = \int_{-1}^{+1} \frac{-dy}{(1+y^2)^2} = -\pi/4 - 1/2.$$

Hence

$$\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = -\pi.$$

(3) A parametrization of C is

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j},\ 0 \leq t \leq 2\pi.$$

Note that the outward unit normal to the circle at $\mathbf{r}(t)$ is the radial vector $\mathbf{n} = \mathbf{r}(t)$. Also,

$$\nabla(x^2 - y^2) = 2x\,\mathbf{i} - 2y\,\mathbf{j}.$$

Thus

$$\begin{aligned}\oint_C \nabla(x^2 - y^2) \cdot d\mathbf{n} &= \int_0^{2\pi} (2\cos t\,\mathbf{i} - 2\sin t\mathbf{j}) \cdot (-\sin t\mathbf{i} + \cos t\mathbf{j})dt \\ &= \int_0^{2\pi} (-2\sin 2t)dt = 0.\end{aligned}$$

(4) Parameterize C as

$$\mathbf{r}(t) = t\,\mathbf{i} + t^3\,\mathbf{j},\ 0 \le t \le 2.$$

Then $\mathbf{r}'(t) = \mathbf{i} + 3t^2\,\mathbf{j}$. Since $\nabla(x^2 - y^2) = 2t\mathbf{i} - 2t^3\mathbf{j}$, we have

$$\int_C \nabla(x^2 - y^2) \cdot d\mathbf{r} = \int_0^2 (2t - 6t^5)dt = 4 - 64 = -60.$$

(5) The required integral is

$$= \int_{C_1} \frac{dx + dy}{|x| + |y|} + \int_{C_2} \frac{dx + dy}{|x| + |y|} + \int_{C_3} \frac{dx + dy}{|x| + |y|} + \int_{C_4} \frac{dx + dy}{|x| + |y|}.$$

Along C_1: $x + y = 1$ and $|x| + |y| = x + y = 1$. Thus

$$\int_{C_1} \frac{dx + dy}{|x| + |y|} = \int_1^0 dx - \int_1^0 dx = 0.$$

Along C_2: $-x + y = 1$ and $|x| + |y| = -x + y = 1$. Thus

$$\int_{C_2} \frac{dx + dy}{|x| + |y|} = \int_0^{-1} dx + \int_0^{-1} dx = -2.$$

Along C_3: $x + y = -1$ and $|x| + |y| = -x - y = 1$. Thus

$$\int_{C_3} \frac{dx + dy}{|x| + |y|} = \int_{-1}^0 dx - \int_{-1}^0 dx = 0.$$

Along C_4: $x - y = 1$ and $|x| + |y| = x - y = 1$. Thus

$$\int_{C_4} \frac{dx + dy}{|x| + |y|} = \int_0^1 dx + \int_0^1 dx = 2.$$

Hence

$$\int_C \frac{dx+dy}{|x|+|y|} = 2-2 = 0.$$

(6)

$$\begin{aligned}
\text{Work W} &= \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (xy\mathbf{i} + x^6y^2\mathbf{j})\cdot(\mathbf{i}dx + \mathbf{j}dy) \\
&= \int_0^1 ax^{b+1}dx + \int_0^1 (a^2x^{2b+6})(abx^{b-1})dx \\
&= \frac{a}{b+2} + \frac{a^3b}{3b+6} \\
&= \frac{a}{b+2}\left(1+\frac{a^2b}{3}\right) = a\left(\frac{3+a^2b}{3(b+2)}\right).
\end{aligned}$$

This will be independent of b iff $\frac{dW}{db} = 0$ iff $0 = \frac{(b+2)a^2-(3+a^2b)}{(b+2)^2}$ iff $a = \sqrt{\frac{3}{2}}$ (as $a>0$).

(7) First we observe that the cylinder is given by

$$\left(x-\frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}.$$

From the equations of the sphere and the cylinder we have that, on the intersection C,

$$z^2 = a^2 - ax.$$

Noting the requirement $z \geq 0$, a parametrization of C is given by

$$x = \frac{a}{2} + \frac{a}{2}\cos\theta, \qquad y = \frac{a}{2}\sin\theta,\ z = a\sin\frac{\theta}{2};\ 0 \leq \theta \leq 2\pi.$$

Then

$$
\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi}[\left(\frac{a^2}{4}\sin^2\theta\right)(-\frac{a}{2}\sin\theta)+(a^2\sin^2\frac{\theta}{2})(\frac{a}{2}\cos\theta) \\
&\qquad +\left(\frac{a^2}{4}+\frac{a^2}{4}\cos^2\theta+\frac{a^2}{2}\cos\theta\right)(\frac{a}{2}\cos\frac{\theta}{2})]d\theta \\
&= \int_0^{2\pi}[-\frac{a^3}{8}\sin^3\theta+\frac{a^3}{2}\sin^2\frac{\theta}{2}\cos\theta+\frac{a^3}{8}\cos\frac{\theta}{2}+\frac{a^3}{8}\cos^2\theta\cos\frac{\theta}{2} \\
&\qquad +\frac{a^3}{4}\cos\theta\cos\frac{\theta}{2}]d\theta \\
&= -\frac{\pi a^3}{4}.
\end{aligned}
$$

(8) $\dfrac{\partial f_1}{\partial y}=3x$, $\dfrac{\partial f_2}{\partial x}=3x^2y$ where $(f_1,f_2)=f$. Now

$$
\begin{aligned}
\frac{\partial f_1}{\partial y}=\frac{\partial f_2}{\partial x} \quad &\text{iff} \quad 3x=3x^2y \\
&\text{iff} \quad \text{either } x=0 \text{ or } xy=1.
\end{aligned}
$$

Since the sets $\{(x,y)|x=0\},\{(x,y)|xy=1\}$ are not open, $\mathbf{F}(x,y)$ is not the gradient of a scalar field on any open subset of $\mathbb{R}^2$.

(9)

$$
\frac{\partial f_1}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}=\frac{\partial f_2}{\partial x} \text{ on } \mathbb{R}^2\setminus\{(0,0)\}.
$$

However, $\mathbf{F}\neq\nabla f$ for any f. Indeed, let C to be the unit circle $x^2+y^2=1$, oriented anticlockwise. Then one has

$$
\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi}(-\sin t\mathbf{i}+\cos t\mathbf{j})\cdot(-\sin t\mathbf{i}+\cos t\mathbf{j})dt \\
&= 2\pi\neq 0
\end{aligned}
$$

(10) Suppose $\mathbf{v}=\nabla\phi$ for some ϕ.

Then

$$
\frac{\partial\phi}{\partial x}=2xy+z^3 \quad\Rightarrow\quad \phi(x,y,z)=x^2y+z^3x+f(y,z)
$$

for some $f(y,z)$. Assuming f has partial derivatives, we get

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= x^2 + \frac{\partial f}{\partial y} = x^2 \\ \text{so that } \frac{\partial f}{\partial y} &= 0\end{aligned}$$

and $f(y,z)$ depends only on z

Let $f(y,z) = g(z)$. Then $\phi(x,y,z) = x^2y + z^3x + g(z) \Rightarrow \dfrac{\partial \phi}{\partial z} = 3z^2x + g'(z) = 3z^2x \Rightarrow g'(z) = 0$. Let us select $g(z) = 0$. It can be checked that $\phi(x,y,z) = x^2y + z^3x$ satisfies $\nabla\phi = \mathbf{v}$.

Hence

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = 0$$

for every smooth closed curve C.

(11) $\mathbf{F}(x,y,z) = f(r)\mathbf{r} = f(r)x\ \mathbf{i} + f(r)y\ \mathbf{j} + f(r)z\ \mathbf{k}$.

Since

$$r = (x^2 + y^2 + z^2)^{1/2},$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

If $\mathbf{F}$ is to be $\nabla\phi$ for some ϕ, then we must have $\phi_x = f(r)x, \phi_y = f(r)y, \phi_z = f(r)z$; that is,

$$\begin{aligned}\phi_x = xf(r) &= \frac{x}{r}rf(r) = \frac{\partial r}{\partial x}rf(r), \\ \phi_y = yf(r) &= \frac{y}{r}rf(r) = \frac{\partial r}{\partial y}rf(r), \\ \phi_z = zf(r) &= \frac{z}{r}rf(r) = \frac{\partial r}{\partial z}rf(r).\end{aligned}$$

Now it can be seen that $\phi(x,y,z) = \displaystyle\int_{t_0}^{r} tf(t)dt$, with some t_0 fixed, satisfies all the desired equations.

Solutions to Tutorial Sheet 10

(1) We have to show that

$$\iint_R (g_x - f_y)\, dxdy = \oint_{\partial R} (f dx + g dy).$$

(i)

$$\text{LHS} = \iint_R 4xy\, dxdy = \int_0^1 \left(\int_0^{1-x^2} 4xy\, dy \right) dx = \frac{1}{3}.$$

Observe that the boundary of R consists of three smooth curves: a segment of the x-axis, a part of the parabola $y = 1 - x^2$ and a segment of the y-axis. The integral on the RHS vanishes on both the axes. We choose the parametrization $t \mapsto (t, 1-t^2)$, $(t \in [0,1])$ for the part of the parabola traced in the opposite direction. This gives

$$\text{RHS} = \oint_{\partial R} (-xy^2\, dx + x^2 y\, dy)$$

$$= -\int_0^1 [-t(1-t^2)^2 + t^2(1-t^2)(-2t)]\, dt = \frac{1}{3}.$$

(ii)

$$\text{LHS} = \iint_R (e^x + 2x - 2x)\, dxdy = \int_0^1 \left(\int_0^x e^x\, dy \right) dx = 1.$$

and

$$\text{RHS} = \oint_{\partial R} [2xy\, dx + (e^x + x^2)\, dy]$$

$$= \int_0^1 (e+1) dy + \int_1^0 (3t^2 + e^t)\, dt = e + 1 - e = 1.$$

(Observe that f and dy are zero on the horizontal segment of the curve, whereas on the vertical segment $dx = 0$.)

(2) (i) Here

$$f(x,y) = y^2; g(x,y) = x.$$

Therefore, the given path integral is equal to

$$\iint_R (1-2y)\,dxdy = \int_0^2 \int_0^2 (1-2y)dy\,dx = 4 - 4\int_0^2 dx = 4-8 = -4.$$

(ii) Here

$$\iint_R (1-2y)\,dxdy = \iint_R dxdy + \int_{-1}^1 \int_{-1}^1 (-2y)dy\,dx = 4+0 = 4.$$

(iii) Here

$$\iint_R (1-2y)\,dxdy = \iint_R dxdy + \int_{-2}^2 \left[\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} (-2y)dy\right] dx = 4\pi + 0.$$

(3) (i) $A = \frac{3\pi a^2}{2}$. (ii) $A = a^2/2$.

(4) (i) The required area is bounded by the curves

$$C_1 : r = a(1-\cos\theta),\ 0 \le \theta \le \pi/2$$

and C_2 which is a portion of the y-axis. In any case, the required area is equal to

$$\frac{1}{2}\oint_C r(\theta)^2\,d\theta.$$

Since θ is a constant along the y-axis, this integral is equal to

$$\frac{1}{2}\int_0^{\pi/2} r^2\,d\theta = \frac{a^2}{8}(3\pi - 8).$$

(ii) The required area is

$$\frac{1}{2}\oint_C xdy - ydx.$$

Here the boundary curve consists of the interval $[0, 2\pi]$ and the cycloid above traced in the opposite direction. But the integrand is zero on the x-axis, since both y and dy vanish there. Hence the required area is

$$-\frac{a^2}{2}\int_0^{2\pi} (t-\sin t)d(1-\cos t) - (1-\cos t)d(t-\sin t) = 2\pi a^2.$$

(iii) Here we use the polar coordinate form as in the previous exercise:

$$A = \frac{1}{2}\oint_C r^2 d\theta.$$

(This formula follows from Green's Theorem.)

Since θ is a constant on the two axes, this integral is equal to

$$\frac{1}{2}\int_0^{\pi/2} (1 - 2\cos\theta)^2 \, d\theta = \frac{1}{2}\left(\frac{3\pi - 8}{2}\right).$$

(5) Observe that

$$xe^{-y^2}dx + (-x^2ye^{-y^2})dy = d(\frac{x^2e^{-y^2}}{2}).$$

Hence the integral of this term along a closed path vanishes. So the given integral is equal to

$$\oint_C \frac{dy}{x^2 + y^2}.$$

We compute this directly. Observe that $dy = 0$ along the two horizontal parts. But then the integral along one vertical segment cancels with that on the other since the integrands are the same and the segments are traced in the opposite direction. So the value of the required integral is equal to 0.

(6) Take $f = -y^3$ and $g = x^3$ and apply Green's theorem. We get

$$\text{RHS} = \iint_R (3x^2 + 3y^2)\, dxdy = 3I_0.$$

(7)

$$\mathbf{u} \cdot \mathbf{v} = (a_x - a_y)a + (b_x - b_y)b.$$

Therefore taking $f = (a^2 + b^2)/2 = g$, we see that

$$\begin{aligned} \textstyle\iint_D \mathbf{u} \cdot \mathbf{v}\, dxdy &= \textstyle\iint_D (g_x - f_y)\, dxdy \\ &= \textstyle\oint_{\partial D}(f\, dx + g\, dy) \\ &= \textstyle\frac{1}{2}\oint_{\partial D} (1 + y^2)(dx + dy) = 0. \end{aligned}$$

(The last equality follows by considering the parametrizaton $(\cos\theta, \sin\theta)$, $0 \leq \theta \leq 2\pi$). Likewise we see that

$$\iint_D \mathbf{u}\cdot\mathbf{w}\,dxdy = \oint_{\partial D}(ab)(dx+dy) = -\int_0^{2\pi}\sin^2\theta\,d\theta = -\pi.$$

(8) Since $\nabla^2(x^2-y^2)=0$, using one of Green's identities (refer to (9),(i)) one has

$$\oint_C \nabla(x^2-y^2).\mathbf{n}ds = \oint_C \frac{\partial(x^2-y^2)}{\partial\mathbf{n}}ds = \iint_R \nabla^2(x^2-y^2)\,dxdy = 0.$$

(9) (a)

$$\nabla^2 w = 0 \quad \text{hence,} \quad \oint_C \frac{\partial w}{\partial n}dS = 0$$

(b) Put $\mathbf{H}=\mathbf{F}-\mathbf{G}$. Then $\operatorname{curl}\mathbf{H}=0$. Since D is simply connected, there exists u such that grad $u=\mathbf{H}$. Now div $\mathbf{H}=0$ implies that $\nabla^2 u=0$, i.e., u is harmonic. Finally $\mathbf{H}\cdot\mathbf{n}=0$, i.e., $\nabla u\cdot\mathbf{n}=\frac{\partial u}{\partial\mathbf{n}}=0$ on the boundary implies, using (9),(ii), that u is a constant. But then $\mathbf{H}=$ grad $u=0$ and hence $F=G$.

(10) (i) There are two distinct cases to be considered:

Case (a): Suppose the curve does not enclose the origin. Take

$$f(x,y)=\frac{y}{x^2+y^2},\, g(x,y)=\frac{x}{x^2+y^2}$$

and apply Green's theorem in the region R boundedby C. So the integral is equal to

$$\iint_R (g_x - f_y)\,dxdy.$$

A simple computation show that $g_x=f_y$ and hence the integral vanishes.

Case (b): Suppose the curve encloses the origin, i.e, $(0,0)\in R$. (Now the above argument does not work!) We choose a small disc D around the origin contained in R and apply Green's theorem in the closure of $R'=$

$R \setminus D$. As before, the double integral vanishes. But since the boundary of R' consists of C and $-\partial D$ it follows that

$$\oint_C \frac{y\,dx - x\,dy}{x^2+y^2} = \oint_{\partial D} \frac{y\,dx - x\,dy}{x^2+y^2}.$$

We can compute this now by using polar coordinates and see that this is equal to -2π.

(ii) Here again, we take D to be a small disc of radius ϵ around the origin and contained in the square R and apply Green's theorem in the closure of $R \setminus D$. Taking

$$f(x,y) = \frac{x^2 y}{(x^2+y^2)^2},\ g(x,y) = \frac{x^3}{(x^2+y^2)^2},$$

we once again observe that the double integral vanishes since $g_x = f_y = \frac{(x^2+y^2)(x^4-3x^2y^2)}{(x^2+y^2)^4}$. Hence the given line integral is equal to the corresponding line integral taken over the boundary of D. This can be computed by using the parametrizaton $(\epsilon\cos\theta, \epsilon\sin\theta),\ 0 \le \theta \le 2\pi$. The answer is $-\pi/4$.

(iii) We have

$$\frac{\partial(\ln r)}{\partial y} = \frac{y}{x^2+y^2} \quad \text{and} \quad \frac{\partial(\ln r)}{\partial x} = \frac{x}{x^2+y^2}.$$

By part (i), the required line integral is -2π.

Solutions to Tutorial Sheet 11

(1) On S,

$$\begin{aligned} z &= \frac{1}{2}(4+y-x) = h(x,y) \text{ so that} \\ \mathbf{r}(x,y) &= x\mathbf{i} + y\mathbf{j} + \frac{1}{2}(4+y-x)\mathbf{k},\ (x,y) \in \mathbb{R}^2 \end{aligned}$$

can be chosen as one parametrization.

The normal vector is

$$\mathbf{r}_x \times \mathbf{r}_y = \frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}.$$

(ii) For $S : y^2 + z^2 = a^2$, a parametrization is

$$r(u,v) = u\mathbf{i} + a\sin v\mathbf{j} + a\cos v\mathbf{k},\ u \in \mathbb{R}, 0 \le v \le 2\pi.$$

The normal vector is

$$\mathbf{r}_u \times \mathbf{r}_v = a\sin v\mathbf{j} + a\ \cos v\mathbf{k}.$$

(iii) If $\mathbf{e} = \dfrac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$, then $\mathbf{e}$ is a unit vector along the axis of the cylinder. Consider the planar cross-section of the cylinder through the origin O. This is a circle C of radius 1. Fix a point P on C. Then $\overrightarrow{OP}$ is a unit vector, say $\mathbf{u}$. Let $\mathbf{v} = \mathbf{e} \times \mathbf{u}$. Then a point on the cylinder is representable as

$$\mathbf{r}(\theta, t) = \cos\theta\mathbf{u} + \sin\theta\mathbf{v} + t\mathbf{e},\ \ 0 \le \theta \le 2\pi,\ t \in \mathbb{R}.$$

The normal vector is

$$\mathbf{r}_\theta \times \mathbf{r}_t = \cos\theta\mathbf{u} + \ \sin\theta\mathbf{v}.$$

(2) (a) The area SA of the surface S with projection R on the xy-plane is given by

$$SA = \iint_R \sec\gamma dxdy$$

where γ is the acute angle between $\mathbf{n}$ and $\mathbf{k}$ at a generic point on the surface. Thus, if this angle is the same at every pointon S, we have

$$SA = \sec\gamma \iint_R dxdy = \sec\gamma SA_{xy},$$

where SA_{xy} is the area of R. Hence,

$$SA_{xy} = SA\ \cos\gamma.$$

(b) By (a) above, one has (for appropriate α, β and γ)

$$\begin{aligned} S_1 &= S\cos\alpha, \\ S_2 &= S\cos\beta, \\ S_3 &= S\cos\gamma. \end{aligned}$$

Thus $S_1^2 + S_2^2 + S_3^2 = S^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = S^2$ in view of the fact that $\cos\alpha,\ \cos\beta,\ \cos\gamma$ are the direction cosines of $\mathbf{n}$.

(3) There are two pieces of the surface - one below and one above the xy-plane, both having the same area. Let S be the upper piece. Then one has

$$\text{Area}(S) = \iint_T \sqrt{1 + z_x^2 + z_y^2} dxdy,$$

where T is the disc

$$\{x^2 + y^2 \leq ay\} = \left\{x^2 + (y - \frac{a}{2})^2 \leq (\frac{a}{2})^2\right\},$$

and $z = \sqrt{a^2 - x^2 - y^2}$. Since

$$z_x = -\frac{x}{z} \text{ and } z_y = -\frac{y}{z},$$

it follows that

$$\text{Area}(S) = \iint_T \frac{adxdy}{z} = \iint_T \frac{adxdy}{\sqrt{a^2 - x^2 - y^2}}.$$

Now T is described in polar coordinates by

$$x = r\cos\theta,\ \ y = r\sin\theta;\ \ 0 \le \theta \le \pi,\ \ 0 \le r \le a\sin\theta.$$

Therefore,

$$\begin{aligned}
\text{Area}(S) &= \int_0^\pi \left(\int_0^{a\sin\theta} \frac{ardr}{\sqrt{a^2 - r^2}} \right) d\theta \\
&= a\int_0^\pi [-\sqrt{a^2 - r^2}]|_0^{a\sin\theta} d\theta \\
&= a\int_0^\pi (-a|\cos\theta| + a)d\theta = (\pi - 2)a^2.
\end{aligned}$$

Thus the required area is $2(\pi - 2)a^2$.

(4) (i) A point (x, y, z) on the surface satisfies $z = x^2 + y^2$. (The surface is thus a portion of a paraboloid of revolution). The given portion lies between the planes $z = 0$ and $z = 16$. $u = c$ gives a horizontal circular section, while $v = c$ gives a profile curve which is the portion of a half parabola.

(ii) $\mathbf{r}_u \times \mathbf{r}_v = -2u^2(\cos v\mathbf{i} + \sin v\mathbf{j}) + u\mathbf{k}$

(iii) $S = \int_{v=0}^{2\pi} \int_{u=0}^{4} |\mathbf{r}_u \times \mathbf{r}_v| dudv = 2\pi \int_0^4 u\sqrt{4u^2 + 1}\ du = \frac{\pi}{6}(65\sqrt{65} - 1)$.

Therefore, $n = 6$.

(5) The area of the paraboloid $x^2 + z^2 = 2ay$ between $y = 0$ and $y = a$ is given by

$$S = \iint_T \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2} dxdz$$

where T is the region $\{z^2 + x^2 \le 2a^2\}$ in the zx-plane. Hence,

$$\begin{aligned}
S &= \iint_T \sqrt{1 + \frac{x^2}{a^2} + \frac{z^2}{a^2}} dxdz \\
&= \int_0^{2\pi} \int_0^{a\sqrt{2}} \sqrt{1 + \frac{r^2}{a^2}}\ rdrd\theta
\end{aligned}$$

$$= \frac{2\pi}{3}(3\sqrt{3} - 1)a^2.$$

(6) We choose the coordinate system in such a way that the center of the sphere is located at the origin and the central axis of the circular cylinder coincides with the z-axis. We consider the case when one plane is cutting the sphere at height h above the xy-plane and the other plane is cutting the sphere at depth k below the xy-plane. (Other cases can be treated similarly). We compute the surface areas S_1 and S_2 of the 'upper' and 'lower' caps of the sphere and subtract their sum from $4\pi a^2$. We are expected to get the result to be $2\pi a(h + k)$.

Note that the plane cutting the sphere at height h above the xy-plane intersects the sphere in the circle $x^2 + y^2 = a^2 - h^2$. A parametrization for the upper cap of the sphere is thus given by $\mathbf{r}(x, y) = (x, y, \sqrt{a^2 - x^2 - y^2})$ with $(x, y) \in D = \{(x, y) : x^2 + y^2 \leq a^2 - h^2\}$. We have then

$$S_1 = \int\int_D \sqrt{1 + (\frac{-x}{\sqrt{a^2 - x^2 - y^2}})^2 + (\frac{-y}{\sqrt{a^2 - x^2 - y^2}})^2}\ dxdy$$

$$\int\int_D \frac{a}{\sqrt{a^2 - x^2 - y^2}}\ dxdy = \int_0^{2\pi} \int_0^{\sqrt{a^2-h^2}} \frac{a}{\sqrt{a^2 - r^2}}\ rdrd\theta$$

$$= 2\pi a(-h + a).$$

Similarly, $S_2 = 2\pi a(-k + a)$; and then $4\pi a^2 - (S_1 + S_2) = 2\pi a(h + k)$, as desired.

(7) (i) Note that $\mathbf{r}_u \times \mathbf{r}_v = -2(\mathbf{i} + \mathbf{j} + \mathbf{k})$ has negative z-component. Thus, using

$$\mathbf{n}dS = -\mathbf{r}_u \times \mathbf{r}_v\ dudv = 2(\mathbf{i} + \mathbf{j} + \mathbf{k})dudv$$

and

$$\mathbf{F} \cdot \mathbf{n}dS = 2(x + y + z)dudv = 2dudv,$$

one has

$$\iint_S \mathbf{F} \cdot \mathbf{n}dS = 2\text{Area}(S^*),$$

where S^* is the parametrizing region in the uv-plane. As the components of $\mathbf{r}$ are affine-linear in u and v, S^* is also a triangle whose vertices are pre-images of the vertices of S. Now the vertices of S^* are $(0,0)$, $(\frac{1}{2}, \frac{1}{2})$, and $(\frac{1}{2}, -\frac{1}{2})$ so that the area of S^* is $\frac{1}{4}$, and hence $\iint_S \mathbf{F} \cdot \mathbf{n} dS = \frac{1}{2}$.

(ii) The surface S satisfies $z = 1 - x - y \geq 0,\ x \geq 0,\ y \geq 0$. Thus, using

$$\mathbf{n} dS = (-z_x, -z_y, -1) dxdy$$

and

$$\mathbf{F} \cdot \mathbf{n} dS = (x, y, z) \cdot (-z_x, -z_y, 1) dxdy = (x + y + z) dxdy = dxdy$$

and $S_1^* = \{x + y \leq 1, x \geq 0, y \geq 0\}$ as the parametrizing region, one has

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_1^*} dxdy = \text{Area}(S_1^*) = \frac{1}{2}.$$

(8) A parametrization of S is

$$\mathbf{r}(u, v) = a \sin v \cos u\ \mathbf{i} + a \sin v \sin u\ \mathbf{j} + a \cos v\ \mathbf{k}, 0 \leq u \leq 2\pi,\ 0 \leq v \leq \pi$$

and

$$\mathbf{r}_u \times \mathbf{r}_v = a \sin v\ \mathbf{r}(u,v)$$

is the outward normal. The integrand is

$$\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = a^4 \sin^3 v \cos v (1 + \cos^2 u).$$

Thus the required integral is

$$\int_{u=0}^{2\pi} \int_{v=0}^{\pi} a^4 \sin^3 v \cos v (1 + \cos^2 u) dudv$$

$$= a^4 \left(\int_0^{\pi} \sin^3 v \cos v\ dv \right) \left(\int_0^{2\pi} (1 + \cos^2 u) du \right) = 0.$$

(9) The hemisphere satisfies

$$z = \sqrt{1 - x^2 - y^2},\ x^2 + y^2 \leq 1.$$

Using

$$\mathbf{n}dS = (-z_x, -z_y, 1)dxdy = (\frac{x}{z}, \frac{y}{z}, 1)dxdy$$

and

$$\mathbf{F} \cdot \mathbf{n}dS = (x, -2x - y, z) \cdot (\frac{x}{z}, \frac{y}{z}, 1) = \frac{(1 - 2xy - 2y^2)}{z}dxdy$$

and $T = \{(x, y) : x^2 + y^2 \leq 1\}$ as the parametrizing region, one has

$$\iint_S \mathbf{F} \cdot \mathbf{n}dS = \iint_T \frac{(1 - 2xy - 2y^2)}{\sqrt{1 - x^2 - y^2}}dxdy$$

$$= \int_0^{2\pi} \int_0^1 \frac{(1 - r^2 \sin 2\theta - 2r^2 \sin^2 \theta)rdrd\theta}{\sqrt{1 - r^2}} = \frac{2\pi}{3}.$$

(10) The flux through the base T is

$$\iint_T \mathbf{F} \cdot (-\mathbf{k})dxdy = 0,$$

as $\mathbf{F} \cdot (-\mathbf{k}) = -z = 0$ along T. The total flux is therefore the same as in the previous problem, namely, $\frac{2\pi}{3}$.

Solutions to Tutorial Sheet 12

(1) The cone $z = \sqrt{x^2+y^2}$ is parametrized as $\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + \sqrt{x^2+y^2}\mathbf{k}$. One has then $\mathbf{n}dS = (-\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1)dxdy$. Further, $\text{curl}\mathbf{F} = (0,0,2)$.

(a) If S is the surface lying on the cone $z = \sqrt{x^2+y^2}$ and bounded by the intersection C of the hemisphere $x^2+(y-a)^2+z^2 = a^2, z \geq 0$ with the cone, then the projection R of S onto the xy-plane is given by $x^2+(y-a/2)^2 \leq a^2/4$.

Thus one has

$$\iint_S \text{curl}\mathbf{F}\cdot\mathbf{n}dS = \iint 2\ dS = 2\iint_R dS = \pi a^2/2.$$

With the choice of the normal $\mathbf{n}$ to S as indicated above, the induced orientation on C is counterclockwise (when viewed from high above). The projection of C onto the xy-plane can then be described by $(\frac{a}{2}\cos\theta, \frac{a}{2}+\frac{a}{2}\sin\theta)$ $(0 \leq \theta \leq 2\pi)$.

Thus,

$$\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{r} &= \oint_C (x-y)dx + (x+z)dy + (y+z)dz \\
&= \oint_C (xdy - ydx) + d(yz) + \frac{1}{2}d(x^2+z^2) \\
&= \oint_C (xdy - ydx) = 2\pi\frac{a^2}{4} = \pi a^2/2.
\end{aligned}$$

Stokes' Theorem now stands verified.

(b) If S is the surface lying on the cone $z = \sqrt{x^2+y^2}$ and bounded by the intersection C of the cylinder $x^2+(y-a)^2 = a^2, z \geq 0$ with the cone, then the projection R of S onto the xy-plane is given by $x^2+(y-a)^2 \leq a^2$.

Thus one has

$$\iint_S \text{curl}\mathbf{F}\cdot\mathbf{n}dS = \iint 2\ dS = 2\iint_R dS = 2\pi a^2.$$

With the choice of the normal $\mathbf{n}$ to S as indicated above, the induced orientation on C is counterclockwise (when viewed from high above). The projection of C onto the xy-plane can then be described by $(a\cos\theta, a+\sin\theta)$ $(0 \leq \theta \leq 2\pi)$.

Thus,

$$\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{r} &= \oint_C (x-y)dx + (x+z)dy + (y+z)dz \\
&= \oint_C (xdy - ydx) + d(yz) + \frac{1}{2}d(x^2+y^2) \\
&= \oint_C (xdy - ydx) = 2\pi a^2.
\end{aligned}$$

Stokes' Theorem now stands verified.

(2) For $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$,

$$\text{curl }(\mathbf{F}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix} = (x-x)\mathbf{i} + (y-y)\mathbf{j} + (z-z)\mathbf{k} = 0.$$

Thus the required line integral is

$$\iint_S \text{curl }(\mathbf{F})\cdot\mathbf{n}dS = 0.$$

(3) By Stokes' theorem, we have

$$\iint_S \text{curl }(\mathbf{v})\cdot\mathbf{n}dS = \oint_{C_1} \mathbf{v}\cdot d\mathbf{s} + \oint_{C_2} \mathbf{v}\cdot d\mathbf{s}$$

where C_1 is the circle $x^2+y^2 = 4, z = -3$ with the counterclockwise orientation when viewed from high above, and C_2 is the circle $x^2+y^2 = 4, z = 0$ with the opposite orientation. Now,

$$\text{curl } \mathbf{v} = (-3zy^2 - 3xz^2)\mathbf{i} + (z^3 - 1)\mathbf{k} \text{ and}$$

$$\mathbf{v} \cdot d\mathbf{s} = ydx + xz^3dy - zy^3dz.$$

Along C_2, $z = 0$; $\mathbf{v} \cdot d\mathbf{s} = ydx$ and $\oint_{C_2} ydx = -\int_0^{2\pi} (-4\sin^2\theta| \, d\theta = 4\pi.$

Along C_1, $z = -3$; $dz = 0$; $\mathbf{v} \cdot d\mathbf{s} = ydx - 27xdy = d(xy) - 28xdy$ and

$$\oint_{C_1} d(xy) - \oint_{C_1} 28xdy = -28\int_0^{2\pi} 4\cos^2\theta d\theta = -112\pi.$$

Hence

$$\iint_S \text{curl } (\mathbf{v}) \cdot \mathbf{n} dS = -108\pi.$$

(4) Note that, to apply Stokes' Theorem, one would have to work inside $U = \mathbb{R}^3 \setminus$ $z-$axis, as $\mathbf{v}$ would not make sense at a point on the z-axis. But there is no surface in $U = \mathbb{R}^3 \setminus z-$axis whose boundary C is. Hence Stokes' theorem cannot be applied.

Using the parametrization $(\cos\theta, -\sin\theta)$, $(0 \leq \theta \leq 2\pi)$ one has

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = \oint_C \frac{-ydx + xdy}{x^2 + y^2} = -\int_0^{2\pi} d\theta = -2\pi.$$

(5) Note that

$$\begin{aligned} \mathbf{F} &= (y^2 - z^2)\mathbf{i} + (z^2 - x^2)\mathbf{j} + (x^2 - y^2)\mathbf{k}, \\ \text{curl } \mathbf{F} &= (-2y - 2z)\mathbf{i} + (-2z - 2x)\mathbf{j} + (-2x - 2y)\mathbf{k}, \\ \text{and } \mathbf{n} &= \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}}. \end{aligned}$$

Thus, along the surface S which is part of the plane $x + y + z = \frac{3a}{2}$ and which is bounded by C, one has

$$\begin{aligned} \text{curl}\mathbf{F} \cdot \mathbf{n} &= -\frac{2}{\sqrt{3}}(y + z + z + x + x + y) \\ &= -\frac{4}{\sqrt{3}}(x + y + z) = -\frac{4}{\sqrt{3}}\frac{3a}{2}. \end{aligned}$$

Hence

$$\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} dS = -2\sqrt{3}a \iint_S dS = (-2\sqrt{3}a)(\text{Area of } S).$$

The surface S is a regular hexagon with vertices $(a/2, 0, a), (a, 0, a/2), (a, a/2, 0), (a/2, a, 0), (0, a, a/2), (0, a/2, a)$. Hence its area is $3\frac{\sqrt{3}}{2}(\text{length of side})^2 = \frac{3\sqrt{3}}{2}\frac{a^2}{2}$.
Stokes' theorem then yields that $\oint_C (y^2 - z^2)dx + (z^2 - x^2)dy + (x^2 - y^2)dz = -\frac{9a^3}{2}$.

(6) We have

$$\begin{aligned} \mathbf{F} &= y\mathbf{i} + z\mathbf{j} + x\mathbf{k}, \\ \text{curl } \mathbf{F} &= -(\mathbf{i} + \mathbf{j} + \mathbf{k}). \end{aligned}$$

Parametrize the surface lying on the hyperbolic paraboloid $z = xy/b$ and bounded by the curve C as $x\mathbf{i} + y\mathbf{j} + \frac{xy}{b}$ $(x^2 + y^2 \leq a^2)$ so that $\mathbf{n}dS = (-\frac{y}{b}, -\frac{x}{b}, 1)dxdy$ and

$$\begin{aligned} \iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} dS &= \frac{1}{b} \iint_{x^2+y^2 \leq a^2} (y + x - b) dxdy \\ &= \frac{1}{b} \int_0^{2\pi} \int_0^a (r \sin\theta + r\cos\theta - b) r dr d\theta = -\pi a^2 \end{aligned}$$

(7) Letting $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and using S to denote the planar area enclosed by C, one has

$$\begin{aligned} I &:= \frac{1}{2} \oint_C a(ydz - zdy) + b(zdx - xdz) + c(xdy - ydx) \\ &= \frac{1}{2} \oint_C \mathbf{n} \times \mathbf{r} \cdot d\mathbf{s} \\ &= \frac{1}{2} \iint_S \nabla \times (\mathbf{n} \times \mathbf{r}) \cdot \mathbf{n} dS \\ &= \frac{1}{2} \iint_S 2\mathbf{n} \cdot \mathbf{n} dS = \iint_S dS = \text{Area}(S). \end{aligned}$$

If C is parametrized as $\mathbf{u}\cos t + \mathbf{v}\sin t \quad (0 \le t \le 2\pi)$, then

$$\begin{aligned}
\text{Area}(S) &= \frac{1}{2}\oint_C \mathbf{n} \times \mathbf{r} \cdot d\mathbf{s} \\
&= \frac{1}{2}\int_0^{2\pi} \mathbf{n} \times (\mathbf{u}\cos t + \mathbf{v}\sin t) \cdot (-\mathbf{u}\sin t + \mathbf{v}\cos t)dt \\
&= \frac{1}{2}\int_0^{2\pi} \mathbf{n} \cdot (\mathbf{u}\cos t + \mathbf{v}\sin t) \times (-\mathbf{u}\sin t + \mathbf{v}\cos t)dt \\
&= \frac{1}{2}\int_0^{2\pi} \mathbf{n} \cdot \mathbf{u} \times \mathbf{v} \; dt
\end{aligned}$$

so that, letting $\mathbf{n} = \frac{\mathbf{u}\times\mathbf{v}}{||\mathbf{u}\times\mathbf{v}||}$, we get

$$\text{Area}(S) = \frac{1}{2}\int_0^{2\pi} ||\mathbf{u} \times \mathbf{v}|| \; dt = \pi||\mathbf{u} \times \mathbf{v}||.$$

Solutions to Tutorial Sheet 13

(1) We have

$$\begin{aligned}
R &= \{(x,y,z) : y^2+z^2 \le x^2, 0 \le x \le 4\} \text{ and} \\
\partial R &= S_1 \bigcup S_2, \text{ where} \\
S_1 : \mathbf{r}(x,\theta) &= x\mathbf{i} + x\cos\theta\mathbf{j} + x\sin\theta\mathbf{k},\ 0 \le \theta \le 2\pi, \\
& S_2 : x = 4, y^2+z^2 \le 16.
\end{aligned}$$

Along S_1, $\mathbf{r_x} \times \mathbf{r}_\theta = x\mathbf{i} - x\cos\theta\mathbf{j} - x\sin\theta\mathbf{k} = x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$ so that the outward normal is $-\mathbf{r_x} \times \mathbf{r}_\theta = -x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Thus

$$\begin{aligned}
\iint_{S_1} \mathbf{F}\cdot\mathbf{n}dS &= \int_0^4 \int_0^{2\pi} (-x^2y^2 + y^2z^2 + z^2x^2)dxd\theta \\
&= \int_0^4 \int_0^{2\pi} x^4(-\cos^2\theta + \cos^2\theta\sin^2\theta + \sin^2\theta)dxd\theta \\
&= \left(\int_0^4 x^4dx\right)\left(\int_0^{2\pi} (-\cos^2\theta + \cos^2\theta\sin^2\theta + \sin^2\theta)\right) \\
&= \frac{4^5}{5}\ \frac{\pi}{4} = 4^4\frac{\pi}{5}.
\end{aligned}$$

Also, along S_2, the outward normal (to $x = 4 \equiv f(y,z)$) is $(1,0,0) = \mathbf{i}$. Thus

$$\begin{aligned}
\iint_{S_2} \mathbf{F}\cdot\mathbf{n}dS &= \iint_{S_2} 4y^2dS = \iint_{y^2+z^2\le 16} 4y^2dydz \\
&= 4\int_0^{2\pi}\int_0^4 r^3\cos^2\theta drd\theta = 4^4\pi.
\end{aligned}$$

Now,

$$\begin{aligned}
\iiint_R \text{div } \mathbf{F}\ dV &= \iiint_{\substack{y^2+z^2\le x^2 \\ 0\le x\le 4}} (x^2+y^2+z^2)dV = \int_0^4 \left(\int_0^x \left(\int_0^{2\pi} (x^2+r^2)rdrd\theta\right) dx\right) \\
&= 2\pi\int_0^4 \left(\int_0^x (x^2r + r^3)dr\right) dx \\
&= 2\pi\int_0^4 \frac{3x^4}{4}dx = 4^4\frac{6\pi}{5}.
\end{aligned}$$

Since $4^4\frac{6\pi}{5} = 4^4\frac{\pi}{5} + 4^4\pi$, the Divergence Theorem is verified.

(2) We have div$\mathbf{F} = (y+z+x)$ and

$$\begin{aligned}
I &= \iiint_R (x+y+z)d(x,y,z) = \\
&= \iiint_R xd(x,y,z) + \iiint_R yd(x,y,z) + \iiint_R zd(x,y,z) \\
&= \int_0^c \int_0^{b(1-\frac{z}{c})} \int_0^{a(1-\frac{y}{b}-\frac{z}{c})} xdxdydz + (\cdots) + (\cdots) \\
&= \frac{a^2bc}{24} + \frac{ab^2c}{24} + \frac{abc^2}{24} \\
&= \frac{abc}{24}(a+b+c).
\end{aligned}$$

Now,

$$\iint_S \mathbf{F}\cdot\mathbf{n}dS = \iint_{S_1} \mathbf{F}\cdot\mathbf{n}dS + \iint_{S_2} \mathbf{F}\cdot\mathbf{n}dS + \iint_{S_3} \mathbf{F}\cdot\mathbf{n}dS + \iint_{S_4} \mathbf{F}\cdot\mathbf{n}dS$$

where

$S_1: \quad z=0; \quad \frac{x}{a}+\frac{y}{b} \le 1, \quad x,y \ge 0$ and

$S_2: \quad y=0; \quad \frac{x}{a}+\frac{z}{c} \le 1, \quad x,z \ge 0$ and

$S_3: \quad x=0; \quad \frac{z}{c}+\frac{y}{b} \le 1, \quad y,z \ge 0$ and

$S_4: \quad \frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1, \quad x,y,z \ge 0.$

Also,

along S_1, $\mathbf{n} = -\mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n} = -xz = 0$ (as $z=0$ on S_1);

along S_2, $\mathbf{n} = -\mathbf{j} \Rightarrow \mathbf{F}\cdot\mathbf{n} = -yz = 0$ (as $y=0$ on S_2);

along S_3, $\mathbf{n} = -\mathbf{i} \Rightarrow \mathbf{F}\cdot\mathbf{n} = -xy = 0$ (as $x=0$ on S_3).

Along S_4, the outward normal (to $z = c(1 - \frac{x}{a} - \frac{y}{b}) \equiv f(x, y)$) is $(\frac{c}{a}, \frac{c}{b}, 1)$ so that

$$\begin{aligned}
\iint_{S_4} \mathbf{F} \cdot \mathbf{n} dS &= \iint_{\frac{x}{a}+\frac{y}{b}\leq 1; x,y\geq 0} \left(\frac{cxy}{a} + \frac{cyz}{b} + zx\right) d(x, y) \\
&= \int_0^a \int_0^{b(1-\frac{x}{a})} \frac{cxy}{a} dxdy + (\cdots) + (\cdots) \\
&= \frac{ab^2c}{24} + \frac{abc^2}{24} + \frac{a^2bc}{24} \\
&= \frac{abc}{24}(a + b + c).
\end{aligned}$$

(3) Consider $\mathbf{F} = uv\mathbf{i}$. By the Divergence Theorem, one has

$$\begin{aligned}
\iint_{\partial R} uvn_x dS &= \iiint_R \frac{\partial}{\partial x}(uv) d(x, y, z) \\
&= \iiint_R u\frac{\partial v}{\partial x} d(x, y, z) + \iiint_R v\frac{\partial u}{\partial x} d(x, y, z)
\end{aligned}$$

Hence

$$\iiint_R u\frac{\partial v}{\partial x} dV = \iint_{\partial R} (uvn_x) dS - \iiint v\frac{\partial u}{\partial x} dV.$$

(4)
$$\begin{aligned}
\nabla \cdot (\phi\nabla\phi) &= \|\nabla\phi\|^2 + \phi\nabla^2\phi \\
\Rightarrow \quad \phi\nabla^2\phi &= 10\phi - 4\phi \\
\Rightarrow \quad \nabla^2\phi &= 6.
\end{aligned}$$

Thus

$$\begin{aligned}
\iint_S \frac{\partial\phi}{\partial\mathbf{n}} dS &= \iint_S \mathrm{grad}\phi \cdot \mathbf{n} dS \\
&= \iiint_V \mathrm{div}(\mathrm{grad}\phi) dV = 6 \iiint_V dV \\
&= 6 \text{ (Volume of the sphere)} = 6 \times \frac{4\pi}{3} = 8\pi.
\end{aligned}$$

(5) Let $\mathbf{F} = x\mathbf{i}$. Using the Divergence Theorem, we get

$$V = \iiint_R d(x, y, z) = \iint_S xn_x dS$$

Similarly, letting $\mathbf{F} = y\mathbf{j}$, we get

$$V = \iint_S y n_y dS$$

and letting $\mathbf{F} = z\mathbf{k}$, we get

$$V = \iint_S z n_z dS.$$

(6) Consider

$$\begin{aligned} I &= \iint_S x^2 dy \wedge dz + y^2 dz \wedge dx + z^2 dx \wedge dy \\ &= \iint_S (\mathbf{F} \cdot \mathbf{n}) dS, \end{aligned}$$

where $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$. We have then, using the Divergence Theorem,

$$I = \iiint (\text{div } \mathbf{F}) dV = \int_0^1 \int_0^1 \int_0^1 2(x+y+z) dx dy dz = 3.$$

(7) The required integral is $I = \iint_{\partial R} (\mathbf{F} \cdot \mathbf{n}) dS$, where $\mathbf{F} = yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k}$.

Since $\text{div}(\mathbf{F}) = 0$, one has

$$I = \iiint_R (\text{div } \mathbf{F}) dV = 0.$$

(8) By the Divergence Theorem, one has

$$\iint_{S \bigcup S_1} (\nabla \times \mathbf{u}) \cdot \mathbf{n} dS = \iiint_R \nabla \cdot (\nabla \times \mathbf{u}) dV = 0,$$

where S_1 is the disc $x^2 + y^2 + z^2 = 1$, $z = 1/2$. Thus

$$\iint_S (\nabla \times \mathbf{u}) \cdot \mathbf{n} dS = \iint_{S_1} (\nabla \times \mathbf{u}) \cdot \mathbf{n} dS,$$

where $\mathbf{n}$ in the R.H.S. integral is the vector $\mathbf{k}$; so R.H.S.$= \iint_{S_1} (\nabla \times \mathbf{u}) \cdot \mathbf{k} dS$.

But the coefficient of $\mathbf{k}$ in $\text{curl}(\mathbf{u})$ is 0. Hence, $\text{curl}(\mathbf{u}) \cdot \mathbf{k} = 0$. Thus

$$\iint_{S_1} (\nabla \times \mathbf{u}) \cdot \mathbf{k} dS = 0.$$

This gives

$$\iint_S (\nabla \times \mathbf{u}) \cdot \mathbf{n} dS = 0.$$

(9) Note that

$$p = \mathbf{n} \cdot \mathbf{r} = \frac{(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2})}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}} \cdot (x, y, z) = \frac{1}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}.$$

(a) Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ $(= \mathbf{r})$. Then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\ dS = \iint_S p\ dS = \iiint_V \operatorname{div}\mathbf{F}\ dV = 3 \iiint_V dV = 3\left(\frac{4\pi}{3}abc\right) = 4\pi abc.$$

(b) Let $\mathbf{F} = \frac{1}{p^2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$. Then

$$\begin{aligned} \iint_S \frac{1}{p}\ dS &= \iint_S \mathbf{F} \cdot \mathbf{n}\ dS = \iiint_V \operatorname{div}\mathbf{F}\ dV \\ &= 5 \iiint_V \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right) d(x, y, z). \end{aligned}$$

Let $x = ar\sin\phi\cos\theta,\ y = br\sin\phi\sin\theta,\ z = cr\cos\phi$. Then

$$\begin{aligned} \iint_S \frac{1}{p}\ dS &= 5abc \int_0^{\pi} \int_0^{2\pi} \int_0^1 r^4 \left(\frac{\sin^3\phi\cos^2\theta}{a^2} + \frac{\sin^3\phi\sin^2\theta}{b^2} + \frac{\sin\phi\cos^2\phi}{c^2}\right) dr d\theta d\phi \\ &= abc \int_0^{\pi} \int_0^{2\pi} \left(\frac{\sin^3\phi\cos^2\theta}{a^2} + \frac{\sin^3\phi\sin^2\theta}{b^2} + \frac{\sin\phi\cos^2\phi}{c^2}\right) d\theta d\phi \\ &= \pi abc \int_0^{\pi} \left(\frac{\sin^3\phi}{a^2} + \frac{\sin^3\phi}{b^2} + \frac{2\sin\phi\cos^2\phi}{c^2}\right) d\phi \\ &= \frac{4}{3}\pi abc \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = \frac{4\pi}{3abc}(b^2c^2 + c^2a^2 + a^2b^2). \end{aligned}$$

(We used the fact that the Jacobian of (x, y, z) with respect to (r, ϕ, θ) is $abcr^2\sin\phi$).

Aliter: If $\psi = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$, then

$$\begin{aligned}
\iint_S \frac{1}{p}\, dS &= \iint_S \left(\frac{\mathbf{n}}{\mathbf{r}\cdot\mathbf{n}}\right)\cdot \mathbf{n}\, dS = \iint_S \frac{\nabla\psi}{\mathbf{r}\cdot\nabla\psi}\cdot \mathbf{n}\, dS \\
&= \iint_S \frac{\nabla\psi}{2}\cdot \mathbf{n}\, dS \\
&= \iiint_V \frac{\nabla^2\psi}{2}\, dV = \left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\iiint_V dV = \frac{4}{3}\pi abc\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right).
\end{aligned}$$

(10) For a simple closed (and sufficiently smooth) plane curve $C : \mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j}$, parametrized by the arc length s, the outward unit normal at $x(s)$ is $\mathbf{n} = y'(s)\mathbf{i} - x'(s)\mathbf{j}$. Let R be the region enclosed by the curve C.

Let $\mathbf{F} = Q\mathbf{i} - P\mathbf{j}$ be a continuously differentiable vector field in a region including $C \bigcup R$. Then

$$\operatorname{div}(\mathbf{F}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \text{ and } \mathbf{F}\cdot\mathbf{n} = Qy' + Px'.$$

Thus one has

$$\oint_C (\mathbf{F}\cdot\mathbf{n})\, ds = \oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d(x,y) = \iint_R \operatorname{div}(\mathbf{F})\, d(x,y).$$