MA-105 Calculus II

Lecture 1

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Welcome to the second half of MA 105!

There is a total of 50 marks to be earned in the second half of this course. The following breakup is tentative*.

Quiz	10 marks
Final	40 marks
Total	50 marks

Academic Honesty: It is obligatory on your part to be honest and not to violate the academic integrity of the Institute. Any form of academic dishonesty, including, but not limited to cheating, plagiarism, submitting as one's own the same or substantially similar work of another, will not be tolerated, and will invite the harshest possible penalties as per institute norms.

Disclaimer: The instructors reserve the right to modify the schedules (e.g. breakup of total marks) and procedures announced in this syllabus. Any such changes will be announced in the class. It is the responsibility of the student to keep informed of such things.

Course objectives

Calculus can be broadly divided into two parts: Differential calculus and integral calculus. This course will be focused on integral calculus of several variables and vector analysis, mainly,

- Double and triple integration, Jacobians and change of variables.
- Parametrisation of curves, vector fields, line integrals.
- Parametrisation of surfaces and surface integrals.
- Gradient of functions, divergence and curl of vector fields, theorems of Green, Gauss, and Stokes and their applications.

Goal: To achieve the rigorous understanding of the above topics along with some techniques and tools which are useful in applications.

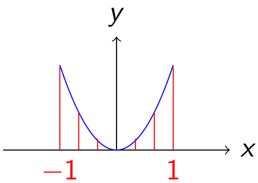
References:

1. [MR] Debanjana Mitra and Ravi Raghunathan, *Lecture slides for MA 105*.
2. [MTW] J.E Marsden, A. J. Tromba, A. Weinstein. *Basic Multivariable Calculus*, South Asian Edition, Springer (2017).
3. [CJ] R. Courant and F. John, *Introduction to Calculus and Analysis, Volumes 1 and 2*, Springer-Verlag (1989).
4. [Apo] T.M. Apostol, *Calculus, Volumes 1 and 2*, 2nd ed., Wiley (2007).

Recall : One variable Integration

Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $a, b \in \mathbb{R}$.

- The area enclosed by the graph of a non-negative function over the region of the interval is $\int_a^b f(t)\ dt$.



The area in the figure on the left is $\int_{-1}^{1} x^2\ dx = 2/3$.

- A partition of the interval $[a, b]$ is a set of points $P = \{a = x_0 \leq x_1 \leq \ldots x_n = b\}$ for some $n \in \mathbb{N}$.
- The lower Darboux integral and upper Darboux integral of f are $L(f) = \sup\{L(f, P) \colon P \text{ is a partition of } [a, b]\}$, and $U(f) = \inf\{U(f, P) \colon P \text{ is a partition of } [a, b]\}$, respectively.
- When $L(f) = U(f)$ then f is Darboux integrable and $\int_a^b f := L(f) = U(f)$.

- Tagged partition: partition P with a set of points $t = \{t_1, \dots t_n\}$, $t_j \in [x_{j-1}, x_j]$ for all $j = 1, \dots, n$. Define $S(f, P, t) = \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1})$ and define the *norm* of a partition P as $\|P\| = \max_j\{|x_j - x_{j-1}|\}, \quad 1 \le j \le n.$
- $f : [a, b] \to \mathbb{R}$ is said to be *Riemann integrable* if for some $S \in \mathbb{R}$ and every $\epsilon > 0$ there exists $\delta > 0$ such that $|S(f, P, t) - S| < \epsilon$, whenever $\|P\| < \delta$. The Riemann integral of f is then S.
- Theorem: The Riemann integral exists iff the Darboux integral exists. Further, the two integrals are equal.
- Unlike the Darboux integral, Riemann integral can be computed as a limit: clearly advantageous in computations!
- $f : [a, b] \to \mathbb{R}$ is bounded, and continuous at all but finitely many points of $[a, b]$. Then f is Riemann integrable on $[a, b]$.
- For computing integrals, we use the Fundamental theorem of calculus. If $f : [a, b] \to \mathbb{R}$ is continuous and $f = g'$ for some continuous function $g : [a, b] \to \mathbb{R}$ which is differentiable on (a, b), then $\displaystyle\int_a^b f = g(b) - g(a).$

Integrating functions on two variables

Any *closed, bounded rectangle R* in $\mathbb{R}^2$:

$$R = [a, b] \times [c, d],$$

the Cartesian product of two closed intervals $[a, b]$ and $[c, d]$.

Consider a real valued function f defined on R i.e.,

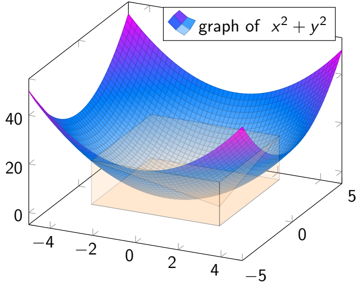
$$f : R \subset \mathbb{R}^2 \to \mathbb{R}.$$

- Graph of f: The subset $\{(x, y, f(x, y)) \in \mathbb{R}^3 \mid (x, y) \in R\}$ in $\mathbb{R}^3$ is called the graph of f.
- Contour line: Fix $c \in \mathbb{R}$. Then the set $\{(x, y, c) \in \mathbb{R}^3 \mid f(x, y) = c, (x, y) \in R\}$ in $\mathbb{R}^3$ is called the contour line of f. It is the intersection of the graph of f by the horizontal plane $z = c$ in $\mathbb{R}^3$.

Double integral of non-negative functions

In particular, let $f(x,y) = x^2 + y^2$, for all $(x,y) \in \mathbb{R}^2$. We can use contour lines to draw the graph of this function by drawing $f(x,y) = c$ for varying values of c.

We want to compute volume of the region below the graph of f over the rectangle $[-3,3] \times [-3,3]$. The volume of the figure in the shaded region is $V := \{(x,y,z) \mid (x,y) \in [-3,3] \times [-3,3], \quad 0 \leq z \leq f(x,y)\}$.



The integral of the non-negative function f over $[-3,3] \times [-3,3]$ can be defined as the volume V; $\int\int_{[-3,3]\times[-3,3]} f(x,y)\,dxdy := \text{Volume} \quad \text{of} \quad V$.

Integration on a Rectangle

Example: Let $g(x, y) = \alpha$, for some non-zero positive constant $\alpha \in \mathbb{R}$. Then for any rectangle $[0, b] \times [0, d]$ it is easy to see that $\int \int_{[0,b]\times[0,d]} g(x, y)\, dxdy = bd\alpha$.

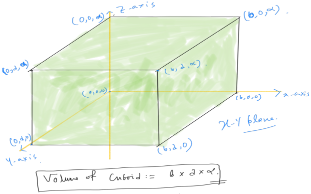


Figure: Cuboid: $[0, b] \times [0, d] \times [0, \alpha]$

Clearly for $f(x, y) = x^2 + y^2$, the computing of the volume is not that simple and we want to be able to define integral for all bounded functions instead of only non-negative ones.

Partitions for rectangles

Partition of R: A partition P of a rectangle $R = [a, b] \times [c, d]$ is the Cartesian product of a partition P_1 of $[a, b]$ and a partition P_2 of $[c, d]$. Let

$$P_1 = \{x_0, x_1, \cdots x_m\}, \quad \text{with} \quad a = x_0 < x_1 < x_2 < \cdots < x_m = b\},$$

$$P_2 = \{y_0, y_1, \cdots y_n\}, \quad \text{with} \quad c = y_0 < y_1 < y_2 < \cdots < y_n = d\},$$

and $P = P_1 \times P_2$ be defined by

$$P = \{(x_i, y_j) \mid i \in \{0, 1, \cdots m\}, \quad j \in \{0, 1, \cdots, n\}\}.$$

The points of P divide the rectangle R into *nm non-overlapping sub-rectangles* denoted by

$$R_{ij} := [x_i, x_{i+1}] \times [y_j, y_{j+1}], \quad \forall\, i = 0, \cdots m-1, \quad j = 1, \cdots, n-1.$$

Note $R = \cup_{i,j} R_{ij}$.

Partitions for rectangles: continued

Example: Let P_1 denote a partition of $[-3,3]$ into 3 equal intervals and P_2 the partition of $[-3,3]$ into 2 equal intervals. Describe the rectangles in the partition $P_1 \times P_2$.

Note $P_1 = \{-3,-1,1,3\}$ and $P_2 = \{-3,0,3\}$ and thus $[-3,3] \times [-3,3]$ is devided into 6 sub-rectangles $R_{00} = [-3,-1] \times [-3,0]$, $R_{01} = [-3,-1] \times [0,3]$, $R_{10} = [-1,1] \times [-3,0]$, $R_{11} = [-1,1] \times [0,3]$, $R_{20} = [1,3] \times [-3,0]$, $R_{21} = [1,3] \times [0,3]$.

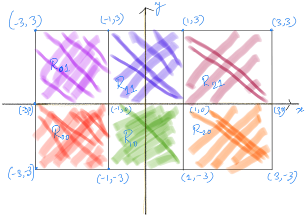


Figure: Partition of $[-3,3] \times [-3,3]$

Partitions for rectangles: continued

The area of each R_{ij}: $\Delta_{ij} := (x_{i+1} - x_i) \times (y_{j+1} - y_j)$, for all $i = 0, \cdots, m-1$, $j = 0, \cdots, n-1$.

Norm of the partition P:
$\|P\| := \max\{(x_{i+1} - x_i), (y_{j+1} - y_j) \mid i = 0, \cdots, m-1, \quad j = 0, \cdots, n-1\}$.

Why do we not define the norm by
$\max\{(x_{i+1} - x_i) \times (y_{j+1} - y_j) \mid i = 0, \cdots, m-1, \quad j = 0, \cdots, n-1\}$?

Darboux integral

Let $f : R \to \mathbb{R}$ be a bounded function where R is a rectangle . Let $m(f) = \inf\{f(x,y) \mid (x,y) \in R\}$, $M(f) = \sup\{f(x,y) \mid (x,y) \in R\}$. For all $i = 0, 1, \cdots, m-1, \quad j = 0, 1, \cdots, n-1$, let,
$m_{ij}(f) := \inf\{f(x,y) \mid (x,y) \in R_{ij}\}$, and
$M_{ij}(f) := \sup\{f(x,y) \mid (x,y) \in R_{ij}\}$. *Lower double sum:*

$L(f,P) := \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} m_{ij}(f)\Delta_{ij}$, and *Upper double sum:*

$U(f,P) := \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} M_{ij}(f)\Delta_{ij}$, Note that for any partition P of R

$$m(f)(b-a)(d-c) \leq L(f,P) \leq U(f,P) \leq M(f)(b-a)(d-c).$$

Lower Darboux integral: $L(f) := \sup\{L(f,P) \mid P \text{ is any partition of } R\}$.
Upper Darboux integral: $U(f) := \inf\{U(f,P) \mid P \text{ is any partition of } R\}$. Note $L(f) \leq U(f)$.

Darboux integral continued

Definition (Darboux integral)

A bounded function $f : R \to \mathbb{R}$ is said to be *Darboux integrable* if $L(f) = U(f)$. The Darboux integral of f is the common value $U(f) = L(f)$ and is denoted by

$$\int\int_R f, \quad \text{or} \quad \int\int_R f(x,y)dA, \quad \text{or} \quad \int\int_R f(x,y)dxdy.$$

Theorem (Riemann condition)

Let $f : R \to \mathbb{R}$ be a bounded function. Then f is integrable if and only if for every $\epsilon > 0$ there is a partition P_ϵ of R such that

$$|U(f, P_\epsilon) - L(f, P_\epsilon)| < \epsilon.$$

Recall the Dirichlet function for one variable:

$$f(x) := \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q} \cap [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Is f integrable over $[0,1]$? Ans. No!

Exercise: Check the integrability of Bivariate Dirichlet function over $[0,1] \times [0,1]$

$$f(x,y) := \begin{cases} 1 & \text{if both } x \text{ and } \quad y \quad \text{are rational numbers,} \\ 0 & \text{otherwise.} \end{cases}$$

Riemann Integral

Riemann integral: Let P be any partition of a rectangle $R = [a, b] \times [c, d]$. We define a tagged partition (P, t) where

$$t = \{t_{ij} \mid t_{ij} \in R_{ij}, \quad i = 0, 1, \ldots m - 1, \quad j = 0, 1, \ldots n - 1\}.$$

The *Riemann sum* of f associated to (P, t) is defined by

$$S(f, P, t) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(t_{ij})\Delta_{ij} \text{ where, } \Delta_{ij} = (x_{i+1} - x_i)(y_{j+1} - y_j)$$

Definition (Riemann integral)

A bounded function $f : R \to \mathbb{R}$ is said to be *Riemann integrable* if there exists a real number S such that for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|S(f, P, t) - S| < \epsilon,$$

for every tagged partition (P, t) satisfying $\|P\| < \delta$ and S is the value of Riemann integral of f.

Riemann Integral contd.

- For any rectangle $R \subseteq \mathbb{R}^2$, let $f : R \to \mathbb{R}^2$ be bounded. The Darboux integrability and Riemann integrability are equivalent.
- A function $f : R \to \mathbb{R}^2$ is called integrable on R if (Darboux or) Riemann integrability condition holds on R.
- In summary, if f is integrable on R, then

$$\int \int_R f(x, y)\ dxdy := S = L(f) = U(f).$$

Examples: Let $R = [a, b] \times [c, d]$.

• The constant function is integrable.

• The projection functions $p_1(x, y) = x$ and $p_2(x, y) = y$ are both integrable on any rectangle $R \subset \mathbb{R}^2$. Why?

• Let $f : R \to \mathbb{R}$ be defined as $f(x, y) = \phi(x)$ where $\phi : \mathbb{R} \to \mathbb{R}$ is a continuous function. Is f integrable? what is $\int \int_R f\ dxdy$?

Regular partitions

However, the current definition isn't truly helpful in making computations. We define *regular* partitions.

The regular partition of R of order any $n \in \mathbb{N}$ is defined by $x_0 = a$ and $y_0 = c$, and for $i = 0, 1, \cdots, n-1, \quad j = 0, 1, \cdots, n-1$,

$$x_{i+1} = x_i + \frac{b-a}{n}, \quad y_{j+1} = y_j + \frac{d-c}{n}.$$

We take $t = \{t_{ij} \in R_{ij} \mid i, j \in \{0, 1, \cdots, n-1\}\}$ any arbitrary tag.
To check the integrability of a function f, it is enough to consider a sequence of regular partitions P_n of R.

Theorem

A bounded function $f : R \to \mathbb{R}$ is Riemann integrable if and only if the Riemann sum

$$S(f, P_n, t) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f(t_{ij}) \Delta_{ij},$$

tends to the same limit $S \in \mathbb{R}$ as $n \to \infty$, for any choice of tag t.

An Example

Example: Let $f(x,y) = x^2 + y^2$. Is it a continuous function on $\mathbb{R}^2$?
Ans. Yes!
Suppose the function is integrable on $[0,1] \times [0,1]$. Compute the integral using the theorem.

Let $R = [0,1] \times [0,1]$ and P_n be a regular partition. Then for tag $t = \{(\frac{i}{n}, \frac{j}{n}) \mid i = 0, \dots n-1, j = 0, \dots, n-1\}$,

$$S(f, P_n, t) = (\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (\frac{i}{n})^2 + (\frac{j}{n})^2) \frac{1}{n^2}.$$

Compute $\lim_{n\to\infty} S(f, P_n, t)$. How would you go about it? Answer is

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Conventions

Based on our definition, we make the following convention: Let $a, b, c, d \in \mathbb{R}$

- If $a = b$ or $c = d$, then $\int\int_{[a,b]\times[c,d]} f(x,y)dxdy := 0$.
- If $a < b$ and $c < d$:
  $\int\int_{[b,a]\times[c,d]} f(x,y)dxdy := -\int\int_{[a,b]\times[c,d]} f(x,y)dxdy$,
  $\int\int_{[a,b]\times[d,c]} f(x,y)dxdy := -\int\int_{[a,b]\times[c,d]} f(x,y)dxdy$,
  $\int\int_{[b,a]\times[d,c]} f(x,y)dxdy := \int\int_{[a,b]\times[c,d]} f(x,y)dxdy$.