MA 105 : Calculus

D4 - Lecture 1

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Aims of the course

First, welcome to IIT Bombay.

- To briefly review the calculus of functions of one variable and to teach the calculus of functions of several variables.

For details about the syllabus, tutorials, assignments, quizzes, exams and procedures for evaluation please refer to the course booklet which is available on moodle: https://moodle.iitb.ac.in

The emphasis of this course will be on the underlying ideas and methods rather than intricate problem solving (though there will be some of that too). The aim is to get you to think about calculus, in particular, and mathematics in general.

Syllabus

- Convergence of sequences and series, power series.
- Review of limits, continuity, differentiability.
- Mean value theorem, Taylor's theorem, maxima and minima.
- Riemann integrals, fundamental theorem of calculus, improper integrals, applications to area, volume.
- Partial derivatives, gradient and directional derivatives, chain rule, maxima and minima, Lagrange multipliers.
- Double and triple integration, Jacobians and change of variables formula.
- Parametrization of curves and surfaces, vector fields, line and surface integrals.
- Divergence and curl, theorems of Green, Gauss, and Stokes.

Sequences

Definition: A sequence in a set X is a function $f : \mathbb{N} \to X$, that is, a function from the set of natural numbers to X.

In this course X will usually be a subset of (or equal to) $\mathbb{R}$, $\mathbb{R}^2$ or $\mathbb{R}^3$, though we will also have occasion to consider sequences of functions sometimes. In later mathematics courses X may be the set of complex numbers $\mathbb{C}$, vector spaces (whatever those maybe), the set of continuous functions on an interval $\mathcal{C}([a, b])$ or other sets of functions.

Rather than denoting a sequence by a function, it is often customary to describe a sequence by listing the first few elements

$$a_1, a_2, a_3, \ldots$$

or, more generally by describing the n^{th} term a_n.

Note that we write a_n rather than $a(n)$. When we want to talk about the sequence as a whole we sometimes write $\{a_n\}_{n=1}^{\infty}$, but more often we once again just write a_n.

Examples of sequences

1. $a_n = n$ (here we can take $X = \mathbb{N} \subset \mathbb{R}$ if we want and f is just the identity function).
2. $a_n = \frac{1}{n}$ (here we can take $X = \mathbb{Q} \subset \mathbb{R}$ if we want, where $\mathbb{Q}$ denotes the set of rational numbers, or we can take $X = \mathbb{R}$ itself).
3. $a_n = \sin\left(\frac{1}{n}\right)$ (here the values taken by a_n are irrational numbers, so it is best to take $X = \mathbb{R}$).
4. $a_n = \frac{n!}{n^n}$.
5. $a_n = n^{1/n}$.
6. $s_n = \sum_{i=0}^{n} r^i$, for some r such that $0 \leq r < 1$.
7. $a_n = \left(n^2, \frac{1}{n}\right)$ (here $X = \mathbb{R}^2$ or $X = \mathbb{Q}^2$).
8. $f_n(x) = \cos nx$ (here X is the space of continuous functions on any interval $[a, b]$ or even on $\mathbb{R}$).
9. $s_n(x) = \sum_{j=0}^{n} \frac{x^j}{j!}$, or writing it out $s_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$. Once again X is a space of functions, for instance the space of continuous functions on $\mathbb{R}$.

Monotonic sequences

For the moment we will concentrate on sequences in $\mathbb{R}$.

Definition: A sequence is said to be a monotonically increasing sequence if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.

Definition: A sequence is said to be a monotonically decreasing sequence if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.

A monotonic sequence is one that is either monotonically increasing or monotonically decreasing.

From the examples in the previous slide, Example 1 ($a_n = n$) is a monotonically increasing sequence, Example 2 ($a_n = 1/n$) is a monotonically decreasing sequence, while Example 3 ($a_n = \sin\left(\frac{1}{n}\right)$) is also monotonically decreasing. How about Examples 4 and 5?

In Example 4 we notice that if $a_n = \frac{n!}{n^n}$,

$$a_{n+1} = \frac{(n+1)!}{(n+1)^{(n+1)}} = a_n \times \frac{(n+1)n^n}{(n+1)^{(n+1)}} \leq a_n,$$

so the sequence is monotonically decreasing.

Eventually monotonic sequences

In Example 5 ($a_n = n^{1/n}$), we note that

$$a_1 = 1 < 2^{1/2} = a_2 < 3^{1/3} = a_3,$$

(raise both a_2 and a_3 to the sixth power to see that $2^3 < 3^2$).

However, $3^{1/3} > 4^{1/4} > 5^{1/5}$. So what do you think happens as n gets larger?

In fact, $a_{n+1} \leq a_n$, for all $n \geq 3$. Prove this fact as an exercise. Such a sequence is called an eventually monotonically decreasing sequence, that is, the sequence becomes monotonically decreasing after some stage. One can similarly define eventually monotonically increasing sequences.

For any fixed non-negative value of r, Example 6 ($s_n = \sum_{j=0}^{n} r^j$) gives a monotonically increasing sequence, while for any fixed non-negative value of x, the sequence $s_n(x) = \sum_{j=0}^{n} \frac{x^j}{j!}$ in Example 9 also gives a monotonically increasing sequence.

Preliminaries

While all of you are familiar with limits, most of you have probably not worked with a rigorous definition. We will be more interested in limits of functions (which is what arise in the differential calculus), but limits of sequences are closely related to the former, and occur in their own right in the theory of Riemann integration.

So what does it mean for a sequence to tend to a limit? Let us look at the sequence $a_n = 1/n^2$. We wish to study the behaviour of this sequence as n gets large. Clearly as n gets larger and larger, $1/n^2$ gets smaller and smaller and seems to approach the value 0, or more precisely

the distance between $1/n^2$ and 0 becomes smaller and smaller.

In fact (and this is the key point), by choosing n large enough, we can make the distance between $1/n^2$ and 0 smaller than any prescribed quantity.

Let us examine the above statement, and then try and quantify it.

More precisely:

The distance between $1/n^2$ and 0 is given by $|1/n^2 - 0| = 1/n^2$.

Suppose I require that $1/n^2$ be less that 0.1 (that is, 0.1 is my prescribed quantity). Clearly, $1/n^2 < 1/10$ for all $n > 3$.

Similarly, if I require that $1/n^2$ be less than $0.0001 (= 10^{-4})$, this will be true for all $n > 100$.

We can do this for any number, no matter how small. If $\epsilon > 0$ is any number,

$$1/n^2 < \epsilon \iff 1/\epsilon < n^2 \iff n > 1/\sqrt{\epsilon}.$$

In other words, given any $\epsilon > 0$, we can always find a natural number N (in this case, any $N > 1/\sqrt{\epsilon}$) such that for all $n > N$, $|1/n^2 - 0| < \epsilon$.

The rigorous definition of a limit

Motivated by the previous example, we define the limit as follows.

Definition: A sequence a_n tends to a limit ℓ, if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - \ell| < \epsilon$$

whenever $n > N$.

This is what we mean when we write

$$\lim_{n \to \infty} a_n = \ell.$$

Equivalently, we say that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to a limit ℓ. If we just want to say that the sequence has a limit without specifying what that limit is, we simply say $\{a_n\}_{n=1}^{\infty}$ converges, or that it is convergent.

A sequence that does not converge is said to diverge, or to be divergent.

Remarks on the definition

1. Note that the N will (of course) depend on ϵ, as it did in our example, so it would have been more correct to write $N(\epsilon)$ in the definition of the limit. However, we usually omit this extra bit of notation.
2. We have already shown that $\lim_{n\to\infty} 1/n^2 = 0$. The same argument works for $\lim_{n\to\infty} 1/n^\alpha$, for any real $\alpha > 0$. We just take N to be any integer bigger than $1/\epsilon^{1/\alpha}$ for a given ϵ. Recall that for $x > 0$, x^α is defined as $e^{\alpha \log x}$.
3. For a given ϵ, once one N works, any larger N will also work. In order to show that a sequence tends to a limit ℓ we are not obliged to find the best possible N for a given ϵ, just some N that works. Thus, for the sequence $1/n^2$ and $\epsilon = 0.1$, we took $N = 3$, but we can also take $N = 10, 100, 1729$, or any other number bigger than 3.
4. Showing that a sequence converges to a limit ℓ is not easy. One first has to guess the value ℓ and then prove that ℓ satisfies the definition. We will see how to get around this in various ways.

More examples of limits

Let us show that $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0$.

For this we note that for $x \in [0, \pi/2]$, $0 \le \sin x \le x$ (try to remember why this is true).
Hence,

$$|\sin 1/n - 0| = |\sin 1/n| \le 1/n.$$

Thus, given any $\epsilon > 0$, if we choose some $N > 1/\epsilon$, $n > N$ implies $1/n < 1/N < \epsilon$. It follows that $|\sin 1/n - 0| < \epsilon$.

Let us consider Exercise 1.1.(ii) of the tutorial sheet. Here we have to show that $\lim_{n\to\infty} 5/(3n+1) = 0$. Once again, we have only to note that

$$\frac{5}{3n+1} < \frac{5}{3n},$$

and if this is to be smaller than ϵ, we must have $n > N > 5/3\epsilon$.

Formulæ for limits

If a_n and b_n are two convergent sequences then

1. $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$.
3. $\lim_{n\to\infty}(a_n/b_n) = \lim_{n\to\infty} a_n / \lim_{n\to\infty} b_n$, provided $\lim_{n\to\infty} b_n \neq 0$.

Implicit in the formulæ is the fact that the limits on left hand side exist.

Note that the constant sequence $a_n = c$ has limit c, so as a special case of (2) above we have

$$\lim_{n\to\infty}(c \cdot b_n) = c \cdot \lim_{n\to\infty} b_n.$$

Using the formulæ above we can break down the limits of more complicated sequences into simpler ones and evaluate them.

The Sandwich Theorem(s)

Theorem 1: If a_n, b_n and c_n are convergent sequences such that $a_n \leq b_n \leq c_n$ for all n, then

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n \leq \lim_{n\to\infty} c_n.$$

A second version of the theorem is especially useful:

Theorem 2: Suppose $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$. If b_n is a sequence satisfying $a_n \leq b_n \leq c_n$ for all n, then b_n converges and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n.$$

Note that we do not assume that b_n converges in this version of the theorem - we get the convergence of b_n for free.

Together with the rules for sums, differences, products and quotients, this theorem allows us to handle a large number of more complicated limits.

An example using the theorems above

Consider Exercise 1.2.(iii) on the tutorial sheet. We have to show that

$$\lim_{n\to\infty} \frac{n^3 + 3n^2 + 1}{n^4 + 8n^2 + 2}$$

exists and to evaluate it.

It is clear that

$$0 < \frac{n^3 + 3n^2 + 1}{n^4 + 8n^2 + 2} \leq \frac{1}{n} + \frac{3}{n^2} + \frac{1}{n^4}.$$

(How do we get this? Note that $n^3/(n^4 + 8n^2 + 2) < n^3/n^4 = 1/n$, and the other two terms can be handled similarly.)

Hence, applying the Sandwich Theorem (Theorem 2) to the sequences

$$a_n = 0, \quad b_n = \frac{n^3 + 3n^2 + 1}{n^4 + 8n^2 + 2} \quad \text{and} \quad c_n = \frac{1}{n} + \frac{3}{n^2} + \frac{1}{n^4}$$

we see that the limit we want exists provided $\lim_{n\to\infty} c_n$ exists, so this is what we must concentrate on proving.

The limit $\lim_{n\to\infty} c_n$ exists provided each of the terms appearing in the sum has a limit and in that case it is equal to the sum of the limits (by the first formula). But each of these limits is quite easy to evaluate.

We already know that

$$\lim_{n\to\infty} 1/n = 0 = \lim_{n\to\infty} 1/n^4,$$

while

$$\lim_{n\to\infty} 3/n^2 = 3 \cdot \lim_{n\to\infty} 1/n^2 = 0$$

where we have used the special case of the second formula (limit of the product is the product of the limits) for the first equality in the equation above. Since all three limits converge to 0, it follows that the given limit is $0 + 0 + 0 = 0$.