Let $|f(x,y)| \le M$ and $|g(x,y)| \le M$ for all $(x,y) \in R$.

For any Q_1 and $Q_2 \in R_{ij}$,

$$|f(Q_1)g(Q_1) - f(Q_2)g(Q_2)|$$

$$= |f(Q_1)g(Q_1) - f(Q_1)g(Q_2) + f(Q_1)g(Q_2) - f(Q_2)g(Q_2)|$$

$$\le M|g(Q_1) - g(Q_2)| + M|f(Q_1) - f(Q_2)|$$

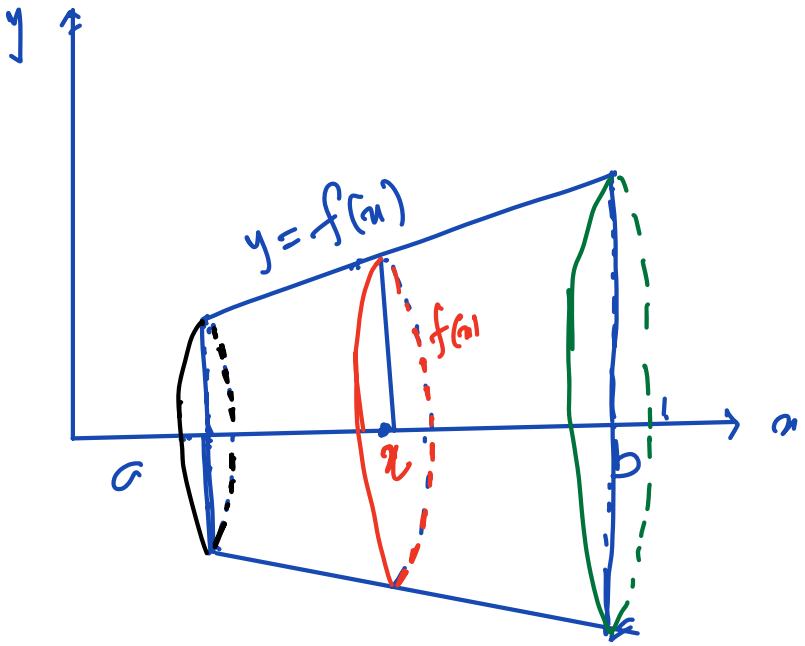
$$\le M\left(M_{ij}(g) - m_{ij}(g)\right) + M\left(M_{ij}(f) - m_{ij}(f)\right)$$

$$\Rightarrow M_{ij}(fg) - m_{ij}(fg) \le M\left[\left(M_{ij}(g) - m_{ij}(g)\right) + \left(M_{ij}(f) - m_{ij}(f)\right)\right].$$

$$\Rightarrow \left|U(fg,P) - L(fg,P)\right| \le M\left[\left(U(g,P) - L(g,P)\right) + \left(U(f,P) - L(f,P)\right)\right]$$

$$< \varepsilon.$$

The Slice Method



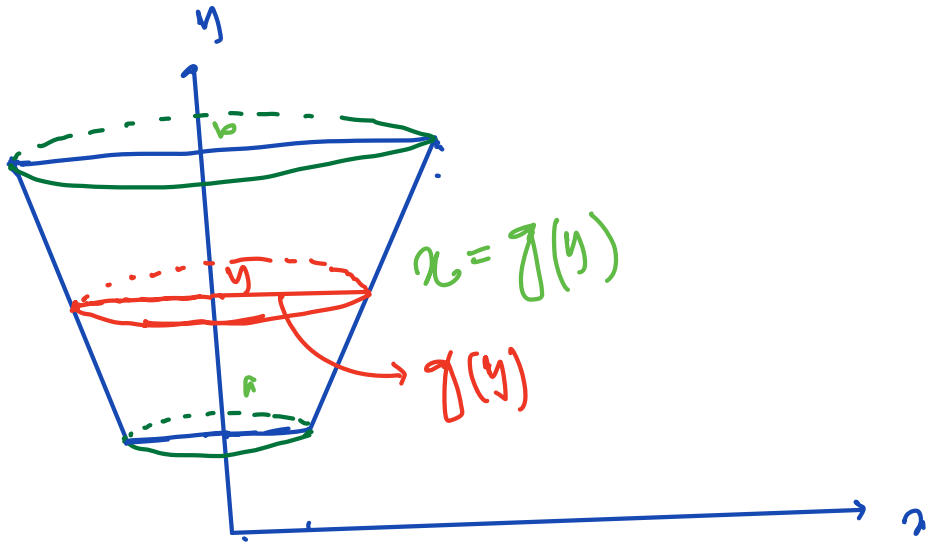
Area of the slice at x

$$A(x) = \pi f(x)^2$$

$$\Downarrow$$

$$V = \int_a^b \pi f(x)^2 \, dx = \pi \int_a^b f(x)^2 \, dx$$

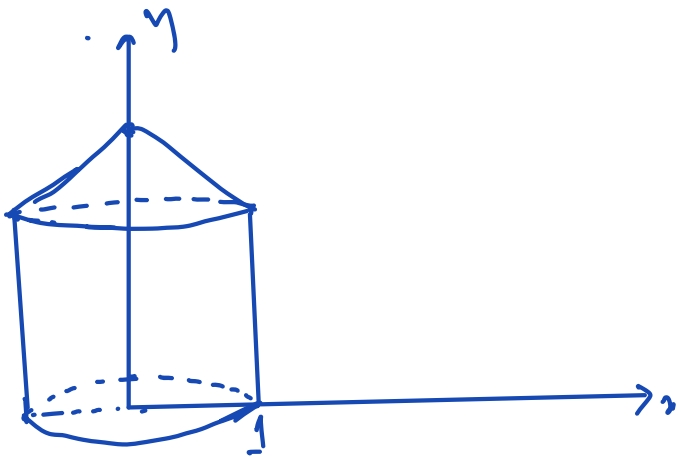
Rotation around y-axis

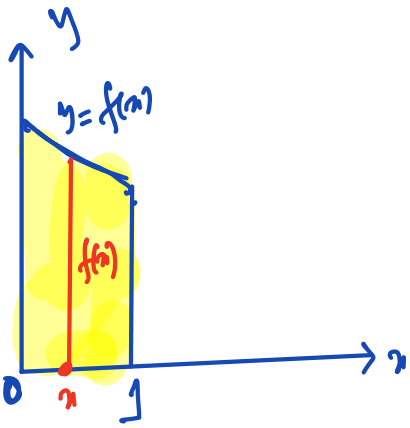


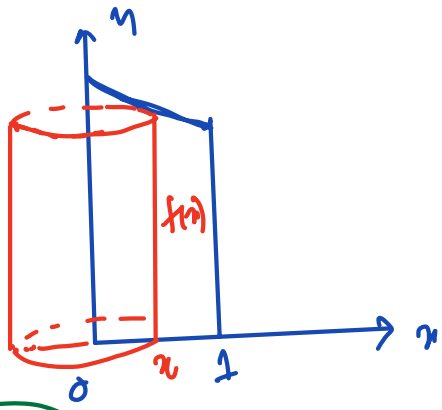
$$A(y) = \pi g(y)^2$$

$$V = \pi \int_a^b g(y)^2 \, dy$$

The Shell Method







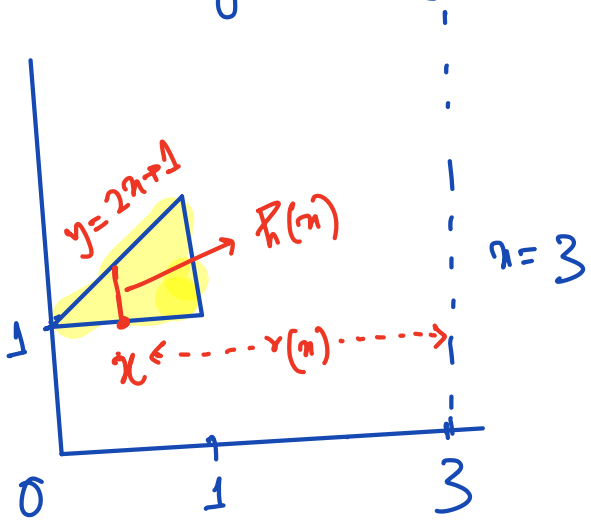
Area of the cylindrical shell at x

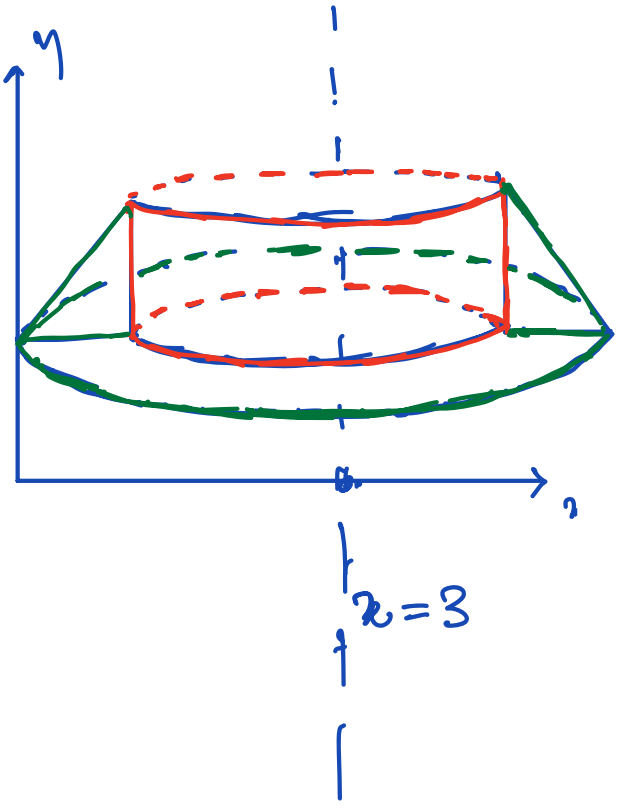
$$A(x) = 2\pi x f(x)$$

$$\Downarrow$$

$$V = \int_0^1 2\pi x f(x)\, dx = 2\pi \int_0^1 x f(x)\, dx.$$

Rotate the triangular region around $x=3$.





$$r(x) = 3-x$$

and $$P_h(x) = 2x+1-1$$
$$= 2x$$

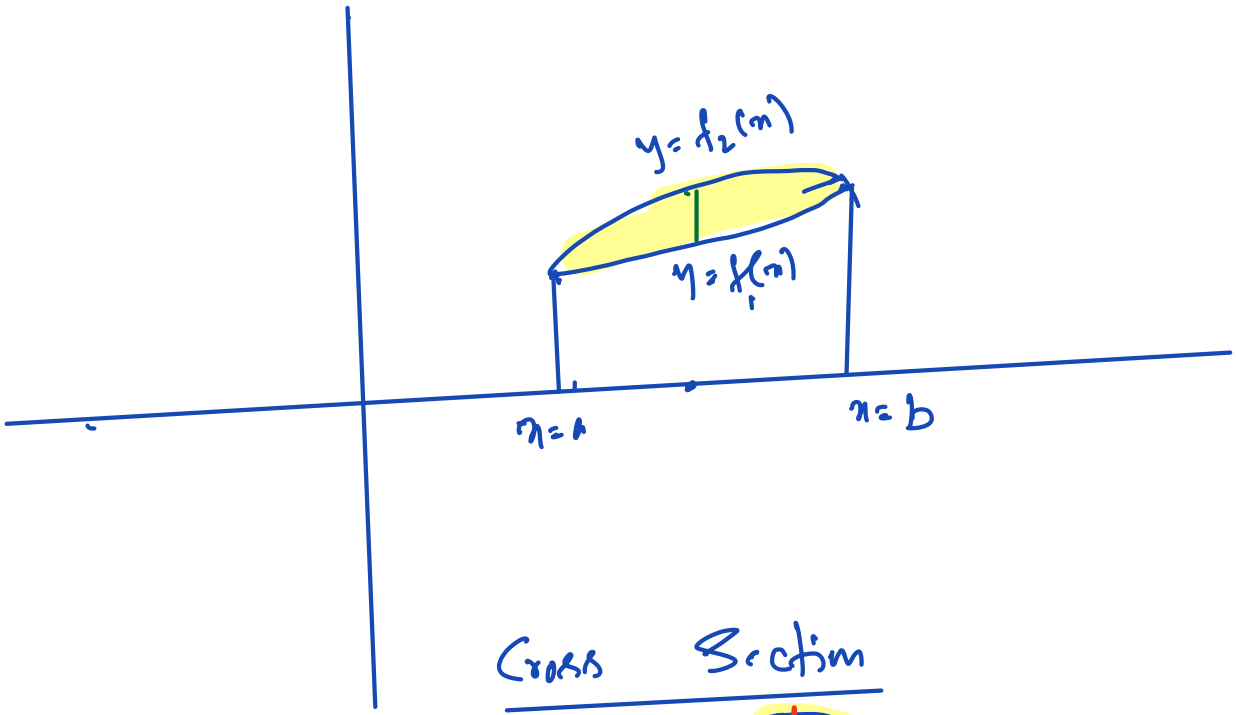
Area of the cylindrical shell at x

$$A(x) = 2\pi\, r(x) P_h(x)$$
$$= 2\pi\,(3-x)\,2x$$

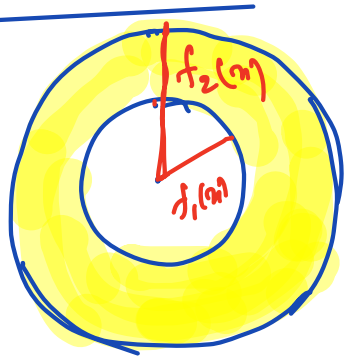
$$V = \int_0^1 A(x)\,dx = 2\pi \int_0^1 (6x-2x^2)\,dx$$

$$= \frac{14}{3}\pi.$$

Washer Method



Cross Section



$$A(x) = \pi\left[f_2(x)^2 - f_1(x)^2\right]$$