MA 105 : Calculus

D4 - Lecture 5

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Maxima and minima

Let $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function (you can think of X as an open, closed or half-open interval, for instance).

Definition: The function f is said to attain a maximum (resp. minimum) at a point $x_0 \in X$ if $f(x) \leq f(x_0)$ (resp. $f(x) \geq f(x_0)$) for all $x \in X$.

Once again, I remind you that, in general, f may not attain a maximum or minimum at all on the set X. The standard example being $X = (0, 1)$ and $f(x) = 1/x$ (can you find an example on the closed interval $[0, 1]$?).

However, if X is a closed and bounded interval and f is a continuous function, Theorem 11 tells us that the maximum and minimum are actually attained. Theorem 11 is sometimes called the Extreme Value Theorem.

Maxima and minima and the derivative

If f has a maximum at the point x_0 and if it is also differentiable at x_0, we can reason as follows.

We know that $f(x_0 + h) - f(x_0) \leq 0$ for every $h \in \mathbb{R}$ such that $x_0 + h \in X$.

Hence, we see (one half of the Sandwich Theorem!) that when $h > 0$,

$$\lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} \leq 0.$$

On the other hand, when $h < 0$, we get

$$\lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h} \geq 0.$$

Because f is assumed to be differentiable at x_0 we know that left and right hand limits must be equal. It follows that we must have $f'(x_0) = 0$. A similar argument shows that $f'(x_0) = 0$ if f has a minimum at the point x_0.

Local maxima and minima

The preceding argument is purely local. Before explaining what this means, we give the following definition.

Definition: Let $f : X \to \mathbb{R}$ be a function and x_0 be in X. Suppose there is a sub-interval $x_0 \in (c, d) \subset X$ such that $f(x_0) \geq f(x)$ (resp. $f(x_0) \leq f(x)$) for all $x \in (c, d)$, then f is said to have a local maximum (resp. local minimum) at x_0.

Sometimes we use the terms global maximum or global minimum instead of just maximum or minimum in order to emphasize the points are not just local maxima or minima. The argument of the previous slide actually proves the following:

Theorem 13: If $f : X \to \mathbb{R}$ is differentiable and has a local minimum or maximum at a point $x_0 \in X$, $f'(x_0) = 0$.

Proof: Exercise.

Rolle's Theorem

Theorem 13 is known as Fermat's theorem. It can be used to prove Rolle's Theorem.

Theorem 14: Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function which is differentiable in (a, b) and $f(a) = f(b)$. Then there is a point x_0 in (a, b) such that $f'(x_0) = 0$.

Proof: Since f is a continuous function on a closed and bounded interval, Theorem 11 tells us that f must attain its minimum and maximum somewhere in $[a, b]$. If both the minimum and maximum are attained at the end points, f must be the constant function, in which case, we know that $f'(x) = 0$ for all $x \in (a, b)$. Hence, we can assume that at least one of the minimum or maximum is attained at an interior point x_0 and Theorem 13 shows that $f'(x_0) = 0$ in this case. □

One easy consequence: If $P(x)$ is a polynomial of degree n with n real roots, then all the roots of $P'(x)$ are also real. (How do we know that polynomials are differentiable?)

Problems centered around Rolle's Theorem

Exercise 2.3: Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose f is differentiable on (a, b). If $f(a)$ and $f(b)$ are of opposite signs and $f'(x) \neq 0$ for all $x \in (a, b)$, then there is a unique point x_0 in (a, b) such that $f(x_0) = 0$.

Solution: Since the Intermediate Value Theorem guarantees the existence of a point x_0 such that $f(x_0) = 0$, the real point of this exercise is the uniqueness.

Suppose there were two points $x_1, x_2 \in (a, b)$ such that $f(x_1) = f(x_2) = 0$. Applying Rolle's Theorem, we see that there would exist $c \in (x_1, x_2)$ such that $f'(c) = 0$ contradicting our hypothesis. This proves the exercise. $\square$

Let us look at Exercise 2.8(i): Find a function f which satisfies all the given conditions, or else show that no such function exists: $f''(x) > 0$ for all $x \in \mathbb{R}$ and $f'(0) = 1, f'(1) = 1$.

Solution: Apply Rolle's Theorem to $f'(x)$ to conclude that such a function cannot exist.

The Mean Value Theorem

The Mean Value Theorem (MVT) is a special case of Rolle's theorem.

Theorem 15: Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function and that f is differentiable in (a, b). Then there is a point x_0 in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(x_0).$$

Proof: Apply Rolle's Theorem to the function

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a} \cdot (x - a)$$

$\square$

Applications of the MVT

Here is an application of the MVT which you have probably always taken for granted:

Theorem 16: If f satisfies the hypotheses of the MVT, and further $f'(x) = 0$ for every $x \in (a, b)$, f is a constant function.

Indeed, if $f(c) \neq f(d)$ for some two points $c < d$ in $[a, b]$,

$$0 \neq \frac{f(d) - f(c)}{d - c} = f'(x_0),$$

for some $x_0 \in (c, d)$, by the MVT. This contradicts the hypothesis. $\square$

Consider Exercise 2.6: Let f be continuous on $[a, b]$ and differentiable on (a,b). If $f(a) = a$ and $f(b) = b$, show that there exist distinct $c_1, c_2 \in (a, b)$ such that $f'(c_1) + f'(c_2) = 2$.

Solution: Split the interval $[a, b]$ into two pieces: $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$ - and apply the MVT to each interval.

Darboux's Theorem

Theorem 17: Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. If c, d, $c < d$ are points in (a, b), then for every u between $f'(c)$ and $f'(d)$, there exists an x in $[c, d]$ such that $f'(x) = u$.

Proof: We can assume, without loss of generality, that $f'(c) < u < f'(d)$, otherwise we can take $x = c$ or $x = d$.

Define $g(t) = ut - f(t)$. This is a continuous function on $[c, d]$ (not only on $[c, d]$ but also on (a, b) and differentiable at all the points in (a, b)) and hence, by Theorem 11 must attain its supremum (also the infimum but we will consider only the supremum).

Since $g'(c) = u - f'(c) > 0$ (as g is differentiable at all the points in (a, b), it is also differentiable at c and d) and by the definition of the derivative of a function, for $\epsilon = \frac{g'(c)}{2} > 0$, $\exists \delta > 0$ such that

$$\left| \frac{g(c+h) - g(c)}{h} - g'(c) \right| < \epsilon$$

whenever $|h| < \delta$.

Proof continued...

That is,

$$g'(c) - \epsilon < \frac{g(c+h) - g(c)}{h} < g'(c) + \epsilon$$

whenever $|h| < \delta$.

In particular, for $0 < h < \delta$ such that $c + h \in (c, d)$,

$$g(c+h) - g(c) > h(g'(c) - \epsilon) = h\frac{g'(c)}{2} > 0,$$

that is,

$$g(c+h) > g(c)$$

and hence $g(c)$ cannot be the $\sup_{x \in [c,d]} g(x)$.

Since $g'(d) = u - f'(d) < 0$, for $\epsilon = -\frac{g'(d)}{2} > 0$, $\exists \delta > 0$ such that

$$\left| \frac{g(d+h) - g(d)}{h} - g'(d) \right| < \epsilon$$

whenever $|h| < \delta$.

Proof continued...

That is,

$$g'(d) - \epsilon < \frac{g(d+h) - g(d)}{h} < g'(d) + \epsilon$$

whenever $|h| < \delta$.

In particular, for $-\delta < h < 0$ such that $d + h \in (c, d)$,

$$g(d+h) - g(d) > h(g'(d) + \epsilon) = h\frac{g'(d)}{2} > 0,$$

that is,

$$g(d+h) > g(d)$$

and hence $g(d)$ cannot be the $\sup_{x \in [c,d]} g(x)$.

Thus there exists $x \in (c, d)$ where g attains its supremum, and hence by Fermat's Theorem, $g'(x) = 0$ which yields $f'(x) = u$. $\square$

Continuity of the first derivative

We have just seen that the derivative satisfies the IVP. Can we find a function which is differentiable but for which the derivative is not continuous?

Here is the standard example:

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

This function will be differentiable at 0 but its derivative will not be continuous at that point. In order to see this you will need to study the function in Exercise 1.13(ii). This will show that $f'(0) = 0$. On the other hand, if we use the product rule when $x \neq 0$ we get

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x},$$

which does not go to 0 as $x \to 0$.

Back to maxima and minima

We will assume that $f : [a, b] \to \mathbb{R}$ is a continuous function and that f is differentiable on (a, b).

A point x_0 in (a, b) such that $f'(x_0) = 0$, is often called a stationary point.

We will assume further that $f'(x)$ is differentiable at x_0, that is, that the second derivative $f''(x_0)$ exists. We formulate the Second Derivative Test below.

Theorem 18: With the assumptions above:

1. If $f''(x_0) > 0$, the function has a local minimum at x_0.
2. If $f''(x_0) < 0$, the function has a local maximum at x_0.
3. If $f''(x_0) = 0$, no conclusion can be drawn.

The proof of the Second Derivative Test

Proof: The proofs are straightforward. For instance, to prove the first part we observe that

$$0 < f''(x_0) = \lim_{h \to 0} \frac{f'(x_0 + h)}{h} \qquad (\text{since } f'(x_0) = 0)$$

that is, for a given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$f''(x_0) - \epsilon < \frac{f'(x_0 + h)}{h} < f''(x_0) + \epsilon$$

whenever $|h| < \delta$. Now, by putting $\epsilon = \frac{f''(x_0)}{2}$ in the above inequality we get

$$0 < \frac{f''(x_0)}{2} < \frac{f'(x_0 + h)}{h}$$

whenever $|h| < \delta$. It follows that for $|h| < \delta$,

$$f'(x_0 + h) < 0 \quad \text{if } h < 0, \text{ and } f'(x_0 + h) > 0 \quad \text{if } h > 0.$$

It follows that $f(x)$ is decreasing to the left of x_0 and increasing to the right of x_0 (why?).

The proof of the Second Derivative Test continued...

Hence, x_0 must be a local minimum. A similar argument yields the second case. $\square$

If the third case of the theorem above occurs, the function may be changing from concave to convex. In this case x_0 is called a point of inflection. An example of this phenomenon is given by $f(x) = x^3$ at $x = 0$.