MA 105 Calculus II

Lecture 9

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Green's Theorem

Theorem (Green's theorem:)

1. *Let D be a bounded region in $\mathbb{R}^2$ with a positively oriented boundary ∂D consisting of a finite number of non-intersecting simple closed piecewise continuously differentiable curves.*
2. *Let Ω be an open set in $\mathbb{R}^2$ such that $\left(D \cup \partial D\right) \subset \Omega$ and let $F_1 : \Omega \to \mathbb{R}$ and $F_2 : \Omega \to \mathbb{R}$ be $\mathcal{C}^1$ functions.*

Then

$$\int_{\partial D} F_1 dx + F_2 dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy.$$

A proof of Green's theorem for regions of special type

We give a proof of Green's theorem when the region D is both of type 1 and type 2 .

Examples: Rectangles, Discs are examples of such region.

Assume that D is of Type 1

$$D = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, \quad \phi_1(x) \leq y \leq \phi_2(x)\},$$

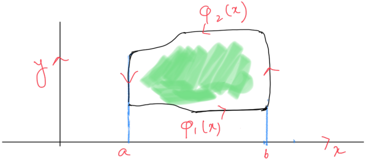
for some continuous functions ϕ_1 and ϕ_2.

Also assume there exist two continuous functions ψ_1 and ψ_2 such that D can be written as Type 2:

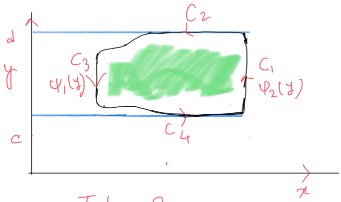
$$D = \{(x, y) \in \mathbb{R}^2 \mid c \leq y \leq d, \quad \psi_1(y) \leq x \leq \psi_2(y)\}.$$

The proof follows two main steps:

- Double integrals can be reduced to iterated integrals.
- Then the fundamental theorem of calculus can be applied to the resulting one-variable integrals.



Type 1.



Type 2.

The proof of Green's theorem, contd.

Step 1: Using the fact that D is a region of Type 2,

$$\iint_D \frac{\partial F_2}{\partial x} = \int_{\partial D} F_2 dy.$$

Step 2: Using the fact that D is a region of Type 1,

$$-\iint_D \frac{\partial F_1}{\partial y} = \int_{\partial D} F_1 dx.$$

Then combining the both equalities, we get our result.
Since D is a region of Type 2, it gives

$$\iint_D \frac{\partial F_2}{\partial x} dxdy = \int_c^d \int_{x=\psi_1(y)}^{\psi_2(y)} \frac{\partial F_2}{\partial x}(x,y) dxdy.$$

Using the Fundamental Theorem of Calculus we get

$$\int_c^d \int_{x=\psi_1(y)}^{\psi_2(y)} \frac{\partial F_2}{\partial x}(x,y) dxdy = \int_c^d F_2(\psi_2(y), y) - F_2(\psi_1(y), y) dy$$

The proof of Green's theorem contd.

Now let us calculate $\int_{\partial D} F_2 dy$. Note that ∂D can be written as union of four curves C_1, C_2, C_3 and C_4 such that

On C_1: $C_1 = \{(\psi_2(y), y) \in \mathbb{R}^2 \mid c \leq y \leq d\}$ with direction upwards. So,

$$\int_{C_1} F_2\, dy = \int_c^d F_2(\psi_2(y), y)\, dy.$$

On C_3: $C_3 = \{(\psi_1(y), y) \in \mathbb{R}^2 \mid c \leq y \leq d\}$ with direction downwards. So,

$$\int_{C_3} F_2\, dy = -\int_{-C_3} F_2\, dy = -\int_c^d F_2(\psi_1(y), y)\, dy.$$

On C_2 and C_4: $C_2 = \{(x, d) \mid \psi_1(d) \leq x \leq \psi_2(d)\}$ going from right to left and $C_4 = \{(x, c) \mid \psi_1(c) \leq x \leq \psi_2(c)\}$ going from left to right. In particular, they are vertical lines and y is constant along these lines. Thus, for any parametrisation of C_2 and C_4, $\frac{dy}{dt} = 0$, and

$$\int_{C_2} F_2\, dy = 0 = \int_{C_4} F_2\, dy.$$

The proof of Green's theorem contd.

$$\int_{\partial D} F_2 dy = \int_{C_1} F_2\, dy + \int_{C_2} F_2\, dy + \int_{C_3} F_2\, dy + \int_{C_4} F_2\, dy,$$

and using previous results, we obtain

$$\int_{\partial D} F_2 dy = \int_c^d F_2(\psi_2(y), y)\, dy - \int_c^d F_2(\psi_1(y), y)\, dy,$$

and thus

$$\iint_D \frac{\partial F_2}{\partial x} dxdy = \int_{\partial D} F_2 dy.$$

Similarly, using that D can be written as a region of Type 1, we get

$$\iint_D \frac{\partial F_1}{\partial y} dxdy = -\int_{\partial D} F_1 dx.$$

Subtracting the two equations above, we get

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \int_{\partial D} F_1 dx + \int_{\partial D} F_2 dy.$$

A more general case

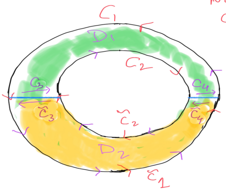
How does one proceed in general, that is for more general regions which may not be of both type 1 and type 2. We can try proceeding as follows:

- Break up D into smaller regions each of which is of both type 1 and type 2 but so that any two pieces meet only along the boundary.
- Apply Green's theorem to each piece.
- Observe that the line integrals along the interior boundaries cancel, leaving only the line integral around the boundary of D.

$D = D_1 \cup D_2$, $\quad \partial D = C_1 \cup C_2$.

$\partial D_1 = C_1 \cup C_3 \cup C_2 \cup C_4$, $\Big|$ $\partial D_2 = \tilde{C}_1 \cup \tilde{C}_4 \cup \tilde{C}_2 \cup \tilde{C}_3$

Note:
$C_3 = -\tilde{C}_3$
$C_4 = -\tilde{C}_4$



Del operator on vector fields

The del operator operates on vector fields as in two different ways. For a vector field $\mathbf{F} = (F_1, F_2, F_3)$ we define the curl of $\mathbf{F}$:

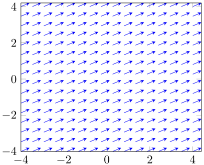
$$\operatorname{curl} \mathbf{F} := \nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

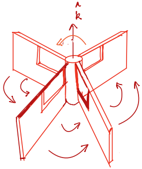
It is often written as a determinant;

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

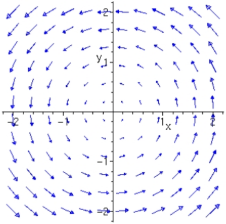
Curl as a measure of rotation

Curl of a vector field is measuring the extent to which the field rotate a particle. For instance ,

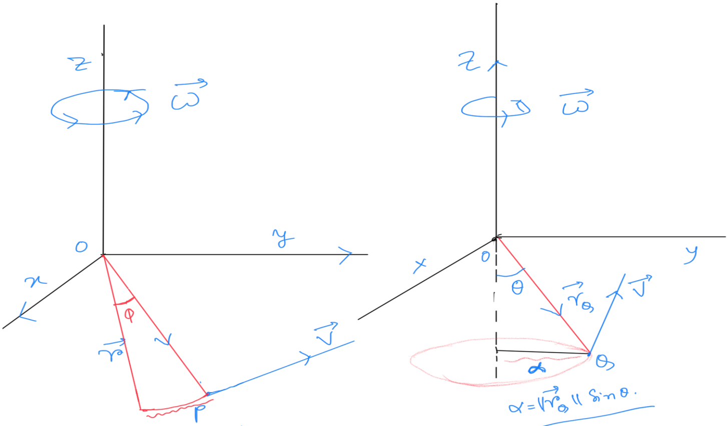




Imagine putting a small paddle wheel as shown in the above figure at any point in the plane with the vector field acting on it and visualize how it will rotate. Clearly in this example it will not rotate.



Angular velocity



$\vec{V} = \vec{\omega} \times \vec{r}$.

$\alpha = \|\vec{r}_{o}\| \sin\theta$.

Angular velocity

Consider a solid body B rotating around the z-axis on the xy-plane.

Let $\mathbf{v}$ denote the velocity vector, $\mathbf{w}$ the angular velocity vector at a point $\mathbf{r}$ in B. Note $\mathbf{w} = \omega \mathbf{k}$, where ω is the angular speed. Further, $\|v\| = \|w\|\|r\| \sin\theta$ where θ is the angle made by $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ with the axis of rotation.

Then $\mathbf{v} = \mathbf{w} \times \mathbf{r} = -\omega y \mathbf{i} + \omega x \mathbf{j}$. Check?

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega \mathbf{k} = 2\mathbf{w}.$$

Thus, the curl of velocity is twice the angular velocity.

If a vector field $\mathbf{F}$ represents the flow of a fluid, then the value of $\nabla \times \mathbf{F}$ at a point is twice the rotation vector of a rigid body that rotates as the fluid does near that point. In particular, $\nabla \times \mathbf{F} = 0$ at a point P means that the fluid is free from the rigid rotations at P.

The curl free vector field is called irrotational field.

The curl of a gradient

Suppose that $\mathbf{F} = \nabla f$ for some scalar function f and f is $\mathcal{C}^2$. Then

$$\nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) \mathbf{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right) \mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \mathbf{k}.$$

Clearly,

$$\nabla \times \mathbf{F} = 0.$$

In particular, this gives a criterion for deciding whether a vector field arises as the gradient of a function. This gives that curl$\mathbf{F} = 0$ is a necessary condition for any smooth vector field $\mathbf{F}$ to be the gradient field.

Is the condition $\nabla \times \mathbf{F} = 0$ sufficient for $\mathbf{F}$ to be a gradient field?

Recall that we have previously looked at the vector field

$$\mathbf{F} = \frac{y}{x^2+y^2} \cdot \mathbf{i} + \frac{-x}{x^2+y^2} \cdot \mathbf{j},$$

Exercise 1: Check that $\nabla \times \mathbf{F} = 0$.
Can you express $\mathbf{F}$ as the gradient of a suitable scalar function? Ans. No!

We can conclude that the Image of ∇ operator on scalar functions defined on $D \subset \mathbb{R}^3$ is a proper subset of

$$\ker(\text{curl}) = \{\mathbf{F} \text{ is a vector field on } D \mid \text{curl}\mathbf{F} = 0\}.$$

Definition (Scalar curl:) If $\mathbf{F} := (F_1, F_2)$ (a vector field in $\mathbb{R}^2$), then we define the curl of F by thinking of it as a vector field in $\mathbb{R}^3$ on the x-y plane with $F_3 = 0$.

$$\operatorname{curl} \mathbf{F} := \nabla \times \mathbf{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathbf{k}.$$

The function $\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$ is called the scalar curl of $\mathbf{F}$.

We can now state a vector valued version of Green's theorem using curl.

Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ be a C^1 vector field on an open connected region D with ∂D be positively oriented. Then

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \iint_D (\operatorname{curl} F \cdot \mathbf{k}) \, dxdy.$$

Other forms of Green's theorem in $\mathbb{R}^2$

Under the hypothesis on the region D and the functions F_1 and F_2 as stated in Green's theorem, we have

$$\int_{\partial D} F_1 dx + F_2 dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy.$$

We assume that ∂D can be parametrised by a single curve - otherwise break up the curve into parametrisable pieces.

Let ∂D be a non-singular, positively oriented curve in $\mathbb{R}^2$, parametrized by $\mathbf{c} : [a, b] \to \mathbb{R}^3$ such that $\mathbf{c}(t) = (x(t), y(t), 0)$. Then the unit tangent to the curve $\mathbf{c}$ and the unit outward normal to the curve are denoted by

$$\mathbf{T}(t) = \frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t)\|}, \quad \mathbf{n}(t) = \mathbf{T}(t) \times \mathbf{k}, \quad \forall\, t \in [a, b].$$

Other form Green's theorem

As consequences of Green's theorem in $\mathbb{R}^2$, we have following results:

Curl form or Tangential form

$$\int_{\partial D} \mathbf{F}.\mathbf{T} ds = \int\int_D (\text{curl}\mathbf{F}).\mathbf{k} dxdy.$$

The Stokes theorem is a 3-dimensional version of the above result.

Outline of its proof: Considering $\mathbf{F} = (F_1, F_2, 0)$ and noting that $ds = \|c'(t)\| dt$, we get

$$\int_{\partial D} \mathbf{F}.\mathbf{T} ds = \int_{\partial D} \mathbf{F}(\mathbf{c}(t)).\mathbf{c}'(t) dt = \int_{\partial D} F_1 dx + F_2 dy.$$

Now using Green's theorem and noting $\text{curl}\mathbf{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}$, the result follows.

Conservative field and its curl in $\mathbb{R}^2$

Theorem

1. *Let Ω be an open, simply connected region in $\mathbb{R}^2$.*
2. *if $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ is such that F_1 and F_2 have continuous first order partial derivatives on Ω.*

Then $\mathbf{F}$ is a conservative field in Ω if and only if

$$\mathit{curl}\,\mathbf{F} = 0, \quad \text{in} \quad \Omega.$$

Outline of the proof: Let the assumptions on Ω and $\mathbf{F}$ in the statement hold.

- If $\mathbf{F}$ is C^1 and a conservative field, i.e., $\mathbf{F} = \nabla f$, for some f is C^2. Then a direct calculation gives curl$F = 0$.
- Now conversely, if $\mathbf{F}$ is C^1 and curl $\mathbf{F} = 0$ on Ω. Then by Green's theorem we can show that the line integral of F over any simple closed curve is 0. That is, the line integral of F in Ω is path independent. Hence the result follows.

Theorem

Let $n = 2, 3$ and let D be an open, simply connected region in $\mathbb{R}^n$.

1. *For $n = 2$, if $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ is such that F_1 and F_2 have continuous first order partial derivatives on D, then $\mathbf{F}$ is a conservative field if and only if*

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \text{on} \quad D.$$

2. *For $n = 3$, if $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is such that F_1, F_2 and F_3 have continuous first order partial derivatives on D, then $\mathbf{F}$ is a conservative field if and only if*

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} \quad \text{on} \quad D.$$

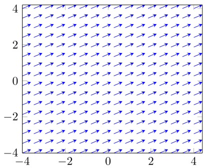
The divergence of a vector field

The del operator can be made to operate on vector fields to give a scalar function as follows.

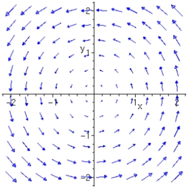
Definition: Let $\mathbf{F} = (F_1, F_2, F_3)$ be a vector field. The divergence of $\mathbf{F}$ is the scalar function defined by

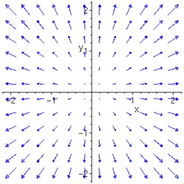
$$\operatorname{div}\mathbf{F} := \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

One way to interpret divergence of a velocity vector field at a point P as the amount of fluid flowing in versus the amount of fluid flowing out.



If $\mathbf{F}$ is a constant vector field then at any point what is flowing in is flowing out and the divergence is 0.





Is the divergence for this vector field 0?

This should have non-zero divergence. But what is it measuring?

Physical interpretation

If $\mathbf{F}$ is the velocity field of a fluid, the divergence of $\mathbf{F}$ gives the rate of expansion of the volume of the fluid per unit volume as the volume moves with the flow. In the case of planar vector fields we get the corresponding rate of expansion of area.

Example : $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$. The flow lines of this vector field point radially outward from the origin, so it is clear that the fluid is expanding as it flows. This is reflected in the fact that

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2 > 0.$$

Example : If we look at the vector field $\mathbf{F} = -x\mathbf{i} - y\mathbf{j}$, we see that $\nabla \cdot \mathbf{F} = -2$. This is consistent with the fact that the flow lines of the vector field all point towards the origin, and the fluid is getting compressed.

Example : $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$. In this case the fluid is moving counterclockwise around the origin - so it is neither being compressed, nor is it expanding. One checks easily that $\nabla \cdot \mathbf{F} = 0$.

The change in area in a flow

Let us assume the vector field $\mathbf{v} = (u, v)$ represents the velocity field of a fluid in $\mathbb{R}^2$. Let us compute the rate of change of unit area of the fluid as it flows along the curve.
We assume that we start at time $t = 0$ at $P = (x, y)$. Let the point evolve under the velocity field $\mathbf{v}$ to a point (X, Y) at time t. In particular,

$$X = X(x, y, t), Y = Y(x, y, t).$$

The change of variables formula tells us how an elementary area changes. Computing the Jacobian determinant for mapping $h(x, y) = (X, Y)$

$$J(x, y, t) = \begin{vmatrix} \frac{\partial X}{\partial x}(x, y, t) & \frac{\partial X}{\partial y}(x, y, t) \\ \frac{\partial Y}{\partial x}(x, y, t) & \frac{\partial Y}{\partial y}(x, y, t) \end{vmatrix}.$$

Now computing $\frac{\partial J(x,y,t)}{\partial t}$,

$$\begin{aligned} \frac{\partial J}{\partial t} &= \frac{\partial^2 X}{\partial x \partial t}\frac{\partial Y}{\partial y} + \frac{\partial X}{\partial x}\frac{\partial^2 Y}{\partial y \partial t} - \left(\frac{\partial^2 X}{\partial y \partial t}\frac{\partial Y}{\partial x} + \frac{\partial X}{\partial y}\frac{\partial^2 Y}{\partial x \partial t}\right) \\ &= (\nabla \cdot \mathbf{v})J. \end{aligned}$$

Divergence free is area preserving

Putting $\frac{\partial X}{\partial t}(x, y, t) = u(X(x, y, t), Y(x, y, t))$ and $\frac{\partial Y}{\partial t}(x, y, t) = v(X(x, y, t), Y(x, y, t))$, $\frac{\partial J}{\partial t}$ is equal to

$$\frac{\partial J}{\partial t} = (\nabla \cdot \mathbf{v})J.$$

Thus, $\nabla \cdot \mathbf{v} = 0$ if and only if J is independent of t. Since at $t = 0$, $J(x, y, 0) = 1$, $J(x, y, t) = J(x, y, 0) = 1$, for all t. There is no change of coordinates and hence Jacobian is trivial.
Clearly, $J = 1$ means that there is no change in the area,

$$Area(D) = \iint_D dXdY = \iint_{D'} |J(x, y)| dxdy = \iint_{D'} dxdy = Area(D').$$

The divergence free vector field is called incompressible field.

The divergence of any curl is zero. In other words, if $\mathbf{G}$ is a $\mathcal{C}^2$ vector field,

$$\operatorname{div}(\operatorname{curl}\mathbf{G}) = \nabla\cdot(\nabla\times\mathbf{G}) = 0.$$

Qn : If $\nabla\cdot\mathbf{F} = 0$, does it imply that $\mathbf{F} = \nabla\times\mathbf{G}$ for some vector field $\mathbf{G}$?

This question is related to the topological properties to of the domain of the vector field as in the case of when a curl free vector field is a gradient vector field. We will be able to show that this is the case when the domain is $\mathbb{R}^n$ for $n = 2, 3$. We postpone it for later.

Next, we consider the Divergence theorem in $\mathbb{R}^2$:
Let ∂D be a non-singular, positively oriented curve in $\mathbb{R}^2$, parametrized by $\mathbf{c} : [a, b] \to \mathbb{R}^3$ such that $\mathbf{c}(t) = (x(t), y(t), 0)$. Then the unit tangent to the curve $\mathbf{c}$ and the unit outward normal to the curve are denoted by

$$\mathbf{T}(t) = \frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t)\|}, \quad \mathbf{n}(t) = \mathbf{T}(t)\times\mathbf{k}, \quad \forall\, t\in[a, b].$$

Divergence form of Green's theorem

Divergence form or Normal form:

$$\int_{\partial D} \mathbf{F}.\mathbf{n} ds = \int\int_D \operatorname{div}\mathbf{F} dxdy.$$

Gauss's divergence theorem is a 3-dimensional analogue of the above.

Outline of its proof: Since $\mathbf{n}(t) = \mathbf{T}(t) \times \mathbf{k}$, for all $t \in [a, b]$, using the definition of $\mathbf{c}(t)$, we get $\mathbf{n}(t) = \left(\frac{y'(t)}{\|\mathbf{c}'(t)\|}, \frac{-x'(t)}{\|\mathbf{c}'(t)\|}, 0 \right)$. Thus, for $\mathbf{F} = (F_1, F_2, 0)$, using $ds = \|\mathbf{c}'(t)\| dt$

$$\begin{aligned} \int_{\partial D} \mathbf{F}.\mathbf{n} ds &= \int_{\partial D} \left[F_1(\mathbf{c}(t)) \frac{y'(t)}{\|\mathbf{c}'(t)\|} - F_2(\mathbf{c}(t)) \frac{x'(t)}{\|\mathbf{c}'(t)\|} \right] ds \\ &= \int_{\partial D} [F_1(\mathbf{c}(t)) y'(t) - F_2(\mathbf{c}(t)) x'(t)] dt = \int_{\partial D} F_1 \, dy - F_2 \, dx. \end{aligned}$$

Now by Green's theorem, we get

$$\int_{\partial D} F_1 \, dy - F_2 \, dx = \int\int_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \right) dxdy = \int\int_D \operatorname{div}\mathbf{F} dxdy.$$

Physical interpretation of Divergence theorem

We can interpret the above theorem in the context of fluid flow. If **F** represents the flow of a fluid, then the left hand side of the divergence theorem represents the net flux of the fluid across the boundary ∂D. On the other hand, the right hand side represents the integral over D of the rate $\nabla \cdot \mathbf{F}$ at which fluid area is being created. In particular if the fluid is incompressible (or, more generally, if the fluid is being neither compressed nor expanded) the net flow across ∂D is zero.

We can talk about volume analogously in the three dimensional case after proving Stokes theorem.

For further studies on curl and divergence of a vector field with physical applications, you can check this source:
https://math.libretexts.org/Bookshelves/Calculus/Book