MA-105 Calculus II

Lecture 4

B.K. Das

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

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1. An application

2. The mean value theorem for double integrals

3. Triple integral

An Applicaton:The integral of the Gaussian

We would like to evaluate the following integral:

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx.$$

What does this integral mean? - so far we have only looked at Riemann integrals inside closed bounded intervals, so the end points were always finite numbers a and b.

An integral like the one above is called an improper integral. We can assign it a meaning as follows. It is defined as

$$\lim_{T\to\infty} \int_{-T}^{T} e^{-x^2}\,dx,$$

provided, of course, this limit exists. We will see how to evaluate this.

The most amazing trick ever

Consider

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \cdot \int_{-\infty}^{\infty} e^{-y^2} dy.$$

We view this product as an iterated integral!

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dxdy.$$

Now under polar coordinates, the plane is sent to the plane. Hence, we can write this as

$$\int_0^{2\pi} \left[\int_0^{\infty} e^{-r^2} rdr \right] d\theta.$$

But we can now evaluate the inner integral. Hence, we get

$$\int_0^{2\pi} \left[-\frac{1}{2} e^{-r^2} \Big|_0^{\infty} \right] d\theta = \int_0^{2\pi} \frac{1}{2} d\theta = \pi$$

The answer

Since $I^2 = \pi$, we see that $I = \sqrt{\pi}$.

Using the above result you can easily conclude that

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}.$$

The integral above arises in a number of places in mathematics - in probability, the study of the heat equation, the study of the Gamma function (next semester) and in many other contexts.

There are many other ways of evaluating the integral I, but the method above is easily the cleverest.

Example Continued

Example: Evaluate $\int\int_D (3x + 4y^2)\, dxdy$, where D is the region in the upper half-plane bounded by the circled $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.
Ans: The region

$$D = \{(x, y) \mid y \geq 0, \quad 1 \leq x^2 + y^2 \leq 4\}.$$

In polar coordinate, after using change of variables $x = r\cos\theta$ and $y = r\sin\theta$, in $r - \theta$ plane, D becomes

$$D^* = \{(r, \theta) \mid 1 \leq r \leq 2, \quad 0 \leq \theta \leq \pi\}.$$

$$\int\int_D (3x + 4y^2)\, dxdy = \int_{\theta=0}^{\pi} \int_{r=1}^{2} (3r\cos\theta + 4r^2\sin^2\theta) r\, drd\theta$$
$$= \int_0^{\pi} [r^3\cos\theta + r^4\sin^2\theta]_{r=1}^2\, d\theta = \int_0^{\pi} [7\cos\theta + 15\sin^2\theta]\, d\theta = \frac{15\pi}{2}.$$

The mean value theorem for double integrals

Theorem

If D is an elementary region in $\mathbb{R}^2$, and $f : D \to \mathbb{R}$ is continuous. There exists (x', y') in D such that

$$f(x', y') = \frac{1}{A(D)} \int\int_D f(x, y) dA.$$

The proof follows using the boundedness of $f(x, y)$ and mean value theorem for continuous functions .

Sketch of Proof Since D is closed and bounded and f is continuous, the function attains its maximum and minimum at some points $(x_0, y_0) \in D$ and $(x_1, y_1) \in D$ respectively. Since D is an elementary region, there exists a path $\gamma : [0, 1] \to \mathbb{R}^2$ such that $\gamma(0) = (x_0, y_0) \in D$ and $\gamma(1) = (x_1, y_1)$.

Now apply the intermediate value theorem function $f \circ \gamma : [0, 1] \to \mathbb{R}$.

Average value contd.

How does one interpret the above statement geometrically?

If $f(x,y) \geq 0$, $f(x_0, y_0)$, the solid region under the graph of f and over the region D is same as the volume of the region over D whose height is the average value or mean value of f defined above.i.e.,

$$f(x_0, y_0) \times A(D) = \int\int_D f(x,y)dxdy.$$

Application: Center of Mass of a thin plate: (Weighted average): Let a plate occupies a region D of the $x - y$ plane and $\rho(x,y)$ be its density at a point (x,y) in D. Let ρ be a positive continuous function on D. The the coordinate of the center of mass $(\bar{x}, \bar{y})$ is given by

$$\bar{x} = \frac{\int\int_D x\rho(x,y)\,dxdy}{\int\int_D \rho(x,y)\,dxdy}, \quad \bar{y} = \frac{\int\int_D y\rho(x,y)\,dxdy}{\int\int_D \rho(x,y)\,dxdy}.$$

Note that for $\rho \equiv 1$, $\bar{x}$ is the average of $f(x,y) = x$ over the region D and $\bar{y}$ is the average of $g(x,y) = y$ over the region D.

Generalizing integration for $n \geq 3$

Recall our definition of Darboux integrals and Reimann integral. Both these definitions have an analogue in dimensions $n \geq 3$.

In this course, we only extend these ideas to functions on 3 variables. Note we already cannot imagine the graph of a function of 3 variables and much of the geometry is lost.

As an exercise you can think about which of the following definitions are specific to $n = 3$ and which can be generalized further.

If we have a bounded function $f : B = [a, b] \times [c, d] \times [e, f] \to \mathbb{R}$, we can integrate it over this rectangular cuboid (which we often refer to as a cuboid.) We divide the rectangular cuboid into smaller ones B_{ijk}, making sure that the length, breadth and height of the subcuboids are all small.

Integrals over rectangular cuboids

In particular, we can use the regular partition of order n to obtain the Riemann sum

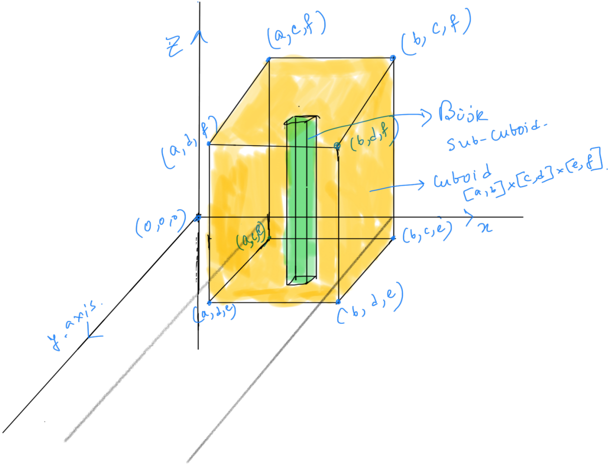
$$S(f, P_n, t) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} f(t_{ijk}) \Delta B_{ijk},$$

where ΔB_{ijk} is the volume of B_{ijk}, and $t = \{t_{ijk} \in B_{ijk}\}$ is an arbitrary tag.

As before we say that f is integrable if $\lim_{n\to\infty} S(f, P_n, t)$ converges to some fixed $S \in \mathbb{R}$ for any choice of tag t. The value of this limit is denoted by

$$\iiint_B f dV, \iiint_B f(x, y, z) dV \quad \text{or} \quad \iiint_B f(x, y, z) dx dy dz.$$

All the theorems for integrals over rectangles go through for integrals over rectangular cuboids.


Z
(a,c,f)
(b, c, f)
(a, d, f)
(b,d,f)
Box
Sub-cuboid.
cuboid
[a,b]×[c,d]×[e,f].
(0,0,0)
(a,c,e)
(b,c,e)
x
(a,d,e)
(b, d, e)
y.axis.

Integrating over bounded regions B in $\mathbb{R}^3$

First, if $f : B \subset \mathbb{R}^3 \to \mathbb{R}$ is bounded and continuous in B, except possibly on (a finite union of) graphs of continuous functions of the form $z = a(x, y)$, $y = b(x, z)$ and $x = c(y, z)$, then it is integrable.

This allows us to define the integral of (say) a continuous function on any bounded region B whose boundary is a set of content zero in $\mathbb{R}^3$. Let B^* be a cuboid enclosing the bounded region and $f^* : B^* \to \mathbb{R}$ be defined as f on B and 0 elsewhere.

Then integral of f over B exists if integral of f^* over B^* exists and

$$\iiint_{B^*} f^* = \iiint_B f.$$

Once we have defined the triple integral in this way, it remains to evaluate it.

Evaluating triple integrals: Fubini's Theorem

Fubini's Theorem can be generalized - that is, triple integrals can usually be expressed as iterated integrals, this time by integrating functions of a single variable three times.

Let f be integrable on the cuboid B. Then any iterated integral that exists is equal to the triple integral; i.e.,

$$\iiint_B f(x,y,z)dxdydz = \int_a^b \int_c^d \int_e^f f(x,y,z)dzdydx,$$

provided the right hand side iterated integral exists.
There are, in fact, five other possibilities for the iterated integrals.

We have a theorem saying if f is integrable, whenever any of these iterated integral exists, it is equal to the value of the integral of f over B.
If f is continuous on B, then f is integrable on B and all iterated integrals exist and their values are equal to the integral of f on B.

Elementary regions in $\mathbb{R}^3$

The triple integrals that are easiest to evaluate are those for which the region W in space can be described by bounding z between the graphs of two functions in x and y with the domain of these functions being an elementary region in two variables.

For example,

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid \gamma_1(x, y) \leq z \leq \gamma_2(x, y), (x, y) \in D\},$$

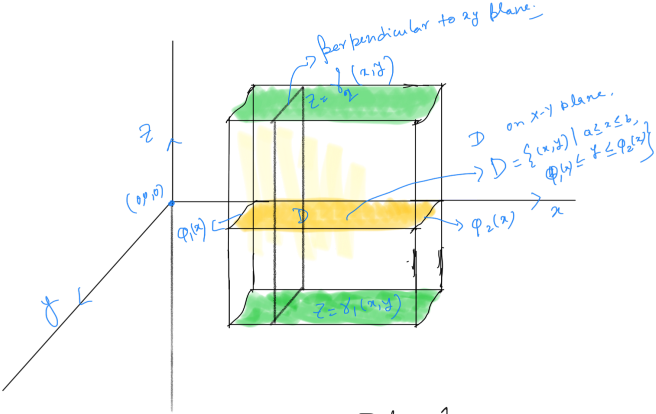
where γ_1 and γ_2 are continuous on $D \subset \mathbb{R}^2$ and D is an elementary region in $\mathbb{R}^2$. For rxample, if D is Type 1, then

$$D = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, \quad \phi_1(x) \leq y \leq \phi_2(x)\},$$

where $\phi_1 : [a, b] \to \mathbb{R}$ and $\phi_2 : [a, b] \to \mathbb{R}$ are continuous functions. The region D can be Type 2 also.

Example:

- The region W between the paraboloid $z = x^2 + y^2$ and the plane $z = 2$.
- The region bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 4$ and $x = z - y - 1$.



When D is Type 1.

Elementary regions (Example)

Suppose that the region W lies between $z = \gamma_1(x, y)$ and $z = \gamma_2(x, y)$. Suppose that the projection of W on the xy plane is bounded by the curves $y = \phi_1(x)$ and $y = \phi_2(x)$ and the straight lines $x = a$ and $x = b$, then for a continuous function f defined over W, we have

$$\iiint_W f(x, y, z) dxdydz = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x, y, z) dzdydx.$$

Example: Let us find the volume of the sphere using the above formula. In other words, let us integrate the function 1 on the region W, where W is the unit sphere, i.e.,

$$\int\int\int_W 1 dxdydz =?, \quad \text{where} \quad W = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}.$$

The volume of the unit sphere

The sphere can be described as the region lying between $z = -\sqrt{1 - x^2 - y^2}$ and $z = \sqrt{1 - x^2 - y^2}$.

The projection of the sphere onto the xy plane gives a disc of unit radius. This can be described as the set of points lying between the curves $-\sqrt{1 - x^2}$ and $\sqrt{1 - x^2}$ and the lines $x = \pm 1$. Thus our triple integral reduces to the iterated integral

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dzdydx.$$

This yields

$$2\int_{-1}^{1} \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)^{1/2} dy \right] dx.$$

After evaluating the inner integral we obtain

$$2\pi \int_{-1}^{1} \frac{1 - x^2}{2} dx = \frac{4}{3}\pi.$$