MA-105 Calculus II

Lecture 3

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1. Integrals over any bounded region in $\mathbb{R}^2$

2. Evaluating integrals over Elementary regions

3. The integral in polar coordinate

Integrals over any bounded region in $\mathbb{R}^2$

So far we have learnt to integrate bounded functions on any rectangle in $\mathbb{R}^2$.

Let D be any bounded subset (not necessarily rectangle) of $\mathbb{R}^2$.

How to define integral of $f : D \to \mathbb{R}$ on D?

Remedy: If D is a bounded subset of $\mathbb{R}^2$, then there exists a rectangle R in $\mathbb{R}^2$ containing D, i.e., $D \subset R$. Why?
Since D is a bounded subset of R^2, there exists $a > 0$ such that any $(x, y) \in D$ satisfies $x^2 + y^2 < a^2$, i.e, $D \subset B_a = \{(x, y) \mid x^2 + y^2 \leq a^2\}$.

Note $B_a \subset [-a, a] \times [-a, a]$

Then the rectangle $R := [-a, a] \times [-a, a]$ contains D.

Extend f from D to R by defining

$$f^*(x,y) := \begin{cases} f(x,y), & (x,y) \in D, \\ 0, & (x,y) \notin D. \end{cases}.$$

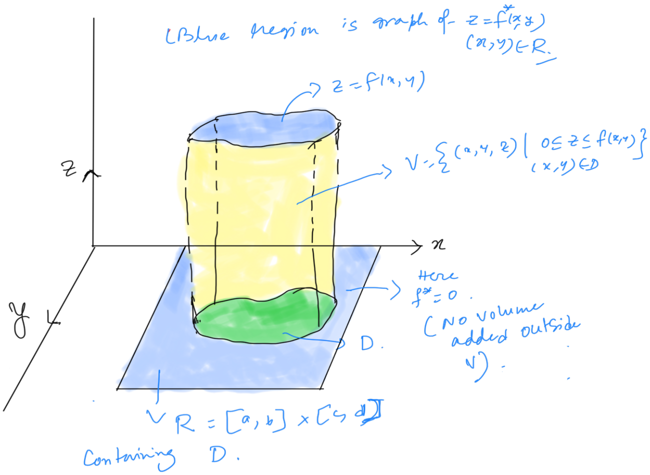
Definition

The function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to be integrable on bounded $D \subset \mathbb{R}^2$, if f^* is integrable on R and the integral of f on D is defined by

$$\int\int_D f(x,y)\,dx\,dy := \int\int_R f^*(x,y)\,dx\,dy.$$

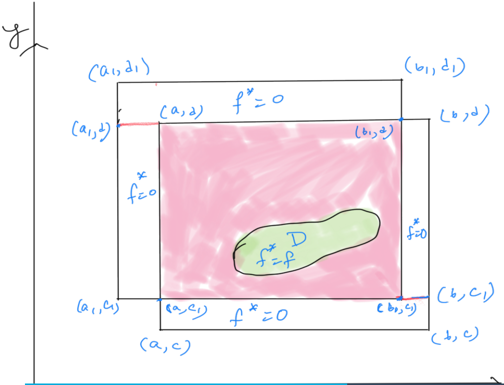
- If $f \geq 0$ on $D \subset \mathbb{R}^2$ and f is integrable on D, then the double integral of f on D is the volume of the solid that lies above D in the x-y plane and below the graph of the surface $z = f(x,y)$ for all $(x,y) \in D$.

$\int\int_D f$ = volume of V



Independent of choice of rectangle

- The choice of rectangle R containing D is not unique.
- But the value of the integral of f on D does not depend on the choice of the rectangle R containing D.
- Use the additivity property of integrals on rectangle and note that only 'zero' is getting added outside D.



Properties of Integrals over bounded sets in $\mathbb{R}^2$

Let D be a bounded subset of $\mathbb{R}^2$. Let $f : D \to \mathbb{R}$ be an integrable function.

- The algebraic properties for integrals on any bounded set D in $\mathbb{R}^2$ hold similarly to those of the case of integrals on rectangle.

Domain additivity property: Let $D \subseteq \mathbb{R}^2$ be a bounded set. Let $D_1, D_2 \subseteq D$ such that $D = D_1 \cup D_2$. Let $f : D \to \mathbb{R}^2$ be a bounded function. If f is integrable over D_1 and D_2 and $D_1 \cap D_2$ has content zero then f is integrable on D and

$$\int\int_D f = \int\int_{D_1} f + \int\int_{D_2} f.$$

Boundary of a bounded region

Let $D \subseteq \mathbb{R}^2$ be a bounded set. A point in the boundary of D is one which has a sequence in D and a sequence in $\mathbb{R}^2 - D$ converging to it. The set of boundary points of D is denoted by ∂D.

Example. $D = \{(x, y) \mid x^2 + y^2 \leq r^2\}$. The boundary of D, $\partial D = \{(x, y) \mid x^2 + y^2 = r^2\}$.

Example. $R = [a, b] \times [c, d]$. The boundary of rectangle R, $\partial R = \{(a, y) \in \mathbb{R}^2 \mid c \leq y \leq d\} \cup \{(b, y) \in \mathbb{R}^2 \mid c \leq y \leq d\} \cup \{(x, c) \in \mathbb{R}^2 \mid a \leq x \leq b\} \cup \{(x, d) \in \mathbb{R}^2 \mid a \leq x \leq b\}$.

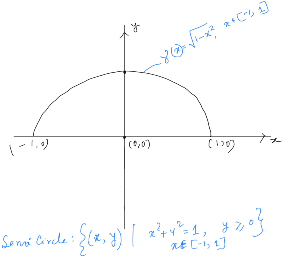
What is the boundary of the set $S = \{(x, y) \mid x, y \in \mathbb{Q}\}$? $\partial S = \mathbb{R}^2$.

Therefore for $f : D \to \mathbb{R}$ to be integrable we need ∂D to be content zero and same should be true for the points of discontinuity of f on D.

Path and Curve

Convention : A *path* γ in $\mathbb{R}^2$ (or $\mathbb{R}^3$) will mean a continuous function $\gamma : [a, b] \to \mathbb{R}^2$ (or $\gamma : [a, b] \to \mathbb{R}^3$) for $a, b \in \mathbb{R}$. It is said to be *closed* if $\gamma(a) = \gamma(b)$.

By a *curve* γ we mean the image of a path γ in $\mathbb{R}^2$ (or $\mathbb{R}^3$).



Existence of Integrals over bounded sets in $\mathbb{R}^2$

Theorem

Let $D \subset \mathbb{R}^2$ be a bounded set whose boundary ∂D is given by finitely many continous closed curve then any bounded and continuous function $f : D \to \mathbb{R}$ is integrable over D.

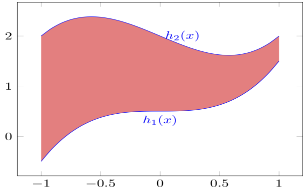
Example. Let $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$ and $f(x, y) = x^2 + y^2, \quad \forall (x, y) \in D$. Then f is integrable over D.

A slightly more general theorem is as follows:
Let D be a bounded set in $\mathbb{R}^2$ such that ∂D is of content zero. Let $f : D \to \mathbb{R}$ be a bounded function whose points of discontinuity have 'content zero'. Then f is integrable over D.

Elementary region: Type 1

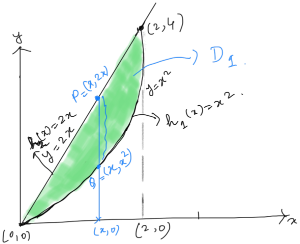
Let $h_1, h_2 : [a, b] \to \mathbb{R}$ be two continuous functions such that $h_1(x) \leq h_2(x)$ for all $x \in [a, b]$. Consider the set of points

$$D_1 = \{(x, y) \mid a \leq x \leq b \text{ and } h_1(x) \leq y \leq h_2(x)\}$$



Such a region is said to be of *Type 1* and for every $x \in \mathbb{R}$ vertical cross-section of D_1 is an interval.

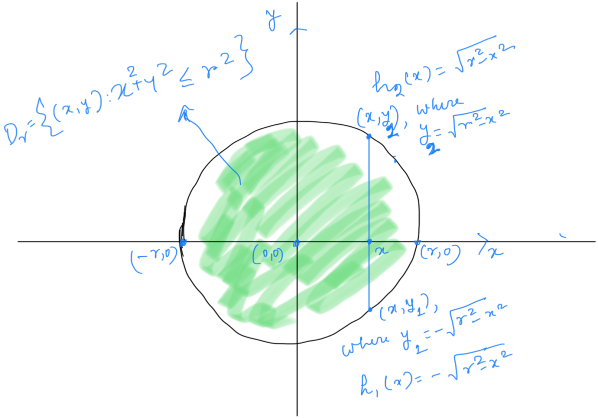
Example I: $D_1 = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2, \quad x^2 \leq y \leq 2x\}$. Here for all $x \in [0, 2]$, $h_1(x) = x^2$ and $h_2(x) = 2x$. Note $h_1(x) \leq h_2(x)$ for $x \in [0, 2]$.



Example II: The closed disc D_r of radius r around the origin,

$$D_r := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq r^2\}.$$

Take $h_1(x) = -\sqrt{r^2 - x^2}$ and $h_2(x) = \sqrt{r^2 - x^2}$. We see that D_r is of Type 1.



Integrability on Type 1 region

For Type 1, when $D_1 = \{(x, y) \,|\, a \leq x \leq b \text{ and } h_1(x) \leq y \leq h_2(x)\}$, the boundary

$$\partial D_1 = \{(a, y) \mid h_1(a) \leq y \leq h_2(a)\} \cup \{(b, y) \mid h_1(b) \leq y \leq h_2(b)\} \\ \cup \{(x, h_1(x)) \mid a \leq x \leq b\} \cup \{(x, h_2(x)) \mid a \leq x \leq b\}.$$

The region D_1 is bounded by continuous curves (the straight lines $x = a$ and $x = b$ and the graphs of the curves $y = h_1(x)$ and $y = h_2(x)$).

Thus ∂D_1 is of 'content zero' in $\mathbb{R}^2$.

Hence any continuous function defined on D_1 is integrable over the elementary region D_1.

Evaluating integrals on regions of Type 1

Let D be a region of Type 1 and assume that $f : D \to \mathbb{R}$ is continuous. Let $D \subset R = [\alpha, \beta] \times [\gamma, \delta]$ and let f^* be the corresponding function on R (obtained by extending f by zero).

The region D is bounded by continuous curves (the straight lines $x = a$ and $x = b$ and the graphs of the curves $y = h_1(x)$ and $y = h_2(x)$). Hence we can conclude that f^* is integrable on R. Applying Fubini's theorem on f^* we get,

$$\int\int_D f(x,y)dxdy := \int\int_R f^*(x,y)dxdy = \int_\alpha^\beta \left[\int_\gamma^\delta f^*(x,y)dy\right] dx.$$

In turn, this gives

$$\int_\alpha^\beta \left[\int_{h_1(x)}^{h_2(x)} f^*(x,y)dy\right] dx = \int_\alpha^\beta \left[\int_{h_1(x)}^{h_2(x)} f(x,y)dy\right] dx,$$

since $f^*(x,y) = 0$ if $y < h_1(x)$ or $y > h_2(x)$.

Examples

Example Let $D = \{(x, y) \mid 0 \le x \le 2, \quad x^2 \le y \le 2x\}$ and $f(x, y) = x + y$. Find $\int\int_D f(x, y)dxdy$.

Ans Since D is a Type 1 region and f is continuous over D, f is integrable over D.

$$\int\int_D f(x,y)\,dxdy = \int_0^2 \Big(\int_{x^2}^{2x} (x+y)\,dy\Big)dx = \int_0^2 [xy + \frac{y^2}{2}]_{y=x^2}^{y=2x}\,dx$$
$$= \int_0^2 [2x^2 + 4\frac{x^2}{2} - x^3 - \frac{x^4}{2}]\,dx$$

Example Let $D = \{(x, y) \mid x^2 + y^2 \le 1, \quad x \ge 0, \quad y \ge 0\}$ and $f(x, y) = \sqrt{1 - y^2}$. Find $\int\int_D f(x, y)dxdy$.

Ans Type 1, i.e, $D = \{(x, y) \mid 0 \le x \le 1, \quad 0 \le y \le \sqrt{1 - x^2}\}$. Then

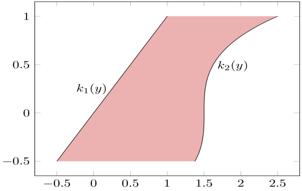
$$\int\int_D f(x,y)\,dxdy = \int_0^1 \Big(\int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\,dy\Big)\,dx.$$

Not easy to compute!

Elementary region: Type 2

Similarly, if $k_1, k_2 : [c, d] \to \mathbb{R}$ are two continuous functions such that $k_1(y) \leq k_2(y)$, for all $y \in [c, d]$. The set of points

$$D_2 = \{(x, y) \mid c \leq y \leq d \text{ and } k_1(y) \leq x \leq k_2(y)\}$$



is called a region of Type 2 and for every $y \in \mathbb{R}$ horizontal cross-section of D_2 is an interval.

Example $D_2 = \{(x, y) \mid x^2 + y^2 \leq 1\}$. If we take $k_1(y) = -\sqrt{1 - y^2}$ and $k_2(y) = \sqrt{1 - y^2}$, we see that D_2 is of Type 2.

Evaluating integrals on regions of type 2

Note that the boundary of D_2 is of content zero in $\mathbb{R}^2$. Hence any continuous function defined on D_2 is integrable over the elementary region.

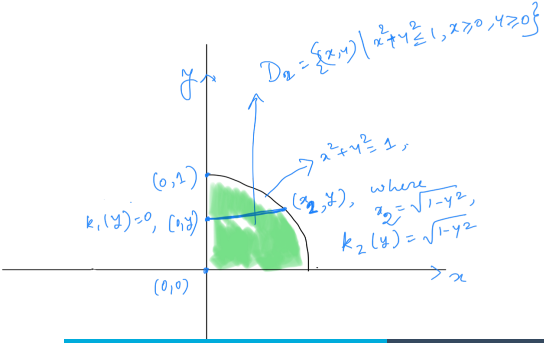
Using exactly the same reasoning as in the previous case (basically, interchanging the roles of x and y) we can obtain a formula for regions of Type 2.

Let D be a bounded set of Type 2 in $\mathbb{R}^2$. Let $f : D \to \mathbb{R}$ be a continuous function on D. We get

$$\int\int_D f(x,y)dxdy = \int_c^d \left[\int_{k_1(y)}^{k_2(y)} f(x,y)dx\right] dy.$$

Example: Let $D = \{(x,y) \mid x^2 + y^2 \leq 1, x \geq 0, y \geq 0\}$. Evaluate $\int\int_D \sqrt{1-y^2}dxdy$.

Ans.
$$\int\int_D \sqrt{1-y^2}dxdy = \int_0^1 \Big(\int_0^{\sqrt{1-y^2}} \sqrt{1-y^2}\, dx \Big)\, dy$$
$$= \int_0^1 [x\sqrt{1-y^2}]_{x=0}^{\sqrt{1-y^2}}\, dy = \int_0^1 (1-y^2)\, dy = \frac{2}{3}.$$



Remark

Both of these formulæ can be viewed as special cases of Cavalieri's principle when $f(x, y) \geq 0$. In the first case we are slicing by planes perpendicular to the x-axis, while in the second case, we are slicing by planes perpendicular to the y-axis.

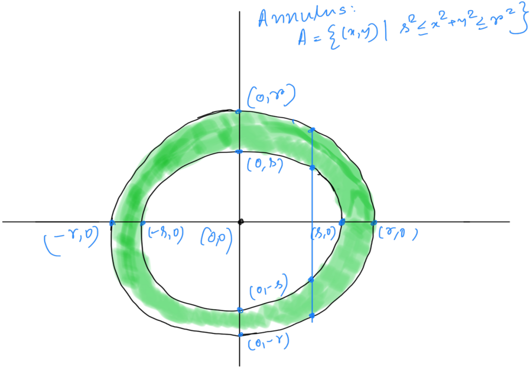
Caution! There exist bounded subsets of $\mathbb{R}^2$ which are not elementary regions; for example, *star-shaped subset* of $\mathbb{R}^2$ or an *annulus*.

Often we can write D as a union of regions of Types 1 and 2 and then we call it a region of type 3.

We could also view the disc as a region of type 3, by dividing it into four quadrants.

What about the *annulus* $A = \{(x, y) \in \mathbb{R}^2 \mid s^2 \leq x^2 + y^2 \leq r^2\}$?

Is it a type 3 region? yes



In summary,

- If D is a bounded region in $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ is bounded, then consider any rectangle R containing the region D in $\mathbb{R}^2$ and extend f to the rectangle by 0 outside D and denote it by f^*. The integral of f over D is defined by the integral of f^* on the rectangle R.
- The above definition is consistent because the definition of integral of f on D is independent of the choice of rectangle R.
- To determine the integrability of f over region D, conditions on f and D? The boundary of D should be 'well-behaved'. The set containing points of discontinuity of f is of 'content zero'.
- Algebraic properties of integrals on D are similar to that of the integrals on rectangle.
- To evaluate the value, use Fubini's theorem.
- To apply Fubini's theorem, hardest part is to determine the lower limit and upper limit of the integration: elementary regions Type 1 and Type 2 or combination of both.

Example1: Compute the integral of $f(x, y) = x^2 + y^2$ on $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

Can we compute this integral using iterated integrals ?

Example2: Compute the integral of $g(x, y) = e^{x^2+y^2}$ on $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

Can we use substitution like we did in one variable?

Let us see what happens when we use polar coordinates.

Polar Coordinates

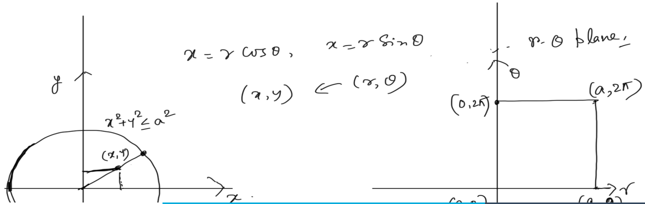
Change of variables from Cartesian coordinate system to polar coordinate system, any $(x, y) \in \mathbb{R}^2$ in Cartesian coordinate can be written as

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad r > 0, \theta \in [0, 2\pi].$$

Transformation of region under change of variables:

Ex. $D := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq a^2\}$ is transformed in polar coordinate system as a rectangle

$$D^* = \{(r, \theta) \mid 0 \leq r \leq a, \quad \theta \in [0, 2\pi]\}.$$



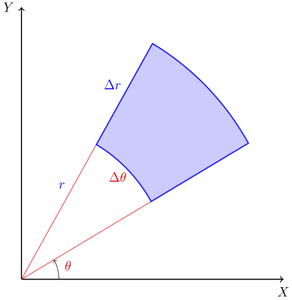
The integral in polar coordinates

Let D^* be a subset of $\mathbb{R}^2$ in polar coordinate system, such that for all $(r,\theta) \in D^*$, $(r\cos(\theta), r\sin(\theta)) \in D$, for $0 \le r \le 1$, and

$$g(r,\theta) := f(r\cos(\theta), r\sin(\theta)), \quad (r,\theta) \in D^*.$$

To integrate the function g on a domain D^* we need to cut up D^* into small rectangles, but these will be rectangles in the r-θ coordinate system.

What shape does a rectangle $[r, r+\Delta r] \times [\theta, \theta + \Delta\theta]$ represent in the x-y plane? A part of a sector of a circle.



Then we will be integrating over this sector instead of rectangle.

What is the area of this part of a sector?

Ans: It is $\frac{1}{2} \cdot [(r + \Delta r)^2 \Delta\theta - r^2 \Delta\theta] \sim r^* \Delta r \Delta\theta$, $r \leq r^* \leq r + \Delta r$.

Partitioning the region into subrectangles is equivalent to partitioning the region into parts of sectors as shown earlier.

It follows that the integral we want is approximated by a sum of the form

$$\sum_i \sum_j g(r_i^*, \theta_j^*) r_i^* \Delta r_i \Delta\theta_j,$$

where $\{(r_i^*, \theta_j^*\}$ is a tag for the partition of the "rectangle" in polar coordinates and

$$\int\int_D f(x, y) dxdy = \int\int_{D^*} f(r\cos\theta, r\sin\theta) r dr d\theta,$$

where D is the image of the region D^*.

This is the change of variable formula for polar coordinates.

Examples

Example1: Integrate $f(x,y) = x^2 + y^2$ on $D = \{(x,y) \mid x^2 + y^2 \le 1\}$.

Solution: Let us use polar coordinates. Let

$$D^* = \{(r,\theta) \mid 0 \le r \le 1, \quad 0 \le \theta \le 2\pi\}.$$

Denoting $x = r\cos\theta$ and $y = r\sin\theta$, the polar coordinates will transform D^* to D and

$$g(r,\theta) = f(r\cos\theta, r\sin\theta) = r^2.$$

$$\begin{aligned}\int\int_D f(x,y)\ dxdy &= \int\int_{D^*} g(r,\theta)\ r\ drd\theta = \int\int_{[0,1]\times[0,2\pi]} r^2.r\ drd\theta \\ &= \int_0^{2\pi}\int_0^1 r^3\ drd\theta = \int_0^{2\pi} \frac{r^4}{4}\Big|_0^1\ d\theta = \frac{\pi}{2}\end{aligned}$$

Examples contd.

☆

Example 2: Integrate $f(x,y) = e^{x^2+y^2}$ on $D = \{(x,y) \mid x^2+y^2 \leq 1\}$.
Solution: Using the same transformation as above

$$x = r\cos\theta, \quad y = r\sin\theta,$$

we get

$$\begin{aligned}
\int\int_D f(x,y)\ dxdy &= \int\int_{D^*} g(r,\theta)\ r\ drd\theta = \int\int_{[0,1]\times[0,2\pi]} e^{r^2} r\ drd\theta \\
&= \int_0^{2\pi}\int_0^1 e^{r^2} r\ drd\theta = \int_0^{2\pi} \frac{e^{r^2}}{2}\Big|_0^1\ d\theta = \pi(e-1)
\end{aligned}$$