Tutorial sheet 1: Sequences, limits, continuity, differentiability

1. Using the (ϵ-N) definition of a limit, prove the following:

   (i) $\lim_{n\to\infty} \frac{10}{n} = 0$

   (ii) $\lim_{n\to\infty} \frac{5}{3n+1} = 0$

   (iii) $\lim_{n\to\infty} \frac{n^{2/3}\sin(n!)}{n+1} = 0$

   (iv) $\lim_{n\to\infty} \left(\frac{n}{n+1} - \frac{n+1}{n}\right) = 0$

2. Show that the following limits exist and find them:

   (i) $\lim_{n\to\infty} \left(\frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n}\right)$

   (ii) $\lim_{n\to\infty} \left(\frac{n!}{n^n}\right)$

   (iii) $\lim_{n\to\infty} \left(\frac{n^3+3n^2+1}{n^4+8n^2+2}\right)$

   (iv) $\lim_{n\to\infty} (n)^{1/n}$

   (v) $\lim_{n\to\infty} \left(\frac{\cos \pi\sqrt{n}}{n^2}\right)$

   (vi) $\lim_{n\to\infty} \left(\sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)\right)$

3. Show that the following sequences are not convergent:

   (i) $\left\{\frac{n^2}{n+1}\right\}_{n\geq 1}$ (ii) $\left\{(-1)^n\left(\frac{1}{2} - \frac{1}{n}\right)\right\}_{n\geq 1}$

4. Determine whether the sequences are increasing or decreasing:

   (i) $\left\{\frac{n}{n^2+1}\right\}_{n\geq 1}$

   (ii) $\left\{\frac{2^n 3^n}{5^{n+1}}\right\}_{n\geq 1}$

   (iii) $\left\{\frac{1-n}{n^2}\right\}_{n\geq 2}$

5. Prove that the following sequences are convergent by showing that they are monotone and bounded. Also find their limits:

   (i) $a_1 = \frac{3}{2}, a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \quad \forall\, n \geq 1$

   (ii) $a_1 = \sqrt{2}, a_{n+1} = \sqrt{2 + a_n} \;\forall\; n \geq 1$

   (iii) $a_1 = 2,\ a_{n+1} = 3 + \frac{a_n}{2} \;\forall\, n \geq 1$

6. If $\lim_{n\to\infty} a_n = L$, find the following: $\lim_{n\to\infty} a_{n+1}$, $\lim_{n\to\infty} |a_n|$.

1b (iii) $\lim_{n\to\infty} \frac{n^{2/3}\sin(n!)}{n+1} = 0$

We need to find $n_0 \in \mathbb{N}$ such that for a given $\varepsilon > 0$ we have $|a_n - l| < \varepsilon$

$\forall \ n \geq n_0$ $\quad [\, l = 0 \,]$

$$a_n = \frac{n^{2/3}\sin(n!)}{n+1}$$

$$|a_n - 0| < \varepsilon$$

$$|a_n| \leq \left|\frac{n^{2/3}}{n+1}\right| \leq \left|\frac{n^{2/3}}{n}\right| = \underbrace{\left|\frac{1}{n^{1/3}}\right|}$$

$(\sin(x) \leq 1 \ \forall x)$ $\qquad$ (decreasing denominator)

If $\left|\frac{1}{n^{1/3}}\right| < \varepsilon$

then $|a_n| < \varepsilon$

so choose such n_0

$$\left|\frac{1}{n^{1/3}}\right| < \varepsilon \qquad \forall \ n \geq n_0$$

$$\frac{1}{\varepsilon} < n^{1/3} \implies n > \frac{1}{\varepsilon^3}$$

Take $n_0 = \left\lfloor \frac{1}{\varepsilon^3} \right\rfloor + 1$

Since, we are able to find n_0 for every ε, we have proved the limit to be zero.

2) (i) $\lim_{n\to\infty} \left(\frac{n}{n^2+1} + \frac{n}{n^2+2} + \dots + \frac{n}{n^2+n} \right)$

The limit can be easily found using the JEE Technique (sandwitch theorem)

$t_r = \frac{n}{n^2+r}$ $(r = 1 \text{ to } n)$

$a_n = \sum_{r=1}^{n} \frac{n}{n^2+r}$ $l = \lim_{n\to\infty} a_n$

$$\underbrace{\frac{n}{n^2+n}}_{\text{increasing denominator}} \le t_r \le \underbrace{\frac{n}{n^2}}_{\text{decreasing denominator}}$$

$$\frac{n}{n^2+n} \times n \le \sum t_r \le \frac{n}{n^2} \times n$$

$$\frac{n}{n+1} \le a_n \le 1$$

$$\lim_{n \to \infty} 1 = 1$$

$$\lim_{n \to \infty} \frac{n}{n+1} = 1$$

$$\left[\left| \frac{n}{n+1} - 1 \right| < \varepsilon \Rightarrow \left| \frac{1}{n+1} \right| < \varepsilon \right.$$
$$\left. \Rightarrow n_0 = \left[\frac{1}{\varepsilon} - 1 \right] + 1 \right]$$

By sandwitch theorem, limit $l = 1$

If $a_n \le b_n \le c_n \quad \forall n \geq n_0$

and $\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = l$

then $\lim_{n \to \infty} b_n = l$

2) (iv) $\lim_{n\to\infty} n^{1/n} = ?$

By guessing, we can get the limit to be 1 $\left[n^{1/n} = e^{\underbrace{\frac{1}{n}\log n}_{0}} \Rightarrow 1 \right]$

So let us assume,

$$n^{1/n} = 1 + \underbrace{h_n}_{\text{variable dependent on } n}$$

So, we have,

$$n = (1+h_n)^n$$

$$\geq \underbrace{1 + nh_n}_{\text{from series expansion}} + \binom{n}{2} h_n^2$$

$$\geq \binom{n}{2} h_n^2 = \frac{n(n-1)}{2} h_n^2$$

$$h_n \leq \left(\frac{2}{n-1}\right)^{1/2}$$

Claim: $\lim_{n\to\infty} h_n = 0$

Proof: $\left(\frac{2}{n-1}\right)^{1/2} < \varepsilon$

$\Rightarrow \quad \frac{2}{\varepsilon^2}+1 < n$

$n_0 = \left\lfloor \frac{2}{\varepsilon^2} + 1 \right\rfloor + 1$

$\Rightarrow$ limit $= 0$

$\lim 1+h_n = 1+0 = 1$

$\lim n^{1/n} = 1+h_n = 1$

3) (ii) $\left\{ (-1)^n \left(\frac{1}{2} - \frac{1}{n} \right) \right\}_{n \geq 1}$

$= \frac{(-1)^n}{2} - \frac{(-1)^n}{n}$

One idea $\Rightarrow$ $n =$ odd and even $\rightarrow$ different limits
$\rightarrow$ not convergent

other $\Rightarrow \lim_{n \to \infty} \frac{(-1)^n}{n} = 0$

$\left| \frac{(-1)^n}{n} \right| < \varepsilon \Rightarrow n > \frac{1}{\varepsilon}$

$\lim_{n \to \infty} \frac{(-1)^n}{2}$ = not convergent

hence, the difference will also not converge

$a_2 > \sqrt{2} = a_1$

5b(ii) $a_1 = \sqrt{2}$

$a_{n+1} = \sqrt{2+a_n}$

Claim: $a_{n+1} \geq a_n$ [monotone]

$\Leftarrow$ $\sqrt{2+a_n} \geq a_n$

$\Leftarrow$ $2+a_n \geq a_n^2$

$\Leftarrow$ $a_n^2 - a_n - 2 \leq 0$

$\Leftarrow$ $(a_n - 2)(a_n + 1) \leq 0$

$\Leftarrow$ $a_n \in [-1, 2]$

Claim: $a_n < 2$

Induction $\Rightarrow$ $a_1 = \sqrt{2} < 2$

Let $a_n < 2$

then $a_{n+1} = \sqrt{2+a_n} < \sqrt{2+2} = 2$

Hence sequence is monotonically increasing and bounded above, thereby convergent

5) $\lim_{n\to\infty} a_n = l$

obvious : $\lim_{n\to\infty} a_{n+1} = l$

claim : $\lim_{n\to\infty} |a_n| = |l|$

$$\big|\, |a_n| - |l| \,\big| \le |a_n - l| < \varepsilon \quad \forall\ n > n_0$$

Hence Proved, $\Rightarrow \lim_{n\to\infty} |a_n| = |l|$