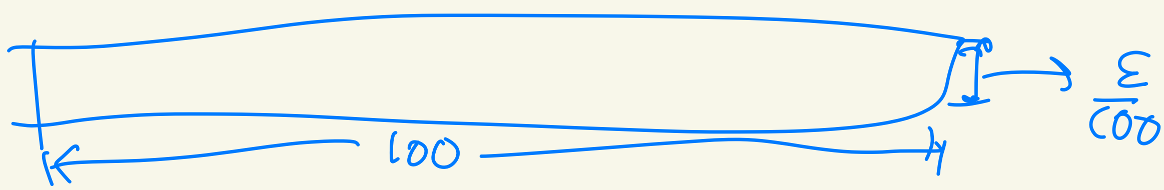
Additional Problems for D2 and D4

1. By considering an example show that the following definition of norm of a partition is not suitable for integration:

$$\|P\| = \max\left\{ (x_{i+1}-x_i) \times (y_{i+1}-y_i) \mid \begin{array}{l} i=0,\dots,m-1, \\ j=0,\dots,n-1 \end{array} \right\}.$$

2. If f is integrable over R then show that $|f|$ is integrable over R.

1) Consider the example of a long rectangle



Area $= \varepsilon$

can be made smaller

but the integral may not converge, because of a wide range over one of the two coordinates

$$\stackrel{2)}{\Rightarrow} \quad \sup_{x \in I} |f(x)| - \inf_{x \in I} |f(x)| \leq \sup f(x) - \inf f(x)$$

$$\Rightarrow \quad U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P) < \varepsilon$$

for P_ε = Partition

$\Rightarrow$ We are done!

MA 105 Part II (IIT Bombay) Tutorial Sheet 1 : Multiple integrals

1. (a) Let $R := [0,1] \times [0,1]$ and $f(x,y) := [x] + [y] + 1$ for all $(x,y) \in R$, where $[u]$ is the greatest integer less than equal to u, for any $u \in \mathbb{R}$. Using the definition of integration over rectangles, show that f is integrable over R. Also, find its value.

   (b) Let $R := [0,1] \times [0,1]$ and $f(x,y) := (x+y)^2$ for all $(x,y) \in R$. Show that f is integrable over R and find its value using Riemann sum.

   (c) Let $R := [a,b] \times [c,d]$ be a rectangle in $\mathbb{R}^2$ and let $f : R \to \mathbb{R}$ be integrable. Show that $|f|$ is also integrable over R.

   (d) Check the integrability of the function f over $[0,1] \times [0,1]$;

$$f(x,y) := \begin{cases} 1 & \text{if both } x \text{ and } y \text{ are rational numbers,} \\ -1 & \text{otherwise.} \end{cases}$$

   What do you conclude about the integrability of $|f|$?

2. (a) Sketch the solid bounded by the surface $z = \sin y$, the planes $x = -1$, $x = 0$, $y = 0$ and $y = \frac{\pi}{2}$ and the xy plane and compute its volume.

   (b) The integral $\int\int_R \sqrt{9 - y^2}\, dxdy$, where $R = [0,3] \times [0,3]$, represents the volume of a solid. Sketch the solid and find its volume.

3. Consider the function $f : [0,1] \times [0,1] \to \mathbb{R}$ defined as

$$f(x,y) = \begin{cases} 1 - 1/q & \text{if } x = p/q \quad \text{where} \quad p, q \in \mathbb{N} \quad \text{are relatively prime and } y \quad \text{is rational,} \\ 1 & \text{otherwise.} \end{cases}$$

   Show that f is integrable but the iterated integrals do not always exist.

4. Consider the function $f : [0,1] \times [0,1] \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{1}{x^2} & \text{if } 0 < y < x < 1, \\ -\frac{1}{y^2} & \text{if } 0 < x < y < 1, \\ 0 & \text{otherwise} \end{cases}$$

   Is f integrable over the rectangle? Do both iterated integrals exist? If they exist, do they have the same value?

5. For the following, write an equivalent iterated integral with the order of integration reversed and verify if their values are equal:

   (a) $\displaystyle\int_0^1 \Big(\int_0^1 \log[(x+1)(y+1)]\, dx \Big)\, dy.$

   (b) $\displaystyle\int_0^1 \Big(\int_0^1 (xy)^2 \cos(x^3)\, dx \Big)\, dy.$

6. (a) Let $R = [a,b] \times [c,d]$ and $f(x,y) = \phi(x)\psi(y)$ for all $(x,y) \in R$, where ϕ is continuous on $[a,b]$ and ψ is continuous on $[c,d]$. Show that

$$\int\int_R f(x,y)\, dxdy = \Big(\int_a^b \phi(x)\, dx \Big) \Big(\int_c^d \psi(y)\, dy \Big).$$

4.b function not bounded → cannot be integrated

$$f(x,y) = \begin{cases} \frac{1}{x^2} & 0 < y < x < 1 \\ \frac{-1}{y^2} & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Let us evaluate the iterated integrals

(i) $\int_0^1 \int_0^1 f(x,y)\,dx\,dy$

For $y = 0$: $\int_0^1 f(x,y)\,dx = 0$

For $y \in (0,1]$: $\int_{x=0}^{1} f(x,y)\,dx = \int_{x=0}^{y} f(x,y)\,dx + \int_{x=y}^{1} f(x,y)\,dx$

$$\int_0^y \frac{-1}{y^2}\,dx + \int_{x=y}^{1} \frac{1}{x^2}\,dx$$

$$= \frac{-1}{y} + \frac{1}{y} - 1 = -1$$

$$A(y) = \begin{cases} 0 & y = 0 \\ -1 & y \in (0,1] \end{cases}$$

$$\iint f(x,y) = \int_0^1 A(y)\,dy = -1$$

(ii) $\iint f(x,y)\,dy\,dx$

$x=0 \quad f(x,y)=0$

$$x\in(0,1] = \int_{y=0}^{x} f(x,y)\,dy + \int_{y=x}^{1} f(x,y)\,dy$$

$$= \int_0^x \frac{1}{x^2}\,dy + \int_x^1 \frac{-1}{y^2}\,dy$$

$$= \frac{1}{x} + 1 - \frac{1}{x} = 1$$

$$B(x) = \begin{cases} 0 & x=0 \\ 1 & x\in(0,1] \end{cases}$$

$$= \int_{y=0}^{1} B(x)\,dx = 1$$

Both the iterated integrals are unequal

3b $f(x,y) = \begin{cases} 1-\frac{1}{q} & \text{if } x = p/q \quad y = \text{rational}, \gcd(p,q)=1 \\ 1 & \text{otherwise} \end{cases}$

Take $\varepsilon > 0$, For some y, define S

$$S = \{x \mid 1 - f(x,y) \geq \varepsilon, \ 0 \leq x \leq 1, \ 0 \leq y \leq 1, \ y \in \mathbb{Q}\}$$

S has all rational numbers $|S| = L$

$0 \leq \frac{p}{q} \leq 1$ and $q \leq \frac{1}{\varepsilon}$ $q \in \mathbb{N}$

$$P_\varepsilon = \{x_0 \dots x_m\} \times \{y_0 \dots y_n\}$$

$x_m = y_n = 1$

$x_j - x_{j-1} < \frac{\varepsilon}{L} \ \forall j$

$y_k - y_{k-1} < \frac{\varepsilon}{L} \ \forall k$

$$\|P\|_\varepsilon < \frac{\varepsilon}{L}$$

$M_{jk} = 1$ between any two numbers there are infinitely many irrational numbers

$m_{jk} = 1 - \frac{1}{q}$

$U(f, P_\varepsilon) - L(f, P_\varepsilon)$

$$= \sum \sum (M_{jk} - m_{jk}) \Delta_{jk}$$

split this into two parts, $x \in S$ and $x \notin S$

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) = \sum_{S \cap [x_{j-1}, x_j] \neq \phi} \sum (M_{jk} - m_{jk}) \Delta_{jk}$$

$$+ \sum_{S \cap [x_{j-1}, x_j] = \phi} \sum (M_{jk} - m_{jk}) \Delta_{jk}$$

In the first part $m_{jk} \leq 1 - \frac{1}{2}$

$$M_{jk} - m_{jk} = \frac{1}{2} \geq \varepsilon$$

$M_{jk} - m_{jk} \leq 1$

$$\sum_{S \cap [x_{j-1}, x_j] \neq \phi} \sum (M_{jk} - m_{jk}) \Delta_{jk} \leq \sum \sum \Delta_{jk} \leq 2L \left(\frac{\varepsilon}{L} \times 1\right)$$

$$\leq 2\varepsilon$$

At most $2L$ number of $\left(\frac{\varepsilon}{L} \times 1\right)$ rectangles such that $S \cap [x_{j-1}, x_j] \neq \phi$

For the second part, $m_{jk} > 1 - \frac{1}{q}$

$$M_{jk} - m_{jk} < \frac{1}{q} = \varepsilon$$

$$\sum \sum (M_{jk} - m_{jk}) \Delta_{jk} \leq \sum \sum \varepsilon \Delta_{jk}$$

$$S \cap (x_{j-1}, x_j) = \phi \qquad \leq \varepsilon$$

Totalsum $\leq 3\varepsilon$

$\Rightarrow$ function = integrable

Now, we calculate the iterated integrals.

- **Case I:**
  Define $\Phi^y(x) := f(x, y)$ for some fixed y.

$$\Phi^y(x) = \begin{cases} \begin{cases} 1 - \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q}, p, q \in \mathbb{N} \\ 1 & \text{otherwise} \end{cases} & y \in \mathbb{Q} \\ 1 & y \notin \mathbb{Q} \end{cases}$$

  Thus, for a given y, $\Phi^y(x)$ is either the constant function 1 or a function that can be derived from Thomae's function in x, both of which we know are integrable for $x \in [0, 1]$. This yields,

$$\int_0^1 \Phi^y(x) dx = \begin{cases} \int_0^1 1 - \mathbf{T}(x) dx & y \in \mathbb{Q} \\ \int_0^1 1 dx & y \notin \mathbb{Q} \end{cases}$$

  Utilizing the fact the integral of $\mathbf{T}(x)$ over any sub-interval of $[0, 1]$ is 0 (Justify), we can write,

$$\int_0^1 \Phi^y(x) dx = 1$$

  Consequently,

$$\int_0^1 \int_0^1 f(x, y) dx dy = \int_0^1 \left(\int_0^1 \Phi^y(x) dx \right) dy = \int_0^1 1 dy = 1$$

- **Case II:**
  Define $\Phi^x(y) := f(x, y)$ for some fixed x.

$$\Phi^x(y) = \begin{cases} \begin{cases} 1 - k_x & y \in \mathbb{Q} \\ 1 & y \notin \mathbb{Q} \end{cases} & x = \frac{p}{q} \in \mathbb{Q}, p, q \in \mathbb{N} \\ 1 & \text{otherwise} \end{cases}$$

  where $k_x = \frac{1}{q}$.
  Observe that for a given x, $\Phi^x(y)$ is either the constant function 1 or $1 - \mathbf{1}_{\mathbb{Q}}(y)$. However, we know that $\mathbf{1}_{\mathbb{Q}}(y), y \in [0, 1]$ is not integrable (Justify). Therefore the iterated integral does not exist. ∎

1) (a) $R = [0,1] \times [0,1]$

$$f(x,y) = [x] + [y] + 1$$

$$f = \begin{cases} 1 & 0 \le x < 1 & 0 \le y < 1 \\ 2 & x = 1 & 0 \le y < 1 \\ 2 & 0 \le x < 1 & y = 1 \\ 3 & x = 1 & y = 1 \end{cases}$$

$$R_{ij} = \left[\frac{i}{n}, \frac{i+1}{n}\right) \times \left[\frac{j}{n}, \frac{j+1}{n}\right]$$

$$i, j \in \{0, 1, 2, \dots, n-1\}$$

Clearly $L(F, P_n) = 1$

$$U(f, P_n) = \sum_{i=0}^{n-2} \sum_{j=0}^{n-2} \frac{1}{n^2} + \sum_{j=0}^{n-2} \frac{2}{n^2} + \sum_{i=0}^{n-2} \frac{2}{n^2} + \frac{3}{n^2}$$

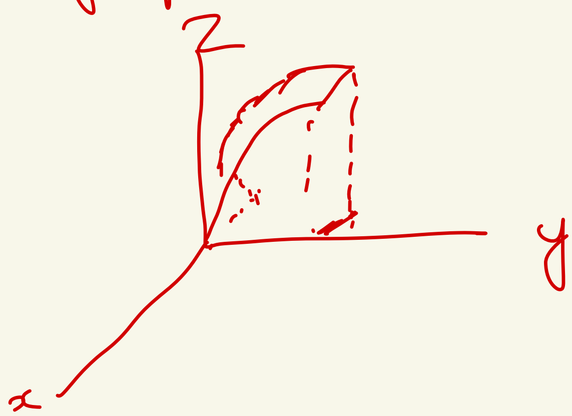
$$= \frac{(n-1)^2 + 4(n-1) + 3}{n^2} = \frac{n(n+2)}{n^2}$$

$$= 1 + \frac{2}{n}$$

$U(f, P_n) - L(f, P_n) = \frac{2}{n} < \varepsilon \quad \forall n > N_0 \Rightarrow$ Done!

2) $z = \sin y \quad x = -1 \quad x = 0 \quad y = 0 \quad y = \frac{\pi}{2}$

xy plane



Volume

$$\int_0^{\pi/2} \int_{-1}^{0} (\sin y - 0)\, dx\, dy$$

$$\int_{-1}^{0} \left(\int_0^{\pi/2} (\sin y)\, dy \right) dx = \quad |x| = 1$$

(b) Compute $\int\int_{[1,2]\times[1,2]} x^r y^s \, dxdy$, for any given $r \geq 0$ and $s \geq 0$.

(c) Compute $\int\int_{[0,1]\times[0,1]} xye^{x+y} \, dxdy$.

7. Evaluate the following integrals:

(a) $\displaystyle\int\int_R (x+2y)^2 \, dxdy$, where $R = [-1,2] \times [0,2]$.

(b) $\displaystyle\int\int_R \left[xy + \frac{x}{y+1}\right] dxdy$, where $R = [1,4] \times [1,2]$.

8. Consider the function f over $[-1,1] \times [-1,1]$:

$$f(x,y) = \begin{cases} x+y & \text{if} \quad x^2+y^2 \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Determine the set of points at which f is discontinuous. Is f integrable over $[-1,1] \times [-1,1]$?

MA 105 Part II (IIT Bombay) Tutorial Sheet 2 : Multiple integrals

1. For the following, write an equivalent iterated integral with the order of integration reversed:

   (a) $\int_0^1 \left[\int_1^{e^x} dy\right] dx$

   (b) $\int_0^1 \left[\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)dx\right] dy$

2. Evaluate the following integrals

   (a) $\int_0^\pi \left[\int_x^\pi \frac{\sin y}{y} dy\right] dx$

   (b) $\int_0^1 \left[\int_y^1 x^2 e^{xy} dx\right] dy$

   (c) $\int_0^2 (\tan^{-1} \pi x - \tan^{-1} x)dx.$

3. Find $\iint_D f(x,y)d(x,y)$, where $f(x,y) = e^{x^2}$ and D is the region bounded by the lines $y = 0,\ x = 1$ and $y = 2x$.

4. (a) Compute the volume of the solid enclosed by the ellipsoid:

   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

   where a, b, c are given real numbers.

   (b) Find the volume of the region under the graph of $f(x,y) = e^{x+y}$ over the region

   $$D := \{(x,y) \in \mathbb{R}^2 \mid |x| + |y| \le 1\}.$$

5. Find

   $$\lim_{r\to\infty} \iint_{D(r)} e^{-(x^2+y^2)} d(x,y),$$

   where $D(r)$ equals:

   (a) $\{(x,y) : x^2 + y^2 \le r^2\}$.

   (b) $\{(x,y) : x^2 + y^2 \le r^2,\ x \ge 0,\ y \ge 0\}$.

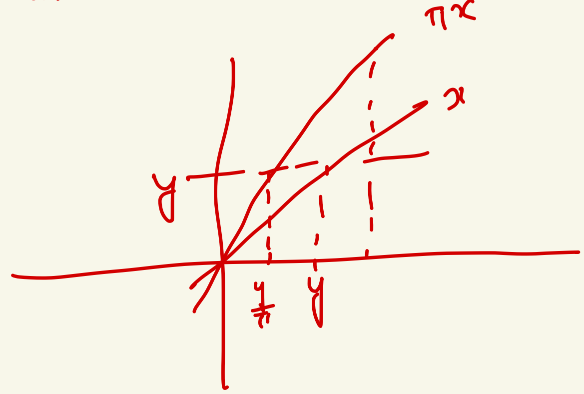
   (c) $\{(x,y) : |x| \le r,\ |y| \le r\}$.

   (d) $\{(x,y) : 0 \le x \le r,\ 0 \le y \le r\}$.

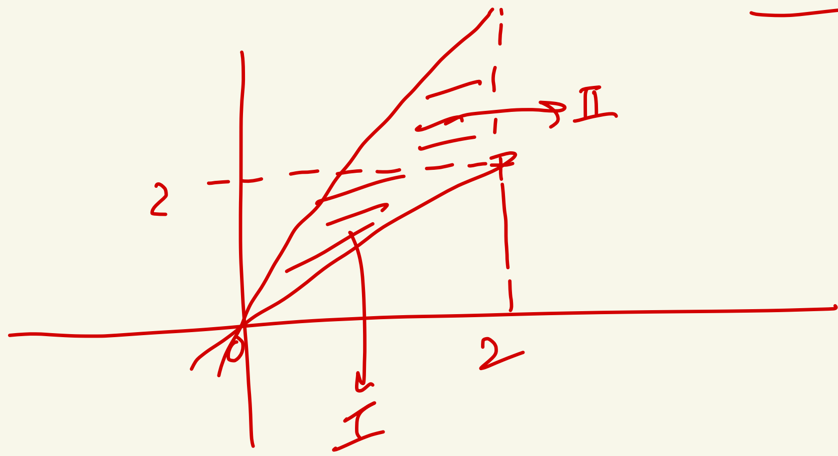
6. Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ using double integral over a region in the plane. (Hint: Consider the part in the first octant.)

7. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ above the region $x^2 + y^2 = 2x$ in $x - y$ plane.

2b (c) $\int_0^2 \left(\tan^{-1}(\pi x) - \tan^{-1}(x)\right) dx$



$$\int_0^2 \left(\int_x^{\pi x} \left(\frac{1}{1+y^2}\right) dy \right) dx$$



$$\int_0^2 \left(\int_{y/\pi}^{y} \left(\frac{1}{1+y^2}\right) dx \right) dy + \int_2^{2\pi} \left(\int_{y/\pi}^{2} \left(\frac{1}{1+y^2}\right) dx \right) dy$$

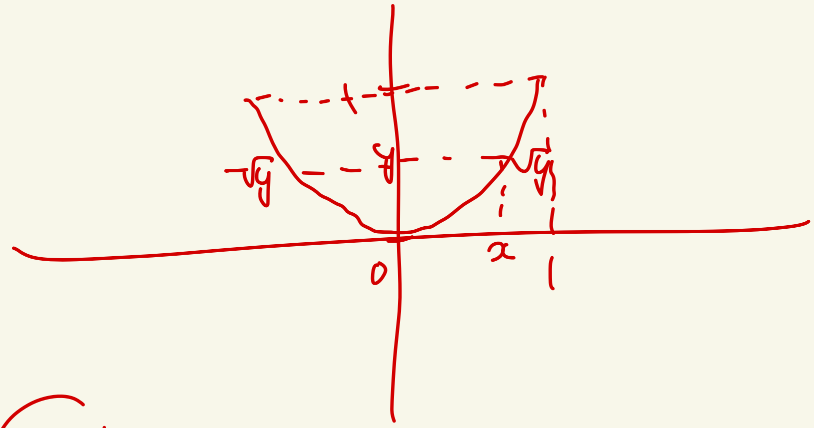
$$\int_0^2 \left(\frac{y - \frac{y}{\pi}}{1+y^2} \right) dy + \int_2^{2\pi} \left(\frac{2 - \frac{y}{\pi}}{1+y^2} \right) dy$$

Now integrate easily

16 (b)

$$\int_0^1 \left(\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\, dx \right) dy \quad \Rightarrow \quad -\sqrt{y} \le x \le \sqrt{y}$$

$$0 \le y \le 1$$



$$\Rightarrow -1 \le x \le 1$$

$$x^2 \le y \le 1$$

$$\int_{-1}^{1} \left(\int_{x^2}^{1} f(x,y)\, dy \right) dx$$

8. Express the solid $D = \{(x, y, z) | \sqrt{x^2 + y^2} \leq z \leq 1\}$ as

$$\{(x, y, z) \mid a \leq x \leq b, \quad \phi_1(x) \leq y \leq \phi_2(x), \quad \xi_1(x, y) \leq z \leq \xi_2(x, y)\}.$$

9. Evaluate

$$I = \int_0^{\sqrt{2}} \left(\int_0^{\sqrt{2-x^2}} \left(\int_{x^2+y^2}^{2} x dz \right) dy \right) dx.$$

Sketch the region of integration and evaluate the integral by expressing the order of integration as $dxdydz$.

10. Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.

11. Describe the solid whose volume is given by the integral

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 \rho^2 \sin \phi d\rho d\phi d\theta,$$

and evaluate the integral.

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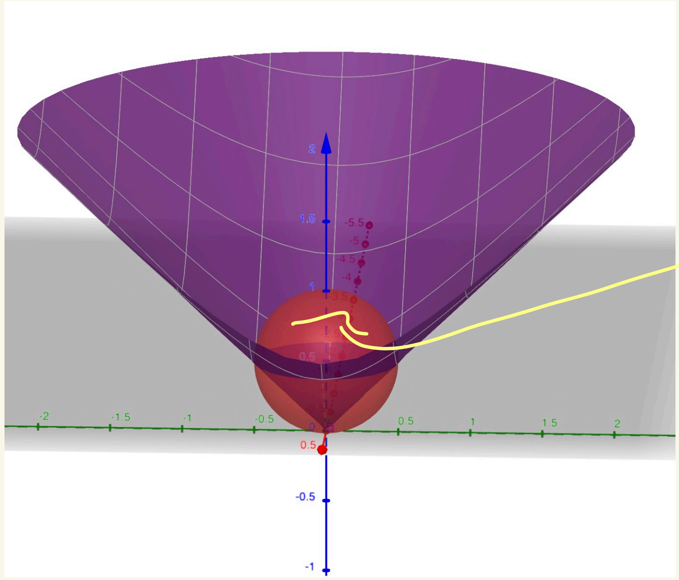


volume

$x = r\cos\phi\sin\theta$

$y = r\sin\phi\sin\theta$

$z = r\cos\theta$

$x^2+y^2+(z-\frac{1}{2})^2 = (\frac{1}{2})^2$

$\Rightarrow z \le \sqrt{\frac{1}{4} - x^2 - y^2} + \frac{1}{2}$

$$D = \left\{ (x,y,z) \;\middle|\; z \ge \sqrt{x^2+y^2},\ z \le \sqrt{\frac{1}{4}-x^2-y^2} + \frac{1}{2} \right\}$$

$r\cos\theta \ge r\sin\theta \Rightarrow 0 \le \theta \le \frac{\pi}{4}$

$r\cos\theta \le \sqrt{\frac{1}{4} - r^2\sin^2\theta} + \frac{1}{2}$

$\cancel{\frac{1}{4}} - r^2\sin^2\theta \ge \cancel{\frac{1}{4}} + r^2\cos^2\theta - r\cos\theta$ $\quad r \ge 0$

$r\cos\theta \ge r^2 \Rightarrow r \le \cos\theta$

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\theta} r^2\,dr\,d\theta\,d\phi$$

8b Clearly : $\underbrace{\sqrt{x^2+y^2}}_{\mathcal{E}_1(x,y)} \le z \le \underbrace{1}_{\mathcal{E}_2(x,y)}$

$$0 \le \sqrt{x^2+y^2} \le 1$$

$$0 \le x^2+y^2 \le 1$$

$$0 \le y^2 \le 1-x^2$$

$$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$$

$$\underbrace{-1 \le x \le 1}_{\text{To satisfy } \sqrt{x^2+y^2} \le 1}$$

MA 105 Part II Tutorial Sheet 3 :
Change of variables, Line integrals, October 16, 2023

I Multiple integrals and change of variables

2. Using a suitable change of variables, evaluate the integral $\int\int_D y\,dxdy$, where D is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geq 0$.
4. Use cylindrical coordinates to evaluate $\int\int\int_W (x^2 + y^2)dzdydx$, where

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid -2 \leq x \leq 2, \quad -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}, \quad \sqrt{x^2 + y^2} \leq z \leq 2\}.$$

6. Find $\iiint_F \frac{1}{(x^2 + y^2 + z^2)^{n/2}} dV$, where F is the region bounded by the spheres with center the origin and radii r and R, $0 < r < R$.
7. Evaluate the integral

$$\iint_D (x - y)^2 \sin^2(x + y) d(x, y),$$

where D is the parallelogram with vertices at $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$ and $(0, \pi)$.
8. Let D be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1$, $xy = 9$ and the lines $y = x$, $y = 4x$. Find $\iint_D dxdy$ by transforming it to $\iint_E dudv$, where $x = \frac{u}{v}$, $y = uv$, $v > 0$.
9. Using suitable change of variables, evaluate the following:
   i.
   $$I = \iiint_D (z^2x^2 + z^2y^2)dxdydz$$
   where D is the cylindrical region $x^2 + y^2 \leq 1$ bounded by $-1 \leq z \leq 1$.

   ii.
   $$I = \iiint_D \exp(x^2 + y^2 + z^2)^{3/2} dxdydz$$
   over the region enclosed by the unit sphere in $\mathbb{R}^3$.

II Vector analysis and line integrals

1. Let f, g be differentiable functions on $\mathbb{R}^2$. Show that
   A. $\nabla(fg) = f\nabla g + g\nabla f$;
   B. $\nabla f^n = nf^{n-1}\nabla f$;
   C. $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$ whenever $g \neq 0$.
2. Let $\mathbf{a}, \mathbf{b}$ be two fixed vectors, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r^2 = x^2 + y^2 + z^2$. Prove the following:
   (i) $\nabla(r^n) = nr^{n-2}\mathbf{r}$ for any integer n.
   (ii) $\mathbf{a} \cdot \nabla \left(\frac{1}{r}\right) = -\left(\frac{\mathbf{a} \cdot \mathbf{r}}{r^3}\right)$.
   (iii) $\mathbf{b} \cdot \nabla \left(\mathbf{a} \cdot \nabla \left(\frac{1}{r}\right)\right) = \frac{3(\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})}{r^5} - \frac{\mathbf{a} \cdot \mathbf{b}}{r^3}$.

4)

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$J = \underbrace{r}$

For cylindrical coordinates

$x^2 + y^2 \le 4$

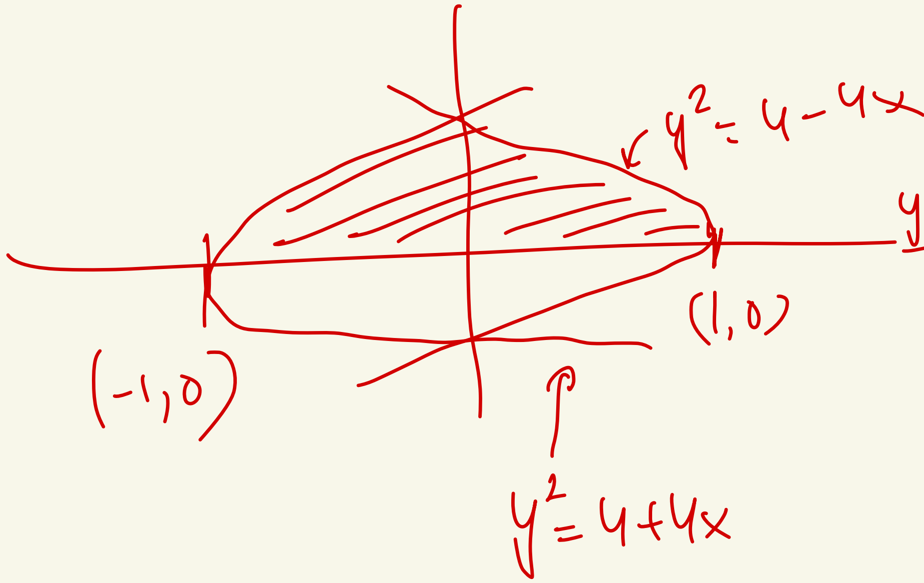
$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$r \le z \le 2$

$$\int_r^2 \int_0^{2\pi} \int_0^2 r^2 \times r \, dr \, d\theta \times dz$$

2b



$$\frac{y^2-4}{4} \le x \le \frac{4-y^2}{4}$$

$$y \geq 0$$

$$\left.\begin{array}{l} y = \sqrt{v} \\ x = 1 + u \end{array}\right\} \text{for shifting}$$

$$J = \begin{vmatrix} 1 & 0 \\ 0 & \frac{1}{2\sqrt{v}} \end{vmatrix} = \frac{1}{2\sqrt{v}}$$

$$0 \le y \le 2$$

$$\Rightarrow 0 \le v \le 4$$

$$\frac{v}{4} - 1 \le 1 + u \le \frac{4-v}{4}$$

now solve

7b $\iint_R (x-y)^2 \sin^2(x+y)\, dx\,dy$

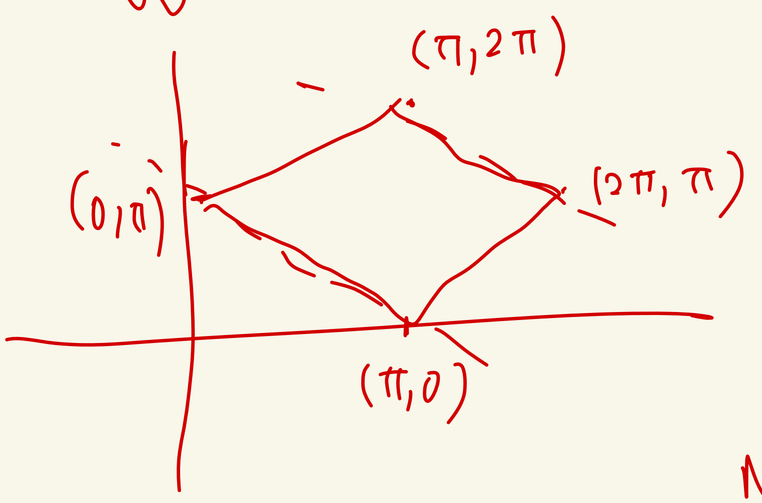
$(\pi, 0)$ $(2\pi, \pi)$ $(\pi, 2\pi)$ $(0, \pi)$

$x - y = u$

$x + y = v$

$$J = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

$$= \iint u^2 \sin^2(v) \times 2 \times du \times dv$$



$\pi \le \underbrace{x+y}_{v} \le 3\pi$

$-\pi \le \underbrace{x-y}_{u} \le \pi$

Now substitute and solve

II 1. (a) $\nabla(fg) = \nabla\left(f(x,y,z)\,g(x,y,z)\right)$

$$\frac{\partial(fg)}{\partial x} = \frac{\partial f}{\partial x} \times g + f \times \frac{\partial g}{\partial x} \qquad \{\text{product rule}\}$$

Similarly, $\dfrac{\partial(fg)}{\partial y} \quad \dfrac{\partial(fg)}{\partial z}$

$$\nabla(fg) = \left(\frac{\partial(fg)}{\partial x} \quad \frac{\partial(fg)}{\partial y} \quad \frac{\partial(fg)}{\partial z}\right)$$

$$= f\nabla g + g\nabla f$$

2) (iii)

$$\mathbf{b}\cdot\nabla\left(\mathbf{a}\cdot\nabla\left(\frac{1}{r}\right)\right)=\frac{3(\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r})}{r^5}-\frac{\mathbf{a}\cdot\mathbf{b}}{r^3}.$$

$$\nabla\left(\frac{1}{r}\right) = \nabla\left(r^{-1}\right) = \nabla\left((x^2+y^2+z^2)^{-1/2}\right)$$

$$\frac{\partial}{\partial x} = (x^2+y^2+z^2)^{-3/2} \times \cancel{2x} \times \frac{-1}{2} = -x\, r^{-3}$$

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} = \left(-x\hat{i} - y\hat{j} - z\hat{k}\right) r^{-3}$$

$$= r^{-3} \times \vec{r}$$

$$a\cdot\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^3}\left(\vec{a}\cdot\vec{r}\right) = -\frac{1}{r^3}\left(a_1x + a_2y + a_3z\right)$$

$$\nabla\left(a\cdot\nabla\left(\frac{1}{r}\right)\right) = -\nabla\left(\frac{a_1x + a_2y + a_3z}{r^3}\right)$$

$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$$

$$\nabla\left(\frac{a_1x + a_2y + a_3z}{r^3}\right) = -\left(\frac{r^3(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) - (3r)\vec{r}\ (\vec{a}\cdot\vec{r})}{r^6}\right)$$

$$= -\left(\frac{r^2(\vec{a}) - 3r(\vec{a}\cdot\vec{r})\vec{r}}{r^6}\right)$$

$$\vec{b}\cdot\nabla\left(\vec{a}\cdot\nabla\left(\frac{1}{r}\right)\right) = -\left(\frac{\vec{a}\cdot\vec{b}}{r^3} - \frac{3}{r^5}(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})\right)$$

Proved !

3. Calculate the line integral of the vector field

$$\mathbf{F}(x,y) = (x^2 - 2xy)\mathbf{i} + (y^2 - 2xy)\mathbf{j}$$

from $(-1,1)$ to $(1,1)$ along $y = x^2$.

4. Calculate the line integral of

$$\mathbf{F}(x,y) = (x^2 + y^2)\mathbf{i} + (x - y)\mathbf{j}$$

once around the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ in the counter clockwise direction.

Remark Often line integral of a vector field $\mathbf{F}$ along a 'geometric curve' C is represented by $\int_C \mathbf{F}.\mathbf{ds}$. A geometric curve C is a set of points in the plane or in the space that can be traversed by a parametrized path in the given direction.
To evaluate $\int_C \mathbf{F}.\mathbf{ds}$, choose a convenient parametrization $\mathbf{c}$ of C traversing C in the given direction and then

$$\int_C \mathbf{F}.\mathbf{ds} := \int_{\mathbf{c}} \mathbf{F}.\mathbf{ds}.$$

'$\oint_C$' means the line integral over a closed curve C.

5. Calculate the value of the line integral

$$\oint_C \frac{(x+y)dx - (x-y)dy}{x^2 + y^2}$$

similar to Q4

$x = a\cos\theta$ $\theta = 0$ to 2π

$y = a\sin\theta$

where C is the curve $x^2 + y^2 = a^2$ traversed once in the counter clockwise direction.

6. Calculate

$$\oint_C ydx + zdy + xdz$$

where C is the intersection of two surfaces $z = xy$ and $x^2 + y^2 = 1$ traversed once in a direction that appears counter clockwise when viewed from high above the xy-plane.

7. Let the curve C be given by $x^2 + y^2 = 1, z = 0$. Let $\mathbf{c}_1$ be a parametrization defined by $\mathbf{c}_1(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$. Find the line integral of $\mathbf{F}(x,y,z) = -y\mathbf{i} + x\mathbf{j}$ along this curve. Also find the line integral along the curve parametrized by $\mathbf{c}_2(t) = (\cos t, -\sin t)$, for $t \in [0, \pi]$.

8. Show that a constant force field does zero work on a particle that moves once uniformly around the circle: $x^2 + y^2 = 1$. Is this also true for a force field $\mathbf{F}(x,y,z) = \alpha(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$, for some constant α.

9. Let $C : x^2 + y^2 = 1$. Find

$$\oint_C \text{grad } (x^2 - y^2) \cdot \mathbf{ds}.$$

10. Evaluate

$$\int_C \text{grad } (x^2 - y^2) \cdot \mathbf{ds},$$

where C is $y = x^3$, joining $(0,0)$ and $(2,8)$.

11. Compute the line integral

$$\oint_C \frac{dx + dy}{|x| + |y|}$$

for the path given $= 1$ Now easy

where C is the square with vertices $(1,0), (0,1), (-1,0)$ and $(0,-1)$ traversed once in the counter clockwise direction.

12. A force $F = xy\mathbf{i} + x^6y^2\mathbf{j}$ moves a particle from $(0,0)$ onto the line $x = 1$ along $y = ax^b$ where $a, b > 0$. If the work done is independent of b find the value of a.

12 a) $\vec{F} = xy\,\hat{i} + x^6 y^2\,\hat{j}$

$x = \tau$

$y = a\tau^b \quad 0 < \tau \le 1$

$c(\tau) = (x, y)$

$c'(\tau) = (1, ab\tau^{b-1})$

$$\int_0^1 \vec{F} \cdot \vec{c}\,'(\tau)\, d\tau$$

$$= \int_0^1 a\tau^{b+1} + \tau^6 \times a^2 \tau^{2b} \times ab \times \tau^{b-1}$$

$$= \frac{a}{b+2} + \frac{a^3 b}{3b+6} = \frac{a^3 b + 3a}{(3b+6)}$$

$$= \frac{a(a^2 b + 3)}{3b+6}$$

$$= \frac{a\left(a^2(b+2) - 2a^2 + 3\right)}{3b+6}$$

$$= \frac{a^3}{3} + \underbrace{\frac{(-2a^2 + 3)}{3b+6}}_{= 0} \Rightarrow a = \sqrt{\frac{3}{2}}$$

$(a > 0)$ given

or set $\frac{\partial}{\partial b} = 0$

6b

$x^2+y^2=1$ ⟶ both satisfy ⟵ $z=xy$

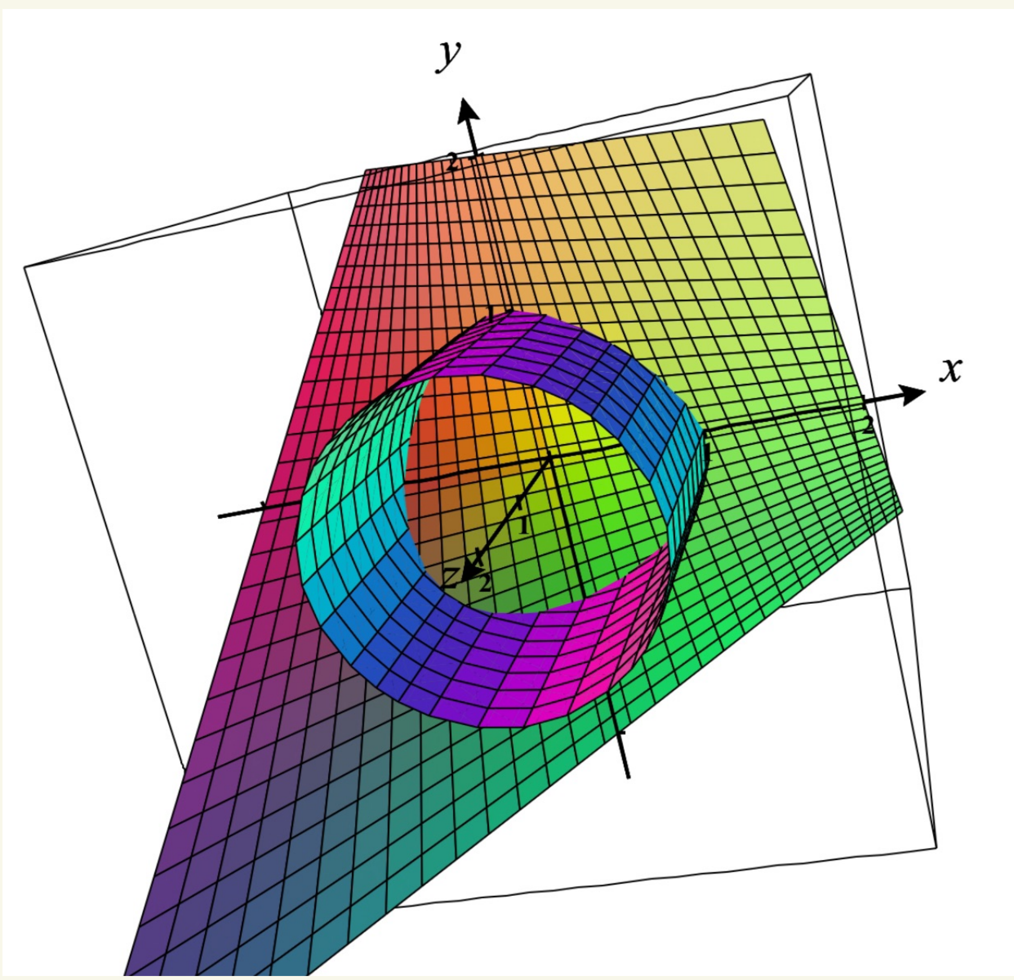
$x=\cos t$ $\qquad$ $z=xy$

$y=\sin t$ $\qquad$ $z=\cos t\sin t$

$$\oint_C y\,dx+z\,dy+x\,dz$$

$$=\int_0^{2\pi} \underbrace{\sin t\,(-\sin t)}_{\text{Average}=-\frac{1}{2}}+\underbrace{(\cos t\sin t)(\cos t)}_{0}+\cos t\,(\underbrace{\cos^2 t}_{0}-\underbrace{\sin^2 t}_{0})\,dt$$

$$=\ -\pi$$



4b $\vec{F}(x,y) = (x^2+y^2)\,\hat{i} + (x-y)\,\hat{j}$

$b^2x^2 + a^2y^2 = a^2b^2$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\quad x = a\cos\theta$

$y = b\sin\theta$

$c(\theta) = (x,y) = (a\cos\theta,\ b\sin\theta)$

$c'(\theta) = (-a\sin\theta,\ b\cos\theta)$

Line integral $= \int \vec{F}\cdot\vec{c}'\,.d\theta$

$$= \int_0^{2\pi} \Big((a^2\cos^2\theta + b^2\sin^2\theta)(-a\sin\theta) + (b\cos\theta)(a\cos\theta - b\sin\theta)\Big)\,d\theta$$

$$= \int_0^{2\pi} \Big(\underbrace{-a^3\cos^2\theta\,\sin\theta - ab^2\sin^3\theta}_{0 \text{ for complete interval}} + \underbrace{ab\cos^2\theta}_{\substack{\text{average value} = \frac{1}{2} \\ \frac{1}{2}\times ab\times 2\pi \\ = \pi ab}} - \underbrace{ab\cos\theta\sin\theta}_{\text{will repeat and get } 0.}\Big)\,d\theta$$

$= \pi ab$

MA 105 Tutorial Sheet 4 :
Line integrals and conservative fields October 24, 2023

1. Determine whether or not the given set is $a)$ open, $b)$ path-connected, and $c)$ simply-connected.

   (a) $D = \{(x, y) \in \mathbb{R}^2 \mid 0 < y < 3\}$, OPS
   (b) $D = \{(x, y) \in \mathbb{R}^2 \mid 1 < |x| < 2\}$, O
   (c) $D = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 4, \quad y \geq 0\}$, PS
   (d) $D = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (1, 4)\}$. OP

2. Determine whether or not the vector field $\mathbf{F}(x, y) = 3xy\mathbf{i} + x^3y\mathbf{j}$ is a gradient on any open subset of $\mathbb{R}^2$.

3. Show that the line integral is path-independent and evaluate the integral:

$$\int_C 2xe^{-y}\, dx + (2y - x^2e^{-y})\, dy$$

   where C is any path from $(1, 0)$ to $(2, 1)$.

4. Is the line integral $\int_C ydx + xdy + xyzdz$ path-independent in $\mathbb{R}^3$?

5. Let $\mathbf{F} = \nabla f$, where $f(x, y) = \sin(x - 2y)$. Find curves C_1 and C_2 that are not closed and satisfy

$$\int_{C_1} \mathbf{F}.\mathbf{ds} = 0, \quad \int_{C_2} \mathbf{F}.\mathbf{ds} = 1.$$

6. Determine whether or not $\mathbf{F}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

   (a) $\mathbf{F}(x, y) = y^2e^{xy}\mathbf{i} + (1 + xy)e^{xy}\mathbf{j}$, for all $(x, y) \in \mathbb{R}^2$.
   (b) $\mathbf{F}(x, y) = (ye^x + \sin y)\mathbf{i} + (e^x + x\cos y)\mathbf{j}$, for all $(x, y) \in \mathbb{R}^2$.
   (c) $\mathbf{F}(x, y) = (2xy + y^{-2})\mathbf{i} + (x^2 - 2xy^{-3})\mathbf{j}$, for all $(x, y) \in \mathbb{R}^2$ and $y > 0$.

7. Let $\mathbf{F}$ be a vector field on $\mathbb{R}^2$. Find a function f such that $\mathbf{F} = \operatorname{grad} f$ and using it evaluate $\int_{\mathbf{c}} \mathbf{F} \cdot \mathbf{ds}$, where $\mathbf{F}$ and $\mathbf{c}$ are given below:

   (a) $\mathbf{F}(x, y, z) = (2xyz + \sin x)\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$ and $\mathbf{c}(t) = (\cos^5 t, \sin^3 t, t^4)$, $0 \leq t \leq \pi$.
   (b) $\mathbf{F}(x, y) = (1 + xy)e^{xy}\mathbf{i} + x^2e^{xy}\mathbf{j}$ and $\mathbf{c}(t) = (\cos t, 2\sin t)$, $0 \leq t \leq \frac{\pi}{2}$.
   (c) $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + (xy + 2z)\mathbf{k}$ and $\mathbf{c}$ is the line segment from $(1, 0, -2)$ to $(4, 6, 3)$.

8. For $\mathbf{v} = (2xy + z^3)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$, show that $\nabla\phi = \mathbf{v}$ for some ϕ and hence calculate $\oint_C \mathbf{v} \cdot d\underline{s}$ where C is any arbitrary smooth closed curve.

9. Let $S = \mathbb{R}^2 \setminus \{(0, 0)\}$. Let

$$\mathbf{F}(x, y) = -\frac{y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j} := F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}.$$

   (a) Show that $\frac{\partial}{\partial y}F_1(x, y) = \frac{\partial}{\partial x}F_2(x, y)$ on S.
   (b) Compute $\int_C \mathbf{F}.\mathbf{ds}$ where C is the circle: $x^2 + y^2 = 1$.

9b $\bar{F} = \dfrac{-y}{x^2+y^2}\,\hat{i} + \dfrac{x}{x^2+y^2}\,\hat{j} = f_1(x,y)\,\hat{i} + f_2(x,y)\,\hat{j}$

(a) $\dfrac{\partial f_1}{\partial y} = \dfrac{-(x^2+y^2) - 2y(-y)}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$

$\dfrac{\partial f_2}{\partial x} = \dfrac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$

(b) $C =$ circle $x^2+y^2=1$

$x = \cos\theta \quad y = \sin\theta$

$c(\theta) = (\cos\theta, \sin\theta)$

$c'(\theta) = (-\sin\theta, \cos\theta)$

$\int_0^{2\pi} \bar{F}\cdot c'(\theta)\,d\theta = \int_0^{2\pi} \left(-\sin\theta(-\sin\theta) + \cos^2\theta\right) = \int_0^{2\pi} 1$

$= 2\pi$

$\neq 0$

(c) Since the line integral on a closed path is not zero, $\bar{F}$ is not conservative

[This is because region S is not simply connected, hence the theorem is no longer valid].

6. (c) $\bar{F}(x,y) = (2xy + y^{-2})\hat{i} + (x^2 - 2xy^{-3})\hat{j}$ $\quad \forall (x,y) \in \mathbb{R}^2 \; y > 0$

This region is open and simply connected

$\Rightarrow$ If $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ $\Rightarrow \bar{F}$ = Conservative

$$\frac{\partial f_1}{\partial y} = 2x - \frac{2}{y^3}$$

$$\frac{\partial f_2}{\partial x} = 2x - \frac{2}{y^3}$$

$\xrightarrow{\text{equal}}$ $\bar{F}$ = Conservative

Now we need to find f such that $\nabla f = \bar{F}$

$$\frac{\partial f}{\partial x} = f_1 = 2xy + y^{-2}$$

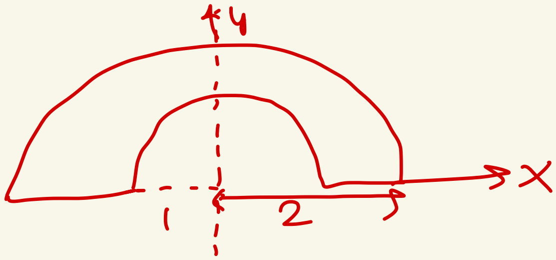
$$\Rightarrow f = x^2 y + xy^{-2} + g(y)$$

$$\frac{\partial f}{\partial y} = x^2 - 2xy^{-3} + g'(y) = f_2$$

$$\Rightarrow g'(y) = 0 \Rightarrow g(y) = C$$

$$f = x^2 y + xy^{-2} + C$$

1) (c)



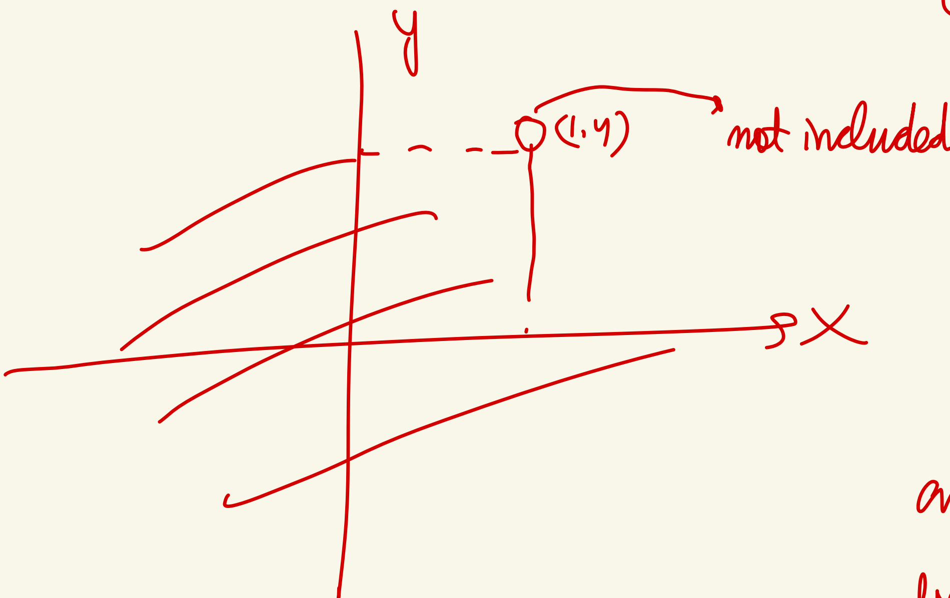
boundary is included → not open

any two points can be joined by a path → Path connected

Any simple closed curve lying in D encloses a region that is in D → Simply connected

(no holes, no separate pieces)

(d)



No boundary → Open ✓

any two points can be joined by a path → Path connected ✓

Curve surrounding (1,4) does not enclose a region in the given set because $(1,4) \notin$ Set

⇒ Not Simply connected

3b $\vec{F} = 2xe^{-y}\,\hat{i} + (2y - x^2e^{-y})\hat{j}$

If we prove $\vec{F} = \nabla f \Rightarrow$ $\frac{\partial f}{\partial x} = 2xe^{-y}$

$\frac{\partial f}{\partial y} = 2y - x^2e^{-y}$

$f = x^2e^{-y} + g(y)$

$\frac{\partial f}{\partial y} = -x^2e^{-y} + g'(y) = 2y - x^2e^{-y}$

$\Rightarrow 2y = g'(y) \Rightarrow g(y) = y^2 + C$

$\Rightarrow f = x^2e^{-y} + y^2 + C$

To evaluate the line integral just do

$f((2,1)) - f((1,0))$

(c) Is $\mathbf{F}$ a conservative field on S?

10. A radial force field is one which can be expressed as $\mathbf{F}(x, y, z) = f(r)\mathbf{r}$ where $\mathbf{r} = (\mathbf{x}, \mathbf{y}, \mathbf{z})$ is the position vector and $r = \|\mathbf{r}\|$. Show that, if f is continuous, $\mathbf{F}$ is conservative in $\mathbb{R}^3$.

(Hint. Try to guess what the potential function could be, provided f is continuous.)

10) Potential $= \phi(r)$

such that $\nabla\phi(r) = \bar{F}(x,y,z)$

$$\frac{\partial \phi(r)}{\partial x} = \frac{\partial}{\partial r}\left(\phi(r)\right)\frac{\partial r}{\partial x}$$

$$= \phi'(r) \times \frac{\partial r}{\partial x} = \phi'(r) \times \frac{x}{r}$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

$$\Rightarrow \nabla\phi = \frac{\phi'(r)}{r}(\vec{r}) = f(r)\vec{r}$$

$$\Rightarrow \phi'(r) = r f(r)$$

$$\Rightarrow \phi(r) = \phi(a) + \int_a^r c f(c)\, dc$$

MA 105 Part II Tutorial Sheet 5 : Green's theorem, October 30, 2023

1. Verify Green's theorem in each of the following cases:

   (i) $F_1(x, y) = -xy$; $F_2(x, y) = xy$; $R: x \geq 0, 0 \leq y \leq 1 - x^2$;

   (ii) $F_1(x, y) = 2xy$; $F_2(x, y) = e^x + x^2$; where R is the triangle with vertices $(0, 0), (1, 0)$, and $(1, 1)$.

2. Use Green's theorem to evaluate the integral $\oint_{\partial R} y^2\, dx + x\, dy$, where $= (1-2y)\,dx\,dy$

   (i) R is the square with vertices $(0, 0), (2, 0), (2, 2), (0, 2)$.

   (ii) R is the square with vertices $(\pm 1, \pm 1)$.

   (iii) R is the disc of radius 2 and center $(0, 0)$ oriented clock-wise. → Use polar coordinates

3. For a simple closed curve given in polar coordinates show using Green's theorem that the area enclosed is given by
$$A = \frac{1}{2} \oint_C r^2 d\theta.$$
Use this to compute the area enclosed by the following curves:

   (i) The cardioid: $r = a(1 - \cos\theta)$, $0 \leq \theta \leq 2\pi$; .

   (ii) The lemniscate: $r^2 = a^2 \cos 2\theta$, ; $-\pi/4 \leq \theta \leq \pi/4$.

4. Find the area of the following regions:

   (i) The area lying in the first quadrant of the cardioid $r = a(1 - \cos\theta)$.

   (ii) The region under one arch of the cycloid
$$\mathbf{r} = a(t - \sin t)\mathbf{i} + a(1 - \cos t)\mathbf{j},\ 0 \leq t \leq 2\pi.$$

   (iii) The region bounded by the limacon on
$$r = 1 - 2\cos\theta,\ \ 0 \leq \theta \leq \pi/2$$
   and the two axes.

5. Let $D = \{(x, y) \in \mathbb{R}^2 \mid a^2 \leq x^2 + y^2 \leq b^2\}$, where $0 < a < b$. Evaluate
$$\int_{\partial D} xe^{-y^2} dx + [-x^2 ye^{-y^2} + 1/(x^2 + y^2)]dy,$$
where ∂D is positively oriented.

6. Let C be a simple closed curve in the xy-plane. Show that

$r^2 = x^2 + y^2$

Apply Green's theorem
$$3I_0 = \oint_C x^3 dy - y^3 dx,$$
where $I_0 = \frac{1}{3} \int \int_D r^2\ dxdy$, D is the region enclosed by C. This I_0 is often called 'polar moment of inertia' of the region D.

Sol

5. Let $D = \{(x, y) \in \mathbb{R}^2 \mid a^2 \le x^2 + y^2 \le b^2\}$, where $0 < a < b$. Evaluate
$$\int_{\partial D} xe^{-y^2}dx + [-x^2ye^{-y^2} + 1/(x^2+y^2)]dy,$$
where ∂D is positively oriented.

$$\int_{\partial D}\left(xe^{-y^2}\,dx - x^2 y\, e^{-y^2} dy\right) + \int_{\partial D}\frac{1}{x^2+y^2}$$

$$= \underbrace{\int_{\partial D}\frac{1}{2}\nabla\left(x^2 e^{-y^2}\right)}_{I} + \int_{\partial D}\frac{1}{x^2+y^2}$$

Conservative vector field

integral on closed path $= 0$

arrows reverse because of positive orientation

For inner path: $x = a\cos t$ $y = a\sin t$

$$\int_0^{2\pi}\frac{1}{a^2}(a\cos t)\,dt = 0$$

Similarly outer $= 0$

Total $= 0$

3) Area of a figure $= \frac{1}{2}\oint x\,dy - y\,dx = \iint 1\,dx\,dy =$ Area

Take $x = r(t)\cos\theta(t)$ converting to polar

$y = r(t)\sin\theta(t)$ parametrization with respect to t

$$\frac{dx}{dt} = r'(t)\cos\theta(t) + (-r(t))\sin\theta(t)\,\theta'(t)$$

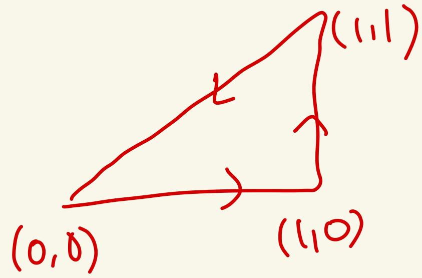
$$\frac{dy}{dt} = r'(t)\sin\theta(t) + r(t)\cos\theta(t)\,\theta'(t)$$

$$\frac{1}{2}\oint r\cos\theta\,(r'\sin\theta + r\cos\theta.\theta') - r\sin\theta\,(r'\cos\theta - r\sin\theta.\theta')$$

$$= \frac{1}{2}\oint r^2(\cos^2\theta + \sin^2\theta)\,\theta'$$

$$= \frac{1}{2}\oint r^2\,d\theta$$

1b (ii) $2xy\,\hat{i} + (e^x + x^2)\,\hat{j}$



$$\oint_{\partial D} \bar{F}\cdot d\bar{r}$$

$$\underbrace{\int_{(0,0)}^{(1,0)} 2xy\,dx + (e^x + x^2)\,dy}_{\substack{y=0\\ dy=0\\ \Rightarrow \text{Integral}=0}} + \int_{(1,0)}^{(1,1)} 2xy\,dx + (e^x+x^2)\,dy + \int_{(1,1)}^{(0,0)} 2xy\,dx + (e^x + x^2)\,dy$$

$x = 1$
$y = t \quad 0 \le t \le 1$
$dy = dt \quad dx = 0$

$$\int_0^1 (e+1)\,dt = e+1$$

$x = y = t$
$t = 1$ to 0
$dx = dy = dt$

$$\int_1^0 (3t^2 + e^t)\,dt$$

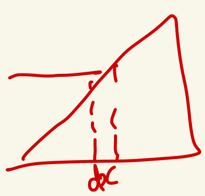
$-1 + 1 - e$

$= 1$

(Green's Theorem)

$$\oint_{\partial D} \bar{F}\cdot d\bar{r} = \iint_D \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy$$

$$= \iint_D (e^x + 2x - 2x)\,dx\,dy = \iint_D e^x\,dx\,dy$$



$0 \le y \le x$

$$\int_0^1 \left(\int_0^x e^x\,dy\right) dx$$

$$= \int_0^1 x e^x\,dx = (x-1)e^x \Big|_0^1$$

$= 1$

$\Rightarrow$ VERIFIED

7. If C is the line segment connecting (x_1, y_1) to the point (x_2, y_2), show that

$$\int_C x\,dy - y\,dx = x_1 y_2 - x_2 y_1.$$

8. Let C be any counterclockwise closed curve in the plane and let $\mathbf{n}$ be the outward unit normal to the curve C. Compute $\displaystyle\oint_C \nabla(x^2 - y^2)\cdot \mathbf{n}ds$.

9. Let D be a region in $\mathbb{R}^2$ with boundary ∂D satisfying the hypothesis stated in the 'Green's theorem'. Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function.

   (i) Show that $\nabla^2\phi = \operatorname{div}(\operatorname{grad}\phi)$, where the operator ∇^2 is defined by

   $$\nabla^2\phi(x, y) = \frac{\partial^2\phi}{\partial^2 x}(x, y) + \frac{\partial^2\phi}{\partial^2 y}(x, y).$$

   The operator ∇^2 is called 'Laplace operator'.

   (ii) Show that the Green's Identity holds:

   $$\iint_D \nabla^2\phi\, d(x, y) = \oint_{\partial D} \frac{\partial\phi}{\partial\mathbf{n}}\, ds,$$

   where $\mathbf{n}$ is the outward unit normal to the curve ∂D .
   (Hint. Use the divergence form of Green's theorem for the vector field $\mathbf{F} = \operatorname{grad}\phi$)

   (iii) Using the above identity, compute

   $$\oint_C \frac{\partial\phi}{\partial\mathbf{n}}\, ds$$

   for $\phi = e^x \sin y$, and D the triangle with vertices $(0, 0), (4, 2), (0, 2)$.

10. Let us consider the region $\Omega = \{(x, y) \mid x^2 + y^2 > 1\}$ and the vector field be defined on Ω. Evaluate the following line integrals where the loops are traced in the counter clockwise sense

   (i)

   $$\oint_C \frac{y\,dx - x\,dy}{x^2 + y^2}$$

   where C is any simple closed curve in Ω enclosing the origin.

   (ii)

   $$\oint_C \frac{y\,dx - x\,dy}{x^2 + y^2}$$

   where C is any simple closed curve in Ω not enclosing the origin.

   (iii) Let C be a smooth simple closed curve lying in Ω. Find

   $$\oint_C \frac{\partial(\ln r)}{\partial y}dx - \frac{\partial(\ln r)}{\partial x}dy.$$

11. Is there a vector field $\mathbf{G}$ in $\mathbb{R}^3$ such that

   i) curl $\mathbf{G}(x, y, z) = (x\sin y)\mathbf{i} + (cosy)\mathbf{j} + (z - xy)\mathbf{k}$.

   ii) curl $\mathbf{G}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

$\nabla\cdot(\nabla\times F) = 0$

$\Rightarrow$ Calculate divergence

If $\neq 0 \to$ No

9b (ii) $\displaystyle\int_{\partial D} \bar{F}\cdot\bar{n}\, ds = \iint_D \nabla\cdot\bar{F}\, dx\,dy$

$$\bar{F} = \nabla\phi$$

$$\left[\nabla\cdot\nabla(\phi) = \nabla^2(\phi)\right]$$

$$\frac{\partial\phi}{\partial n} = \nabla\phi\cdot\vec{n} \quad \text{(directional derivative rule)}$$

Hence the result written is a form of Green's Theorem

8b)

$$\int_{\partial D} \bar{F}\cdot\bar{n}\, ds = \iint_D \nabla\cdot\bar{F}\, dx\, dy$$

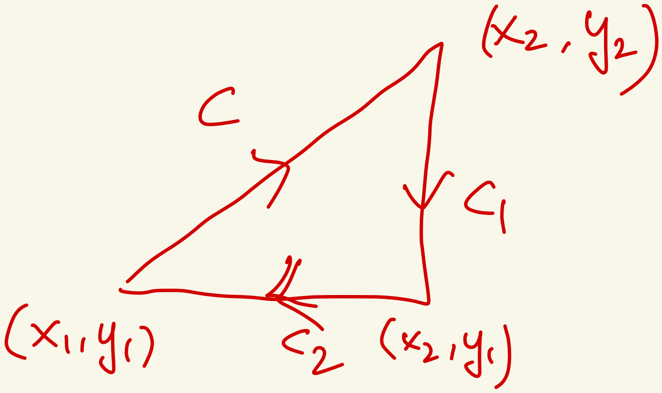
$$\bar{F} = 2x\,\hat{i} - 2y\,\hat{j}$$

$$\nabla\cdot\bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}$$

$$= 2 - 2 = 0$$

$$\Rightarrow RHS = 0 \Rightarrow LHS = 0$$

Q)



$$\oint_{\partial D} x\,dy - y\,dx = -2\times \text{Area}$$

because direction is clockwise

$$C + C_1 + C_2 = -2\times\text{Area}$$
$$= -2\times\frac{1}{2}\times(y_2-y_1)\times(x_2-x_1)$$
$$= -(y_2-y_1)(x_2-x_1)$$

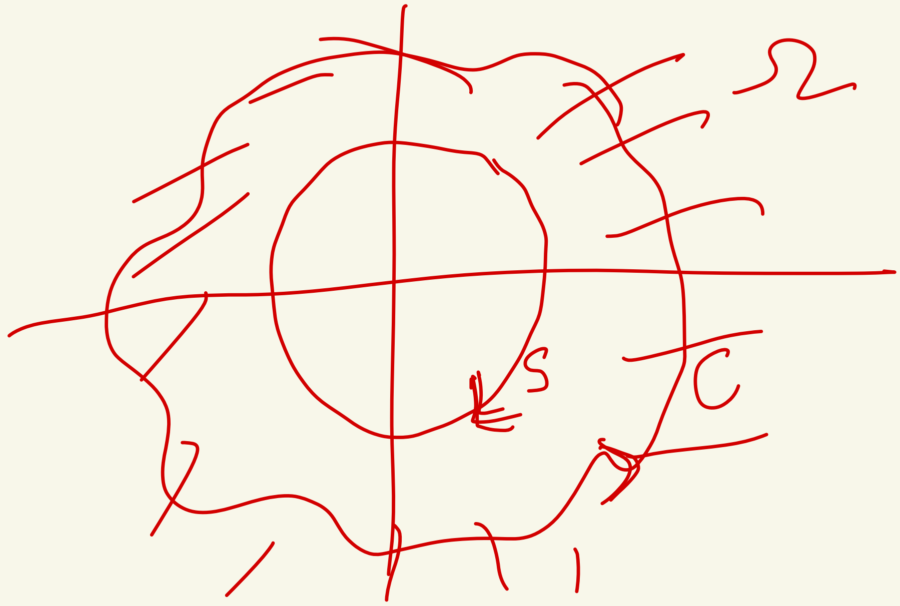
$$C_1 \to \quad dx=0 \quad x=x_2 \quad x_2(y_1-y_2)$$
$$C_2 \to \quad dy=0 \quad y=y_1 \quad y_1(x_1-x_2)$$

$$C \to -(y_2-y_1)(x_2-x_1) - x_2(y_1-y_2) + y_1(x_1-x_2)$$

$$= \cancel{-x_2y_2} + \cancel{x_2y_1} + x_1y_2 - \cancel{x_1y_1} - \cancel{x_2y_1} + \cancel{x_2y_2} + \cancel{x_1y_1} - x_2y_1$$

$$= x_1y_2 - x_2y_1$$

10b



S = curve circle

C = another curve enclosing origin.

$$\oint_{C \cup S} \bar{F} \cdot ds = 0$$ [because of Green's in an open set]

$$\oint_{S} \bar{F} \cdot d\bar{s} = 2\pi$$ [Calculated before]

$$\Rightarrow \oint_{C} \bar{F} \cdot ds = -2\pi$$

(ii) If C does not enclose origin we can directly apply Green's theorem [simply connected]

→ Area = 0

(iii) $\ln r = \ln\left(\sqrt{x^2+y^2}\right)$ → Make cases

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(x) & g(y) & h(z) \end{vmatrix} = 0$$

12. Show that any vector field defined of the form

$$\mathbf{F}(x, y, z) = f(x)\,\mathbf{i} + g(y)\,\mathbf{j} + h(z)\,\mathbf{k}, \quad \text{in} \quad \mathbb{R}^3,$$

where f, g, h are differentiable functions, is irrotational, i.e., curl $\mathbf{F} = 0$.

13. Show that any vector field defined of the form

$$\mathbf{F}(x, y, z) = f(y, z)\,\mathbf{i} + g(x, z)\,\mathbf{j} + h(x, y)\,\mathbf{k}, \quad \text{in} \quad \mathbb{R}^3,$$

where f, g, h are differentiable functions, is incompressible, i.e., div $\mathbf{F} = 0$.

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= 0 + 0 + 0 = 0$$

MA 105 Tutorial Sheet 6 :
Surface integrals, Stokes theorem, Gauss divergence theorem
November 7, 2023

I Surface and surface integrals

1. Find a suitable parametrization $\Phi(u, v)$ and the normal vector $\Phi_u \times \Phi_v$ for the following surface:
   (i) The plane $x - y + 2z + 4 = 0$.
   (ii) The right circular cylinder $y^2 + z^2 = a^2$.
2. Find the tangent plane to the surface with parametric equations $x = u^2$, $y = v^2$ and $z = u + 2v$ at the point $(1, 1, 3)$.
3. Compute the surface area of that portion of the sphere $x^2 + y^2 + z^2 = a^2$ which lies within the cylinder $x^2 + y^2 = ay$, where $a > 0$.
4. Compute the area of that portion of the paraboloid $x^2 + z^2 = 2ay$ which is between the planes $y = 0$ and $y = a$.
5. Let S denote the plane surface whose boundary is the triangle with vertices at $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, and let $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Let $\mathbf{n}$ denote the unit normal to S having a nonnegative z-component. Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$.

1. Application of Stokes theorem

1. Consider the vector field $\mathbf{F} = (x - y)\mathbf{i} + (x + z)\mathbf{j} + (y + z)\mathbf{k}$. Verify Stokes theorem for $\mathbf{F}$ where S is the surface of the cone: $z^2 = x^2 + y^2$ intercepted by
   (a) $x^2 + (y - a)^2 + z^2 = a^2 : z \geq 0$ (b) $x^2 + (y - a)^2 = a^2$
2. Using Stokes Theorem, evaluate the line integral
$$\oint_C yz\, dx + xz\, dy + xy\, dz$$
   where C is the curve of intersection of $x^2 + 9y^2 = 9$ and $z = y^2 + 1$ with clockwise orientation when viewed from the origin.
3. Find the integral of $\mathbf{F}(x, y, z) = z\mathbf{i} - x\mathbf{j} - y\mathbf{k}$ around the triangle with vertices $(0, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 2)$.
4. Let C be the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$. Let C be oriented so that when it is projected onto the xy-plane the resulting curve is traversed counterclockwise. Evaluate
$$\int_C -y^3 dx + x^3 dy - z^3 dz.$$
5. Let $\boldsymbol{F}(x, y, z) := (y, -x, e^{xz})$ for $(x, y, z) \in \mathbb{R}^3$, and let $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z - \sqrt{3})^2 = 4 \text{ and } z \geq 0\}$, be oriented by the outward unit normal vectors. Find $\iint_S (\text{curl}\, \boldsymbol{F}) \cdot d\boldsymbol{S}$.

III Application of Gauss divergence theorem

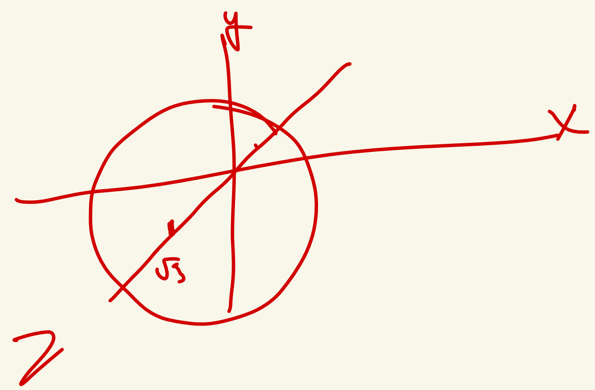
1. Calculate the flux of $\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ through the unit sphere.

$\iiint_{\text{volume}} \nabla \cdot F \, dV \leftarrow$ use polar coordinates

5. Let $\boldsymbol{F}(x, y, z) := (y, -x, e^{xz})$ for $(x, y, z) \in \mathbb{R}^3$, and let $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z - \sqrt{3})^2 = 4 \text{ and } z \geq 0\}$, be oriented by the outward unit normal vectors. Find $\iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S}$.

5b

$x^2 + y^2 + (z - \sqrt{3})^2 = 4 \qquad z \geq 0$



Just see the projection on xy plane for ∂S

$\Rightarrow \underbrace{x^2 + y^2 = 1}_{\partial S \text{ boundary}}$

$$\iint_S (\bar{\nabla} \times \bar{f}) \cdot ds = \int \bar{F} \cdot ds$$

$$\bar{F} = \left(\sin\theta, -\cos\theta, e^{\cos\theta \times 0} \right)$$

$$ds = (-\sin\theta, \cos\theta)$$

$$\bar{F} \cdot ds = \int_0^{2\pi} -1 = -2\pi$$

Application of stoke's theorem

2)

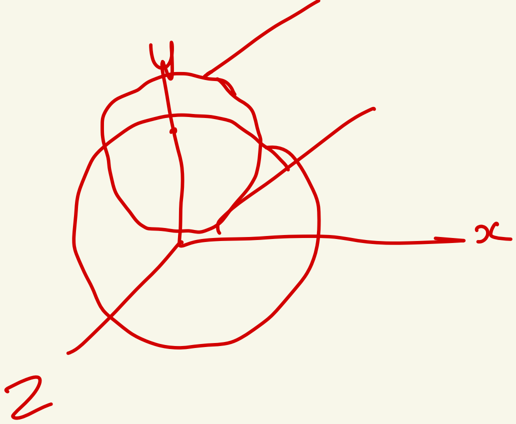
$$\int_{\partial S} \bar{F} \cdot ds = \iint_S (\nabla \times \bar{F})\, dS$$

$$\bar{F} = yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k}$$

$$\nabla \times \bar{F} = 0$$

$\Rightarrow$ Integral $= 0$

3b



sphere $\Rightarrow$ $x = u \quad y = v$

$$z = \sqrt{a^2 - x^2 - y^2} = \sqrt{a^2 - u^2 - v^2} = f(u,v)$$

$$f_u = \frac{-u}{\sqrt{a^2-u^2-v^2}}$$

$$f_v = \frac{-v}{\sqrt{a^2-u^2-v^2}}$$

inside the cylinder $\Rightarrow$ $x^2 + y^2 \le ay$

$$\Rightarrow u^2 \le av - v^2$$

$$\Rightarrow -\sqrt{av - v^2} \le u \le \sqrt{av - v^2}$$

$$\iint_S dS = \iint |\phi_u \times \phi_v| \, du\, dv$$

$$= \iint \sqrt{1 + f_u^2 + f_v^2} \; du\, dv$$

$$= \iint \frac{a}{\sqrt{a^2 - u^2 - v^2}} \, du\, dv$$

$u = r\cos\theta$

$v = r\sin\theta$

$|J| = r$

Jacobian

$$= \iint \frac{a}{\sqrt{a^2 - r^2}} |J| \, dr\, d\theta$$

$$= \iint \frac{ar}{\sqrt{a^2 - r^2}} \, dr\, d\theta$$

$$r\cos\theta \le \sqrt{ar\sin\theta - r^2\sin^2\theta}$$

$$r^2\cos^2\theta \le ar\sin\theta - r^2\sin^2\theta$$

$$\Rightarrow 0 \le r \le a\sin\theta$$

$\theta = 0$ to π

Now double the answer you get for $z < 0$

1b (i) $x = u \quad y = v$

$$u - v + 2z + 4 = 0$$

$$z = \frac{v - u - 4}{2}$$

$$\phi_u = \left(1, 0, \frac{-1}{2}\right) \quad \phi_v = \left(0, \quad 1, \frac{1}{2}\right)$$

$$\phi_u \times \phi_v = \left(\frac{1}{2}, \quad \frac{-1}{2}, 1\right)$$

2. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = xy^2\mathbf{i} + x^2y\mathbf{j} + y\mathbf{k}$ and S is the surface of the 'can' W given by $x^2 + y^2 \leq 1$, $-1 \leq z \leq 1$.

3. Evaluate $\int\int_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = xy\mathbf{i} + (y^2 + e^{xz^2})\mathbf{j} + \sin(xy)\mathbf{k}$$

and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$ and $y + z = 2$.

4. Find out the flux of $F = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$ outward through the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, $z = 1$. → calculate surface integral across 3 surfaces

5. Is $\mathbf{F}(x, y, z) = x\mathbf{i} - 2y\mathbf{j} + z\mathbf{k}$ defined in $\mathbb{R}^3$ the curl of a vector filed? If yes, find a vector field $\mathbf{G}$ such that $\mathbf{F} = \text{curl } \mathbf{G}$ in $\mathbb{R}^3$.

$\bar{F} = \nabla \times \bar{G}$

$\nabla \cdot \bar{F} = 1 - 2 + 1 = 0 \Rightarrow \bar{G}$ = will exist

Try with $G = (a_1xy + a_2yz + a_3zx,\ b_1xy + b_2yz + b_3zx,\ c_1xy + c_2yz + c_3zx)$

$\bar{G} = (-yz, 0, xy)$

2)

$$\iint \bar{F} \cdot d\bar{s} = \iiint (\bar{\nabla} \cdot \bar{F})\, dv$$

surface = cylinder

$$\bar{\nabla} \cdot \bar{F} = x^2 + y^2$$

$$\iiint (x^2+y^2)\, dx\, dy\, dz$$

$x = r\cos\theta$ $\quad z = z$

$y = r\sin\theta$ $\quad |J| = r\, dr\, d\theta$

$$\int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{1} r^2 \; r\, dr\, d\theta\, dz$$

$$\frac{1}{4} \times 2\pi \times 2 = \pi$$