# MA 105 Calculus II

## Lecture 10

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1. Parametrized surfaces

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# Recall

Let $D$ be a region in $\mathbb{R}^2$ which satisfy the hypothesis of Green's theorem. With the induced positive orientation on $\partial D$, let $\mathbf{c} : [a, b] \to \mathbb{R}^3$ such that $\mathbf{c}(t) = (x(t), y(t), 0)$ be a non-singular parametrization of $\partial D$. Then the unit tangent to the curve $\mathbf{c}$ and the unit outward normal to the curve are defined as

$$\mathbf{T}(t) = \frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t)\|}, \quad \mathbf{n}(t) = \mathbf{T}(t) \times \mathbf{k}, \quad \forall\, t \in [a, b].$$

- Curl form or Tangential form

$$\int_{\partial D} \mathbf{F}.\mathbf{T} ds = \int\int_D (\text{curl}\mathbf{F}).\mathbf{k} dxdy.$$

- Divergence form or Normal form:

$$\int_{\partial D} \mathbf{F}.\mathbf{n} ds = \int\int_D \text{div}\mathbf{F} dxdy.$$

# Surfaces : Definition

A curve is a "one-dimensional" object. Intuitively, this means that if we want to describe a curve, it should be possible to do so using just one variable or parameter.

To do line integration, we further required some extra properties of the curve - that it should be $\mathcal{C}^1$ and non-singular.

We will now discuss the two dimensional analog, namely, surfaces. In order to describe a surface, which is a two-dimensional object, we clearly need two parameters.

## Definition

Let $D$ be a path connected subset in $\mathbb{R}^2$. A parametrised surface is a continuous function $\mathbf{\Phi} : D \to \mathbb{R}^3$.

This definition is the analogue of what we called paths in one dimension and what are often called parametrized curves.

# Geometric parametrised surfaces

As with curves and paths, we will distinguish between the surface $\mathbf{\Phi}$ and its image. Similarly, the image $S = \mathbf{\Phi}(D)$ will be called the geometric surface corresponding to $\mathbf{\Phi}$.

Note that for a given $(u, v) \in D$, $\mathbf{\Phi}(u, v)$ is a vector in $\mathbb{R}^3$. Each of the coordinates of the vector depends on $u$ and $v$. Hence we write

$$\mathbf{\Phi}(u, v) = (x(u, v), y(u, v), z(u, v)),$$

where $x$, $y$ and $z$ are scalar functions on $D$.

The parametrized surface $\mathbf{\Phi}$ is said to be a smooth parametrized surface if the functions $x$, $y$, $z$ have continuous partial derivatives in a open subset of $\mathbb{R}^2$ containing $D$.

# Examples

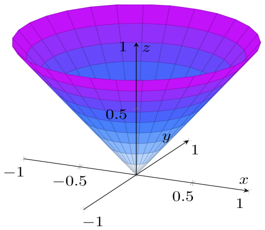
Example 1: Graphs of real valued functions of two independent variables are parametrised surfaces.
Let $f(x, y)$ be a scalar function and let $z = f(x, y)$, for all $(x, y) \in D$, where $D$ is a path connected region in $\mathbb{R}^2$. We can define the parametrised surface $\mathbf{\Phi}$ by

$$\mathbf{\Phi}(u, v) = (u, v, f(u, v)), \quad \forall\,(u, v) \in D.$$

More specifically, we have $x(u, v) = u$, $y(u, v) = v$ and $z(u, v) = f(u, v)$.

Example 2: Consider the cylinder, $x^2 + y^2 = a^2$. Then this is parametrized surface defined by $\mathbf{\Phi} : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}^3$ defined as
$\mathbf{\Phi}(u, v) = (a \cos u, a \sin u, v)$.

Example 3: Consider the sphere of radius $a$,
$S = \{(x, y, z) \mid x^2 + y^2 + z^2 = a^2\}$. Is it a parametrized surface? Recall using spherical coordinates we can represent it using the following parametrization, $\mathbf{\Phi} : [0, 2\pi] \times [0, \pi] \to \mathbb{R}^3$ defined as
$\mathbf{\Phi}(u, v) = (a \cos u \sin v, a \sin u \sin v, a \cos v)$.



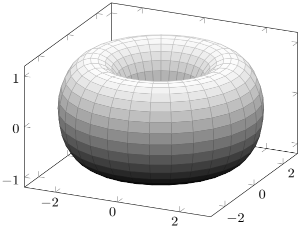

Example 4: The graph of $z = \sqrt{x^2 + y^2}$ can also be parametrized. We use the idea that at each value of $z$ we get a circle of radius $z$. We can describe the cone as the parametrized surface $\mathbf{\Phi} : [0, \infty) \times [0, 2\pi] \to \mathbb{R}^3$ as $\mathbf{\Phi}(u, v) = (u \cos v, u \sin v, u)$.

Example 5: If we have parametrized curve on the $z$-$y$-plane $(0, y(u), z(u))$ which we rotate around $z$-axis, we can parametrise it as follows:

$$x = y(u) \cos v, \quad y = y(u) \sin v, \quad \text{and} \quad z = z(u).$$

Here $a \le u \le b$ if $[a, b]$ is the domain of the curve, and $0 \le v \le 2\pi$.

# Surfaces of revolution around the $z$-axis




For instance we can parametrize a torus by taking a circle in the $y$-$z$ plane with center $(0, a, 0)$ of radius $b$. This is given by the curve $(0, a + b\cos u, b\sin u)$.

Then the parametrization of the torus is then $\mathbf{\Phi}(u, v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u)$ where $0 \leq u \leq 2\pi$ and $0 \leq v \leq 2\pi$.

Parametrised surfaces are more general than graphs of functions.

# Tangent vectors for a parametrised surface

Let $\mathbf{\Phi}(u, v)$ be a smooth parametrised surface. If we fix the variable $v$, say $v = v_0$, we obtain a curve $\mathbf{c}(u, v_0)$ that lies on the surface. Thus

$$\mathbf{c}(u) = x(u, v_0)\mathbf{i} + y(u, v_0)\mathbf{j} + z(u, v_0)\mathbf{k}.$$

Since this curve is $C^1$ we can talk about its tangent vector at the point $u_0$. This is given by

$$\mathbf{c}'(u_0) = \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}.$$

We can *define* the partial derivative of a vector valued function as

$$\mathbf{\Phi}_u(u_0, v_0) = \frac{\partial \mathbf{\Phi}}{\partial u}(u_0, v_0) := \mathbf{c}'(u_0).$$

Similarly, by fixing $u$ and varying $v$ we obtain a curve $\mathbf{l}(u_0, v)$ and we can set

$$\mathbf{\Phi}_v(u_0, v_0) = \frac{\partial \mathbf{\Phi}}{\partial v}(u_0, v_0) := \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}.$$

# The tangent plane

Let for any given point on the surface, $P_0 = (x_0, y_0, z_0) := \mathbf{\Phi}(u_0, v_0)$ for some $(u_0, v_0) \in D$.

The two tangent vectors $\mathbf{\Phi}_u(u_0, v_0)$ and $\mathbf{\Phi}_v(u_0, v_0)$ at $P_0$ define a plane. We call this plane as the tangent plane to the surface at $P_0$.

The normal to this plane at $P_0$, $\mathbf{n}(u_0, v_0) = \mathbf{\Phi}_u(u_0, v_0) \times \mathbf{\Phi}_v(u_0, v_0)$.

Thus for a given point $(x_0, y_0, z_0) = \mathbf{\Phi}(u_0, v_0)$ in $\mathbb{R}^3$ the equation of the tangent plane is given by

$$\mathbf{n}(u_0, v_0) \cdot (x - x_0, y - y_0, z - z_0) = 0,$$

provided $\mathbf{n} \neq 0$.

In particular, if $\mathbf{\Phi}_u(u_0, v_0) \times \mathbf{\Phi}_v(u_0, v_0) = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$, then the equation of the tangent plane at $P_0$ is given by

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Let us find the equation of the tangent plane at points on the various parametrised surfaces we have already looked at.

Example 1: Let $D$ be a path-connected subset of $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ be a $C^1$ function. The surface given by the graph of the function $z = f(x, y)$ is parametrized by $\mathbf{\Phi}(x, y) = (x, y, f(x, y))$. In this case, at $P_0 = \mathbf{\Phi}(x_0, y_0)$ for $(x_0, y_0) \in D$,

$$\mathbf{\Phi}_x(x_0, y_0) = \mathbf{i} + \frac{\partial f}{\partial x}(x_0, y_0)\mathbf{k} \quad \text{and} \quad \mathbf{\Phi}_y(x_0, y_0) = \mathbf{j} + \frac{\partial f}{\partial y}(x_0, y_0)\mathbf{k}.$$

Hence,

$$\mathbf{n}(x_0, y_0) = \mathbf{\Phi}_x(x_0, y_0) \times \mathbf{\Phi}_y(x_0, y_0) = \left(-\frac{\partial f}{\partial x}(x_0, y_0), -\frac{\partial f}{\partial y}(x_0, y_0), 1\right).$$

Thus the equation of the tangent plane is

$$(x - x_0, y - y_0, z - z_0) \cdot \left(-\frac{\partial f}{\partial x}(x_0, y_0), -\frac{\partial f}{\partial y}(x_0, y_0), 1\right) = 0;$$

which yields,

$$z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$

# Tangent Plane: Examples

Example 2: Let us consider a cylinder parametrized as

$$\mathbf{\Phi}(u,v) = (a\cos u, a\sin u, v), \quad \forall (u,v) \in [0,2\pi]\times[0,h],$$

where $a > 0$. Then

$$\mathbf{\Phi}_u(u,v)\times\mathbf{\Phi}_v(u,v) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin u & a\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = (a\cos u, a\sin u, 0).$$

Since this is non-zero on $[0,2\pi]\times[0,h]$ for any $h > 0$, we can define the tangent plane to $\mathbf{\Phi}$ at any point $P_0 = (x_0, y_0, z_0) = \mathbf{\Phi}(u_0, v_0)$ as

$$(a\cos u_0, a\sin u_0, 0).(x - x_0, y - y_0, z - z_0) = 0.$$

Now using $(x_0, y_0, z_0) = \mathbf{\Phi}(u_0, v_0) = (a\cos u_0, a\sin u_0, v_0)$, we get the equation for the tangent plane to $\mathbf{\Phi}$ at $P_0$ is

$$(\cos u_0)x + (\sin u_0)y = a.$$

Example 3: The sphere: $x^2 + y^2 + z^2 = a^2$, for some $a > 0$. Let us consider the parametrization

$$\mathbf{\Phi}(u, v) = (a \cos u \sin v, a \sin u \sin v, a \cos v), \quad \forall (u, v) \in [0, 2\pi] \times [0, \pi].$$

Check $\mathbf{\Phi}_u(u, v) \times \mathbf{\Phi}_u(u, v) = (a \sin v)\mathbf{\Phi}(u, v)$, for all $(u, v) \in [0, 2\pi] \times [0, \pi]$.

Note for $(u_0, v_0) \in [0, 2\pi] \times (0, \pi)$, $\mathbf{\Phi}_u(u_0, v_0) \times \mathbf{\Phi}_u(u_0, v_0) \neq (0, 0, 0)$ and the tangent plane at $P_0 = \mathbf{\Phi}(u_0, v_0)$ is

$$(\sin v_0 \cos u_0)x + (\sin v_0 \sin u_0)y + (\cos v_0)z = a.$$

Example 4: This was the example of the right circular cone. The parametric surface was given by

$$\mathbf{\Phi}(u, v) = (u \cos v, u \sin v, u), \quad (u, v) \in [0, \infty) \times [0, 2\pi].$$

In this case we get

$$\mathbf{\Phi}_u(u, v) = \cos v\mathbf{i} + \sin v\mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{\Phi}_v(u, v) = -u \sin v\mathbf{i} + u \cos v\mathbf{j},$$

where $\mathbf{n}(u, v) = \mathbf{\Phi}_u(u, v) \times \mathbf{\Phi}_v(u, v) = (-u \cos v, -u \sin v, u)$.

For any $(u_0, v_0) \in (0, \infty) \times [0, 2\pi]$, $\mathbf{n}(u_0, v_0) \neq (0, 0, 0)$ and the tangent plane check

$$(\cos v_0)x + (\sin v_0)y = z.$$

Note that if $(u, v) = (0, 0)$, then $\mathbf{n}(0, 0) = 0$, so the tangent plane is not defined at the origin. However, it is defined at any other point.

# Non-singular surfaces

In analogy with the situation for curves, we will call $\mathbf{\Phi}$ a regular or non-singular parametrised surface if $\mathbf{\Phi}$ is $C^1$ and $\mathbf{\Phi}_u \times \mathbf{\Phi}_v \neq 0$ at all points.

As we just saw, the right circular cone is not a regular parametrised surface.

For a regular surface parametrized by $\mathbf{\Phi} : D \to \mathbb{R}^3$, the unit normal $\hat{\mathbf{n}}$ to the surface at any point $P_0 = \mathbf{\Phi}(u_0, v_0)$ is defined by

$$\hat{\mathbf{n}}(u_0, v_0) := \frac{\mathbf{\Phi}_u(u_0, v_0) \times \mathbf{\Phi}_v(u_0, v_0)}{\|\mathbf{\Phi}_u(u_0, v_0) \times \mathbf{\Phi}_v(u_0, v_0)\|}.$$

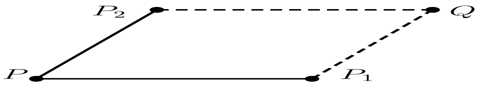
# Surface Area

Let $\mathbf{\Phi} : E \to \mathbb{R}^3$ be a smooth parametrized surface, where $E$ is a path-connected, bounded subset of $\mathbb{R}^2$ having a non-zero area. Also assume $\partial E$, the boundary of $E$, is of content zero.

Let $(u, v) \in E$. For $h, k \in \mathbb{R}$ with $|h|, |k|$ small, assuming $\mathbf{\Phi}$ is $C^1$ we can get the following approximations;

$$P := \mathbf{\Phi}(u, v), \quad P_1 := \mathbf{\Phi}(u + h, v) \approx \mathbf{\Phi}(u, v) + h\, \mathbf{\Phi}_u(u, v),$$

$$P_2 := \mathbf{\Phi}(u, v + k) \approx \mathbf{\Phi}(u, v) + k\, \mathbf{\Phi}_v(u, v), \quad Q := \mathbf{\Phi}(u + h, v + k).$$




Area of the parallelogram with sides $PP_1$ and $PP_2$

$$= \|(P_1 - P) \times (P_2 - P)\| \approx \|\mathbf{\Phi}_u(u, v) \times \mathbf{\Phi}_v(u, v)\|\, |h||k|.$$

In view of this approximation, we define

$$\text{Area}(\mathbf{\Phi}) := \iint_E \|(\mathbf{\Phi}_u \times \mathbf{\Phi}_v)(u,v)\| \, du \, dv.$$

Since the subset $E$ of $\mathbb{R}^2$ is bounded with boundary $\partial E$ which is of content zero and the function $\|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|$ is continuous on $E$, the integral in the definition of Area($\mathbf{\Phi}$) is well-defined.

In analogy with the differential notation $ds = \|\gamma'(t)\| dt$, we introduce the following differential notation:

$$dS = \|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\| \, dudv.$$

Thus Area($\mathbf{\Phi}$) $:= \iint_E dS$.

Examples

• Graph of a function: Given a subset $E$ of $\mathbb{R}^2$ have an area, $f : E \to \mathbb{R}$ be a smooth function, and $\mathbf{\Phi}(u,v) = (u, v, f(u,v))$ for $(u,v) \in E$. Then

$$\begin{aligned} \text{Area}(\mathbf{\Phi}) &= \iint_E \|(-f_u, -f_v, 1)\| dudv \\ &= \iint_E \sqrt{1 + f_u^2 + f_v^2} \, dudv \end{aligned}$$

Example: Let $E := [0, 2\pi] \times [0, h]$, $\mathbf{\Phi}(\theta, z) := (a\cos\theta, a\sin\theta, z)$, and $\mathbf{\Psi}(\theta, z) := (a\cos 2\theta, a\sin 2\theta, z)$ for $(\theta, z) \in E$. Then

$$\begin{aligned} \text{Area}(\mathbf{\Phi}) &= \iint_E \|\mathbf{\Phi}_\theta \times \mathbf{\Phi}_z\| d\theta dz = \iint_E a\, d\theta dz = 2\pi a\, h, \\ \text{Area}(\mathbf{\Psi}) &= \iint_E \|\mathbf{\Psi}_\theta \times \mathbf{\Psi}_z\| d\theta dz = \iint_E 2a\, d\theta dz = 4\pi a\, h. \end{aligned}$$

We note that $\mathbf{\Psi}(E) = \mathbf{\Phi}(E)$, but $\text{Area}(\mathbf{\Psi}) = 2\,\text{Area}(\mathbf{\Phi})$.

Example: Let $E := [0, \pi] \times [0, 2\pi]$, and $\mathbf{\Phi}(\varphi, \theta) = (a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi)$ for $(\varphi, \theta) \in E$. Then

$$\begin{aligned} \text{Area}(\mathbf{\Phi}) &= \iint_E \|\mathbf{\Phi}_\varphi \times \mathbf{\Phi}_\theta\| d\varphi d\theta = \iint_E a^2 \sin\varphi\, d\varphi d\theta \\ &= \int_0^{2\pi} \left( \int_0^{\pi} a^2 \sin\varphi\, d\varphi \right) d\theta = 4\pi a^2. \end{aligned}$$

Let $C$ be a smooth curve in $\mathbb{R}^2 \times \{0\}$ given by $\boldsymbol{\gamma}(t) := (x(t), y(t))$, $t \in [\alpha, \beta]$. If $C$ lies on or above the $x$-axis, and $C$ is revolved about the $x$-axis, then it generates a surface parametrized by

$$\boldsymbol{\Phi}(t,\theta) := (x(t),\ y(t)\cos\theta,\ y(t)\sin\theta) \quad \text{for } (t,\theta) \in E,$$

where $E := [\alpha,\beta]\times[0,2\pi]$. For all $(t,\theta) \in E$,

$$\begin{aligned}(\boldsymbol{\Phi}_t \times \boldsymbol{\Phi}_\theta)(t,\theta) &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x'(t) & y'(t)\cos\theta & y'(t)\sin\theta \\ 0 & -y(t)\sin\theta & y(t)\cos\theta \end{vmatrix} \\ &= (y(t)y'(t), -x'(t)y(t)\cos\theta, -x'(t)y(t)\sin\theta).\end{aligned}$$

By the Fubini theorem, we obtain

$$\begin{aligned}\text{Area}(\boldsymbol{\Phi}) &= \iint_E \sqrt{y(t)^2y'(t)^2 + x'(t)^2y(t)^2}\, d(t,\theta) \\ &= 2\pi\int_\alpha^\beta y(t)\sqrt{x'(t)^2 + y'(t)^2}dt,\end{aligned}$$

Note: $\boldsymbol{\Phi}$ is non-singular $\iff \boldsymbol{\gamma}$ is non-singular and $y(t) \neq 0$ for $t \in [\alpha,\beta]$.

# The area vector of an infinitesimal surface element

We see that $\mathbf{\Phi}$ takes the small rectangle $R$ to the parallelogram given by the vectors $\mathbf{\Phi}_u \Delta u$ and $\mathbf{\Phi}_v \Delta v$.

It follows that the 'area vector' $\Delta \mathbf{S}$ of this parallelogram is

$$\Delta \mathbf{S} = (\mathbf{\Phi}_u \times \mathbf{\Phi}_v) \Delta u \Delta v.$$

Thus the surface 'area vector' is to be thought of as a vector pointing in the direction of the normal to the surface and in differential notation:

$$d\mathbf{S} = (\mathbf{\Phi}_u \times \mathbf{\Phi}_v)\, du\, dv.$$

The magnitude of the surface 'area vector' is given by

$$dS = \|d\mathbf{S}\| = \|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|\, du\, dv.$$

If the parametric surface $\mathbf{\Phi}$ is non-singular, we can write

$$d\mathbf{S} = \hat{\mathbf{n}} dS,$$

where $\hat{\mathbf{n}}$ is the unit vector normal to the surface.

# The magnitude of the area vector

It remains to compute the magnitude $dS$. To do this we must find $\|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|$. Writing this out in terms of $x$, $y$ and $z$, we see that

$$d\mathbf{S} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} dudv.$$

$$dS = \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2} dudv,$$

where $\frac{\partial(y,z)}{\partial(u,v)}$, $\frac{\partial(x,z)}{\partial(u,v)}$, $\frac{\partial(x,y)}{\partial(u,v)}$ are the determinant of corresponding Jacobian matrix. For example

$$\frac{\partial(y,z)}{\partial(u,v)} = \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v},$$

$$\frac{\partial(x,z)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial z}{\partial u}, \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

# The surface area integral

Because of the calculations we have just made, the surface area is given by the double integral

$$\iint_S dS = \iint_E \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2} dudv.$$

The area is nothing but the integral of the constant function 1 on the surface $S$. We integrate any bounded scalar function $f : S \to \mathbb{R}$:

$$\iint_S fdS = \iint_E f(x,y,z)\sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2} dudv,$$

provided the R.H.S double integral exists. If $\Sigma$ is a union of parametrised surfaces $S_i$ that intersect only along their boundary curves, then we can define

$$\iint_\Sigma fdS = \sum_i \iint_{S_i} fdS.$$

# The surface integral of a vector field

Let $\mathbf{F}$ be a bounded vector field (on $\mathbb{R}^3$) such that the domain of $\mathbf{F}$ contains the non-singular parametrised surface $\mathbf{\Phi} : E \to \mathbb{R}^3$. Then the surface integral of $\mathbf{F}$ over $S$ is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} := \iint_E \mathbf{F}(\mathbf{\Phi}(u,v)) \cdot (\mathbf{\Phi}_u \times \mathbf{\Phi}_v) du dv,$$

provided the R.H.S double integral exists. This can also be written more compactly as

$$\iint_S \mathbf{F} \cdot d\mathbf{S} := \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} dS,$$

which is the surface integral of the scalar function given by the normal component of $\mathbf{F}$ over $S$.

## Examples

(i) Let a subset $E$ of $\mathbb{R}^2$ have an area, and let $f : E \to \mathbb{R}$ be a smooth function. Let the smooth parametrized surface $\Phi : E \to \mathbb{R}^3$ represent the graph of $f$, and let $\boldsymbol{F} : \boldsymbol{\Phi}(E) \to \mathbb{R}^3$ be a continuous vector field. If $\boldsymbol{F} := (P, Q, R)$, then

$$\iint_{\boldsymbol{\Phi}} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_E (-P\, f_x - Q\, f_y + R) d(x, y)$$

since $d\boldsymbol{S} = (\boldsymbol{\Phi}_x \times \boldsymbol{\Phi}_y) dxdy = (-f_x, -f_y, 1) dxdy$.

Using above result, let $E := [0,1] \times [0,1]$, $f(x,y) := x + y + 1$ for $(x, y) \in E$. If $\boldsymbol{F}(x,y,z) := (x^2, y^2, z)$ for $(x, y, z) \in \mathbb{R}^3$, then

$$\begin{aligned} \iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S} &= \iint_E \big( - x^2 - y^2 + (x + y + 1)\big) d(x, y) \\ &= \int_0^1 \left( \int_0^1 (x + y + 1 - x^2 - y^2) dy \right) dx \\ &= = \frac{1}{2} + \frac{1}{2} + 1 - \frac{1}{3} - \frac{1}{3} = \frac{4}{3}. \end{aligned}$$

## Examples Contd.

(ii) Let $E := [0, 2\pi] \times [0, h]$, and $\boldsymbol{\Phi}(u, v) := (a\cos u, a\sin u, v)$ for $(u, v) \in E$. If $\boldsymbol{F}(x, y, z) := (y, z, x)$ for $(x, y, z) \in \mathbb{R}^3$, then

$$\iint_{\boldsymbol{\Phi}} \mathbf{F} \cdot d\boldsymbol{S} = \iint_{E} (a^2 \cos u \sin u + v\, a \sin u + 0) dudv = 0,$$

since $d\boldsymbol{S} = (\boldsymbol{\Phi}_u \times \boldsymbol{\Phi}_v) dudv = (a\cos u, a\sin u, 0) dudv$.