MA 105 : Calculus

D4 - Lecture 4

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Continuity - the definition

Definition: If $f : [a, b] \to \mathbb{R}$ is a function and $c \in [a, b]$, then f is said to be continuous at the point c if and only if

$$\lim_{x \to c} f(x) = f(c).$$

Thus, if c is one of the end points, we require only the left or right hand limit to exist.

A function f on (a, b) (resp. $[a, b]$) is said to be continuous if and only if it is continuous at every point c in (a, b) (resp. $[a, b]$).

If f is not continuous at a point c we say that it is discontinuous at c, or that c is a point of discontinuity for f.

Intuitively, continuous functions are functions whose graphs can be drawn on a sheet of paper without lifting the pencil off the sheet of paper. That is, there should be no "jumps" in the graph of the function.

Continuity of familiar functions: polynomials

What are the functions we really know or understand? What does "knowing" or understanding a function $f(x)$ even mean? Presumably, if we understand a function f, we should be able to calculate the value of the function $f(x)$ at any given point x. But if you think about it, for what functions $f(x)$ can you really do this?

One class of functions is the polynomial functions. More generally we can understand rational functions, that is, functions of the form $R(x) = P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are polynomials, since we can certainly compute the values of $R(x)$ by plugging in the value of x. How do we show that polynomials or rational functions are continuous (on $\mathbb{R}$)?

It is trivial to show from the definition that the constant functions and the function $f(x) = x$ are continuous. Because of the rules for limits of functions, the sum, difference, product and quotient (with non-zero denominator) of continuous functions are continuous. Applying this fact we see easily that $R(x)$ is continuous whenever the denominator is nonzero.

Continuity of other familiar functions

What are the other (continuous) functions we know? How about the trigonometric functions? Well, here it is less clear how to proceed. After all we can only calculate $\sin x$ for a few special values of x ($x = 0, \pi/6, \pi/4, \dots$ etc.). How can we show continuity when we don't even know how to compute the function?

Of course, if we define $\sin x$ as the y-coordinate of a point on the unit circle it seems intuitively clear that the y-coordinate varies continuously as the point varies on the unit circle, but knowing the precise definition of continuity this argument should not satisfy you.

Note that $\lim_{x\to 0} \sin x = 0 = \sin 0$, which can be proved easily by using the inequality $|\sin x| \leq |x|$, for all $x \in [-\pi/2, \pi/2]$

and by using the formula $\sin(a+h) = \sin a \cos h + \cos a \sin h$ and $\lim_{x\to a} \sin x = \lim_{h\to 0} \sin(a+h) = \sin a \cos 0 + \cos a \sin 0 = \sin a$, the continuity of the function $\sin x$ at any point $a \in \mathbb{R}$ can be shown (here we have used the continuity of $\cos x$ at 0).

How can we show that $\cos x$ is continuous at each $a \in \mathbb{R}$?

The composition of continuous functions

Theorem 8: Let $f : (a, b) \to (c, d)$ and $g : (c, d) \to (e, f)$ be functions such that f is continuous at x_0 in (a, b) and g is continuous at $f(x_0) = y_0$ in (c, d). Then the function $g(f(x))$ (also written as $g \circ f(x)$ sometimes) is continuous at x_0. So the composition of continuous functions is continuous.

Exercise 4: Prove the theorem above starting from the definition of continuity.

Using the theorem above we can show that $\cos x$ is continuous if we show that $\sqrt{x}$ is continuous, since $\cos x = \sqrt{1 - \sin^2 x}$ and we know that $1 - \sin^2 x$ is continuous since it is the product of the sums of two continuous functions ($(1 + \sin x)$ and $(1 - \sin x)$).

Once we have the continuity of $\cos x$ we get the continuity of all the rational trigonometric functions, that is, functions of the form $P(x)/Q(x)$, where P and Q are polynomials in $\sin x$ and $\cos x$, provided $Q(x)$ is not zero.

The continuity of the square root function

Thus in order to prove the continuity of $\cos x$ (assuming the continuity of $\sin x$) we need only prove the continuity of the square root function.

The main observation is that continuity is a local property, that is, only the behaviour of the function near the point being investigated is important.

Let $x_0 \in [0, \infty)$. To show that the square root function is continuous at x_0 we need to show that $\lim_{x \to x_0} \sqrt{x} = \sqrt{x_0}$, that is, we need to show that $|\sqrt{x} - \sqrt{x_0}| < \epsilon$ whenever $0 < |x - x_0| < \delta$ for some δ. First assume that $x_0 \neq 0$. Then

$$|\sqrt{x} - \sqrt{x_0}| = \left| \frac{x - x_0}{\sqrt{x} + \sqrt{x_0}} \right| \leq \frac{|x - x_0|}{\sqrt{x_0}}.$$

If we choose $\delta = \epsilon\sqrt{x_0}$, we see that

$$|\sqrt{x} - \sqrt{x_0}| < \epsilon,$$

which is what we needed to prove. When $x_0 = 0$, I leave the proof as an exercise. $\square$

The intermediate value theorem

One of the most important properties of continuous functions is the Intermediate Value Property (IVP). We will use this property repeatedly to prove other results.

Theorem 9: Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function. For every u between $f(a)$ and $f(b)$ there exists $c \in [a, b]$ such that $f(c) = u$.

Functions which have this property are said to have the Intermediate Value Property. Theorem 9 can thus be restated as saying that continuous functions have the IVP.

We will not be proving this property - it is a consequence of the completeness of the real numbers. Intuitively, this is clear. Since one can draw the graph of the function without lifting one's pencil off the sheet of paper, the pencil must cut every line $y = u$ with u between $f(a)$ and $f(b)$.

The IVT in a picture

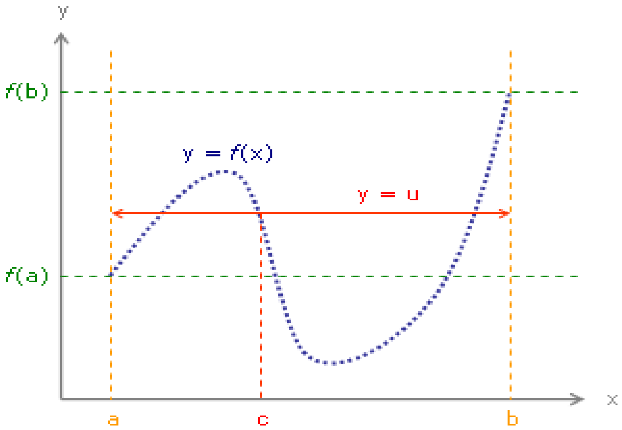

Image created by Enoch Lau see http://en.wikipedia.org/wiki/File:Intermediatevaluetheorem.png (Creative Commons Attribution-Share Alike 3.0 Unported license).

Zeros of functions

One of the most useful applications of the intermediate value property is to find roots of polynomials, or, more generally, to find zeros of continuous functions, that is to find points $x \in \mathbb{R}$ such that $f(x) = 0$.

Theorem 10: Every polynomial of odd degree has at least one real root.

Proof: Let $P(x) = a_n x^n + \ldots + a_0$ be a polynomial of odd degree. We can assume without loss of generality that $a_n > 0$. By using the fact that $\lim_{x \to \pm\infty}(P(x)/x^n) = a_n$, it is easy to see that if we take $x = b > 0$ large enough, $P(b)$ will be positive, and by taking $x = a < 0$ small enough, we can ensure that $P(a) < 0$. Since $P(x)$ is continuous, it has the IVP, so there must be a point $x_0 \in (a, b)$ such that $P(x_0) = 0$. $\square$

The IVP can often be used to get more specific information. For instance, it is not hard to see that the polynomial $x^4 - 2x^3 + x^2 + x - 3$ has a root that lies between 1 and 2.

Continuous functions on closed and bounded intervals

The other major result on continuous functions that we need is the following. A closed and bounded interval is one of the form $[a, b]$, where $-\infty < a$ and $b < \infty$.

Theorem 11: A continuous function on a closed and bounded interval $[a, b]$ is bounded and attains its infimum and supremum, that is, there are points x_1 and x_2 in $[a, b]$ such that $f(x_1) = m$ and $f(x_2) = M$, where m and M denote the infimum and supremum respectively.

Again, we will not prove this, but will use it quite often. Note the contrast with open intervals. The function $1/x$ on $(0, 1)$ does not attain a maximum - in fact it is unbounded. Similarly the function $1/x$ on $(1, \infty)$ does not attain its minimum, although, it is bounded below.

Exercise 5: In light of the above theorem, can you find a continuous function $g : (a, b) \to \mathbb{R}$ for part (i) of Exercise 1.11, with $c \in (a, b)$? (Exercise 1.11.(i). Show that the statement $\lim_{x\to c} f(x) = 0 \Rightarrow \lim_{x\to c} f(x)g(x) = 0$, for any g, is false.)

The function $\sin\frac{1}{x}$

Let us look at Exercise 1.13 part (i).
Consider the function defined as $f(x) = \sin\frac{1}{x}$ when $x \neq 0$, and $f(0) = 0$. The question asks if this function is continuous at $x = 0$.
How about $x \neq 0$? Why is $f(x)$ continuous? Because it is a composition of the sin function and a rational function $1/x$. Since both of these are continuous when $x \neq 0$, so is $f(x)$.

Let us look at the sequence of points $x_n = \frac{2}{(2n+1)\pi}$. Clearly $x_n \to 0$ as $n \to \infty$.
For these points, $f(x_n) = \pm 1$. This means that no matter how small I take my δ, there will be a point $x_n \in (0, \delta)$, such that $|f(x_n)| = 1$.

But this means that $|f(x) - f(0)| = |f(x)|$ cannot be made smaller than 1 no matter how small δ may be. Hence, f is not continuous at 0. The same kind of argument will show that there is no value that we can assign $f(0)$ to make the function $f(x)$ continuous at 0.

You can easily check that $f(x)$ has the IVP. However, we have proved that it is not continuous. So IVP $\not\Rightarrow$ continuity.

Sequential continuity

The preceding example showed that in order to demonstrate that a function, say $f(x)$, is not continuous at a point x_0 it is enough to find a sequence x_n tending to x_0 such that the value of the function $|f(x_n) - f(x_0)|$ remains large.

Suppose it is not possible to find such a sequence. Does that mean the function is continuous at x_0? The following theorem answers the question affirmatively.

Theorem 12: A function $f(x)$ is continuous at a point a if and only if for every sequence x_n converging to a, the sequence $f(x_n)$ converges to $f(a)$.

A function that satisfies the property that for every sequence $x_n \to a$, $\lim_{n\to\infty} f(x_n) = f(a)$ is said to be sequentially continuous.

The theorem says that sequential continuity and continuity are the same thing. Indeed, it is clear that a continuous function is necessarily sequentially continuous. It is the reverse that is harder to prove.

Proof of Theorem 12

Proof. Let $I \subseteq \mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ be a function. Let $a \in I$.

$(\Rightarrow)$. Let f be continuous at a. That is, for a given $\epsilon > 0$, $\exists \delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$.

Since the sequence (x_n) converges to a, for the above δ, $\exists N \in \mathbb{N}$ such that $|x_n - a| < \delta$ whenever $n \geq N$.

Hence $|f(x_n) - f(a)| < \epsilon$ whenever $n \geq N$.

$(\Leftarrow)$. We will show this part by using the method of contradiction.

For, let if possible, the function f is not continuous at a, that is, there exists $\epsilon > 0$ such that $\forall\ \delta > 0$, $\exists\ x_\delta \in I$ with $|x_\delta - a| < \delta$ and $|f(x_\delta) - f(a)| \geq \epsilon$.

Now we find a sequence (x_n) converging to a, for which, $(f(x_n))$ does not converge to $f(a)$.

Proof continued...

For $n \in \mathbb{N}$ and for the same ϵ, if we take $\delta_n = \frac{1}{n}$ then $\exists\, x_n \in I$ such that $|x_n - a| < \frac{1}{n}$ and $|f(x_n) - f(a)| \geq \epsilon$.

It is now clear that the sequence (x_n) converges to a but for the ϵ above, there does not exist $N \in \mathbb{N}$ such that $|f(x_n) - f(x)| < \epsilon$, whenever $n \geq N$, that is, the sequence $(f(x_n))$ does not converge to $f(a)$ which is a contradiction.

Hence the function f is continuous. $\square$

Remark: Theorem 12 (continuity is same as sequential continuity) goes through without any problems even when the range and/or domain of the function are/is in $\mathbb{R}^2$ or $\mathbb{R}^3$. Exactly the same proof works in this case. Note that we have not yet defined the continuity of functions having range and/or domain in $\mathbb{R}^2$ or $\mathbb{R}^3$. You can try defining it and proving the above theorem in this case.

Differentiability: The definition

Recall that $f:(a,b)\to\mathbb{R}$ is said to be differentiable at a point $c\in(a,b)$ if

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

exists.

In this case the value of the limit is denoted $f'(c)$ and is called the derivative of f at c. The derivative may also be denoted by $\frac{df}{dx}(c)$ or by $\frac{dy}{dx}|_c$, where $y=f(x)$.

In general, the derivative measures the rate of change of a function at a given point. Thus, if the function we are studying is the position of a particle on the x-coordinate, then $x'(t)$ is the velocity of the particle.

The slope of the tangent

If the function we are studying is the velocity $v(t)$ of the particle, then the derivative $v'(t)$ is the acceleration of the particle.

From the point of view of geometry, the derivative $f'(c)$ gives us the slope of the curve, that is, the slope of the tangent to the curve $y = f(x)$ at $(c, f(c))$.

This becomes particularly clear if we rewrite the derivative as the following limit:

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

The expression inside the limit obviously represents the slope of a line passing through $(c, f(c))$ and $(x, f(x))$, and as x approaches c this line obviously becomes tangent to $y = f(x)$ at the point $(c, f(c))$.

Examples

Exercise 1.16: Let $f : (a, b) \to \mathbb{R}$ be a function such that

$$|f(x+h) - f(x)| \leq c|h|^{\alpha}$$

for all $x, x+h \in (a, b)$, where c is a constant and $\alpha > 1$. Show that f is differentiable on (a, b) and compute $f'(x)$ for $x \in (a, b)$.

Solution: By the Sandwich theorem

$$\lim_{h\to 0} \left| \frac{f(x+h) - f(x)}{h} \right| \leq c \lim_{h\to 0} |h|^{\alpha-1} = 0 \implies f'(x) = 0$$

$(\lim_{h\to 0} |g(h)| = 0 \iff \lim_{h\to 0} g(h) = 0)$. $\square$

Note: Functions that satisfy the property above for $\alpha > 0$ (not necessarily greater than 1) are said to be Lipschitz continuous with exponent α.

Calculating derivatives

As with limits, all of you are already familiar with the rule for calculating the derivatives of the sums, differences, products and quotients of differentiable functions. You should try and remember how to prove these.

You should also recall the chain rule (for $F(x) = f(g(x))$, $F'(x) = f'(g(x))g'(x)$) for calculating the derivative of the composition of functions and try to prove it as an exercise using the $\epsilon - \delta$ definition of a limit.

Exercise: Show that the differentiable functions are continuous.

Use the simple observation that

$$f(x) = f(a) + (x-a)\frac{f(x) - f(a)}{(x-a)}$$

and then apply the limit rule.