MA 105 : Calculus

D4 - Lecture 7

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The Taylor Series: Some Notation

We will first introduce some notation.

The space $\mathcal{C}^k(I)$ will denote the space of k times continuously differentiable functions on an interval I, for some fixed $k \in \mathbb{N}$, that is, the space of functions for which k derivatives exist and such that the k-th derivative is a continuous functions.

The space $\mathcal{C}^\infty(I)$ will consist of functions that lie in $\mathcal{C}^k(I)$ for every $k \in \mathbb{N}$. Such functions are called smooth or infinitely differentiable functions.

From now on we will denote the k-th derivative of a function $f(x)$ by $f^{(k)}(x)$.

Our aim will be to enlarge the class of functions we understand using the polynomials as stepping stones.

The Taylor polynomials

Given a function $f(x)$ which is n times differentiable at some point x_0 in an interval I, we can associate to it a family of polynomials $P_0(x), P_1(x), \ldots, P_n(x)$ called the Taylor polynomials of order $0, 1, \ldots, n$ at x_0 as follows.

We let $P_0(x) = f(x_0)$,

$$P_1(x) = f(x_0) + f^{(1)}(x_0)(x - x_0),$$

$$P_2(x) = f(x_0) + f^{(1)}(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2,$$

we can continue in this way to define

$$P_n(x) = f(x_0) + f^{(1)}(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Taylor's Theorem

The Taylor polynomials are rigged exactly so that the order n Taylor polynomial has the same first n derivatives at the point x_0 as the function $f(x)$ has, that is, $P^{(k)}(x_0) = f^{(k)}(x_0)$ for all $0 \leq k \leq n$, where $f^{(0)}(x_0) = f(x_0)$ by convention.

Taylor's Theorem says that we can recover a lot of information about the function from the Taylor polynomials.

Theorem 19: Let $f \in \mathcal{C}^n(I)$ for some open interval I containing a, and suppose that $f^{(n+1)}$ exists on this interval. Then for each $b \neq a \in I$, there exists c between a and b such that

$$f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

where P_n denotes the Taylor polynomial of order n at a.

It is customary to denote the function $f(b) - P_n(b)$ by $R_n(b)$.

Taylor's Theorem gives us a simple formula for $R_n(b)$. If we can make $R_n(b)$ small, we can approximate our function $f(x)$ by a polynomial.

The proof of Taylor's theorem

Proof: Consider the function

$$F(x) = f(b) - f(x) - f^{(1)}(x)(b-x) - \frac{f^{(2)}(x)}{2!}(b-x)^2 - \cdots - \frac{f^{(n)}(x)}{n!}(b-x)^n.$$

Clearly $F(b) = 0$, and

$$F^{(1)}(x) = -\frac{f^{(n+1)}(x)(b-x)^n}{n!} \tag{1}$$

Observe that $F(a) = f(b) - P_n(b)$ which need not be zero in general but it is zero if f is a polynomial of degree less or equal to n (as in this case $P_n(b) = f(b)$) and in this case $f^{(n+1)}(c) = 0$ for all c and the theorem gets proved!

We compute $F(a)$ by using the Rolle's Theorem. For, we consider

$$g(x) = F(x) - \left(\frac{b-x}{b-a}\right)^{n+1} F(a)$$

(this is similar to the method by which we proved the MVT using Rolle's Theorem), and we see that $g(b) = g(a) = 0$.

Applying Rolle's Theorem we see that there is a c between a and b such that $g'(c) = 0$.

This yields

$$F^{(1)}(c) = -(n+1)\left(\frac{(b-c)^n}{(b-a)^{n+1}}\right)F(a) \tag{2}$$

We can eliminate $F^{(1)}(c)$ using (1). This gives

$$-(n+1)\left(\frac{(b-c)^n}{(b-a)^{n+1}}\right)F(a) = -\frac{f^{(n+1)}(c)(b-c)^n}{n!},$$

from which we obtain

$$f(b) - P_n(b) = F(a) = \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

This proves what we want. $\square$

Remarks on Taylor's Theorem and some examples

Remark 1: When $n = 0$ in Taylor's Theorem we get the MVT. When $n = 1$, Taylor's Theorem is called the Extended Mean Value Theorem.

Remark 2: The Taylor polynomials are nothing but the partial sums of the Taylor Series associated to a $\mathcal{C}^\infty$ function about (or at) the point a:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(b-a)^k.$$

We can show that this series converges to $f(b)$ provided we know that the difference $f(b) - P_n(b) = R_n(b)$ can be made less than any $\epsilon > 0$ when n is sufficiently large. We will see how to do this for certain simple functions like e^x or $\sin x$.

The Taylor series for e^x

Let us show that the Taylor series for the function e^x about the point 0 is a convergent series for any value of $x = b \geq 0$ and that it converges to the value e^b (a similar proof works for $b < 0$).

In this case, at any point a, $f^{(n)}(a) = e^a$, so at $a = 0$ we obtain $f^{(n)}(0) = 1$. Hence the series about 0 is

$$\sum_{k=0}^{\infty} \frac{b^k}{k!}.$$

If we look at $R_n(b) = e^b - P_n(b)$, we obtain

$$|R_n(b)| = \frac{e^c b^{n+1}}{(n+1)!} \leq \frac{e^b b^{n+1}}{(n+1)!},$$

since $c \leq b$. (In case when $b < 0$, we get $b \leq c \leq 0$ and $e^c < 1$).

As $n \to \infty$ this clearly goes to 0. This shows that the Taylor series of e^b converges to the value of the function at each real number b.

Appendix: The ratio test for the convergence of a series

Theorem: Let $\sum_{k=1}^{\infty} a_k$ be an infinite series and let

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = L.$$

Then, there are three possibilities:

1. If $L < 1$, then the above series is convergent.
2. If $L > 1$, then the above series is divergent.
3. If $L = 1$, then the test is inconclusive.

Proof: Exercise (Hint: 1. The geometric series $\sum_{k=1}^{\infty} a_N r^{k-1}$ is convergent for $|r| < 1$. 2. If $L > 1$, then $\lim_{k\to\infty} a_k \neq 0$. 3. Try to find an example of a convergent series for which $L = 1$. Also, find an example of a divergent series for which $L = 1$).

Defining functions using Taylor series

Instead of finding the Taylor series of a given function we can reverse the process and define functions using convergent series.

Thus, one can define the function e^x as

$$e^x := \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

In this case, we have to first show that the series on the right hand side converges for a given value of x, in which case the definition above makes sense.

We show the convergence of such series by showing that they are Cauchy series, that is, the sequence of the n-th partial sum of the series is a Cauchy sequence. This means that we do not have to guess a value of the limit.

One can use the ratio test too to conclude the convergence of such series!

Power series

As we have explained in the previous slide, the "correct" (both from the point of view of proofs and of computation) way to define a function like e^x is via convergent series involving non-negative integer powers of x.

Such series are called power series and such functions should be viewed as the natural generalizations of polynomials.

The nice thing about power series is that once we know that they converge in some interval $(a - r, a + r)$ around a, it is not hard to show that the functions that they define are continuous functions.

In fact, it is not too hard to show that they are smooth functions (that is, that all their derivatives exist).

Thus when functions are given by convergent power series, we can automatically conclude they are smooth. This is the advantage of defining functions in this way.

Computing the values of functions

A calculator or a computer program calculates the values of various common functions like trigonometric polynomials and expressions in $\log x$ and e^x by using Taylor series.

The great advantage of Taylor series is that one can estimate the error since we have a simple formula for the error which can be easily estimated.

For instance, for the function $\sin x$, the n-th derivative is either $\pm \sin x$ or $\pm \cos x$, so in either case $|f^{(n)}(x)| \leq 1, \ \forall n \geq 1$.

Hence,

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

If we take $x = 1$, and we want to compute $\sin 1$ to an error of less than 10^{-16}, we need only make sure that $(n+1)! > 10^{16}$, which is achieved when $n \geq 21$.

Computing the values of $\sin x$ for general $x \in \mathbb{R}$

First, remember that $\sin x$ is periodic, so we only have to look at the values of x between $-\pi$ and π.

But we can do better, because $\sin(-x) = -\sin x$. So we only have to bother about the interval $[0, \pi]$.

We can do still better! Once we know $\sin x$ in $[0, \pi/2]$, we can easily figure out what it is in $[\pi/2, \pi]$.

So finally, it is enough to find the desired value of n for $x \in [0, \pi/2]$.

Computing the values of $\sin x$

We know that the remainder term satisfies

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}.$$

Hence, we need

$$\frac{|x|^{n+1}}{(n+1)!} < 10^{-16}.$$

We know that it is enough to look at $0 \leq x \leq \pi/2$. Let us be a little careless and allow $x \leq 2$ (so we won't get the best possible n, maybe).

We already know that $1/(n+1)! < 10^{-16}$ if $n \geq 21$. Now $|x|^{22} \leq 2^{22}$. If we take $n = 31$, we see that $|x|^{32} \leq 2^{22} \cdot 2^{10}$,

$$1/(n+1)! = 1/32! < 10^{-16} \cdot 10^{-10} \cdot 2^{-10}.$$

Hence $|R_n(x)| = \frac{|x|^{n+1}}{(n+1)!} < 10^{-16}$ for $n = 31$. □

Smooth functions and Taylor series

Given a smooth function $f(x)$ on on an open interval $I \subseteq \mathbb{R}$, we can write down its associated Taylor polynomials $P_n(x)$ around any point a in I.

Here are some natural questions that arise. Let us take $a = 0$ in what follows.

Question 1. When $x = 0$, obviously $P_n(0) = f(0)$ for all n. Do the Taylor polynomials $P_n(x)$ (around 0, say) always converge as $n \to \infty$ for $x \neq 0, x \in I$? at least for all x in some sub-interval $(c, d) \ni 0$?

Question 2. If $P_n(x)$ converges as $n \to \infty$, does it necessarily converge to $f(x)$?

We will answer the second question.

Smooth but not approximated by Taylor polynomials

The standard example is the function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-1/x} & \text{if } x > 0. \end{cases}$$

Notice that $f^{(k)}(0) = 0$ for all $k \geq 0$. Hence $P_n(x) = 0$ for all $n \geq 0$. Hence, $\lim_{n\to\infty} P_n(x) = 0$. Thus the Taylor polynomials $P_n(x)$ around 0 converge to 0 for any $x \in \mathbb{R}$.

But obviously, they do not converge to the value of the function, since $f(x) > 0$ if $x > 0$.

In this case, the Taylor series does a very poor job of approximating the function. Indeed, the remainder term $R_n(x) = f(x)$ for all $x > 0$.

Thus, when we use Taylor series to approximate a function in an interval I, we must make sure that $R_n(x) \to 0$ as $n \to \infty$, for all $x \in I$.

L'Hôpital's rule

Suppose f and g are $\mathcal{C}^1$ functions in an interval I containing 0. By the MVT, for $x \in I$,

$$f(x) = f(0) + f^{(1)}(c_1)x \quad \text{and} \quad g(x) = g(0) + g^{(1)}(c_2)x$$

for $0 < c_1, c_2 < x$. If $f(0) = g(0) = 0$,

$$\lim_{x \to 0} f(x)/g(x) = \lim_{x \to 0} f^{(1)}(c_1)x/g^{(1)}(c_2)x = \lim_{x \to 0} f^{(1)}(c_1)/g^{(1)}(c_2).$$

But $f^{(1)}$ and $g^{(1)}$ are continuous functions and as $x \to 0$, $c_1, c_2 \to 0$. Hence,

$$\lim_{x \to 0} f^{(1)}(c_1)/g^{(1)}(c_2) = f^{(1)}(0)/g^{(1)}(0).$$

If the functions are in $\mathcal{C}^n$, and $f^{(k)}(0) = g^{(k)}(0) = 0$ for all $k < n$, we can apply the MVT repeatedly (or we can apply Taylor's theorem directly) to get $f^{(n)}(0)/g^{(n)}(0)$ as the limit.

Partitions

Definition: Given a closed interval $[a, b]$, a partition P of $[a, b]$ is simply a collections of points

$$P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}.$$

We can think of the points of the partition as dividing the original interval $I = [a, b]$ into sub-intervals $I_j = [x_{j-1}, x_j]$, $1 \leq j \leq n$.

Indeed, $I = \cup_j I_j$ and if two sub-intervals intersect, they have at most one point in common. Hence, the notation "partition".

Definition: A partition $P' = \{a = x'_0 < x'_1 < \cdots < x'_m = b\}$ is said to be a refinement of the partition P if for each $x_i \in P$, there exists an $x'_j \in P'$ such that $x_i = x'_j$.

Intuitively, a refinement P' of a partition P will break some of the sub-intervals in P into smaller sub-intervals.

Any two partitions P_1 and P_2 have a common refinement $P = P_1 \cup P_2$.

Lower and Upper sums

Given a partition $P = \{a = x_0 < x_1 < \cdots < x_{b-1} < x_n = b\}$ and a bounded function $f : [a, b] \to \mathbb{R}$, we define two associated quantities.

First, we set:

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) \quad \text{and} \quad m_i = \inf_{x \in [x_{i-1}, x_i]} f(x), \quad 1 \leq i \leq n.$$

Definition: We define the Lower sum as

$$L(f, P) = \sum_{j=1}^{n} m_j (x_j - x_{j-1}).$$

Similarly, we can define the Upper sum as

$$U(f, P) = \sum_{j=1}^{n} M_j (x_j - x_{j-1}).$$

In case the words "infimum" and "supremum" bother you, you can think "minimum" and "maximum most of time since we will usually be dealing with continuous functions on $[a, b]$.

The Darboux integrals

For any partition $P = \{a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\}$ of $[a, b]$, $m_j \leq M_j$, $\forall\ 1 \leq j \leq n$ and hence

$$L(f, P) \leq U(f, P).$$

Since the function f is bounded on $[a, b]$, there exists $m, M \in \mathbb{R}$ such that

$$m \leq f(x) \leq M$$

for all $x \in [a, b]$ and hence

$$m(b-a) \leq L(f, P) \leq U(f, P) \leq M(b-a)$$

for any partition P of $[a, b]$.

We now define the lower Darboux integral of f by

$$L(f) = \sup\{L(f, P) \colon P \text{ is a partition of } [a, b]\},$$

where the supremum is taken over all partitions of $[a, b]$.

The Darboux integrals

Similarly, the upper Darboux integral of f is defined by

$$U(f) = \inf\{U(f, P) \colon P \text{ is a partition of } [a, b]\},$$

and again the infimum is over all partitions of $[a, b]$.

(This time there is no escaping inf and sup!)

If $L(f) = U(f)$, then we say that f is Darboux-integrable and define

$$\int_a^b f(t)\, dt := U(f) = L(f).$$

This common value of the two integrals is called the Darboux integral.

One basic example

In order to illustrate what we are saying we will take the following basic example. Let $[a, b] = [0, 1]$ and let $f(x) = x$.

One of the most natural partitions of an interval is a partition that divides the interval into sub-intervals of equal length.

For $[0, 1]$, this is

$$P_n = \{0 < 1/n < 2/n < \cdots < (n-1)/n < 1\}.$$

On the interval $I_j = [\frac{j-1}{n}, \frac{j}{n}]$, where does the function $f(x) = x$ take its minimum?

Clearly, the minimum $m_j = \frac{j-1}{n}$ is attained at $\frac{j-1}{n}$ and the maximum $M_j = \frac{j}{n}$ at $\frac{j}{n}$. And finally, $\frac{j}{n} - \frac{j-1}{n} = \frac{1}{n}$, for all $1 \leq j \leq n$.

An example of a refinement of P_n is P_{2n}, or, more generally, P_{kn} for any natural number k.

The $L(f, P_n)$ and $U(f, P_n)$ for $f(x) = x$ on $[0, 1]$

Let us calculate $L(f, P_n)$ and $U(f, P_n)$ for the example we gave in the last slide.

$$L(f, P_n) = \sum_{j=1}^{n} \frac{(j-1)}{n} \cdot \frac{1}{n} = \sum_{j=0}^{n-1} \frac{j}{n^2}.$$

This can be evaluated explicitly:

$$L(f, P_n) = \frac{n(n-1)}{2} \cdot \frac{1}{n^2} = \frac{1}{2} - \frac{1}{2n}.$$

Similarly, we can check that

$$U(f, P_n) = \sum_{j=1}^{n} \frac{j}{n} \cdot \frac{1}{n} = \sum_{j=1}^{n} \frac{j}{n^2} = \frac{n(n+1)}{2} \cdot \frac{1}{n^2} = \frac{1}{2} + \frac{1}{2n}.$$

Can we conclude that the Darboux integral is $\frac{1}{2}$ by letting $n \to \infty$? Unfortunately, no, as of now. But we will see soon that the function $f(x) = x$ is Darboux integrable on any finite interval.

An example of a function that is not Darboux integrable

Here is a function that is not Darboux integrable on $[0, 1]$. Let

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

It should be clear that no matter what partition one takes the infimum on any sub-interval in the partition will be 0 and the supremum will be 1.

From this, one can see immediately that

$$L(f, P) = 0 \neq 1 = U(f, P),$$

for every P, and hence that $L(f) = 0 \neq 1 = U(f)$.