7. If $\lim\limits_{n\to\infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that

$$|a_n| \geq \frac{|L|}{2}, \ \forall\, n \geq n_0.$$

8. If $a_n \geq 0$ and $\lim\limits_{n\to\infty} a_n = 0$, show that $\lim\limits_{n\to\infty} a_n^{1/2} = 0$. State and prove a corresponding result if $a_n \to L > 0$.

9. For given sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, prove or disprove the following:

   (i) $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent.

   (ii) $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent and $\{b_n\}_{n\geq 1}$ is bounded.

10. Show that a sequence $\{a_n\}_{n\geq 1}$ is convergent iff both the subsequences $\{a_{2n}\}_{n\geq 1}$ and $\{a_{2n+1}\}_{n\geq 1}$ are convergent to the same limit.

11. Let $f, g : (a, b) \to \mathbb{R}$ be functions and suppose that $\lim_{x\to c} f(x) = 0$ for $c \in [a, b]$. Prove or disprove the following statements.

   (i) $\lim\limits_{x\to c}[f(x)g(x)] = 0$. Take counter example

   (ii) $\lim\limits_{x\to c}[f(x)g(x)] = 0$, if g is bounded.

   (iii) $\lim\limits_{x\to c}[f(x)g(x)] = 0$, if $\lim\limits_{x\to c} g(x)$ exists.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $\lim\limits_{x\to\alpha} f(x)$ exists for some $\alpha \in \mathbb{R}$. Show that

$$\lim_{h\to 0}[f(\alpha + h) - f(\alpha - h)] = 0.$$

Apply triangular inequality

   Analyze the converse. → Counter Example

13. Discuss the continuity of the following functions:

   (i) $f(x) = \sin\dfrac{1}{x}$, if $x \neq 0$ and $f(0) = 0$

   (ii) $f(x) = x\sin\dfrac{1}{x}$, if $x \neq 0$ and $f(0) = 0$ $|f(x)| \leq |x|$

   (iii) $f(x) = \begin{cases} \dfrac{x}{[x]} & \text{if} \quad 1 \leq x < 2, \\ 1 & \text{if} \quad x = 2, \\ \sqrt{6 - x} & \text{if} \quad 2 < x \leq 3. \end{cases}$

14. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at 0, show that f is continuous at every $c \in \mathbb{R}$.

15. Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that f is differentiable on $\mathbb{R}$. Is f' a continuous function?

↓ Use definition

↓ Use sequences method of 13(i)

7)

$a - x + x = a \Rightarrow |a| \leq |x| + |a-x|$

$\Rightarrow |a| - |x| \leq |a-x|$

$x - a + a = x \Rightarrow |x-a| + |a| \geq |x|$

$\Rightarrow |x-a| \geq |x| - |a|$

$|x-a| \geq ||x| - |a||$

$\Rightarrow \lim_{n \to \infty} a_n = L \Rightarrow ||a_n| - |L|| \leq |a_n - L| < \varepsilon \quad \forall n > n_0$

$\Rightarrow ||a_n| - |L|| < \varepsilon \quad \forall n > n_0$

Take $\varepsilon = \frac{|L|}{2}$ $\quad ||a_n| - |L|| < \frac{|L|}{2}$

$-\frac{|L|}{2} < |a_n| - |L| < \frac{|L|}{2}$

$\Rightarrow |a_n| \geq \frac{|L|}{2}$

9) (i) (ii) $a_n = 1 \quad b_n = (-1)^n \quad n$

$\Rightarrow$ Not convergent

Disproved using a counter example

10) $\{a_n\}$ is convergent $\Leftrightarrow$ $\{a_{2n}\}$ and $\{a_{2n+1}\}$ convergent

Proof of $\Rightarrow$

$|a_n - L| < \varepsilon \quad \forall n > n_0$

let there be m_1 such that $2m_1 > n_0$

similarly m_2 such that $2m_2 + 1 > n_0$

$\{ |a_n - L| < \varepsilon \quad \forall\ n > 2m_1$

$\{ |a_n - L| < \varepsilon \quad \forall\ n > 2m_2 + 1$

$\{a_{2n}\}$ convergent $\qquad \{a_{2n+1}\}$ convergent

Proof of $\Leftarrow$ Let there be

$|a_{2n+1} - L| < \varepsilon \quad \forall\ n > n_1$

$|a_{2n} - L| < \varepsilon \quad \forall\ n > n_2$

Let $n_0 = \max\{2n_1 + 1,\ 2n_2\}$

$\Rightarrow \quad |a_n - L| < \varepsilon \quad \forall\ n > n_0$

$\Rightarrow \quad \lim_{n \to \infty} a_n = L$

12) $f(\alpha+h) - f(\alpha-h)$

$$= f(\alpha+h) - f(\alpha) + f(\alpha) - f(\alpha-h)$$

$$\leq |f(\alpha+h) - f(\alpha)| + |f(\alpha) - f(\alpha-h)|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall\, h \in (-\delta, \delta)$$

$\Rightarrow$ Limit proved

Converse = false

$$f(x) = \begin{cases} 1 & x = 0 \\ \frac{1}{|x|} & x \neq 0 \end{cases}$$

11b (i) $\lim_{x \to c} f(x) = 0$

$|g(x)| \leq M$ (bounded) $(M > 0)$

$|f(x) - 0| < \varepsilon/M$

$\forall x \in (c-\delta, c+\delta)$

[$\frac{\varepsilon}{M}$ = random positive quantity]

$\lim_{x \to c} f(x)\, g(x) = ?$

$|f(x)g(x) - 0| < \frac{\varepsilon}{M} \times M = \varepsilon$

$\forall x \in (c-\delta, c+\delta)$

$\Rightarrow$ limit $= 0$.

13b (i) A countinuous function on x at $x = c$ will be continuous on every sequence of x approaching c.

Take sequences = $\{x_n\} = \frac{1}{n\pi}$

$$\{y_n\} = \frac{1}{2n\pi + \frac{\pi}{2}}$$

Both of them approach 0 as $n \to \infty$

$$f(x) = \sin\frac{1}{x}$$

$$f(x_n\}) = \sin n\pi = 0$$

$$f(y_n) = \sin\left(2n\pi + \frac{\pi}{2}\right) = 1$$

Since, the two limits are not equal, the function is not convergent.

14) $f(x+y) = f(x) + f(y)$

$$\lim_{h\to 0} f(h) = 0 \quad [\text{continuous at } 0]$$

To prove: continuity at c

To prove: $\lim_{h\to 0} f(c+h) = f(c)$

$$\lim_{h\to 0} f(c) + f(h) = f(c) + 0 = f(c)$$