MA 105 Tutorial Sheet 6 :
Surface integrals, Stokes theorem, Gauss divergence theorem
November 7, 2023

I Surface and surface integrals

1. Find a suitable parametrization $\Phi(u, v)$ and the normal vector $\Phi_u \times \Phi_v$ for the following surface:
   (i) The plane $x - y + 2z + 4 = 0$.
   (ii) The right circular cylinder $y^2 + z^2 = a^2$.
2. Find the tangent plane to the surface with parametric equations $x = u^2$, $y = v^2$ and $z = u + 2v$ at the point $(1, 1, 3)$.
3. Compute the surface area of that portion of the sphere $x^2 + y^2 + z^2 = a^2$ which lies within the cylinder $x^2 + y^2 = ay$, where $a > 0$.
4. Compute the area of that portion of the paraboloid $x^2 + z^2 = 2ay$ which is between the planes $y = 0$ and $y = a$.
5. Let S denote the plane surface whose boundary is the triangle with vertices at $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, and let $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Let $\mathbf{n}$ denote the unit normal to S having a nonnegative z-component. Evaluate the surface integral $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$.

1. Application of Stokes theorem

1. Consider the vector field $\mathbf{F} = (x - y)\mathbf{i} + (x + z)\mathbf{j} + (y + z)\mathbf{k}$. Verify Stokes theorem for $\mathbf{F}$ where S is the surface of the cone: $z^2 = x^2 + y^2$ intercepted by
   (a) $x^2 + (y - a)^2 + z^2 = a^2 : z \geq 0$ (b) $x^2 + (y - a)^2 = a^2$
2. Using Stokes Theorem, evaluate the line integral
$$\oint_C yz\, dx + xz\, dy + xy\, dz$$
   where C is the curve of intersection of $x^2 + 9y^2 = 9$ and $z = y^2 + 1$ with clockwise orientation when viewed from the origin.
3. Find the integral of $\mathbf{F}(x, y, z) = z\mathbf{i} - x\mathbf{j} - y\mathbf{k}$ around the triangle with vertices $(0, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 2)$.
4. Let C be the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$. Let C be oriented so that when it is projected onto the xy-plane the resulting curve is traversed counterclockwise. Evaluate
$$\int_C -y^3 dx + x^3 dy - z^3 dz.$$
5. Let $\boldsymbol{F}(x, y, z) := (y, -x, e^{x\,z})$ for $(x, y, z) \in \mathbb{R}^3$, and let $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z - \sqrt{3})^2 = 4 \text{ and } z \geq 0\}$, be oriented by the outward unit normal vectors. Find $\iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S}$.

III Application of Gauss divergence theorem

1. Calculate the flux of $\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ through the unit sphere.

2. Evaluate $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = xy^2\mathbf{i} + x^2y\mathbf{j} + y\mathbf{k}$ and S is the surface of the 'can' W given by $x^2 + y^2 \leq 1$, $-1 \leq z \leq 1$.
3. Evaluate $\int\int_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = xy\mathbf{i} + (y^2 + e^{xz^2})\mathbf{j} + \sin(xy)\mathbf{k}$$

and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$ and $y + z = 2$.
4. Find out the flux of $F = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$ outward through the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, $z = 1$.
5. Is $\mathbf{F}(x, y, z) = x\mathbf{i} - 2y\mathbf{j} + z\mathbf{k}$ defined in $\mathbb{R}^3$ the curl of a vector filed? If yes, find a vector field $\mathbf{G}$ such that $\mathbf{F} = \text{curl } \mathbf{G}$ in $\mathbb{R}^3$.