Problems and Solutions Book

December 5, 2024

Contents

Chapter 1

Questions

Question 1: Place or Take

Topic: probability **Difficulty**: hard
You are playing a one-player game with two opaque boxes. At each turn, you can choose to either "place" or "take". "Place" places $1 from a third party into one box randomly. "Take" empties out one box randomly and that money is yours. This game consists of 100 turns where you must either place or take. Assuming optimal play, what is the expected payoff of this game? Note that you do not know how much money you have taken until the end of the game.

Question 2: Collecting Toys II

Topic: probability **Difficulty**: hard
Every box of cereal contains one toy from a group of 5 distinct toys, each of which is mutually independent from the others and is equally likely to be within a given box. How many distinct toys can you expect to collect if you bought 7 boxes?

Question 3: Chess Tournament I

Topic: probability **Difficulty**: medium
A chess tournament has 128 players, each with a distinct rating. Assume that the player with the higher rating always wins against a lower rated opponent and that the winner proceeds to the subsequent round. What is the probability that the highest rated and second-highest rated players will meet in the final?

Question 4: Poker Hands I

Topic: probability **Difficulty**: easy
A poker hand consists of five cards from a standard deck of 52 cards. If p is the probability that you have a four-of-a-kind (four of the five cards have the same rank), find the reciprocal of p.

Question 5: Free Sundae

Topic: probability **Difficulty**: hard
You are in line at an ice cream shop when the manager announces that she will give a free sundae to the first person in line whose birthday is the same as someone who has already bought an ice cream. Assuming that you do not know anyone else's birthday and that all birthdays are uniformly distributed across the 365 days in a normal year, what position in line will you choose to maximize your probability of receiving the free sundae?

Question 6: Rubik's Cube Stickers

Topic: brainteasers **Difficulty**: easy
A $4 \times 4 \times 4$ Rubik's Cube is composed of $1 \times 1 \times 1$ mini-cubes. The Rubik's Cube has stickers on its outer layer to denote the colors of the cube. How many mini-cubes have at least one sticker on them?

Question 7: Mental Arithmetic

Topic: brainteasers **Difficulty**: medium
Find the pair of positive integers (a, b) that minimizes $a + b$ and satisfies $a^2 + b^2 + ab = c^2$ for an integer c. Find c^2.

Question 8: Observance Range

Topic: probability **Difficulty**: medium
Find the smallest value of n such that if we generate n IID Unif$(0, 1)$ random variables, the probability that at least one of them lies in the interval $(0.48, 0.52)$ is at least 99%.

Question 9: River Length

Topic: brainteasers **Difficulty**: easy
You are in a river of unknown length and velocity. You know that if you are stationary on the river, then it takes you 6 seconds for you to reach the other

end. If you swim down the river at a rate of 3 feet per second, then it takes you 4 seconds to reach the other end. Find the length of the river (in feet).

Question 10: Heaven 37

Topic: brainteasers **Difficulty**: hard
Let x be a 5 digit integer in the form $37abc$ such that $37abc, 37bca$, and $37cab$ are all divisible by 37. How many possible values of x are there?

Question 11: Basketball Practice II

Topic: probability **Difficulty**: medium
Frank is shooting free throws. He makes his first free throw and misses his second free throw. For $n \geq 3$, the probability of making the nth free throw is equal to the proportion of free throws he made during his first $n-1$ attempts. How many free throws can Frank expect to make in 100 attempts?

Question 12: Increasing Uniform Chain II

Topic: probability **Difficulty**: medium
Let $X_1, X_2, \cdots \sim$ Unif$(0,1)$ IID. Let N be the first index n where $X_n \neq \max\{X_1, \ldots, X_n\}$. Find $\mathbb{E}[N]$. The answer will be in the form $a+be$ for integers a and b. Note here that e is Euler's constant. Find $a+b$.

Question 13: Candleburn

Topic: brainteasers **Difficulty**: easy
We have two candles of the same size and uniform density. One burns in 2 hours, while the other burns in 1 hour. At what time should we simultaneously light these candles in order to have one candle be double the length of the other at 4 PM? Answer in military time. For example if the answer is 8 : 20 PM, answer 2020.

Question 14: Ten Consecutive Heads

Topic: probability **Difficulty**: medium
On average, how many times must a fair coin be flipped to obtain 10 consecutive heads?

Question 15: Consecutive Tails

Topic: probability **Difficulty**: medium
Suppose a coin is flipped 10 times and the outcomes are recorded. Find the probability that any tails occur only in consecutive pairs. For example, with 4 flips, $TTHH$, $TTTT$, and $HHHH$ are both valid, but $HTHH$ and $TTTH$ are not valid.

Question 16: Bowl of Cherries II

Topic: probability **Difficulty**: medium
Amy has a bowl of 5 red cherries and 8 purple cherries. She takes out cherries one at a time until there are no cherries left. What is the probability that when the last red cherry is drawn, there are exactly 2 purple cherries left?

Question 17: Specific Card Pull II

Topic: probability **Difficulty**: medium
A deck of cards is shuffled well. The cards are dealt one-by-one, until the two of hearts appears. Find the probability that exactly one king, queen and jack appear before the two of hearts.

Question 18: Straddle Delta

Topic: finance **Difficulty**: easy
Let's consider a straddle on underlying S with strike K and expiry T. You have a put option on S, also with strike K and expiry T. This put option has $\Delta = -0.31$. What is the Δ of the straddle? Assume Black-Scholes dynamics.

Question 19: Mixing Glasses

Topic: brainteasers **Difficulty**: medium
You have a glass of orange juice and a glass of water with equal volumes. You pour some water into the orange juice glass and then pour some of the mixed fluids back into the water glass such that their volumes are again equal. Let x be the percent concentration of water in the glass of diluted water, and y be the percent concentration of orange juice in the glass of diluted orange juice. What is $x - y$?

Question 20: Cheese Lover II

Topic: probability **Difficulty**: easy
Jon loves cheese. He decides to make 100 blocks of cheese. The distribution of the weight (in grams) of each block he makes follows IID Exp$\left(\frac{1}{250}\right)$ distribution. Let W_i denote the weight of the ith block of cheese, and T_{100} represent the total weight of the 100 blocks of cheese. Using Chebyshev's Inequality, what is an upper bound on $\mathbb{P}[T_{100} > 30000]$?

Question 21: Beer Bottles

Topic: probability **Difficulty**: hard
Bob is singing the traditional song, âN Bottles of Beer.â With each verse, he counts down the number of bottles. The first verse contains the lyrics âN bottles of beer,â the second verse contains the lyrics â$N-1$ bottles of beer,â and so on. The last verse contains the lyrics â1 bottle of beer.â Thereâs just one problem: Bob has early onset Alzheimer's. When completing any given verse, he has a tendency to forget which verse he's on. When this happens, he finishes the verse he is currently singing and then goes back to the beginning of the song (with N bottles) on the next verse.

For each verse, suppose you have a $\frac{1}{N}$ chance of forgetting which verse you are currently singing. Let K denote the expected number of verses in the song. Compute $\lim_{N\to\infty} \frac{K}{N}$. The answer is in the form $ae - b$ for integers a and b. Find $a^2 + b^2$.

Question 22: Consecutive Children

Topic: brainteasers **Difficulty**: hard

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children of distinct ages (in years) were born with a fixed interval of time between consecutive children. The sum of the squares of the ages of all the children is equal to the square of their father's age. Assuming that the father is aged at most 60, what is the father's age?

Question 23: Fleeing Flea

Topic: brainteasers **Difficulty**: medium
There is a flea being chased by a snake and it needs to get safety as fast as possible. The flea is sitting on one of the corners of a right parrallelpiped with side lengths of 2, 2, and 3 and can run at a pace of 2.5 units per second. It is

trying to reach the safety of the direct opposite corner, how many seconds until it can reach the other corner?

Question 24: Determination II

Topic: statistics **Difficulty**: hard
Given three data sets X_1, X_2, and Y , we run two linear regressions to obtain $y \sim \alpha_1 + \beta_1 x_1$ and $y \sim \alpha_2 + \beta_2 x_2$. The R^2 value for both regressions is 0.05. Find the lowest upper bound on R^2 value of the regression $y \sim \alpha + \beta' x_1 + \beta'' x_2$.

Question 25: Coin on Chess Board

Topic: probability **Difficulty**: easy
A chess board consists of 2 inch by 2 inch squares. We toss a coin (diameter of 1 inch) that lands somewhere on the board randomly. What is the probability that the coin is completely within one of the 2 inch by 2 inch squares (not on more than 1 square)?

Question 26: Shattering Orbs

Topic: probability **Difficulty**: hard

$$7$$

orbs are labeled $1 - 7$ and are linked linearly in a vertical stack from the ceiling with orb 1 being a part of the ceiling and orb 7 being closest to the floor. Each orb is attached to adjacent orbs by a chain link. At each time step, one of the remaining links is going to be uniformly at random selected and cut. As a result, all the orbs below that link will fall and shatter. What is the expected number of cuts needed until orb 1 is the only remaining orb?

Question 27: Matrix Exponential

Topic: pure math **Difficulty**: medium
Find trace(e^A) to 3 decimal points, where A is defined as

$$\begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$$

The answer will be in the form $e^a + e^b$ for integers a and b. Find ab.

Question 28: Consecutive Pairs

Topic: probability **Difficulty**: hard
Consider the set of 10 consecutive integers $\{1, 2, \ldots, 10\}$. How many subsets contain exactly 1 pair of consecutive integers? For example, $\{3, 5, 6, 9\}$ contains exactly 1 pair of consecutive integers.

Question 29: Die Multiple II

Topic: probability **Difficulty**: medium
You roll a fair $6-$sided die until the sum of all upfaces is a multiple of 6. Find the expected number of rolls performed.

Question 30: Thick Coin

Topic: probability **Difficulty**: hard
Let the radius of a penny be 1. Assume that the thickness of the penny is non-negligible so that a flipped penny can land on its side. Find the thickness of the penny that, when flipped, has a $\frac{1}{3}$ chance of landing on its side. The answer is in the form $\frac{1}{\sqrt{k}}$ for an integer k. Find k.

Note: There is no definite answer for this question. Take the approach of inscribing the penny in a sphere.

Question 31: Arithmetic Mean

Topic: statistics **Difficulty**: easy
The arithmetic mean of 5, 6, 11, x, and y is 20. What is the arithmetic mean of x and y?

Question 32: Intersecting Chords

Topic: probability **Difficulty**: medium

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chords with uniformly randomly chosen endpoints are drawn on a circle. What is the expected number of intersections?

Question 33: Particle Reach I

Topic: probability **Difficulty**: medium
Consider a particle that performs a random walk on the integers starting at position 0. At each step, the particle moves from position i to position $i+1$ with probability p, while the probability it moves from i to $i-1$ is $1-p$. If $p = 1/3$, find the probability the particle ever reaches position 1.

Question 34: Silly SDE

Topic: pure math **Difficulty**: hard
Let W_t be a standard Brownian Motion. Let X_t be a process satisfying the SDE

$$dX_t = \kappa(\theta - X_t)dt - \sigma\sqrt{X_t}dW_t$$

with $X_0 = x > 0$. It can be shown (you do not need to do this) that if $\frac{2\kappa}{\theta} > \sigma^2$, $X_t > 0$ with probability 1, so $\sqrt{X_t}$ is defined almost surely. For $T > 0$, $\mathbb{E}[X_T]$ can be written as a function of $x, \kappa, \theta, \sigma$, and T. Evaluate this function when $T = 10, \kappa = 0.2, \theta = 2, x = 5$, and $\sigma = 0.1$. The answer will be in the form $A + Be^C$ for integers A, B, and C. Find ABC.

Question 35: St. Petersburg Paradox

Topic: probability **Difficulty**: easy
Suppose you are offered to play a game where you flip a fair coin until you obtain a heads for the first time. If the first heads occurs on the nth flip, you are paid out $\$2^n$. What is the fair value of this game? If your answer is infinite, enter -1.

Question 36: Correlation Ranges

Topic: probability **Difficulty**: medium
Suppose that X, Y, and Z are three random variables. We know that $\text{Corr}(X, Y) = \frac{5}{13}$ and $\text{Corr}(Y, Z) = \frac{12}{13}$. The range of possible values for $\text{Corr}(X, Z)$ is an interval in the form $[0, b]$, where b is a fraction in fully reduced form. Find b.

Question 37: Coefficient Swap

Topic: statistics **Difficulty**: medium
Suppose that we have two datasets X and Y with $\text{Var}(X) = 10$ and $\text{Var}(Y) = 20$. We perform the linear regression $y \sim \alpha_x + \beta_x x$ and obtain $\beta_x = 1$. Suppose now that we perform the regression $x \sim \alpha_y + \beta_y y$. Find β_y. If the value can't be determined, enter -100.

Question 38: Colorful Socks I

Topic: probability **Difficulty**: easy

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pairs of socks, each with a distinct color, are in a drawer. You draw out 2 socks at random. Find the probability that you obtain a matching pair.

Question 39: Slippery Ladder II

Topic: pure math **Difficulty**: medium
A 50−ft ladder is placed against a vertical wall of a large building. The base of the ladder is in oil, which makes the base slip and the tip of the ladder slide down the wall. The base of the ladder slips away from the wall at a constant rate of 4 feet per minute. Find the rate at which the angle between the ladder and ground is decreasing (in radians per minute) when the base of the ladder is 30 feet away from the wall.

Question 40: Bull Call Spread I

Topic: finance **Difficulty**: easy
Consider the following asset S, with initial price $S_0 = 7$. In this bull call spread, you will long a call at strike $K = 5$ and short a call at strike $K = 10$. What is the maximum and minimum *payoff* of this contract?

Give the result in the form: $\max^2 + \min^2$

Question 41: Limiting Values I

Topic: probability **Difficulty**: easy
You roll a fair die until a value other than 1 appears and are paid the amount on the die that appears on the last roll. What is the fair value of this game?

Question 42: Absolute Difference

Topic: probability **Difficulty**: easy
Let $x_0 = 0$ and let $x_1, \ldots, x_{10}$ satisfy that $|x_i - x_{i-1}| = 1$ for $1 \leq i \leq 10$ and $x_{10} = 4$. How many such sequences are there satisfying these conditions?

Question 43: Counting Digits

Topic: brainteasers **Difficulty**: medium
How many digits are in 99 to the 99th power?

Question 44: Horse Arbitrage

Topic: finance **Difficulty**: easy
There are three horses numbered $1, 2$, and 3. You have \$1 to make bets. You may make bets in fractions of dollars. If horse 1 wins, you get \$2 back for a \$1 bet. If horse 2 wins, you receive \$4 back for a \$1 bet. If horse 3 wins, you receive \$6 back for a \$1 bet. Note that you do not receive your initial bet back. There is an arbitrage here. Find the maximum guaranteed profit that can be made from this arbitrage.

Question 45: Ants on a Triangle

Topic: probability **Difficulty**: easy
There are three ants each on their own side of an equilateral triangle. Each picks one adjacent vertex to move to with equal probability. What is the probability that no two ants will intersect at a corner?

Question 46: McQueen Speeding

Topic: brainteasers **Difficulty**: easy
Lightning McQueen, Doc Hudson, and Chick Hicks all race in the Piston Cup to run it back. They race around an 1 kilometer track. It turns out that Lightning McQueen is first, Doc Hudson is second, and Chick Hicks is third. Lightning McQueen finishes his lap 200 meters ahead of Doc Hudson. Furthermore, Doc Hudson finishes his lap 200 meters ahead of Chick Hicks. Assuming that each car moves with a constant speed, at the moment when Lightning McQueen finishes his lap, how many meters is Chick Hicks away from finishing his lap?

Question 47: The Perfect Hedge I

Topic: finance **Difficulty**: medium
You have two assets. We will call them asset 1 and asset 2. Asset 1 has an expected return of 4% and a variance of 15%. Asset 2 has an expected return of 2% and a variance of 4%. They have a correlation $\rho = -1$.

We want to create a risk-free portfolio using assets 1 and 2. We will denote w_1 and w_2 as the weights of asset 1 and 2 in the portfolio respectively. Assume that $w_1 + w_2 = 1$. What is w_1? Give the answer to 2 decimal points.

Question 48: First Ace

Topic: probability **Difficulty**: medium
On average, how many cards in a normal deck of 52 playing cards do you need to flip over to observe your first ace?

Question 49: 4 Die Sum

Topic: probability **Difficulty**: easy
Calculate the probability that when we roll 4 fair 6−sided dice, the sum of their upfaces is 20.

Question 50: Common Ball Draw

Topic: probability **Difficulty**: hard

$$k \geq 2$$

people play a game as follows: Each of the k people go up one-at-a-time to draw $1 \leq r \leq n$ balls without replacement from an urn that contains n balls labelled $1 - n$. Once a given person has drawn their r balls, they note them, and then put them back in the urn for the next player to draw. Let $p(k, n, r)$ be the probability that all k of the people draw at least one ball in common under the specifications above. Find $p(4, 17, 5)$. Round your answer to three decimal places.

Question 51: Diagonal Eigenvalue

Topic: pure math **Difficulty**: medium
Consider the matrix 30×30 matrix A with $A_{ii} = 30$ for $1 \leq i \leq 30$ and $A_{ij} = 1$ for $i \neq j$. Let $\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$ be the vector of distinct eigenvalues $\lambda_1 \neq \lambda_2$ of A and $g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$ be the vector of geometric multiplicities corresponding to λ_1 and λ_2, respectively. Find $||\lambda + g||^2$

Question 52: Triangle Area

Topic: probability **Difficulty**: easy

$$X, Y \overset{\text{iid}}{\sim} \text{Unif}(\{1, 2, 3, 4, 5\})$$

. A triangle is drawn with vertices $(0, 0), (X, 0), (0, Y)$. Compute the expected area of the triangle.

Question 53: Bond Practice I

Topic: finance **Difficulty**: easy
Say you have a generic bond that has a 5 year maturity, $3000 face value, an annual coupon of 4$

Question 54: Hand Meet

Topic: brainteasers **Difficulty**: easy
It is presently $12:00$ PM. The minute and hour hands meet there. After how many hours will the minute and hour hands meet again? Enter a fraction if it is not an integer amount of hours.

Question 55: Positive Brownian II

Topic: pure math **Difficulty**: hard
Let B_t be a standard Brownian Motion. Find $\mathbb{P}[B_2 > 0, B_8 > 0]$.

Question 56: The Price of a Garden

Topic: brainteasers **Difficulty**: easy
A neighbor told you he was offered a triangular piece of ground for a garden. Its sides were 55 yards, 62 yards, and 117 yards. The price was \$10 per square yard. What will its cost be?

Question 57: 3 Larger Die

Topic: probability **Difficulty**: easy
You have two fair $10-$sided dice that are colored green and yellow with values $1-10$ on each side. You roll both dice simultaneously. If they show the same value, then the game is over. If the two dice are not equal, but the yellow die shows a value at least 3 larger than the green die, you receive \$1 and can roll both dice again. In all other cases, you receive nothing and can roll again. The game only ends once the two dice show the same value. What is your expected payout on this game?

Question 58: Odd Before Even

Topic: probability **Difficulty**: easy
Suppose you are continually rolling a standard fair $6-$sided die. Find the probability that all of the odd values show up before the first even value.

Question 59: Pharmaceutics II

Topic: statistics **Difficulty**: medium
A pharmaceutical company has researched a drug that they claim will enhance focus for 80% of people suffering from attention deficit disorder. After examining the drug, the FDA believes that their claims regarding the effectiveness of the drug are inflated. In an attempt to disprove the company's claim, the FDA administers the drug to 20 people with attention deficit disorder and observe X, the number for whom the drug dose induces focus. More formally, the FDA is testing the null hypothesis $H_0 : p = 0.8$ against the alternative hypothesis $H_a : p < 0.8$. Assuming the rejection region $x \leq 12$ is used, what is power of this test when $p = 0.6$?

Question 60: Matching Die Trio

Topic: probability **Difficulty**: medium
Three fair $6-$sided dice are rolled and their upfaces are recorded. Find the probability that the values showing upon rolling all three dice again is the same as the original three values recorded.

Question 61: Put-Call Parity I

Topic: finance **Difficulty**: medium
You are a trader trying to find an arbitrage opportunity. Your friend tells you to look at the \$100 strike on the \$$TSLA$ April options chain, as he thinks there may be a mispricing. Currently, \$$TSLA$ is trading at \$110, and the put sells for \$2. Your friend tells you if the call is priced at anything under \$14 then the market is mispricing the option and you should buy as many calls as possible. Is your friend right?

If so, input 14. If not, input the whole number the call would need to be priced under to present an arbitrage opportunity.

(TSLA does not pay a dividend)

Question 62: Differ By 2

Topic: probability **Difficulty**: medium
How many times do we have to roll a fair 6-sided die till we roll two numbers in a row that differ by 2?

Question 63: Pass the Ball

Topic: probability **Difficulty**: medium
You and 4 other people are sitting in a circle. You are given a ball to start the game. Every second of this game, the person with the ball has three choices they can make. They can either pass the ball to the left, pass the ball to the right, or keep the ball (all with equal probability). This game goes on till someone keeps the ball. What is the probability that you are the person to end the game and keep the ball?

Question 64: Unlucky Seven II

Topic: probability **Difficulty**: medium
You are given a fair $6-$sided die and you roll it. You can either choose to keep your roll and receive the observed value in dollars. Alternatively, you are allowed to roll again, but if the sum of your two rolls is at least 7, you pay the value equal to your first roll. If the sum of the two rolls is less than 7, you receive the sum of the two observed values in dollars. Assuming optimal play, what is your expected payout?

Question 65: 3 Heads 3 Tails II

Topic: probability **Difficulty**: medium
Anna and Brenda are playing a game. They repeatedly toss a coin. Anna wins if 3 heads appear in a row. Brenda wins if 3 tails appear in a row. What is the expected number of coin tosses for a winner to be determined?

Question 66: Cats and Dogs I

Topic: probability **Difficulty**: medium
Six dogs and six cats are sitting at a circular table uniformly at random. Find the probability that there are at least four dogs in a row somewhere in the circle.

Question 67: Increasing Dice Order I

Topic: probability **Difficulty**: medium
You throw three fair dice one by one. What is the probability that you obtain three numbers in strictly increasing order?

Question 68: Non-Zero Eigenvalue

Topic: pure math **Difficulty**: hard
Let $x_n = \begin{bmatrix} 1 & 2 & \dots & n \end{bmatrix}^T \in \mathbb{R}^n$. Define $A_n = x_n x_n^T$. For any fixed n, there is a single non-zero eigenvalue. Call this eigenvalue λ_n. Find the value of k such that $\dfrac{\lambda_n}{n^k}$ converges to a finite non-zero value.

Question 69: Car Crashes

Topic: probability **Difficulty**: easy
On a given busy intersection, the probability of at least one car crash in a 1 hour time interval is $\dfrac{8}{9}$. Assuming that car crashes occur independently of one another and at a constant rate throughout time, find the probability of at least one car crash in a 30 minute interval.

Question 70: Prime First

Topic: brainteasers **Difficulty**: hard
Consider the set of integers $S = \{2, 3, \dots, 30\}$. Alice and Bob play a game where they take turns selecting integers from S. The first player to select an integer that shares a common factor with a previously picked integer loses. Alice has the ability to determine if she wants to select first or second. Assume both players play optimally. Let $p = 1$ if Alice should choose to go first and $p = 2$ if she should go second. There are 6 optimal selections $v_1, \dots, v_5$ for Alice's first turn. Find $100p + \dfrac{v_1 + v_2 + v_3 + v_4 + v_5 + v_6}{6}$.

Question 71: Decorrelation

Topic: probability **Difficulty**: easy
Let X and Y be random variables with finite expectation. Suppose we know $\mathrm{Var}(X) = 16$ and $\mathrm{Cov}(X, Y) = 8$. Find a constant k so that X and $Y - kX$ are uncorrelated.

Question 72: Doda Rectangle

Topic: probability **Difficulty**: medium
Each vertex of a regular dodecagon is randomly colored either red or blue. A "dodeca-rectangle" is defined as a rectangle whose 4 identically-colored vertices belong to the regular dodecagon. Compute the probability that a dodeca-rectangle cannot be formed.

Question 73: Multinomial Expansion

Topic: probability **Difficulty**: medium
How many terms are there in the expansion of $(x_1 + x_2 + x_3 + x_4 + x_5 + x_6)^{18}$ after all like terms have been combined?

Question 74: Dice Products

Topic: probability **Difficulty**: easy
Take the product of 2 fair dice rolls. What is the probability that it is divisible by 6?

Question 75: Tennis Deuces II

Topic: probability **Difficulty**: easy
Andrew and Beth are playing a game of tennis. Tennis scoring works off these rules:

1. Scoring starts at 0-0 and when someone wins a point, their score goes up by 1 (thus either 1-0 or 0-1).

2. If a player gets 4 points before the other player, they win unless the score was 3-3 (deuce). If the score was 3-3, a player has to win by two points to win the game.

How many ways are there to get to a score of 4-4?

Question 76: Red and Black Urn II

Topic: probability **Difficulty**: easy

Complete Red and Black Urn I Prior!

An urn initially consists of $r > 0$ red and $b > 0$ black balls inside. Balls are drawn one-by-one. If the ball drawn is red, return it to the urn. If the ball drawn is black, replace it by a red ball in the urn. The probability that the nth ball drawn is red is a function $f(r, b, n)$. Find $f(8, 12, 11)$ to the nearest hundredth.

Question 77: Covariance of BM

Topic: pure math **Difficulty**: easy
Let W_t be a standard Brownian Motion. Compute $\text{Cov}(W_1, W_2)$.

Question 78: Conditional Head Starter

Topic: probability **Difficulty**: medium
A fair coin is flipped n times. It turns out that no heads appear consecutively in the sequence. Given this information, in terms of n, find the probability that the first flip was a heads. You should get a function $p(n)$. Evaluate $p(10)$.

Question 79: Car Bidding II

Topic: statistics **Difficulty**: easy
Fred is selling his old car. He will sell it to the first bidder that places a bid of at least \$9000. He receives bids for the car that are all independent and identically distributed exponential random variables with mean \$5000. Given that Fred sells his car, what is the probability that he sells the car for at least \$15000? The answer is in the form e^a for a rational number a. Find a.

Question 80: 1 or Bust

Topic: probability **Difficulty**: medium
You roll a die. If it is 1, 2, or 3, you obtain 1 and roll again. If you roll 4 or 5, you cash out and the game ends. If the dice is 6, you lose your gains and the game ends. How much would you pay to play this game?

Question 81: Row Your Boat

Topic: brainteasers **Difficulty**: medium
A group of friends travels in a canoe. They travel upstream from their campsite for three hours. After a while, they want to go back home. However, they travel five hours downstream and end up 32 miles downstream below their initial campsite. The next morning, they travel back up the river to their campsite and arrive at 7:00 PM. If the river flows downstream at a constant rate of 2 miles

per hour and the friends row their canoe at a constant speed, what time did they leave to go back home the next morning? Enter your answer in military time. For example, if the time is 4:30 AM, then enter 430 as your answer. If the answer is 4:30 PM, enter 1630 as your answer.

Question 82: Ancient Births

Topic: brainteasers **Difficulty**: medium
Suppose we have two people B and C. B died 129 years after C was born. At least one of B or C was alive for exactly 100 years. C died in 30 B.C. When was B born? If the answer is in BC, enter it as a negative number. For example, if the answer was 100 B.C., enter it as -100.

Question 83: Windless Mile

Topic: brainteasers **Difficulty**: easy
A man riding on a bike can travel 1 mile in 3 minutes if the wind blows in the direction he is moving and in 4 minutes if it is against him. How fast can the man travel 1 mile if there is no wind?

Question 84: Number Concatenate

Topic: brainteasers **Difficulty**: medium
Write out the decimal expansions of 2^{1000} and 5^{1000} adjacent to one another. Concatenate them to form a new number, say x. Find the amount of digits in x.

Question 85: 3 Heads 3 Tails I

Topic: probability **Difficulty**: easy
Anna and Brenda are playing a game. They repeatedly toss a coin. Anna wins if 3 heads appear. Brenda wins if 3 tails appear. The heads and tails do not need to be consecutive. What is the expected number of coin tosses for a winner to be determined?

Question 86: Multinomial Sum

Topic: probability **Difficulty**: medium
Find the sum of all multinomial coefficients $\binom{7}{b_1, b_2, b_3, b_4}$, where $b_1 + \cdots + b_4 = 7$ and each $b_i \geq 0$ is an integer.

Question 87: Put-Call Arbitrage

Topic: finance **Difficulty**: easy
You have access to a stock with price $S_0 = 10$, a European call on S with price $C_0 = 4$, a European put on S with price $P_0 = 3$, both at strike $K = 4$, and bonds paying 1 at time T. Assume interest rates are 0. Find the arbitrage. You are allowed to long or short assets.

Give the answer in the format of # Stock + # Call + # Put + # Bonds

Question 88: High-Low Guess

Topic: brainteasers **Difficulty**: easy
You and your friend play a game where your friend first selects a random number between 1 to 1000. Afterwards, you must guess the number. At each turn, your friend must reveal if your guess is higher or lower than the actual number he guessed. Let n be the minimum number of guesses you must perform to find your friend's number, regardless of what they selected. Find n.

Question 89: Ten Ten

Topic: probability **Difficulty**: easy
Adam is rolling a fair $10-$sided die. He gets to roll repeatedly and may decide when to stop at any time. If he obtains a value that is not 10 on each roll, he adds the upface to his total monetary sum. If he rolls a 10, he loses all of his money. If Adam plays optimally, he should stop once he has at least \$$k$ total. Find k.

Question 90: Voter Mayhem I

Topic: probability **Difficulty**: hard
Two candidates, say A and B, are running for office. Candidate A received n votes, while Candidate B received m votes, with $n > m$. The $n + m$ votes are thrown into a box and shuffled around. Then, the votes are drawn without replacement one-by-one. A running tally of the number of votes for each candidate is kept. The probability that Candidate A is always strictly ahead in the voting count (excluding the initial state where both have 0) is a function $P(n, m)$. Find $P(100, 80)$.

Question 91: Say Your Color

Topic: brainteasers **Difficulty**: easy
If WHITE=000, RED-101, BLUE-110, and PURPLE-100, then what three-digit string corresponds to YELLOW?

Question 92: Basic Gamma

Topic: finance **Difficulty**: easy
What is the gamma of a deep out-of-the-money call?

Question 93: Bivariate Covariance

Topic: probability **Difficulty**: medium
Let (X, Y) follow a Bivariate Normal distribution with X and Y marginally standard normal and $\mathrm{Corr}(X, Y) = \rho$. Compute $\mathrm{Cov}(X, Y^2)$.

Question 94: Replacement Orbs

Topic: probability **Difficulty**: easy
An urn containing 2 red and 1 blue orb is in front of you. At each step, you will take out an orb uniformly at random and replace it with a blue orb, regardless of the color selected. Find the expected amount of draws needed until the urn only contains blue orbs.

Question 95: Illegible Dice

Topic: brainteasers **Difficulty**: easy
You are given two fair dice: one is a normal six-sided die and the other is an illegible, six-sided die. Note that the values on this other die do not necessarily need to be $1 - 6$. Whenever you toss the two dice, it is equally likely for the sum of the faces to be any integer between 1 and 12, inclusive. What is the sum of the sides of the illegible die?

Question 96: Sharpe Marbles

Topic: statistics **Difficulty**: medium
Siblings Alice and Bob play a game with marbles. Each player has one red and one blue marble and shows one marble to the other uniformly at random. If both show blue, Alice wins \$1. If both show red, Alice wins \$3. Else, Bob wins \$2. Note that the winnings come from their mother, not the other player. Let

A_r and B_r define the ratio between the expected return and variance of Alice's and Bob's payoffs, respectively. What is $B_r - A_r$?

Question 97: Dollar Cent Switch

Topic: brainteasers **Difficulty**: medium
A man went to his local bank to cash a check. When giving his money to the cashier, by mistake, gave him dollars as cents and cents as dollars. He didn't examine the money and just walked home and spent a nickel on some candy. He then found that he possessed exactly twice as much money as the check was worth. He had no money in his pocket before going to the bank. How much was the check for?

Question 98: Questionable Values

Topic: brainteasers **Difficulty**: easy
Sean is creating questions for QuantGuide. Each easy question is worth \$3, while each medium question is worth \$6. He has a question storage saved up of easy and medium questions. If Sean picks a question at random, the probability it is a medium is 1/4. The total value of Sean's question bank is a perfect square. What is the smallest possible value of his question bank (in dollars)?

Question 99: Poisson Review II

Topic: probability **Difficulty**: easy
On average, 7 customers arrive at the QuantGuide gift shop per hour following a Poisson process. Suppose it takes 10 minutes to serve each customer. Compute the sum of the mean and variance of total service time in hours for customers arriving in a given 1 hour period.

Question 100: Cylindrical Intersection

Topic: pure math **Difficulty**: medium
Two cylinders of radius 1 intersect at right angles to one another. Furthermore, their central axes also intersect. Find the volume of the enclosed region.

Question 101: Word Shift II

Topic: probability **Difficulty**: medium
Find the number of anagrams of $BOOLAHUBBOO$ that have at least 2 B's before the first O.

Question 102: Graph Search I

Topic: probability **Difficulty**: medium
You are given an undirected graph with 10 nodes. From every node, you are able to access any other node (including itself), all with an equal probability of 1/10. What is the expected number of steps to reach all nodes at least once (rounded to the nearest step)?

Question 103: Black or Yellow

Topic: brainteasers **Difficulty**: easy
There are four balls, two black and two yellow, in a box. You pick out one at a time at random without replacement. Before picking each one out, you guess at the color, and if you're right, you receive in dollars the number of balls left in the container before you chose it. For example, if you guess the first ball correctly, you would receive $4 because there were 4 balls remaining in the box before you picked one out. Assuming optimal play, what is the expected payoff for this game?

Question 104: Options Gamma

Topic: finance **Difficulty**: easy
We have a European call and put option at strike K. The call option has a Γ of 0.02. What is the Γ of the put option?

Question 105: Complex Exponential

Topic: brainteasers **Difficulty**: easy
Find the principal branch value of i^i. The answer is in the form $e^{-c\pi}$ for a constant c. Find c.

Question 106: Stone Ripple

Topic: pure math **Difficulty**: easy
A stone is thrown into a pond and sends out a circular ripple whose radius increases at a rate of 2 feet per second. After 10 seconds, how fast is the area of the ripple increasing (in square feet per second)? The answer is in the form $k\pi$ for an integer k. Find k.

Question 107: Random Angle II

Topic: probability **Difficulty**: easy
A right triangle is being formed with legs labeled A and B. The random lengths of legs A and B are both IID Unif$(0, 1)$ RVs. Let θ be the angle (of random degree measure) that is opposite of side A. Find the probability $\theta > \frac{\pi}{3}$. The answer is in the form $\frac{a}{b\sqrt{c}}$ for integers a, b, c that are relatively prime. Find abc.

Question 108: Compound Interest I

Topic: brainteasers **Difficulty**: easy
You start with \$100 in your bank account today. You invest in a stock that yields 1% interest that is compounded daily. To the nearest dollar, how much will you have in your bank account after 100 days?

Question 109: Negative Correlated Sum

Topic: probability **Difficulty**: easy
Suppose X and Y are two random variables with respective variances 9 and 16. Find the standard deviation of $X + Y$ if $\rho(X, Y) = -3/8$.

Question 110: Orthogonal Cosine

Topic: pure math **Difficulty**: easy
Let x and y be vectors in $\mathbb{R}^n$ with angle 75 between them. Let A be an orthogonal $n \times n$ matrix. Find the angle between Ax and Ay.

Question 111: 112 Appearance

Topic: probability **Difficulty**: medium
Suppose you roll a fair $6-$sided die until you either obtain a 2 for the first time or 2 consecutive 1s. Find the expected number of rolls you perform.

Question 112: Field Imperfection

Topic: probability **Difficulty**: easy
Suppose that a field for the World Cup is being constructed. The field is intended to have dimensions of 300 feet long by 200 feet wide. However, due to human imperfection, the error in the length measurement is a RV $X \sim$ Unif$(-15, 15)$ and the error in the width measurement is a RV $Y \sim$ Unif$(-5, 15)$, independent

of X. Hence, the true dimensions are $300 + X$ and $200 + Y$. Find the expected area of the field.

Question 113: Brussels Sprouts

Topic: probability **Difficulty**: easy
Your mother plays a game with you. She allows you to roll a fair dice and if the value is at least three, you will eat that many Brussels sprouts and the game ends. Else, you will eat that many Brussels sprouts and roll again until the game ends. On average, how many Brussels sprouts will you eat?

Question 114: Coloring Components II

Topic: probability **Difficulty**: medium
Consider a line of 25 adjacent colorless squares. Color in each individual square black with probability $\frac{3}{4}$ or white with probability $\frac{1}{4}$, independent of all other squares. A connected component is a maximal sequence of adjacent squares all with the same color. For example, BBWBWWWBBW has 6 connected components. Find the expected number of connected components in our line.

Question 115: Multiple Divisors II

Topic: probability **Difficulty**: medium
Find the probability that a uniformly randomly selected integer from the set of divisors of 20! is divisible by 20.

Question 116: Prime Subset

Topic: brainteasers **Difficulty**: hard
Consider all subsets of $S = \{1, 2, 3, \ldots, 30\}$ so that each pair of numbers in that subset are coprime. Find the subset $\Omega \subseteq S$ satisfying the previous condition whose elements have the largest sum. What is the sum of all the elements in Ω?

Question 117: Red and Black Urn I

Topic: probability **Difficulty**: medium
An urn initially consists of $r > 0$ red and $b > 0$ black balls inside. Balls are drawn one-by-one. If the ball drawn is red, return it to the urn. If the ball drawn is black, replace it by a red ball in the urn. Let R_n be the expected number of red balls in the urn after n drawings have been done. Find a closed form function for $R_n = f(r, b, n)$. Calculate $f(8, 12, 10)$ to the nearest hundredth.

Question 118: Eigenspace Intersection

Topic: pure math **Difficulty**: easy
Let A be a $n \times n$ matrix with two distinct eigenvalues $\lambda_1 \neq \lambda_2$. Let E_{λ_1} and E_{λ_2} be the eigenspaces corresponding to the two eigenvalues. Find the cardinality ("size") of $E_{\lambda_1} \cap E_{\lambda_2}$. If this intersection is not finite, enter -1.

Question 119: Light Switch

Topic: probability **Difficulty**: easy
A stoplight displays the colors red (stop), yellow (slow down), and green (go). The colors go in the sequence green $\rightarrow$ yellow $\rightarrow$ red. The light stays green for 40 seconds until it switches to yellow for 4 seconds. Afterwards, it turns red for 40 seconds before turning green again and repeating the cycle. Find the probability that the color of the stoplight switches in a 4 second interval.

Question 120: 60-40 Split

Topic: probability **Difficulty**: easy
You have a fair $40-$sided and fair $60-$sided die in front of you. If you roll both, find the probability the $60-$sided die shows a strictly larger number than the $40-$sided die.

Question 121: Company Purchase I

Topic: finance **Difficulty**: easy
How much would you pay for a company that generates \$100 of cash flow every single year, until you sell, if you have a targeted yield of 16%?

Enter your solution rounded to the nearest whole number.

Question 122: Double Data Trouble II

Topic: statistics **Difficulty**: easy
Suppose that you run linear regression on some dataset and obtain the coefficients $\hat{\beta}_{OLS}$. Recall that if X is the data and σ^2 is the variance of the IID normal errors, then $\mathrm{Var}\left(\hat{\beta}_{OLS}\right) = \sigma^2(X^TX)^{-1}$. If you were to run linear regression again on the dataset where you duplicate each point in your original dataset and obtain new coefficients $\hat{\beta}'_{OLS}$, find the constant c such that $\mathrm{Var}\left(\hat{\beta}'_{OLS}\right) = c\mathrm{Var}\left(\hat{\beta}_{OLS}\right)$. If no such constant exists, enter -1.

Question 123: Hours of Labor

Topic: probability **Difficulty**: medium
A factory wants to maximize the number of hours their employees are not working each year. If none of the employees has a birthday on a given day of the year, then every employee must come to work all 24 hours that day. If at least one person has a birthday on a given day, then none of the employees need to work that day. This factory has good work-life balance, so they want to hire the number of people that would maximize the expected total number of hours that employees are NOT working each year. How many employees should the factory hire? If the answer is infinite, answer -1.

Question 124: Overlapping Subsets

Topic: probability **Difficulty**: medium
Consider the set $\Omega = \{1, 2, \ldots, 20\}$. Two subsets A and B of Ω are uniformly at random selected from the power set of Ω independently. It is possible that $A = B$. Define $N = |A \cap B|$. Compute $\dfrac{\mathrm{Var}(N)}{\mathbb{E}[N]}$.

Question 125: Error and Residual

Topic: statistics **Difficulty**: easy
We run an OLS regression and observe that SSE = 120 and SSR = 80. What is R^2 for this regression?

Question 126: Prime Sum

Topic: probability **Difficulty**: easy
Two distinct prime integers between 1 and 20, inclusive, are selected uniformly at random. Find the probability their sum is even.

Question 127: Always Profit II

Topic: brainteasers **Difficulty**: easy
QuantGuide stock will either double its value or go down 50% by tomorrow. You can also bet whether the stock goes up or down with your friend at 1 : 1 payout. Let A and B be the lowest round bet (can only bet whole dollars) amount you long/short the stock and bet on the stock going up/down with your friend respectively. Assume that if you are betting, you want to make the same amount profit no matter the outcome of the stockâs movement tomorrow (bank balance will be the same no matter the outcome). How much do you profit from betting A and B? If you canât guarantee profit, enter 0.

Question 128: Bowl of Cherries V

Topic: probability **Difficulty**: hard
There are two red cherries and five purple cherries in bowl A. There are six red cherries and three purple cherries in bowl B. If possible, Jenny randomly transfers a cherry from bowl B into bowl A. Then, she randomly picks a cherry from bowl A to eat. She repeats this process until all cherries are eaten. What is the probability that the last cherry she eats is red? Round your answer to 5 decimal places.

Question 129: Birthday Off

Topic: probability **Difficulty**: medium
Let $b(n)$ be the expected number of distinct birthdays (number of days of the year where exactly one person has a birthday) among n people. There are two integer values of n that maximize $b(n)$. Find the sum of the two values.

Question 130: Car Line

Topic: brainteasers **Difficulty**: medium
Alice, Bob, and Carter are driving down a one-lane road in the same direction. Each car moves at some constant speed. In some instant, Alice is a distance d behind Bob and Carter is a distance $2d$ in front of Bob. Alice passes Bob 7 minutes after this instant. 5 minutes after that, Alice passes Carter. How many seconds after Alice passes Carter would Bob pass Carter?

Question 131: Comparing Flips II

Topic: probability **Difficulty**: medium
You and your friend are playing a game with a fair coin, tossing it and writing down the outcomes. You win if HTH appears before HHT, else your friend wins. What is the probability that your friend wins?

Question 132: Colorful Bracelet

Topic: probability **Difficulty**: easy
How many unique bracelet configurations can be made with 3 red and 3 blue beads? Configurations that can be made as rotations of other configurations are considered indistinguishable. Do not consider symmetry over flipping the bracelet. For example,

$$RRGRGG \neq RRGGRG$$

Question 133: Options Rho

Topic: finance **Difficulty**: easy
If interest rates increase, how does this affect the price of a European call option?

Enter 1 if it increases, 0 if no change, or -1 if it decreases.

Question 134: Car Question

Topic: probability **Difficulty**: hard
Suppose that we have 2 cars parked in a line occupying spaces 1 and 2 of a parking lot. Spaces 3 and 4 are initially empty. Every minute, a car is considered eligible to move forward one space if a) the space in front of them is empty prior to any potential moves and b) label of the space they are on is less than 4. Suppose that every minute, if a car is eligible to move, it moves forward one space with probability $1/2$. What is the expected amount of minutes before the cars occupy spaces 3 to 4?

Note: The space in front must be empty before any moves. For example, on the first turn, only the car in space 2 is eligible to move.

Question 135: 77 Multiple I

Topic: brainteasers **Difficulty**: easy
What is the smallest multiple of 77 that is at least $70,000$?

Question 136: Basic Die Game VI

Topic: probability **Difficulty**: medium
Alice rolls a fair $6-$sided die with the values $1-6$ on the sides. She sees that value showing up and then is allowed to decide whether or not she wants to roll again. Each re-roll costs \$1. Whenever she decides to stop, Alice receives a payout equal to the upface of the last die she rolled. Note that there is no limit on how many times Alice can re-roll. Assuming optimal play by Alice, what is her expected payout on this game?

Question 137: Alternating Sum

Topic: brainteasers **Difficulty**: easy
Find the value of $100^2 - 99^2 + 98^2 - 97^2 + \cdots + 2^2 - 1^2$

Question 138: All Equal

Topic: statistics **Difficulty**: easy
The median, mean, and mode of 100, 40, 200, 50, 90, 60, and x are all equal to x. What is the value of x?

Question 139: Spacious Uniform Values II

Topic: probability **Difficulty**: hard
You sample 101 uniformly random numbers in the interval $(0, 1)$. Find the expected length of the shortest distance between any two selected points.

Question 140: Digit Sum

Topic: brainteasers **Difficulty**: medium
What is the sum of the digits from 1 to 1 million, inclusive? For example, the sum of the digits of 36 is 9.

Question 141: Simple Delta Hedge I

Topic: finance **Difficulty**: easy
You currently have 100 call options of an underlying with $\Delta = 0.33$. You want to delta-hedge your portfolio. How many units of the underlying should you buy/sell? Enter $-x$ if you are looking to sell x units.

Question 142: Leap Frog

Topic: probability **Difficulty**: medium
A frog starting at position 0 is going to jump on the integers. At each step, the frog will choose to jump forward 1 or 2 steps with equal probability. Let p_k be the probability that the frog hits position $k > 0$. Find the largest value of p_k.

Question 143: Stack Double

Topic: brainteasers **Difficulty**: hard

7

men labeled $A-G$ play a coin flip game that starts with player A. At each turn, the player will flip a coin. If it appears heads, then the player must double the stack of each other player that plays the game, taking the money from their own stack. In other words, the payout to each of the other 6 players is the value of their current stack. Otherwise, nothing happens. The sequence $HHHHHHH$

is observed and each of the 7 players ends up with \$1.28 as their final stack. If $x_1, \ldots, x_7$ represent the amount that each player started with, find the value of $10000(x_1^2 + \cdots + x_7^2)$. For example, if player A starts with \$5.31, then $x_1 = 5.31$.

Question 144: 2 Below I

Topic: probability **Difficulty**: medium
You and friend play a game where you both select an integer $1 - 100$. The winner receives \$1 from the loser. The winner is the player who selects the strictly higher number. If there is a tie, then nothing happens. However, a player can also win by selecting a value exactly 2 below the larger integer. For example, if you select 80 and your friend selects 82, you are the winner in this case. Assume both you and your friend play optimally. The optimal strategy here is a mixed strategy, where you select a random value X from some appropriately determined distribution. Find $\mathrm{Var}(X)$.

Question 145: Private Documents

Topic: brainteasers **Difficulty**: medium
A quant firm wants to protect their IP. The firm only has 7 employees. They want to ensure that a safe can be opened only when at least 4 of them want to open it. Therefore, they put some locks on the safe. All of the locks must be unlocked to open the safe. Let n be the minimum number of locks needed to achieve this goal. Let m be the number of keys each of the 7 people carries. Find $10m + n$.

Question 146: 7 Multiple

Topic: probability **Difficulty**: easy
You and your friend play a game in which you take turns rolling a fair $6-$sided die and keep a running tally of the sum of the upfaces obtained in each roll. The winner of the game is the person who most recently rolled the die when the running sum first becomes a multiple of 7. You get to decide whether to go first or second. Under rational strategy from you, what is your probability of winning?

Question 147: Fancy Factorial

Topic: brainteasers **Difficulty**: easy

$$\frac{10!}{6!} = n!$$

for some integer n. What is n?

Question 148: Dice Profits

Topic: probability **Difficulty**: hard
Suppose you have a fair $20-$sided die. You must select a number of times to re-roll the die (or to not re-roll) before seeing any of the outcomes. However, each re-roll costs \$1. When the rolling process is complete, you receive $\$M$, where M is the maximum value that appeared among all your rolls. Assuming you re-roll an optimal amount of times, find the expected profit.

Question 149: Socks and Shelves

Topic: probability **Difficulty**: easy
There are two shelves, the top and bottom. The top shelf contains red socks with probability 0.4 and black socks with probability 0.6. The bottom shelf contains red socks with probability 0.7 and black socks with probability 0.3. I pick a shelf randomly. From that self, I picked 2 red socks. What is the probability that those 2 red socks come from the top shelf? Assume you have so many socks that taking from either shelf does not change the probability of future socks that are drawn.

Question 150: Non-Consecutive Sequence

Topic: probability **Difficulty**: hard
Suppose that you flip a coin n times and it turns out that no consecutive heads appear in the sequence. Let $E(n)$ be the expected number of heads that appear given this information. Compute $\lim_{n\to\infty} \frac{E(n)}{n}$. The answer is in the form $\frac{a}{b+\sqrt{c}}$ for integers a, b, and c with c minimal. Find abc.

Question 151: Straddle Gamma

Topic: finance **Difficulty**: easy
Let's consider a straddle on underlying S with strike K and expiry T. You have a call option on S, also with strike K and expiry T. This call option has $\Gamma = 0.03$. What is the Γ of the straddle? Assume Black-Scholes dynamics.

Question 152: Power to the Matrix

Topic: pure math **Difficulty**: medium
Let $A = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}$ Find $A^{10}v$, where $v = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. The answer is in the form

$$\begin{bmatrix} a \cdot b^p + c \\ d \cdot b^p - g \end{bmatrix}$$

For integers a, b, c, d, p. Find $bp + a + c + d + g$.

Question 153: Nearest Circular Neighbor

Topic: probability **Difficulty**: medium
Suppose that you select 20 IID points on the circumference of the unit circle, say $X_1, \ldots, X_{20}$. Find the expected arc length (in degrees) between X_1 and the point nearest to it.

Question 154: Solo or Pair

Topic: brainteasers **Difficulty**: medium
You are given the option between two games involving fair standard 6−sided dice. Which one gives you a higher probability of winning? Answer 1 for the first game, 2 for the second game, or 3 if both give you equal probability of winning.

Game 1: You are given 4 rolls of a single die and you must roll 6 at least once.

Game 2: You are given 24 rolls of a pair of dice and you must roll 66 at least once.

Question 155: Statistical Test Review VIII

Topic: statistics **Difficulty**: easy
Consider a random sample of size 25 from the distribution $\mathcal{N}(160, 900)$. To the nearest ten thousandth, what is the probability that the sample mean is 165 or greater?

Question 156: Close Dice II

Topic: probability **Difficulty**: hard
On average, how many times does a fair 6−sided die need to be rolled to obtain two consecutive rolls that differ by at most 1?

Question 157: Magic Doors

Topic: probability **Difficulty**: easy
John is trapped in a room with five indistinguishable doors. Behind two of the

doors are paths to freedom; one path is two miles long, and the other path is five miles long. Behind three of the doors are paths that magically lead back to the room; one path is three miles long, another path is four miles long, and the final path is one mile long. If John returns to the room, the order of the doors are scrambled and the five doors are once again indistinguishable. How many miles will John travel before reaching freedom?

Question 158: Birthday Twins II

Topic: probability **Difficulty**: medium
Assuming 365 days in a year, how many people do we need in a class to make the probability that at least two people have the same birthday more than $\frac{1}{2}$?

Question 159: Hitting MGF

Topic: pure math **Difficulty**: medium
Let W_t be a standard Brownian Motion and define $T_a = \inf\{t > 0 : |W_t| > a\}$ for $a > 0$. Find $\mathbb{E}[e^{-\lambda T_a}]$ as a function of λ and a. Evaluate this function with $\lambda = 4$ and $a = \ln(2)$.

Question 160: Machine Variance

Topic: probability **Difficulty**: easy
A pizza shop consists of 25 independent workers that make pizzas. The standard deviation of the number of pizzas produced by each worker in a day is 60. Find the standard deviation of the total number of pizzas produced by the shop per day.

Question 161: Paired Pumpkins II

Topic: brainteasers **Difficulty**: medium
Dracula has 5 pumpkins. When he pairs any two pumpkins, they weigh $21, 22, \ldots, 30$ pounds. What's the sum of the weights of all the pumpkins?

Question 162: Squared GBM

Topic: pure math **Difficulty**: medium
Let S_t be a geometric Brownian Motion with drift and volatility parameters both 1. Furthermore, let $S_0 = 1$. Find $\mathbb{E}[S_2^2]$. The answer is in the form e^k for a constant k. Find k.

Question 163: Statistical Test Review IV

Topic: statistics **Difficulty**: easy
QuantGuide's competitor, QuantChaperone, offers three plans: A, B, C. 400 of the first 1000 plans sold of plan A. Can one conclude that customers have a preference for plan A? Respond 1 if yes, 0 if no.

Question 164: Poisson Review IV

Topic: probability **Difficulty**: easy
Gabe makes rugs. The number of imperfections follows a Poisson distribution with an average of 4 per square yard. The probability that a 3-square-yard rug sample contains at least 1 imperfection can be expressed in the form $1 + \frac{a}{e^b}$. Compute $a + b$.

Question 165: Throwing Darts I

Topic: probability **Difficulty**: easy
A dart lands uniformly randomly on a dartboard composed of three concentric circles with radii of 1ft, 2ft, and 3ft. Assuming the dart lands on the dartboard, what is the probability that it lands in the central ring, between the 1ft and 2ft circles?

Question 166: 10 Die Sum

Topic: probability **Difficulty**: medium
Find the probability of obtaining a sum of 10 when rolling three fair $6-$sided dice.

Question 167: Eight Dice

Topic: brainteasers **Difficulty**: medium

$$8$$

fair standard 6-sided dice are thrown, the probability that the sum of the numbers on the top faces is 12 can be written as $\frac{x}{6^8}$. What is x?

Question 168: ABC Sum

Topic: brainteasers **Difficulty**: medium
Find the sum of all numbers in the form $0.abcabcabc\ldots$, where each of a, b, c are distinct integers from 0 to 9.

Question 169: Modified Even Coins

Topic: probability **Difficulty**: easy

$$n$$

coins are laid out in front of you. One of the coins is fair, while the other $n-1$ have probability $0 < \lambda < 1$ of showing heads. If all n coins are flipped, find the probability of an even amount of heads.

Question 170: Peaky Poisson

Topic: probability **Difficulty**: easy

Let $X \sim \text{Poisson}(13.4)$. Find the largest value of k such that $\dfrac{\mathbb{P}[X = k]}{\mathbb{P}[X = k-1]} \geq 1$. Note that k must be in the support of X.

Question 171: Dice Order III

Topic: probability **Difficulty**: hard

You roll three fair dice. On average, what will the minimum of the three rolls be?

Question 172: Random Angle I

Topic: probability **Difficulty**: easy

A right triangle is being formed with legs labeled A and B. The random lengths of legs A and B are both IID $\text{Unif}(0, 1)$. Let θ be the angle that is opposite of side A. Find the probability $\theta > \dfrac{\pi}{4}$.

Question 173: Butterfly Payoff

Topic: finance **Difficulty**: easy

You buy a butterfly spread with wings at $K = 35$ and $K = 40$ and body at $K = 37.5$. What would you like the price of the underlying at expiry to be? In other words, what do you want S_T to be?

Question 174: 60 Heads

Topic: probability **Difficulty**: easy

A fair coin is tossed 100 times. Using the Central Limit Theorem without continuity correction, estimate the probability that at least 60 heads are observed. Round to 5 digits after the decimal point.

Question 175: Specific Partition

Topic: probability **Difficulty**: hard
Let $S = \{1, 2, 3, \ldots, 22\}$. Find the number of ways in which S can be partitioned into eleven subsets such that each subset contains exactly two elements of S and the absolute difference between the two elements of a subset is 1 or 11.

Question 176: Triangle of Primes

Topic: brainteasers **Difficulty**: hard
20 points are selected from a circle and labeled 1-20 in clockwise order. Line segments are drawn between every pair of points whose labels differ by a prime number. How many triangles are formed whose vertices are among the original 20 points?

Question 177: Cylindrical Cone

Topic: probability **Difficulty**: easy
Suppose that a point (X, Y, Z) is uniformly randomly selected from the upwards-facing cylinder of radius r and height h whose lower base is centered at $(0, 0, 0)$. Find the probability that this point also lies in the cone whose base and height is the same as the cylinder.

Question 178: Russian Roulette IV

Topic: probability **Difficulty**: easy
You're playing a game of Russian Roulette with a friend. The six-chambered revolver is loaded with two consecutively placed bullets. Initially, the cylinder is spun to randomize the order of the chambers. Your friend goes first, and lives after the first trigger pull. You are then given the choice to either spin the barrel or not before pulling the trigger. What is the different in probability of you surviving between not spinning and spinning the barrel?

Question 179: Confused Ant II

Topic: probability **Difficulty**: medium
An ant walks the corner of a 3D cube and moves to one of the three adjacent vertices with equal probability at each step. Find the expected number of steps needed for the ant to return to the vertex it started at.

Question 180: Perfect Square

Topic: probability **Difficulty**: hard
Let p_n be the probability that $c+d$ is a perfect square when the integers c and d are selected independently at random from the set $\{1,\ldots,n\}$. Express this limit in the form $r(\sqrt{s}-t)$ where s and t are integers and r is a rational number.

Question 181: Crazy Covariance

Topic: statistics **Difficulty**: medium
Let $X \sim \text{Exp}(1)$ and $Y \mid X = x \sim \text{LogNorm}(0,x)$. Find $\text{Cov}(X,Y)$. Note that we say a random variable $R \sim \text{LogNorm}(\mu,\sigma^2)$ if $\log(R) \sim N(\mu,\sigma^2)$.

Question 182: First Right

Topic: probability **Difficulty**: easy
You and your opponent flip a coin. If the first player gets heads, the second player pays him $30. If he flips tails, the coin goes to the second player. If the second player flips heads, he wins $30 from the first player. If the second player also gets tails, the process repeats. What is the maximum amount (in dollars) you would pay for the right to go first?

Question 183: Clock Angle I

Topic: brainteasers **Difficulty**: easy
Find the angle (in degrees) between the minutes and hours hands at $9:45$ PM on a standard clock.

Question 184: Greedy Bob

Topic: statistics **Difficulty**: medium
Alice and Bob are playing a game. Alice spins a spinner and wins X dollars, with mean 12 and variance 2. Bob flips a coin, and if the coin lands on heads, then Bob takes from Alice whatever she won, else Bob wins nothing. What is $\text{Var}(Y)$, where Y denotes the amount Bob wins?

Question 185: Bond Question II

Topic: finance **Difficulty**: easy
Say you have a generic bond that has a 3 year maturity, $1000 face value, and a **quarterly** coupon rate of 2

Question 186: Gamma Review III

Topic: probability **Difficulty**: easy
Let $X \sim \text{Gamma}(2, 4)$. Note that this is the shape-scale parameterization of Gamma. Evaluate $\mathbb{E}[X^6]$. The answer is in the form $a^b \cdot c!$ for integers a, b, c, with a minimal. Find $a + b + c$.

Question 187: Spacious Uniform Values I

Topic: probability **Difficulty**: hard
You sample 101 uniformly random numbers in the interval $(0, 1)$. Find the probability that no two of the values selected are within a distance of $\frac{1}{1000}$ of one another. The answer should be in the fully reduced form $\left(\frac{a}{b}\right)^c$. Find $a + b + c$.

Question 188: Better in Red V

Topic: probability **Difficulty**: medium
The surfaces of a $3 \times 3 \times 3$ cube (initially white) are painted red. The large cube is then cut up into 27 $1 \times 1 \times 1$ small cubes. One of the small cubes is selected uniformly at random and is rolled twice. The face appearing is red both times. What is the probability the cube selected was a corner cube?

Question 189: Checkmate

Topic: probability **Difficulty**: medium
Andy is playing chess. In order to receive a prize, he must win at least two consecutive games out of three. Andy may either play Michael, then Aaron, then Michael (option 1), or he may play Aaron, then Michael, then Aaron (option 2). Aaron is better at chess than Michael. Which option, 1 or 2, should Andy pick in order to maximize his chances at receiving a prize?

Question 190: Bubbly Sort

Topic: probability **Difficulty**: medium
Suppose that you run one iteration of Bubble Sort on some random permutation of $(1, 2, \ldots, 100)$. Find the probability that the list is now sorted in order. The probability is in the form

$$\frac{a^b}{c!}$$

for integers a, b, and c. Find $a + bc$.

Question 191: Game Arbitrage I

Topic: finance **Difficulty**: hard

Consider the following group with the following teams. The contracts are given in the format of (Team, Time-0 Price). The price of 0.67 means that if the team wins, you will get paid 1. If the team loses, you will get paid 0.

$$(\text{Team } 1, 0.84)$$
$$(\text{Team } 2, 0.73)$$
$$(\text{Team } 3, 0.25)$$
$$(\text{Team } 4, 0.15)$$

The group works as follows: 2 teams are guaranteed to make it out. Find the arbitrage. You are allowed to long or short contracts. You are also allowed to long or short bonds, which pay 1 at expiry. Assume interest rates are 0, so $Z_0 = 1$.

Give the answer in the form of the initial credit you will receive in the arbitrage.

Question 192: Duplicating Data

Topic: statistics **Difficulty**: easy

You use OLS regression model the relationship between news sentiment, X, and AAPL share price, Y, and find that $Y = 156.3 + 10.21X$ with a correlation coefficient of 0.78. You duplicate the data and re-run the OLS regression. How much did the correlation coefficient, ρ, increase by?

Question 193: Minimal Variance

Topic: probability **Difficulty**: easy

Let X_1 and X_2 be two independent random variables with variance 25 and 100, respectively. Find the value of $0 \leq c \leq 1$ that minimizes the variance of $Y = cX_1 + (1-c)X_2$.

Question 194: Hit Or Miss

Topic: probability **Difficulty**: medium

A particle starts at (4, 4). Each turn, it moves either 1 unit in the $-x$ direction, 1 unit in the $-y$ direction, or 1 unit in each of the $-x$ and $-y$ directions. During a turn, the particle decides how it will move randomly such that the probability of each option is $\frac{1}{3}$. The particle repeatedly takes turns until it hits the x or y axes for the first time. Compute the probability that the particle hits the origin.

Question 195: Dollar Bills

Topic: brainteasers **Difficulty**: easy
You have to deposit money into your bank account. All of the bills are either \$10 or \$20. You know that you deposited a total of 150 bills and the bank teller said you deposited \$2150. How many \$20 bills did you deposit?

Question 196: Half Cycle

Topic: probability **Difficulty**: hard
Let C_n count the number of cycles in a random permutation of $\{1, 2, \ldots, 2n\}$ that are larger than n in length. Compute $\lim_{n\to\infty} \mathbb{E}[C_n]$. The answer is in the form $\ln(q)$ for some rational number q. Find q.

Question 197: Limiting Values II

Topic: probability **Difficulty**: medium
You roll two fair dice until a sum other than 7 appears and are paid that amount. What is the fair value of this game?

Question 198: Graph Search II

Topic: probability **Difficulty**: medium
You are given an undirected graph with 11 nodes. From every node, you are able to access any other node (not including itself), all with an equal probability of 1/10. What is the expected number of steps to reach all nodes at least once (rounded to the nearest step)?

Question 199: Lucky Genie

Topic: probability **Difficulty**: easy
A genie rolls a fair 6−sided die. You can bet on the outcome of the die being even or odd. For every \$4 you bet on the die being even, you will receive \$6 back if correct. Similarly, for every \$6 you bet on the die being odd, you will receive \$9 back. Playing optimally, how much should you bet on this game?

Question 200: Penny Stack

Topic: brainteasers **Difficulty**: medium
You are given 100 pennies. You may arrange these pennies into however many stacks you want and put as many pennies into each stack as you would like. You are paid out the product of the number of pennies among all of the stacks. For example, if you make three stacks of sizes $10, 20$, and 70, your payout is $10 \cdot 20 \cdot 70 = 14000$. The optimal arrangement of the pennies will give you a payout of $a \cdot b^c$, where a and b are relatively prime. Find abc.

Question 201: Perfect Seating I

Topic: probability **Difficulty**: easy
Seven people with distinct ages randomly sit down at a circular table with seven seats. What is the probability that the people sit themselves in increasing order of age, irrespective of direction?

Question 202: Straddle Price

Topic: finance **Difficulty**: medium
What is the price of an at-the-money straddle with underlying $S = 20$, time-until-expiry $T = 0.36$, and volatility $\sigma = 0.4$? Round to 2 decimal points.

Question 203: No Arithmetic

Topic: brainteasers **Difficulty**: medium
Consider the following sequence: $3, 4, 5, a, b, 30, 40, 50$. How many different ordered pairs of integers (a, b) satisfy the condition that the sequence is strictly increasing and there does not exist 4 numbers belonging to the sequence that form an arithmetic sequence in any order?

Question 204: Uniform Order II

Topic: probability **Difficulty**: easy
Let $X_1, X_2, \ldots, X_{20}$ be IID Unif$(0, 1)$ random variables. Compute $\mathbb{P}[X_1 > X_{10} \mid X_{10} < X_{20}]$.

Question 205: Valuable Hearts

Topic: probability **Difficulty**: easy
Suppose that each card in a standard deck is given a value equal to it's face, with $A = 1, J = 11, Q = 12$, and $K = 13$. However, any card that is heart is

doubled in value (ex: Ace of Hearts is worth 2). Find the expected value of a uniformly at random selected card from the deck.

Question 206: Birthday Guessing

Topic: brainteasers **Difficulty**: easy
A group of colleagues know that their manager Alice's birthday is one of the following 10 dates:

March 4, March 5, March 8June 4, June 7September 1, September 5December 1, December 2, Decembe

Alice only told Bob the month of her birthday and Charlie the day. After that, Bob first said, "I know that neither I nor Charlie knows Alice's birthday." After hearing this, Charlie replies, "I didn't know Alice's birthday, but now I do!" Bob smiles gently, and comments, "Now I do, as well!" What is Alice's birthday (as an integer in the format mmdd, including any leading zeros)?

Question 207: Put Fly I

Topic: finance **Difficulty**: medium
We have the underlying S with initial price $S_0 = 15$. We have access to the following calls. The calls are given in the format of (Strike K, Price C_0)

$$(10, 8.1)$$
$$(15, 4.2)$$
$$(20, 1.4)$$

What is the time-0 price of a 10/15/20 put-fly?

Question 208: Make Your Martingale IV

Topic: pure math **Difficulty**: easy
Let W_t be a standard Brownian Motion. Define $X_t = W_t^{10} - \int_0^t a_s ds$ for some process $\{a_t\}_{t\geq 0}$ adapted to the natural filtration of the probability space. The process a_t that makes X_t a martingale can we written as $a_t = AW_t^B$ for some real constants A and B. Find AB.

Question 209: Coloring Components III

Topic: probability **Difficulty**: hard
Consider a line of 20 adjacent colorless squares. Color in each individual square black or white with equal probability, independent of all other squares. A

connected 5-component is a group of 5 consecutive black squares. Note that overlapping components are not counted. For example, WBBBBBWWB and WBBBBBBWB have 1 connected 5-component, but WBBBBBBBBBBW has 2 connected 5-components. Find the expected number of connected 5-components in our line. The answer can be written as a simplified fraction of the form $\frac{p}{q}$. Find $p + q$.

Question 210: Unfriendly

Topic: probability **Difficulty**: easy
John has 8 friends. He will invite 5 to his party. However, two of his friends have beef and they refuse to attend together. How many possible combinations of guests are possible given this constraint?

Question 211: Unfair Roulette

Topic: probability **Difficulty**: medium
Gabe has a revolver with six chambers, and he convinces his friend Miriam to play a round of modified Russian roulette with him. He loads two bullets into the gun such that there is exactly one empty chamber between the two loaded chambers. He points the gun at his forehead, pulls the trigger, and survives. He offers Miriam the gun and gives her a choice to either point the gun at herself and pull the trigger (Option 1) or add a third bullet to the gun, randomly spin the cylinder, and then point the gun at herself and pull the trigger (Option 2). Which option should she choose to optimize her probability of survival? Enter 1 for Option 1, 2 for Option 2, or 3 if the two options are equally optimal.

Question 212: Dividing Nuggets

Topic: probability **Difficulty**: medium
Mr. Garrison has four students in his elementary school math class: Eric, Stan, Kyle, and Kenny. While on a field trip, Mr. Garrison's class stops at a fast food restaurant for lunch. Mr. Garrison purchases 50 chicken nuggets in bulk for his four students to share. Stan, Kyle, and Kenny each want at least 6 nuggets, while Eric wants at least 18 nuggets. Mr. Garrison randomly partitions the 50 nuggets into four piles such that each partition has an equal chance of occurring. Each student has at least 5 nuggets after the partition. The probability that Mr. Garrison's students are all satisfied can be expressed in the form

$$\frac{\binom{a}{b}}{\binom{c}{d}}$$

Find $\min(a + 2b + c + 2d)$.

Question 213: Covered Calls II

Topic: finance **Difficulty**: medium
You are entering a covered call position on $TSLA. You buy 200 shares at $230, sell 100 $240 Dec call @10.50, and 100 $250 Dec call @9.50 (same expiry). What is your max profit for this strategy (assuming that your options will be exercised if they are above the strike at expiration)?

Question 214: Counting Nash Equillibria

Topic: probability **Difficulty**: hard
A treasure chest is up for auction and there are 10 participants participating in the auction. The value of the treasure chest is determined as follows: behind a curtain 20 fair coins are independently flipped and for each head that shows, $1 is added to the chest (participants cannot see behind the curtian but are aware of the way the chest is valued).

Each person chooses a non-negative integer number of dollars to submit as a sealed bid. The chest goes to a uniformly random person who was among the highest bidders, in return for the amount bid and everyone else has their bid returned. Count how many pure Nash equilibria there are for this game. (You may assume everyone's utility for money is linear in the range $0 to $10.)

Note: A pure Nash equilibrium means the strategy of each player must choose a single bid value with probability 1 (e.g. a player's strategy cannot be to bid 3 with probability 0.4 and bid 4 with probability 0.6).

Question 215: Difference of Four

Topic: probability **Difficulty**: medium
You roll three fair dice. What is the probability that the difference between the highest and lowest values rolled is exactly four?

Question 216: Colorful Line

Topic: probability **Difficulty**: easy
How many distinct ways can you arrange 3 red balls, 7 blue balls, and 9 green balls in a line?

Question 217: 4 Before 2

Topic: probability **Difficulty**: easy
Jed is rolling two fair 6$-$sided dice repeatedly. Find the probability that Jed

obtains a sum of 4 before a sum of 2?

Question 218: Prime Janitors

Topic: brainteasers **Difficulty**: hard
Janitors at a large trading company come to work to see 100 open doors. These doors are labeled 1 to 100. Throughout the day, janitors open/close doors based off of their number. The first janitor closes every door that's a multiple of 2. Then the next janitor closes every door that's a multiple of 3 unless its currently closed, in which case he opens it back up. This will continue with by having the kth janitor open/close all door numbers that are a multiple of the kth positive prime integer. How many doors are open at the end of the day?

Question 219: Ant in a Hurry

Topic: brainteasers **Difficulty**: medium
An ant sits at one corner of a unit cube, wishing to travel to the corner farthest from it. Let d be the minimal distance the ant must travel to achieve its goal. Find d^2. Note that the ant must always be in contact with a surface and cannot fly.

Question 220: Better in Red I

Topic: probability **Difficulty**: medium
A $10 \times 10 \times 10$ cube is painted red on the surface and then cut into 1000 $1 \times 1 \times 1$ cubes and one is selected uniformly at random. Find the expected number of red faces on this cube.

Question 221: Resell Painting

Topic: probability **Difficulty**: easy
You are currently bidding for a painting. You know that the value of the painting is between \$0 and \$100,000 uniformly. If your bid is greater than the value of the painting, you win and sell it to an art museum at a price of 1.5 times the value. What's your bid to max your profit? If you can not profit, bid \$0.

Question 222: Pick Your Opponent

Topic: brainteasers **Difficulty**: easy
You are in a tennis tournament with Alice and Bob. The tournament consists of 3 games. You win the tournament if you win two consecutive games. You have a better chance of beating Bob than Alice. If A represents Alice and B

represents Bob, would you prefer your three games to be, in order, ABA or BAB? Answer 1 and 2 for ABA and BAB, respectively.

Question 223: Bull Call Spread I

Topic: finance **Difficulty**: easy
Consider the following asset S, with initial price $S_0 = 7$. In this bull call spread, you will long a call at strike $K = 5$ and short a call at strike $K = 10$. What is the maximum and minimum *payoff* of this contract?

Give the result in the form: $\max^2 + \min^2$

Question 224: Red Card Deal

Topic: probability **Difficulty**: medium
A dealer presents to you the following game: The dealer is going to deal cards from the top of a standard deck one card at a time and turn them over on a table for you to see. You can tell the dealer to stop dealing at any time. If the next card is red, you win. Assuming optimal play, what is the highest probability you can achieve of being correct?

Question 225: Eigenshift

Topic: pure math **Difficulty**: easy
Let A be a $n \times n$ matrix with eigenvalues 5 and 7. Find the sum of the eigenvalues of $B = (A - 3I_n)^{-3}$. We write M^{-3} to mean $(M^{-1})^3$.

Question 226: Statistical Test Review III

Topic: statistics **Difficulty**: easy
Trader wages have mean 132 and standard deviation 25. QuantHomies, a proprietary trading firm, employs 40 traders and pays them a wage of 122. Using an $\alpha = 0.01$ level test (note that $z_{0.01} = -2.326$), determine whether QuantHomies can be accused of paying wages that are below the industry standard. Respond 1 if yes, 0 if no.

Question 227: Miscalculated Fibonacci

Topic: brainteasers **Difficulty**: medium
Suppose that you are trying to calculate the terms in the Fibonacci sequence. You calculate the values of $F_0, F_1, \ldots, F_{99}$ correctly. However, you say that the

value of F_{100} is $G_{100} = 1 + F_{100}$. Assuming you carry over this error into your subsequent calculations of terms of $F_{101}, \ldots, F_{110}$, let G_{110} be the value that you calculate for F_{110}. Find $G_{110} - F_{110}$.

Question 228: Make Your Martingale III

Topic: pure math **Difficulty**: medium
Let $X_1, X_2, \cdots \sim \text{Exp}(\lambda)$ IID. Define the process $\{M_n\}_{n\geq 0}$ by $M_0 = 1$ and $M_n = M_{n-1} \cdot \frac{1}{2} e^{\frac{\lambda}{2} X_n}$ with the natural filtration $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. If $0 < a < p < b$ for some real values a and b, then $\{M_n^p\}_{n\geq 0}$ is a sub-martingale. Find $a + b$.

Question 229: Exact 5 II

Topic: probability **Difficulty**: medium
Abd continually rolls a fair 6−sided die until he obtains his first 6. Compute the expected number of times Abd obtains the value 5 before he stops rolling.

Question 230: Snowman Surface

Topic: probability **Difficulty**: easy
A snowman is formed by placing three spherical snowballs of radii $8, 11$, and 14 inches on top of one another. Assume that every point on the surface of each snowball is visible. A uniformly random point is selected on the surface of the snowman. Find the probability that this point is at most 39 inches above the ground.

Question 231: Three Repeat II

Topic: probability **Difficulty**: easy
A fair coin is flipped 6 times. Find the probability of obtaining exactly 3 consecutive heads somewhere in the 6 flips.

Question 232: Card Shuffling

Topic: brainteasers **Difficulty**: hard
The rudimentary method of shuffling a pack of cards is to take the pack face downwards in the left hand and then transfer them one by one to the right hand, putting the second on top of the first, the third under, the fourth above, and so on until all are transferred.

If you do this with any even number of cards and keep on repeating the shuffle in the same way, the cards will in due time return to their original order. Try with 4 cards, you will find the order is restored in 3 shuffles. In face, where the number of cards is $2, 4, 8, 16, 32, 64$, the number of shuffles to get them back to the original arrangement is $2, 3, 4, 5, 6, 7$ respectively.

How many shuffles are necessary in the case of 14 cards?

Question 233: Red Tower

Topic: probability **Difficulty**: easy
A tower of 12 blocks tall is going to be comprised of either $1-$tall or $2-$tall red blocks. Assume you have at least 12 of each type of block. How many different tower configurations can be made?

Question 234: Compound Game

Topic: probability **Difficulty**: easy
You are given a fair $6-$sided die and play the following game: You are paid the upface of every roll. If you roll an odd value, the game is over. If you roll an even value, you flip a fair coin. If it lands on tails, the game is over. If it lands on heads, you roll again and repeat the same coin flip process after your roll. Find the expected payout from this game.

Question 235: Poisson Review I

Topic: probability **Difficulty**: easy
On average, 7 customers arrive at the QuantGuide gift shop per hour following a Poisson process. To the nearest thousandth, what is the probability that at least 4 customers arrive in a given hour?

Question 236: Cube Pack

Topic: probability **Difficulty**: easy
How many $3 \times 3 \times 3$ cubes can be fit inside a cube of dimensions $20 \times 20 \times 20$?

Question 237: Regional Manager III

Topic: statistics **Difficulty**: medium
The regional sales manager of a large paper corporation is attempting to detect a difference equal to one deal in the average number of deals closed per week by

his employees. He runs a statistical test with the null hypothesis $H_0 : \mu = 15$ against his alternative hypothesis $H_a : \mu = 16$. Assuming $\sigma^2 = 6$, what sample size will ensure that $\alpha = \beta = 0.05$? Assume simple random sampling, variance homogeneity, and that the number of deals closed is approximately normally distributed.

Question 238: Binomial Maximizer

Topic: probability **Difficulty**: easy
Suppose that $X \sim \text{Binom}(12, p)$ with $0 < p < 1$. Find the value of p that maximizes $\mathbb{P}[X = 8]$.

Question 239: Five Below

Topic: probability **Difficulty**: medium
You roll a die until you observe a 5. What is the expected minimum number rolled?

Question 240: Close Dice I

Topic: probability **Difficulty**: medium
On average, how many times does a fair 6−sided die need to be rolled to obtain two consecutive rolls that differ by exactly 1?

Question 241: Student Appointment

Topic: probability **Difficulty**: easy
You have 5 students that you want to see 3 times each over the next month. There are 15 time slots available for the students to select from. However, one of the students can't attend in the first time slot. How many different ways can these students be scheduled?

Question 242: Mean Babysitter

Topic: probability **Difficulty**: medium

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kids are really hungry! Their babysitter has 12 units of food to give. However, she decides she only wants to give 4 of the children food. How many ways can she distribute the food units such that 6 of the children are hungry (receive no food), and the other 4 children receive at least 1 unit of food each?

Question 243: Big Mod II

Topic: pure math **Difficulty**: easy
Compute $15^{2021} \bmod 17$.

Question 244: Extra Coin

Topic: probability **Difficulty**: easy
Alice has $n+1$ fair coins and Bob has n fair coins. What is the probability that Alice will flip more heads than Bob if both flip all of their coins?

Question 245: Dice-Coin Paradigm

Topic: probability **Difficulty**: medium
We flip a fair coin until we obtain our first tails. Given the first tails occurs on the nth flip, we roll a fair $6-$sided die n times. Find the probability that the die value 1 is observed in the rolls.

Question 246: Alternating Sum

Topic: brainteasers **Difficulty**: easy
Find the value of $100^2 - 99^2 + 98^2 - 97^2 + \cdots + 2^2 - 1^2$

Question 247: Random Scale

Topic: probability **Difficulty**: easy
We have 6 weights of mass $201, 202, \ldots, 206$ grams. We randomly arrange 3 weights on each side of a scale. Find the probability that the side with the 206 gram weight is heavier.

Question 248: Arbitrage Detective II

Topic: finance **Difficulty**: easy
You are a forex trader analyzing the markets and think you have spotted an arbitrage opportunity. You have \$1000 USD to start with, and can trade the following exchanges at the following rates.

$$\text{EUR/USD} = .85$$
$$\text{EUR/GBP} = 1.5$$
$$\text{USD/GBP} = 1.8$$

Assuming optimal trades, how much profit do you make (round to the nearest hundredth)?

Question 249: Paper Draw

Topic: probability **Difficulty**: medium
You have two urns presented to you. One of them has the values $1-8$ written on 8 different pieces of paper. The other one only has the values 1 and 2 on 4 pieces of paper each. You select one urn uniformly at random and then select a piece of paper from it uniformly at random. You see that the paper selected has the value 2 on it. You then replace the paper in the urn you selected from. If you select from the same urn 40 more times with replacement between trials, find the expected number of times 2 would appear in these 40 draws.

Question 250: Wandering Ant II

Topic: probability **Difficulty**: hard
An ant starts at the origin in the plane. At each step, with probability $\frac{1}{4}$, the ant will move one unit north, south, east, or west. Find the expected number of steps until the ant first hits the square with vertices at $(\pm 2, \pm 2)$.

Question 251: In Order

Topic: statistics **Difficulty**: medium
Suppose that $X_1, X_2, X_3, X_4 \sim \text{Unif}(0,1)$ IID and let O_1, O_2, O_3, and O_4 be the order statistics corresponding to these random variables. Find $\mathbb{E}[O_3 \mid O_4 = 0.9, O_1 = 0.3]$.

Question 252: Dominated Turtle

Topic: probability **Difficulty**: hard
Two turtles, Tort and Bort, are going to perform independent simple symmetric random walks on the integers starting at positions 0 and 4, respectively. Compute the probability after 10 steps, Tort and Bort are back at their initial positions and that Tort was strictly behind Bort at all 10 steps. The answer will be in the form

$$\frac{\binom{a}{b}^p - \binom{c}{d}^p}{2^r}$$

where $b \leq \frac{a}{2}$ and $d \leq \frac{c}{2}$. Find $a^2 + b^2 + c^2 + d^2 + p^2 + r^2$.

Question 253: Flip Again

Topic: probability **Difficulty**: medium
4 fair coins are laid out in front of you and are flipped. You receive the amount

of dollars equal to the number of heads that appear. However, you also have the option to re-flip all 4 coins at once one time for \$1. If you re-flip whenever you get 0 or 1 heads on the initial flip, what is the fair value of this game?

Question 254: Lily Pads II

Topic: brainteasers **Difficulty**: medium
Lily pads double in area each day, and each lily pad is one square foot in area. How many days pass until a 20480-square foot pond that initially has 10 lily pads is covered in lily pads? Assume the region each lily pad covers is disjoint from the others.

Question 255: Prime Pair

Topic: probability **Difficulty**: easy
You roll two fair $6-$sided dice. Each of the two dice have the first 6 prime numbers on the sides. Find the probability that the sum of the two upfaces is also prime?

Question 256: Make Your Martingale II

Topic: pure math **Difficulty**: easy
Let W_t be a standard Brownian Motion and let $X_t = W_t^3 - ctW_t$ for some constant c. Find the value of c that makes X_t a martingale.

Question 257: Geometrical Progression

Topic: brainteasers **Difficulty**: hard
Write out a series of whole numbers in geometrical progression with at least 3 terms, starting from 1, so that the numbers add up to a square. The common ratio must be strictly larger than 1.

For example, an example of a geometrical progression (not the correct one):

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 1 + 2 + 4 + 8 + 16 + 32 = 63$$

Give the answer in the form of the smallest square number in which a progression can be written.

Question 258: Coefficient Sum

Topic: brainteasers **Difficulty**: easy
Evaluate
$$12\binom{3}{3} + 11\binom{4}{3} + 10\binom{5}{3} + \cdots + 2\binom{13}{3} + \binom{14}{3}.$$

Question 259: Normal Activities

Topic: statistics **Difficulty**: easy
Suppose that X and Y are independent standard normal random variables. Find $\mathbb{P}[Y > 4X]$.

Question 260: Hatching Eggs I

Topic: probability **Difficulty**: easy
Amy has a chicken. The number of eggs laid by the chicken in a month follows a Poisson process with $\lambda = 6$. The probability that an egg hatches is 0.3. Eggs hatch independently of one another. Compute the expected number of hatched eggs.

Question 261: Sum Exceedance II

Topic: probability **Difficulty**: hard
Let $X_1, X_2, \cdots \sim \text{Unif}(0,1)$ IID and $N_2 = \min\{n : X_1 + \cdots + X_n > 2\}$. Find $\mathbb{E}[N_2]$. Your answer will be in the form $ae^2 + be$ for integers a and b. e here is Euler's constant. Find $a + b$.

Question 262: 0DTE Option

Topic: finance **Difficulty**: medium
You have a call option on the underlying, S with a current price of $S_0 = 5$. You have a call option with initial price $C_0 = 0.3$ and strike $K = 25$. The option expires in an hour and has a charm of -0.04. If the underlying moves down by \$0.5 in the next minute, what is an approximation for the price of the option?

Question 263: Optimal Card Pick

Topic: brainteasers **Difficulty**: medium
You are given a standard 52 card deck. You may select any 5 cards from the deck to be your hand as long as they form a full house (3 cards of one rank and 2 cards or another). Then, your opponent receives 5 random cards from the

remaining 47. Let Jack, Queen, King, and Ace have the values $11-14$. The value of a full house is the sum of all the values of the cards that comprise it. For example, 22299 has value 24. Find the sum of the values of all optimal full houses. We define optimal here as maximizing the probability of winning with respect to standard poker rules.

Question 264: Sum and Difference

Topic: probability **Difficulty**: medium
Let $X, Y \sim \text{Unif}(0,1)$ IID. Compute $\mathbb{P}[X+Y \geq 2|X-Y|]$.

Question 265: Double Data Trouble IV

Topic: statistics **Difficulty**: easy
Suppose that you run linear regression on some dataset and obtain an R^2 value of 0.48. If you were to run linear regression again on the dataset where you double all of the values of each point in the dataset (for example, if $(2,5)$ were in the original dataset, it would now be $(4,10)$), what would R^2 be? If it can't be determined, enter -1.

Question 266: Dog Days

Topic: probability **Difficulty**: easy
Every day is either good (G) or bad (B). If today was G, tomorrow is also G with probability $\frac{3}{5}$. If today was B, tomorrow is also B with probability $\frac{7}{10}$. You woke up to find that today is a bad day. Find the expected number of days until another bad day.

Question 267: Balanced Beans IV

Topic: brainteasers **Difficulty**: hard
There are 90 beans; one weighs slightly heavier or lighter than the others. What is the minimum number of times a balance scale must be used to guarantee the determination of the abnormal bean?

Question 268: Sample Size for Z

Topic: statistics **Difficulty**: easy
When deciding on using the Z test, as the sample size increases, the sampling distribution is considered to be approximately normally distributed. At what sample size does the Z test become approximately normal and thus viable, under the Central Limit Theorem?

Question 269: Terminating Sum

Topic: brainteasers **Difficulty**: hard

Evaluate $\sum_{k \in S} \frac{1}{k^3}$, where S is the set of all positive integers such that $\frac{1}{k}$ has a terminating decimal expansion. For example, $2 \in S$, as $\frac{1}{2} = 0.5$ has a finite expansion. However, $7 \notin S$, as $\frac{1}{7} = 0.\overline{142857}$ does not terminate.

Question 270: Shoe Manufacturing

Topic: statistics **Difficulty**: easy

A shoe manufacturer offers rock climbing shoes in black, blue, and pink. Of the first 1000 shoes sold, 400 were black. Calculate the value of the appropriate test statistic to determine if customers have a preference for black rock climbing shoes. Assume random sampling, variance homogeneity, and that preference is approximately normally distributed.

Question 271: Uniform Equilibrium I

Topic: probability **Difficulty**: easy

Two players, say 1 and 2, simultaneously pick real numbers in the interval $[0, 1]$. The payoff of Player 1 (equal to the loss of Player 2) is the absolute distance between those numbers. Find the number of pure-strategy Nash equilbria.

Question 272: Good Grid I

Topic: probability **Difficulty**: easy

Suppose that two integers a and b are uniformly at random selected from $S = \{-10, -9, \ldots, 9, 10\}$. Find the probability that $\max\{0, a\} = \min\{0, b\}$.

Question 273: Beer Barrel I

Topic: brainteasers **Difficulty**: medium

A 120-quart beer barrel was discovered by Anna's parents. Furious, they plan to dump the barrel. Anna begs her parents to let her keep some of the beer. They say that Anna may do so if she is able to measure out an exact quart into each of a 7-quart and 5-quart vessel. Define a transaction as a pour of liquid from one container into another. What is the smallest number of transactions needed to accomplish the challenge? If impossible, respond with -1.

Question 274: Picking Candy

Topic: probability **Difficulty**: easy
Your older brother places 5 good candies and 5 bad candies in front of you, as well as two boxes. He asks you to put the candies into these two boxes, and he will, blindfolded, choose a random candy from a random box, which you get to keep. You strategize to put the candies into the boxes in such a way that maximizes your chances of receiving a good candy. With this strategy, what is the probability that you receive a good candy?

Question 275: No Adjacent Evens

Topic: probability **Difficulty**: medium
How many permutations of $\{1, 2, \ldots, 7\}$ have no adjacent even digits?

Question 276: The Ten Cards

Topic: brainteasers **Difficulty**: medium
Place any ten playing cards in a row face up. There are two players. The first player may turn face down any single card he chooses. Then the second player can turn face down any single card or any 2 adjacent cards. And so on. Thus the first player must turn face down a single, but afterwards either player may turn down either a single or two adjacent cards. The player who turns down the last card wins. Should the first or second player win? Answer 1 for the first player or 2 for the second player.

Question 277: Multiple Likely Coin

Topic: probability **Difficulty**: medium
Coins 1 and 2 are weighted such that Coin 1 has probability $0 < p_1 < 1$ of landing heads up and Coin 2 has probability $0 < p_2 < 1$ of landing heads up. It is known that $p_1 + p_2 = 1$ and that $p_2 > p_1$. One of the coins is chosen randomly and then flipped twice. Both of the flips resulted in heads. If we know that it is now 4 times as likely that we chose Coin 2 as opposed to Coin 1 given this information, find p_2.

Question 278: Pairwise Digit Sums II

Topic: brainteasers **Difficulty**: medium
Let A be the set of 5 digit integers such that all pairwise sums of digits are unique. For example, a three digit number with this property is 174. Let x and y be the minimal and maximal elements of A, respectively. Find $y - x$.

Question 279: Make Your Martingale V

Topic: pure math **Difficulty**: easy

Let W_t be a standard Brownian Motion. Define a process $Y_t = \int_0^t s dW_s$. Furthermore, define a process $Z_t = tY_t - a_t$ for some process a_t. Let a_t be specifically selected so that Z_t is a martingale. The form of a_t is $a_t = k \int_0^t Y_s ds$ for a constant k. Find k.

Question 280: Longest Rope II

Topic: probability **Difficulty**: medium

Suppose you have a rope that is 1 meter in length. You cut the rope uniformly at random along the length. Find the variance of the length of the longer piece (in meters).

Question 281: Straddle Arbitrage I

Topic: finance **Difficulty**: medium

You have a straddle with initial price $V_0 = 5.4$ with strike $K = 17$, a European vanilla put with initial price $P_0 = 4.2$, a European vanilla call with $C_0 = 1.4$, both at strike $K = 17$, the underlying stock S with initial price $S_0 = 14$, and bonds that pay 1 at time-T and initial price $B_0 = 0.9$.

Find the arbitrage. Give the answer in the format of # Stock + # Call + # Put + # Bonds + # Straddle

Question 282: Poisson Review V

Topic: probability **Difficulty**: easy

Gabe makes rugs. The number of imperfections follows a Poisson distribution with an average of 4 per square yard. It costs 10 to repair each imperfection. Find the sum of the mean and variance of the repair cost for a 10-square-yard rug.

Question 283: Sock Drawer I

Topic: probability **Difficulty**: easy

A drawer contains blue and green socks only. When two socks are drawn at random from the drawer without replacement, the probability that both are blue socks is $\frac{1}{2}$. What is the minimum number of socks that could be in the drawer?

Question 284: Thank You, Quant!

Topic: probability **Difficulty**: medium

Two quant firms have client inflow that is well-modeled by independent Poisson processes. The respective intensity parameters are 6 and 10. Clients give outstanding reviews about the service with respective probabilities $\frac{1}{6}$ and $\frac{1}{5}$ for the two firms, independent of one another. The clients that give outstanding reviews are sent a thank you card. Find the expected time between clients that receive thank you cards.

Question 285: Fixed Point Limit I

Topic: probability **Difficulty**: easy

Let F_n be the number of fixed points of a random function $f : S_n \to S_n$, where $S_n = \{1, 2, \ldots, n\}$. Find $\lim_{n\to\infty} \mathbb{P}[F_n = 5]$. The answer is in the form $\frac{c}{e}$, where e is Euler's constant and c is a constant. Find c.

Question 286: Movie Arrivals

Topic: probability **Difficulty**: medium

At the new Avengers movie, people arrive according to a Poisson Process with parameter 1/min. Each person's gender is independent of all others and has probability $\frac{2}{3}$ of being male. Let N be the number of people that arrive until (and including) the first woman. Find the probability that none of the first N interarrival times are within 1 minute of each other. The answer is in the form $\frac{a}{be - c}$ for relatively prime integers a, b, c. Find abc

Question 287: Deviation Probability

Topic: probability **Difficulty**: easy

$$X$$

is a Gaussian random variable with $\mu = 50$, $\sigma^2 = 4$. what is $\mathbb{P}(X > 54)$? Round to the nearest thousandth.

Question 288: Greedy Pirates

Topic: brainteasers **Difficulty**: medium

There are five pirates with distinct seniority. Each is rational in that they prioritize staying alive first and getting as much gold as possible second. The five pirates agree on the following method to divide 100 gold coins. The most senior pirate will propose a distribution of the coins. All pirates five pirates vote on whether or not to pass the proposition. If at least 50

Question 289: Exponential Maximum Asymptotic

Topic: probability **Difficulty**: medium
Let $X_1, X_2, \dots$ be IID Exp(1) random variables and $M_n = \max\{X_1, \dots, X_n\}$. Define p_n as $\mathbb{P}[M_n \leq \ln(n)]$. Evaluate $\lim_{n\to\infty} p_n$. Your answer should be in the form $\frac{b}{e}$, where b is some constant and e is Euler's constant. What is b?

Question 290: Segment Traversal

Topic: probability **Difficulty**: hard
You select a uniformly random starting point on the circumference of a circle, say P_1. Then, you select another uniformly random point on the circumference, P_2, and draw a line segment between P_1 and P_2. You then continue to select uniformly random points on the circumference of the circle, say $P_3, \dots, P_n$, and draw a line segment between P_i and P_{i+1} for each $1 \leq i \leq n-1$. Lastly, you draw a line segment from P_n back to P_1. Find the expected number of intersections between line segments on the interior of the circle as a function of n. Evaluate this for $n = 12$.

Question 291: Expecting HTH

Topic: probability **Difficulty**: medium
On average, how many tosses of a fair coin does it take to see HTH?

Question 292: Mental Black Scholes

Topic: finance **Difficulty**: medium
What is the value of a three-month at-the-money call option on a $100 stock when the implied vol is 40? Assume that $r = 0$ and that the stock follows GBM dynamics. Do not use a calculator.

Question 293: Zero Volatility Returns

Topic: finance **Difficulty**: easy
Suppose that $dS_t = \mu S_t dt + \sigma S_t \sqrt{dt}$ with $\sigma = 0$ and $\mu = 1$. Find $\frac{S_2}{S_0}$. The answer is in the form e^a for some a. Find a.

Question 294: Coin Toss Game I

Topic: probability **Difficulty**: easy
Alice and Bob are playing a game where they toss a fair coin. The first person to flip both a heads and a tails (not necessarily in that order) wins the game. Alice starts and then gives the coin to Bob and the cycle repeats until one of them has flipped both a heads and a tails. What is the probability that Alice wins this game?

Question 295: The Last Airbender

Topic: probability **Difficulty**: easy
Four cards labelled water, earth, fire, and air are placed in front of you faced down. You flip each of the cards over one at a time. You win if you flip over both the air and water cards before the fire card. Otherwise, you lose. What is the probability that you win?

Question 296: Observing Cars

Topic: probability **Difficulty**: easy
You are standing at a bus stop watching cars go by. The probability of observing at least one accident in an hour interval is $\frac{3}{4}$. What is the probability of observing at least one accident within thirty minutes? Assume that the probability of observing an accident at any moment within an hour interval is uniform.

Question 297: Writing to Recruiters

Topic: probability **Difficulty**: easy
You have N letters for recruiters at N firms. You forgot which recruiter works for which firm so you randomly assign each letter to a firm (only one letter going to each firm). What is the expected number of letters going to their correct firm?

Question 298: Chess Tournament III

Topic: probability **Difficulty**: medium
A chess tournament has 2^n players, each with a distinct rating. Assume that the player with the higher rating always wins against a lower rated opponent with probability $0.5 < p < 1$. The winner proceeds to the subsequent round. Since the tournament's structure resembles that of a knockout bracket, n total rounds are played, including the final. The probability that the second-highest rated player defeats the highest rated player in the final round can be expressed as a function of n and p. Determine this function evaluated at $n = 4, p = \frac{3}{4}$.

Question 299: Egg Drop II

Topic: brainteasers **Difficulty**: hard
You are holding three identical eggs in a really large building with n stories. If an egg is dropped at an elevation under story X, then the egg will survive; else, the egg breaks. It is known that the egg will break at some floor number $1 - n$. What is the maximum value of n such that at most 9 drops are required to determine X in the worst-case scenario?

Question 300: Bowl of Cherries III

Topic: probability **Difficulty**: medium
Amy has 100 red cherries and 300 purple cherries in bowl A. She has 300 red cherries and 100 purple cherries in bowl B. She randomly transfers half of the cherries in bowl A into bowl B. What is the probability that a randomly picked cherry from bowl B is red?

Question 301: Put Option Price Estimate

Topic: finance **Difficulty**: hard
Consider a European vanilla put option with strike $K = 15$. You have access to the following binary puts. A binary put pays 1 at expiry if $S_T <= K$ and 0 otherwise. You have access to the following binary puts and their time-0 prices, given in the format of (Strike K, Price P_0)

$$\begin{aligned}(15, 0.83)\\(12.5, 0.64)\\(10, 0.42)\\(7.5, 0.31)\\(5, 0.20)\end{aligned}$$

Give the best estimate for the time-0 price of the strike $K = 15$ put.

Question 302: Median Roll

Topic: probability **Difficulty**: easy
Mike rolls three fair 6-sided dice at once. What is the expected median value of the dice?

Question 303: You Got Mail

Topic: probability **Difficulty**: easy
A letter is going to be sent to one of three mailboxes uniformly at random.

These mailboxes are labelled $1-3$. If the mail ends up in mailbox i, then there is a $\frac{i}{3}$ probability that the mail is actually read. Given that you checked Mailbox 1 and didn't read any letter, find the probability that the mail ended up in Mailbox 1.

Question 304: Bowl of Cherries I

Topic: probability **Difficulty**: medium
Amy has a bowl of 5 red cherries and 8 purple cherries. She takes out cherries one at a time until there are no red cherries left. What is the probability that the bowl is empty?

Question 305: Bear and Bull Market

Topic: brainteasers **Difficulty**: easy
The price of a stock starts at $100. The price then drops 1

Question 306: Counting Up

Topic: brainteasers **Difficulty**: medium
Jay and Kay play a game where Jay calls out a number between 1 and 8 to start. Afterwards, Kay gets to pick a number that is between 1 and 8 larger than the number Jay keeps. They repeatedly select in this fashion. The first player to call out the number 200 wins. There is a value that Jay can choose to start with so that if he plays optimally, he will always win. What is this value? If this is not possible, answer "-1".

Question 307: Simulation Scheme

Topic: probability **Difficulty**: easy
Suppose we want to simulate an event with probability $\frac{1}{3}$. This is possible via the following scheme: Flip a fair coin twice. If the outcome is HH, we say the event happened. If the outcome is HT or TH, we say that the event did not happen. If the outcome is TT, we re-run the 2 flips experiment under the same rules. Find the expected amount of coin flips needed to simulate this event.

Question 308: Poor Odds

Topic: probability **Difficulty**: easy
Angelina is playing a game which she wins with probability 0.1. She must pay 10 to play, and if she wins, she receives 80. If Angelina starts out with 30, to the

nearest thousandth, what is the probability that she wins exactly once before losing it all?

Question 309: Colosseum Fight I

Topic: probability **Difficulty**: hard
Alice and Bob are in Roman times and have 4 gladiators each. The strengths of each of Alice's gladiators are $1-4$, while Bob's gladiators have strengths $4, 5, 9$, and 12. The tournament is going to consist of Alice and Bob picking gladiators to fight against one another one-at-a-time. Then, the two gladiators fight to the death with no ties. If the two gladiators are of strengths x and y, respectively, then the probability that the gladiator with strength x wins is $\frac{x}{x+y}$. The winning gladiator also inherits the strength of its opponent. This means that if a gladiator of strength x wins against a gladiator of strength y, the winner now has strength $x+y$.

Alice is going to pick first for each fight among her remaining gladiators. Afterwards, Bob can select his gladiator (assuming he has one) to go against the one Alice selected. The winner of the tournament is the person who has at least one gladiator left at the end. Assuming Bob plays optimally, what is his probability of winning the tournament?

Question 310: Restack Rings

Topic: brainteasers **Difficulty**: medium
You have a three identical poles in front of you however one has rings in the form of a tower labeled $1-10$ (1 being on the top, then 2, all the way down to 10 on the bottom). You are trying to make this same tower on one of the other poles. There are a couple constraints though. You can not move more than one ring at a time and you can't have rings with higher numbers on top of smaller numbers. How many moves does it take to remake the tower on another pole?

Question 311: Lily Pads I

Topic: brainteasers **Difficulty**: easy
A single lily pad sits in an empty pond. Everyday, the lily pad doubles in area until the whole pond is covered- it is known that it would take 10 days for the single lily pad to cover the entire pond. Imagine instead starting with 8 lily pads on the first day. How many days will it take for the surface of the pond to be covered?

Question 312: Say 50!

Topic: brainteasers **Difficulty**: medium
You and your friend are playing a game where you each pick an integer from 1 to 10 inclusive and add that to a running total. The person that states 50 is the winner. You get to go first, what number do you say first?

Question 313: Basic Dice Game III

Topic: probability **Difficulty**: easy
A casino offers you a game with a six-sided die where you are paid the value of the roll. If you roll a $1, 2$, or 3, the game stops and you are paid out. Else, you add the value to your total sum and get to continue rolling. What is the fair value of this game?

Question 314: Coin Flipping Competition III

Topic: probability **Difficulty**: hard
Ty, Guy, and Psy are all flipping fair until they respectively obtain their first heads. Let T, G, and P represent the number of flips needed for Ty, Guy, and Psy, respectively. Find $\mathbb{P}[T < G < P]$.

Question 315: Power Grid

Topic: probability **Difficulty**: medium
A 3×3 grid of light bulbs is formed. Then, each light bulb is powered on with probability $\frac{1}{2}$. Find the probability that no two adjacent (grid cells that share a common side) light bulbs are powered on.

Question 316: Standing Table

Topic: probability **Difficulty**: hard
We make a table from a circular disk and three legs. We attach the three legs to the circumference of the circular disk. What is the probability that the table stands up?

Question 317: Largest Ball

Topic: probability **Difficulty**: medium
There are 20 balls in an urn labeled from 1 through 20. You pick 10 balls out of this urn. What is the expected maximum value of the 10 balls you picked out?

Question 318: Doubling Bacterium

Topic: brainteasers **Difficulty**: easy
A bacteria population doubles every 10 minutes. At $t_1 = 55$ minutes, the population consists of 200 bacteria. How many bacteria does the population consist of at $t_2 = 95$ minutes?

Question 319: Rolls in a Row

Topic: probability **Difficulty**: easy
On average, how many rolls of a fair 6-sided die must be rolled on average to get two 6s in a row?

Question 320: Fibonacci Limit II

Topic: probability **Difficulty**: medium
Let F_n be the Fibonacci sequence. Compute $\lim_{n\to\infty} \frac{F_{n+2}}{F_n}$. Your answer should be in the form $\frac{a+\sqrt{b}}{c}$, where all of a, b, and c are pairwise relatively prime. Find abc.

Question 321: Sum Leak I

Topic: probability **Difficulty**: easy
Let $X_1, X_2, \dots, X_{40}$ be IID random variables with $\mathbb{E}[X_1] = 2$ and $\mathbb{E}\left[\frac{1}{X_1}\right] = 1$. Define $S_n = X_1 + \cdots + X_n$. Compute $\mathbb{E}\left[\frac{S_{20}}{S_{40}}\right]$.

Question 322: Adult Concert

Topic: brainteasers **Difficulty**: medium

$$\frac{5}{12}$$

of concert attendees are adults. A bus carrying 50 people arrives at the concert. Now, $\frac{11}{25}$ of concert attendees are adults. What is the minimum number of adults who could have been at the concert before the bus arrived?

Question 323: American Call Arbitrage

Topic: finance **Difficulty**: easy
We have 2 call options on the underlying S with initial price $S_0 = 24$ and strike $K = 21$. One option is European and the other is American. A European option is one where you can only exercise it at expiry while an American option can be exercised any time. The European option is valued at 3.21 while the American option is valued at 3.15. You also have access to bonds with discount rate $Z_0 = 0.9$. The underlying pays no dividends.

What is the arbitrage? Give the answer in the form of the initial credit you receive (round to 2 decimal points). Answer -1 if there is no arbitrage.

Question 324: Straddle Output

Topic: finance **Difficulty**: medium
We have a straddle with strike $K = 0$. The underlying asset price is $S \sim N(0, 1)$. What is the value of v, the expected value of the straddle? v^2 can be written in the form $\frac{a}{\pi}$ for a rational number a. Find a.

Question 325: Paired Values I

Topic: probability **Difficulty**: medium
Suppose we have the values $1-6$ in a bowl. We draw them without replacement, noting the order in which we selected them. We multiply the first two values together, the next two values together, and the last two values together. Lastly, we add the three products above. Find the probability the sum is odd.

Question 326: Uniform Product I

Topic: probability **Difficulty**: easy
Let $X, Y, Z \sim \text{Unif}(0, 1)$ IID. Find $\mathbb{P}[X > YZ]$.

Question 327: Circular Hop

Topic: probability **Difficulty**: medium
You are in a circle with 100 points labelled $0 - 99$ clockwise. You start at 1. At each turn, you move one unit left or right with equal probability. What is the expected time to hit 0?

Question 328: Skater Boy

Topic: probability **Difficulty**: medium
Michael is taking his remote control skateboard around campus. Assume the front of the Hopkins sign is the origin $(0,0)$, and that movement to the right is positive horizontally, and movement into campus (upwards) is positive vertically. Each second, Michael chooses a uniformly random angle between 0 and 2π, and then moves his skateboard a distance of 1 foot at that angle from the skateboards previous position. After 16 seconds, what is the expected squared distance Michael's skateboard is away from the Hopkins sign?

Question 329: Short Wood

Topic: probability **Difficulty**: hard
A lumberjack cuts a piece of wood of 1 meter in length at 2 uniformly randomly selected locations along the length of the wood. Find the probability the shortest piece of wood is at most 5 centimeters.

Question 330: Brownian Bridge

Topic: probability **Difficulty**: medium
Let W_t be a standard Brownian Motion. We define the Brownian Bridge on $[0,1]$ as $X_t = W_t - tW_1$. Find $\mathbb{E}[X_{1/2} \mid X_{3/4} = 3]$.

Question 331: Sheep Sharing

Topic: brainteasers **Difficulty**: easy
An Australian farmer dies and leaves his sheep to his three sons. Alfred is to get 20 percent more than John, and 25 percent more than Charles. John's share is 3600 sheep. How many sheep does Charles get?

Question 332: Green Ball Draw

Topic: probability **Difficulty**: easy

3

green and 7 blue balls are in an urn. You draw out 2 balls without replacement and note the second is blue. Find the probability that the first was green.

Question 333: Exercise Gamma

Topic: finance **Difficulty**: easy
You have 3 at-the-money options on some different underlyings with strikes $K = 10$, $K = 20$, $K = 30$, all with expiry T. Assuming Black-Scholes dynamics, which option (strike) has the largest gamma?

Question 334: 2D Paths I

Topic: brainteasers **Difficulty**: easy
You are playing a 2D game where your character is trapped within a 6×6 grid. Your character starts at $(0, 0)$ and can only move up and right. How many possible paths can your character take to get to $(6, 6)$?

Question 335: Keg of Wine

Topic: brainteasers **Difficulty**: easy
Alex has a 10-gallon keg of wine and a big ladle. On Monday, Alex drew off a ladle-full of wine and filled the keg back up with water. On Tuesday, he repeated the process, making sure to mix the contents in the keg around first. On Wednesday morning, he realized that the keg now contains equal parts of wine and water. How many gallons can the ladle hold? The answer is in the form $a - b\sqrt{c}$ with $a = bc$. Find $a + b + c$.

Question 336: Light Bulb

Topic: brainteasers **Difficulty**: easy
You stand outside of a room that contains a light bulb. Next to you are four switches, one of which powers the light bulb. What is the least number of times you need to enter the room to figure out which switch powers the light bulb?

Question 337: Counting Odds

Topic: brainteasers **Difficulty**: medium
How many odd numbers from 1 to 2023, inclusive, are divisible by 3?

Question 338: Balanced Beans II

Topic: brainteasers **Difficulty**: hard
There are 12 beans; one weighs slightly heavier or lighter than the others. What is the minimum number of times a balance scale must be used to guarantee the determination of the abnormal bean?

Question 339: Reciprocal SDE

Topic: pure math **Difficulty**: medium
Let W_t be a standard Brownian Motion. Suppose that X_t is some process such that $\frac{1}{X_t}$ satisfies the SDE $d\left(\frac{1}{X_t}\right) = \frac{1}{X_t}(2dt - dW_t)$. The SDE that X_t satisfies is in the form $dX_t = X_t(adt + bdW_t)$ for some integers a and b. Find ab.

Question 340: Marble Mischief

Topic: probability **Difficulty**: medium
Sean has 200 red, 400 blue, and 600 green marbles in an urn. He draws out the marbles one-at-a-time without replacement. Find the probability that there is at least 1 blue and 1 green marble left in the urn right after the last red marble is selected.

Question 341: Picture Day

Topic: brainteasers **Difficulty**: medium
Ten students of distinct heights are lining up for a picture. The photographer requires that the two tallest students stand in the two center positions and that the remaining students line up such that the heights strictly descend outwards. How many ways can the students line up?

Question 342: Full Solutions

Topic: pure math **Difficulty**: easy
Find all pairs of integers (x, y) such that $3x + 7y = 10000$. All of the solutions can be written in the form $x = a + 7n$ and $y = b - 3n$, where n is any arbitrary integer and a, b are integers such that b is a minimal positive integer. Find ab.

Question 343: Arbitrage Detective IV

Topic: finance **Difficulty**: hard
There are five options on the $TSLA$ options chain right now, all of which are expiring at the end of the month, and with strikes 165, 170, 175, 180, and 185. Suppose the put options respectively cost \$9, \$12, \$14, \$14.5, and \$15 and the call options cost \$15, \$14, \$13, \$12, and \$9. Is there an arbitrage opportunity? If so, enter the minimum amount you are guaranteed to make, if no opportunity exists, enter -1.

Question 344: Trinomial Call Pricing I

Topic: finance **Difficulty**: easy
We want to price a European call option of strike $K = 8$ using a 1-period trinomial tree. The initial stock price is $S_0 = 8$ and has a 50% chance of having a 0% increase, a 30% chance of a 50% decrease, and a 20% chance of a 30% increase.

What is the time-0 price of the option?

Question 345: Russian Roulette I

Topic: probability **Difficulty**: medium
You're playing a game of Russian Roulette with a friend. The six-chambered revolver is loaded with one bullet. Initially, the cylinder is spun to randomize the order of the chambers. The two of you take turns pulling the trigger until the person who fires the gun loses. Given that the cylinder is not re-spun after each turn, what is the probability that you win if your friend goes first?

Question 346: Big Smalls

Topic: probability **Difficulty**: medium
Alice and Bob play the following game: Alice generates a uniformly random integer between 1 and 100, inclusive of both. Bob can pick any number that he wants between 1 and 100, inclusive of both. Whoever has the larger number must pays out the amount of the smaller number to the person with the smaller number. For example, if Alice selects 70 and Bob selects 50, Bob will receive 50 from Alice. If they select the same number, nobody pays out anything. Assuming optimal play by Bob, what is his expected payout?

Question 347: Beta Difference

Topic: probability **Difficulty**: easy
Let $X_1, \ldots, X_{324}, Y_1, \ldots, Y_{324} \sim \text{Beta}(1, 2)$ IID. Define $S_{324} = X_1 + \cdots + X_{324}$ and $T_{324} = Y_1 + \cdots + Y_{324}$. Approximate $\mathbb{P}[S_{324} - T_{324} > 10]$. Your answer should be in the form $\Phi(a)$, where Φ is the standard normal CDF. Find a.

Question 348: Dice Upon Dice

Topic: probability **Difficulty**: easy
We have a fair 6−sided die. Roll it once and call the upface N. We then roll N

fair 6−sided dice and sum the upfaces of all N dice. Call this sum M. Lastly, we roll M fair 6−sided dice and call the resulting sum of the upfaces S. Find $\mathbb{E}[S]$.

Question 349: Probability of Unfair Coin II

Topic: probability **Difficulty**: easy
Flip 98 fair coins, one double-headed coin, and 1 double-tailed coin and observe the first coin tossed. A coin is selected uniformly at random and you see it shows heads. What is the probability that this coin is the double-headed coin?

Question 350: Voter Mayhem II

Topic: probability **Difficulty**: hard
Two candidates, say A and B, are running for office. Candidate A received n votes, while Candidate B received m votes, with $n > m$. The $n + m$ votes are thrown into a box and shuffled around. Then, the votes are drawn without replacement one-by-one. A running tally of the number of votes for each candidate is kept. The probability that Candidate A is never behind in the voting count (excluding the initial state where both have 0) is a function $Q(n, m)$. Find $Q(100, 80)$.

Question 351: Matrix Editor

Topic: probability **Difficulty**: medium
Suppose you start with the 2×2 identity matrix I_2. At each step, select one of the four elements of the matrix uniformly at random. If the element you select is a 1, change it to 0. If the element you select is a 0, change it to 1. Find the expected number of steps needed to obtain a singular matrix (i.e. the determinant is 0).

Question 352: Fish Capture

Topic: brainteasers **Difficulty**: easy
You are at a pond. One day, you decide to catch 50 fish and tag them. The next day, you capture 40 fish and 10 of them have tags. What is your best guess at the number of fish in the pond?

Question 353: Delayed Ruin

Topic: probability **Difficulty**: hard
A gambler starts with $\$n$, where $n \geq 1$ is an integer. Each round, the gambler

wins or loses \$1 with respective probabilities p and $1-p$. Once the gambler has \$0 left, he leaves the table. Find the probability that the gambler ruins after $n+2k$ rounds of playing the game, where $k \geq 0$ is an integer. The probability can be written as a function $f(p,n,k)$. Compute $f(1/3,5,3)$ to the nearest ten-thousandth.

Question 354: Defining Standard Deviation

Topic: statistics **Difficulty**: easy
What is the sample standard deviation of $(1,2,3,4,5)$? Assume that we have drawn independent and identically distributed samples and are estimating the population parameter.

Question 355: Bacterial Survival II

Topic: probability **Difficulty**: medium
There is one bacterium on a cell plate. After every minute, the bacterium may die, stay the same, split into two, or split into three with equal probabilities. Assuming all bacteria behave the same and independent of other bacteria, what is the probability that the bacterial population will die out? The answer is in the form $\sqrt{a}-b$ for integers a and b. Find $a+b$.

Question 356: Flash Drive Finders

Topic: probability **Difficulty**: medium
Gabe lost a flash drive on which \$500000 of Bitcoin is stored. If he doesn't find it in a week, he'll lose all his money. Luckily, professional flash drive finders are up for hire at a rate of \$5000 per week. Suppose each flash drive finder has an 90% chance of locating the lost flash drive. How many flash drive finders should Gabe hire?

Question 357: Central Containment

Topic: probability **Difficulty**: hard
What is the probability that three random points on a unit circle would form a triangle that includes the center of the unit circle?

Question 358: Ascending Stairs

Topic: brainteasers **Difficulty**: easy
You are at the bottom of a staircase with 5 steps. You can either take one or

two steps at a time. How many different ways are there for you to ascend the staircase?

Question 359: RPS Galore

Topic: brainteasers **Difficulty**: easy
Jack and Jane play Rock (R) Paper (P) Scissors (S) 10 times. It is known that each round had a winner. Additionally, it is known that Jack played R, P, and S, respectively, $3, 1$, and 6 times, while Jane played R, P, and S, respectively, $2, 4$, and 4 times. How many wins did Jack have? If it is not possible to know, answer "-1".

Question 360: Defining Mean

Topic: statistics **Difficulty**: easy
Let $X \sim N(2, 4)$ and $Y \sim N(1, 4)$. Furthermore, let $Z = 3X - 2Y$. What is the mean of Z?

Question 361: Determination I

Topic: statistics **Difficulty**: easy
Given three data sets X_1, X_2, and Y , we run two linear regressions to obtain $y \sim \alpha_1 + \beta_1 x_1$ and $y \sim \alpha_2 + \beta_2 x_2$. The R^2 value for both regressions is 0.05. Find the greatest lower bound on R^2 value of the regression $y \sim \alpha + \beta' x_1 + \beta'' x_2$.

Question 362: High Roller

Topic: probability **Difficulty**: medium
Two fair $6-$sided dice are rolled. You can either keep the product of the observed values or re-roll both dice once for \$4. What is your expected payout on this game under a rational strategy?

Question 363: 2D Paths II

Topic: brainteasers **Difficulty**: medium
You are playing a 2D game where your character is trapped within a 6×6 grid. Your character starts at $(0, 0)$ and can only move up and right. There is a power-up located at $(2, 3)$. How many possible paths can your character take to get to $(6, 6)$ such that it can collect the power-up?

Question 364: Card Turner

Topic: probability **Difficulty**: medium
Suppose you have 20 cards with the values $1 - 10$ each appearing twice in a deck. You draw 2 cards at a time uniformly at random from the 20 cards. If they match in value, you remove them from the deck. Otherwise, they are put back into the deck. The game finishes once there are no more cards to draw. Each drawing of two cards is a turn. Find the expected number of turns needed to finish the game.

Question 365: Stacked Derivative

Topic: pure math **Difficulty**: medium
Find the derivative of $f(x) = (\ln(x))^x$ at $x = e$.

Question 366: Make a Market I

Topic: finance **Difficulty**: easy
Let's say you have product A in which you are quoting 4 @ 5 and product B in which you are quoting 10 @ 12. X @ Y means that X is our bid and Y is our ask. We want to make a market on the product $A + B$. What is the bid-ask spread you will quote? Give the answer in the format of $Y^2 - X^2$

Question 367: 9 Sum I

Topic: probability **Difficulty**: medium
How many non-negative $6-$digit integers have digits that sum to 9? 116001 and 801000 are two examples.

Question 368: Overlapping Segments

Topic: probability **Difficulty**: medium
Suppose that $X_1, X_2, X_3, X_4 \sim \text{Unif}(0, 1)$ IID. Draw two line segments: One between X_1 and X_2 and one between X_3 and X_4. Find the probability that there is overlap between the two line segments.

Question 369: Digit Match

Topic: probability **Difficulty**: easy
Let X be a uniformly at random selected integer from the collection of all $11-$digit integers. Find the expected amount of digits of X that are equal to 8.

Question 370: Make Your Martingale I

Topic: pure math **Difficulty**: easy
Let $X_1, X_2, \dots$ be IID random variables that take the values ± 1 with equal probability. Fix a constant c. Let $M_0 = 1$ and $M_n = M_{n-1}e^{X_n + c}$. There is a unique value of c such that M_n is a martingale with respect to the natural filtration $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. Find c. The answer will be in the form $-\log\left(ae^b + ce^d\right)$ for integers a, b, c, d. log here refers to the natural logarithm. Find $a + b + c + d$.

Question 371: Regional Manager I

Topic: statistics **Difficulty**: easy
The regional sales manager of a large paper corporation claims that his salespeople are averaging no more than 15 deals per week, which is subpar compared to the neighboring regions. To check this, he records the number of deals that 36 random salespeople make on a random week. He finds the mean and variance of the number of deals that the 36 random salespeople made to be 17 and 9, respectively. With these values, he runs a statistical test to see if his employees are making significantly more deals than 15. What is the value of the appropriate test statistic he used? Assume simple random sampling, variance homogeneity, and that the number of deals closed is approximately normally distributed.

Question 372: 2 In A Row

Topic: probability **Difficulty**: medium
You have a coin where you can decide the probability p of obtaining a heads before flipping the coin 4 times. What value of p maximizes the probability that you obtain exactly two consecutive heads in your 4 flips? For example, $HHHT$ and $HHHH$ are not allowed, but $HHTH$ is allowed. The answer is in the form $\dfrac{a - \sqrt{b}}{c}$ for integers $a, b,$ and c with c minimal. Find $a + b + c$.

Question 373: Missing Million II

Topic: probability **Difficulty**: easy
You are on a game show with 3 doors in front of you. One of the doors has \$1 million inside, while the other two are empty. In the final round, the host lets you spin a wheel two times that may reveal which door the \$1 million is in. On each spin, this wheel will tell you the location of the \$1 million 3/5 of the time, independent between spins. Otherwise, it tells you nothing about the location of the money. After two spins, if the wheel has not revealed to you the location of the \$1 million, you must guess uniformly at random. What is the probability you locate the \$1 million door?

Question 374: Fishy Bear

Topic: probability **Difficulty**: easy
A bear wants to catch 3 fish from a river. When the bear has caught 3 fish, it will leave. The bear catches each fish with probability $\frac{1}{2}$. Find the probability the 5th fish is caught.

Question 375: Coin Flipping Competition I

Topic: probability **Difficulty**: medium
Ty and Guy are both flipping fair coins until the respectively obtain their first heads. Find the probability that it takes Ty at least 4 times as many flips to obtain his first heads as Guy.

Question 376: Basic Dice Game IV

Topic: probability **Difficulty**: medium
You roll a fair die. Then, you get to roll until you obtain a value that differs from your first roll. You get paid out the value equal to the sum of all of your rolls (including the first one). What is the fair value of this game?

Question 377: Expected Chord Length

Topic: probability **Difficulty**: hard
Two points are uniformly at random selected from the circumference of the unit circle. Find the expected length of the chord (line segment) between the two points. The answer is in the form $\frac{a}{\pi}$ for some rational number a. Find a.

Question 378: Rabbit Hop II

Topic: probability **Difficulty**: easy
A rabbit starts at the floor in front of a staircase of 10 stairs. The rabbit can hop up any amount of stairs at each move, but it must make an even amount of hops to get to the top. How many distinct paths are there from the floor to the top of the staircase (i.e. to the top of the 10th stair)?

Question 379: Primitive Preference

Topic: brainteasers **Difficulty**: easy
On Earth II, every couple prefers to have a baby boy over a baby girl. Assume that there is an equally likely chance of having a boy or girl, and that the genders

of children are mutually independent. If each couple continues on having more children until they have a boy, in which they will stop having more children, what does the percentage of boys in this society approach over time?

Question 380: Statistical Test Review V

Topic: statistics **Difficulty**: easy
Rishab wants to use a Z test to test a hypothesis involving the population mean. Which of the following conditions must be met in order for Rishab to go through with the Z test? (1) Population variance must be known, (2) sample must be a random sample of the population, (3) distribution of sample mean must be approximately normal, (4) the sample size is at least 30. Please respond with the conditions concatenated in increasing order. For example, if 1 and 3 must be met, respond with 13.

Question 381: Arbitrage Detective III

Topic: finance **Difficulty**: medium
There are 5 strikes available on the QG June $TSLA$ options chain (expiring in one month). There are also 5 more strikes available on the July $TSLA$ options chain, with the same strikes at 165, 170, 175, 180, and 185. We will only consider the call side in this question.

Suppose we are expecting some volatility in the coming earnings report, and our June call chain is priced at $\$21, \$17, \$14, \$12, \$10$ (increasing in strike price). We look out at the July chain and our prices are $\$20, \$17, \$14, \$13, \$11$.

Is there an arbitrage opportunity? If so, enter the minimum amount you are guaranteed to make, if no opportunity exists, enter -1.

Question 382: Lollipop Mix

Topic: probability **Difficulty**: easy
How many distinguishable permutations of LOLLIPOP are there that start and end with the same letter?

Question 383: 4 and 5 Sum

Topic: brainteasers **Difficulty**: easy
How many positive integers at most 1000 can be written as the sum of some amount of 4s and/or 5s? For example, $21 = 4 + 4 + 4 + 4 + 5$

Question 384: Strictly Better

Topic: probability **Difficulty**: medium
Suppose Jimmy and Simon are selecting uniformly random numbers. Jimmy selects from the set $\{1, 2, ..., 1000\}$ and Simon selects from $\{1, 2, ..., 3000\}$. Compute the probability that Simon chooses a strictly larger number than Jimmy

Question 385: Jellybean Jar I

Topic: probability **Difficulty**: easy
A pack contains 6 red and 10 blue jellybeans. A child wants to eat 4 jellybeans and grabs into the pack to select 4 jellybeans one-by-one uniformly at random without replacement. Find the probability that the first two jellybeans are red and the last two are blue.

Question 386: Playlist

Topic: probability **Difficulty**: easy
Drew is creating a playlist. There are 147 songs in the playlist, of which 4 are written by DaBaby and 6 are written by Rae Sremmurd. Drew randomly orders all of the songs in the playlist. Find the probability all of the Rae Sremmurd songs in the playlist come before the second DaBaby song.

Question 387: Missing 7

Topic: probability **Difficulty**: medium

$$7$$

people uniformly at random select a random integer from the set $\{1, 2, \ldots, 7\}$. Given that nobody selected 7, find the probability that 7 is the only value nobody selected.

Question 388: Random Particles

Topic: probability **Difficulty**: hard

$$1000$$

infinitesimal particles, each of which travels at a constant rate of one meter per second, are randomly placed along a line of meter length. When two particles collide, both immediately change direction and continue traveling at a constant rate of one meter per second. How long will it take for all of the particles to fall off of the line on average?

Question 389: Shuffled Deck

Topic: brainteasers **Difficulty**: hard
Suppose we have a standard deck of cards arranged in some order. We shuffle the cards by cutting the deck into two halves: Cards in positions $1-26$ and $27-52$. Then, we alternate cards from each half of the deck. Namely, the cards in positions $1-26$ are now in positions $2, 4, \ldots, 52$. and the cards in positions $27-52$ are now in positions $1, 3, \ldots, 51$. What is the minimum number of shuffles needed before the deck returns to it's original state?

Question 390: Water(melon) Weight

Topic: brainteasers **Difficulty**: easy
A 100-pound watermelon is 99% water. After spending some time in the sun, it has only dried down to 98% water. How much does the watermelon weigh now?

Question 391: Dice Order I

Topic: probability **Difficulty**: medium
You roll two fair dice. What is the probability that the highest value rolled is a four?

Question 392: Basic Eigenvalues

Topic: pure math **Difficulty**: easy
You have the following matrix

$$\begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$$

There are 2 eigenvalues, a and b. What are the two eigenvalues? Give the answer in the format $a^2 + b^2$.

Question 393: Digit Multiplier

Topic: probability **Difficulty**: easy
Four numbers from $\{1, 2, \ldots, 9\}$ are selected with repetition allowed. Find the probability that the product of the four integers has a units digit of $1, 3, 7$, or 9.

Question 394: Parameter Picker

Topic: probability **Difficulty**: medium
Let $X \sim \text{Beta}(a, b)$ for some unknown parameters a and b. Suppose we know

that for some $c, d \in \mathbb{R}$, the support of $Y = cX + d$ is $[-0.02, 0.04]$, $\mathbb{E}[Y] = \frac{1}{400}$, and $\mathrm{Var}(Y) = \frac{3}{32000}$. What is $a - b$?

Question 395: Triangle Walk

Topic: probability **Difficulty**: medium
Consider a triangular grid composed of equilateral triangles with length 1 unit. Suppose, during each turn, Joey randomly selects the next grid line independently and uniformly. Compute the probability that, after 5 turns, Joey had ventured beyond 1 unit from his starting point at least once?

Question 396: Thorough Frog

Topic: probability **Difficulty**: medium
A frog has 100 lily pads arranged in a circle. Each one has a distinct number $0, 1, \ldots, 99$ on it, arranged in increasing order counter-clockwise starting from 0. The frog starts at lily pad 0 and hops to the closest lily pad on the left or right of its current position with equal probability per hop. Find the probability that when the frog lands on lily pad 50 for the first time, it has visited every other lily pad not labeled 50.

Question 397: Disc EV

Topic: probability **Difficulty**: easy
A point (X, Y) is selected uniformly at random from the unit disc. Find $\mathbb{E}[XY]$.

Question 398: Bond Practice IV

Topic: finance **Difficulty**: easy
Calculate the price of a bond with these characteristics. The coupon rate is 0.06, coupon payments are made every six months (twice per year), and the par value of the bond is 1,000. There are 8.0 years to maturity and a market interest rate of 0.03

Question 399: Rain Chance I

Topic: probability **Difficulty**: easy
This weekend, there is a 40% chance it rains on Saturday and a 70% chance it rains on Sunday. Assuming independence, what is the probability it does not rain this weekend?

Question 400: Prime Product

Topic: brainteasers **Difficulty**: easy

$$3$$

integers are in the form $a, a+k$, and $a+2k$ for some integers a and k. The product of these three integers is prime. Find a^2+k^2. Note that prime integers are positive by definition.

Question 401: Covered Calls I

Topic: finance **Difficulty**: easy
You are entering a covered call position on $TSLA. You buy 100 shares at $230 and sell the Dec call @10.50. What is your breakeven for this strategy?

Question 402: Increasing Dice Order II

Topic: probability **Difficulty**: medium
You roll a standard 10-sided dice 5 times. The probability that the values are strictly increasing can be written in the form:

$$\frac{x}{10^5}$$

What is x?

Question 403: Series Moment

Topic: probability **Difficulty**: medium
Let X be a random variable with MGF $M(\theta) = 2e^{\frac{\theta}{2}} - 1$. Find $\mathbb{E}\left[\frac{1}{1-X}\right]$. If this expectation can't be computed or is infinite, enter -100.

Question 404: Particle Reach V

Topic: probability **Difficulty**: medium
Consider a particle that performs a random walk on the integers starting at position 0. At each step, the particle moves from position i to position $i+1$ with probability p, while the probability it moves from i to $i-1$ is $1-p$. If $p = 1/3$, find the expected number of steps until the particle reaches 1. If the answer is infinite, answer -1.

Question 405: Nested Root

Topic: brainteasers **Difficulty**: easy
Find the value of $\sqrt{6+\sqrt{6+\sqrt{6+\dots}}}$.

Question 406: Seating Drama

Topic: brainteasers **Difficulty**: easy
Five people, denoted $1, 2, 3, 4, 5$, sit around a circular table of 5 seats. 1 does not want to sit next to either one of 2 or 5. Additionally, 4 wishes to sit immediately to the right of 2 while 2 sits immediately to the right of 5. Who is to the left of 5?

Question 407: Cube Shadow

Topic: brainteasers **Difficulty**: medium
Consider a cube is floating in front of you in the air, and there is a light shining on it projecting it's shadow on the wall. You are free to rotate the cube around, and want to find out what's the largest sided polygon you can create with the shadow. Enter your answer as n where n is the largest n-sided polygon you can create with it's shadow.

Question 408: Covariance Review II

Topic: probability **Difficulty**: easy
You are given the following information:

$$\begin{aligned} \mathbb{E}[X_1] &= -3 \\ \mathbb{E}[X_2] &= 7 \\ \mathrm{Var}[X_1] &= 24 \\ \mathrm{Var}[X_2] &= 6 \end{aligned}$$

Compute the maximum possible value for $\mathrm{Cov}(X_1, X_2)$.

Question 409: Equal Money

Topic: brainteasers **Difficulty**: medium

$$9$$

boys and 3 girls have some amount of money that they plan to equally share. Every boy gave an equal sum to every girl, and every girl gave another equal sum to every boy. Girls start out with more money than boys. Every child then possessed exactly the same amount. What is the smallest possible amount that every person possesses at the end? The money can be as granular as individual cents.

Question 410: Rainbow Selection

Topic: probability **Difficulty**: medium
An urn contains 20 balls colored each of the 7 colors of the rainbow (140 total balls). We select balls one-by-one without replacement. Given that in the first 70 draws we selected 5 more red balls than yellow, find the probability the 71st ball drawn is yellow.

Question 411: Circular Cut

Topic: probability **Difficulty**: hard
Three points are uniformly at random selected from the circumference of the unit circle. These three points divide the circle into three arcs. Find the expected length of the arc that contains the point $(1, 0)$. The answer is in the form $q\pi$ for a rational number q. Find q.

Question 412: Trading Cards

Topic: probability **Difficulty**: hard
Dan loads up a box with $N \sim \text{Poisson}(6)$ trading cards. He wants to choose 4 cards from this box to take with him to a convention. Find the expected numbers of ways that Dan can select the 4 cards. Note that $\binom{n}{k} = 0$ if $n < k$.

Question 413: Maximal Variance

Topic: probability **Difficulty**: medium
Let X be any random variable only defined on $[-1, 1]$. Find the maximum possible value of $\text{Var}(X)$.

Question 414: Increasing Uniform Chain I

Topic: probability **Difficulty**: hard
Let $X_1, X_2, \cdots \sim \text{Unif}(0, 1)$ IID. Let N be the first index n where $X_n \neq \max\{X_1, \ldots, X_n\}$. Find $\mathbb{E}[X_{N-1}]$ i.e. the largest value among the first N values selected. The answer will be in the form $a + be$ for integers a and b. Note here that e is Euler's constant. Find $a + b$.

Question 415: Repetitious Game I

Topic: probability **Difficulty**: easy
Audrey is playing a game. She flips a fair coin repeatedly until she gets two heads in a row, after which she finishes the game. Determine the expected number of coin flips.

Question 416: Presidential Options

Topic: finance **Difficulty**: medium
You have access to 4 different contracts that correspond to the 2020 US Presidential election. These contracts pay 1 if the outcome occurs and 0 otherwise. You are allowed to short contracts, in which the payout is -1 if the event occurs and 0 otherwise. The current value of the contract is the cost to enter the contract (or the credit you gain if you short).

Contract 1: Trump wins the 2020 election (yes, no): \$0.17, \$0.83

Contract 2: Which party wins Arizona (Democrat, Republican): \$0.80, \$0.20

Contract 3: Which party wins Georgia (Democrat, Republican): \$0.56, \$0.44

Contract 4: Which party will win Pennsylvania (Democrat, Republican): \$0.84, \$0.16

For example, if you buy the Trump does not win the 2020 election contract, it will cost 0.83, which will be worth \$1.00 if he does not win, and \$0.00 if he does win. Also, assume that Trump will only win the election if he wins **all three states:** Arizona, Georgia, and Pennsylvania. Find the arbitrage opportunity. Give the answer in the format of the absolute value of the credit (in dollars) you obtain to enter the position.

Question 417: Reflip

Topic: probability **Difficulty**: easy
You are given 8 fair coins and flip all of them at once. Afterwards, you are allowed to reflip as many coins as you would like one time each. At the end, you are given \$1 per head that appears. Assuming optimal play, find the fair value of this game.

Question 418: Breakeven Price I

Topic: finance **Difficulty**: easy
You buy a call option for \$3 at a strike price of \$4. What is the breakeven price?

Question 419: Put Call Futures Parity

Topic: finance **Difficulty**: easy
If the 120 call on an underlying is priced at 3. The underlying future is priced at 110, what price should the corresponding put at the 120 strike be?

Question 420: Unique IDs

Topic: probability **Difficulty**: medium
There are IDs for employees of a large trading company that consists of 10 distinct digits (0-9). Suppose the number follows the pattern $ABCDEFGHIJ$ where $A > B > C > D > E$ and $E < F < G < H < I < J$. How many unique IDs can be made?

Question 421: Casino War

Topic: probability **Difficulty**: medium
You are playing a game with the casino. You and the dealer are each dealt a card from a fair, shuffled deck of 52 cards. If you have a strictly larger number than the dealer's, you win- else, you lose. What is the probability you win?

Question 422: 100 Factorial Digits

Topic: brainteasers **Difficulty**: medium
How many digits are in 100!?

Question 423: Conditional Uniform

Topic: probability **Difficulty**: easy
Suppose $X, Y \sim \text{Unif}(0, 1)$ IID. Compute $\mathbb{P}[X - Y > 1/2 \mid X + Y < 1]$.

Question 424: Six Card Sum

Topic: probability **Difficulty**: medium
Jamie is told there are 3 aces and 3 jacks in a pile. Each turn, a card is drawn without replacement; Jamie earns \$1 if he guesses the drawn card correctly. Jamie plays 6 turns under the optimal strategy. How much money should Jamie expect to earn?

Question 425: Same Heads

Topic: probability **Difficulty**: easy

$$2$$

people flip a fair coin 4 times each and record their flips. What is the probability that the two people flipped the same number of heads?

Question 426: No Heads

Topic: probability **Difficulty**: easy
Let $N \sim \text{Poisson}(2)$. We flip a fair coin N times. Find the probability that no heads are obtained by the end of N flips. Your answer should be in the form e^a for a rational number a. Find a.

Question 427: Taylor Sum

Topic: pure math **Difficulty**: medium
Evaluate $f^{(7)}(0)$, where $f(x) = (1 + x + 2x^2 + 3x^3)e^{x^2}$.

Question 428: Visible Number of Heads

Topic: probability **Difficulty**: easy
Suppose 20 people whose heights follow some unknown continuous distribution are arranged in a single-file line. We then stand at the front of the line and observe that we can see someone's head if they are taller than everyone that comes before. Let X be the number of visible heads. Compute $\mathbb{E}[X]$? Round to the nearest tenths.

Question 429: Infinite Exponents

Topic: brainteasers **Difficulty**: easy
If $x^{x^{\cdots}} = 2$, what is x (rounded to the nearest whole number)?

Question 430: Oil Transport

Topic: brainteasers **Difficulty**: medium
An oil tanker is tasked with the goal of transporting 3000 gallons of oil from Port A to Port B which are 1000 miles apart. However, the oil tanker loses 1 gallon of oil for every mile it travels from spillage (at a constant rate) and can carry a max of 1000 gallons at any time. The tanker can also dump any oil that it is carrying at any number of storage ports on the way from Port A to

Port B and pick it up later. Assuming an optimal travel plan where you decide where to place any number of storage ports and how to carry the oil, how many gallons can you transport form Port A to Port B (round to the nearest gallon)?

Question 431: ITM Expiration

Topic: finance **Difficulty**: easy
Consider a vanilla call and a vanilla put option on an underlying stock S, with strike $K = 15$ following GBM dynamics. The risk-neutral probability that the put expires in the money is 0.42. What is the risk-neutral probability that the call expires in-the-money?

Question 432: Least Multiple of 15

Topic: brainteasers **Difficulty**: easy
What is the least positive multiple of 15 whose digits consist of 1's and 0's only?

Question 433: Russian Roulette III

Topic: probability **Difficulty**: easy
You're playing a game of Russian Roulette with a friend. The six-chambered revolver is loaded with two randomly placed bullets. Initially, the cylinder is spun to randomize the order of the chambers. The two of you take turns pulling the trigger until the person who fires the gun loses. Your friend goes first, and lives after the first trigger pull. You are then given the choice to either spin the barrel or not before pulling the trigger. What is the different in probability of you surviving between spinning and not spinning the barrel?

Question 434: Marketing Claims

Topic: statistics **Difficulty**: easy
A designer watch brand claims that at least 25% of the public prefer their product over their competitor's. To check this claim, you sample 80 people and ask which product they prefer. With $\alpha = 0.05$, how small would the sample percentage need to be before the claim could be rejected? Round to the nearest hundredth. Assume simple random sampling, variance homogeneity, and that sentiment is approximately normally distributed.

Question 435: Deriving Log-Price Dynamics

Topic: pure math **Difficulty**: medium
Let's say S_t follows the dynamics:

$$dS_t = 3S_t dt + 4S_t dW_t$$

where W_t is a standard Brownian motion.
Find the dynamics of $X_t = \ln(S_t)$, where we can write the dynamics as:

$$dX_t = a dt + b dW_t$$

Find $a^2 + b^2$

Question 436: Equal Flip Timer

Topic: probability **Difficulty**: easy
Brad and Chad both flip coins with probability p of heads on each flip. Let B be the event that it takes strictly less flips for Brad to get his first heads than Chad, C be the event that it takes Chad strictly less flips than Brad to get his first heads than Brad, and E be the event that it takes them an equal number of flips to obtain their first heads. Find the value of p such that $\mathbb{P}[E] = \mathbb{P}[C]$.

Question 437: Poisoned Kegs I

Topic: brainteasers **Difficulty**: medium
A king has 8 kegs of liquor. 1 of the kegs is poisonous. If someone drinks any amount of liquor from the poisoned keg, they will die in exactly 1 month. 3 servants have volunteered to risk their lives for the king and test the kegs. What is the minimum number of months needed before the poisoned keg can be identified?

Question 438: Russian Dolls

Topic: probability **Difficulty**: easy
Russian Dolls are dolls of decreasing size that can be opened up to have smaller dolls placed inside. Kaushik has 7 nesting dolls of distinct sizes. A valid nesting of the dolls requires that at least 2 of the 7 dolls are used and that each of the dolls are placed inside one another with decreasing size as you move from the outer doll to the inner doll. How many valid nestings can Kaushik create?

Question 439: Option Arbitrage

Topic: finance **Difficulty**: easy
You have 3 calls at the following strikes and prices:

1000 @ 4

1010 @ 3.5

1020 @ 2.75

There is an arbitrage. What amount of money are you guaranteed to make (in dollars)?

Question 440: The Picking Hat

Topic: probability **Difficulty**: hard
A hat contains the numbers 1-100. The rules to the game are as follows. Each round, AJ draws a number out of the hat, writes it down, and puts the number back in the hat. The last number written down is the number of dollars awarded to AJ. AJ may play as many rounds as he would like, but each round costs $1. Assuming optimal play, what is the fair value of this game?

Question 441: Proper Tables

Topic: probability **Difficulty**: hard
Three points are randomly selected uniformly at random on the circumference of a circular slab. Then, 3 table legs are attached vertically at those three points. For this table to be "proper", no two legs can be within 90 degrees of one another. What is the probability that this table is "proper"?

Question 442: Optimal Ball Draw

Topic: probability **Difficulty**: medium
There are 2 white and 3 black balls in an urn. Rajiv randomly draws balls out of the urn without replacement; he has the option to stop drawing at any time. For every white ball he draws, he earns $1, but for every black ball he draws, he loses $1. Suppose Rajiv draws balls following the optimal strategy. What is his expected loss or gain?

Question 443: Particle Reach VII

Topic: probability **Difficulty**: medium
Consider a particle that performs a random walk on the integers starting at position 0. At each step, the particle moves from position i to position $i+1$ with probability p, while the probability it moves from i to $i-1$ is $1-p$. If $p = 1/2$, find the expected number of steps until the particle reaches 1. If the answer is infinite, answer -1.

Question 444: Dice Roll Intuition

Topic: probability **Difficulty**: medium
Do not use a pen for the question. We are playing a game; we roll a fair, 6-sided die repeatedly. We stop playing the game when we get two specific numbers in a row; the payoff is equal to the total number of rolls of the die. For our two specific numbers, should we have a preference for (case 1) 4 then 5, or (case 2) 4 then 4, or (case 3) would it not matter? Respond with the number of the case.

Question 445: Odd Stars

Topic: probability **Difficulty**: medium
How many non-negative odd integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 96$? The answer is in the form $\binom{n}{k}$ for $k < \frac{n}{2}$. Find nk.

Question 446: Dart Distance

Topic: probability **Difficulty**: easy
A circular dartboard of radius 1 is placed on a ball. Joan throws a dart such that it lands uniformly at random on the surface of the dartboard. Find the expected radial distance that the dart lands from the center.

Question 447: Put Theta

Topic: finance **Difficulty**: medium
The underlying is currently at price $S_0 = 35$. You buy a put at strike $K = 33$. What will happen to theta if the price of the underlying increases? Answer -1 if theta decreases, 0 if theta stays the same, and 1 if it increases.

Question 448: Coin Toss Game II

Topic: probability **Difficulty**: medium
Alice and Bob are playing a game where they toss a fair coin. The first person to flip two heads (not necessarily sequentially) wins the game. Alice starts and then gives the coin to Bob and the cycle repeats until one of them has flipped two heads. What is the probability that Alice wins this game?

Question 449: Bakery Boxes

Topic: probability **Difficulty**: medium
A bakery manager uniformly at random selects an integer (name it k) between 1 and 4, inclusive. He then chooses from k distinct desert types at his shop to create a gift basket consisting of k items. Find the expected number of ways that the manager can create the gift basket.

Question 450: Tic Tac Toe

Topic: probability **Difficulty**: medium
In a standard 3×3 tie tac toe grid, a player wins when they have 3 of their symbol (either X or O) in a row, column, or upon one of the two diagonals, yielding 8 possible winning paths. How many winning paths are there in $4\times4\times4$ 3D tic tac toe grid? The winning condition is that a player must have 4 of their symbol in a line, which includes diagonally and vertically down.

Question 451: Combinatorial Electrician

Topic: probability **Difficulty**: medium
An electrician sets up 12 light bulbs on a circle. Each pair of light bulbs has a wire connecting them. Each individual wire is active with probability $\frac{1}{2}$, independent of all other wires. A collection of $2 \leq k \leq 12$ light bulbs is said to form a complete $k-$circuit if between any 2 light bulbs in the circuit, we have an active wire. Find the expected number of $4-$circuits in our electrician's arrangement of light bulbs.

Question 452: Collecting Toys I

Topic: probability **Difficulty**: medium
Every box of cereal contains one toy from a group of five distinct toys, each of which is mutually independent from the others and is equally likely to be within a given box. On average, how many boxes of cereal will you need to open in order to collect all five toys?

Question 453: Word Shift I

Topic: probability **Difficulty**: medium
Find the number of anagrams of $BOOLAHUBBOO$ that have all of the B's before the first O.

Question 454: Soccer Practice

Topic: probability **Difficulty**: hard
You're practicing soccer by taking 100 penalty kicks. Assume that you have made the first goal but missed the second. For each of the following kicks, the probability that you score is the fraction of goals you've made thus far. For example, if you made 17 goals out of the first 30 attempts, then the probability that you make the 31st goal is $\frac{17}{30}$. After 100 attempts, including the first two, what is the probability that you score exactly 66 penalty kicks?

Question 455: Leg Count

Topic: brainteasers **Difficulty**: easy
A group of chickens (2 legs), cows (4 legs), and spiders (8 legs) are hanging out. There are a total of 440 legs between all of them. There are twice as many chickens as cows and twice as many spiders as chickens. How many spiders are there?

Question 456: Sum Exceedance III

Topic: probability **Difficulty**: hard
Let $X_1, X_2, \cdots \sim \text{Unif}(0,1)$ IID and $N_1 = \min\{n : X_1 + \cdots + X_n > 1\}$. Let $S_n = X_1 + \cdots + X_n$. Compute $\mathbb{E}[S_{N_1}]$. The answer will be in the form ce for a rational number c. Find c.

Question 457: Identical Hands

Topic: probability **Difficulty**: medium
A dealer is dealing a deck with 12 cards to 4 players. The deck consists of 4 Kings, 4 Queens, and 4 Jacks. What is the probability that every person receives the same hand of 1 of each card?

Question 458: Returning Books

Topic: statistics **Difficulty**: easy
Library patrons may either return their book at their local library or another library. If the book is returned to the local library, the system takes 2 hours to register this return action. If the book is returned to another library, the system takes 5 hours to register this return action. Suppose 70% of patrons return their books to their local library. What is variance in the number of hours the system takes to register a return action?

Question 459: Slippery Ladder I

Topic: pure math **Difficulty**: easy

A 50−ft ladder is placed against a vertical wall of a large building. The base of the ladder is in oil, which makes the base slip and the tip of the ladder slide down the wall. The base of the ladder slips away from the wall at a constant rate of 4 feet per minute. Find the rate at which the tip of the ladder is sliding down the wall (in feet per minute) when the base of the ladder is 30 feet away from the wall.

Question 460: 3 Card Straight

Topic: probability **Difficulty**: easy

$$8$$

standard 52−card decks are combined together and shuffled well. Find the probability that 3 cards drawn uniformly at random from this deck form a three card straight. Note that we do not count a straight flush in this.

Question 461: Crossing Pairs

Topic: probability **Difficulty**: medium

Suppose 6 points are chosen uniformly at random along the perimeter of the circle. The 6 points are then divided into three pairs, and a chord is drawn between every pair of points. What is the probability none of the chords intersect?

Question 462: Minecraft Mine

Topic: probability **Difficulty**: medium

Suppose that Steve in Minecraft is trapped in a dungeon and has a diamond pickaxe. There are 3 paths that he can take: Path 1 leads back to his original position after 2 days. Path 2 leads back to his original position after 4 days. Path 3 leads him to the surface after 8 days. Steve selects doors completely at random and independently between selections. Let N be the number of times Steve must select a door before reaching safety and T be the total amount of time it takes for Steve to get out. Furthermore, let X_i represent the amount of time that Steve takes to explore the path chosen at the ith turn. Let $A_n = \mathbb{E}\left[T \mid N = n\right]$ and $B_n = \mathbb{E}\left[\sum_{k=1}^{n} X_i\right]$ Find $\lim_{n\to\infty} \frac{A_n}{B_n}$. If the limit doesn't exist, enter −1.

Question 463: Pick Your Urn Wisely

Topic: probability **Difficulty**: medium
You have 2 indistinguishable urns in front of you. One of the urns has 4 \$1 chips and 6 \$10 chips. The other urn has 3 \$1 chips and 7 \$10 chips. You reach into one urn at random and select a \$1 chip. You are given the opportunity to pick another chip at random either from same urn as the first chip or at random from the other urn. Your payout is the value of the second chip you select. Under optimal gameplay, what is your expected payout?

Question 464: Minimum Variance Portfolio

Topic: probability **Difficulty**: easy
Suppose you purchase a 1 share of stock A. A share of stock A has mean return 2 and return variation 4. You also want to purchase stock B. 1 share of B has mean return 3 and return variation 9. Additionally, the returns of A and B are correlated with coefficient $-1/2$. How many shares of stock B should you purchase so that the variance of your portfolio consisting of both A and B is minimum?

Question 465: Matching Socks I

Topic: brainteasers **Difficulty**: easy
Your drawer contains 2 red socks, 4 yellow socks, and 5 blue socks. In a rush, you randomly grab socks from the drawer. What is the minimum number of socks you need to grab to guarantee a pair of socks of the same color?

Question 466: Coin Runs

Topic: probability **Difficulty**: easy
You toss a fair coin 100 times and record the outcomes. How many runs will you have on average? A run is classified as the longest continuous flips of heads or tails.

Question 467: Company Purchase II

Topic: finance **Difficulty**: medium
QuantGuide Inc. generates \$100 of cash flow per year, and its cash flow is expected to grow at 5% per year until you sell. If you invested in another similar company, you could earn 10% per year. How much would you pay to buy QuantGuide Inc.?

Question 468: Die Multiple I

Topic: probability **Difficulty**: easy
A fair 6−sided die is rolled until the sum of all upfaces is even. Find the expected number of rolls performed.

Question 469: Positive 25

Topic: probability **Difficulty**: easy
How many ways can you sum three positive integers to get a total of 25?

Question 470: Swift Betting

Topic: probability **Difficulty**: easy
Juliana is playing "Fifteen" by Taylor Swift. The length of the song is 234 seconds. Juliana offers you the following bet: You may guess a value $0 < x < 1$. Juliana will then stop the song at a uniformly random point along its duration. If the point at which she stops the song is at most $234x$ seconds into the song, you must pay Juliana $2x$ monetary units. Otherwise, she pays you x monetary units. Find the value of x that maximizes your expected profit.

Question 471: Real Roots

Topic: probability **Difficulty**: medium
Suppose that $B \sim \text{Rayleigh}(1)$ i.e. B has the PDF $f(b) = be^{-b^2/2} I_{(0,\infty)}(b)$ and $C \sim \text{Exp}(2)$ are independent. Find the probability that the equation $x^2 + Bx + C = 0$ has no real solutions.

Question 472: Contracts and Pricing III

Topic: probability **Difficulty**: medium
After being caught in an embarrassing high-profile data fabrication scandal, Calambya University decides to stop sharing data with the national college ranking organization, effective in 2023. Rival school Mahogany University is overjoyed and offers each student a tuition refund worth X thousand dollars, where X = Calambya's 2023 Ranking − Mahogany's 2023 Ranking. One Mahogany student, Justin, offers to sell the following contract to his friend Kevin for 0.70 thousand dollars, before X is known: Justin will give Kevin the right but not the obligation to purchase Justin's tuition refund for only 0.5 thousand dollars. Suppose $X \sim \text{Exp}(\beta)$, where β is the scale. Compute β to the nearest tenth under the assumption that the contract is priced fairly.

Question 473: Cooked Steaks

Topic: brainteasers **Difficulty**: easy

$$4$$

steaks are being cooked on a grill. The grill can cook one side at a time, and both sides must be cooked to be considered edible. Each side of the steak takes 2 minutes to cook. The grill can hold up to 3 steaks at a time. What is the minimum number of minutes it takes to cook all 4 steaks?

Question 474: Sharing Resources

Topic: brainteasers **Difficulty**: medium
Alice, Bob, and Charlie live together in the wilderness and share everything equally. Today, Alice brought home five bushels of wheat, Bob brought home three bushels of wheat, and Charlie brought home none. To compensate, Charlie gives $8 instead for Alice and Bob to share. How much more money does Alice receive than Bob?

Question 475: Basic Dice Game I

Topic: probability **Difficulty**: easy
A casino offers you a game with a six-sided die where you are paid the value of the roll. The casino lets you roll the first time. If you are happy with your roll, you can cash out. Else, you can choose to re-roll and cash out on this second value. What is the fair value of this game?

Question 476: Shootout

Topic: brainteasers **Difficulty**: easy

$$n$$

pirates all stole some gold! However, all of them are greedy, so they each are loaded with 2 cannons. Each of the n pirates has 2 cannons that they point at 2 distinct pirates among the other $n-1$. What is the minimum value of n such that it is possible to have a situation where no two pirates are mutually pointing cannons at one another?

Question 477: Captive Marbles

Topic: probability **Difficulty**: easy
You are on death row presently. The judge offers you the following game: You

have 50 white balls, 50 black balls, and 2 empty urns in front of you. You can distribute all 100 balls between the 2 urns in any way you please. Then, you are blindfolded and must pick one of the urns to select a ball from. If you pick a white ball, you are free. However, you are executed on the spot if you draw a black ball. Assuming optimal strategy, what is your chance of survival?

Question 478: Random Triangle

Topic: probability **Difficulty**: hard
Three points are selected uniformly at random on the circumference of the unit circle and are labelled points A, B, and C. Find the expected perimeter of the triangle ABC. The answer is in the form $\frac{a}{\pi}$ for a rational number a. Find a.

Question 479: Odd Coefficients

Topic: brainteasers **Difficulty**: medium
Find the sum of all binomial coefficients $\binom{n}{k}$ with n and k being positive odd integers at most 1000. Your answer is in the form $\frac{a^b - c}{d}$, where a, b, c, and d are integers and b is maximal. Find $abcd$.

Question 480: Exact 5 I

Topic: probability **Difficulty**: medium
Abd continually rolls a fair $6-$sided die until he obtains his first 6. Compute the probability that Abd obtains the value 5 exactly four times before he stops rolling.

Question 481: Poker Hands IV

Topic: probability **Difficulty**: medium
A poker hand consists of five cards from a fair deck of 52 cards. What is the probability that you have a single pair (two cards of the same value and the other three cards of different unique values)?

Question 482: Hasty Horseman

Topic: brainteasers **Difficulty**: medium
A linear army of people that is 50 miles long walks 50 miles while a horseman gallops from the back of the line to the front and then gallops to the back

again. How far (in miles) has the horseman travelled? The answer is in the form $a(b+\sqrt{c})$ for integers a, b, and c with a maximal. Find abc.

Question 483: Balanced Beans III

Topic: brainteasers **Difficulty**: medium
There are 18 beans, 17 of which are identical and one that is heavier than the others. What is the minimum number of times a balance scale must be used to guarantee the determination of the abnormal bean?

Question 484: Coin Pair IV

Topic: probability **Difficulty**: medium
Four fair coins appear in front of you. You flip all four at once and observe the outcomes of the coins. After seeing the outcomes, you may flip any pair of tails again. You may not flip a single coin without flipping another. You can iterate this process as many times as there are at least two tails to flip. Find the expected number of coin flips needed until you are unable to better your position.

Question 485: Second Moment

Topic: probability **Difficulty**: easy
Given that $\mathrm{Var}(X) = 25$ and $\mathbb{E}[X] = 3$, find $\mathbb{E}[X^2]$.

Question 486: Binary Call

Topic: finance **Difficulty**: easy
A binary put is a derivative that pays 1 if $S_T < K$ at expiry and 0 otherwise. Similarly, a binary call is a derivative that pays 1 if $S_T \geq K$ at expiry and 0 otherwise. The binary put has an initial price of $P_0 = 0.36$. What is the time-0 price of the binary call? Assume a discount factor of 0.9 for a bond paying 1 at expiry.

Question 487: Overlapping Data

Topic: statistics **Difficulty**: medium
Suppose that $X_1, \ldots, X_{150}$ are all IID random variables with variance 1. Define $U = 3(X_1 + \ldots X_{50}) + X_{51} + \cdots + X_{100}$ and $V = X_{51} + \cdots + X_{100} + 3(X_{101} + \ldots X_{150})$. Find $\mathrm{Corr}(U, V)$.

Question 488: Limiting Product

Topic: probability **Difficulty**: medium
Let $X_1, X_2, \ldots$ be a sequence of IID Unif$(0,1)$ random variables. Let $Y_n = (X_1 X_2 \ldots X_n)^{\frac{1}{n}}$. It can be shown easily that this limit exists almost surely. Compute $\lim_{n\to\infty} Y_n$. The answer is in the form e^a for some a. Find a.

Question 489: Generational Wealth II

Topic: probability **Difficulty**: medium
Suppose that you generate a uniformly random number in the interval $(0,1)$. You can generate additional random numbers as many times as you want for a fee of \$0.05 per generation. The number of additional generations must be made prior to your first number that you generate. Your payout is the maximum of all the numbers you generate. Assuming optimal strategy, what is your expected payout of this game?

Question 490: Basic Die Game VII

Topic: probability **Difficulty**: easy
A player rolls a standard fair $6-$sided die. If the player rolls a 6, the game ends and the player receives no payout. Otherwise, the player may quit and receive $\$k$, where k is the upface of the previous roll. Otherwise, the player may roll again under the same rules. Assuming optimal play from the player, what is their expected payout?

Question 491: Needy Friends

Topic: brainteasers **Difficulty**: hard
A rich man sets aside a fixed weekly allowance to distribute to his friends in need. He remarks that if there are 5 fewer people that need money next week, then each person will receive \$2 more than this week. Unfortunately, there were 4 more people in the following week that needed money compared to this week and everyone received \$1 less. How much money did each person receive in the present week (the week that had 4 less people apply)?

Question 492: Deja Vu

Topic: probability **Difficulty**: medium
Find the expected number of distinct faces observed when rolling a fair $6-$sided die before rolling a face previously observed.

Question 493: Game Show III

Topic: probability **Difficulty**: medium
You're on a game show and are given the choice of 10 doors to choose from. Behind one door is \$1000 and behind the other nine are goats. You pick one of the doors at random and announce your choice to the game show host. The game show host, knowing which prize is behind each door, opens three doors that you did not choose and shows goats behind all of them after hearing your initial choice. He offers you the opportunity to either keep your original door or switch to the other closed door. What is the expected value of this game in dollars? Assume goats are worth \$0.

Question 494: Connected Origin

Topic: probability **Difficulty**: medium
Suppose that in the integer lattice $\mathbb{Z}^2$, each pair of vertices of Euclidean distance 1 has a path connecting them with probability p. Using the bound provided by sub-additivity, the interval of values of p so that the probability that the origin is in an infinite connected component is 0 is in the form $(0, a)$. Find a.

Question 495: Integral Variance V

Topic: pure math **Difficulty**: easy
Let W_t be a standard Brownian Motion. Compute the variance of $X_t = \int_0^t s dW_s$ as a function of t. The answer will be in the form kt^3 for a constant t. Find k.

Question 496: Sum Covariance

Topic: probability **Difficulty**: easy
Let X, Y, and Z be uncorrelated random variables with respective variances $64, 36$, and 225. Let $U = X + Y$ and $V = X + Z$. Find $\text{Corr}(U, V)$.

Question 497: Coin Priors

Topic: probability **Difficulty**: medium
We have a coin with unknown probability p of heads. We prescribe a prior distribution to p of $P \sim \text{Beta}(10, 10)$. If $X_1, \dots, X_{80}$ are indicator random variables of the event that the ith flip is a heads, $1 \leq i \leq 80$, compute $\mathbb{E}\left[P \,\middle|\, \sum_{i=1}^{80} X_i = 50\right]$.

Question 498: Doubly Blue

Topic: probability **Difficulty**: easy
Three bowls are presented to you. One has two blue balls, one has two red balls, and the last has one red ball and one blue ball. You select one bowl uniformly at random and note that you drew a blue ball from it. You then draw from the same bowl again after replacing the ball you selected. Find the probability that you would draw a blue ball on this draw.

Question 499: Builders

Topic: brainteasers **Difficulty**: medium
Alice and Bob can each build a pipe on their own in 120 and 100 minutes, respectively. Instead, they work together on building a pipe. After 40 minutes of building, Charlie assists them in building the pipe. Together, they finish building the pipe 10 minutes after Charlie joins. In how many minutes could Charlie build the pipe on his own?

Question 500: Valid Expressions

Topic: probability **Difficulty**: easy
How many valid arithmetic expressions using only $*$ and $+$ are there that include the values 123456789 in order? Some examples include $1+2+3456+7*89$, $123+456*78+9$, and 123456789.

Question 501: Hidden Code

Topic: probability **Difficulty**: medium
A random word generator generates strings of 5 letters from the set $\{a, b, c, d, e\}$ with repetition allowed. However, you also know that no character is used more than twice and that no two consecutive characters are the same. How many strings are there?

Question 502: First Pair

Topic: probability **Difficulty**: easy
Find the probability that the top two cards of a well-shuffled standard deck form a pair (two cards of the same rank).

Question 503: Single Double Sum

Topic: probability **Difficulty**: medium
A 6−sided die with values $1-6$ is weighed such that the probability of rolling the values $2, 3, 4, 5$, and 6 are the same whether we roll this die once or if we roll the die twice and take the sum of the two outcomes. Let p_1 be the probability that the value 1 is obtained on this die. p_1 is the solitary positive solution to the polynomial equation

$$\sum_{k=1}^{6} c_k p_1^k = 1$$

for some real constants $c_1, \ldots, c_6$. Find $\sum_{k=1}^{6} c_k$.

Question 504: Taxman

Topic: brainteasers **Difficulty**: medium
We start with the numbers $1-10$ on a board. You pick a number. The taxman then takes all the numbers that divide it evenly. We do this until there are no numbers left. However, you can only pick numbers that have at least one factor that hasn't been taken yet. If there are any unclaimed numbers at the end, the taxman takes them. Your score is the total of the numbers that you take. Following optimal strategy, what is your max possible score?

Question 505: Digit Halving

Topic: brainteasers **Difficulty**: easy
Which two digit integer has the unique property that the product of its digits is equal to half of the integer?

Question 506: Throwing Darts II

Topic: probability **Difficulty**: easy
You throw 3 darts that land uniformly randomly on a circular dartboard composed of three concentric circles with radii 1ft, 2ft, and 3ft. Assuming the dart lands on the dartboard, what is the probability that at least one of the darts land in the inner ring (within the 1ft circle)?

Question 507: Variance Product

Topic: statistics **Difficulty**: easy
Suppose that X and Y are independent random variables with $\mathbb{E}[X] = 4, \mathbb{E}[Y] =$

$-3, \mathbb{E}[X^2] = 20$, and $\mathbb{E}[Y^2] = 45$. Compute $\mathrm{Var}(XY)$.

Question 508: Turducken Hunt

Topic: probability **Difficulty**: hard
Mordecai and Rigby are hunting for a Turducken. They both start at $x = 0$ and see a Turducken 1 meter away. Mordecai and Rigby start walking at the same rate towards $x = 1$. They each have only one bullet in their guns and they can take it out and shoot at any point along the walk. Given that they are at position x, Mordecai has probability $m(x) = x$ of hitting the Turducken with his shot. Rigby has probability $r(x) = x^2$ of hitting the Turducken. If they shoot at the same time and both miss or hit the Turducken, then they just start again with a new Turducken. If exactly one of them hits the Turducken, then that person gets to keep it for Thanksgiving dinner. They don't neccesarily need to shoot at the same time. Assuming both Mordecai and Rigby apply an optimal strategy, what is Mordecai's chance of keeping the Turducken? The answer is in the form $\dfrac{\sqrt{a} - b}{c}$ for integers $a, b, c > 0$ that are pairwise relatively prime. Find $a + b + c$. Note that Mordecai and Rigby both know $m(x)$ and $r(x)$.

Question 509: Temperature Conversion

Topic: statistics **Difficulty**: easy
In June, New York City's daily maximum temperature has a mean of 80°F and standard deviation of 5°F. What is the absolute difference in mean and variance in °C? Round your answer to the nearest hundredth.

Question 510: Absolutely Brownian

Topic: pure math **Difficulty**: medium
Let W_t be a standard Brownian Motion and let $T_4 = \inf\{t > 0 : |W_t| > 4\}$. Find $\mathrm{Var}(T_4)$.

Question 511: Incomplete Deck

Topic: brainteasers **Difficulty**: easy
A standard deck of cards (52 cards total) has a few cards removed from it. If you divide the the number of cards in the deck by 3, you get a remainder of 1. If you divide the deck by 5, you get a remainder of 3. Finally, if you divide the deck by 4, you get a remainder of 3. How many cards are in the deck?

Question 512: Birth Paradox

Topic: probability **Difficulty**: medium
A family has 2 children, of which we know one is a boy born on a Friday. Assuming that each birthday and gender is equally likely, as well as independent of one another, find the probability that the family has 2 boys.

Question 513: Probability Discussion

Topic: probability **Difficulty**: medium
Gabe and a student have decided to meet up to discuss probability. The student is very prompt and will show up at a uniformly random time between 4 and 5 PM. Gabe is a tad late, so he will show up at a uniformly random time between $4:30$ and 5 PM. For whichever person gets there first, they will wait up to 10 minutes, and if the other person has not shown up, they will leave. Find the probability that the meeting occurs.

Question 514: Minimal Flipping

Topic: probability **Difficulty**: easy

$$3$$

people flip a weighted coin with probability of heads $p = 0.25$ until they obtain their first tails. Find the probability that none of the three people flip more than twice.

Question 515: Balanced Beans I

Topic: brainteasers **Difficulty**: easy
There are 8 beans; one is slightly heaver than the others. What is the minimum number of times a balance scale must be used to determine the heavier bean?

Question 516: Dinner at Dorsia

Topic: probability **Difficulty**: medium
Two quants are planning for dinner at Dorsia. Assume that each independently arrives at some uniformly random time between 8:00pm and 9:00pm, for which they stay for exactly 10 minutes before leaving. What is the probability that they will meet each other and stay for dinner?

Question 517: Unknown Starter

Topic: probability **Difficulty**: hard
You have 3 cards, each labeled n, $n+1$, $n+2$, and you don't know n. All cards start face down. You flip one card and observe the value. If you say "stay", your payout is equal to that card's value. If you don't "stay", then you flip another card. Again, choose to "stay" (and receive payout equal to the 2nd card's value) or flip the final card and receive payout equal to the final card's value. Design the optimal strategy. The expected out payout of this strategy is $n+c$ for some constant c. Find c.

Question 518: Family Ties

Topic: brainteasers **Difficulty**: easy
A grandmother has two grandchildren and their ages so conveniently line up that the age of each grandchild is one of the digits of the grandmother's age. Additionally, the sum of all three ages is 98. What is the age of the grandmother?

Question 519: Plane Partition

Topic: probability **Difficulty**: medium
Find the maximum number of disjoint regions the plane can be divided into by 10 non-parallel lines. For example, one line splits the plane into 2 regions.

Question 520: Conditional Expectation I

Topic: statistics **Difficulty**: medium
Consider the following joint pdf:

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} c(1-x_2) & 0 \le x_1 \le x_2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

where c is a constant such that f_{X_1,X_2} is a valid joint pdf. Compute $\mathbb{E}[X_1]$.

Question 521: Racecar Speed

Topic: brainteasers **Difficulty**: easy
A racecar is going around a racetrack three times. The first lap was at a speed of 100 mph, the second lap was at 150 mph, and the final lap was at a speed of 200 mph. What was the average speed (in mph) of the racecar throughout all three laps?

Question 522: Ordering at Chipotle

Topic: brainteasers **Difficulty**: easy
There are 10 vegetarian toppings at Chipotle to choose from. How many distinct vegetarian bowl combinations can you make from these toppings (a bowl with no toppings does not count)?

Question 523: Likely Targets I

Topic: probability **Difficulty**: hard
Two linear targets, say A and B, of radius $\varepsilon << 1$ are placed on an infinitely long line. The targets are centered at $x_A = -1$ and $x_B = 3$. In other words, target A covers the interval $[1 - \varepsilon, 1 + \varepsilon]$, and similarly with target B. You have one dart to shoot at the line. Your goal is to maximize your probability of hitting one of the targets. You can choose where to center your throw on the line. If you select to center your dart at μ, the actual position your dart lands at is $X \sim N(\mu, 4)$. Find the value of μ that maximizes your chances of hitting a target. If necessary, round your answer to the nearest hundredth.

Question 524: Bowl of Cherries IV

Topic: probability **Difficulty**: easy
There are four red cherries and five purple cherries in bowl A. There are six red cherries and three purple cherries in bowl B. Jenny randomly transfers a cherry from bowl A into bowl B. Then, she randomly picks a cherry from bowl B to eat. What is the probability that she eats a red cherry?

Question 525: Particle Reach VI

Topic: probability **Difficulty**: medium
Consider a particle that performs a random walk on the integers starting at position 0. At each step, the particle moves from position i to position $i + 1$ with probability p, while the probability it moves from i to $i - 1$ is $1 - p$. If $p = 2/3$, find the expected number of steps until the particle reaches 1. If the answer is infinite, answer -1.

Question 526: Carded Pair

Topic: probability **Difficulty**: hard
Cards are dealt one-by-one from a standard deck of 52 cards without replacement. The card dealing ends when either 2 kings appear OR at least 1 king and 1 ace appear, whichever comes first. Find the expected number of cards that are dealt in the game.

Question 527: Deriving Put-Call Parity II

Topic: finance **Difficulty**: medium
You have access to the underlying S, which has an initial price $S_0 = 8$, bonds that pay 1 at time-T, where $T = 1$. The interest rates are 0.02, continuously compounded. Finally, you have access to three different put options of varying strikes. The puts are given in the format of (Strike K, Price C_0)

$$(5, 0.4)$$
$$(10, 3.2)$$
$$(15, 5.6)$$

Find the time-0 price of a call option with strike $K = 10$. Round to two decimal points.

Question 528: Probability of Unfair Coin I

Topic: probability **Difficulty**: easy
Among 1000 coins, 999 are fair and 1 has heads on both sides. You randomly choose a coin and flip it 10 times. Miraculously, all 10 flips turns up heads. What is the probability that you chose the unfair coin?

Question 529: Dollar Draw

Topic: probability **Difficulty**: easy
A bin consists of 4 \$10 bills, 3 \$20 bills, and 1 \$100 bill. You select 7 bills out of this bin uniformly at random without replacement. Find the expected value of the sum of all the bills you select.

Question 530: Ace Distribution

Topic: probability **Difficulty**: medium
Assume the dealer holds a fair deck of 52 cards. She shuffles the deck and gives each of you and your three friends at the table 13 cards. What is the probability that each of you receive an ace?

Question 531: Green Light

Topic: probability **Difficulty**: easy
There are 4 green balls and 11 red balls in a bin. We draw out balls without replacement. Find the probability that the 4th green ball is picked out on the 9th draw.

Question 532: Team Winners

Topic: probability **Difficulty**: easy
A tournament consistent of 32 teams plays in a round-robin style tournament. This means that every single team plays every other team exactly one time. Suppose that every team is equally skilled so that every team is equally likely to beat any other team in a given match. Each team keeps track of how many wins they have. Find the probability that every team has won a distinct number of games at the end of the tournament. The answer will be in the form $\frac{a!}{2^b}$ for integers a and b, where b is maximal. Find $a + b$.

Question 533: Partial Derivatives

Topic: probability **Difficulty**: medium
Let $f : \mathbb{R}^{10} \to \mathbb{R}$, where $f = f(x_1, \ldots, x_{10})$ is a smooth function (i.e. partial derivatives exist of all orders). How many 5th order partial derivatives does f have? For example, $\frac{\partial f}{\partial x_1^5}$ and $\frac{\partial f}{\partial x_1^2 \partial x_2^2 \partial x_6}$ are both 5th order partial derivatives.

Question 534: Spherical Coodinates

Topic: probability **Difficulty**: hard
A random point $(X_1, X_2, \ldots, X_{10})$ is uniformly at random selected from the $10-$ball that has radius 12 centered at the origin. Find the variance of X_1, the first coordinate.

Question 535: English and History Majors

Topic: probability **Difficulty**: easy
At a local university, 30% of students major in English, 45% major in History, and 15% double major in both. What proportion of students major in neither English nor History?

Question 536: Limiting Random Variable I

Topic: probability **Difficulty**: easy
Let $X_1, X_2, \ldots$ be a sequence of IID random variables with mean 5 and variance 20. It is also known the third and fourth moments of X_1 are finite. Let

$$Y_n = \frac{X_1^2 + X_2^2 X_3 X_4 + X_5^2 + X_6^2 X_7 X_8 + \cdots + X_{4n-3}^2 + X_{4n-2}^2 X_{4n-1} X_{4n}}{n}$$

Find $\lim_{n\to\infty} \mathbb{E}[Y_n]$. If this limit does not exist, enter -1.

Question 537: Confident Double Heads

Topic: probability **Difficulty**: easy
A bag has 19 fair coins and one double-headed coin. We pick a coin uniformly at random from the bag. What is the minimum number of times this coin would need to appear heads in a row to be at least 95% certain that the coin we have is the double-headed coin?

Question 538: Sequence Probability

Topic: probability **Difficulty**: easy
What is the probability that you flip a fair coin 4 times and you get the sequence $HTHT$ in that order?

Question 539: Conditional 8 Sum

Topic: probability **Difficulty**: easy
Two fair $6-$sided dice were rolled and the sum of the upfaces is 8. Find the probability that one of the dice showed 6.

Question 540: Doubly 5 II

Topic: probability **Difficulty**: medium
Jenny has a fair $6-$sided die with numbers $1-6$ on the sides. Jenny continually rolls the die and keeps track of the outcomes in the order they appear. Jenny rolls until she sees two 5s (not necessarily consecutive) OR both 4 and 6. Find the probability Jenny stops rolling due to seeing two 5s.

Question 541: High or Die

Topic: probability **Difficulty**: easy
Francisco rolls a fair die and records the value he rolls. Afterwards, he continues rolling the die until he obtains a value at least as large as the first roll. Let N be the number of rolls after the first he performs. Find $\mathbb{E}[N]$.

Question 542: High or Die

Topic: probability **Difficulty**: easy
Francisco rolls a fair die and records the value he rolls. Afterwards, he continues rolling the die until he obtains a value at least as large as the first roll. Let N be the number of rolls after the first he performs. Find $\mathbb{E}[N]$.

Question 543: Squared Matrix

Topic: pure math **Difficulty**: easy

Suppose that $A = \begin{bmatrix} 1 & x & -1 \\ 2 & -2 & y \\ 0 & 3 & 0 \end{bmatrix}$ and $A^2 = \begin{bmatrix} 9 & -7 & 11 \\ -2 & 21 & -8 \\ 6 & -6 & 9 \end{bmatrix}$. Find xy.

Question 544: Leaked IP

Topic: probability **Difficulty**: easy

Someone leaked IP from a quant firm! A witness claims that the IP was leaked by a trader. Suppose that witnesses are correct 2/3 of the time. At this particular firm, 2/3 of the employees are traders, while the other 1/3 are researchers. Taking into account the witness statement, what is the probability that the person who leaked IP was a trader?

Question 545: Jane's Children

Topic: probability **Difficulty**: easy

There is a conference for parents with at least one daughter. Jane, a mother with two children, is invited. What is the probability that Jane has two daughters?

Question 546: Tricky Bob I

Topic: probability **Difficulty**: medium

Bob proposes a game where Alice and Bob are each given a coin. Each is allowed decide the probability of heads for their coin. Afterwards, they both flip their coins. If it comes up HH, Alice gives Bob \$6. If it comes up TT, Alice gives Bob \$4. Otherwise, Bob gives Alice \$5. However, Alice must tell Bob the success probability p_1 they have chosen before Bob decides his p_2. The interval of values for p_1 such that regardless of what Bob selects, Bob has non-positive expected value on the game is in the form $[a, b]$, where a and b are rational numbers. Find $b - a$.

Question 547: Equicorrelated

Topic: probability **Difficulty**: medium

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random variables $X_1, \ldots, X_7$ are all identically distributed with mean 0 and variance 1. However, they also all have the same pairwise correlation, say ρ. Find the minimum possible value of ρ.

Question 548: Stamp Sum

Topic: brainteasers **Difficulty**: easy
You have infinitely many stamps of values 5 and 21. Find the largest value that is unable to be expressed as a combination of these stamps.

Question 549: Poker Hands V

Topic: probability **Difficulty**: medium
A poker hand consists of five cards from a fair deck of 52 cards. What is the probability that you have three-of-a-kind (three cards of the same rank and two cards of two other ranks)?

Question 550: Binary String

Topic: probability **Difficulty**: easy
A computer is interpreting an $8-$bit binary string, which consists of 8 characters that are either 1 or 0. How many such strings begin with 1 or end with the two characters 00?

Question 551: Kiddie Pool

Topic: brainteasers **Difficulty**: easy
You are trying to fill up an inflatable kiddie pool. One hose can fill it up in 20 minutes. Another hose can fill it up in 10 minutes. If the pool is full, it takes 15 minutes to drain out the water after the drain is opened. How long does it take in minutes to fill the pool with both hoses and the drain being open?

Question 552: Good Grid II

Topic: probability **Difficulty**: hard
Let a, b, c, and d be uniformly at random selected from $S_n = [-n, n] \cap \mathbb{Z}$, the integers that are at most n. Define the two closed intervals $I_1 = [\min\{a, b\}, \max\{a, b\}]$ and $I_2 = [\min\{c, d\}, \max\{c, d\}]$. Let $p(n)$ be the probability (as a function of n) that I_1 is completely contained inside I_2. This means that

$$\min\{c, d\} < \min\{a, b\} \leq \max\{a, b\} < \max\{c, d\}$$

Define $p = \lim_{n\to\infty} p(n)$. Compute $\dfrac{p - p(10)}{p}$.

Question 553: Mathematical Birthday

Topic: brainteasers **Difficulty**: medium
A man who lived in the 19th and 20th century once said that he was $a^2 + b^2$ years old in the year $a^4 + b^4$, $2m$ years old in the year $2m^2$, and $3n$ years old in the year $3n^4$. All variables here are integers. Find $abmn$.

Question 554: Swapping X and Y

Topic: statistics **Difficulty**: easy
Consider a simple univariate linear regression. We have the following X and Y values.

$$(5, 5), (4, 0), (6, 10)$$

We regress Y onto X and obtain some β. Now, we regress X onto Y. Without doing any calculation, how does β change? Enter 1 for increase, 0 for stay the same, and -1 for decrease.

Question 555: Geometric PGF

Topic: probability **Difficulty**: easy
Instead of the Moment Generating Function (MGF), for non-negative integer-valued discrete random variables X, we can define the *Probability Generating Function* (PGF), which is given by $p(z) = \mathbb{E}[z^X]$ for $|z| \leq 1$. Find the PGF of $X \sim \text{Geom}(p)$. Evaluate this function for $z = \frac{1}{3}$ and $p = \frac{2}{3}$.

Question 556: Fibonacci Sum

Topic: brainteasers **Difficulty**: medium
Evaluate $\displaystyle\sum_{k=0}^{\infty} \frac{F_k}{10^{k+1}}$, where F_k is the kth Fibonacci number defined by $F_0 = 0$, $F_1 = 1$, and $F_k = F_{k-1} + F_{k-2}$ for $k \geq 2$.

Question 557: Integral Variance III

Topic: pure math **Difficulty**: easy
Let W_t be a standard Brownian Motion. Compute $\text{Var}\left(\displaystyle\int_0^t W_s^2 dW_s\right)$. The answer will be in the form kt^3 for a constant k. Find k.

Question 558: 9 Appearance

Topic: brainteasers **Difficulty**: easy
How many times does the digit 9 occur when counting 1 to 1000? Note that the number 919, for example, would count as 2 9s.

Question 559: Shiny Pennies

Topic: probability **Difficulty**: easy
There are 3 shiny and 4 dull pennies in a box. Pennies are drawn from the box without replacement. Compute the probability that it takes more than 4 draws for the third shiny penny to appear.

Question 560: Three Riflemen

Topic: probability **Difficulty**: easy
Three riflemen A, B, and C take turns shooting at a target. A shoots first, B second, and C third, after which the cycle repeats again with A, until one of the riflemen hits the target. Each shot hits the target with probability 1/2, independent of other shots. Find the probability A wins.

Question 561: Call Option Change

Topic: finance **Difficulty**: medium
You bought an at-the-money call option with initial price $C_0 = 2.3$ for an underlying S, which has initial price $S_0 = 13$. The gamma of this call is $\Gamma = 0.03$. If the underlying S moves up by \$2, give an approximation for the new value of the call option.

Question 562: 9 For 1

Topic: brainteasers **Difficulty**: medium
An $8-$digit number is going to be created by a two-step process. First, select a uniformly random integer $k \in \{0, 1, \ldots, 8\}$. Afterwards, we select one $8-$digit number uniformly at random from the collection of $8-$digit numbers with exactly k 1s and the other $8 - k$ digits are 9s. Call this selected number X. Find $\mathbb{E}[X]$.

Question 563: Lognormal I

Topic: probability **Difficulty**: easy
Suppose that $\ln(X) \sim N(0, 1)$. Find $\ln(\mathbb{E}[X])$.

Question 564: Coin-Die Oddity

Topic: probability **Difficulty**: easy
Abdelrahman flips 2 fair coins. For each head that results, Abdelrahman rolls one fair $6-$sided die. Find the probability that the sum of all the upfaces is odd.

Question 565: Dollar Break

Topic: brainteasers **Difficulty**: easy
What is the maximum amount of money in coins you can have in your pocket and not be able to make change for a dollar? The coin denominations in the USA are pennies (0.01 dollars), nickels (0.05 dollars), dimes (0.10 dollars), and quarters (0.25 dollars).

Question 566: Multiple Divisors I

Topic: probability **Difficulty**: easy
Find the probability that a randomly selected integer from the set of positive divisors of 10^{99} is divisible by 10^{80}.

Question 567: Infinite Maturity

Topic: finance **Difficulty**: easy
What's the value of a vanilla European call option of infinite maturity, and a given strike, volatility, interest rate, and spot price?

Question 568: The Big Three

Topic: probability **Difficulty**: easy
Suppose we roll 5 standard fair dice and sum the upfaces of the largest 3 values showing. Find the probability that the sum is 18.

Question 569: 3 Ace

Topic: probability **Difficulty**: easy
You are dealt 5 cards from a standard deck. You are told at least 3 of the cards are of the same rank. Find the probability your hand contains all 4 cards of some rank.

Question 570: 20-30 Die Split II

Topic: probability **Difficulty**: medium
Alice and Bob have fair 30−sided and 20−sided dice, respectively. The goal is to obtain the largest possible value. Alice rolls her die one time. However, Bob can roll his die twice and keep the maximum of the two values. In the event of a tie, Bob is the winner. Find the probability Alice is the winner.

Question 571: Measuring Water

Topic: brainteasers **Difficulty**: medium
The maid was given the task of fetching exactly 2 pints of water from the brook using two vessels, one measuring 7 pints and the other measuring 11 pints. To accomplish this, what is the minimum number of transactions required? A "transaction" can involve filling a vessel, emptying it, or pouring water from one vessel to another.

Question 572: Rook on a Chessboard

Topic: brainteasers **Difficulty**: medium
Consider a standard chessboard where there is a rook in the bottom right corner. Here, the rook can move in straight lines upwards or to the left. You and your friend are playing a game where you take turns moving the rook. The winner of the game will be the one that moves the rook to the top-left square. If you are to guarantee a win, should you go first or second? Enter 1 for first and 2 for second.

Question 573: Deviating Sums

Topic: probability **Difficulty**: easy
Two identical sets of cards are in front of you. Each set has 5 cards labeled $0-4$. You pick a card from each set and add them together. What is the standard deviation of sum of the cards chosen?

Question 574: Tennis Deuces I

Topic: probability **Difficulty**: easy
How many ways are there to get a score of 40-40 (deuce) in tennis? Scores in tennis follow $0, 15, 30, 40$.

Question 575: Proper Statement

Topic: probability **Difficulty**: medium
A proper statement of length $2n$ is a string of characters consisting of $(,), \{$ and $\}$, where each of parentheses and curly braces are individually properly opened and closed. For example, $()\{\}$ and $\{()\}$ are proper statements, while $\{(\})$ and $((\})$ are not proper statements. Let P_n be the number of proper statements of length $2n$. Find P_5.

Question 576: Lead Count

Topic: probability **Difficulty**: medium
Suppose you flip a fair coin 10 times. We say a certain outcome (heads or tails) is leading after n flips if there are strictly larger than $\frac{n}{2}$ flips of that outcome within the first n flips. Find the probability that the outcome of the first flip of the coin is leading after 10 flips.

Question 577: 18 Sides

Topic: brainteasers **Difficulty**: easy
There is a fair 18-sided die where each side is either a 6 or a 3. The expected value of a roll is 4. How many 6s are on the 18-sided die?

Question 578: Defining Regression

Topic: statistics **Difficulty**: easy
Let X and Y be random variables such that $E[X] = 6, V[X] = 2, E[Y] = 8, V[Y] = 10$, and $Cov[X, Y] = 0$. Suppose you do a simple linear regression of X on Y that results in a model of $Y = aX + b$. What is $a + b$?

Question 579: Binomial Contract Pricing I

Topic: finance **Difficulty**: medium
We want to price a derivatives contract on the underlying S, which has payoff S_T^2 at expiry, where $T = 2$. At each step, the underlying will either double or half. Assuming the initial price of the underlying is $S_0 = 5$, what is the time-0 price of the contract?

Question 580: ATM Implied Volatility

Topic: finance **Difficulty**: medium
You have an at-the-money European call option with an expiration of 3 months

and a price $C_0 = 2.3$ of the underlying S, with an initial price of $S_0 = 12$. What is an approximation of the implied volatility (annualized)? Round to 3 decimal points.

Question 581: Perfect Correlation II

Topic: probability **Difficulty**: medium
Suppose that X and Y are perfectly correlated i.e. $\rho(X,Y) = 1$ with $\sigma_Y < \sigma_X$. Find the correlation of $X+Y$ and $X-Y$.

Question 582: Bus Wait I

Topic: probability **Difficulty**: easy
You are waiting for a bus to get to QuantGuide on time! The bus runs on a fixed schedule of appearing every 10 minutes. If you arrive at a uniformly random time throughout the day, what is the expected time until the next bus appears (in minutes)?

Question 583: 100 Factorial Digits

Topic: brainteasers **Difficulty**: medium
How many digits are in 100!?

Question 584: Missing Product

Topic: brainteasers **Difficulty**: medium
You have 4 positive real numbers a, b, c, and d. When you multiply each pair of the numbers together, you get some element of the 6 element set $\{9, 10, 12, 27, 30, x\}$. Find x.

Question 585: Modified RNG

Topic: probability **Difficulty**: medium
Jimmy picks a number uniformly at random from $(0,1)$. If Jimmy chooses x, then Jon picks a number from $(x,1)$ uniformly at random. If Y represents the number Jon selects, in simplest form, find $\dfrac{\mathbb{E}[Y]}{\mathrm{Var}(Y)}$.

Question 586: Hatching Eggs II

Topic: probability **Difficulty**: easy
Amy has a chicken. The number of eggs laid by the chicken in a month follows

a Poisson process with $\lambda = 6$. The probability that an egg hatches is 0.3. Eggs hatch independently of one another. Compute the variance of the number of hatched eggs.

Question 587: Egg Drop I

Topic: brainteasers **Difficulty**: medium
You are holding two identical eggs in a 100-story building. If an egg is dropped at an elevation under story X, then the egg will survive; else, the egg breaks. What is the minimum number of drops required to determine X in the worst-case scenario?

Question 588: Casino Combo

Topic: probability **Difficulty**: easy
A fair $6-$sided die, a standard roulette wheel (has 38 values with two zeroes and $1 - 36$), and a standard deck of cards are in a casino. Letting Aces count as ones, what is the probability that when the roulette wheel is spun, the die is rolled, and a card is dealt from the deck, all of them show the same value?

Question 589: Price an Option II

Topic: finance **Difficulty**: medium
You have access to the European call options at the following strikes and T_0 prices, the underlying S which has initial price $S_0 = 7$, and a bond that has initial price 0.9, paying 1 at time T. The calls are given in the format of $(\text{Strike } K, \text{Price } C_0)$

$$(5, 4.2)$$
$$(10, 1.4)$$
$$(15, 0.7)$$
$$(20, 0.1)$$

Find the time-0 price of a contract that pays $\max(3S_T, 15)$ at time T.

Question 590: Fair Load

Topic: probability **Difficulty**: easy
Emma has two $6-$sided dice with the values $1 - 6$ on the sides. One of the dice is fair, while the other is loaded such that each side appears in proportion to the value on the side. Emma rolls both dice. Find the probability that the sum of the outcomes will odd.

Question 591: Exercise Vega

Topic: finance **Difficulty**: easy
You have 3 at-the-money options on some different underlyings with strikes $K = 10$, $K = 20$, $K = 30$, all with expiry T. Assuming Black-Scholes dynamics, which option (strike) has the largest vega?

Question 592: OLS Review III

Topic: statistics **Difficulty**: easy
Given the following data, (x, y): $(-2, 0), (-1, 0), (0, 1), (1, 1), (2, 3)$, Bob does a simple linear regression modeling $Y = \beta_0 + \beta_1 x + \epsilon$, where $\mathbb{E}[\epsilon] = 0$ and $\text{Var}[\epsilon] = \sigma^2$. Estimate σ^2.

Question 593: Friendly Competition

Topic: probability **Difficulty**: easy
Players 1 and 2 respectively start with \$1 and \$2. They flip a coin with probability $\dfrac{2}{3}$ of heads on each toss. Player 1 receives \$1 from Player 2 if the coin shows heads. Otherwise, Player 2 receives \$1 from Player 1. Find the probability Player 2 goes broke first i.e. Player 1 wins.

Question 594: Weird Die

Topic: probability **Difficulty**: easy
A 6−sided die shows the values $1-6$ with respective probabilities $1/6, 1/5 - x, 1/10 + 2x, 1/5 - x, 1/6$, and $1/6$. Find the value of x that maximizes the probability of obtaining a sum of 7 when you roll this die twice.

Question 595: Digit Difference

Topic: brainteasers **Difficulty**: medium
Find the average of all ten-digit base-ten positive integers $\overline{d_9 d_8 \dots d_1 d_0}$ that satisfy the property $|d_i - i| \leq 1$ for all $i \in \{0, 1, \dots, 9\}$.

Question 596: Safe Cracking

Topic: brainteasers **Difficulty**: hard
An electronic safe has a three digit passcode. You are given three constraints regarding the code. Firstly, the code is not an odd number. Secondly, the code does not contain the number six. Lastly, one of the digits appears more than once. How many possible three digit entries satisfy these three requirements?

Question 597: Relatively Prime Coins

Topic: statistics **Difficulty**: medium

You flip a fair coin n times. Let X denote the number of heads, and let Y denote the number of tails. $\mathbb{E}[XY]$ can be expressed in the form

$$\frac{an^2 + bn + c}{d}$$

for some integers a, b, c, d, where a, b, c are each relatively prime with d. What is $a + b + c + d$?

Question 598: Oil Profits I

Topic: probability **Difficulty**: easy

A drilling company must decide whether or not to drill oil at some site. The company can drill the site for a price of $\$350,000$ paid up front. Afterwards, the company will learn if there is actually oil at this site. If there is oil, the company will generate $\$1,400,000$ in revenue. If there is no oil, there will be no future profits. Let p be the probability there is oil at this site. What is the minimum value of p such that the company should drill if they act rationally?

Question 599: Double Data Trouble III

Topic: statistics **Difficulty**: easy

Suppose that you run linear regression on some dataset and obtain the coefficients $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$. Let t_0 be the $t-$statistic obtained when testing the null hypothesis $H_0 : \beta_1 = 0$ against $H_A : \beta_1 \neq 0$. Suppose that we duplicate our data and then run linear regression again on this new data set. Let $\hat{y}_i = \hat{\beta}'_0 + \hat{\beta}'_1 x_i$ be the new coefficients that are obtained. If t_1 is the corresponding $t-$statistic obtained when testing the null hypothesis $H_1 : \beta'_1 = 0$ against $H_A : \beta'_1 \neq 0$, find $\frac{t_1^2}{t_0^2}$. If it can't be determined, enter -1.

Question 600: Asset Dynamics

Topic: pure math **Difficulty**: easy

Let X denote an asset that has the following dynamics with $X_0 = 2$.

$$dX_t = 5dt + 2dW_t$$

where dW_t is a standard Brownian motion.

Find the mean μ_T and variance σ_T^2 of X_T, where $T = 0.2$. Give the answer in the format of $\mu_T * \sigma_T^2$.

Question 601: Precise N Sum

Topic: probability **Difficulty**: medium
Suppose $N > 0$ is an integer. Let X_1, X_2, X_3 be independently selected uniformly at random from the set $\{0, 1, \ldots, N\}$. Let $p_N = \mathbb{P}[X_1 + X_2 + X_3 = N]$. Find $\lim_{N\to\infty} Np_N$. If this limit does not exist, enter -1.

Question 602: Maximum Volatility

Topic: finance **Difficulty**: easy
Given an European call option, what would the price of our contract go to as the volatility tends towards infinity?

Enter -1 for negative infinity, 0 for zero, and 1 for infinity.

Question 603: Option Dice II

Topic: finance **Difficulty**: easy
Pretend you have a simple options chain on the expected value of the product of two dice rolls. What would the put option at 4 be priced at in this market?

Question 604: 1900 Age

Topic: brainteasers **Difficulty**: medium
A man's age at death was 1/29th the year of his birth. The man lived to see his birthday in 1900 but died before the end of World War II in 1945. How old did he turn on this birthday?

Question 605: Uniformly Correlated

Topic: probability **Difficulty**: easy
Suppose that $X, Y \sim \text{Unif}(0, 1)$ IID. Define $U = 1 - X^2 - Y^2$ and $V = X^2 + Y^2$. Find $\text{Cov}(U, V)$.

Question 606: Random Bivariate

Topic: probability **Difficulty**: medium
Suppose that M and S are IID Exp(1) random variables. Given that $M = \mu$ and $S = \sigma$, $(X, Y) \sim \text{BVN}(\mu, \mu, \sigma^2, \sigma^2, \rho)$, where $|\rho| \leq 1$ is fixed. Compute $\text{Var}(Y)$.

Question 607: Rebuy Gambler

Topic: probability **Difficulty**: easy
Alice and Bob start with \$15 and \$10, respectively. They bet \$1 at a time on opposite outcomes of a fair coin. If Alice bankrupts, she has the ability to buy in again with \$15, while Bob has the stack he won. If she bankrupts again, she loses the game. Find the probability Bob loses the game.

Question 608: Find the Triangle

Topic: brainteasers **Difficulty**: hard
The sides and height of a triangle are four consecutive whole numbers. What is the area of the triangle?

Question 609: Rectangles on Chess Board

Topic: probability **Difficulty**: medium
How many rectangles can be formed from the squares in an 8×8 square chess board? Squares also count as rectangles.

Question 610: Lowest Target

Topic: probability **Difficulty**: easy
An archer is shooting at targets. There are 4 columns of identical targets, each with $2, 3, 3,$ and 4 targets stacked vertically, respectively. The archer first selects a column and then shoots the lowest-hanging target in that column that is not broken. Once a column is out of targets, it is no longer able to be selected. In how many ways can the archer break all the targets?

Question 611: Squares on a Chess Board

Topic: brainteasers **Difficulty**: easy
How many squares can be formed from the squares in an 8×8 square chess board?

Question 612: Peeled Dice

Topic: probability **Difficulty**: medium
Suppose you roll a fair $6-$sided die and peel that rolled value off the die so that there are five remaining sides with values on them. You roll until you obtain one of the other five values. What is the expected sum of the two rolls?

Question 613: Ranged Stars and Bars

Topic: probability **Difficulty**: hard
Find the number of non-negative integer solutions to $6 \leq x_1 + \cdots + x_5 \leq 10$.

Question 614: Car Crash

Topic: probability **Difficulty**: hard

$$N$$

employees are driving their individual cars on their way to work at QuantGuide. All cars are reasonably spaced apart, and they all travel at distinct speeds which are randomly assigned. When a faster car catches up to a slower car, it assumes the slower car's speed. After a very long period of time has passed, all N cars have settled into K clusters traveling at distinct speeds. What is the expected value of K when $N = 10$?

Question 615: Inheritance Split

Topic: brainteasers **Difficulty**: easy
A father leaves a \$100 inheritance to his two sons, Abe and Ben. If one-third of Abe's inheritance is to be taken from one-fourth of Ben's inheritance, the difference would be \$11. How much did Abe receive?

Question 616: Two Puts

Topic: finance **Difficulty**: medium
We have the underlying stock S with initial value $S_0 = 32$ and a vanilla put with strike $K_1 = 35$ with initial value $P_0 = 4.5$. We want to price a vanilla put with strike $K_2 = 33$. Give the best estimate for the price of the $K_2 = 33$ put. Give the answer to 2 decimal points.

Question 617: 9 Sum II

Topic: probability **Difficulty**: medium
How many integers at most 100000 have digits that sum to 9? Some examples include 81, 144, and 13212.

Question 618: Paired Values II

Topic: probability **Difficulty**: medium
Suppose we have the values $1-6$ in a bowl. We draw them without replacement,

noting the order in which we selected them. We multiply the first two values together, the next two values together, and the last two values together. Lastly, we add the three products above. How many unique sums are possible?

Question 619: Variance of Uniform

Topic: probability **Difficulty**: easy
Let $X \sim \text{Unif}(0, 1)$. Find $\text{Var}(X)$.

Question 620: Colorful Socks II

Topic: probability **Difficulty**: easy

$$10$$

pairs of socks, each with a distinct color, are in a drawer. You draw out 3 socks at random. Find the probability that you obtain a matching pair.

Question 621: Sticky Delta

Topic: finance **Difficulty**: medium
We have a call option on the underlying S with initial price $S_0 = 4$ and strike $K = 4$. The Γ of the underlying is 0.03. The implied volatility follows a sticky-delta model. Let $\sigma(x) = (x - 0.5)^2 + .3$, where x represents the Δ of the option and σ the implied volatility.

Give an approximation for the implied volatility of the $K = 6$ call option. Round to 2 decimal points.

Question 622: Pizza Munching

Topic: probability **Difficulty**: easy
Garrett loves eating pizza, but enjoys savoring the slices too. The time it takes for him to eat a pizza slice is uniformly distributed between 1 and 5 minutes. If he has been eating for 2 minutes and still isn't done, find the probability that he finishes the slice in the next minute.

Question 623: Soccer Bets

Topic: probability **Difficulty**: easy
You are betting on the outcome of a soccer match. Team Quant has $2:1$ odds

(you get \$3 back if you win, \$2 profit) and Team Guide has 3 : 1 odds. You can also bet that both teams will tie which has 10 : 1 odds. Let A, B, and C be the lowest round bet (can only bet whole dollars) amount you bet on Team Quant, Team Guide, and a tie respectively. Assume that if you are betting, you want to make the same amount no matter the outcome of the game. How much do you profit? If you shouldnât bet on this game, enter 0.

Question 624: Dollar Cent Switch II

Topic: brainteasers **Difficulty**: medium
A man entered a store and spent half of the money in his pocket. When he left, he had just as many cents as he had dollars when he went in and half as many dollars as he had cents when he went in. His pocket was not empty upon arriving. How much money did he have on him when he entered?

Question 625: Product Over Sum

Topic: probability **Difficulty**: easy
Roll two fair standard 6−sided dice. Find the probability that the sum of the upfaces is at least the product of the upfaces.

Question 626: Numerical Triangle

Topic: brainteasers **Difficulty**: hard
The integers 9 integers $1 - 9$ are put on the sides and vertices of an equilateral triangle. Any integer put on a vertex of the triangle will count for both sides intersecting at that vertex. If all three vertices have an integer fixed on them and the sum of the integers on each side (including the vertices) is 17, find number of distinct triangles that can be formed. Any triangle that can be formed as a rotation of another triangle is not considered distinct. Order of the integers on the sides is irrelevant as well.

Question 627: Prime or Not

Topic: brainteasers **Difficulty**: easy
Is 1027 a prime number (1 for yes and 0 for no)?

Question 628: Bond Practice V

Topic: finance **Difficulty**: easy
Calculate the price of a bond with these characteristics. The coupon rate is 0.06, coupon payments are made every six months (twice per year), and the

par value of the bond is 1,000. There are 12.0 years to maturity and a market interest rate of 0.06

Question 629: Pizza Passcode

Topic: probability **Difficulty**: medium
Pizza Hut gives new users a 7−digit code when they sign up. There are no restrictions on how many times a digit can be used, and digits are allowed to start with 0. How many such codes are non-decreasing? In other words, each digit in the passcode is at least as large as the last. 1122345 and 9999999 are such examples, whereas 2345657 is not.

Question 630: Pharmaceutics I

Topic: statistics **Difficulty**: easy
A pharmaceutical company has researched a drug that they claim will enhance focus for 80% of people suffering from attention deficit disorder. After examining the drug, the FDA believes that their claims regarding the effectiveness of the drug are inflated. In an attempt to disprove the company's claim, the FDA administers the drug to 20 people with attention deficit disorder and observe X, the number for whom the drug dose induces focus. More formally, the FDA is testing the null hypothesis $H_0 : p = 0.8$ against the alternative hypothesis $H_a : p < 0.8$. Assuming the rejection region $x \leq 12$ is used, what is α, the level of significance?

Question 631: Factorial Zeros

Topic: brainteasers **Difficulty**: easy
How many zeros does the expansion of 100! have?

Question 632: Sum Leak II

Topic: probability **Difficulty**: medium
Let $X_1, X_2, \ldots, X_{40}$ be IID random variables with $\mathbb{E}[X_1] = 2$, $\mathbb{E}\left[\frac{1}{S_{20}}\right] = \frac{1}{10}$. Define $S_n = X_1 + \cdots + X_n$. Compute $\mathbb{E}\left[\frac{S_{40}}{S_{20}}\right]$.

Question 633: Marble Runs

Topic: probability **Difficulty**: hard
You repeatedly draw marbles from a bag containing 50 red and 50 blue marbles

until there are no more marbles left, recording the order of red and blue marbles drawn. You then count the number of "runs," where a run is defined as any number of consecutive marbles of the same color. For example, $RBBRRRBRR$ contains 5 runs. What is the expected number of runs that you observe?

Question 634: Elliptical Area

Topic: probability **Difficulty**: easy
An ellipse is centered at the origin. Let A and B be the horizontal and vertical radii lengths of the ellipse, respectively. Assume that $A, B \sim \text{Exp}(1)$ and are independent. Recall that the area of an ellipse with radii a and b is given by πab. A circle with diameter A is created as well. Find the probability the area of the circle is larger than the area of the ellipse.

Question 635: Generous Banker

Topic: probability **Difficulty**: medium
You are at the bank and it is your lucky day. The banker is going pick random positive integer and you are too. You are both allowed to determine the probability distribution on the positive integers that you select from. If you and the banker select the same value, say n, you receive n^2 dollars. Otherwise, you receive nothing. Neither party knows the distribution the other selected. Assuming optimal strategy from both parties, what is your expected payout from this game? The answer will be in the form $\frac{a}{\pi^b}$ for integers a and b. Find ab.

Question 636: Arbitrage Detective IV

Topic: finance **Difficulty**: hard
There are five options on the $TSLA$ options chain right now, all of which are expiring at the end of the month, and with strikes 165, 170, 175, 180, and 185. Suppose the put options respectively cost \$9, \$12, \$14, \$14.5, and \$15 and the call options cost \$15, \$14, \$13, \$12, and \$9. Is there an arbitrage opportunity? If so, enter the minimum amount you are guaranteed to make, if no opportunity exists, enter -1.

Question 637: Ferry Stops

Topic: brainteasers **Difficulty**: easy
There are an unknown amount of people on a ferry. After the first stop, $\frac{3}{4}$ of them get off and 7 people get on. This happens again at 2 more stops. After this

process, what is the minimum amount of possible people that could be aboard the ferry?

Question 638: Racecar Driver

Topic: probability **Difficulty**: easy

Andy is a good driver. The probability that he runs any given stoplight is less than half. The probability the first stoplight Andy runs is the second one he sees is $\frac{9}{100}$. Find the probability that the fourth stoplight is the first one Andy runs. Assume that each stoplight Andy runs is independent of all others and that the probability is constant.

Question 639: Diner Dash

Topic: probability **Difficulty**: easy

Four friends agree to get lunch together at Jimmy's Diner. However, Jimmy's Diner is a chain with five locations. Suppose each friend goes to one of the locations at random. Compute the probability that they all end up at different locations.

Question 640: Investment Arbitrage

Topic: finance **Difficulty**: easy

You are in a market where you can invest in a stock or bet at a casino in \$100 increments. Call these increments units. Note that you may not purchase fractions of a unit. If you invest in the stock, then it goes up or down. If it goes up in value, you gain an additional 100 dollars per unit. If it goes down, you lose 50 dollars per unit. Additionally, you can bet on whether the stock will go up or down at the casino, where you either win 100 per unit if you're right or lose all 100 if not. Devise a strategy where you will always make a profit. Find the guaranteed profit if you hold a minimal amount of total units.

Question 641: Binomial Pricing I

Topic: finance **Difficulty**: medium

You have a stock, S, which has an initial price of $S_0 = 10$. You want to price a European call option with a strike of $K = 5$. The stock will either increase by 100% or decrease by 50%. What is the price of the time-0 call option, C_0? Assume interest rates are 0.

Question 642: 30 Side Difference

Topic: probability **Difficulty**: easy
Let X and Y be the upfaces of two independent rolls of fair $30-$sided dice with values $1-30$ on the sides. Find $\mathbb{E}[|X-Y|]$.

Question 643: Head-Tail Product

Topic: probability **Difficulty**: easy
Billy flips a fair coin n times. Let X denote the number of heads, and let Y denote the number of tails. $\mathbb{E}[XY]$ can be expressed in the form

$$\frac{an^2+bn+c}{d}$$

for some integers a,b,c,d, where a,b,c are each relatively prime with d. Compute $a+b+c+d$.

Question 644: Fixed Point Variance

Topic: probability **Difficulty**: hard
Consider a uniformly random permutation of $(1,2,\dots,1000)$. Let X be the number of fixed points and Y be the number of non-fixed points. Compute $\text{Var}(X-Y)$.

Question 645: Going Extinct

Topic: probability **Difficulty**: easy
An endangered species currently has a population size of 10000. Every year, the species has a 90% chance of doubling in size and a 10% chance of going extinct. At this rate, what is the expected number of years before extinction?

Question 646: Cubic Sum

Topic: brainteasers **Difficulty**: easy
Find the integer x such that $x^3=441^3+588^3+735^3$.

Question 647: 2D Paths IV

Topic: probability **Difficulty**: hard
How many paths are there where you can only move up or right one unit at each step that go from $(0,0)$ to $(5,3)$ without crossing the line $y=x$? You are allowed to touch it.

Question 648: Uniform Product II

Topic: probability **Difficulty**: easy
Let $X, Y \sim \text{Unif}(0,1)$ IID. Find $\mathbb{P}\left[XY > \frac{1}{2}\right]$. Round your answer to the nearest hundredth.

Question 649: Positively Normal

Topic: probability **Difficulty**: medium
Suppose that X and Y are two IID standard normal random variables. Compute $\mathbb{P}\left[Y > \sqrt{3}X \mid Y > 0\right]$.

Question 650: 7 Multiple

Topic: probability **Difficulty**: easy
You and your friend play a game in which you take turns rolling a fair $6-$sided die and keep a running tally of the sum of the upfaces obtained in each roll. The winner of the game is the person who most recently rolled the die when the running sum first becomes a multiple of 7. You get to decide whether to go first or second. Under rational strategy from you, what is your probability of winning?

Question 651: Geometric Distribution

Topic: probability **Difficulty**: easy
Suppose that $X \sim \text{Geom}\left(\frac{1}{4}\right)$. Compute $\mathbb{P}[X \leq 8 \mid X \geq 5]$.

Question 652: Defeating Dragons

Topic: probability **Difficulty**: medium
A 3-headed dragon is attacking your village! You, the mightiest of knights, are tasked with taking down this dragon with your trusty sword. The dragon is defeated if you chop off all of its heads. However, this dragon uses magic and has the ability to grow heads over time! If you attack the dragon, one of three events can happen. You either chop of 2 of its heads, you chop of 1 head (which will ALWAYS grow back), or you miss entirely and the dragon grows 1 head. The probability you chop of 2 heads and miss the dragon are the same (assuming the dragon has 2 or more heads). You know that if the dragon has 5 or more heads, the dragon is too strong for you to take care of (in which case you and the rest of the villagers will perish). What is the probability you defeat the dragon?

Question 653: Binary Strangle

Topic: finance **Difficulty**: easy
Let's define a binary strangle as an option that pays 1 if $S_T >= K_2$ or if $S_T <= K_1$. Here, we have $K_1 = 15$ and $K_2 = 20$. You have access to the following binary calls. What is the time-0 price of the binary strangle? Assume a discount factor of $Z_0 = 0.9$. The binary calls are given in the format of $(\text{Strike } K, \text{Price } C_0)$

$$(15, 0.73)$$
$$(17.5, 0.34)$$
$$(20, 0.13)$$

Question 654: Adjacent Birthdays

Topic: probability **Difficulty**: medium
Find the average number of people needed for there to be a pair of people with adjacent birthdays. Round your answer to the nearest tenth. You may assume the standard assumptions of the birthday problem i.e. 365 days in a year, birthdays are independent, etc. We count December 31 and January 1 to be adjacent. It may be good to use computer software to evaluate the answer.

Question 655: Comparing Flips I

Topic: probability **Difficulty**: medium
Audrey repeatedly flips a coin until she sees a pattern of HHT or HTT. What is the probability that HHT occurs before HTT?

Question 656: Plane Boarding

Topic: probability **Difficulty**: medium
100 people are in line waiting to board a plane. Let us define the ith passenger in line as having a ticket for plane seat i. Intoxicated, the first passenger in line picks a random seat on the plane to sit on with equal probability. The other 99 passengers are sober, and will sit in their assigned seat unless taken, in which they will sit in a random free seat. You are the last person in line. What is the probability that you end up in your assigned seat?

Question 657: Circular Delete

Topic: brainteasers **Difficulty**: medium
The numbers $1, 2, \ldots, 2000$ are put on the circumference of a circle in clockwise

increasing order. You start at the integer 1 and delete it. Then you, move clockwise to 2 and keep it. Afterwards, you are going to repeat in this fashion of alternating between deleting integers and keeping integers repeatedly until you reach the last integer of a given rotation. Afterwards, you start again by deleting the first integer in the new cycle, keeping the second, etc. Find the last integer to be deleted from the circle.

Question 658: Multiplicative Mix

Topic: brainteasers **Difficulty**: easy
We have 4 positive integers a, b, c, and d that satisfy $ab = 21, bc = 15$, and $cd = 45$. It is also known all of the integers are strictly larger than 1. Find $a^2 + b^2 + c^2 + d^2$.

Question 659: Triangular Change

Topic: pure math **Difficulty**: easy
ABC is a triangle that is changing with respect to time. The length of side AC is increasing at 3 units per second, the length of side AB is decreasing at 2 units per second, and the length of side BC is decreasing at 2 units per second. At time $t = t_1$, sides AC, AB, BC have side lengths $12, 14$, and 16 respectively. Calculate the rate of change of the perimeter of ABC at time t_1 in units per second.

Question 660: Pocket Aces

Topic: probability **Difficulty**: easy
You are dealt two cards from a fair deck of 52 cards. What is the probability that you have a pair of aces?

Question 661: Ping Pong Tournament II

Topic: probability **Difficulty**: medium

$$50$$

people are competing in a ping pong tournament where there is only one ping pong table. The competitors are numbered 1 through 50. Suppose that if two competitors meet, the one with the larger number wins. Two competitors are chosen at random, and the loser is removed from the tournament. The winner moves on to the next round, where their opponent is chosen at random. This process is repeated until one person is left (a total 49 rounds will be played). Compute the probability that competitor 49 is still in the tournament after the first 10 rounds have been played.

Question 662: Square Cross

Topic: probability **Difficulty**: medium
A square of side length 20 is formed in front of you. You select a point uniformly at random from the interior of the square. You then proceed to form a circle of radius $R \sim \text{Unif}(0, 10)$, independent of the point selected. Find the probability that the circle you create does not intersect the square at any point.

Question 663: Boy Chairs

Topic: probability **Difficulty**: easy

3

boys and 3 girls are in a group. 3 of these children are randomly selected to be seated in 3 chairs. Given that all 3 children selected are all of the same gender, what is the expected amount of boys?

Question 664: Leading Sum

Topic: pure math **Difficulty**: medium
Written in closed form, the expression $1^{1000} + 2^{1000} + \cdots + n^{1000}$ is a polynomial in n with leading term an^{1001} for some a. Find a.

Question 665: Cheater

Topic: probability **Difficulty**: easy
As a good test taker, on a multiple choice exam with 5 options per question, Gabe either knows the answer to a question beforehand or chooses an answer completely at random. If he knows the answer beforehand, he selects the correct answer. If the probability that Gabe knows the answer to any given question is 0.6, find the probability that an answer that was correct was one for which he knew the answer.

Question 666: Statistical Test Review I

Topic: statistics **Difficulty**: easy
We want to test whether or not a coin is balanced based on the number of heads, X, that appear after 36 tosses of the coin. Let's say we use the rejection region $|x - 18| \geq 4$. What is the value of α to the nearest thousandth?

Question 667: Expecting Heads

Topic: probability **Difficulty**: easy
Suppose we are flipping a fair coin 5 times. Find the expected number of heads we obtain given that the first and last flips are of opposite parity (i.e. one is heads and one is tails).

Question 668: Signed Correlation

Topic: statistics **Difficulty**: easy
Let X and Y be two random variables with $\text{Corr}(X, Y) = -0.4$. If possible, find $\text{Corr}(3X + 1, 1 - Y)$. If it is not possible, enter 100.

Question 669: Leftwards Frog

Topic: probability **Difficulty**: hard
A frog is hopping between 8 lily pads labeled $1 - 8$ from left to right. The frog starts on lily pad 1. At each turn, the frog hops to a lily pad that it has not been to yet. Find the probability the frog made exactly 1 leftwards hop after visiting all of the stones.

Question 670: Tigers Love Sheep

Topic: brainteasers **Difficulty**: easy
Tigers on this magical grass island are rational and prioritize survival first and eating sheep over grass second. Assume that at each time step, only one tiger can eat one sheep, and that tiger itself will become a sheep after it eats the sheep. One hundred tigers and one sheep are on the magic grass island. How many sheep are left?

Question 671: Sphere Slicer

Topic: probability **Difficulty**: hard
Consider the set of vertices

$$V = \{(0,0,1), (1,0,0), (0,1,0), (-1,0,0), (0,-1,0), (0,0,-1)\}$$

on the unit sphere. Line segments are drawn between each pair of vertices. If we choose a normal vector uniformly at random, then the corresponding plane with that normal vector passing through the origin splits the sphere into two parts. Find the expected number of edges that the plane passes through.

Question 672: Derivative Difference

Topic: brainteasers **Difficulty**: medium
Let $f(x)$ be a function satisfying the differential equation $f(x)-f'(x) = (x+1)^3$ with initial condition $f(0) = 16$. Find $f(9)$.

Question 673: Primes and Composites

Topic: probability **Difficulty**: easy
There is a game where an integer is picked uniformly between 2 and 10 inclusive. You win \$$n$ if the integer is prime and lose \$$\frac{n}{2}$ if its composite (where n is the integer picked). How much would you pay to play this game? If you donât wish to play, enter 0.

Question 674: DisCard Game

Topic: brainteasers **Difficulty**: easy
A casino offers a card game using a normal deck of 52 cards. The rule is that you turn over two cards each time. For each pair, if both cards are black, they go to the dealer's pile; if both are red, they go to your pile; else, they are discarded. The process is repeated 26 times until all cards are exhausted. If you have more cards in your pile, you win 10$.Else, includingties, yougetnothing.Whatisthefairvalueofthisgame$?

Question 675: Bacterial Survival I

Topic: probability **Difficulty**: easy
Suppose that a colony of bacteria starts with 1 cell. Each generation, each living cell, independently of all other cells, will either die with no offspring or produce some random positive integer number of offspring. The probability that the colony with one bacteria will go extinct at some point in the future is $\dfrac{1}{3}$. Find the probability that the colony starting with 3 cells goes extinct at some point.

Question 676: Modified Deck Pair

Topic: probability **Difficulty**: easy
Marissa has a deck of 40 cards: 4 1's, 4 2's, …, 4 10's. A matching pairâ2 cards with the same numberâis removed from the deck without replacement. If Marissa randomly draws 2 more cards, what is the probability that she draws another matching pair?

Question 677: Double Data Trouble VI

Topic: statistics **Difficulty**: easy
Suppose that you run linear regression on some dataset and obtain the coefficients $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$. Let t_0 be the $t-$statistic obtained when testing the null hypothesis $H_0 : \beta_1 = 0$ against $H_A : \beta_1 \neq 0$. Suppose that we double the values of each of our data points and then run linear regression again on this new data set. Let $\hat{y}_i = \hat{\beta}'_0 + \hat{\beta}'_1 x_i$ be the new coefficients that are obtained. If t_1 is the corresponding $t-$statistic obtained when testing the null hypothesis $H_1 : \beta'_1 = 0$ against $H_A : \beta'_1 \neq 0$, find $\frac{t_1^2}{t_0^2}$. If it cannot be determined, enter -1.

Question 678: Optimal Bidders I

Topic: probability **Difficulty**: medium
Carter has come into contact with a bounty of gold. He takes it to an auction shop. The auction shop says that each person that bids will place a bid that is uniformly distributed between \$500 and \$1000. They also state that they can recruit people to bid for a price of \$5 per person. Assuming Carter selects the optimal number of people to bid on his gold, what is his expected payout?

Question 679: Egg Drop II

Topic: brainteasers **Difficulty**: hard
You are holding three identical eggs in a really large building with n stories. If an egg is dropped at an elevation under story X, then the egg will survive; else, the egg breaks. It is known that the egg will break at some floor number $1 - n$. What is the maximum value of n such that at most 9 drops are required to determine X in the worst-case scenario?

Question 680: Big or Small Deck?

Topic: brainteasers **Difficulty**: medium
A standard deck is split up into two subdecks of 39 and 13 cards. You and a friend may choose one of the decks to draw cards from the top of one-by-one. The goal is to minimize the expected number of cards needed to be drawn to obtain your first ace. Whoever needs to draw fewer cards to see their first ace wins the round. You get to select first. Which deck do you select? Answer 13 if you should select the deck of 13 to draw from, 39 if you should select the deck of 39 to draw from, and -1 if it doesn't matter. Note that if a given deck has no aces, we consider the other deck faster and that we disregard ties i.e. neither person wins.

Question 681: Double Data Trouble V

Topic: statistics **Difficulty**: easy
Suppose that you run linear regression on some dataset and obtain the coefficients $\hat{\beta}_{OLS}$. Recall that if X is the data and σ^2 is the variance of the IID normal errors, then $\text{Var}\left(\hat{\beta}_{OLS}\right) = \sigma^2(X^TX)^{-1}$. If you were to run linear regression again on the dataset where you double the values of each point in your original dataset and obtain new coefficients $\hat{\beta}'_{OLS}$, find the constant c such that $\text{Var}\left(\hat{\beta}'_{OLS}\right) = c\text{Var}\left(\hat{\beta}_{OLS}\right)$. If no such constant exists, enter -1.

Question 682: Basic Die Game V

Topic: probability **Difficulty**: easy
Alice rolls a fair 6−sided die with the values $1-6$ on the sides. She sees that value showing up and then is allowed to decide whether or not she wants to roll again. The re-roll costs \$1. If she decides to stop, Alice receives a payout equal to the upface of the die. If she rolls again, she receives a payout equal to the upfaces of the second roll. Assuming optimal play by Alice, what is her expected payout on this game?

Question 683: Pairwise Digit Sums I

Topic: brainteasers **Difficulty**: medium
Find the largest value of n such that the there exists a n digit number with all unique pairwise sums of digits. For example, if $n = 3$, 174 is a valid integer, since $1 + 7 \neq 1 + 4 \neq 7 + 4$.

Question 684: Rainbow Trains

Topic: probability **Difficulty**: easy
There are three train cars: a blue one, a red one, and a green one. This train has a conductor whose job is to disconnect the back train cars and put it in front. They can do this as many or as few times as they want. If the three train cars are randomly ordered, what is the probability that the conductor can rearrange it into rainbow order? (Note: Here, rainbow order means red, then green, then blue).

Question 685: Big Bubble I

Topic: pure math **Difficulty**: easy
A circle has a radius that is increasing at a constant positive rate from 0. What

is the radius at the moment when the circumference and area of the circle are increasing at the same rate?

Question 686: German Tanks

Topic: statistics **Difficulty**: medium
Suppose that German tanks are assigned distinct serial numbers $1, 2, \ldots, N$. You observe 6 tanks with serial numbers $38, 17, 59, 42, 97$, and 120. Under a frequentist approach, what is the best guess for N?

Question 687: Same Side

Topic: probability **Difficulty**: easy
What is the probability of flipping the same side of a coin 4 times a row?

Question 688: Prime Hunter

Topic: probability **Difficulty**: easy
A standard fair $6-$sided die is rolled 6 times. We'll define a successful roll as rolling a prime number; otherwise, itâs a failure. What is the probability that there are exactly 3 successful rolls?

Question 689: Sum Standard Deviation

Topic: probability **Difficulty**: easy
Let X and Y be random variables with standard deviations 1 and 3, respectively. The standard deviation of $X + Y$ is 4. Find $\mathrm{Corr}(X, Y)$.

Question 690: Building Blocks

Topic: probability **Difficulty**: medium
Max is building a tower from three distinct colors of building blocks (red, blue, and yellow). He has 7 blocks of each color and randomly picks a block out at a time to build his tower of 5 blocks in height. What is the probability the tower has composed of exactly two different colors?

Question 691: Existent MGF

Topic: probability **Difficulty**: medium
Let X be a random variable with MGF $M(\theta)$ such that $M(1) = 7$. Find the smallest possible value for $M(4)$.

Question 692: Cube Slice

Topic: brainteasers **Difficulty**: medium
A lumberjack is cutting a $3 \times 3 \times 3$ cubic piece of wood into 27 $1 \times 1 \times 1$ cubes of wood. The lumberjack can reposition the cube and individual pieces after each cut. Find the minimum number of cuts the lumberjack needs to make to form the smaller cubes.

Question 693: Fireball Thrower

Topic: probability **Difficulty**: medium
Mario and Luigi are throwing Fireballs. Each one throws a fireball twice. Let M_1 and M_2 represent the distance of Mario's 2 attempts, while L_1 and L_2 represent the distance of Luigi's 2 attempts. We have that $M_i \sim \text{Exp}(1)$ while $L_i \sim \text{Exp}(2)$. All attempts are independent of all other attempts. Let $M = \max\{M_1, M_2\}$ and $L = \max\{L_1, L_2\}$. Find $\mathbb{P}[M > 2L]$.

Question 694: Laser Game

Topic: probability **Difficulty**: easy
Mordecai has a large target of radius 10 and a laser pointer. He is clumsy, so he points the laser at a uniformly random point on the target. Let D represent the random distance from the laser to the center of the target. He offers you to play the following game: Mordecai points the laser at the target one time. Before doing so, you fix a value $0 \leq r \leq 10$. If your laser is within distance r away from the center, you must pay Mordecai $\$(10 - r)$. Otherwise, Mordecai pays you $\$r$. Find the value of r that maximizes your expected winnings from this game.

Question 695: Contracts and Pricing II

Topic: probability **Difficulty**: medium
Steve wants to buy Amy's house 6 months in the future. However, house prices have been volatile lately; the value of Amy's house in 6 months is equal to its current market value scaled by a constant C, where $C \sim \text{Unif}([0.6, 1.6])$. Amy offers to sell the following contract: Steve receives the right but not the obligation to, in 6 months, purchase Amy's house at its current market value instead of its value then. Amy is willing to negotiate the price of the contract. What is the fair price for Amy's contract as a proportion of her home's current market value?

Question 696: Bowl of Cherries VI

Topic: probability **Difficulty**: easy
Each of 16 distinct cherries has an equal chance of being placed into one of 4 distinct bowls. Let a equal the probability that one bowl has 3 cherries, another has 5 cherries, and the remaining bowls contain 4 cherries each. Let b denote the probability that all bowls contain 4 cherries each. What is $\frac{a}{b}$?

Question 697: Infected Dinner I

Topic: brainteasers **Difficulty**: easy
There are 1000 people having dinner at a grand hall. One of them is known to be sick, while the other 999 are healthy. Each minute, each person talks to one other person in the room at random. There is no limit on how many times a person talks to another person. If one is sick and one is healthy, the healthy person is infected and becomes sick. Once a person becomes sick, they are assumed to be sick for the rest of the dinner. Find the minimum amount of time (in minutes) until every person in the hall becomes sick.

Question 698: Confused Ant I

Topic: probability **Difficulty**: medium
An ant is standing on one vertex of a cube and can only walk along the edges. The ant is confused and moves randomly along the edges at random. How many edges, on average, will the ant travel to reach the opposite vertex of the cube?

Question 699: Acceptance-Rejection Sampling

Topic: probability **Difficulty**: medium
Suppose we want to sample from the PDF of $X \sim \text{Beta}(3,2)$, which has PDF $f(x) = 12x^2(1-x)I_{(0,1)}(x)$. Since you know how to generate uniformly random pairs in the rectangle $R_a = [0,1] \times [0,a]$ for some $a > 0$, you decide to sample from R_a. You will count your random point $(X,Y) \in R_a$ as valid for the PDF if it lies in the region bounded by the $x-$axis and $f(x)$. Otherwise, it is rejected. If a is selected to minimize the number of rejected pairs of points per sample, find the expected amount of rejected pairs per accepted pair.

Question 700: Smallest Factorizaiton

Topic: brainteasers **Difficulty**: medium
The number 1234567890 is not prime, so it can be written in the form ab for two positive integers a and b. Find a^*+b^*, where a^* and b^* satisfy $a^*b^* = 1234567890$

and for any other pair (a, b) such that $ab = 1234567890$, $|a - b| \geq |a^* - b^*|$. It may be helpful to consider the prime factorization.

Question 701: Zero Appearance

Topic: brainteasers **Difficulty**: easy
How many positive integers at most 10000 have the number 0 somewhere?

Question 702: Particle Reach IX

Topic: probability **Difficulty**: medium
Consider a particle that performs a random walk on the integers starting at position 0. At each step, the particle moves from position i to position $i+1$ with probability p, while the probability it moves from i to $i-1$ is $1-p$. If $p = 2/3$, find the variance of the number of steps until the particle reaches 1. If the answer is infinite, answer -1.

Question 703: Greater Than

Topic: brainteasers **Difficulty**: easy

$$X, Y, Z \overset{\text{iid}}{\sim} \text{Unif}(0, 1)$$

. What is the probability that $X \geq Y \geq Z$?

Question 704: Complex Circle

Topic: probability **Difficulty**: medium
Suppose that Z is a uniformly random point selected from the boundary of the unit circle in the complex plane $|z| = 1$. Compute $\mathbb{E}[Z^2]$.

Question 705: Expecting Jacks

Topic: probability **Difficulty**: medium
Bill draws from a deck of cards repeatedly, with replacement. What is the expected number of draws to get four jacks in a row?

Question 706: Option Dice III

Topic: finance **Difficulty**: medium
Pretend you have a simple options chain on the expected value of the product

of two dice rolls. Make a 2 unit wide market on the call at 19.

Enter your answer as the sum of the bid and ask. For example, if your market was 5 @ 7, you would enter 12.

Question 707: Mod Mondays

Topic: brainteasers **Difficulty**: easy
Let $1, 2, \ldots, 7$ be the days of the week (Sunday, Monday, ..., Saturday). What day of the week will it be 100 days after Monday (as an integer)?

Question 708: Bolt Variance II

Topic: statistics **Difficulty**: easy
A manufacturer produces nuts and bolts. A random sample of twelve bolts returned a sample variance of .0003, while a random sample of twelve bolts from a competitor returned a sample variance of .0001. They wish to test if there is sufficient data to indicate a smaller variation in bolt diameter from their competitor. What is the value of the relevant test statistic? Assume independence, variance homogeneity, and that diameter variance is approximately normally distributed.

Question 709: Uniform Summation

Topic: statistics **Difficulty**: medium
By considering $U \sim \text{Unif}(0, 0.5)$ and $\mathbb{E}\left[\dfrac{1}{1-U}\right]$, compute $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n(n+1)}$. The answer should be in the form $\ln(a)$ for some integer a. Find a.

Question 710: Sock Drawer II

Topic: probability **Difficulty**: medium
In a drawer, Sandy has 5 pairs of socks, each pair a different color. On monday, Sandy selects two individual socks at random from the 10 socks in the drawer. On Tuesday, Sandy selects two of the remaining 8 socks at random, and on Wednesday, two of the remaining 6 socks at random. Find the probability that Wednesday is the first day Sandy selects matching socks.

Question 711: 5 Pairwise Sum

Topic: brainteasers **Difficulty**: hard
Let $S = \{a, b, c, d, e\}$ be a set of 5 (possibly repeated) real numbers. The pair sums of elements in S are:

$$5, 11, 11, 13, 13, 14, 16, 19, 22, 22$$

Find $a^2 + b^2 + c^2 + d^2 + e^2$.

Question 712: Finite Coin Equalizer

Topic: probability **Difficulty**: hard
Suppose you flip a coin that has probability $0 < p < 1$ of landing heads on each flip. You flip the coin until you have an equal amount of heads and tails appearing for the first time (of course, after the initial state of 0 for both). For any integer $n \geq 1$, find the probability that you flip the coin exactly $2n$ times. The probability can be written as a function $f(p, n)$. Compute $f(1/3, 5)$ to the nearest ten-thousandth.

Question 713: Bus Wait II

Topic: probability **Difficulty**: medium
You are waiting for a bus to get to QuantGuide on time! The bus runs on a fixed schedule of appearing every 10 minutes. However, the driver, independently between appearances, may want to refill on gas. The driver refills on gas with 10% probability per trial, independently between trials. If the driver fills up on gas, 1 hour is added to his travel time. If you arrive at a uniformly random time throughout the day, what is the expected time until the next bus appears (in minutes)?

Question 714: Circular Slice II

Topic: probability **Difficulty**: medium
A random angle $\theta_1 \sim \text{Unif}(0, 2\pi)$ is selected. Then, the arc of the unit circle that sweeps out θ_1 radians is marked red going counterclockwise starting from $(1, 0)$. Two other angles $\theta_2, \alpha \sim \text{Unif}(0, 2\pi)$ IID are also selected. Afterwards, an arc of length θ_2 radians starting from the point that is α radians counterclockwise of the origin is swept out and colored blue. When the blue and red regions intersect, they form a purple region. Given that there is at least one purple region, find the probability there is exactly one purple region.

Question 715: Convergent intervals

Topic: probability **Difficulty**: medium
Suppose you continually randomly sample nested intervals from $[0, 1]$, halving the size each time. That is, the next interval is $[x, x+0.5]$, where $x \sim U(0, 0.5)$, and so on. What is the variance of the point this converges to?

Question 716: Parking Rush

Topic: probability **Difficulty**: hard
There are 10 parking spots in front of QuantGuide headquarters. By 9:00 AM, all 10 parking spots are occupied. QuantGuide's 3 executives head home early in the afternoon, vacating their parking spots at random times distributed independently and uniformly between 12:00 PM and 3:00 PM. Andy is one of QuantGuide's 9 lazy employees; lazy employees arrive at the office at random times distributed independently and uniformly between 12:00 PM and 3:00 PM. If a parking spot is vacant when a lazy employee arrives, then that employee will occupy that spot until 5:00 PM. Otherwise, the lazy employee will call in sick and return home. What is the probability that Andy calls in sick?

Question 717: Perfect Correlation III

Topic: probability **Difficulty**: medium
Suppose that X and Y are perfectly negatively correlated i.e. $\rho(X, Y) = -1$ with $\sigma_Y > \sigma_X$. Find the correlation of $X + Y$ and $X - Y$.

Question 718: 100 Lights

Topic: brainteasers **Difficulty**: medium
There are 100 light bulbs in a room, each initially off and with its switch by its side. The first person enters and flips every switch. The second person enters and flips every other switch. The third person enters and flips every third switch. This process continues until the 100-th person flips every 100-th switch, or the last switch. How many lights are on?

Question 719: Careful Coin Question

Topic: probability **Difficulty**: medium
Your friend has a fair coin. They flip it 100 times consecutively and record the sequence of outcomes. Your goal is to guess the sequence that your friend flipped. You may ask a single Yes/No question to your friend to help you determine the sequence. The maximum probability you can achieve of guessing

your friend's sequence is in the form a^{-b} for integers a and b, where b is maximal. Find ab.

Question 720: Price an Option III

Topic: finance **Difficulty**: medium
You have access to the European call options at the following strikes and T_0 prices, the underlying S which has initial price $S_0 = 7$, and a bond that has initial price 0.9, paying 1 at time T. The calls are given in the format of $(\text{Strike } K, \text{Price } C_0)$

$$\begin{aligned} (5, 4.2) \\ (10, 1.4) \\ (15, 0.7) \\ (20, 0.1) \end{aligned}$$

Find the time-0 price of a contract that pays $|S_T - 15|$ at time T.

Question 721: Poisoned Kegs II

Topic: brainteasers **Difficulty**: medium
A king has 10 servants that bravely risk their lives to test whether or not the wine in n kegs is poisonous. It is known that exactly one of the n kegs is poisonous. If someone drinks any amount of liquor from the poisoned keg, they will die in exactly 1 month. Otherwise, the servant will be fine. The servants only agree to participate in the wine tasting for 1 month. What is the maximum value of n such that the king is guaranteed to determine which keg among the n is poisoned?

Question 722: The Last Roll

Topic: brainteasers **Difficulty**: easy
A fair $6-$sided die is continually rolled until the sum of all the numbers is at least 1000. What value on the die was most likely to show up on the last roll?

Question 723: Basic Delta Hedging

Topic: finance **Difficulty**: easy
A stock S currently has an initial price of 25. You purchase 10 units of an at-the-money call option ($K = 25$). In order to delta-hedge the position, how many units of the underlying should you hold? Note that if you plan to short the underlying, answer with a negative value.

Question 724: Conditioned Heads

Topic: statistics **Difficulty**: easy
Let H_n be the number of heads that have occurred in the first n flips of a fair coin. Compute $\mathbb{P}[H_9 = 6 \mid H_6 = 5]$.

Question 725: Discrete Walker

Topic: probability **Difficulty**: easy
Consider a random walk on a discrete line of 11 points (0 through 10). Supposed you have equal probability of stepping up or down at each movement. If you reach either 10 or 0 you must stop. If you start at point 6, what is the probability that you arrive at 10 before you arrive at 0?

Question 726: Cone Combo

Topic: probability **Difficulty**: easy
A point is uniformly at random selected from a cone with radius 1 and height 2. Find the probability that the point is at most a height 1 away from the circular base.

Question 727: Theater Seating

Topic: brainteasers **Difficulty**: easy
Five boys and five girls are seated in a row at the movie theater. To ensure that the children are engaged during the movie, the teacher mandates that no two children of the same gender can sit next to each other. How many arrangements are possible?

Question 728: Ant Collision II

Topic: brainteasers **Difficulty**: easy
Alice and Bob raised their own ants. Alice has 40 ants, while Bob has 80 ants. Alice and Bob stand at opposite ends of an infinitesimally wide string and let all of their ants crawl on the string. Whenever one ant runs into another ant, they both bounce back and change direction. Each of the ants can only move forward. Find the number of collisions between ants that take place.

Question 729: Pair Die Sum

Topic: probability **Difficulty**: easy

2

red and 2 blue fair 6−sided dice are rolled and the values are recorded. Find the probability that the sum of the upfaces of the two blue dice is larger than the sum of the upfaces of the two red dice.

Question 730: 20-30 Die Split III

Topic: probability **Difficulty**: hard
Alice and Bob have fair 30−sided and 20−sided dice, respectively. The goal for each player is to have the largest value on their die. Alice and Bob both roll their dice. However, Bob has the option to re-roll his die in the event that he is unhappy with the outcome. He can't see Alice's die beforehand. Bob then keeps the value of the new die roll. In the event of a tie, Bob is the winner. Assuming optimal play by Bob, find the probability Alice is the winner.

Question 731: Pricing Put Spread

Topic: finance **Difficulty**: medium
We have two puts: a put at strike $K_1 = 20$ and a put at $K_2 = 24$. We want to create a put spread. In other words, we will short the $K = 20$ put and long the $K = 24$ put. Find the best upper bound for the time-0 price of the spread. You have access to the underlying, with $S_0 = 23$ and bonds that pay 1 at time-T with initial price $B_0 = 0.9$.

Question 732: Gambling Addiction

Topic: probability **Difficulty**: easy
Gabe has a crippling gambling addiction. As such, he plays a game where the winning number on the nth game, where $n \geq 1$, is uniformly at random selected from the set $\{1, 2, \ldots, n + 2\}$. Let W be the random variable representing how many times Gabe plays before he wins. Find $\mathbb{E}[W]$. If the answer is infinite, answer -1.

Question 733: Clock Angle II

Topic: brainteasers **Difficulty**: easy
What is the angle between the hands of a clock at 3:15 PM (in degrees)?

Question 734: Particle Reach VIII

Topic: probability **Difficulty**: easy

Complete Particle Reach VI First!

Consider a particle that performs a random walk on the integers starting at position 0. At each step, the particle moves from position i to position $i+1$ with probability p, while the probability it moves from i to $i-1$ is $1-p$. If $p = 2/3$, find the expected number of steps until the particle reaches 7. If the answer is infinite, answer -1.

Question 735: Squid Game II

Topic: probability **Difficulty**: medium

$$10$$

contestants are arranged into a line on a bridge and in front of them lay ten left tiles and ten right tiles side by side. In order to cross the bridge, the contestants must cross 10 tiles, and at each step, the person in front must pick either the left or right tile to step on. However, for each left right tile pair, there is exactly one sturdy tile and one faulty tile, but the contestants cannot tell them apart. The contestants cross the bridge in their assigned order with the first person picking either the left or right tile, and continuing to lead unless either a faulty tile is picked (resulting in elimination) or person one reaches the other side. If the first person is eliminated before reaching the other side, the person second in line assumes the lead picking until he/she is eliminated (or reaches the other side), and so on. The winner of the game is the *first* person to reach the other side. Let p_i be the probability that the ith contestant in the line wins. Find $\sup_i p_i$.

Question 736: Limiting Random Variable II

Topic: probability **Difficulty**: easy

Let $X_1, X_2, \ldots$ be a sequence of IID random variables with mean 5 and variance 20. It is also known the third and fourth moments of X_1 are finite. Let

$$Y_n = \frac{X_1^2 + X_2^2 X_3 X_4 + X_5^2 + X_6^2 X_7 X_8 + \cdots + X_{4n-3}^2 + X_{4n-2}^2 X_{4n-1} X_{4n}}{n}$$

Find $\lim_{n\to\infty} \mathrm{Var}(Y_n)$. If this limit does not exist, enter -1.

Question 737: Unique-ish Solution

Topic: pure math **Difficulty**: medium
Let A be a $3 \times n$ matrix for some $n \geq 2$ and $\alpha = \{e_1, e_2, e_3\}$ be the canonical basis of $\mathbb{R}^3$. Suppose that $Ax = e_1$ has no solution and $Ax = e_2$ has a unique solution. Find n.

Question 738: Digit Multiplication II

Topic: brainteasers **Difficulty**: medium
Let A be the set of all integers whose digits multiply to 96. Furthermore, let x and y be the minimal and maximal elements of A, respectively. What is $y - x$? Note that no digit can be 1, so that A is finite.

Question 739: Intersecting Intervals

Topic: probability **Difficulty**: hard
Five intervals are each selected according to the following procedure: two points are sampled from $U[0, 1]$, the larger becoming the right endpoint and the smaller becoming the left endpoint. What is the probability that there exists a point of intersection between all five intervals? The answer can be written as a simplified fraction of the form $\frac{p}{q}$. Find $p + q$.

Question 740: Stranded at Sea

Topic: probability **Difficulty**: easy
You are boating through the Mediterranean Sea when your motor breaks down. You remember learning from your divemaster that the probability of seeing at least one boat within a given hour is 73.7856

Question 741: Regional Manager II

Topic: statistics **Difficulty**: medium
The regional sales manager of a large paper corporation is attempting to detect a difference equal to one deal in the average number of deals closed per week by his employees. To check this, he records the number of deals that 36 random salespeople make on a random week. He finds the variance of the number of deals that the 36 random salespeople made to be 9. He runs a statistical test with $\alpha = 0.05$ to test the null hypothesis $H_0 : \mu = 15$ against his alternative hypothesis $H_a : \mu = 16$. What is the probability of a Type II error? Assume simple random sampling, variance homogeneity, and that the number of deals closed is approximately normally distributed.

Question 742: Random Determinant

Topic: probability **Difficulty**: easy
Let A be a random 2×2 matrix formed by having each element ij set by $A_{ij} \sim \text{Bernoulli}(0.5)$. Find $\mathbb{E}[\det(A - A^T)]$.

Question 743: Triangular Selection II

Topic: probability **Difficulty**: easy
There are 6 points in some space, no three of which lie on the same line. Matt and Aaron each uniformly at random select 3 distinct points each in the space and draw a triangle with those three points as vertices. Note that while Matt can't choose the same point twice among his selections, Aaron and Matt can select the same point. What is the probability they share at least one vertex in common?

Question 744: Colosseum Fight II

Topic: brainteasers **Difficulty**: hard
Alice and Bob are in Roman times and have 4 gladiators each. The strengths of each of Alice's gladiators are $1-4$, while Bob's gladiators have strengths $4, 5, 9$, and 12. The tournament is going to consist of Alice and Bob picking gladiators to fight against one another one-at-a-time. Then, the two gladiators fight to the death with no ties. If the two gladiators are of strengths x and y, respectively, then the probability that the gladiator with strength x wins is $\frac{x}{x+y}$. The winning gladiator will maintain the same strength after the round is over.

Alice is going to pick first for each fight among her remaining gladiators. Afterwards, Bob can select his gladiator (assuming he has one) to go against the one Alice selected. The winner of the tournament is the person who has at least one gladiator left at the end. Does Bob have an optimal strategy S^* such that the probability Bob wins the tournament with strategy S^* is strictly larger than with any other strategy $S \neq S^*$? If so, answer 1. If not, answer 0.

Question 745: Double Dice Payoff

Topic: probability **Difficulty**: easy
You roll two fair dice. If you get double 6s, you receive \$100. If you get a 6 and a non-6, you lose \$$x$. If you get anything else, you reroll both dice until you get double 6s or a 6 and non-6. What is the maximum value of x where the game still has non-negative expected value?

Question 746: Always Profit I

Topic: brainteasers **Difficulty**: easy
QuantGuide stock will either double its value or go down 50% by tomorrow. You can also bet whether the stock goes up or down with your friend at 1 : 1 payout. Let A and B be the lowest round bet (can only bet whole dollars) amount you long/short the stock and bet on the stock going up/down with your friend respectively. Assume that if you are betting, you want to make the same gross profit no matter the outcome of the stockâs movement tomorrow. How much do you profit from betting A and B? If you canât guarantee profit, enter 0.

Question 747: Covariance Review IV

Topic: probability **Difficulty**: easy

$$X_1$$

and X_2 have a correlation coefficient $\rho_{X_1,X_2} = 0.2$. $Y_1 = 1 + 2X_1$, and $Y_2 = 3 - 4X_2$. Compute ρ_{Y_1,Y_2}, the correlation coefficient for Y_1 and Y_2.

Question 748: Option Dice I

Topic: finance **Difficulty**: easy
Pretend you have a simple options chain on the expected value of the product of two dice rolls. What would the call option at 30 be priced at in this market?

Question 749: Soccer Jerseys

Topic: probability **Difficulty**: easy
Liverpool FC sells player jerseys at random prices. The price of a Mohamed Salah jersey is normally distributed with mean \$80 and standard deviation \$15. The price of a Sadio Mane jersey is normally distributed with mean \$70 and standard deviation \$20, independent of Salah jerseys. Gabe wants to be 97.5% sure that he can purchase both jerseys. How much money (in dollars) should Gabe bring with him? For the purposes of this question, take $\Phi^{-1}(0.95) = 1.65$ and $\Phi^{-1}(0.975) = 1.96$

Question 750: Upface Correlation

Topic: probability **Difficulty**: medium
Suppose that you roll a standard fair 6−sided die n times. Let X be the random variable representing the number of times you roll a 1 and Y be the random variable representing the number of times you roll a 5. Calculate $\text{Corr}(X, Y)$.

Question 751: Bullseye

Topic: probability **Difficulty**: easy
Fred throws darts at a dartboard. There is a 10% chance that he hits the bullseye on any given throw, independent between throws. How many throws n must Fred perform to have at least a 90% chance of hitting the bullseye at least once in n throws?

Question 752: Head-Tail Equality

Topic: probability **Difficulty**: hard
Aaron flips a fair coin until he receives an equal amount of heads and tails. For example, HHTHTT would correspond to 6 flips. What is the reciprocal of the expected number of flips Aaron does?

Question 753: Bread Slicer I

Topic: probability **Difficulty**: easy
Gabe has 2 loaves of bread of lengths 5 and 8. He has no control of where he cuts the length 8 bread piece, so he cuts it at a uniformly random position along its length. What is the probability that the loaf of length 5 and the two resulting pieces of bread from the cut form a triangle?

Question 754: Gamma Review I

Topic: probability **Difficulty**: easy
Compute $\dfrac{\Gamma\left(\dfrac{9}{4}\right)}{\Gamma\left(\dfrac{1}{4}\right)}$.

Question 755: Mile Rate

Topic: brainteasers **Difficulty**: easy
A racecar driver drives a one mile track at a constant rate of 30 miles per hour on his first lap. Let x be the constant speed of the racecar driver on the second lap. What value of x should the racecar driver drive at for the second lap if he wants his average speed between the two laps to be 60 miles per hour? If there is no such value of x, enter -1.

Question 756: Simple Delta Hedge II

Topic: finance **Difficulty**: medium
You see that call options have $\Delta = 0.25$ at strike $K = 24$. The underlying has price $S_0 = 21$. You sell 100 put options at strike $K = 24$. You want to delta-hedge your portfolio. How many units of the underlying should you buy/sell? Enter $-x$ if you should sell x units.

Question 757: Empty Boxes

Topic: probability **Difficulty**: medium

$$n$$

balls are being dropped into n bins uniformly at random. Let P_n be the proportion of bins that are empty after this dropping process is done with n balls. Find $\lim_{n\to\infty} \ln(\mathbb{E}[P_n])$. If this limit does not exist, enter 2.

Question 758: Lawn Teamwork

Topic: brainteasers **Difficulty**: easy
There are two lawnmowers: Mr. Rabbit can mow a lawn in 1 hour, Mr. Turtle needs 1 hour and 15 minutes. If they work together, how long does it take to mow one lawn (in minutes)?

Question 759: Couples Reunited

Topic: probability **Difficulty**: easy

$$9$$

married couples (1 male and 1 female) are standing in a line. They all scatter around and form 9 pairs uniformly at random. Find the expected number of couples that are paired up together. Note that pairs can also consist of members of the same gender.

Question 760: Even 7

Topic: probability **Difficulty**: easy
Three fair 6−sided dice are rolled. Given that at least 2 of the dice show an even number, find the probability that the sum is at most 7.

Question 761: Particle Reach III

Topic: probability **Difficulty**: easy

Complete Particle Reach I and II First!

Consider a particle that performs a random walk on the integers starting at position 0. At each step, the particle moves from position i to position $i+1$ with probability p, while the probability it moves from i to $i-1$ is $1-p$. If $p = 1/3$, find the probability the particle ever reaches position 7.

Question 762: All Attainable Values

Topic: probability **Difficulty**: medium
How many $6-$sided dice with values on each side in the set $\{1, 2, 3, 4, 5, 6\}$ are there with the property that when rolled twice, for each integer $2 \leq k \leq 12$, there is positive probability that the sum is exactly k? Note that not every value in the set necessarily needs to be used and that two dice are considered indistinguishable if they contains the exactly same amount of faces corresponding to each value in the set, regardless of the labelling of the sides. Assume each side appears with equal probability.

Question 763: Better in Red III

Topic: probability **Difficulty**: medium
A $3 \times 3 \times 3$ cube is composed of 27 $1 \times 1 \times 1$ cubes that are white by default. All the surfaces of the $3 \times 3 \times 3$ cube are painted red and then the cube is disassembled such that all orientations of all cubes are equally likely. You select a random small cube and notice that all 5 sides that are visible to you are white. What is the probability the last face not visible to you is red?

Question 764: Meeting on a Grid

Topic: probability **Difficulty**: medium
Alan is sitting at $(0, 0)$ and Barbara is sitting at $(4, 4)$. Each minute, Alan moves up or to the right one unit with equal probability, unless he is on the border of the 4×4 grid centered at $(2, 2)$, in which case he moves in the only possible direction that keeps him within range. Barbara behaves similarly, but moves down or to the left instead. What is the probability that the two meet before they reach the opposite corner?

Question 765: Die To Number

Topic: probability **Difficulty**: easy
A fair die is tossed 16 times, and the values are recorded in the order they appear. The values are then treated as individual digits and appended together. What is the probability that the resulting number is divisible by 8?

Question 766: Exponential + Uniform

Topic: probability **Difficulty**: easy
Let $U \sim \text{Unif}(0, 1)$ and $V \sim \text{Exp}(1)$ be independent. Find $\mathbb{P}[U + V > 1]$. The answer is in the form $a - e^b$ for integers a and b. Find ab.

Question 767: Exact Bills I

Topic: brainteasers **Difficulty**: easy
Given the denominations of \$1, \$5, \$20, and \$100 dollar bills, what is the fewest amount of bills needed to form any amount from \$1 to \$100 by taking a subset of the bills? In other words, how many bills do you need to form any number from \$1 to \$100 where you can reuse bills?

Question 768: Uniform Movement

Topic: probability **Difficulty**: easy
Let $X, Y \sim \text{Unif}(3, 4)$ IID. Find $\mathbb{E}[|X - Y|]$.

Question 769: Unlucky Seven I

Topic: probability **Difficulty**: medium
You are given a fair $6-$sided die and you roll it. You can either choose to keep your roll and receive the observed value in dollars. Alternatively, you are allowed to roll again, but if the sum of your two rolls is at least 7, you pay the value equal to your first roll. If the sum of the two rolls is less than 7, you receive the second observed value in dollars. Assuming optimal play, what is your expected payout?

Question 770: Ramanujan's Run

Topic: probability **Difficulty**: medium
Ramanujan is running late to teach his lecture. He must run a mile across Cambridge to the lecture hall within ten minutes in order to get to his students on time. Beginning with attempt $n = 1$, he runs a distance of x_n miles in t_n

minutes towards the lecture hall and repeats the process if he has (1) not yet arrived at the hall and (2) still has time to spare; the maximum value of n is 3. The values of x_n are chosen independently and uniformly at random between 0 and 1 miles, while the values of t_n are chosen independently and uniformly at random between 0 and 10 minutes. If Ramanujan runs out of time during his j-th attempt, then time magically stops, allowing him to finish traveling the remainder of the x_j miles. If Ramanujan runs equal to or more than a mile total within the ten minutes, then he arrives at the hall on time. What is the probability that Ramanujan gets to the lecture hall on time?

Question 771: Placing Dots

Topic: probability **Difficulty**: easy
You place three dots along four edges of a square at random. What is the probability that the dots lie on distinct edges?

Question 772: Sum Exceedance I

Topic: probability **Difficulty**: hard
Let $X_1, X_2, \ldots$ be IID Unif$(0,1)$ random variables and let $N = \min\{n : X_1 + \cdots + X_n > \ln(2)\}$. Find $\mathbb{E}[N]$.

Question 773: Limited Urns

Topic: probability **Difficulty**: hard
Suppose that there are n identical urns each containing white and black balls. The ith urn, where $1 \leq i \leq n$, contains 1 white ball and $2^i - 1$ black balls. You randomly select an urn and then draw one ball at random from it. The ball is white. Let $p(n)$ be the probability that if you replace this ball and again draw a ball at random from the same urn then the ball drawn on the second occasion is also white. Compute $\lim\limits_{n\to\infty} p(n)$.

Question 774: Binary Zeroes

Topic: brainteasers **Difficulty**: medium
In the binary expansion of 142!, how many trailing zeroes are there?

Question 775: Base Exponent Square

Topic: brainteasers **Difficulty**: medium
How many integers $100 \leq k \leq 400$ are such that k^k is a perfect square?

Question 776: Three Sides

Topic: probability **Difficulty**: medium
On average, how many tosses of a fair coin does it take until three heads or three tails are observed, consecutive or otherwise?

Question 777: Party Groups II

Topic: probability **Difficulty**: medium
There are 50 guests at a party and they are making groups for a game. To do this they each write their name on a piece of paper and put it into a hat. One by one, each guest picks a name from the hat. Each guest will be a part of a group with the guest they pulled the name out of from the hat. If a guest pulls out their own name, they are in a group all by themselves. If Guest A pulls out Guest B's name, Guest B pulls out Guest C's name, and Guest C pulls out Guest A's name, they are all a part of the same closed group and no one else will be able to join them. What is the average size of a group? Round your answer to the nearest tenth.

Question 778: Sequence Terminator

Topic: probability **Difficulty**: hard
A fair $6-$sided die is rolled repetitively, forming a sequence of values, under the following rules: If any even value is rolled, add it to the current sequence. If a 3 or 5 is rolled, discard the entire sequence and don't add the 3 or 5 to the start of the new sequence. If a 1 is rolled, add the 1 to the current sequence and end the game. Find the expected length of the sequence at the end of the game.

For example if the sequence of rolls is 34265241, we would reset to an empty sequence at the first roll, reset to an empty sequence at the 5, and our final sequence would be 241, which is of length 3.

Question 779: 1 Head Up

Topic: probability **Difficulty**: easy

$$3$$

fair coins are flipped, and you are told that at least one of the coins showed heads. Find the probability exactly one coin showed heads.

Question 780: ATM Option Pricing

Topic: finance **Difficulty**: medium
You have an at-the-money call option with maturity $T = 1000$ years of a stock S with initial value 190. What is the price of this option at time-0 assuming Black-Scholes dynamics? Do not use a calculator.

Question 781: Surface Rotation

Topic: pure math **Difficulty**: medium
Let $0 < r < 1$ be fixed. Consider the curve $y = \sqrt{r^2 - x^2}$ over the interval $r^2 \le x \le r$. Let S_r be the surface area of the surface obtained by rotating this curve above the $x-$axis. Define $S = \sup_{0<r<1} S_r$. S can be written as $q\pi$ for a rational number q. Find q.

Question 782: Golden Pancakes

Topic: probability **Difficulty**: easy
You have three pancakes in front of you. One is toasted on both sides, one is toasted on one side, and one is not toasted at all. You hungrily choose a pancake at random and notice the top is toasted. What is the probability that the other side of your pancake is also toasted?

Question 783: Cube Colorer

Topic: probability **Difficulty**: easy
Dyann paints the outer faces of a $5 \times 5 \times 5$ cube green and then cuts this large cube up into 125 $1 \times 1 \times 1$ cubes. One of the cubes is then uniformly at random selected and rolled. Find the probability that this cube shows a green face up.

Question 784: Significant Others

Topic: statistics **Difficulty**: hard
Suppose that we have a set of data $\{(x_1, y_1), \ldots, (x_n, y_n)\}$, where both x_i and y_i are real-valued and mean 0. We fit a linear regression line without intercept to regress y onto x. This yields a line $y = \hat{\beta}_0 x + \varepsilon$, ε being an error term. Given that $\mathrm{Corr}(X, Y) = \frac{1}{10\sqrt{3}}$, find the smallest value of n so that the results of the hypothesis test $H_0 : \beta_0 = 0$ vs. $H_A : \beta_0 > 0$ are significant at the 95% level. You may use the fact that the 95 and 97.5 percentile values are approximately 1.645 and 1.96.

Question 785: Ant Collision I

Topic: brainteasers **Difficulty**: easy
Alice and Bob raised their own ants. Alice has 40 ants, while Bob has 80 ants. Alice and Bob stand at opposite ends of an infinitesimally wide string and let all of their ants crawl on the string. Whenever one ant runs into another ant, they both bounce back and change direction. Each of the ants can only move forward. Let x be the number of ants that reach Alice and y be the number of ants that reach Bob. Find $x - y$.

Question 786: Bond Practice VI

Topic: finance **Difficulty**: easy
Calculate the price of a bond with these characteristics. The coupon rate is 0.06, coupon payments are made every six months (twice per year), and the par value of the bond is 1,000. There are 20.0 years to maturity and a market interest rate of 0.09

Question 787: Breakeven Price II

Topic: finance **Difficulty**: easy
You write a call option for \$2.50 at a strike price of \$12. What is the breakeven price?

Question 788: Covariance Review V

Topic: probability **Difficulty**: easy
Consider the following joint pdf:

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} c(1-x_2) & 0 \le x_1 \le x_2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

where c is a constant such that f_{X_1,X_2} is a valid joint pdf. Compute $\mathrm{Var}(2X_1 - 4X_2)$.

Question 789: Repeated Flipper

Topic: brainteasers **Difficulty**: medium

$$25$$

fair coins are lined up and flipped once. Afterwards, remove all of the coins that showed tails, and repeat the aforementioned process until you have no coins left or all of the coins show up heads. Find the probability that you end the game by having all of the coins show heads.

Question 790: Lucky Chuck

Topic: probability **Difficulty**: easy
You are playing a game in which you may bet on any number 1 through 6. Three dice are then rolled. If the selected number appears on one, two, or three of the dice, you receive respectively one, two, or three times your original bet plus your money back; else you lose your bet. What is your expected payoff of playing this game with a bet size of \$20.

Question 791: Pulling Cards in Order

Topic: probability **Difficulty**: easy
You have a 100 card deck in front of you containing the numbers from 1 to 100. You randomly pick one card after the other 4 times (without replacement). What is the probability that the cards you picked out are in ascending order?

Question 792: Doubly 5 I

Topic: probability **Difficulty**: hard
Jenny has a fair $6-$sided die with numbers $1-6$ on the sides. Jenny continually rolls the die and keeps track of the outcomes in the order they appear. Jenny rolls until she sees both 4 and 6. Find the probability Jenny observed exactly 2 5s while rolling her die.

Question 793: Independent Children

Topic: probability **Difficulty**: easy
Consider a family of n children with $n \geq 2$. Let A be the event that the family has at least one child of each gender and B be the event that there is at most one girl in the family. Assume each child is equally likely to be born a boy or girl. Find the unique value of n such that A and B are independent.

Question 794: Distinct Biased Coins

Topic: probability **Difficulty**: easy
Suppose we have a bag of distinct biased coins such that a randomly chosen coinâs probability of heads has density $f(t) = 2t$ on $(0, 1)$. You choose a coin at random, toss it 6 times, and win if you get 6 heads. What is your probability of winning?

Question 795: Exact Bills II

Topic: brainteasers **Difficulty**: easy
What is the fewest amount of bills (denominations of \$1, \$5, \$20, and \$100) you need to always have exact change for a transaction of \$$x$ (paid using a \$100 bill) given that x is a positive integer less than 100?

Question 796: Double Data Trouble I

Topic: statistics **Difficulty**: easy
Suppose that you run OLS regression on some dataset and obtain an R^2 value of 0.56. If you were to run linear regression again on the dataset where you duplicate each point in your original dataset, what would R^2 be? If it cannot be determined, enter -1.

Question 797: Bowl Distributions

Topic: probability **Difficulty**: medium
How many ways can 7 white balls and 8 black balls be distributed into 5 bowls such that each bowl receives exactly 3 balls?

Question 798: Brownian Difference

Topic: pure math **Difficulty**: medium
Let B_t be a standard Brownian Motion. Fix $t > 0$, and define $\Delta_{m,n} = B_{tm2^{-n}} - B_{t(m-1)2^{-n}}$. Evaluate $\mathbb{E}\left[\left(\sum_{m=1}^{2^n} \Delta_{m,n}^2 - t\right)^2\right]$ as a function of n and t. Evaluate this function with $t = 1$ and $n = 5$.

Question 799: Optimal Bidders II

Topic: probability **Difficulty**: medium
Carter has come into contact with a bounty of gold. He takes it to an auction shop. The auction shop says that each person that bids will place a bid that is uniformly distributed between \$500 and \$1000. They also state that they can recruit people to bid for a price of \$10 per person. However, the auction shop also states that Carter will only receive the payment of the second highest bid. Assuming Carter selects the optimal number of people to bid on his gold, what is his expected payout?

Question 800: Poisoned Kegs III

Topic: brainteasers **Difficulty**: medium
A king has 10 servants that bravely risk their lives to test whether or not the wine in n kegs is poisonous. It is known that exactly one of the n kegs is poisonous. If someone drinks any amount of liquor from the poisoned keg, they will die in exactly 1 month. Otherwise, the servant will be fine. The servants only agree to participate in the wine tasting for 1 month. What is the maximum value of n such that the king is guaranteed to have at least a 10% chance to determine which keg among the n is poisoned?

Question 801: Integer Polygon

Topic: brainteasers **Difficulty**: easy
How many values of n are there such that a regular $n-$gon has interior angles that (in degrees) are integer-valued?

Question 802: Binary Option

Topic: finance **Difficulty**: medium
Consider an underlying with $S_0 = 25$. We have the following derivatives contract: this contract pays 1 if $S_T \geq 24$ and -1 otherwise. We have access to the following binary calls (a contract paying 1 if $S_T \geq K$ and 0 otherwise). The calls are given in the format of (Strike K, Price C_0)

$$(25, 0.6)$$
$$(24, 0.73)$$
$$(23, 0.88)$$

Give the time-0 price of the derivatives contract. Bonds pay 1 at time-T and have time-0 price $B_0 = 0.9$.

Question 803: Even Blocks

Topic: probability **Difficulty**: medium
Hannah has 12 blocks, 2 each of 6 distinct colors. She randomly arranges the blocks in a straight line. What is the probability that there are an even number of blocks between every identically colored pair of blocks?

Question 804: Planets Aligned

Topic: brainteasers **Difficulty**: easy
There is a solar system with three planets orbiting around the sun. One of them

has a translation period of 40 years, another one of 60 years and another one of 90 years. Today, the three planets are aligned with the sun. When is the next time, in years, the three planets will be aligned with the sun?

Question 805: Coin Pair III

Topic: probability **Difficulty**: medium
Four fair coins appear in front of you. You flip all four at once and observe the outcomes of the coins. After seeing the outcomes, you may flip any pair of tails again. You may not flip a single coin without flipping another. You can iterate this process as many times as there are at least two tails to flip. Find the expected number of heads that we end up with after this process is complete.

Question 806: Too Many Primes to Count

Topic: probability **Difficulty**: easy
We are playing a game with two players. The first player rolls a $6-$sided die 10000 times and the second player 10001 times. What are the odds of the second player getting more prime numbers than the first one?

Question 807: Infected Dinner II

Topic: brainteasers **Difficulty**: hard
There are 1000 people having dinner at a grand hall. One of them is known to be sick, while the other 999 are healthy. Each minute, each person talks to one other person in the room at random. However, as everyone is social, nobody talks to people they have previously talked to. In each pair, if one is sick and one is healthy, the healthy person is infected and becomes sick. Once a person becomes sick, they are assumed to be sick for the rest of the dinner. Find the maximum amount of time (in minutes) until every person in the hall becomes sick.

Question 808: Exponential Brownian Motion

Topic: pure math **Difficulty**: easy
Let W_t be a standard Brownian Motion. Find the sum of the factorials of the values of n where $X_t = W_t^n$ is a martingale.

Question 809: Sheep Stealing

Topic: brainteasers **Difficulty**: easy
Some sheep stealers made a raid and carried off one-third of the flock of sheep,

and one-third of a sheep. Another party stole one-fourth of what remained, and one fourth of a sheep. Then, a third party of raiders carried off one-fifth of the remainder and three-fifths of a sheep, leaving 409 behind. What was the number of sheep in the flock?

Note: The question is intentionally worded hard to understand to test your ability to parse through and interpret information.

Question 810: Make Your Martingale VI

Topic: pure math **Difficulty**: medium
Let $N(t)$ be a Poisson Process with rate $\lambda > 0$ and fix $c > 0$. Consider the process $S(t) = e^{N(t)\log(c) - \lambda t}$. Find the constant c such that $S(t)$ is a martingale.

Question 811: Bags of Fruit

Topic: brainteasers **Difficulty**: easy
You are given three bags of fruits. One has apples in it; one has oranges in it; one has a mix of apples and oranges. Each bag has a label on it ("apple", "orange", or "mix"). Unfortunately, your teacher tells you that all bags are mislabeled. In order to identify the bags correctly, you develop a strategic way of picking fruits from the mislabeled bags. What is the minimum number of fruits that we need to pick from the bags to correctly identify the bags?

Question 812: Minimal Shade

Topic: brainteasers **Difficulty**: hard
Consider an array of white unit squares arranged in a rectangular grid with r rows of unit squares and c columns of unit squares. What is the smallest possible value of c such that, if we shade s unit squares in each column black, then there must necessarily be some row with at least t black unit squares? You should receive a function $f(r, t, s)$. Evaluate $f(30, 21, 14)$.

Question 813: Beer Barrel II

Topic: brainteasers **Difficulty**: medium
A 120-quart beer barrel was discovered by Anna's parents. Furious, they plan to dump the barrel. Anna begs her parents to let her keep some of the beer. They say that Anna may do so if she is able to measure out an exact quart into each of a 7-quart and 5-quart vessel. You have exactly 1 of each type of vessel. Note that Anna is allowed to pour beer from a vessel back into the beer barrel.

Define a transaction as a pour of liquid from one container into another. What is the smallest number of transactions needed to accomplish the challenge? If impossible, respond with -1.

Question 814: Show The Face

Topic: probability **Difficulty**: medium
Roll a fair standard 6−sided die until a 6 appears. Given that the first 6 occurs before the first 5, find the expected number of times the die was rolled.

Question 815: Cyclic 4

Topic: brainteasers **Difficulty**: hard
What is the smallest positive integer that ends with the digit 4 such that moving its last digit to the first position (i.e. 1234 =¿ 4123) multiplies the original integer by exactly 4?

Question 816: Perfect Correlation I

Topic: probability **Difficulty**: medium
Suppose that X and Y are perfectly correlated i.e. $\rho(X, Y) = 1$ with $\sigma_Y > \sigma_X$. Find the correlation of $X + Y$ and $X - Y$.

Question 817: Defining Variance

Topic: statistics **Difficulty**: easy
Let $X \sim N(2, 4)$ and $Y \sim N(1, 4)$ be independent. Furthermore, let $Z = 3X - 2Y$. What is the variance of Z?

Question 818: Whole Lotta Dice

Topic: probability **Difficulty**: medium
We have 100 fair 100−sided dice in front of us and we toss them all at the same time. What is the probability that the resulting sum of the faces is divisible by 100?

Question 819: Fixed Point Limit II

Topic: probability **Difficulty**: medium
Let F_n be the number of fixed points of a random permutation $f : S_n \to S_n$,

where $S_n = \{1, 2, \ldots, n\}$. Find $\lim_{n\to\infty} \mathbb{P}[F_n = 5]$. The answer is in the form $\frac{c}{e}$, where e is Euler's constant and c is a constant. Find c.

Question 820: Fair Bounty I

Topic: probability **Difficulty**: easy
You're on a game show! The host tells you that \$200 is located behind one of 7 doors. The other 6 doors have nothing behind them. You have no idea which door the money is behind. You keep selecting randomly until you find the money. You do not select any doors that you have selected prior. If each door opening costs \$$x$ and the game is fair, what is x?

Question 821: OLS Review II

Topic: statistics **Difficulty**: easy
Use the method of least squares to fit a straight line to the following dataset of ordered pairs (x, y): $(-2, 0), (-1, 0), (0, 1), (1, 1), (2, 3)$. Find the sum of the slope and the intercept.

Question 822: Absolute Normal Difference

Topic: probability **Difficulty**: medium
Suppose that $X \sim N(0, 1)$ and $Y \sim N(0, 4)$ are independent. Compute $\mathbb{E}[|Y - X|]$. The answer will be in the form $\left(\frac{K}{\pi}\right)^b$ for rational b and K. Find bK.

Question 823: Meek Mill

Topic: probability **Difficulty**: easy
Meek Mill is performing at Spring Fair. The administration does not know where he is, and they estimate there is an 80% chance he is in Philadelphia and 20% chance he is in Baltimore. If he is in Baltimore, there is a 80% chance he will perform. If he is in Philadelphia, there is a 10% chance he will perform. Find the probability that if Meek Mill performs, he came from Philadelphia.

Question 824: Likely Targets III

Topic: probability **Difficulty**: hard
Three linear targets, say A, B, and C, of respective radii $\varepsilon, 2\varepsilon$, and 3ε, where $\varepsilon << 1$, are placed on an infinitely long line. The targets are centered at $x_A = -1, x_B = 3$, and $x_C = 5$. In other words, target A covers the interval $[1 - \varepsilon, 1 + \varepsilon]$, target B covers the interval $[3 - 2\varepsilon, 3 + 2\varepsilon]$, and target C covers

the interval $[5-3\varepsilon, 5+3\varepsilon]$. You have one dart to shoot at the line. Your goal is to maximize your probability of hitting one of the targets. You can choose where to center your throw on the line. If you select to center your dart at μ, the actual position your dart lands at is $X \sim N(\mu, 4)$. Find the value of μ that maximizes your chances of hitting a target. If necessary, round your answer to the nearest hundredth.

Question 825: Lots of LOTUS

Topic: statistics **Difficulty**: medium

Suppose that $X \sim N(0,1)$ and define $Z = e^{\theta X - \frac{1}{2}\theta^2}$. What is $\mathbb{E}[Z]$?

Question 826: Minimax Box

Topic: probability **Difficulty**: hard
You have 2 boxes and 100 distinct cards numbered $1-100$. At each turn, you deal a card from the top of the deck and place the card in a box uniformly at random. What is the expected value of the smallest numbered card in the box that has card 100 in it? The answer is in the form $a(1-a^{-b})$ for integers a and b. Find ab.

Question 827: Lock and Key Pair

Topic: brainteasers **Difficulty**: easy
There are 5 keys that unlock exactly one of 5 locks. Assuming an optimal strategy, what is the maximum number of times you need to try the locks to identify which key unlocks each lock?

Question 828: Maximize Head Ratio II

Topic: probability **Difficulty**: hard
Say that you are flipping a fair coin where you can stop flipping whenever you want. Your goal is to maximize the ratio between the number of heads you get with the total number of flips. The ratios are in the form $a:1$. What is the expectation of a when you follow the strategy of stopping when you have more heads than tails? Round your answer to the nearest thousandths.

Question 829: Mean Cards

Topic: probability **Difficulty**: easy
Suppose you have a deck of 100 cards labeled with values $1-100$. The deck

is thoroughly shuffled and cards are dealt one-by-one from the top of the deck. For every card that is dealt, you may either keep it or discard it. Discarding a card costs \$1. At the end, you are paid out the average value of all the cards you keep. What is the optimal strategy and expected profit on this game?

Question 830: Mixing Wine

Topic: brainteasers **Difficulty**: easy
There are two glasses: the first one is one-third full of wine, and the second one, which has the same capacity, is one-fourth full of wine. Both glasses are then filled with water, and their contents are combined in a jug. Subsequently, half of this mixture is poured back into one of the glasses. What proportion of the mixture in the glass consists of wine?

Question 831: Random Minimal Sum

Topic: probability **Difficulty**: hard
Let $X_1, X_2, \cdots \sim \text{Unif}(0,1)$ IID. Consider the summation $S = \sum_{i=1}^{N} \frac{X_i}{2^i}$, where N is the first index k where $X_k < X_{k+1}$. If no such index exists, then $k = \infty$. Compute $\mathbb{E}[S]$. The answer will be in the form $ae^b + c$, where a and c are integers and b is a rational number in fully reduced form. Find $a^2 + c^2 + 4b$.

Question 832: Paying Bail

Topic: probability **Difficulty**: medium
You have been arrested and need to post bail. You have a bag of \$100 bills of which one half are real and the other half are fake. You need to pay \$500 for bail. Every time you give the jail a fake bill, the funds you have given are given back to you and you need to start again. How many bills will you have to pull out of the bag on average? Assume the probability of drawing each type of bill is constant per trial.

Question 833: Dual Die View

Topic: probability **Difficulty**: medium
You have two fair $6-$sided dice. At each round, you roll both dice and record the upfaces. Find the expected number of rounds you need to perform to observe all 6 faces of the dice.

Question 834: Position Guess

Topic: probability **Difficulty**: medium
You are given 3 IID Unif$(0, 1)$ random variables but can't see the values. One of the values is revealed to you at random among the three. You want to determine if the revealed value is the minimum, median, or maximum of the three values. Assuming you guess optimally, what is your probability of being correct?

Question 835: Child Births

Topic: statistics **Difficulty**: easy
We suppose that the number of children that a family has in the USA is $N \sim \text{Poisson}(\lambda)$ for some λ. We prescribe a prior distribution of $\lambda \sim \text{Gamma}(32, 10)$ to the distribution, where we use the Shape-Rate parameterization. We then observe 34 families who have a total of 100 kids combined. Find the posterior mean for λ.

Question 836: Broken Trading System

Topic: finance **Difficulty**: easy
We want to price an options contract. Our trading system is broken and isn't telling us the expected price. However, we are given the following details. How much is the option?

$$\begin{aligned}
\Theta &= -1.9 \\
\Delta &= 0.3 \\
\Gamma &= 0.03 \\
r &= 0.03 \\
\sigma^2 &= 0.2 \\
S &= 25
\end{aligned}$$

Give the answer to 2 decimal points.

Question 837: Close Couples

Topic: probability **Difficulty**: medium

$$n$$

married couples (n husbands and n wives) sit at a round table of $2n$ seats at random. The seating scheme is such that the seats must alternate between men and women. Find the number of couples that will sit together $\mathbb{E}[N]$ on average. Assume $n > 2$.

Question 838: Dangerous Doubles

Topic: probability **Difficulty**: easy
Two fair coins are flipped at once. You receive \$2 if exactly one heads that appears, but you lose \$7 if you flip two heads. What is your expected profit/loss on this game if you play once?

Question 839: Fair Bounty II

Topic: probability **Difficulty**: medium
You're on a game show! The host tells you that \$200 is located behind 2 of 7 doors. The other 5 doors have nothing behind them. You have no idea which doors the money is behind. You keep selecting randomly until you find the first door with money behind it. You do not select any doors that you have selected prior. If each door opening costs $\$x$ and the game is fair, what is x?

Question 840: Non-Disjoint Subsets

Topic: probability **Difficulty**: hard
Let A and B be uniformly at random selected subsets of $\{1,2,3,4,5\}$. Find the probability that A and B are not disjoint.

Question 841: Uniform Triangle

Topic: probability **Difficulty**: medium
Let $X, Y, Z \sim \text{Unif}(0,1)$ IID. Find the probability that a triangle of side lengths $X, Y,$ and Z can be formed.

Question 842: First Flip

Topic: probability **Difficulty**: hard
Jay and John each flip fair coins until they obtain their first heads, respectively. Given that it takes strictly fewer flips for Jay to get his first heads than John, compute the expected number of flips Jay performed.

Question 843: Josephus' Dilemma

Topic: brainteasers **Difficulty**: hard

$$2000$$

soldiers get captured in war, including yourself. Rather than feeding information to the opposing side, you and your fellow soldiers decide it is honorable to kill one another until there is one person standing and they can kill themselves. However, you know that if you are the last person still alive, thereâs a good chance that you can escape and be free. The process in killing other soldiers is as follows: all soldiers stand in a circle and Person 1 starts with a sword. Person 1 kills Person 2 and gives the next soldier the sword (in this case Person 3). This process continues going around the circle till thereâs one person left. What position in this circle should you stand to be the last person alive?

Question 844: Largest Inscribed Circle

Topic: pure math **Difficulty**: medium
What is the area of the largest circle that can be inscribed inside a triangle with side lengths of 16, 16, and 24? The answer can be written in the form $\frac{p\pi}{q}$, where $\frac{p}{q}$ is a simplified fraction. Find $p+q$.

Question 845: Dot Removal

Topic: probability **Difficulty**: easy
A fair $6-$sided die has the values $1-6$ on the sides. The values of one of the sides (selected uniformly at random) is decreased by 1. Find the probability an even value is rolled on this die.

Question 846: Clockwise Murder

Topic: brainteasers **Difficulty**: hard
There are N people standing in a circle labeled 1 through N, such that the smallest power of two less than or equal to N is X. Starting with the person 1, they take a sword and kill the person to their left. Then, they pass the sword to the living person on their left, and the process continues until one person is left. The survivor's number can be expressed as $aN+bX+c$. Compute $a+2b+3c$.

Question 847: Coloring Components I

Topic: probability **Difficulty**: medium
Consider a line of 25 adjacent colorless squares. Color in each individual square black or white with equal probability, independent of all other squares. A connected component is a maximal sequence of adjacent squares all with the same color. For example, BBWBWWWBBW has 6 connected components. Find the expected number of connected components in our line.

Question 848: Bank Arbitrage

Topic: finance **Difficulty**: easy
You have a bond and bank account that pay 5% and 3% compounded annually. You are allowed to buy or borrow the bond (needing to repay it back later). Similarly, you are able to borrow money from the bank (needing to pay the principal and interest at the end) or deposit money into the bank. There is an arbitrage opportunity. How much profit will you receive at the end of 1 year if you have $200 total? Round to the nearest cent.

Question 849: Grid Filling I

Topic: probability **Difficulty**: medium
The integers 1 through 9 are randomly placed into the 9 squares of a 3 x 3 grid such that each square has one integer and each integer is used once. What is the probability that the sum of each row and each column is even?

Question 850: Midrange

Topic: probability **Difficulty**: medium
Suppose $X_1, X_2 \sim \text{Exp}(1)$ IID. Let O_1 and O_2 be the minimum and maximum order statistics, respectively. Find the probability that the range of X_1 and X_2 is larger than the midpoint of X_1 and X_2.

Question 851: Maximize Head Ratio I

Topic: brainteasers **Difficulty**: easy
Say that you are flipping a fair coin where you can stop flipping whenever you want. Your goal is to maximize the ratio between the number of heads you get with the total number of flips. The ratios are in the form $a : 1$. What is the expectation of âaâ when your strategy is to target a ratio of 0.5 : 1 in a long run of flips but if its greater than 0.5 : 1, you immediately stop there.

Question 852: Poisson Review III

Topic: probability **Difficulty**: easy
The number of defects per foot of rope, X, satisfies $X \sim \text{Poisson}(2)$. The profit per foot of rope is $50 - 2X - X^2$. Compute the expected profit per foot of rope.

Question 853: Picky Primes

Topic: probability **Difficulty**: easy
4 distinct integers are drawn from the set of the first 16 positive prime integers. Find the probability that the sum is even.

Question 854: OLS Review I

Topic: statistics **Difficulty**: easy
Consider the following ordered pairs (x, y): $(-2, 3), (-1, 2), (0, 1), (1, 1), (2, 0.5)$. Compute the sum of the least-squares estimators for the parameters β_0, β_1 in the model $Y = \beta_0 + \beta_1 x + \epsilon$, where $\mathbb{E}[\epsilon] = 0$.

Question 855: Square Billy

Topic: probability **Difficulty**: easy
Billy is at $(1, 2)$ and begins a 2D symmetric random walk (each turn, Billy has a $\frac{1}{4}$ chance each of going up, down, right, or left). There is a square with vertices $(0, 0), (0, 4), (4, 4), (4, 0)$. Billy ends his random walk when he hits the square. What is the probability that Billy ends up on a vertical side of the square?

Question 856: Basic Put Delta

Topic: finance **Difficulty**: easy
What is the Δ of a deep in-the-money put?

Question 857: 29 Divide

Topic: brainteasers **Difficulty**: hard
Which of the digits from $0 - 9$ (inclusive of both) does 2^{29} not have? You may use the fact that 2^{29} has 9 digits.

Question 858: Options Rho II

Topic: finance **Difficulty**: easy
If interest rates increase, how does this affect the price of a European put option?

Enter 1 if it increases, 0 if no change, or -1 if it decreases.

Question 859: Points on a Circle I

Topic: probability **Difficulty**: medium
Three points are selected randomly at uniform around a circle. What is the probability that the three points form an obtuse triangle?

Question 860: Longest Rope I

Topic: statistics **Difficulty**: hard
A rope that is one meter long is divided into three segments by two random points. What is the expected length of the longest segment?

Question 861: Bet Size I

Topic: finance **Difficulty**: easy
Say you are betting on a game where you win 75% where you get paid $1:1$ on it. Assuming optimal strategy, what percentage of your bankroll should you bet on this game?

Question 862: Splitwise

Topic: brainteasers **Difficulty**: easy
You and seven friends go out for lunch. The subtotal is 182.30. You all decide to add a generous 60

Question 863: Fresh Fruits

Topic: brainteasers **Difficulty**: medium
Some fresh fruit was weighted for some domestic purpose. It was found that the apples, pears, and plums exactly balanced each other as follows: One pear and three apples weight the same as 10 plums; and one apple and six plums weigh the same as one pear.

How many plums alone would weigh the same as one pear?

Question 864: Integral Limit

Topic: pure math **Difficulty**: hard
Fix $\varepsilon > 0$. Compute $\lim_{n\to\infty} n \int_1^{1+\varepsilon} \frac{1}{1+x^n} dx$. Your answer will be in the form $\ln(k)$ for a constant k. Find k.

Question 865: Mixed Set I

Topic: probability **Difficulty**: easy
How many subsets of $\{1, 2, \ldots, 10\}$ contain exactly 2 elements from $\{1, 2, 3\}$?

Question 866: Colorful Apples

Topic: probability **Difficulty**: medium
There are 4 green and 50 red apples in a basket. They are removed one-by-one, without replacement, until all 4 green ones are extracted. What is the expected number of apples that will be left in the basket?

Question 867: Clarence's Bread

Topic: probability **Difficulty**: hard
Clarence is getting this bread. Literally. Clarence has 7 small loaves and 3 large loaves of bread in a bag. He draws in his bag and pulls out a loaf of bread uniformly at random, one-by-one. If it is a small loaf, Clarence eats it. If it is a large loaf, Clarence cuts it into two small loaves. He eats one one of them and puts the other one back in the bag. Find the expected number of small loaves that Clarence eats until the bag contains only small loaves left, inclusive of the small loaf he eats as a result of the last large loaf he draws.

Question 868: Normal LOTUS I

Topic: probability **Difficulty**: medium
Let X and Y be jointly continuous random variables with joint PDF given by $f(x, y) = xe^{-x(y+1)} I_{(0,\infty)}(x) I_{(0,\infty)}(y)$. Evaluate $\mathbb{E}\left[\dfrac{X}{Y+1}\right]$.

Question 869: Coordinate Jumper

Topic: probability **Difficulty**: easy
How many paths are there from $(0, 0, 0)$ to $(3, 4, 5)$ in 3D space if we move only right, forward or up one unit at each step?

Question 870: Good Accuracy

Topic: probability **Difficulty**: easy
A dartboard consists of 3 concentric circular regions of radii $1, 2,$ and 3. If you shoot 3 darts that land in a uniformly random spot on the dartboard, find the probability that each lands in a distinct region.

Question 871: Make a Market II

Topic: finance **Difficulty**: medium
Let's say you have product A in which you are quoting 4 @ 5 and product B in which you are quoting 10 @ 12. X @ Y means that X is our bid and Y is our ask. We want to make a market on the product $A - B$. What is the bid-ask spread you will quote? Give the answer in the format of $Y^2 - X^2$

Question 872: Straddle Arbitrage II

Topic: finance **Difficulty**: medium
You have 2 straddles on stock S, V_1, which has a strike of $K = 5$ and the other, V_2, which has a strike of $K = 8$. What is the smallest amount of money you must receive as a credit for there to be an arbitrage? Assume that the straddles themselves must have non-negative prices.

Question 873: Real Solutions

Topic: probability **Difficulty**: hard
Let M and N be IID Unif$(0,1)$ random variables. Let $p(c)$ be the probability that $Mx^3 + Nx = c$ has a real solution in the interval $(0,1)$, where $0 < c < 2$. $p(c)$ can be written in the form

$$1 - \left[(ac^2 + bc + d)I_{(0,1)}(c) + (rc^2 + sc + t)I_{(1,2)}(c)\right]$$

where $I_{(a,b)}(x) = 1$ if $x \in (a,b)$ and 0 otherwise. Furthermore, all of a, b, d, r, s, t are rational numbers. Find $a + b + d + r + s + t$.

Question 874: Equalizer

Topic: probability **Difficulty**: easy
Dave and Mike each have a fair 6−sided die. They both roll at once. If they roll the same value, Mike wins. If Mike rolls a strictly larger value than Dave, then Dave wins. If Dave rolls a strictly larger value than Mike, then they both re-roll until one of the other two outcomes occurs. Find the probability that Mike wins this game.

Question 875: Russian Roulette II

Topic: probability **Difficulty**: medium
You're playing a game of Russian Roulette with a friend. The six-chambered revolver is loaded with one bullet. Initially, the cylinder is spun to randomize the order of the chambers. The two of you take turns pulling the trigger until

the person who fires the gun loses. Given that the cylinder is re-spun after each turn, what is the probability that you win if your friend goes first?

Question 876: Contracts and Pricing I

Topic: probability **Difficulty**: easy
QuantGuide has a 20% chance of being acquired between Wednesday evening and Thursday morning. One share of its stock is worth \$30 on Wednesday. If QuantGuide is acquired, then there is a 50% chance that one share of its stock is worth \$50 and a 50% chance that one share of its stock is worth $\$x$. Otherwise, there is a 60% chance that one share of its stock is worth \$0 and a 40% chance that one share of its stock is worth \$10. A fair contract that awards the buyer the right but not the obligation to buy the stock at \$40 on Thursday is worth \$15. Compute x.

Question 877: Basketball Practice

Topic: probability **Difficulty**: medium
Frank is shooting free throws. He makes his first free throw and misses his second free throw. For $n \geq 3$, the probability of making the nth free throw is equal to the proportion of free throws he made during his first $n-1$ attempts. What is the probability that Frank makes exactly 50 free throws in 100 attempts?

Question 878: Game Show I

Topic: probability **Difficulty**: medium
You're on a game show and are given the choice of three doors to choose from. Behind one door is a car and behind the other two are goats. You pick one of the doors at random and announce your choice to the game show host. The game show host, knowing which prize is behind each door, opens a door that you did not choose and shows a goat after hearing your initial choice. He offers you the opportunity to either keep your original door or switch to the other closed door. What is the probability that you win the car if you switch?

Question 879: No Rock

Topic: probability **Difficulty**: easy
You play rock, paper, scissors with an opponent, but your opponent cannot play rock. Every time you win, you receive \$1, every time you lose, you lose \$1, and every time you draw, nothing happens. Assuming optimal play, what is your expected profit per round?

Question 880: Stages of Life

Topic: brainteasers **Difficulty**: easy
An old man is looking back on his life. He spent 1/4 of his life as a boy, 1/5 of his life as a youth, 1/3 of his life as a man, and 13 years in old age. How old is the man?

Question 881: Digit Multiplication I

Topic: brainteasers **Difficulty**: medium
What is the smallest positive integer whose digits multiply to 10000?

Question 882: Consecutive 1s

Topic: probability **Difficulty**: easy
You toss a dice 5 times and record the outcomes each time. What is the probability of throwing at least 3 consecutive ones within those 5 rolls?

Question 883: Busted 6 I

Topic: probability **Difficulty**: medium
Suppose you play a game where you continually roll a die until you obtain either a 5 or a 6. If you receive a 5, then you cash out the sum of all of your previous rolls (excluding the 5). If you receive a 6, then you receive no payout. You do not have the decision to cash out mid-game. What is your expected payout?

Question 884: Coin Pair V

Topic: probability **Difficulty**: medium
Four fair coins appear in front of you. You flip all four at once and observe the outcomes of the coins. After seeing the outcomes, you select any pair of coins to reconsider. One of the two coins can be turned over, while the other must be flipped again. If you already have 3 heads, you still must perform both actions. You iterate this process until you end up with all 4 coins being heads. A movement of a coin is when you either turn over or flip it. Find the expected number of coin movements needed to obtain 4 heads.

Question 885: Ranged Max

Topic: probability **Difficulty**: easy
Let $X_1, X_2, X_3, X_4 \sim \text{Unif}(0, 4)$ IID. Find the probability that the maximum of these 4 random variables is in the interval $(2, 3]$.

Question 886: Ship Meeting

Topic: brainteasers **Difficulty**: medium
A pirate ship sets sail from island A to island B. At the same time, another pirate ship starts sailing from island B to island A. Assume each ship stays along the same path between the two islands. The pirate ships pass other at $1:00$ PM and continue on their paths. One of them arrives at $5:00$ PM, the other at $10:00$ PM. At what time did the pirate ships set sail? Put your answer in military time, excluding the 0 if the hour is only one digit. For example, if the answer is $4:30$ PM, answer 1630.

Question 887: No Marble Missing

Topic: probability **Difficulty**: easy
You have a box of 6 marbles of distinct colors, two of which are blue and green. You draw 4 marbles one-at-a-time with replacement. Find the probability that there is at least one green and one blue marble among the 4 you selected.

Question 888: Fair Match

Topic: brainteasers **Difficulty**: medium
Suppose that Andy has m 6−sided dice and Bandy has n 8−sided dice. If S_m is the sum of the upfaces of all m of Andy's dice and T_n is the sum of the upfaces of all n of Bandy's dice, find the product m^*n^*, where m^* and n^* are the smallest integer values of m and n such that $\mathbb{P}[S_{m^*} < T_{n^*}] = \mathbb{P}[S_{m^*} > T_{n^*}]$.

Question 889: Equivariant

Topic: statistics **Difficulty**: easy
Let X and Y be any random variables with finite mean and variance 16. Find $\text{Corr}(X+Y, X-Y)$.

Question 890: Matching Socks II

Topic: probability **Difficulty**: medium
6 distinct pairs of socks are in a drawer. You're in a hurry to pack for a trip, so you draw out 8 socks uniformly at random from the drawer. Find the expected number of pairs of socks that remain in the drawer.

Question 891: Game Show II

Topic: probability **Difficulty**: medium
You're on a game show and are given the choice of three doors to choose from. Behind one door is a car and behind the other two are goats. You pick one of the doors at random and announce your choice to the game show host. The game show host then chooses an audience member at random, who is unaware of the prizes behind the doors, to open one of the two doors you did not choose at random. The audience member opens a door that you did not choose and happens to reveal a goat. The game show host offers you the opportunity to either keep your original door or switch to the other closed door. What is the probability that you win the car if you switch?

Question 892: Colorful Cube

Topic: probability **Difficulty**: easy
Consider coloring the 6 sides of a cube each a distinct color among red, blue, green, yellow, purple, and pink. Find the probability that the sides that are colored red, green, and blue all share a vertex of the cube (i.e. corner) in common.

Question 893: Colorless Sides

Topic: probability **Difficulty**: medium
A $3\tilde{A}3\tilde{A}3$ cube that is colored red on the outside is cut into 27 $1\tilde{A}1\tilde{A}1$ smaller cubes. A random cube is selected from these 27 and you see five sides without any color. What is the probability that the cube has one colored side?

Question 894: Rotation Matrix Ranked

Topic: pure math **Difficulty**: medium
Let $R_{\alpha,\beta,\gamma}$ be the rotation matrix in $\mathbb{R}^3$ by α, β, and γ radians CCW about the x, y, and z axes, respectively. For any fixed α, β, and γ, compute $(\text{rank}(R_{\alpha,\beta,\gamma}))^2 + (\text{null}(R_{\alpha,\beta,\gamma}))^2$.

Question 895: Die Rank

Topic: probability **Difficulty**: medium
Let $X_1, \ldots, X_5$ be the upfaces of 5 independent rolls of a fair 6−sided die with values $1-6$ on each side. Rank each of the following from largest to smallest in terms of value: a) $\mathbb{E}[X_1X_2]$; b) $\mathbb{E}[X_1^2]$; c) $\mathbb{E}[X_{(3)}^2]$, where $X_{(3)}$ is the median of the 5 die rolls.

Let $a = 1, b = 2$, and $c = 3$. Answer with the integer corresponding to the concatenation of the order from largest to smallest. For example, if you believe $c > b > a$, answer with 321.

Question 896: Circular Birthdays

Topic: probability **Difficulty**: easy

$$7$$

people sit around a circular table uniformly at random. All of them have a distinct age. Find the probability that they sit down at the table in age order. Note that the ages can be increasing in either the clockwise or counter-clockwise directions.

Question 897: Uniform Equilibrium II

Topic: probability **Difficulty**: medium
Two players, say 1 and 2, simultaneously pick real numbers in the interval $[0, 1]$. The payoff of Player 1 (equal to the loss of Player 2) is the absolute distance between those numbers. There exists a non-pure Nash equilibrium. Under rational play from both players, find the expected payoff for Player 1.

Question 898: Lapping

Topic: brainteasers **Difficulty**: easy
Alice and Bob now are in NASCAR. They continually race around a circular track. Alice can drive a lap in 6 minutes, while Bob takes 9 minutes to drive a lap. After how many minutes will Alice lap Bob i.e. catch back up to Bob a lap ahead of him?

Question 899: Coin Pair II

Topic: probability **Difficulty**: easy
Four fair coins appear in front of you. You flip all four at once and observe the outcomes of the coins. After seeing the outcomes, you may flip any pair of coins again unlimited amounts of times. You may not flip a single coin without flipping another. Assuming optimal play, find the expected number of heads that appear.

Question 900: Triangular Partition

Topic: probability **Difficulty**: medium
A point P is chosen within triangle $\triangle ABC$. Compute the probability that the area of $\triangle ABP$, $\triangle BCP$, or $\triangle CAP$ is no more than one fourth that of $\triangle ABC$.

Question 901: Cats and Dogs II

Topic: probability **Difficulty**: medium
Six dogs and six cats are sitting at a circular table uniformly at random. Find the probability that there are exactly four dogs in a row somewhere in the circle.

Question 902: Big Bubble II

Topic: pure math **Difficulty**: easy
A spherical bubble has a radius increasing at a rate of $\frac{9}{4\pi}$ inches per second. At the moment when the volume of the bubble is increasing at 36 cubic inches per second, what rate is the surface area increasing at (in square inches per second)?

Question 903: 9 Toss 4

Topic: probability **Difficulty**: easy
Yungeun rolls 9 fair dice simultaneously. Compute the probability that the product of the 9 values obtained is divisible by 4.

Question 904: Exponential Ball Draw

Topic: brainteasers **Difficulty**: hard
There are 100 balls in an urn. Alice and Bob pick the balls in turn, for each round they can pick 2^k balls for any choice of $k \geq 0$ as long as there are at least that many balls remaining. Whoever draws the last remaining ball in the urn loses. Alice can choose whether to go first or second. Let p be the position of Alice ($p = 1$ if she should go first, $p = 2$ if she should go second). Let b be the maximum number of balls Alice should select on her first turn if Bob chooses 32 balls on his first turn. Find $100p + b$.

Question 905: Hide and Seek

Topic: probability **Difficulty**: easy
You are playing a game of hide-and-seek with two friends. While one of your

friends counts, you and your other friend are given a chance to hide in one of five possible hiding spots. You are each allowed to pick a hiding spot, and are permitted to share a hiding spot. Your friend finishes counting and checks one of the five hiding spots. Assuming that everyone's decisions are made uniformly at random, what are the chances that your friend does not find anyone in the first spot that they check?

Question 906: Specific Card Pull I

Topic: probability **Difficulty**: easy
A deck of cards is shuffled well. The cards are dealt one-by-one, until the two of hearts appears. Find the probability that no kings, queens or jacks appear before the two of hearts.

Question 907: Split 100

Topic: probability **Difficulty**: easy
The integers $1 - 100$ are split into two groups containing 30 and 70 numbers respectively. Compute the probability that numbers 6 and 66 are in different groups.

Question 908: Rain Chance II

Topic: probability **Difficulty**: easy
This weekend, there is a 40% chance it rains on Saturday and a 70% chance it rains on Sunday. Without assuming independence, find the tightest upper bound on the probability it does not rain this weekend.

Question 909: Independent Zeta

Topic: probability **Difficulty**: medium
Let X and Y be independent Zeta(2) random variables. This means that they have PMF $\mathbb{P}[X = k] \propto \dfrac{1}{k^2}$ for all positive integers k. This means that the PMF is a constant multiple of $1/k^2$ for each k. Find $\text{Cov}(X, Y)$. If it is $+\infty$, enter 123. If it is $-\infty$, enter -123. If it does not exist, enter 12345.

Question 910: Football Bets

Topic: probability **Difficulty**: easy
Connor and Calvin bet \$10 each on opposing outcomes of a football game. During the match, Connor raises the bet to \$20. If Calvin doesn't match Connor's

bet, he automatically loses what he has bet thus far. Let p be the probability that Calvin's team will win the game when Connor raises. Find the smallest value of p so that Calvin should accept Connor's bet.

Question 911: Cat Dog Line

Topic: brainteasers **Difficulty**: medium
You're walking down an infinite street, and you notice that six out of every seven cats on a sidewalk are followed by a dog, while one out of every four dogs is followed by a cat. What proportion of animals on the sidewalk are dogs?

Question 912: Slippery Snail

Topic: brainteasers **Difficulty**: easy
A snail is 30 feet away from some food. The snail moves at a rate of 4 feet per hour. However, at the end of each hour, a large gust of wind blows and the snail moves back 2 feet. The snail eats the food instantaneously once it reaches it. How many hours does it take for the snail to reach the food?

Question 913: Bull Call Spread II

Topic: finance **Difficulty**: medium
Consider the following asset S, with initial price $S_0 = 7$. Now, let's look at a bull call spread with $K_1 = 5$ and $K_2 = 10$. We want to price the bull call spread, but we do not have access to a computer. The only information we know is that there exists a discount factor of 0.9 and that the initial price is 7. Give the best upper-bound for the price of this bull call spread.

Question 914: Power Digits

Topic: probability **Difficulty**: medium

$$m$$

is randomly selected from $\{111, 133, 155, 177, 199\}$. n is randomly selected from $\{2004, 2005, \ldots, 2023\}$. What is the probability that m^n has a 1 as the units digit?

Question 915: Game Time

Topic: probability **Difficulty**: hard people are looking to see a tennis match. The match costs \$5 to enter. 19 of the people only have \$5 bills, while the other 19 only have \$10 bills. The cashier currently has no change. If the 38 people arrange themselves in line completely at random, find the probability that all of the people can successfully purchase a ticket without reordering in line.

Question 916: Mean Difference

Topic: probability **Difficulty**: medium
Suppose that $X_1, \ldots, X_{10}$ are independent random variables such that $\mu_i = \mathbb{E}[X_i] = 2i - 1$ and $\mathrm{Var}(X_i) = 100$ AKA there is a common variance 100 but not a common mean. Define $\overline{\mu} = \frac{1}{10}\sum_{i=1}^{10} \mu_i$. Compute $\mathbb{E}[(X_1 - \overline{X})^2]$.

Question 917: Log Comparison

Topic: probability **Difficulty**: medium
Let $X, Y \overset{\text{iid}}{\sim} \mathrm{Unif}(0,1)$. Compute the probability that $\lfloor \log_2 X \rfloor = \lfloor \log_2 Y \rfloor$.

Question 918: Extrinsic Value II

Topic: finance **Difficulty**: easy
We have an underlying S with price $S_0 = 17$. You have a European put option on S with strike $K = 15$, currently with value $P_0 = 7.2$. What is the intrinsic (I) and extrinsic (E) value of the option?

Give the answer in the format of $I^2 + E^2$.

Question 919: Unit Fraction Representation

Topic: brainteasers **Difficulty**: medium
Assume, without proof, that every rational number $0 < q < 1$ can be represented as the sum of unit fractions. For example,

$$\frac{4}{5} = \frac{1}{2} + \frac{1}{4} + \frac{1}{20}$$

Find the largest denominator in the unit fraction representation of $\frac{179}{720}$ that has the fewest terms and largest possible maximum denominator. In the example above, the answer would be 10.

Question 920: Gamma Review IV

Topic: probability **Difficulty**: medium
Suppose that $X \sim \text{Gamma}\left(8, \frac{1}{3}\right)$. Compute $\text{Var}(e^X)$. The answer is in the form $a^b - \left(\frac{a}{2}\right)^c$ for integers a, b, and c. Find abc.

Question 921: Three Consecutive Heads

Topic: probability **Difficulty**: medium
On average, how many fair coin tosses will it take to observe three consecutive heads?

Question 922: Squid Game I

Topic: probability **Difficulty**: easy
10 contestants are arranged into a line on a bridge and in front of them lay ten left tiles and ten right tiles side by side. In order to cross the bridge, the contestants must cross 10 tiles, and at each step, the person in front must pick either the left or right tile to step on. However, for each left right tile pair, there is exactly one sturdy tile and one faulty tile, but the contestants cannot tell them apart. The contestants cross the bridge in their assigned order with the first person picking either the left or right tile, and continuing to lead unless either a faulty tile is picked (resulting in elimination) or person one reaches the other side. If the first person is eliminated before reaching the other side, the person second in line assumes the lead picking until he/she is eliminated (or reaches the other side), and so on. The winner of the game is the *first* person to reach the other side. Which position in line (1-indexed) should a contestant choose if they are playing optimally to win?

Question 923: Ship Stops

Topic: probability **Difficulty**: easy
You are sailing from one island to another. Uninterrupted travel between the two islands takes 12 minutes. However, there are 5 wave checks between the islands. Four of the wave checks will stop you with 25% probability each. These wave checks will add 1 minute to your total travel time if you are stopped. However, the last wave check stops you with 75% probability. This wave check will add 4 minutes to your travel time if you are stopped. Find the expected duration (in minutes) of the trip between the two islands.

Question 924: Normal LOTUS II

Topic: probability **Difficulty**: easy
Let X and Y be jointly continuous with joint PDF $f(x, y) = 10ye^{-5x}I_{(0,\infty)}(x)I_{(0,1)}(y)$. Compute $\mathbb{E}\left[\frac{X^2}{Y}\right]$.

Question 925: 3rd Head

Topic: probability **Difficulty**: easy
Find the expected number of flips needed to observe 3 heads on a coin with probability $\frac{1}{4}$ of heads per flip.

Question 926: Different Variance

Topic: probability **Difficulty**: easy
Let X and Y be random variables with $\text{Var}(X + Y) = 18$ and $\text{Var}(X - Y) = 10$. Find $\text{Cov}(X, Y)$.

Question 927: Last Light Bulb Standing

Topic: probability **Difficulty**: medium
Suppose you have 10 independent light bulbs labelled $1 - 10$ whose lifetime is modelled by $T_i \sim \text{Exp}(1/10)$ for each $1 \leq i \leq 10$. Furthermore, suppose you have an 11th light bulb whose lifetime is modelled by $X \sim \text{Exp}(1/90)$. Find $\mathbb{P}[X > T_1 + \cdots + T_{10}]$. The answer is in the form q^b, where $0 < q < 1$ is a rational number and b is an integer with b maximal. Find qb.

Question 928: Colorful Candy

Topic: probability **Difficulty**: easy
A bag of 10 candies has 6 blue, 2 red, 1 green, and 1 yellow candies. 3 candies are randomly selected. Find the expected number of colors among the selected candies.

Question 929: Chord on a Square

Topic: probability **Difficulty**: medium
Two points p_1 and p_2 are placed uniformly at random along the border of a square which has side length 1. The probability that the distance between p_1 and p_2 is greater than 1 can be expressed as $\frac{a-\pi}{b}$. What is $\frac{a}{b}$?

Question 930: Binary Dot

Topic: probability **Difficulty**: medium
Let v_1 and v_2 be two vectors of length 10. The 10 elements of v_1 are IID Bernoulli $\left(\frac{1}{2}\right)$ random variables. The 10 elements of v_2 are IID Bernoulli $\left(\frac{3}{4}\right)$ random variables. Find the probability that $v_1 \cdot v_2$ is odd, where the $\cdot$ is dot product of two vectors. The probability is in the form $\frac{1}{2} - \frac{1}{a^n}$, where a is prime. Find an.

Question 931: Quintuple Value

Topic: probability **Difficulty**: easy
35 cards are in a deck numbered $1-35$. One card is dealt uniformly at random and you are paid out 5 times the value on the card. Find the expected payout if you draw 2 cards from this deck.

Question 932: ATM Expiration

Topic: finance **Difficulty**: easy
You have a European call option with a $\Delta = 0.67$ of strike K and expiry T. Approximate the probability for a put of strike K and expiry T of expiring in-the-money.

Question 933: Very Very Normal

Topic: probability **Difficulty**: easy
Let $X, Y, Z \sim N(12, 12)$ IID. Compute $\mathbb{P}[X - Y > Z]$. The answer is in the form $\Phi(a)$ for some a. Find a.

Question 934: Portfolio Returns

Topic: probability **Difficulty**: easy
You have an equally-weighted portfolio consisting of three assets with the following annual returns: Asset A has a 30

Question 935: Salary Covariance

Topic: statistics **Difficulty**: easy
In a population of male-female couples, male annual earnings, denoted M, have a mean of \$60k per year and a standard deviation of \$13k. Female annual

earnings, denoted F, have a mean of \$55k per year and a standard deviation of \$11k. The correlation between male and female annual earnings is 0.85. What is the covariance between male and female annual earnings (in thousands of dollars squared)?

Question 936: Deriving Put-Call Parity I

Topic: finance **Difficulty**: easy
A forward contract is a derivative that pays $S_T - K$ at expiration, where K is the strike price. More specifically, you will get paid if $S_T > K$. If $S_T < K$, then you must pay the other party.

You have access to the underlying S, which has an initial price $S_0 = 3$. You also have access to bonds, which pay 1 at time-T. The interest rates are 0.04 continuously compounded.

Find the time-0 price of the forward contract with strike $K = 2$ and $T = 1$. Round to 2 decimal points.

Question 937: Last Lightbulb

Topic: probability **Difficulty**: medium
Let $X_1, X_2, X_3, X_4 \sim \text{Exp}(1)$ IID represent the lifetimes of 4 lightbulbs before burning out. Find the expected time until the last lightbulb burns out.

Question 938: Car and Fly II

Topic: brainteasers **Difficulty**: medium
A road of length 400 miles has two cars, say A and B, starting at opposite ends of the road. Cars A and B respectively drive towards at each at constant rates of 120 miles per hour and 60 miles per hour. A fly starts on Car B and flies towards Car A at a rate of 180 miles per hour. Then, once it hits Car A, it immediately starts to fly back to Car B at the same rate. This repeats until Car A and Car B crash into one another. Assuming the fly repeats this pattern and doesn't fly out of the way before the cars crash, how many miles will the fly travel in total?

Question 939: Which Sum First?

Topic: probability **Difficulty**: easy
When rolling two dice continuously, what's the probability that you roll a sum of 5 before you roll a sum of 2?

Question 940: Tennis Gambling

Topic: probability **Difficulty**: medium

The rules of tennis are the following: The scores go $0 - 15 - 30 - 40$ to start. If one player has 40 points and the other has less than 40 and the player with 40 points wins the next serve, this player wins the game. If both players have 40 points, the player who wins the next serve gets advantage. If the player with advantage wins again, they win the game. Otherwise, if the advantaged player loses, the game goes back to having no advantages (40-40). Imagine Rob and Bob are in a tennis match that starts out $30 - 30$. Rob has a probability 0.6 of winning each serve, independent between serves. Find the probability Rob wins the game.

Question 941: Life Is A Raceway

Topic: brainteasers **Difficulty**: easy

Lightning McQueen goes for a race. He places 14 places ahead of last place and 6 places below the upper half of competitors. What place was Lightning McQueen in?

Question 942: Local Maxima

Topic: probability **Difficulty**: medium

$$14$$

pieces of paper labelled $1 - 14$ are placed in a line at random. We spot i is a local maxima if the paper at the ith position is strictly larger than all of it's adjacent papers. Find the expected number of local maxima in the sequence. For example, with 6 numbers, 513246 has 3 local maxima at the first, third, and last spots.

Question 943: Particle Reach IV

Topic: probability **Difficulty**: medium

Complete Particle Reach I and III First!

Consider a particle that performs a random walk on the integers starting at position 0. At each step, the particle moves from position i to position $i + 1$ with probability p, while the probability it moves from i to $i - 1$ is $1 - p$. Suppose that $p = 1/3$, the particle is currently at position 4, and the particle eventually reaches position 7. Find the probability that the particle moves to position 5 in the next step.

Question 944: Jellybean Jar II

Topic: probability **Difficulty**: easy
A child has a pack of 6 red and 10 blue jellybeans. The child wants to eat 4 jellybeans, so they grab 4 jellybeans one-by-one uniformly at random without replacement from the pack. Find the probability that the first two and last two jellybeans agree in colors (not necessarily in order). For example, $RBBR$ and $RRRR$ are both valid, but $RRBR$ is not.

Question 945: Grid Filling II

Topic: probability **Difficulty**: medium
The integers 1 through 9 are randomly placed into the 9 squares of a 3 x 3 grid such that each square has one integer and each integer is used once. What is the probability that the sum of each row and each column is odd?

Question 946: Baby Boy

Topic: probability **Difficulty**: easy
A hospital nursery has 3 baby boys and an unknown number of baby girls. A mother just gave birth to a child of unknown gender and it is added to the nursery. The doctor then picks up a random child in the nursery and it is a baby boy. Assuming that births of boys and girls are equally likely, find the probability that the mother just gave birth to a boy.

Question 947: Variance of Sum of BM

Topic: pure math **Difficulty**: easy
Let W_t be a standard Brownian Motion. Compute $\text{Var}(W_1 + W_2)$.

Question 948: Defining Correlation

Topic: statistics **Difficulty**: easy
The correlation between X and Y is 0.56. What is the correlation between $8X$ and $Y + 7$?

Question 949: Sticky Strike

Topic: finance **Difficulty**: medium
We are pricing an option for an underlying S with initial price $S_0 = 12$. However, we are not under the assumptions of Black-Scholes. More specifically, we assume that the implied volatility follows a sticky-strike model.

For an option at strike $K = 13$, the implied volatility is $\sigma = 0.3$. At time 1, the underlying price is $S_1 = 15$. What is the implied volatility of the $K = 13$ strike option?

Question 950: Estimating Pi

Topic: probability **Difficulty**: medium
You decide to estimate π by generating N independent samples from the unit square $[0, 1] \times [0, 1]$. Let I_i represent the indicator random variable of the event that the ith sample, $1 \leq i \leq N$, lands inside the region enclosed by $x^2 + y^2 = 1$ in the first quadrant. Using

$$T_N = \frac{I_1 + \cdots + I_N}{N}$$

in your estimate for π, what is variance of your estimate for π? Note that T_N is not necessarily an estimator for π itself. The answer is in the form

$$\frac{a\pi^2 + b\pi}{N}$$

for integers a and b. Find $a + b$. Note that this variance is somewhat useless given that you need π to calculate your variance.

Question 951: Square Ratio

Topic: probability **Difficulty**: hard
Suppose that $X, Y \sim \text{Unif}(0, 1)$ IID. Compute the probability that $\lceil \frac{Y}{X} \rceil$ is a perfect square. The answer is in the form $q - \dfrac{\pi^2}{a}$ for a rational q and an integer a. Find aq.

Question 952: Dice Strike Price

Topic: probability **Difficulty**: easy
A fair 12−sided die has the values $1, 3, \ldots, 23$ on it. What is the fair price of an option with strike price 12, where you are paid out the numerical value in dollars of the resultant face of the die upon rolling it?

Question 953: Optimal Marbles II

Topic: probability **Difficulty**: hard
Two players, say A and B, play the following game: Both players have 100

marbles and may put anywhere between 1 and 100 marbles in the box each. This decision is not revealed to the other player. Then, they draw 1 marble. If the marble belongs to A, then assuming that A put a marbles in the box, A is paid $100 - a$ monetary units from player B. Similarly if the marble belongs to B, then assuming B put b marbles in the box, B is paid $100 - b$ monetary units from player A. Assume both players play optimally. How many marbles should player A select?

Question 954: Rabbit Hop V

Topic: probability **Difficulty**: medium
A rabbit starts at the floor in front of a staircase of 10 stairs. The rabbit can hop up any odd amount of stairs at each movement. How many distinct paths are there from the floor to the top of the staircase (i.e. to the top of the 10th stair)?

Question 955: Normal Conditions

Topic: probability **Difficulty**: medium
Suppose that $X_1 \sim N(0, 9)$ and $X_2 \sim N(0, 16)$ are independent. Compute $\mathbb{E}[X_1 \mid X_1 + X_2 = 5]$.

Question 956: Put Arbitrage

Topic: finance **Difficulty**: medium
You have access to a stock with price $S_0 = 14$, a put at strike $K = 17$ with initial price $P_0 = 5.9$, a put at strike $K = 14$ with initial price $Q_0 = 2.1$, and bonds that pay out 1 at time T with initial value $B_0 = 0.9$. Find the arbitrage.

Give the answer in the format of # Stock + # Put (K = 17) + # Put (K = 14) + # Bonds

Question 957: Graph Shading

Topic: probability **Difficulty**: easy
Two values X and Y are chosen uniformly at random between 0 and 1. What is the probability that the ratio $\frac{X}{Y}$ is in between 1 and 2?

Question 958: Poisoned Kegs IV

Topic: brainteasers **Difficulty**: hard
A king has 5 servants that bravely risk their lives to test whether or not the wine

in n kegs is poisonous. It is known that exactly one of the n kegs is poisonous. If someone drinks any amount of liquor from the poisoned keg, they will die in exactly 1 month. Otherwise, the servant will be fine. The servants agree to participate in the wine tasting for 3 months. What is the maximum value of n such that the king is guaranteed to determine which keg among the n is poisoned?

Question 959: 6 Die Stopper

Topic: probability **Difficulty**: medium
A standard fair $6-$sided die is rolled up to (and including) when the first 6 is rolled. Find the probability that the sum of all upfaces viewed before stopping is even.

Question 960: Rabbit Hop I

Topic: probability **Difficulty**: easy
A rabbit starts at the floor in front of a staircase of 10 stairs. The rabbit can hop up any amount of stairs at each move. How many distinct paths are there from the floor to the top of the staircase (i.e. to the top of the 10th stair)?

Question 961: Shifty Cylinder

Topic: pure math **Difficulty**: easy
The radius of a cylinder is growing at a rate of 2 meter per hour, and the height of the cylinder is decreasing at a rate of 5 meters per hour. At what rate is the volume of the cylinder changing (in cubic meters per hour) at the instant where the base radius is 3 meters and the height is 6 meters? The answer is in the form $k\pi$ for a constant k. Find k.

Question 962: Salah Goal

Topic: probability **Difficulty**: easy
Mo Salah is lining up to take a penalty kick. He decides to perform $3n$ motions. Each motion is either 2 steps forward or 1 step backwards. Find the probability that after 9 motions, Salah is exactly where he started.

Question 963: Gamma Review II

Topic: probability **Difficulty**: easy
Evaluate $\int_0^\infty xe^{-x^n}\,dx$, where $n > 0$. The answer will be in the form $\frac{a}{n} \cdot \Gamma\left(\frac{b}{n}\right)$

for integers a and b. Find $a+b$.

Question 964: Allocating Capital

Topic: probability **Difficulty**: easy
You are the portfolio manager of a \$50 million fund with five potential asset classes to invest in: equity options, commodity futures, fixed income, FX, and cryptocurrencies. How many ways can you invest the capital into these five asset classes in increments of \$1 million? For example, one asset allocation strategy (in millions) is $(\$25;\$15;\$5;\$5;\$0)$.

Question 965: Soda Machines

Topic: statistics **Difficulty**: easy
A soda machine is designed to release not more than 7 ounces of soda per cup, on average. To test this machine, you fill 8 cups of soda and measure their volumes. The mean and standard deviation of the 8 measurements were 7.1 ounces and 0.12 ounce, respectively. What is value of the appropriate test statistic to validate the machine's claim? Assume random sampling, variance homogeneity, and that dispensing volume is approximately normally distributed.

Question 966: Integral Variance IV

Topic: pure math **Difficulty**: medium
Let W_t be a standard Brownian Motion. Find the variance of $X = \int_0^2 \sqrt{t}e^{\frac{W_t^2}{8}}\,dW_t$.

Question 967: Casted Shadow

Topic: pure math **Difficulty**: medium
A person who is 6 feet tall is walking away from a lamp post at the rate of 40 feet per minute. When the person is 10 feet from the lamp post, his shadow is 15 feet long. Find the rate at which the shadow's length is increasing (in feet per minute) when he is 40 feet from the lamp post.

Question 968: RNG on RNG

Topic: probability **Difficulty**: hard
You generate a uniformly random number in the interval $(0,1)$. You can generate additional random numbers as many times as you want for a fee of \$0.02 per generation. This decision can be made with the information of all of the previous values that have been generated. Your payout is the maximum of all the

numbers you generate. Under the optimal strategy, find your expected payout of this game.

Question 969: Heads and Tails I

Topic: probability **Difficulty**: medium
On average, how many flips of a fair coin will it take until you observe heads immediately followed by tails?

Question 970: Uniform Order III

Topic: probability **Difficulty**: medium
Let $X_1, X_2, \dots, X_{20}$ be IID Unif$(0, 1)$ random variables. Compute

$$\mathbb{P}[X_{16} = \min\{X_1, \dots, X_{16}\} \mid X_1 = \max\{X_1, \dots, X_{16}\}, X_{17} < X_{18}]$$

Question 971: Ronaldo's House

Topic: probability **Difficulty**: easy
Cristiano Ronaldo lives in Portugal in a house numbered 122577. A hurricane hits Portugal, and two random numbers on Ronaldo's house fall off. After the hurricane passes, he sees the numbers on the ground, but has amnesia, so he doesn't remember what his house number is. Therefore, he places the two numbers up in the two empty spots uniformly at random. Find the probability Ronaldo creates the number 122577 again.

Question 972: Game Arbitrage II

Topic: finance **Difficulty**: hard
Consider the following group with the following teams. The contracts are given in the format of (Team, Time-0 Price). The price of 0.67 means that if the team wins, you will get paid 1. If the team loses, you will get paid 0.

$$(\text{Team } 1, 0.95)$$
$$(\text{Team } 2, 0.72)$$
$$(\text{Team } 3, 0.35)$$
$$(\text{Team } 4, 0.03)$$

The group works as follows: 2 teams are guaranteed to make it out. Find the arbitrage. You are allowed to long or short contracts. You are also allowed to long or short bonds, which pay 1 at expiry. Assume interest rates are 0, so $Z_0 = 1$.

Give the answer in the form of the initial credit you will receive in the arbitrage.

Question 973: Card Diff

Topic: brainteasers **Difficulty**: hard
You have all the clubs from a standard deck (13 cards) and you can choose 2 from the deck and get paid the product of their values. All face cards are considered to have value 0 and Ace is considered to have value 1. You can pay \$1 to reveal the difference of any two cards you choose. You can exercise this "difference" option between any two cards as many times as you would like. However, in the end, you must select two cards be to the ones you take the product of for your payout. Under a rational strategy, what is the maximum guaranteed profit you can achieve?

Question 974: Hidden Prisms

Topic: brainteasers **Difficulty**: medium
You are presented with a clear 5x5x5 cube. How many unique rectangular prisms can you see within the cube? Uniqueness does not include orientation. For example, there are 8 unique 3x3x3 cubes within a 4x4x4 cube.

Question 975: Vasicek Equation

Topic: pure math **Difficulty**: hard
Let W_t be a standard Brownian Motion. Let R_t satisfy the SDE

$$dR_t = (a - bR_t)dt + \sigma dW_t$$

for $a, b, \sigma > 0$ and $R_0 = r > 0$. $\mathbb{E}[R_T]$ can be written as a function of a, b, r, and T. Evaluate this function with parameters $T = 10, a = 0.2, b = 0.1, \sigma = 0.1$, and $r = 1$. The answer will be in the form $A + Be^C$ for integers A, B, and C. Find $A + B + C$.

Question 976: Median Uniform

Topic: probability **Difficulty**: medium
Let $X_1, X_2, X_3 \sim \text{Unif}(0, 3)$ IID. Find the probability that $X_{(2)}$, the median of the three random variables, is between 1 and 2.

Question 977: Party Groups I

Topic: probability **Difficulty**: hard
There are 50 guests at a party and they are making groups for a game. To do this they each write their name on a piece of paper and put it into a hat. One by one, each guest picks a name from the hat. Each guest will be a part of a group with the guest they pulled the name out of from the hat. If a guest pulls out their own name, they are in a group all by themselves. If Guest A pulls out Guest B's name, Guest B pulls out Guest C's name, and Guest C pulls out Guest A's name, they are all a part of the same closed group and no one else will be able to join them. How many groups will there be on average? Round your answer to the nearest tenth.

Question 978: Bold Subsets

Topic: probability **Difficulty**: easy
Erica intends to construct a subset T of $S = \{I, J, K, L, M, N\}$, but if she is unsure about including an element x of S in T, she will write x in bold and include it in T. For example, $\{I, J\}$, $\{J, \mathbf{K}, L\}$, and $\{\mathbf{I}, \mathbf{J}, \mathbf{M}, \mathbf{N}\}$ are valid examples of T, while $\{I, J, \mathbf{J}, K\}$ is not. Find the total number of such sets T that Erica can construct.

Question 979: Strangle Delta

Topic: finance **Difficulty**: medium
The underlying is currently at price $S_0 = 2$. You buy a strangle at strikes $K = 3$ and $K = 5$. What direction would you prefer the underlying to go towards? Answer 1 for up, -1 for down, and 0 for stay where it's at.

Question 980: Support Center

Topic: statistics **Difficulty**: easy
An IT support center receives 270 calls per hour, on average. What is the probability that more than 40 calls will be received in the next 10 minutes? Round your answer to the nearest thousandth.

Question 981: Doubly Winner

Topic: probability **Difficulty**: easy
James flips a sequence of coins labelled $1 - n$, $n \geq 2$. The kth coin has probability $\frac{1}{k}$ of landing up heads. James will receive \$1 for each time he obtains 2 consecutive heads. These sequences can overlap, so HHH would yield a payout

of \$2. Let $p(n)$ be the expected payout when James has n coins. Evaluate $\lim_{n\to\infty} p(n)$. If this limit does not exist, enter -1.

Question 982: Coin Duel

Topic: probability **Difficulty**: medium
Bill and Bob are simultaneously flipping coins with probability $1/3$ of heads per flip. Find the probability that Bill obtains his first heads in strictly less flips than Bob.

Question 983: Summed Brownians

Topic: probability **Difficulty**: easy
Let B_1, B_2, and B_3 be standard Brownian Motions. Find $\text{Cov}(B_1(1)+B_2(1), B_2(2)+B_3(2))$.

Question 984: The Perfect Hedge II

Topic: finance **Difficulty**: medium
You have two assets. We will call them asset 1 and asset 2. Asset 1 has an expected return of 4% and a variance of 15%. Asset 2 has an expected return of 2% and a variance of 4%. They have a correlation $\rho = -1$.

We want to create a risk-free portfolio using assets 1 and 2. We will denote w_1 and w_2 as the weights of asset 1 and 2 in the portfolio respectively. What is the expected return of this portfolio? Round the answer to three significant figures.

Question 985: Unknown Baby

Topic: probability **Difficulty**: easy
There are 5 babies in a room: 2 boys and 3 girls. one baby with unknown sex is added. One baby is random selected and it is a boy. What's the probability that the added baby is a boy? You may assume that each gender is equally likely at birth.

Question 986: Covariance Review VI

Topic: statistics **Difficulty**: easy
Suppose that $Z \sim N(0,1)$ and $Y \sim \chi^2(\nu)$ are independent. Define $W = Z/\sqrt{Y}$. Compute $\text{Cov}(Y, W)$.

Question 987: Correlation Variance

Topic: statistics **Difficulty**: easy
Suppose we have that $\sigma_X = 4$, $\sigma_Y = 7$, and $\sigma_{X+Y} = 11$. What is $\rho(X, Y)$?

Question 988: Increasing Chains

Topic: probability **Difficulty**: easy
Fix positive integers n and k, and suppose $X_1, \dots, X_n \sim \text{Unif}(0,1)$ IID. We say that there is an *increasing k-chain* starting from position i if $X_i < X_{i+1} < \cdots < X_{i+(k-1)}$. Find n such that the expected number of increasing 6−chains among $X_1, \dots, X_n$ is 1.

Question 989: Averaging Squares

Topic: statistics **Difficulty**: medium
Let $X = [1, 2, ..., 2023, 1^2, 2^2, ..., 2023^2]$. What is the median of X?

Question 990: Quick Summation

Topic: brainteasers **Difficulty**: easy
What is the sum of the integers from 1 to 50, inclusive?

Question 991: Curious Multiplicand

Topic: brainteasers **Difficulty**: medium
What integer can be multiplied by $1, 2, 3, 4, 5$, and 6 and no new digits appear in any of the results? For example, $100 \cdot 10 = 1000$ has no new digits appearing in the result after multiplication.

Question 992: Take and Roll I

Topic: probability **Difficulty**: medium
You are given a fair 20−sided die and 100 actions in a game. The die starts with upface 1. The two options you can perform are to roll and to take. Performing a roll re-rolls the current upface of the die. Performing a take allows you to cash out the current upface of the die. Note that the game does not end when you perform a take and that you do not have to roll between takes. Therefore, for example, you can just perform 100 takes on the initial \$1 upface and walk away with \$100 guaranteed. Your strategy is to cash out the upface when you roll at least some threshold n for the first time. You fix this n at the beginning

of the game. Assuming rational strategy in selecting n, what is your expected payout on this game?

Question 993: Continuous Blackjack

Topic: probability **Difficulty**: hard
Two players, a gambler and dealer, play continuous blackjack. The gambler goes first and generates a uniformly random number in $(0, 1)$. At any time, they can choose to stop or play again. If they play again, they generate another independent uniform random number in $(0, 1)$. If the sum of the generated numbers exceeds 1, the gambler busts and the dealer wins. Otherwise, if the gambler stops at some value $0 \le a \le 1$, then the dealer begins generating independent uniform random numbers in $(0, 1)$. The dealer keeps adding the values up until he either obtains a sum in the interval $(a, 1)$ (in which case the dealer wins) or obtains a sum larger than 1 (in which case the gambler wins).

The strategy of the gambler is going to be to stop when the sum is at least α and hit (generate another random number) if the sum is below α. The α that maximizes the probability of the gambler winning solves $e^\alpha = \dfrac{e+\alpha}{x+y\alpha}$ for integers x and y. e here is Euler's constant. Find $x^2 + y^2$.

Question 994: Decreasing Uniform Chain

Topic: probability **Difficulty**: hard
Let $X_1, X_2, \cdots \sim \text{Unif}(0,1)$ IID. Let N be the first index n where $X_n \neq \min\{X_1, \dots, X_n\}$. Find $\mathbb{E}[X_{N-1}]$ i.e. the smallest value among the first N values selected. The answer will be in the form $a + be$ for integers a and b. Note here that e is Euler's constant. Find $a + b$.

Question 995: 77 Multiple II

Topic: probability **Difficulty**: easy
What is the smallest multiple of 77 that is at least $700,000$?

Question 996: 8 Card Heart

Topic: probability **Difficulty**: easy

$$8$$

cards are dealt out from a standard deck. Find the variance of the number of hearts in the 8 cards.

Question 997: Circular Charlie

Topic: probability **Difficulty**: easy
There are 14 boys standing around in a circle, of which one of them is Charlie. 10 girls walk up and join the circle at random to form a larger circle. Each girl must be between 2 boys. If the girls arrange such that all ways of forming a larger circle are equally likely, find the probability Charlie is still standing between two boys.

Question 998: Integral Variance II

Topic: pure math **Difficulty**: medium
Let W_t be a standard Brownian Motion. Compute $\text{Var}\left(\int_0^t W_s dW_s\right)$. The answer is in the form kt^2 for a constant k. Find k.

Question 999: Double Evens

Topic: probability **Difficulty**: easy
A fair 6−sided die is rolled. You are paid the face value if the value is odd and twice the face value if it is even. You are allowed to re-roll the die one time if you want and keep the last value rolled. Find the expected payoff of this game under a rational strategy.

Question 1000: Unit Distance

Topic: pure math **Difficulty**: easy
Let x and y be two unit length orthogonal vectors in $\mathbb{R}^n$. Compute $||x - y||$. This answer is in the form $\sqrt{a}$ for an integer a. Find a.

Question 1001: Empty Urn

Topic: probability **Difficulty**: hard
We flip a fair coin until we obtain our first heads. If the first heads occurs on the kth flip, we are given k balls. We put them into 3 bins labeled 1,2, and 3 at random. Find the probability that none of the three bins are empty.

Question 1002: Big Mac

Topic: brainteasers **Difficulty**: easy
In the year 2000, Big Macs sold for 89 cents each. Let x be the number of Big

Macs you can buy with a \$20 bill and y be the change (in cents) you have left over. Find xy. Ignore tax or other expenses.

Question 1003: High-Low

Topic: probability **Difficulty**: medium
You roll a fair 100−sided die with values $1 - 100$ on the sides and observe the value that appears. You now must guess whether the second roll will be at least as large as the first roll or not. If the second value that appears is y and you guess correctly, you receive \$$y$ payout. Otherwise, you receive nothing. Assuming optimal play, find the smallest integer t at which we guess that the second roll is lower than the first.

Question 1004: Min Card

Topic: probability **Difficulty**: easy
John has a standard deck of 52 cards. John identifies the ranks of the cards as $1 - 13$, where Ace is 1 and King is 13. He selects 4 cards uniformly at random from the deck. He then looks at the 4 cards he selected and removes the lowest ranked card from the subset of 4 and discards it. Afterwards, he selects another card uniformly at random from the remaining 48 card deck to replace the minimum. Find the expected value of the card John puts into the subset.

Question 1005: Fibonacci Ratio

Topic: brainteasers **Difficulty**: easy
Let F_n be the nth Fibonacci number. Compute $\dfrac{F_1 + F_2 + F_3 + \cdots + F_{300}}{F_3 + F_6 + F_9 + \cdots + F_{300}}$.

Question 1006: Wire Connection

Topic: probability **Difficulty**: medium
10 wires each of length 1 mile lay parallel to each other. Electrician A can only see one side of the wires, while Electrician B can only see the other side of the wires. They randomly connect pairs of distinct wire ends until all ends are accounted for. What is the probability that the connections form a closed circuit with total length 10 miles?

Question 1007: Projection Matrix Ranked

Topic: pure math **Difficulty**: medium

Fix a non-zero vector $v \in \mathbb{R}^n$. Define the projection matrix onto v as $P_v = \frac{vv^T}{||v||^2}$. Compute $(\text{rank}(P_v))^2 + (\text{null}(P_v))^2$ as a function of n. Your answer should be in the form $an^2 + bn + c$ for integers a, b, and c. Find $a + b + c$.

Question 1008: Coin Flipper Ruin

Topic: probability **Difficulty**: easy

Vishnu and Jason have \$1000 and \$500 in their respective bank accounts. They flip a fair coin repeatedly. If heads, Vishnu gives Jason \$1. Else, Jason gives Vishnu \$1. What is the probability that Jason runs out of money first?

Question 1009: Diverse Distributions

Topic: statistics **Difficulty**: easy

Let $X \sim \text{Unif}(0, \lambda)$ and $Y \sim \text{Exp}(\lambda)$, where $\lambda > 0$ is a constant. For what value of λ do X and Y have the same mean? Find λ^2

Question 1010: Sum1

Topic: brainteasers **Difficulty**: medium

Find two real numbers x and y composed of only 1s in their decimal expansion such that $xy = x + y$. What is xy?

Question 1011: Circular Partition

Topic: probability **Difficulty**: medium

Suppose X_1, X_2, X_3, and X_4 are all independent random points selected uniformly from the perimeter of the unit circle. Draw a chord between X_1 and X_2. Draw another chord between X_3 and X_4. Find the expected number of disjoint regions the circle is partitioned into.

Question 1012: Life Support

Topic: pure math **Difficulty**: easy

The population of a space station on a recolonization mission is kept alive by its expandable life support system. Given they are trying to repopulate, the population aboard the station quadruples every 54 years, but the life support system capacity can only be doubled every 54 years. Suppose the initial population is 2 people and the initial life support system has a capacity for 16384 people. How many years will pass before the population reaches the capacity of the support system?

Question 1013: Wedding Handshakes

Topic: brainteasers **Difficulty**: easy
At a wedding of 100 people, each person shakes hands with every other person once. How many handshakes occur in total?

Question 1014: Couple Pairs

Topic: probability **Difficulty**: medium
3 heterosexual couples (6 people) are having a nice dinner. The 6 people are randomly paired up with no restrictions. Find the probability at least one of the couples are paired up together.

Question 1015: Divisible Dice

Topic: brainteasers **Difficulty**: medium
You roll 10 fair $6-$sided dice and sum the values. What is the probability that the sum is divisible by 6?

Question 1016: Idempotent Eigenvalues

Topic: pure math **Difficulty**: medium
A $n \times n$ matrix A is called idempotent is $A^2 = A$. Let A be a 10×10 non-defective idempotent matrix with $\text{null}(A) = 4$. Let λ be the only possible non-zero eigenvalue of A and r be the dimension of the eigenspace generated by that eigenvalue. Find $\lambda^2 + r^2$.

Question 1017: King Activity

Topic: probability **Difficulty**: medium
The dealer is known to deal cards from a deck that is comprised of 10 standard card decks. This means that there are 40 of each rank of card and 520 total cards. The dealer plays a game where he shuffles all 10 decks together randomly, and then he starts turning over cards from the top of the deck. Once he turns over 5 kings, the game is over. You win the game if you are closest to how many cards total (including the final king) the dealer turns over before the game ends.

How many cards would you guess to **maximize your chances**? Note that this is different than the expected value.

Question 1018: Die Roll LCM

Topic: probability **Difficulty**: hard
You have a 10-sided die. What is the expected number of times you must roll until the least common multiple of your die rolls exceeds 2000? The answer can be written as a simplified fraction of the form $\frac{p}{q}$. Find $p + q$.

Question 1019: Coin Streak

Topic: probability **Difficulty**: easy
Find the expected number of flips of a fair coin needed to either have 10 more heads flipped than tails or 7 more tails flipped than heads.

Question 1020: Expected Returns

Topic: probability **Difficulty**: hard
A frog performs a simple symmetric random walk on the integers, starting at position 5 and hopping 1 unit up or down at each step with equal probability. Find the expected of number of times that the frog lands on position 1000 before landing on position 0.

Question 1021: Rolls in a Row II

Topic: probability **Difficulty**: easy
How many rolls of a fair 6-sided die must be rolled on average to get 5 and 6 in a row in that order?

Question 1022: Fish Slice

Topic: probability **Difficulty**: medium
Noelle is preparing a tuna fish of length L in anticipation of dinner. She selects 2 cutting points uniformly at random and independently along the length of the fish so that the cutting points are on opposite sides of the midpoint of the fish. Find the probability that the distance between the two points Noelle selects is greater than $\frac{L}{3}$.

Question 1023: 2 Lead

Topic: probability **Difficulty**: easy
Alice and Bob are playing basketball. Both start from 0 points. Each game, Alice has a 30% chance of winning, independent between games. In each game, the winner gains 1 point and the loser loses 1 point. The first person to 2 points wins the set. Find the probability Alice wins the set.

Question 1024: 12-8 Showoff

Topic: probability **Difficulty**: easy
Suppose you have a fair $8-$sided die and roll it once. You can either keep the value showing on that die or roll a fair $12-$sided die once. You receive the payout of the last die you roll. Find your expected payout under optimal play.

Question 1025: View All Sides

Topic: probability **Difficulty**: easy
On average, how many times do you need to roll a standard fair $6-$sided die to observe all of the sides?

Question 1026: Sine Condition

Topic: probability **Difficulty**: medium

$$X \sim \text{Unif}([0, \pi])$$

. Find $\cos(\mathbb{E}[X \,|\, \sin(X)])$.

Question 1027: Cats and Mice

Topic: brainteasers **Difficulty**: hard
A number of cats (more than one) killed between 999919 mice, and every cat killed an equal number of mice (more than one). Each cat killed more mice than there were cats. How many cats were there?

Question 1028: Random Tic Tac Toe

Topic: probability **Difficulty**: medium
We toss 3 pebbles at a tic-tac-toe board at the same time. Each of those pebbles will land in one of the nine squares of the board with uniform probability. What is the probability that the pebbles will land in such a way that they form a tic-tac-toe (all pebbles in a row either vertically, horizontally, or diagonally)?

Question 1029: Likely Targets II

Topic: probability **Difficulty**: hard
Two linear targets, say A and B, of respective radii ε and 2ε, where $\varepsilon << 1$, are placed on an infinitely long line. The targets are centered at $x_A = -1$ and $x_B = 3$. In other words, target A covers the interval $[1-\varepsilon, 1+\varepsilon]$, while target B covers the interval $[3-2\varepsilon, 3+2\varepsilon]$. You have one dart to shoot at the line. Your goal is to maximize your probability of hitting one of the targets. You can choose where to center your throw on the line. If you select to center your dart at μ, the actual position your dart lands at is $X \sim N(\mu, 4)$. Find the value of μ that maximizes your chances of hitting a target. If necessary, round your answer to the nearest hundredth.

Question 1030: Queen First

Topic: probability **Difficulty**: easy
Hannah shuffles a standard deck of cards. She wins \$100 if the last queen appears before the last king. Her friend Alissa looks at the card order and tells Hannah that there are exactly two queens before the first king. Given this additional information, how much should Hannah expect to earn?

Question 1031: Competitive Sampling

Topic: probability **Difficulty**: hard
You and your opponent each draw a number from $U(0,1)$ without revealing it. You each have the option to redraw and keep that value instead, and the other person doesn't know which choice they made. The person with the higher number wins. The optimal strategy is of the form reroll if the number is less than k. Find k to 3 decimal places.

Question 1032: Price an Option I

Topic: finance **Difficulty**: medium
You have access to European call options at the following strikes and T_0 prices, as well as the underlying S with an initial price of $S_0 = 21$. The calls are given in the format of $(\text{Strike } K, \text{Price } C_0)$

$$(15, 4.2)$$
$$(20, 1.4)$$
$$(25, 0.7)$$
$$(30, 0.1)$$

Find the time-0 price of a contract that pays $\min(S_T, 25)$ at time T.

Question 1033: Horse Racing

Topic: brainteasers **Difficulty**: medium
There are 25 horses, each of which runs at a distinct and constant speed. The track has five lanes, and thus can race at most five horses at once. By simply racing different sets of horses and observing the placing of the horses within those races without access to a stopwatch, what is the minimum number of races needed to uniquely identify the three fastest horses?

Question 1034: Car and Fly I

Topic: brainteasers **Difficulty**: easy
A road of length 400 miles has two cars, say A and B, starting at opposite ends of the road. Cars A and B respectively drive towards at each at constant rates of 120 miles per hour and 60 miles per hour. A fly starts on Car B and flies towards Car A at a rate of 180 miles per hour. Then, once it hits Car A, it immediately starts to fly back to Car B at the same rate. To the nearest minute, how long after the cars start driving will the fly return to Car B?

Question 1035: Poker Hands III

Topic: probability **Difficulty**: easy
A poker hand consists of five cards from a fair deck of 52 cards. What is the probability that you have two pairs (two cards of the same value and another two cards of the same value)?

Question 1036: Revolver Time

Topic: brainteasers **Difficulty**: easy
Sandra is firing her revolver, which has 6 chambers. The time between the first bullet and last is 1 minute. Assuming a uniform firing rate, in seconds, how long would it take Sandra to fire 3 shots?

Question 1037: The Orchard Problem

Topic: brainteasers **Difficulty**: medium
A market gardener was planting a new orchard. The young trees were arranged in rows so as to form a square, and it was found that there were 146 trees unplanted. To enlarge the square by an extra row each way, he had to buy 31 additional trees. How many trees were in the orchard when it was finished?

Question 1038: Exponent Reverse

Topic: brainteasers **Difficulty**: medium
Is π^e or e^π larger? Answer 1 and 2 for π^e and e^π, respectively.

Question 1039: Bike Count

Topic: brainteasers **Difficulty**: medium
You know that a bike shop has some weird bikes − their bikes are identical, and each bikeâs front and back wheels has at least one spoke each, but front and back wheels may or may not have different numbers of spokes. You know there are between 200 and 300 spokes in total in the shop and at least 2 bikes.

If you were told the exact number of spokes, you would be able to figure out the number of bikes. Unfortunately, you do not know the exact number of spokes, but just knowing that you could figure it out with this information is sufficient for you to deduce the answer. How many bikes are in the shop?

Question 1040: Picky Casino

Topic: probability **Difficulty**: medium
A dealer at a casino holds a game that is played as follows: The player first pays \$$d$ to the dealer, where d is some positive integer less than 100. The dealer then intends to flip a coin d times. If at any point two consecutive heads are flipped, then the dealer stops flipping the coin, and the player wins and is awarded \$100. Otherwise, the player loses (and is awarded nothing). A player pays the dealer \$29. The dealer weights the coin before flipping it, changing the probability of flipping heads such that the player's expected net profit is non-positive. The player is not aware of this re-weighing. Find the maximum probability that the dealer should choose.

Question 1041: Chess Tournament II

Topic: probability **Difficulty**: medium
A chess tournament has 128 players, each with a distinct rating. Assume that the player with the higher rating always wins against a lower rated opponent and that the winner proceeds to the subsequent round. Since the tournament's structure resembles that of a knockout bracket, 7 total rounds are played, including the final. What is the probability that the highest rated and the third-highest rated players will meet in the final?

Question 1042: Matching Die Pair

Topic: probability **Difficulty**: easy
Two fair 6−sided dice are rolled and their upfaces are recorded. Find the probability that if both dice are rolled again, the values rolled on the second trial are the same as on the first trial.

Question 1043: Particle Reach II

Topic: probability **Difficulty**: medium
Consider a particle that performs a random walk on the integers starting at position 0. At each step, the particle moves from position i to position $i+1$ with probability p, while the probability it moves from i to $i-1$ is $1-p$. If $p = 2/3$, find the probability the particle ever reaches position 1.

Question 1044: Even Steven

Topic: probability **Difficulty**: medium
Steven and his friend are playing a game where the first person to flip a heads wins. Steven's friend sadly only has a biased coin with a probability of 0.4 to get flip a heads. Since the friend is at a disadvantage, they decided that he would get the chance to flip first and then they alternate who flips their coin. Steven has a special coin where he can choose the probability it comes up heads. Because Steven is a fair man, he wants this game to be fair to both players. What probability of heads should Steven set his magical coin at to make this game fair?

Question 1045: The Sum Is Right

Topic: probability **Difficulty**: hard
Suppose that we generate $X_1, X_2, \ldots, X_n \sim \text{Unif}(0,1)$ IID. Let $U = \min\{X_1, \ldots, X_n\}$ and $V = \max\{X_1, \ldots, X_n\}$. Compute $\mathbb{P}[U + V > 1]$.

Question 1046: Coin Identifier

Topic: brainteasers **Difficulty**: easy
Brian walks to the convenience store with \$0.60 in his pocket. His money is made of solely nickels, dimes, and quarters. If Brian has N coins in his pocket, find the smallest value of N such that the quantities of each type of coin in Brian's pocket can't be uniquely identified.

Question 1047: Coin Pair I

Topic: probability **Difficulty**: easy
Four fair coins appear in front of you. You flip all four at once and observe the outcomes of the coins. After seeing the outcomes, you may turn any pair of coins over. You may not flip over a single coin without flipping over another. You can iterate this process as many times as you would like. Assuming optimal play, find the expected number of heads that appear.

Question 1048: Archery Accuracy

Topic: probability **Difficulty**: easy
An archer is shooting at a dartboard. If she has a $\frac{3}{4}$ chance of hitting any given shot, find the probability that she hits at least one of her next three shots.

Question 1049: Contracts and Options IV

Topic: probability **Difficulty**: easy
An energy company is drilling for oil. There is a 30% chance the well will be successful and a 70% chance it will fail. If they find oil, the stock is worth \$100. If they fail its worth \$20. You own a right (but not an obligation) to buy the stock for \$50 the day after they find out if the well is successful. What is the fair value of this option?

Question 1050: Uniform Order I

Topic: probability **Difficulty**: easy
Let $X_1, X_2, \ldots, X_{20}$ be IID $\text{Unif}(0,1)$ random variables. Compute $\mathbb{P}[X_1 > \max\{X_2, X_{10}, X_{15}\}]$

Question 1051: Bank Account Arbitrage

Topic: finance **Difficulty**: easy
Let's say you have two bank accounts, B_1 with interest rate 0.04 and B_2 with interest rate 0.02 (annual interest rates). Both are compounded continuously. Both bank accounts start with initial value $B_1 = B_2 = 1$. What is the minimum amount of money you are guaranteed to make in 1 year, assuming you can only deposit / borrow integer numbers in both bank accounts? You have \$0 initially and cannot just deposit your money into one of the bank accounts. Round to 2 decimal points.

Question 1052: Sum Exceedance IV

Topic: probability **Difficulty**: hard
A fair 5−sided die with the values $1 - 5$ on the sides is rolled repeatedly until the sum of all upfaces is at least 5. Find the expected number of times the die needs to be rolled.

Question 1053: Birds of a Feather

Topic: probability **Difficulty**: easy
Two blue and six yellow birds are flying around the sky and then land on a telephone wire in uniformly random order. Assuming no bird sits on another bird, what is the probability that the two blue birds are adjacent to each other?

Question 1054: R-Squared Range

Topic: statistics **Difficulty**: medium
Using OLS, we regress y onto X_1 and find that the model has an R^2 of 0.15. We also regress y onto X_2 but this time the model has an R^2 of 0.2. Let $[\min, \max]$ denote the lower and upper-bound of the R^2 of a model which regresses y onto X_1, X_2. Express your answer as $\frac{\max}{\min}$.

Question 1055: Take And Roll II

Topic: probability **Difficulty**: hard
You are given a fair 20−sided die and 100 actions in a game. The die starts with upface 1. The two options you can perform are to roll and to take. Performing a roll re-rolls the current upface of the die. Performing a take allows you to cash out the current upface of the die. Note that the game does not end when you perform a take. However, you must roll the die again before doing another take. Your strategy is to accept any number that is at least some threshold n. This n must be decided in advance and is fixed for the entire game. Assuming rational play in selecting n, find your expected payout.

Question 1056: Subsequent First Ace

Topic: probability **Difficulty**: medium
A deck is well-shuffled and cards are dealt out from the top one-by-one. The first ace is the 27th card dealt. Find the probability that the card right after this ace is the 2 of hearts.

Question 1057: Double Aces

Topic: probability **Difficulty**: easy
What is the probability of taking out 13 cards from a well shuffled standard 52 card deck and getting exactly 2 Aces? Round your answer to the nearest hundredths.

Question 1058: Last Love

Topic: probability **Difficulty**: medium
On average, how many cards do you need to flip over in a standard $52-$card deck to obtain your last heart-suited card?

Question 1059: Statistical Test Review II

Topic: statistics **Difficulty**: easy
We want to test whether or not a coin is balanced based on the number of heads, X, that appear after 36 tosses of the coin. Let's say we use the rejection region $|x - 18| \geq 4$. If $p = 0.7$, what is the value of β to the nearest ten thousandth?

Question 1060: Call Vega

Topic: finance **Difficulty**: easy
The underlying is currently at price $S_0 = 42$. You sell a call at strike $K = 36$. What will happen to vega if the price of the underlying decreases? Answer -1 if vega decreases, 0 if vega stays the same, and 1 if it increases.

Question 1061: Forming a Triangle

Topic: probability **Difficulty**: hard
A stick of unit length is randomly broken into three pieces. Assuming each break follows a uniform distribution along the stick, what is the probability that the three segments can form a triangle?

Question 1062: Rabbit Hop IV

Topic: probability **Difficulty**: medium
A rabbit starts at the floor in front of a staircase of 10 stairs. The rabbit can hop up any amount of steps strictly larger than 1 at each movement. In particular, this means that the rabbit can't go up a one stair staircase. How many distinct paths are there from the floor to the top of the staircase (i.e. to the top of the 10th stair)?

Question 1063: 2D Paths III

Topic: probability **Difficulty**: medium
You are playing a 2D game where your character is trapped in a 6×6 grid. Your character starts at $(0, 0)$ and can only move up and right. There are two power-ups located at $(2, 3)$ and $(4, 6)$. How many possible paths can your character take to get to $(6, 6)$ such that it can collect at least one power-up?

Question 1064: Horse Results

Topic: probability **Difficulty**: easy

$$6$$

equally-skilled swimmers labeled $1 - 6$ are racing. Find the probability that swimmer 2 ends up in 2nd place and swimmer 5 is in the top 3 somewhere.

Question 1065: Poisson Process Covariance

Topic: probability **Difficulty**: easy
Let $N(t)$ be a Poisson Process with rate 5. Compute $\text{Cov}(N(5), N(15))$.

Question 1066: Odd Coin Flips

Topic: probability **Difficulty**: medium
You flip 100 fair coins simultaneously. What is the probability that you observe an odd number of heads?

Question 1067: Points on a Circle II

Topic: probability **Difficulty**: medium

$$n$$

points are selected randomly at uniform around a circle. What is the probability that all n points are on the same semicircle for $n = 100$? The answer is in the form

$$\frac{a}{b^c}$$

for integers $a, b, c > 0$ with b minimal. Find $a + b + c$.

Question 1068: Expected Increase

Topic: probability **Difficulty**: medium
Let $X_1, X_2, \ldots$ be a sequence of IID random variables with some continuous PDF $f(x)$. Let N be the time at which the sequence stops decreasing i.e. the first value n such that $X_1 \geq X_2 \geq \cdots \geq X_{n-1}$ but $X_{n-1} < X_n$. Find $\ln(\mathbb{E}[N])$.

Question 1069: Car Bidding I

Topic: statistics **Difficulty**: easy
Fred is selling his old car. He will sell it to the first bidder that places a bid of at least \$9000. He receives bids for the car that are all independent and identically distributed exponential random variables with mean \$5000. Find the expected number of bids that Fred receives before selling his car. This is inclusive of the bid that he receives that makes him sell the car. The answer is in the form e^a for a rational number a. Find a.

Question 1070: Python or R

Topic: statistics **Difficulty**: medium
Two sets of developers learn to code in Python or R, 50 to each language. At the end of the instructional period, a coding performance test yielded the results $\bar{x_1} = 74, \bar{x_2} = 71, s_1 = 9, s_2 = 10$. What is the attained significance level (to 4 significant figures) of a test to conclude whether or not there is a difference in performance between the developers using the two languages? Assume simple random sampling, variance homogeneity, and that performance is approximately normally distributed.

Question 1071: Win by N

Topic: brainteasers **Difficulty**: easy
Person A has a 60% chance to win a round of a game against Person B. To win the entire game, one of the players must win N more rounds than the other person and they keep playing till someone wins. What is the probability Person A wins the entire game as N approaches infinity?

Question 1072: 2 Coin More

Topic: probability **Difficulty**: medium
Tim has 20 fair coins and Jeff has 22 fair coins. They both flip their coins. What is the probability that Jeff has more heads than Tim? Round to the nearest thousandth.

Question 1073: Compound Interest II

Topic: brainteasers **Difficulty**: easy
You start with \$100 in your bank account today. You invest in a stock that yields 1% interest that is compounded daily. To the nearest dollar, how much will you have in your bank account after 15 days?

Question 1074: Conditional Expectation II

Topic: statistics **Difficulty**: easy
Suppose $X \sim \text{Pois}(\lambda)$, where $\lambda \sim \text{Exp}(1)$. Compute $\mathbb{E}[X]$.

Question 1075: Annie's Coin

Topic: probability **Difficulty**: easy
Annie's coin lands on heads with probability $\frac{1}{3}$. Brittany's coin lands on heads with probability $\frac{2}{5}$. Annie and Brittany take turns tossing coins; Annie goes first. The first person to get a heads wins. What is the probability that Annie wins?

Question 1076: Same Month

Topic: brainteasers **Difficulty**: easy
What is the minimum amount of people needed in a group to guarantee at least two of them have a birthday in the same month?

Question 1077: Spaced Darts

Topic: probability **Difficulty**: hard
Nicole is throwing two darts at a dartboard of radius R. Let R_1 be the distance from the center to where Nicoleâs first dart lands. It is known that the distance from the center of the dartboard that the first dart lands is uniformly distributed. Once Nicole throws her first dart, it is known that she always throws her second dart a further distance away from the center than the first dart, and its location is uniformly distribution throughout the region that is further distance away than the first dart. Find the probability that her second dart is at least $\dfrac{R}{2}$ distance away from the center. The answer is in the form $\dfrac{a + \ln(b)}{c}$, where the fraction is fully reduced. Find $a + b + c$.

Question 1078: Highest Drawn Card

Topic: probability **Difficulty**: medium
Imagine you are playing a game in which you draw four cards from a standard shuffled 52 card deck. What is the expected value of the highest card. Round your answer to the nearest hundredth.

Note: Aces have a value of 1, Jacks a value of 11, Queens a value of 12, and Kings a Value of 13.

Question 1079: Busted 6 II

Topic: probability **Difficulty**: medium
Suppose you play a game where you continually roll a die until you obtain either a 5 or a 6. If you receive a 5, then you cash out the sum of all of your previous rolls (excluding the 5). If you receive a 6, then you receive no payout. You also have the decision to cash out mid-game for the sum of all your previously obtained rolls. Assuming optimal play, what is your expected payout? Round your answer to the nearest hundredth.

Question 1080: Triangular Selection I

Topic: probability **Difficulty**: easy
There are 8 points in some space, no three of which lie on the same line. Matt and Aaron each uniformly at random select 3 points in the space. Matt and Aaron respectively draw a triangle whose vertices are the 3 points they selected. What is the probability they end up with the same triangle?

Question 1081: Lognormal II

Topic: probability **Difficulty**: medium
Suppose that $\ln(X) \sim N(0,1)$. Find $\text{Var}(X^4)$. Your answer will be in the form of $e^a - e^b$ for positive integers a and b. Find $a + b$.

Question 1082: Three-Way Tile

Topic: probability **Difficulty**: hard
How many ways can you tile a 3×8 grid with 1×2 and 2×1 tiles?

Question 1083: Grid Filling III

Topic: probability **Difficulty**: medium
We place the integers $1 - 9$ (inclusive, no replacement) randomly on a 3×3 grid. What is the probability each row, column and diagonal add up to an odd number?

Question 1084: Ramen Bowl

Topic: probability **Difficulty**: hard
There are 100 noodles in your bowl of ramen. You take the ends of two noodles uniformly at random and connect the two, putting the connected noodle back into the bowl and continuing until there are no ends left to connect. On average, how many circles will you create? Round to the nearest whole number.

Question 1085: Dinner Party

Topic: brainteasers **Difficulty**: easy
At a dinner party, everyone shakes hands with everyone else exactly once. How many people are at the party if there are exactly 120 handshakes?

Question 1086: Farming Emergency

Topic: brainteasers **Difficulty**: easy
A farming company has been hired to till 11 plots of land. 4 farmers are initially hired. It takes them a total of 5 hours to till 2 of the plots. The farming company decides to enlist more farmers to finish up the job within the next 12 hours. What is the minimum number of additional farmers the company could have sent so that the other 9 plots of land are tilled in the next 12 hours? You may assume all plots takes the same time to till, individual farmers work at equal constant rates, and that they will work at the same efficiency regardless of the number of farmers there.

Question 1087: Complementary Dice

Topic: probability **Difficulty**: easy
Two players are playing a game by rolling a single standard die. Player 1 rolls first. If the top surface of the die shows 1 or 2, then he wins. If not, then Player 2 will roll the die. If the top surface of the die shows 3, 4, 5, or 6, then he wins. Else, the game is continues until there is a winner. Let p_1 and p_2 represent the probabilities that the two players win, respectively. Find $\max\{p_1, p_2\}$.

Question 1088: Smallest Probability

Topic: brainteasers **Difficulty**: easy
Suppose that we want to simulate an event E with $\mathbb{P}[E] = p$. We only have two rolls of a fair die and 1 flip of a fair coin. Find the smallest value of p for which we can simulate E.

Question 1089: Weighted Dice

Topic: probability **Difficulty**: easy
A 6−sided die is weighted so that the probability of rolling a side with n dots on it $(1 \leq n \leq 6)$ is the proportion of the number of dots on that side to the total number dots on the die. Find the probability that if this die is rolled twice, the sum will be a 7.

Question 1090: The Swarm of Bees

Topic: brainteasers **Difficulty**: easy
The square root of half the number of bees in a swarm has flown out upon a bush; eight-ninths of the whole swarm has remained behind; Then, one female and one male bee exit the bush to be caught by a bird. How many total bees were there at the start?

Question 1091: Face Value

Topic: probability **Difficulty**: easy
Ten cards with values $1 - 10$ are face down in front of you. You select one card at random and look at it. You can either choose the payout of \$3.50 or the face value of the card. What is the fair value of this game?

Question 1092: Circular Slice I

Topic: probability **Difficulty**: hard
A random angle $\theta_1 \sim \text{Unif}(0, 2\pi)$ is selected. Then, the arc of the unit circle that sweeps out θ_1 radians is marked red going counterclockwise starting from $(1, 0)$. Two other angles $\theta_2, \alpha \sim \text{Unif}(0, 2\pi)$ IID are also selected. Afterwards, an arc of length θ_2 radians starting from the point that is α radians counterclockwise of $(1, 0)$ is swept out and colored blue. Find the probability that the blue and red regions are disjoint.

Question 1093: Mississippi

Topic: probability **Difficulty**: easy
How many permutations of the word MISSISSIPPI are palindromes? A palindrome means that the string creates reads the same both forwards and backwards.

Question 1094: Poker Hands II

Topic: probability **Difficulty**: medium
A poker hand consists of five cards from a fair deck of 52 cards. What is the probability that you have a full house (three cards of the same value and another two cards of the same value)?

Question 1095: Doubly Stochastic

Topic: probability **Difficulty**: medium
We say that a transition matrix P is *doubly stochastic* if 1) every entry is non-negative and 2) every row and column sums to 1. Suppose that you have a Markov Chain with a doubly stochastic transition matrix P on $S = \{1, 2, \dots, 100\}$. Let π be the stationary distribution of this Markov Chain. Find $||\pi||$.

Question 1096: Non-Uniform Fix

Topic: probability **Difficulty**: medium
Let $T, X_1, X_2, \dots \sim \text{Beta}(12, 8)$ IID. Then, let $N = \min\{n \in \mathbb{N} : X_n > T\}$. Find $\mathbb{E}[N]$. If this is not finite, enter -1.

Question 1097: Fibonacci Limit I

Topic: brainteasers **Difficulty**: medium
Let F_n be the Fibonacci sequence. Compute $\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$. Your answer should be in the form $\frac{a+\sqrt{b}}{c}$, where all of a, b, and c are pairwise relatively prime. Find abc.

Question 1098: 9 Digit Sum

Topic: brainteasers **Difficulty**: medium
Find the sum of all $9-$digit integers using all of the digits $1-9$. The answer is

in the form $k \cdot a! \cdot (10^b - 1)$ for a, b, and k integers such that a and b are maximal. Find abk.

Question 1099: Betting Orbs

Topic: probability **Difficulty**: medium
You are presented with a bag of 4 orbs. You know that 2 are blue and 2 are red. You start drawing them without replacement from the bag one-by-one. Before each draw, you are given the opportunity to guess the color of the orb that is about to be drawn. If you get it correct, you earn \$1. Under an optimal strategy, what is your expected profit of this game?

Question 1100: Cow, Goat, and Goose

Topic: brainteasers **Difficulty**: easy
A farmer found that his cow and goat would eat all the grass in a certain field in 45 days, that the cow and the goose would eat it in 60 days, but that it would take the goat and the goose 90 days to eat it down.

Now, if he had turned cow, goat, and goose into the field together, how long would it have taken them to eat all the grass?

Question 1101: Better In Red IV

Topic: probability **Difficulty**: medium
The surfaces of a $3 \times 3 \times 3$ cube (initially white) are painted red and then is cut up into 27 $1 \times 1 \times 1$ small cubes. One of the cubes is selected uniformly at random and is rolled. The face appearing is red. What is the probability the cube selected was a corner cube?

Question 1102: Uniform Fix

Topic: probability **Difficulty**: easy
Let $U \sim \text{Unif}(0,1)$ and $X_1, X_2, \cdots \sim \text{Unif}(0,1)$ IID. Define $N = \min\{n \in \mathbb{N} : X_n > U\}$. Find $\mathbb{E}[N]$. Enter -1 if the answer is infinite.

Question 1103: Product and Sum

Topic: brainteasers **Difficulty**: easy
Suppose x and y satisfy $x + y = 10$ and $xy = 20$. Find the value of $x^4 + y^4$.

Question 1104: Better in Red II

Topic: probability **Difficulty**: medium
A $10 \times 20 \times 30$ rectangular prism is painted red on the surface and then cut into 6000 $1 \times 1 \times 1$ cubes and one is selected uniformly at random. Find the expected number of red faces on this cube.

Question 1105: Points on a Circle III

Topic: probability **Difficulty**: easy
There are 8 points on a circle. You draw all possible chords that can be formed by connecting pairs each pair of points. Then, you randomly selects 4 chords. What is the probability that the 4 chords form a convex quadrilateral?

Question 1106: Cheese Lover I

Topic: probability **Difficulty**: easy
Jon loves cheese. He decides to make 100 blocks of cheese. The distribution of the weight (in grams) of each block he makes follows IID $\text{Exp}\left(\frac{1}{250}\right)$ distribution. Let W_i denote the weight of the ith block of cheese, and T_{100} represent the total weight of the 100 blocks of cheese. Using Markov's Inequality, what is an upper bound on $\mathbb{P}[T_{100} > 26000]$?

Question 1107: Couple Handshakes

Topic: brainteasers **Difficulty**: easy
A room of four couples greet each other by shaking hands. If each person shakes the hand of every other person besides their partner, how many handshakes occur?

Question 1108: Bond Practice III

Topic: finance **Difficulty**: easy
Calculate the price of a bond with these characteristics. The coupon rate is 0.06, coupon payments are made every six months (twice per year), and the par value of the bond is 1,000. There are 5.0 years to maturity and a market interest rate of 0.05

Question 1109: Remainders

Topic: brainteasers **Difficulty**: hard
What is the smallest positive integer that has a remainder of i when divided by $i+1$ for all $1 \leq i \leq 9$?

Question 1110: Square Normal

Topic: probability **Difficulty**: medium
Let $X, Y \sim N(0,1)$ IID. Compute $\mathbb{E}[X \mid X^2 + Y^2]$.

Question 1111: Tricky Bob II

Topic: probability **Difficulty**: medium
Alice and Bob are each given a coin. Each is allowed decide the probability of heads for their coin. Afterwards, they both flip their coins. If it comes up HH, Alice gives Bob \$6. If it comes up TT, Alice gives Bob \$4. Otherwise, Bob gives Alice \$5. However, Alice must tell Bob the success probability p_1 they have chosen before Bob decides. Suppose $p_1 = \frac{3}{4}$. The range of probabilities p_2 for which Bob has positive expected value on the game is in the form $(a, 1)$, where a is a fraction in reduced form. Find a.

Question 1112: Breaking Even

Topic: probability **Difficulty**: medium
Julie and Judy bet on a series of 8 successive coin tosses. After each toss, assuming that both players have at least \$1, Julie gives Judy \$1 if it is heads, and Judy gives Julie \$1 if it is tails. They both start out with \$4 each. Compute the probability that they break even.

Question 1113: Distinct Date I

Topic: brainteasers **Difficulty**: medium
Find the next date where all of the digits, when expressed in the form $MM/DD/YYYY$ are distinct. For example, 01/23/4567 would be a valid date. Express your answer in the form $MMDDYYYY$.

Question 1114: Even Doubles

Topic: probability **Difficulty**: easy
Andre flips a fair coin until he obtains two consecutive heads or tails for the first time. Find the probability that Andre flips the coin an even amount of times.

Question 1115: Cubic Difference

Topic: brainteasers **Difficulty**: easy
Find the smallest positive integer y such that $y^2 = a^3 - b^3$ for some integers a and b.

Question 1116: Optimizing Aces

Topic: probability **Difficulty**: hard
Aaron picks an integer $k \in [1, 52]$. Then, he draws the first k cards from a standard, shuffled 52-card deck. Aaron wins a prize if the last card he draws is an ace and if there exists exactly one ace in the remaining cards. What k should Aaron pick?

Question 1117: Distinct Date II

Topic: brainteasers **Difficulty**: medium
Find the most recent date where all of the digits, when expressed in the form $MM/DD/YYYY$ are distinct. For example, 01/23/4567 would be a valid date. Express your answer in the form $MMDDYYYY$.

Question 1118: Missing Million I

Topic: probability **Difficulty**: easy
You are on a game show with 3 doors in front of you. One of the doors has $1 million inside, while the other two are empty. In the final round, the host lets you spin a wheel that may reveal which door the $1 million is in. This wheel will tell you the location of the $1 million 3/5 of the time. Otherwise, it tells you nothing, and you must guess uniformly at random. What is the probability you locate the $1 million door?

Question 1119: First and Last Heads

Topic: probability **Difficulty**: easy
A fair coin is flipped 5 times. Find the probability that the number of heads in the first 3 flips is equal to the number of heads in the last 2 flips.

Question 1120: Left Corner

Topic: probability **Difficulty**: easy
A pawn on an 8×8 chess board starts in the top left corner. In each move, the pawn is allowed to move one square in any direction, including diagonally, as long

as it stays on the board. Each legal move on a given turn has equal probability of being selected. Find the probability that in exactly two movements the pawn is back in the top left corner.

Question 1121: Optimal Marbles I

Topic: probability **Difficulty**: hard
Two players, say A and B, play the following game: Both players have 100 marbles and may put anywhere between 1 and 100 marbles in the box each. This decision is not revealed to the other player. Then, they draw 2 marbles with replacement between trials. If the marble belongs to A, then assuming that A put a marbles in the box, A is paid $100 - a$ monetary units from a third party. Similarly if the marble belongs to B, then assuming B but b marbles in the box, B is paid $100 - b$ monetary units from a third party. Assume both players play optimally. Find the expected total payout of player A.

Question 1122: Rowdy Root

Topic: brainteasers **Difficulty**: easy
Evaluate $\sqrt{120 \cdot 121 \cdot 122 \cdot 123 + 1}$.

Question 1123: First Quarter

Topic: brainteasers **Difficulty**: easy
You play a game with a friend. You both take turns to put a quarter on a round table (the table can be any size, but must be of symmetrical shape). The goal is to cover the table up with quarters. Quarters can't overlap with other quarters. A player loses when there's no place for them to put down a quarter. Should you be the first person to play? Answer 1 if yes and 2 if no.

Question 1124: Wood Chop

Topic: probability **Difficulty**: easy
We have a stick of length 1 that we are cutting into 6 parts with 5 uniform and random cuts. What is the probability that no piece is greater than $\frac{1}{2}$ in length?

Question 1125: Paired Pumpkins I

Topic: brainteasers **Difficulty**: easy
Dracula has 3 pumpkins, labeled $1 - 3$. He knows the mass of each pair of pumpkins is (in kgs) is $19, 21$, and 28. Let w_1, w_2, and w_3 be the weights of the three pumpkins. If possible, find $w_1^2 + w_2^2 + w_3^2$. If this is impossible, enter -1.

Question 1126: Brownian Supremum

Topic: pure math **Difficulty**: hard
Let W_t be a standard Brownian Motion. Define $M_t = \sup_{s \in [0,t]} W_s$. For any $t > 0$, find $\mathbb{P}[M_t > 0]$.

Question 1127: Card Up

Topic: probability **Difficulty**: easy
There are 83 cards in a deck. Each is labeled with a distinct value $1 - 83$. A dealer randomly shuffles them up and deals the top 5 cards of the deck are dealt. If the top 5 cards are in strictly ascending or descending order, your payout is $\$x$; otherwise, you receive nothing. If the cost is \$1 to play this case, what must x be for the game to be fair?

Question 1128: Covariance Review I

Topic: statistics **Difficulty**: medium
Consider the following joint pdf:

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} c(1 - x_2) & 0 \leq x_1 \leq x_2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

where c is a constant such that f_{X_1,X_2} is a valid joint pdf. Compute $\text{Cov}(X_1, X_2)$.

Question 1129: Sharpe Maximization

Topic: finance **Difficulty**: hard
Suppose you have two assets, say A and B. It is known that asset A has expected excess returns of 7% and asset B has expected excess returns of 4%. The covariance matrix of the excess returns of A and B is given by $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Constraining to holding 1 unit of assets total (i.e. if you buy a units of asset A and b units of asset b, $a + b = 1$), how many units of asset A should be held to maximize the expected Sharpe Ratio of the portfolio consisting of some combination of assets A and B?

Question 1130: Ken Flipping Coins

Topic: probability **Difficulty**: medium
Ken flips a fair coin until he observes a head. He receives the minimum of \$64 and $\$2^n$, where n is the number of times that Ken flips the coin. What is his expected payoff?

Question 1131: Red Blue Equality

Topic: probability **Difficulty**: easy

$$3$$

red and n blue socks are in a drawer. We know that when two socks are drawn uniformly at random without replacement, the probability both socks are red is $\frac{1}{2}$. Find n.

Question 1132: A Low Median

Topic: probability **Difficulty**: easy
Suppose we draw 3 numbers from a Unif$[0, 1]$ distribution independently. We then sort the numbers in ascending order. What is the probability that the median is less than $2/3$?

Question 1133: Corner Meet

Topic: probability **Difficulty**: medium
You have a 5×5 checkerboard. You randomly place a checker in the center 3×3 grid. The checkers can move in two equally random ways: up to the left, and up to the right. If the checker is on the boundary, then it can only make valid moves. What is the probability that the checker ends up in the top-left or top-right corner?

Question 1134: Generational Wealth I

Topic: probability **Difficulty**: easy
Suppose that you generate a uniformly random number in the interval $(0, 1)$. You can either keep this number or generate a new number once. Your payout is the last number generated. Find your expected payout under a rational strategy.

Question 1135: Bad Bagel

Topic: probability **Difficulty**: easy
A bagel factory has a conveyor belt that delivers bagels continuously. Every bagel is either good (G) or bad (B). If a given bagel was G, the next bagel is also G with probability $\frac{2}{5}$. If a given bagel was B, the next bagel is also B with probability $\frac{3}{5}$. You woke up to find that the first bagel is a bad one. Find the expected number of bagels that pass until another bad bagel.

Question 1136: Animal Farm

Topic: brainteasers **Difficulty**: medium
A farmer goes to the market and spends \$1000 on 100 animals. The farmer buys at least one of each animal. The cost of each cow is \$50. The cost of each sheep is \$10. The cost of each chicken is \$0.50. Let x_1, x_2, and x_3 represent the number of cows, sheep, and chickens that the farmer bought, respectively. Find $x_1 x_2 x_3$.

Question 1137: Birthday Twins I

Topic: probability **Difficulty**: easy
What is the probability that at least two people in a group of three friends were born on the same day of the week?

Question 1138: Circular Covariance

Topic: probability **Difficulty**: easy
Suppose that $(X, Y) = (\cos\theta, \sin\theta)$, and θ is discrete uniform over the set $\left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \ldots, \frac{7\pi}{4}\right\}$. Find $\mathrm{Cov}(X, Y)$.

Question 1139: Bombing Campaign

Topic: probability **Difficulty**: easy
A bomb lands uniformly at random within a circle with radius 1 mile such that the target is at the center of the circle. When the bomb explodes, it wipes out everything within $\frac{1}{2}$ miles of its landing spot. Compute the probability that the target is not destroyed.

Question 1140: Color Swap

Topic: probability **Difficulty**: medium
Two urns A and B are presented before you. Urn A has 100 green and 400 red balls. Urn B has 400 green and 100 red balls. 300 of the balls from Urn A are randomly selected and moved to Urn B. Then, one ball in Urn B is selected uniformly at random from the 800 balls. Find the probability that this ball is red.

Question 1141: Particle Reach X

Topic: probability **Difficulty**: medium
Consider a particle that performs a random walk on the integers starting at position 0. At each step, the particle moves from position i to position $i+1$ with probability p, while the probability it moves from i to $i-1$ is $1-p$. If $p = 2/3$, find the variance of the number of steps until the particle reaches 7. If the answer is infinite, answer -1.

Question 1142: Repetitious Game II

Topic: probability **Difficulty**: medium
Audrey repeatedly draws cards from a standard 52-card deck with replacement. Compute the expected number of draws for Audrey to get at least one card from each suit.

Question 1143: Basic Dice Game II

Topic: probability **Difficulty**: medium
A casino offers you a game with a six-sided die where you are paid the value of the roll. The casino lets you roll the first time. If you are happy with your roll, you can cash out. Else, you can choose to roll a second time. If you are happy with your roll, you can cash out on this second value. Else, you can choose to roll for your third and final time and cash out on this third value. What is the fair value of this game?

Question 1144: Rabbit Hop III

Topic: probability **Difficulty**: medium
A rabbit starts at the floor in front of a staircase of 10 stairs. The rabbit can hop up 1 or 2 stairs at each step. How many distinct paths are there from the floor to the top of the staircase (i.e. to the top of the 10th stair)?

Question 1145: Odd Valued Roll

Topic: probability **Difficulty**: easy
On average, how many times must a fair $12-$sided die with values $1-12$ be rolled to obtain an odd value showing up?

Question 1146: Toasting Bread

Topic: brainteasers **Difficulty**: easy

Your frying pan can hold up to three slices of bread for toasting. It takes one minute to toast one side of bread, and both sides of a slice of bread must be toasted in order for it to be considered toasted. Assuming that the time required to flip a slice of bread is negligible, how many minutes will it take you to toast four slices of bread?

Question 1147: Big Mod I

Topic: pure math **Difficulty**: medium

Find the remainder of 11^{2360} when it is divided by 50.

Question 1148: The Arithmetical Cabby

Topic: brainteasers **Difficulty**: medium

The driver of the taxi-cab was wanting in civility, so Mr. Wilkins asked him for his number. "You want my number, do you?" said the driver. "Well, work it out for yourself. If you divide by number by $2, 3, 4, 5, 6$, you will find there is always 1 over, but if you divide it by 11, there ain't no remainder. What's more, there's no other driver with a lower number who can say the same." What was the fellow's number?

Question 1149: Contract Arbitrage

Topic: finance **Difficulty**: medium

We have a contract, V, that pays $\min(2S_T, 4)$ at expiry with initial price $V_0 = 3.4$, the following European puts, given in the format of (Strike K, Price C_0).

$$(4, 2.4)$$
$$(2, 0.8)$$

We also have access to the underlying, S, with an initial price of $S_0 = 3$ and bonds that pay 1 at expiry, with an initial price of $Z_0 = 0.9$. Find the arbitrage. Give the answer in the format of # Contract (V) + # Put (K = 4) + # Put (K = 2) + # Bonds

Question 1150: The Weight of the Fish

Topic: brainteasers **Difficulty**: easy

A man caught a fish. The tailed weighed 9 ounces. The head weighed as much as the tail and half the body, and the body weighed as much as the head and tail together. What is the weight of the fish? Give the answer in ounces.

Question 1151: Bolt Variance I

Topic: statistics **Difficulty**: easy
A manufacturer produces nuts and bolts that have a diameter variance no larger than .0002 centimeters. A random sample of twelve bolts returned a sample variance of .0003. What is the value of the appropriate test statistic to test, at the 5% level, $H_0 : \sigma^2 = .0002$ against $H_a : \sigma^2 > .0002$? Assume independence, variance homogeneity, and that diameter variance is approximately normally distributed.

Question 1152: Log Square

Topic: pure math **Difficulty**: medium
Consider solving the SDE $dM = \sigma M dW_t$. The two methods below yield different answers in calculating $d \ln M^2$, so one of them is incorrect. Let $a = 1$ if the first method is incorrect and $a = 2$ if the second is incorrect. Let l be the first line number in the solution that is incorrect. Find $100a + l$.

Method 1

$$\begin{aligned}
dM^2 &= \left(M^2\right)' dM + \frac{1}{2}\left(M^2\right)'' d[M,M] \\
dM^2 &= 2M dM + d[M,M] \\
dM^2 &= 2\sigma M^2 dW_t + \sigma^2 M^2 dt \\
\frac{dM^2}{M^2} &= 2\sigma dW_t + \sigma^2 dt \\
d \ln M^2 &= 2\sigma dW_t + \sigma^2 dt
\end{aligned}$$

Method 2

$$\begin{aligned}
d \ln M^2 &= \left(\ln M^2\right)' dM + \frac{1}{2}\left(\ln M^2\right)'' d[M,M] \\
d \ln M^2 &= \frac{2M}{M^2} dM + \frac{1}{2}\left(\frac{2M}{M^2}\right)' d[M,M] \\
d \ln M^2 &= \frac{2}{M} dM + \frac{1}{2}\left(\frac{2}{M}\right)' d[M,M] \\
d \ln M^2 &= \frac{2}{M} dM - \frac{1}{2}\frac{2}{M^2} d[M,M] \\
d \ln M^2 &= \frac{2}{M} dM - \frac{1}{M^2} d[M,M] \\
d \ln M^2 &= 2\sigma dW - \sigma^2 dt
\end{aligned}$$

Question 1153: Ping Pong Tournament I

Topic: probability **Difficulty**: easy

$$50$$

people are competing in a ping pong tournament where there is only one ping pong table. The competitors are numbered 1 through 50. Suppose that if two competitors meet, the one with the larger number wins. Two competitors are chosen at random, and the loser is removed from the tournament. The winner moves on to the next round, where their opponent is chosen at random. This process is repeated until one person is left (a total 49 rounds will be played). Compute the probability that competitors 49 and 50 play in the final round.

Question 1154: Skewed

Topic: statistics **Difficulty**: medium

Suppose that X is a continuous random variable with PDF $f(x) = \dfrac{\lambda}{2}e^{-\lambda|x-\mu|}$ for all some real constant λ and $\mu > 0$. Compute the skewness of X i.e. $\mathbb{E}\left[\left(\dfrac{X-\mu_X}{\sigma_X}\right)^3\right]$, where μ_X and σ_X are the mean and standard deviation of X. You may assume without proof the first three moments of X are finite.

Question 1155: 20-30 Die Split I

Topic: probability **Difficulty**: medium

Alice and Bob have fair $30-$sided and $20-$sided dice, respectively. Both roll their die, and the person with the higher value showing wins. The loser must pay the winner the value showing on the winner's die. In the event of a tie, Bob is the winner. Find the expected payout for Alice.

Question 1156: Sum Over Min Die

Topic: probability **Difficulty**: medium

You have been tasked to find the expected value of a die game in which you are rolling 2 dice at a time. Your first roll of the dice will all be summed up and will be your starting score. Your next, and final, roll of the 2 dice will be your divisor, in which you are going to pick the optimal die out of the two to divide your total by, which will be your final score.

Calculate the expected value of playing this game optimally.

Question 1157: Mixed Set II

Topic: probability **Difficulty**: easy
How many subsets of $\{1, 2, \ldots, 10\}$ contain exactly 2 elements from $\{1, 2, 3\}$ AND only have odd elements?

Question 1158: Water Measurement

Topic: brainteasers **Difficulty**: hard
A maid was sent to the brook with two vessels that exactly measured 7 pints and 11 pints exactly. She had to bring back exactly 2 pints of water. What is the smallest possible number of transactions necessary? A transaction is filling a vessel, emptying it, or pouring from one vessel to another. In other words, you cannot transfer a proportion of the vessel. You must transfer all of it.

Question 1159: Conditionally Normal

Topic: probability **Difficulty**: medium
Suppose that $X, Y \sim N(0, 1)$ IID. Find $\mathbb{P}[X > 0 \mid X + Y > 0]$.

Question 1160: Three Repeat I

Topic: probability **Difficulty**: easy
A fair coin is flipped 5 times. Find the probability of obtaining exactly 3 consecutive heads somewhere in the 5 flips.

Question 1161: Origin In Between

Topic: probability **Difficulty**: easy
Two points are uniformly at random selected between -1 and 2. What is the probability that the origin lies between those two points?

Question 1162: Matching Jar Colors

Topic: probability **Difficulty**: easy
Jar A has 3 blue and 4 red balls. Jar B has 6 blue and 4 red balls. If one ball is selected uniformly at random from each jar, what is the probability that the two selected balls match in color?

Question 1163: Delta Decay

Topic: finance **Difficulty**: hard
Suppose you have a European call option with strike $K = 23$ on an underlying S with initial price $S_0 = 25$. The Δ of this call option is currently 0.74. We delta-hedge our position such that the overall delta of our portfolio is 0. Suppose that our call option expires in 4 hours.

Give an approximation for the Δ of our portfolio after 2 hours, provided that the underlying stays at the same price. Round to 2 decimal points.

Question 1164: Team Division

Topic: probability **Difficulty**: easy
There are 12 employees at QuantGuide. 3 employees will work on the Software Engineering Team, 4 employees will work on the Content Development Team, and 5 employees will work on the Marketing team. Before assigning the employees, Michael and Nuo Wen admit that they don't want to be on the Marketing team. Under this restriction, how many ways can the employees at QuantGuide be assigned to teams?

Question 1165: Secrets

Topic: probability **Difficulty**: medium
Consider n people $P_1, \ldots, P_n$. P_1 receives binary information "yes" or "no" and will pass this information along to P_2. More generally, P_i will pass information along to P_{i+1}. However, P_i will transfer the information that they hear with probability p and transfer the opposite information with probability $1-p$, where $0 < p < 1$, independent of all other people. Let A_i be the event that P_i transfers the original information (i.e. what P_1 received) to P_{i+1}, and let the probability of this be p_i. Compute $\lim\limits_{n\to\infty} p_n$. If this limit does not exist, answer -1.

Question 1166: Numerous Uniforms

Topic: probability **Difficulty**: hard
Suppose that $X_1, \ldots, X_7 \sim \text{Unif}(0,1)$. Find the PDF of $Y = X_1 X_2 \ldots X_7$. The answer is in the form $f(x) = \dfrac{(-\log(x))^a}{b} I_{(0,1)}(x)$ for integers a and b. Find $a + b$.

Question 1167: Integral Variance I

Topic: pure math **Difficulty**: medium

Let W_t be a standard Brownian Motion. Find $\text{Var}\left(\int_0^t W_s ds\right)$ as a function of t. The answer is in the form kt^3 for a constant k. Find k.

Question 1168: Radioactive Decay

Topic: probability **Difficulty**: easy

Plutonium decays at an average rate of 1 time per second. Find the probability an atom of Plutonium doesn't decay in next 3 seconds. The answer is in the form e^a for some a. Find a.

Question 1169: Goat Search

Topic: probability **Difficulty**: easy

A farmer wants one male and one female goat for breeding. He goes to Mount Everest to find goats. Each goat is known to be male with probability $\frac{3}{4}$, independent of all other goats. The farmer takes a goat, examines it, and then leaves with the first two goats that are of opposite genders. If N is the total number of goats that the farmer examines until he observes two goats of opposite genders, find $\mathbb{E}[N]$.

Question 1170: Binomial Pricing a Binary Call

Topic: finance **Difficulty**: medium

We want to find the fair-value of a European binary call on an underlying S with initial value $S_0 = 4$. At each time step, the underlying can either grow by 90% or shrink by 30%. At expiry ($T = 2$), the call will pay 1 if $S > 2$ and 0 otherwise. Round to 3 decimal points.

Question 1171: Dice Labels

Topic: probability **Difficulty**: medium

How many distinct ways can you label a 6 sided dice if you wipe off all the numbers? Arrangements that can be formed by rotating the die around are not considered distinct.

Question 1172: Statistical Test Review VII

Topic: statistics **Difficulty**: medium
Andy claims that 20% of the public loves QuantGuide. Aaron dislikes Andy and wants to prove him wrongâspecifically, he thinks Andy's estimate is too large. He takes a random sample of 100 people to test his claim. It turns out that 15 of the 100 people love QuantGuide, so Aaron argues that 15% of the public loves QuantGuide. Compute, to the nearest thousandth, the value of β for Aaron's alternative hypothesis under a 0.05-level test (note that $z_{0.05} = -1.645$).

Question 1173: DeMorgan's Birthday

Topic: brainteasers **Difficulty**: easy
Augustus DeMorgan (yes, from DeMorgan's Laws) died in 1871. He used to brag that he was x years old in the year x^2. Assuming a lifespan of no longer than 120 years, what is x?

Question 1174: Conditional First Ace

Topic: probability **Difficulty**: hard
Suppose we deal out cards from a standard deck. Find expected number of cards after the first 2 and before the first ace given that the first 2 appears before the first ace.

Question 1175: Options Delta

Topic: finance **Difficulty**: easy
We have a European call and put option at strike K. The call option has a Δ of 0.21. What is the Δ of the put option?

Question 1176: Identical Alpha

Topic: statistics **Difficulty**: hard
Let X_1 and X_2 be i.i.d. uniform distributions over $[\theta, \theta + 1]$. We are testing the null hypothesis $H_0 : \theta = 0$ against the alternative hypothesis $H_a : \theta > 0$ with two possible tests. The first test will reject H_0 if $X_1 > 0.95$ and the second test will reject H_0 if $X_1 + X_2 > c$, for some constant c. For what value of c will the two tests have the same α?

Question 1177: Pascal Ratio

Topic: brainteasers **Difficulty**: medium
Row n of Pascal's Triangle contains three successive entries with ratio $3:4:5$. What is the smallest value of n?

Question 1178: Postgame

Topic: probability **Difficulty**: medium
Marc is exiting a function and walks out the door to go home. Each minute, he takes either one step left or one step right, independent of all other steps, with equal probability. Let D_n be the distance (number of steps left or right) Marc is from the door after n minutes. Find $\mathbb{E}[D_{10}^2]$.

Question 1179: Delta Decay II

Topic: finance **Difficulty**: hard
You currently have a European straddle (not necessary at-the-money) at strike $K = 25$ and with underlying price $S_0 = 27$.

The call option has $\Delta = 0.72$ and the put option has $\Delta = -0.28$. The straddle expires in exactly 1 hour. Currently, you are delta-hedged by buying (or selling) the underlying such that the overall portfolio Δ is 0. The straddle expires in exactly 1 hour. Let's say 30 minutes pass and the underlying remains at the same price, how many units of the underlying do you need to buy/sell to remain delta-hedged? Enter $-x$ if your answer is to sell x units. Fractional quantities are allowed.

Question 1180: Prepped Offer?

Topic: probability **Difficulty**: easy

$$45\%$$

of students donât use QuantGuide, 25% are heavy prep users and 30% are light prep users. If the heavy prep users are twice as likely to get an offer as light prep users, and light prep users are twice as likely to get an offer as those that donât use the platform, then what is the probability that if someone got an offer that they were a heavy prep user of QuantGuide?

Question 1181: Colorful Socks III

Topic: probability **Difficulty**: easy socks are in a drawer. There are 10 colors of socks in the drawer. 9 of the colors just have a pair of socks of that color. The remaining color has 3 socks of that color in the drawer. You draw out 2 socks at random. Find the probability that you obtain a matching pair.

Question 1182: Extrinsic Value I

Topic: finance **Difficulty**: easy
We have an underlying S with price $S_0 = 5$. You have a European call option on S with strike $K = 2$, currently with value $C_0 = 4.7$. What is the intrinsic (I) and extrinsic (E) value of the option?

Give the answer in the format of $I^2 + E^2$.

Question 1183: Up or Down

Topic: brainteasers **Difficulty**: easy
If you have an asset that goes up or down 20% each day from its price the day prior with equal probability, whatâs the probability that after 10 days the assetâs price is unchanged?

Question 1184: Crossing the River

Topic: brainteasers **Difficulty**: medium
During the Turkish stampede in Thrace, a small detachment found itself confronted by a wide and deep river. However, they discovered a boat in which two children were rowing about. It was so small that it would only carry the two children, or one grown up person. There are a total of 358 officers and soldiers. How many times does the boat pass from shore to shore if all the officers and soldiers need to get across, and that the two children are the last ones in joint possession of the boat?

Question 1185: Red-Blue Die Match

Topic: probability **Difficulty**: easy
You have three fair $6-$sided dice in front of you. One of the dice is red, while the other two are blue. Find the probability exactly one of the blue dice matches the red die in value.

Question 1186: Dice Order II

Topic: probability **Difficulty**: medium
You roll three fair dice. What is the probability that the highest value rolled is a four?

Question 1187: Coin Flipping Competition II

Topic: probability **Difficulty**: hard
Ty, Guy, and Psy are all flipping fair coins until they respectively obtain their first heads. Let T, G, and P represent the number of flips needed for Ty, Guy, and Psy, respectively. Find $\mathbb{P}[T \leq G \leq P]$.

Question 1188: Head Tail Sequence

Topic: probability **Difficulty**: easy

$$5$$

fair coins are flipped. Given that you obtained 3 heads, find the probability that you obtained the sequence $THHTH$.

Question 1189: Number 50

Topic: brainteasers **Difficulty**: medium
How many 10 digit numbers are there whose digits are prime and whose product of its digits is exactly 100000?

Question 1190: Boys with Girls

Topic: probability **Difficulty**: medium
There are 15 boys and 10 girls in a class. They line up in a row in random order. What is the expected amount of times a boy and a girl are standing next to each other? Ex: $BGBBGGGBGGBBBBG$ has 7 occurrences where a boy and girl are next to each other.

Question 1191: Shopping Habits

Topic: statistics **Difficulty**: hard
A researcher is investigating the temporal effect of shopping habits at the mall. She independently and randomly selects 20 weekend and 20 weekday shoppers and finds that weekend shoppers spend \$78 on average with a standard deviation of \$22, while weekday shoppers spend \$67 on average with a standard deviation

of \$20. The researcher would like to test if there is sufficient evidence to claim that there is a significant difference in the average amount spent by weekend and weekday shoppers. What is the attained significance level? Round to 3 significant figures. Assume simple random sampling, variance homogeneity, and that spending is approximately normally distributed.

Question 1192: Familiar Neighbors

Topic: probability **Difficulty**: medium

$$2n$$

people, $n > 1$, are seated randomly at a circular table with $2n$ seats for the 2023 Ballon D'Or Ceremony. The $2n$ people have distinct labels $0, 1, \ldots, 2n-1$, where the person with label i knows exactly i of the other $2n-1$ people at the table. Note that just because person A knows person B, that does not imply person B knows person A. Find the expected proportion of people that know at least one of the people on either side of them.

Question 1193: Mossel's Dice

Topic: probability **Difficulty**: medium
You roll a fair $6-$sided die until you get 6. What is the expected number of rolls (including the roll giving 6) performed conditioned on the event that all rolls show even numbers?

Question 1194: Random Subsets

Topic: probability **Difficulty**: hard
Subsets A and B are chosen uniformly at random from the collections of all subsets of a set X of cardinality 5. What is the probability that A is a subset of B?

Question 1195: Bridge Crossing

Topic: brainteasers **Difficulty**: easy
Alice, Bob, Charlie, and Daniel need to get across a river. The only way to cross the river is by an old wooden bridge, which holds at most two people at a time. Because it is dark, they can't cross the bridge without a lantern, of which they only have one. Each pair can only walk at the speed of the slower person. They need to get all of them across to the other side as quickly as possible. Alice takes 10 minutes to cross; Bob takes 5 minutes; Charlie takes 2 minutes; Daniel takes 1 minute. What is the minimum time to get all of them across to the other side?

Question 1196: Wandering Ant I

Topic: probability **Difficulty**: medium

An ant starts at the origin in the plane. At each step, with probability $\frac{1}{4}$, the ant will move one unit north, south, east, or west. What is the probability that the ant returns to the origin before hitting any point on the boundary of the square with vertices at $(\pm 2, \pm 2)$?

Question 1197: Statistical Test Review VI

Topic: statistics **Difficulty**: easy

Andy claims that 20% of the public loves QuantGuide. Aaron dislikes Andy and wants to prove him wrongâspecifically, he thinks Andy's estimate is too large. He takes a random sample of 100 people to test his claim. With $\alpha = 0.05$ (note that $z_{0.05} = -1.645$), what is the maximum number of people in the sample that may love QuantGuide such that Aaron can refute Andy's claim?

Question 1198: Bakugan and Beyblade

Topic: probability **Difficulty**: medium

Two machines independently manufacture Bakugan and Beyblade toys. The time it takes each machine to produce a toy is Exp(1) distributed. The manufacturing time is independent of all other toys. Assume there is no delay between when a machine produces a toy to when it starts the next toy. Find the probability that the third Bakugan is manufactured before the second Beyblade.

Question 1199: Positive Brownian I

Topic: pure math **Difficulty**: medium

Let W_t be a standard Brownian Motion. Find $\mathbb{P}[W_1 > 0, W_2 > 0]$.

Question 1200: No More Than Four

Topic: probability **Difficulty**: easy

Alex tosses two dice n times. The probability that the sum of the values on the two dice faces is at most 4 in at least one of the n tosses is p. Find the maximum value of n such that $p < 0.5$.

Question 1201: Square Shade

Topic: brainteasers **Difficulty**: hard

Consider a 2023×2023 grid. The grid squares from the middle row are shaded in. What is the probability that a randomly selected rectangle contains at least one shaded square?

Question 1202: Covariance Review III

Topic: probability **Difficulty**: easy

$$Z \sim \mathcal{N}(0,1), X = Z, Y = Z^2$$

. Compute $\text{Cov}(X, Y)$.

Question 1203: 1 Glove Off

Topic: probability **Difficulty**: hard
You have 5 pairs of gloves that each have a distinct number $1-5$. The 10 gloves are randomly paired up. Find the probability that the gloves are paired up such that the values of any pair differ by at most 1.

Question 1204: Magic 11

Topic: brainteasers **Difficulty**: medium
A number is magical if it ends with the digits 11. Suppose $x \in \{1, \ldots, 10^{100}\}$. What is the probability that x^3 is magical?

Question 1205: Side Add

Topic: probability **Difficulty**: easy
Ab rolls a fair standard $6-$sided die. Afterwards, given he rolls k on the first roll, Ab rolls a fair $(6+k)-$sided die for the second roll with the values $1, 2, \ldots, 6+k$. Find the expected value that appears on Ab's second roll.

Question 1206: Choose Your Profit

Topic: probability **Difficulty**: easy
You play a game where you initially start with \$1. For each roll of a die where you don't obtain a 1, you will multiply your current payout by α. At any point you are allowed to stop the game and take your payout. Otherwise, if you don't stop and roll a 1, you receive no payout. Find the value of α such that your expected payout on the game is constant regardless of the number of times you roll.

Question 1207: 12 Left

Topic: probability **Difficulty**: easy
Three players, say A, B, and C, are going to draw 5 chips each (without replacement) out of a bag of 15 chips numbers $1-15$. The order of selection is $A \to B \to C$. Given that player A only selected even-numbered chips, find the probability that player B selected Chip 12.

Question 1208: Equal Unequal Game

Topic: probability **Difficulty**: easy
Abby and Ben are playing a coin flipping game. Abby has a coin with probability p of showing heads on each flip and Ben has a fair coin. Abby flips her coin first, and then they alternate until one of them shows a heads. The person who flips the first heads is declared the winner. If it is known that Abby and Ben have equal chances of winning the game, find p.

Question 1209: Careful Side Choice

Topic: probability **Difficulty**: medium
You and a friend are each given a die. Both of you are allowed to independently determine how many sides each of your individual dice have. You may select $1, 2, 3, 4, 5$, or 6 sides. The sides are labeled $1-k$ if you select k sides. Each side is equally likely to appear among the number of sides you select. Then, both of you roll your dice. If the sum of the two upfaces is at most 7, you receive a payout equal to the upface of your die from a third party. Otherwise, neither of you receive anything. Assume both players play optimally. Let s be the number of sides you select and p be the expected payout. Find sp.

Question 1210: Cheese Lover III

Topic: probability **Difficulty**: easy
Jon loves cheese. He decides to make 100 blocks of cheese. The distribution of the weight (in grams) of each block he makes follows IID $\text{Exp}\left(\frac{1}{250}\right)$ distribution. Let W_i denote the weight of the ith block of cheese, and T_{100} represent the total weight of the 100 blocks of cheese. Using Central Limit Theorem, what is an approximation for $\mathbb{P}[T_{100} > 26000]$? The answer is in the form $\Phi(a)$ for some real a. Φ here is the CDF of the standard normal distribution. Find a.

Chapter 2

Solutions

Solution to Question 1: Place or Take

There are two aspects to this game that should be noted: the ordering of the places and takes, and the number of places and takes. We can tackle the former aspect first. Intuitively, no places should come after a take. Greedily so, you might as well put a take after all places such that you have a higher expected value of that take (considering a single take, every place you put after the take is \$1 less that could be taken); thus, all takes should be stacked at the end of the 100 turns. This can also be proved with induction, and this will be left as an exercise to the reader. The next aspect is regarding the number of places and takes. If there are p places, then there must be $100-p$ takes, and we know that the p places must occur before the $100-p$ takes. In order to find the value of p, we can calculate at what position does adding an additional place action not improve our expected payoff. Consider the p-th action where $1 < p < 100$ and where this is the last place before we begin taking, or the next action $(p+1)$ will be the last pace before we begin taking. What is the expected value of the game if action p is the last place? There are p dollars in aggregate and we have $100-p$ takes. Our expected payoff will be $p \times \sum_{i=1}^{100-p} \frac{1}{2^i}$, which can be seen from the linearity of expectation- at every take, we will take half of what is remaining on average. What is the expected value of the game if action $p+1$ is the last place? There are $p+1$ dollars in aggregate and we have $100-(p+1)$ takes. Thus, our expected payoff is $(p+1) \times \sum_{i=1}^{100-p-1} \frac{1}{2^i}$. We are looking for the largest value of p such that the payoff where the p-th place is the last place is greater the payoff where the incremental $p+1$-th place is the last place. Thus:

$$p \sum_{i=1}^{100-p} \frac{1}{2^i} > (p+1) \sum_{i=1}^{100-p-1} \frac{1}{2^i}$$

$$p \sum_{i=1}^{100-p} \frac{1}{2^i} > p \sum_{i=1}^{99-p} \frac{1}{2^i} + \sum_{i=1}^{99-p} \frac{1}{2^i}$$

$$\frac{p}{2^{100-p}} > \sum_{i=1}^{99-p} \frac{1}{2^i}$$

$\sum_{i=1}^{99-p} \frac{1}{2^i}$ approaches 1, and thus $p > 2^{100-p} \Rightarrow p > 93$. With 94 places, the expected payoff is $94 \sum_{i=1}^{6} \frac{1}{2^i} \approx 92.5$. To check this is the maximum payoff, we can look at the 93 and 95 placing cases. With 93 places, the expected payoff is $93 \sum_{i=1}^{7} \frac{1}{2^i} \approx 92.3$. With 95 places, the expected payoff is $95 \sum_{i=1}^{5} \frac{1}{2^i} \approx 92.0$. Both surrounding values are less than our expected payoff, and hence the optimal number of times we should place is 94 with an expected payoff of $\frac{2961}{32} \approx 92.5$.

Solution to Question 2: Collecting Toys II

Let X be the number of distinct toys you collect from the set of 5 toys; we are looking for $E[X]$.

Let I_i $(i = 1, 2, ..., 5)$ be the indicator random variable where:

$$I_i = \begin{cases} 1 & \text{if the } i\text{-th toy type exists in the set of seven boxes} \\ 0 & \text{otherwise} \end{cases}$$

We can define X as $X = I_1 + I_2 + ... + I_5 = \sum_{i=1}^{5} I_i$. For a particular toy type, the probability that it is not observed in the first box is $\frac{4}{5}$, and the probability that it is observed in none of the seven boxes is $(\frac{4}{5})^7$ since each box is independent. Therefore, the probability that it is observed in at least one of the seven boxes is $1 - (\frac{4}{5})^7$. Applying the linearity of expectation and the fact that each toy type follows the same distribution, we can solve for $E[X]$:

$$E[X] = E\left[\sum_{i=1}^{5} I_i\right] = 5E[I_1] = 5 \times \left[1 \times \left(1 - \left(\frac{4}{5}\right)^7\right) + 0 \times \left(\frac{4}{5}\right)^7\right] = \frac{61741}{15625}$$

Solution to Question 3: Chess Tournament I

The highest rated player (P_1) will always make it to the final since no other player can knock him or her out. The second-highest rated player (P_2) makes it to the final if and only if he or she does not play P_1 beforehand, in which the P_2 will be knocked out before the final. In the first round, the 128 players are divided into two subgroups of 64. P_2 will make it to the final if he or she is in a different subgroup as P_1. More concretely, P_2 has 127 possible match-ups, 64 of which are not in the same subgroup as P_1. Thus, the probability that P_1 and P_2 meet in the final is

$$\frac{64}{127}$$

Solution to Question 4: Poker Hands I

There are a total of $\binom{52}{5}$ total hand combinations. To count the number of hands that contain a four-of-a-kind, we can break the problem down into the four cards that have the same value and the fifth card. The value that is shared amongst four of them can be any of the 13 faces. The fifth card can be any of the 48 remaining cards left in the deck. Thus, the probability that you have a four-of-a-kind is:

$$\frac{13 \times 48}{\binom{52}{5}} = \frac{1}{4165}$$

Therefore, the answer to the question is 4165.

Solution to Question 5: Free Sundae

You choose to be the n-th person in line. In order for you to get the free sundae, all of the first $n-1$ people ahead of you must have different birthdays, and your birthday needs to be the same as one of these individuals. Thus, the probability that you receive the free sundae is $P(n) = \frac{365 \times 364 \times ... \times (365-n+2)}{365^{n-1}} \times \frac{n-1}{365}$. You may feel free to find the derivative with respect to n and solve for n. Another approach would be to notice that the n that we are solving for must be such that $P(n) > P(n-1)$ and $P(n) > P(n+1)$; or in other words, the value of n maximizes $P(n)$.

$$P(n-1) = \frac{365}{365} \times \frac{364}{365} \times ... \times \frac{365-(n-3)}{365} \times \frac{n-2}{365}$$

$$P(n) = \frac{365}{365} \times \frac{364}{365} \times ... \times \frac{365-(n-2)}{365} \times \frac{n-1}{365}$$

$$P(n+1) = \frac{365}{365} \times \frac{364}{365} \times ... \times \frac{365-(n-1)}{365} \times \frac{n}{365}$$

Thus,

$$P(n) > P(n-1) \Rightarrow n^2 - 3n - 363 < 0$$

$$P(n) > P(n+1) \Rightarrow n^2 - n - 365 > 0$$

We find that the value of n that satisfies these two inequalities is 20.

Solution to Question 6: Rubik's Cube Stickers

There are a total of 4^3 mini-cubes that compose the Rubik's Cube. Of these, the core 2^3 mini-cubes that make up the inside of the cube and that are not showing do not have stickers on it. Thus, the total number of mini-cubes that do have stickers on it is:

$$4^3 - 2^3 = 56$$

Solution to Question 7: Mental Arithmetic

Let a and b be the numbers in question.

We have $a^2 + ab + b^2 = (a - mb)^2$ for some m.

Then then solve for b, obtaining $b = \frac{a(2m+1)}{m^2-1}$.

m can be any integer greater than 1, and a is chosen to make b an integer. Set $m = 2$, $a = 3$ and $b = 5$. Plugging these values into the formula, we obtain 49. However, $a = 8$ and $b = 7$ also works.

Solution to Question 8: Observance Range

Let p_n be the probability in question when we have n IID Unif$(0,1)$ random variables. We want to find the smallest n such that $p_n \geq 0.99$. The event that at least one of the random variables is in this interval is the complement of the event that all n of the random variables are outside this interval. This latter probability is much easier to compute, as we have intersections of events now instead of unions.

Therefore, the probability of the complement is $1 - p_n$, so we want to find the smallest n such that $1 - p_n \leq 0.01$. Now, we want to find an expression for $1 - p_n$, the probability of the complement.

If the complement occurs, none of the n IID random variables are in $(0.48, 0.52)$. This is saying $\mathbb{P}[X_1 \notin (0.48, 0.52), \ldots X_n \notin (0.48, 0.52)] = \mathbb{P}[X_1 \notin (0.48, 0.52)] \ldots \mathbb{P}[X_n \notin (0.48, 0.52)]$ by independence. As all of the random variables have identical distributions, we can evaluate one of the probabilities and raise it to the nth power, so this is $(\mathbb{P}[X_1 \notin (0.48, 0.52)])^n$. The probability X_1 is in $(0.48, 0.52)$ is just $\dfrac{0.04}{1} = 0.04$. This is because we have a uniform density on $(0,1)$, so the probability our random variable belongs to some sub-interval is just the length of that interval over the total length of our supported interval. As the probability X_1 is in the interval is 0.04, the probability it is not in the interval is 0.96.

All of this work above has led us to finding the smallest n such that $(0.96)^n \leq 0.01$. Namely, we can solve this with logarithms to be $n = \text{ceil}\left(\dfrac{\ln(0.01)}{\ln(0.96)}\right) = 113$. We need to take ceilings as n must be an integer and the floor would be too small and not satisfy the condition.

Solution to Question 9: River Length

Let v be the velocity of the river and l be the length. Then we know that $l = 6v$ from the first part. Furthermore, we have that $l = 4(v+3)$ from the part where we swim. By equating these, we have that $6v = 4(v+3) \iff 2v = 12 \iff v = 6$. Therefore, we have that

$$l = 6 \cdot 6 = 36$$

Solution to Question 10: Heaven 37

Note that $x = 37abc = 37000 + abc$. As 37 already divides 37000, we just need to find integers abc such that all of abc, bca, and cab are divisible by 37. Consider $abc - bca$. We will show that in fact any three digit integer abc divisible by 37 will automatically have the other two forms divisible by 37 as well.

Let abc be an integer divisible by 37. Then $abc = 100a + 10b + c$ by expanding out the digits. For the purposes of this solution, we will prove that if abc is divisible by 37, then bca also is. The proof for cab is also very similar.

By expanding digits again, $bca = 100b + 10c - a$. We have that

$$abc - bca = (100a+10b+c) - (100b+10c+a) = 99a - 90b - 9c = -9(-11a+10b+c)$$

Note that the form of the interior parenthesis is the same as it was originally but we have $-11a$ instead of $100a$. The difference between these is $-9 \cdot 111a = -999a$. As 999 is divisible by 37, we can subtract $999a$ and get that $abc - bca \equiv -9(100a + 10b + c)$ in mod 37. However, $abc = 100a + 10b + c$ is divisible by 37 by assumption, so $abc - bca \equiv 0$ in mod 37. As abc is divisible by 37, this implies bca is also divisible by 37.

With this in mind, the question is really just asking how many integers abc at most 1000 are divisible by 37. In other words, how many non-negative integer values of k are there such that $37k \leq 1000$. We have that 999 is divisible by 37 from before and $999 = 37 \cdot 27$. Therefore, $k = 0, 1, \ldots, 27$, all satisfy this, meaning there are 28 such values total.

Solution to Question 11: Basketball Practice II

Let E_n denote the expected number of free throws made after n total attempts. Then, $E_n = \frac{E_{n-1}}{n-1} + E_{n-1}$. This is because there is a probability of $\frac{E_{n-1}}{n-1}$ that the n-th free throw attempt is made, increasing the total by 1. Simplifying the expression, we find $E_n = \frac{n}{n-1}E_{n-1}$. Note that $E_3 = \frac{3}{2}$. The recurrence relation can be rewritten in terms of n: $E_n = \frac{n}{2}$. So, for $n = 100$, we have $E_{100} = 50$. This can also be seen by the first iteration of this question, as our total throw we make is uniform on $\{1, 2, \ldots, 99\}$, which has mean 50.

Solution to Question 12: Increasing Uniform Chain II

We can use the tail-sum formula for non-negative integer-valued random variables $\mathbb{E}[N] = \sum_{n=0}^{\infty} \mathbb{P}[N > n]$. Clearly $\mathbb{P}[N > 0] = \mathbb{P}[N > 1] = 1$, as we need at least 2 random variables to compare to get one that is not the max. Suppose that $n \geq 2$. Then the event $\{N > n\}$ means that $X_1 < X_2 < \cdots < X_n$, which occurs with probability $\dfrac{1}{n!}$, as this is one specific ordering of the indices among $n!$ possible orderings. Therefore, $\mathbb{P}[N > n] = \dfrac{1}{n!}$. In particular, this holds for $n = 0, 1$ as well, which is ideal. Summing up,

$$\mathbb{E}[N] = \sum_{n=0}^{\infty} \frac{1}{n!} = e$$

The answer is therefore $0 + 1 = 1$.

Solution to Question 13: Candleburn

After x hours, a proportion of $1 - \dfrac{x}{2}$ would be remaining from the 2 hour candle, while $1 - x$ would be remaining from the 1 hour candle. We want to find x such that $1 - \frac{x}{2} = 2(1 - x)$. This is because the 1 hour candle burns faster. Therefore, solving this, we see that $x = \dfrac{2}{3}$. This means we should light both candles 40 minutes before $4:00$ PM, which is $3:20$ PM. Thus, our answer is 1520 in military time.

Solution to Question 14: Ten Consecutive Heads

We are going to solve this more generally for n consecutive heads. Let f_k be the expected number flips of the coin needed to obtain n heads when you have $0 \leq k \leq n$ heads consecutively obtained already. Our boundary condition is that $f_n = 0$, as you already have all n heads obtained. We can see that $f_{n-1} = \dfrac{1}{2} \cdot 1 + \dfrac{1}{2} \cdot (1 + f_0) = 1 + \dfrac{1}{2} f_0$, as with probability $1/2$, you reach n heads, and with probability 1/2, you start over. Similarly, we have that

$$f_{n-2} = 1 + \frac{1}{2}f_{n-1} + \frac{1}{2}f_0 = 1 + \frac{1}{2}\left(1 + \frac{1}{2}f_0\right) + \frac{1}{2}f_0 = \frac{3}{2} + \frac{3}{4}f_0$$

By doing this again, we obtain

$$f_{n-3} = 1 + \frac{1}{2}f_{n-2} + \frac{1}{2}f_0 = 1 + \frac{1}{2}\left(\frac{3}{2} + \frac{3}{4}f_0\right) + \frac{1}{2}f_0 = \frac{7}{4} + \frac{7}{8}f_0$$

A pattern now arises. By recursively applying this process, we can see that

$$f_{n-k} = \left(2 - \frac{1}{2^{k-1}}\right) + \left(1 - \frac{1}{2^k}\right) f_0$$

By plugging in $k = n$, we obtain

$$f_0 = f_0\left(1 - \frac{1}{2^n}\right) + 2 - \frac{1}{2^{n-1}} \iff \frac{1}{2^n}f_0 = 2 - \frac{1}{2^{n-1}} \iff f_0 = 2^{n+1} - 2$$

In particular, for $n = 10$, our answer is $2^{11} - 2 = 2046$.

Solution to Question 15: Consecutive Tails

Consider this problem with n coin flips. Let t_n be the number of sequences of length n where this is satisfied. For any such valid sequence of length n, it must end with either H or TT. These subcases are both disjoint.

If our sequence ends with H, we can have any satisfactory sequence for the first $n-1$ flips. Therefore, there are t_{n-1} such sequences of flips satisfying this. If our sequence ends with TT, we can have any satisfactory sequence for the first $n-2$ flips. Therefore, there are t_{n-2} such sequences of flips satisfying this case. Therefore, we obtain the recurrence $t_n = t_{n-1} + t_{n-2}$.

We now need some initial conditions. For $n = 1$, there is 1 sequence satisfying this: H. For $n = 2$, there are 2 sequences satisfying this: HH and TT. Therefore, we can fully recognize this as $t_n = F_{n+1}$, where F_n is the nth Fibonacci number. Thus, our probability is $\frac{F_{n+1}}{2^n}$. For $n = 10$, we get that our answer is $\frac{89}{1024}$.

Solution to Question 16: Bowl of Cherries II

The only way for a bowl to have k cherries left is if the remaining k cherries are all purple and the $k+1$-th cherryâcounting backwardsâis red. We now have a simple ordering problem: how many ways can m red cherries and n purple cherries be ordered such that the last k cherries are purple and the $k+1$-th cherry from the end is red?

We begin by assigning the last k cherries as purple and the $k+1$-th cherry from the end as red. We are left with $m-1$ red cherries and $n-k$ purple cherries to order however we'd like. Correcting for overcounting (since red cherries are indistinguishable from each other and purple cherries are indistinguishable from each other), we find that there are

$$\frac{(m+n-k-1)!}{(m-1)!\,(n-k)!}$$

possible orderings.

There are a total of

$$\frac{(m+n)!}{m!\,n!}$$

possible orderings of the m red and n purple cherries without the restriction.

Thus, the probability is

$$\frac{(m+n-k-1)!}{(m-1)!\,(n-k)!} \cdot \frac{m!\,n!}{(m+n)!} = \frac{70}{429}$$

Alternatively, we know that the last three spots must be RPP, so the probability of this sequence is $\dfrac{5}{13} \cdot \dfrac{8}{12} \cdot \dfrac{7}{11} = \dfrac{70}{429}$

Solution to Question 17: Specific Card Pull II

We have 13 cards of interest in this scenario, and we can ignore the other 39. Initially, we have a $\frac{12}{13}$ chance to draw one of the cards that we need (being one of the 4 kings, 4 queens, or 4 jacks). On our next draw, let's imagine we drew a king, then we have an $\frac{8}{12}$ chance to draw one of the cards that we need (being one of the 4 queens or 4 jacks). For our third draw, imagine we just drew a queen and the pattern continues, meaning we have a $\frac{4}{11}$ chance to draw one of the cards we need (being the 4 jacks). For our last draw, we need the ace and just the ace, and we have a $\frac{1}{10}$ chance of doing so.

Combining all of the above probabilities, we get

$$\frac{12}{13} \cdot \frac{8}{12} \cdot \frac{4}{11} \cdot \frac{1}{10} = \frac{16}{715}$$

For a more combinatorial approach, there are 3! ways to arrange around the jack, queen, and king that come before the two of hearts. There are 4 ways to pick each of the suits from the three ranks above. After these three, the next card must be the two of hearts. Thus, there are $3! \cdot 4^3$ such sequences that satisfy our condition in the first four cards, and there are $13 \cdot 12 \cdot 11 \cdot 10$ total ways to draw the first four cards, so the probability is

$$\frac{3! \cdot 4^3}{13 \cdot 12 \cdot 11 \cdot 10} = \frac{16}{715}$$

Solution to Question 18: Straddle Delta

We can replicate a straddle by going long 1 unit of a call option and long 1 unit of a put option at the same strike K. By replication, the Δ of a portfolio will be equal to the Δ of the components. We know that $\Delta_C = 1 + \Delta_P = 1 - 0.31 = 0.69$. So, we have total portfolio $\Delta = 0.69 - 0.31 = 0.38$.

This is an interesting result: this straddle has positive delta, not 0 delta as someone may naively think.

Solution to Question 19: Mixing Glasses

Let V be the initial and final volumes (in mL) of the two glasses and w be the volume of water (in mL) that is ultimately in the orange juice glass after all transferring/mixing. Thus, there are $V - w$ mL of water in the water glass, and since we know the final volume is V, the remaining w mL of volume must be taken up by orange juice. Similarly, since there are w mL of water in the orange juice glass and the final volume is V, then the remaining $V - w$ mL of volume must be water. Hence, the difference in concentrations between the two glasses is $\frac{V-w}{V} - \frac{V-w}{V} = 0$.

Solution to Question 20: Cheese Lover II

Chebyshev's Inequality states that for any random variable X with mean μ and variance σ^2, $\mathbb{P}[|X - \mu| > c] \leq \dfrac{\sigma^2}{c^2}$. In this case, we first need to center T_{100}.

Namely,

$$\mathbb{P}[T_{100} > 30000] = \mathbb{P}[T_{100} - 25000 > 5000] \leq \mathbb{P}[|T_{100} - 25000| > 5000] \leq \frac{\mathrm{Var}(T_{100})}{5000^2}$$

The inequality comes from the fact that $\{T_{100} - 25000 > 5000\} \subseteq \{|T_{100} - 25000| > 5000\}$. By the independence of the individual blocks of cheese, $\mathrm{Var}(T_{100}) = 100\mathrm{Var}(T_{100}) = 100 \cdot 250^2$, so we get the upper bound on the probability as $\dfrac{100 \cdot 250^2}{5000^2} = \dfrac{1}{4}$.

Solution to Question 21: Beer Bottles

It is easy to construct a transition graph. Let μ_j denote the expected number of times Bob sings verses when the current verse contains the lyrics âj bottles of beer.â Then, we can write the following equations modeling absorption time:

$$\begin{aligned}
\mu_N &= 1 + \frac{1}{N}\mu_N + \frac{N-1}{N}\mu_{N-1} \\
\mu_{N-1} &= 1 + \frac{1}{N}\mu_N + \frac{N-1}{N}\mu_{N-2} \\
&\vdots \\
\mu_j &= 1 + \frac{1}{N}\mu_N + \frac{N-1}{N}\mu_{j-1} \\
&\vdots \\
\mu_2 &= 1 + \frac{1}{N}\mu_N + \frac{N-1}{N}\mu_1 \\
\mu_1 &= 1
\end{aligned}$$

We simply need to solve for $K = \mu_N$. Let's begin substituting.

$$\begin{aligned}
\mu_N &= 1 + \frac{1}{N}\mu_N + \frac{N-1}{N}\mu_{N-1} \\
&= 1 + \frac{1}{N}\mu_N + \frac{N-1}{N}\left(1 + \frac{1}{N}\mu_N + \frac{N-1}{N}\mu_{N-2}\right) \\
&= \left(1 + \frac{N-1}{N}\right)\left(1 + \frac{\mu_N}{N}\right) + \left(\frac{N-1}{N}\right)^2 \mu_{N-2} \\
&= \left(1 + \frac{N-1}{N} + \left(\frac{N-1}{N}\right)^2\right)\left(1 + \frac{\mu_N}{N}\right) + \left(\frac{N-1}{N}\right)^3 \mu_{N-3} \\
&= \left(1 + \frac{N-1}{N} + \left(\frac{N-1}{N}\right)^2 + \ldots + \left(\frac{N-1}{N}\right)^{N-2}\right)\left(1 + \frac{\mu_N}{N}\right) + \left(\frac{N-1}{N}\right)^{N-1} \mu_1 \\
&= \left(1 + \frac{N-1}{N} + \left(\frac{N-1}{N}\right)^2 + \ldots + \left(\frac{N-1}{N}\right)^{N-2}\right)\left(1 + \frac{\mu_N}{N}\right) + \left(\frac{N-1}{N}\right)^{N-1}
\end{aligned}$$

Note that

$$1+\frac{N-1}{N}+\left(\frac{N-1}{N}\right)^2+\ldots+\left(\frac{N-1}{N}\right)^{N-2}=1\cdot\left(\frac{1-\left(\frac{N-1}{N}\right)^{N-1}}{1-\frac{N-1}{N}}\right)$$
$$=N-N\left(\frac{N-1}{N}\right)^{N-1}$$

Substituting and solving for μ_N, we find

$$\mu_N=N-N\left(\frac{N-1}{N}\right)^{N-1}+\left(1-1\left(\frac{N-1}{N}\right)^{N-1}\right)\mu_N+\left(\frac{N-1}{N}\right)^{N-1}$$
$$\mu_N=\frac{N-N\left(\frac{N-1}{N}\right)^{N-1}}{\left(\frac{N-1}{N}\right)^{N-1}}+\left(\frac{N-1}{N}\right)^{N-1}$$

Our final task is to compute

$$\lim_{N\to\infty}\frac{1}{N}\left(\frac{N-N\left(\frac{N-1}{N}\right)^{N-1}}{\left(\frac{N-1}{N}\right)^{N-1}}+\left(\frac{N-1}{N}\right)^{N-1}\right)$$

Let's solve.

$$\lim_{N\to\infty}\frac{1}{N}\left(\frac{N-N\left(\frac{N-1}{N}\right)^{N-1}}{\left(\frac{N-1}{N}\right)^{N-1}}+\left(\frac{N-1}{N}\right)^{N-1}\right)=\lim_{N\to\infty}\left(\frac{1-\left(\frac{N-1}{N}\right)^{N-1}}{\left(\frac{N-1}{N}\right)^{N-1}}\right)+\lim_{N\to\infty}\left(\frac{1}{N}\cdot\left(\frac{N-1}{N}\right)^{N-1}\right)$$
$$=\lim_{N\to\infty}\left(\frac{N^{N-1}-(N-1)^{N-1}}{(N-1)^{N-1}}\right)+\lim_{N\to\infty}\left(\frac{1}{N}\cdot\left(\frac{N-1}{N}\right)^{N-1}\right)$$
$$=\lim_{N\to\infty}\left(\frac{N}{N-1}\right)^{N-1}-1+\lim_{N\to\infty}\left(\frac{1}{N}\cdot\left(\frac{N-1}{N}\right)^{N-1}\right)$$

Recall

$$e=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$
$$=\lim_{x\to\infty}\left(\frac{x+1}{x}\right)^x$$

Let $x=N-1$.

$$e=\lim_{N\to\infty}\left(\frac{N}{N-1}\right)^{N-1}$$

Similarly,

$$\frac{1}{e}=\lim_{N\to\infty}\left(\frac{N-1}{N}\right)^{N-1}$$

This means the last term goes to 0, as we have the extra $\frac{1}{N}$ term out back. Substituting, we find

$$\lim_{N\to\infty}\frac{K}{N}=\boxed{e-1}$$

Our answer is $1^2+1^2=2$.

Solution to Question 22: Consecutive Children

Let m be the age of the middle child, s be the spacing between the children's births, and d be the age of the dad. Then our condition says that $(m-4s)^2+(m-3s)^2+\cdots+m^2+(m+s)^2+\cdots+(m+4s)^2=d^2$. After expanding and cancelling, this reduces to $d^2=9m^2+60s^2$. Since the RHS is a multiple of 3, we have that $3 \mid d^2$, which means $3 \mid d$. Therefore, let $d=3r$ for an integer r. This means that $9r^2=9m^2+60s^2$, which equivalent is $(r-m)(r+m)=\dfrac{20}{3}s^2$. Since the LHS is an integer, this means that $3 \mid s$ as well.

For the youngest child to be born, we must have that $m-4s \geq 0$, which means $m \geq 4s$. Therefore, we have that $d^2 \geq 9(4s)^2+60s^2=204s^2 \geq (14s)^2$. Therefore, $d \geq 14s$. Since $3 \mid s$ and $d \geq 60$, we must have that $s=0,3$. Since they are not of the same age, $s=3$. Plugging this in, we have that $(r-m)(r+m)=60$. We now need to find factorizations of 60 that have an integer midpoint, as the midpoint of $r-m$ and $r+m$ is r. Namely, we see that $10 \cdot 6$ and $30 \cdot 2$ both have integer midpoints. If we select the former, then $r=8$ and $m=2$, meaning that the middle child has age 2. As $s=3$, we would get some negative ages. Therefore, we must have that the midpoint is $r=16$ and $m=14$. This is now consistent, as the children would be aged $2,5,8,11,14,17,20,23,$ and 26. Since the father's age is $3r$, we get that the father is aged $3 \cdot 16=48$.

Solution to Question 23: Fleeing Flea

One simple way to attack this problem is to lay the $3D$ object out into a $2D$ map. From there, we can see there are two quick straight paths the flea can take. Both paths are along the hypotenuse of two separate right triangles, one with side lengths of $(2,5)$ and the other with side lengths of $(3,4)$. This means our two shortest paths total $\sqrt{29}$ units and 5 units.

Our shortest path of the two is 5 units, meaning our flea can flee to the other corner in $\frac{5}{2.5}=2$ seconds.

Solution to Question 24: Determination II

Consider the dataset Y that is sampled perfectly from a parabola far from the origin. Let X_2 be the dataset where each value of X_1 is squared. If Y is far enough from the origin, $y \sim \alpha_1+\beta_1 x_1$ has low R^2 since it is linear, while Y is quadratic. Furthermore, $y \sim \alpha_2+\beta_2 x_2=\alpha_2+\beta_2 x_1^2$ is also low R^2 because although it is a parabola, it is not able to be shifted around horzontally to match Y. Therefore, individually, these linear regressions have low R^2. However, the model $y \sim \alpha + \beta' x_1 + \beta'' x_2$ has $R^2 = 1$, as you can now shift the parabola around in the plane to perfectly match the non-noisy dataset Y.

Solution to Question 25: Coin on Chess Board

Because of the layout and symmetry of squares, we only need to consider if the coin fits in a 2 inch by 2 inch square. If it doesnât, we know that it overlaps onto a neighboring square and thus doesnât fit the criteria. In order to find the probability that a coin is fully within a square, we can calculate the area that the center of the coin can land in and divide it by the total area of a square. Given the radius of the coin is 0.5 inches, it canât be any closer than 0.5 inches to any edge of the square. This gives a total area that the center can land in of $(2 - 0.5 - 0.5)^2 = 1$ square inch. The total area of the square (and thus total area that the center of the coin can land in) is $2^2 = 4$ square inches. This gives us the answer of $\frac{1}{4} = 0.25$.

Solution to Question 26: Shattering Orbs

We are going to solve this for the general case of when there are n links. Let a_n be the expected number of cuts needed with n links. We want to find a recurrence relation for a_n. To do this, let's consider what happens after the first cut. If we cut at link k, $1 \leq k \leq n$, where link 1 connects orbs 1 and 2, then this means we have $k-1$ links above it still standing. Thus, we have added 1 to our count and want to now find the same expectation but starting from $k-1$ links instead of n. Mathematically, this means that $a_n = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n}(1 + a_{k-1})$, as with probability $\dfrac{1}{n}$, we cut each given link on the first step. This is just Law of Total Expectation above, where we have conditioned on the position of the first cut. We index shift this sum by 1 to get $a_n = \dfrac{1}{n}\displaystyle\sum_{k=0}^{n-1}(1 + a_k) = 1 + \dfrac{1}{n}\displaystyle\sum_{k=0}^{n-1} a_k$.

Now, consider $a_n - a_{n-1}$. Using the recurrence relation above, we obtain that

$$a_n - a_{n-1} = 1 + \frac{1}{n}\sum_{k=0}^{n-1} a_k - \left(1 + \frac{1}{n-1}\sum_{k=0}^{n-2} a_k\right) = \frac{1}{n}a_{n-1} + \left(\frac{1}{n} - \frac{1}{n-1}\right)\sum_{k=0}^{n-2} a_k$$

Note that $\frac{1}{n} - \frac{1}{n-1} = -\frac{1}{n(n-1)}$, so the above reduces to

$$a_n - a_{n-1} = \frac{1}{n}a_{n-1} - \frac{1}{n}\left[\frac{1}{n-1}\sum_{k=0}^{n-2} a_k\right]$$

The term in brackets is just $a_{n-1} - 1$ by our original recurrence relation. Therefore, $a_n - a_{n-1} = \frac{1}{n}a_{n-1} - \frac{1}{n}(a_{n-1} - 1) = \frac{1}{n}$. Rearranging, $a_n = a_{n-1} + \frac{1}{n}$. Our initial condition on this that makes physical sense is that $a_1 = 1$, as we only have one link to select from and we cut it on our first trial.

With this initial condition, we can recurse and clearly see that

$$a_n = a_{n-1} + \frac{1}{n} = a_{n-2} + \frac{1}{n-1} + \frac{1}{n} = \cdots = a_1 + \frac{1}{2} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$$

Therefore, this is our solution for general n, and our specific case is when $n = 6$, as we have 6 links. Thus, the final result is $\frac{49}{20}$.

Solution to Question 27: Matrix Exponential

Recall that e^A can be rewritten as

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!}A^k$$

In addition, note the following:

$$\begin{aligned}
\text{trace}(e^A) &= \text{trace}\left(\sum_{k=0}^{\infty} \frac{1}{k!}A^k\right) \\
&= \sum_{k=0}^{\infty} \text{trace}\left(\frac{1}{k!}A^k\right) \\
&= \text{trace}\left(\begin{bmatrix} \sum_{k=0}^{\infty} \frac{3^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{6^k}{k!} \end{bmatrix}\right) \\
&= e^3 + e^6
\end{aligned}$$

Therefore, our answer is $3 \cdot 6 = 18$

Solution to Question 28: Consecutive Pairs

We can avoid a really messy casework solution by thinking about the problem creatively. Suppose we have all 10 integers written out in order on paper. Our goal is to cover some of the integers with tiles such that only two tiles are side-by-side; covered integers belong in a subset of interest. Perhaps we can invent our own special tiles of different shapes that prevent some integers from being next to each other by design; then, we can simply find the number of arrangements of the tiles. Specifically, we can have a few tiles labeled U for uncovered, a few tiles labeled UC for uncovered-covered, and exactly one tile labeled UCC for uncovered-covered-covered (this tile ensures that exactly one pair of consecutive integers is in the subset). In order to account for all cases, however, we need to have an additional slot at 0. We have the following cases:

Case 1: There are 6 total integers in the subset. In other words, there are 4 UC tiles and 1 UCC tile. There are $\binom{5}{4} = 5$ ways to order these 5 tiles.

Case 2: There are 5 total integers in the subset. In other words, there are 3 UC tiles, 1 UCC tile, and 2 U tiles. There are $\binom{6}{3,1,2} = 60$ ways to order these 6 tiles.

Case 3: There are 4 total integers in the subset. In other words, there are 2 UC tiles, 1 UCC tile, and 4 U tiles. There are $\binom{7}{2,1,4} = 105$ ways to order these 7 tiles.

Case 4: There are 3 total integers in the subset. In other words, there is 1 UC tile, 1 UCC tile, and 6 U tiles. There are $\binom{8}{1,1,6} = 56$ ways to order these 8 tiles.

Case 5: There are 2 total integers in the subset. In other words, is 1 UCC tile, and 8 U tiles. There are $\binom{9}{8} = 9$ ways to order these 9 tiles.

Adding it all up, we conclude there are $5 + 60 + 105 + 56 + 9 = 235$ possible subsets that contain exactly 1 pair of consecutive integers.

Solution to Question 29: Die Multiple II

Let $S_k = X_1 + \cdots + X_k$ represent the sum of the first k die rolls, with X_i being the value of the ith die roll. On each roll, regardless of what was rolled prior, there is a $\frac{1}{6}$ probability that the sum will be divisible by 6. This is because exactly 1 of the 6 values on each roll makes the sum divisible by 6, no matter what the current sum is. Therefore, the number of rolls needed is Geom $\left(\frac{1}{6}\right)$, which has mean 6. Therefore, our answer is 6.

Solution to Question 30: Thick Coin

When we inscribe the coin in the sphere, we should take the sphere and the coin to be concentric. The coin can be regarded as a right cylinder. Then, we select a uniformly random point on the surface of the sphere. If the radius that is drawn from the point that we select to the center strikes the cylinder, then we say that the coin landed on its side.

A theorem in geometry helps simplify this problem. When two parallel planes cut a sphere, the surface of the sphere between the two planes is called a zone. The theorem states that the zone is proportional to the distance between the planes. This means that our coin should be $\frac{1}{3}$ as thick as the sphere's diameter. In this case, the "planes" are the head and tail faces of the coin, as we inscribe our cylinder into a sphere.

The question here is how thick is the sphere? Let's call the radius R. The Pythagorean Theorem gives that

$$R^2 = 1^2 + \left(\frac{R}{3}\right)^2$$

This is because the distance from the center of the coin to the upper face is the radius of the sphere. Additionally, we know that the radius of the coin is 1 and the thickness (height) of the coin is $\frac{R}{3}$ from before. Solving this yields $R = \frac{3}{\sqrt{8}}$. However, we want $\frac{1}{3}(2R) = \frac{1}{\sqrt{2}}$. We want $2R$ because that is the diameter, and hence how thick the coin should be. Our answer is $k = 2$.

Solution to Question 31: Arithmetic Mean

By the definition of the arithmetic mean:

$$\frac{5+6+11+x+y}{5} = 20 \Rightarrow x+y = 78$$

Thus, the arithmetic mean of x and y is $\frac{x+y}{2} = 39$.

Solution to Question 32: Intersecting Chords

Consider indicator random variables on each pair of chords. Any pair of chords will have a total of 4 endpoints, and each of the $\binom{4}{2} = 6$ ways to pair them up into chords are equally likely, of which only 2 result in an intersection. Alternatively, this can be seen by choosing two points for the first chord and computing the probability that two randomly chosen points would lie on the same side of the chord via integrating over the arc angle between the points of the first chord. Hence for each of the $\binom{10}{2} = 45$ pairs of chords, there are 1/3 expected intersections for a total number of 15.

Solution to Question 33: Particle Reach I

We are going to solve this for more general p. Let x_1 be the probability that the particle ever reaches position 1. The key is to condition on the move at the first step. If the particle moves right, which occurs with probability p, then position 1 is reached. Otherwise, the particle reaches position -1. From -1, the particle first needs to reach 0 again, which occurs with probability x_1, and then from there, reach position 1, which occurs with probability x_1 as well. Therefore, given the particle moves left, the probability it reaches position 1 is x_1^2. This gives rise to the equation

$$x_1 = p + (1-p)x_1^2$$

We can solve for this quadratic in x_1 to get that $x_1 = 1, \dfrac{p}{1-p}$. Since $x_1 \leq 1$, we know that for $p \geq 1/2$, $x_1 = 1$, as the other root would be larger than 1. For $p < 1/2$, the random walk is biased down, so it is not probability 1 that the particle ever reaches 1. This argument can be made more rigorous using more mathematically advanced tools, but those are not of concern here. Therefore, the answer for $p < 1/2$ is $\dfrac{p}{1-p}$. Since $p = 1/3$ in this question, our answer is $\dfrac{1/3}{1-1/3} = \dfrac{1}{2}$.

Solution to Question 34: Silly SDE

First, we are going to calculate $d(e^{\kappa t}X_t)$. By the product rule, we have that

$$d(e^{\kappa t}X_t) = \kappa e^{\kappa t}X_t dt + e^{\kappa t}dX_t = \kappa e^{\kappa t}X_t + e^{\kappa t}\left(\kappa(\theta - X_t)dt - \sigma\sqrt{X_t}dW_t\right)$$

Distributing everything, we get the first term cancels with the $-\kappa X_t dt$ term inside the parentheses, so we are left with

$$d(e^{\kappa t}X_t) = \kappa\theta e^{\kappa t}dt - \sigma e^{\kappa t}\sqrt{X_t}dW_t$$

We can use this to get $e^{\kappa t}X_t$ very simply by integrating both sides. Namely,

$$e^{\kappa t}X_t - X_0 = \int_0^t \kappa\theta e^{\kappa s}ds - \int_0^t \sigma e^{\kappa s}\sqrt{X_s}dW_s$$

Rearranging to isolate X_t, we have that

$$X_t = X_0 e^{-\kappa t} + \int_0^t \kappa\theta e^{\kappa(s-t)}ds - \int_0^t \sigma e^{\kappa(s-t)}\sqrt{X_s}dW_s$$

Now, we know that $X_0 = x$, so we can substitute that in. In addition, the second integral is Ito and the function inside is "sufficiently nice" to conclude that the mean of it is 0. Thus,

$$\mathbb{E}[X_T] = xe^{-\kappa T} + \int_0^T \kappa\theta e^{\kappa(s-T)}ds = xe^{-\kappa T} + \theta e{-}\kappa T e^{\kappa s}\Big|_0^T = xe^{-\kappa T} + \theta\left(1 - e^{-\kappa T}\right)$$

Evaluating this at the specific values, our expectation evaluates to $5e^{-2} + 2(1 - e^{-2}) = 2 + 3e^{-2}$. The answer is thus $3 \cdot 2 \cdot (-2) = -12$.

Solution to Question 35: St. Petersburg Paradox

The number of flips needed to see the first heads is $N \sim \text{Geom}(1/2)$. We are looking for $\mathbb{E}[2^N]$, which is our expected payout. Namely,

$$\mathbb{E}[2^N] = \sum_{n=1}^{\infty} 2^n \mathbb{P}[N = n] = \sum_{n=1}^{\infty} 2^n \cdot \left(\frac{1}{2^n}\right) = \sum_{n=1}^{\infty} 1 = \infty$$

Therefore, the answer is infinite, meaning the input is -1.

Solution to Question 36: Correlation Ranges

The key idea here is that the correlation matrix for any collection of random variables must be positive semi-definite. Denoting $\text{Corr}(X, Z) = \rho$, the correlation matrix for (X, Y, Z), say C, is a 3×3 matrix

$$\begin{bmatrix} 1 & \frac{5}{13} & \rho \\ \frac{5}{13} & 1 & \frac{12}{13} \\ \rho & \frac{12}{13} & 1 \end{bmatrix}$$

A condition to show that a matrix is positive definite is that each of the submatrices of order $1 \times 1, 2 \times 2, \ldots, n \times n$, where n is a size of the matrix, originating from the top left corner have non-negative determinant. In this case, it is easy to see that the top left 2×2 matrix has determinant $C_{11}C_{22} - C_{12}C_{21} = \frac{144}{169} > 0$, so we only need to check that the entire matrix C has non-negative determinant.

Evaluating this determinant as a function of ρ, we obtain that it is $\frac{120}{169}\rho - \rho^2$. Setting this equal to 0, we obtain that $\rho = 0, \frac{120}{169}$. As this polynomial had a negative leading coefficient for ρ^2, it follows that the parabola (i.e. determinant) must have been positive between the two roots, so $\rho^* = \frac{120}{169}$ is the largest value where this determinant is non-negative.

Solution to Question 37: Coefficient Swap

Let r be the Pearson Correlation Coefficient of X and Y. Then $\beta_x = r\frac{\sigma_y}{\sigma_x}$ and $\beta_y = r\frac{\sigma_x}{\sigma_y}$. Therefore,

$$\frac{\beta_y}{\beta_x} = \frac{\sigma_x^2}{\sigma_y^2} \iff \beta_y = \beta_x \frac{\sigma_x^2}{\sigma_y^2} = 1 \cdot \frac{10}{20} = \frac{1}{2}$$

Solution to Question 38: Colorful Socks I

Let the first sock be arbitrarily drawn. We can do this since there are the same amount of socks of each color in the drawer. The second sock must match the color of the first sock. Of the 19 remaining socks, only 1 is of the matching color, so our probability is $\frac{1}{19}$.

Solution to Question 39: Slippery Ladder II

We can view the ladder as the hypotenuse of a triangle whose sites are the ground and the wall. Using Pythagorean Theorem, we find that

$$30^2 + y^2 = 50^2 \iff y = 40$$

Therefore, the tip of the ladder is 40 feet above ground at that time. Using the result from Slippery Ladder I that $y' = -3$ at this instant, we can find the rate of change of the angle by using $\tan\theta = \frac{y}{x}$ Taking the derivative of both sides, we see that

$$\sec^2\theta\theta' = \frac{xy' - yx'}{x^2}$$

We know that $y = 40$ and $x = 30$ at this instant, so $\sec^2\theta = 1 + \tan^2\theta = 1 + \frac{y^2}{x^2} = \frac{25}{9}$. With all of this information, we plug in and get

$$\frac{25}{9}\theta' = \frac{30(-3) - 40(4)}{30^2} \iff \theta' = -\frac{1}{10}$$

Solution to Question 40: Bull Call Spread I

If we look at the payoff diagram of a bull call spread, we have $V_T = 0, S_T \in [0,5]$, $V_T = S_T - 5, S_T \in [5,10]$ and $V_T = 10, S_T \in [5,\infty)$.

So, in the worst case, we will gain 0 and in the best case, we will gain 5. In general, a bull call spread can cap the upside, but can also limit the downside loss. This option contract is the best when an investor expects a moderate increase in the stock (i.e it stays in the range of K_1 and K_2).

Solution to Question 41: Limiting Values I

The expected value of this uniform distribution is $\frac{2+3+4+5+6}{5} = 4$.

Solution to Question 42: Absolute Difference

Let's put this problem in a different light. Consider a walk where you move up 1 for heads and move down 1 for tails. The condition states that in the 10 flips, we have 4 more heads than tails, so this means that there are 7 movements up and 3 movements down. In other words, each valid sequence corresponds to some ordering of $UUUUUUUDDD$. There are $\binom{10}{3} = 120$ such arrangements, so this is our answer.

Solution to Question 43: Counting Digits

There are 198 digits in 99^{99}.

$$99^{99} = 99^{99} \times (\frac{100}{100})^{99} = 100^{99} \times (\frac{99}{100})^{99} = 10^{198} \times (1 - \frac{1}{100})^{99} \approx 10^{198} \times (1 - \frac{1}{100})^{100} \approx 10^{198} \times \frac{1}{e}$$

Thus, 99^{99} has 198 digits.

Solution to Question 44: Horse Arbitrage

The implied odds of the respect horses are $1/2, 1/4$, and $1/6$, respectively. These sum to strictly less than 1, so there is an arbitrage. We want to make a constant amount of money regardless of the outcome. Namely, if we bet our money in proportions $\$6/11, \$3/11$, and $\$2/11$ in order on the three horses, we see that regardless of the outcome, we get a payout of $12/11. However, we only bet $1, yielding a profit of $1/11 always.

We obtain our proportions because the sum of the probabilities is 11/12, so all we need to do is bet in a way that yields a constant positive expected payout relative to the probabilities of each horse winning.

Solution to Question 45: Ants on a Triangle

In order for the ants to not meet at any corner, all three ants have to travel in the same direction (either clockwise or counter-clockwise). The probability they all travel clockwise is $(\frac{1}{2})^3 = \frac{1}{8}$. The probability they all travel counter-clockwise is also $(\frac{1}{2})^3 = \frac{1}{8}$. Thus, the probability the ants are on their unique corners is $\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$.

Solution to Question 46: McQueen Speeding

To solve this, we must obtain the relative speed of Lightning McQueen to Chick Hicks. Let s be the speed Chick Hicks. Doc Hudson must move at a speed of $\dfrac{5}{4}s$, as Doc Hudson moves 1000 meters in the time that Chick Hicks moves 800 meters. Similarly, Lightning McQueen must move at a rate of $\dfrac{5}{4}\left(\dfrac{5}{4}s\right) = \dfrac{25}{16}s$ by the same logic of comparing his distance travelled to Doc Hudson.

Since Lightning McQueen moves at a speed $\dfrac{25}{16}$ times as large as Chick Hicks, this says that if Lightning McQueen covers a distance d, Chick Hicks covers a distance of $\dfrac{16}{25}d$. In particular, $d = 1000$ here, so Chick Hicks has travelled 640 meters in the lap, meaning he is 360 meters away from finishing his lap.

Solution to Question 47: The Perfect Hedge I

Let's denote $w_1 = w$ and $w_2 = 1 - w$. In a 2 asset world, we can write the variance of a portfolio as: $\sigma^2 = w^2\sigma_1^2 + 2w(1-w)\rho\sigma_1\sigma_2 + (1-w)^2\sigma_2^2$. For a risk-less portfolio, we require $\sigma = 0$. Plugging in $\rho = -1$ and solving for $w_1 = w$. We obtain:

$$w = \frac{\sigma_2}{\sigma_1 + \sigma_2}$$

This gives us $w = \frac{\sqrt{.04}}{\sqrt{.15}+\sqrt{.04}} \approx 0.34$.

We see that the expected returns do not matter. We can obtain a riskless portfolio from two assets if they have a correlation of -1 and weight them using the above manner. This rarely happens in real life.

Solution to Question 48: First Ace

The four aces are dispersed throughout the deck and cut the 48 remaining cards into 5 distinct sections, each of some random length $X_i \in [0, 48]$ where $1 \leq i \leq 5$; that is, $\sum_{i=1}^{5} X_i = 48$. Furthermore, by symmetry, $E[X_1] = E[X_2] = \cdots = E[X_5]$, as in the absence of any additional information, none of the sections is expected to be any larger or smaller than any other. Thus, by linearity of expectation:

$$E\left[\sum_{i=1}^{5} X_i\right] = 5 \times E[X_1] = 48 \Rightarrow E[X_1] = \frac{48}{5}$$

We have found that the expected number of cards in the first section is $\frac{48}{5}$, so we will observe the first ace on the next card. Hence, we expect to flip $\frac{48}{5} + 1 = 10.6$ cards to observe the first ace.

Solution to Question 49: 4 Die Sum

Clearly there are $6^4 = 1296$ total outcomes of the 4 dice rolls. We need to divide this up into all of the permutations that sum to 20, as well as count all of their arrangements. The permutations that sum to 20 are

$$6, 6, 6, 2; 6, 6, 5, 3; 6, 6, 4, 4; 6, 5, 5, 4; 5, 5, 5, 5$$

by brute force. Note that we now need to count all of the arrangements for each combination, as we created our sample space so that it include all 4-tuples. For $6, 6, 6, 2$, there are 4 permutations corresponding to where the 2 is put. For $6, 6, 5, 3$, there are 12 permutations corresponding to just dividing 4! (total permutations) by 2! for the two sixes. By the same idea, $6, 6, 4, 4$ has 6 permutations, $6, 5, 5, 4$ has 12 permutations, and $5, 5, 5, 5$ has 1 permutation. Adding all of these up, we get 35 possible permutations that add to 20. Therefore, the probability is $\frac{35}{1296}$.

Solution to Question 50: Common Ball Draw

Let A_i be the event that ball i, $1 \leq i \leq r$, is selected in common between all k people. We want $\mathbb{P}\left[\bigcup_{i=1}^{r} A_i\right]$. The events A_i are not mutually exclusive, so we need to use inclusion-exclusion here to calculate this probability. As all of the balls are exchangeable, all individual probabilities and intersection probabilities are the same regardless of which intersection we consider. Therefore, by the inclusion-exclusion formula, we have that

$$\mathbb{P}\left[\bigcup_{i=1}^{r} A_i\right] = \sum_{i=1}^{r}(-1)^{i+1}\binom{n}{i}\mathbb{P}\left[\bigcap_{m=1}^{i} A_m\right]$$

We get the $\binom{n}{i}$ term from the fact that there are $\binom{n}{i}$ ways to select i of the n balls to be in common to everyone. The $(-1)^{i+1}$ term comes from the inclusion-exclusion formula itself. All that remains is to compute the intersection probability.

The event $\bigcap_{m=1}^{i} A_m$ means that all of the first i balls are in common to all k people selecting. Therefore, fix those i balls. We now need to select $r - i$ balls from the remaining $n - i$ in the urn per person, which can be done in $\binom{n-i}{r-i}$ ways. The total number of ways to select r balls from n is $\binom{n}{r}$, so the probability that any individual person selects the first i balls is

$$\frac{\binom{n-i}{r-i}}{\binom{n}{r}}$$

As this needs to be done for all k people, we raise the previous term to the kth power. Therefore, we get our final answer of

$$p(k, n, r) = \sum_{i=1}^{r}(-1)^{i+1}\binom{n}{i}\left[\frac{\binom{n-i}{r-i}}{\binom{n}{r}}\right]^k$$

Plugging in the respective values, we get that

$$p(4, 17, 5) = 103566035/840373563 \approx 0.123$$

Solution to Question 51: Diagonal Eigenvalue

A fact from Linear Algebra states that if M is $n \times n$ with eigenvalues (repeated according to multiplicity) $\{\lambda_i\}_{i=1}^n$, then the eigenvalues of $M + kI_n$ is $\{\lambda_i + k\}_{i=1}^n$.

Note that we can write $A = 1_{30} + 29I_{30}$, where 1_{30} is the 30×30 matrix of all ones.

To find the eigenvalues of 1_{30}, we can simply note that every row is dependent on the first, so the geometric multiplicity of the eigenvalue 0 is 29. Since the sum of the eigenvalues is the trace, we can see that the algebraic multiplicity is also 29 and that the last eigenvalue must be 30, as $\mathrm{tr}(1_{30}) = 30$. Therefore, the eigenvalues of A are 29 with geometric multiplicity 29 and $30 + 29 = 59$ with geometric multiplicity 1.

This means that $\lambda + g = \begin{bmatrix} 58 \\ 60 \end{bmatrix}$, so $||\lambda + g||^2 = 58^2 + 60^2 = 6964$

Solution to Question 52: Triangle Area

Note that X and Y are independent. The expected area of the triangle is $\mathbb{E}[\frac{1}{2}XY] = \frac{1}{2}\mathbb{E}[X]\mathbb{E}[Y] = \frac{1}{2}\left(3\right)^2 = \frac{9}{2}$.

Solution to Question 53: Bond Practice I

The first step we want to take is calculating our cash flows. In order to do this, we need to calculate how much money we lose to inflation each year we are paid out. To start, we calculate how much we are paid out per period, which is our face value * our coupon rate. In this case $3000 \cdot 0.04 = 120$. We then want to calculate how much value we are giving up to inflation. This is given by the formula:

$$\sum_{n=1}^{5} \frac{120}{1.07^n}$$

Here 120 is our coupon payment, 1.07 represents our inflation rate of 7%, and we iterate through all 5 years. Calculating here we get 492.02.

The second step is calculating our final face value payment, which is given by the face value and then dividing by the inflation rate to the power of however many years we were losing money to inflation. In this case, it will be $3000/1.07^5$, which equals 2138.98.

Finally, add up the cash flows and the final face value to get our answer of 2630.98.

Solution to Question 54: Hand Meet

Let h be the number of hours. The minutes hand will move $360h$ degrees. In addition, we know that the hours hand moves $30h$ degrees. However, as we need to account for rotation, the first time they meet, the minutes hand will have travelled an additional 360 degrees beyond the hours hand, as it completes a rotation. Therefore, we get the equation $360h = 30h + 360$, which yields $h = \frac{12}{11}$.

Solution to Question 55: Positive Brownian II

We are going to start off with a similar trick to Positive Brownian I. We are instead going to compute $\mathbb{P}[B_2 > 0, B_8 < 0]$. Then, we can receive our result of $\mathbb{P}[B_2 > 0, B_8 < 0] = \mathbb{P}[B_2 > 0] - \mathbb{P}[B_2 > 0, B_8 < 0]$. The first term of this is clearly just $\frac{1}{2}$, as $B_2 \sim N(0, 2)$, which is symmetric about 0.

We are going to solve this question for all $0 \leq s < t$. Set $X_t = B_t - B_s$ and $Y_s = -B_s$. $X_t \sim N(0, t - s)$ by the stationarity of Brownian Motion. Furthermore, by independent increments, we know that X_t and Y_s are independent. Letting I be our probability of interest, we have that

$$I = P\left[B_t > 0, B_s < 0\right] = P\left[B_t - B_s > -B_s, -B_s > 0\right] = P\left[X_t > Y_s, Y_s > 0\right]$$
$$I = \frac{1}{2\pi\sqrt{s(t-s)}} \int_0^\infty \int_y^\infty \exp\left(-\frac{y^2}{2s} - \frac{x^2}{2(t-s)}\right) dxdy$$

The integral arises from plugging in the PDFs of X_t and Y_s and then integrating first where $x > y$ and then where $y > 0$. This integral looks difficult to evaluate. We know that normal distributions have radial symmetry in the plane, so attempting to convert to polar coordinates is a good move. To get rid of the denominators in the exponentials, we use the standard $x = r\cos\theta$ and $y = r\sin\theta$, but with extra terms of $\sqrt{s}$ and $\sqrt{t-s}$ to cancel once they are squared. Namely, we have

$$y = r\sqrt{s}\sin\theta$$
$$x = r\sqrt{t-s}\cos\theta$$

As we are attempting to switch to a different coordinate system, we need to evaluate the Jacobian of our transformation. Namely, this is

$$
\begin{aligned}
dydx &= J dr d\theta \\
dydx &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} dr d\theta \\
dydx &= \begin{vmatrix} \sqrt{s}\cos\theta & -\sqrt{s}r\sin\theta \\ \sqrt{t-s}\sin\theta & \sqrt{t-s}r\cos\theta \end{vmatrix} dr d\theta \\
dydx &= \left(\sqrt{s(t-s)}r\cos^2\theta + \sqrt{s(t-s)}r\sin^2\theta\right) dr d\theta = \sqrt{s(t-s)} r dr d\theta
\end{aligned}
$$

Next, we need to find our integral bounds. It's fairly clear to see that $0 < r < \infty$, as there is no bound in our region. As $x > y$ and $y > 0$, this implies

$$\sqrt{s}r\sin\theta < \sqrt{t-s}r\cos\theta$$

By rearranging the above to isolate θ, we see

$$\tan\theta < \sqrt{\frac{t-s}{s}}$$

Our bound must be

$$\theta < \tan^{-1}\left(\sqrt{\frac{t-s}{s}}\right) = \cos^{-1}\left(\sqrt{\frac{s}{t}}\right)$$

The second equality just comes from writing out the triangle with side lengths $\sqrt{t-s}$ opposite, $\sqrt{s}$ adjacent, and $\sqrt{t}$ on the hypotenuse. Plugging everything into the integral above, our new integral becomes

$$
\begin{aligned}
I &= \frac{1}{2\pi}\int_0^{\cos^{-1}\left(\sqrt{\frac{s}{t}}\right)}\int_0^{\infty} r\exp\left(-\frac{r^2}{2}\right) dr d\theta \\
I &= \frac{1}{2\pi}\int_0^{\cos^{-1}\left(\sqrt{\frac{s}{t}}\right)} -\exp\left(-\frac{r^2}{2}\right)\Bigg|_0^{\infty} d\theta \\
I &= \frac{1}{2\pi}\int_0^{\cos^{-1}\left(\sqrt{\frac{s}{t}}\right)} d\theta = \frac{1}{2\pi}\cos^{-1}\left(\sqrt{\frac{s}{t}}\right)
\end{aligned}
$$

Plugging in our specific values of $s = 2$ and $t = 8$, our answer is $\frac{1}{2\pi}\cos^{-1}\left(\frac{1}{2}\right) = \frac{1}{2\pi}\cdot\frac{\pi}{3} = \frac{1}{6}$.

Therefore, the probability in question for us is $\frac{1}{2} - \frac{1}{6} = \frac{1}{3}$.

Solution to Question 56: The Price of a Garden

From the triangle inequality, we see that $55+62 = 117$. This violates the triangle inequality, which says that any two sides must sum to a number strictly larger than the third side. This is impossible and the triangle will have 0 area, so the answer is 0.

Solution to Question 57: 3 Larger Die

Let e be the expected payout. The yellow die is at least 3 larger than the green die with probability $\frac{7}{25}$. You can verify this by conditioning on the value of the green die and noting the number of outcomes that are satisfactory if the green die shows k is $8 - k$. Summing those up, you get 28 of 100 equally-likely outcomes. The dice show the same value with probability $\frac{1}{10}$. Therefore, they show different values with probability $\frac{31}{50}$. Applying the Law of Total Expectation on the outcome of the first roll,

$$e = \frac{7}{25} \cdot (e+1) + \frac{31}{50}e$$

If we roll a value at least 3 larger on the yellow than the green, then it is as if our game restarts with our money being 1 instead of 0. Solving for e, we get that $\frac{1}{10}e = \frac{7}{25}$, so $e = \frac{14}{5}$.

Solution to Question 58: Odd Before Even

We need three things to happen: The first odd must show up before the first even, the second odd must show up before the first even, and the final odd must show up before the first even. This is really a chain of conditional probabilities, as the second odd showing up must occur given the first odd has already shown up.

The probability the first odd shows up before the first even is $\frac{1}{2}$, as the first roll is either odd or even with equal probability, so the first roll must appear odd. Afterwards, given the first odd has appeared, we now have reduced our sample space to 5 outcomes of interest. If we roll the odd that has already appeared, it is as if we ignore it. Therefore, given one of the other 5 outcomes occurs, the probability it is odd is $\frac{2}{5}$, as two of the remaining 5 outcomes are odd integers. By the same logic, once the second odd has occurred, if we see either of those two odds again, we ignore. Conditional on one of the other 4 outcomes appearing, the probability that it is the final odd is $\frac{1}{4}$, as 1 of the last 4 outcomes is odd.

This implies that the probability of this event is $\frac{1}{2} \cdot \frac{2}{5} \cdot \frac{1}{4} = \frac{1}{20}$.

Alternatively, consider all $6! = 720$ ways to arrange the orders of the appearance of the 6 values. We must have $1, 3, 5$ appear in the first 3 spots (in any order), while $2, 4, 6$ must appear in the last three spots in any order. There are $3! \cdot 3! = 6 \cdot 6 = 36$ ways to arrange the values satisfying the condition, so the probability of this occurring is $\frac{36}{720} = \frac{1}{20}$.

Solution to Question 59: Pharmaceutics II

Power is $1 - \beta$, where β is the probability that the test statistic is not in the rejection region when H_a is true. In order to calculate power, we must solve for β.

$$\beta = P(\text{failing to reject } H_0 \text{ when } H_a \text{ is true}) = P(x > 12 \mid p = 0.6)$$

$$= \sum_{i=13}^{20} \binom{20}{i} \times 0.6^i \times 0.4^{20-i} \approx 0.4159$$

Thus, the power of this test is:

$$1 - \beta \approx 0.5841$$

Solution to Question 60: Matching Die Trio

With three dice being rolled originally, we can either have all show the same value, have two distinct values, or have all three be distinct values. We condition on the the number of distinct values showing after the first roll.

The probability all of the values are the same is $\frac{6}{6^3} = \frac{1}{36}$, as you can roll the first die and the other two must match that value, which occurs with probability $\frac{1}{6}$ per die. The probability that all three values are distinct is easily calculated to be $\frac{6 \cdot 5 \cdot 4}{6^3} = \frac{5}{9}$. Therefore, the probability that exactly two distinct values appear is $1 - \frac{1}{36} - \frac{5}{9} = \frac{5}{12}$. Now, we must calculate the probability in each case that the second values match the first values.

If all values are the same, the probability the second rolling matches the first only happens when all three of the dice share that same value, which occurs with probability $\frac{1}{6^3} = \frac{1}{216}$. If we have two distinct values, say $a \neq b$, then there are 3 sequences that would result in the same values as the first rolling, which are aab, aba, and baa. Therefore, the probability in this case is $\frac{3}{6^3} = \frac{1}{72}$. Lastly, if all three values are distinct, say $a \neq b \neq c$, there are $3! = 6$ outcomes that result in the same values appearing, so the probability in this case is $\frac{3!}{6^3} = \frac{1}{36}$.

By Law of Total Probability, the probability of matching values between the rolls is thus $\frac{1}{36} \cdot \frac{1}{216} + \frac{5}{12} \cdot \frac{1}{72} + \frac{5}{9} \cdot \frac{1}{36} = \frac{83}{3888}$

Solution to Question 61: Put-Call Parity I

The put-call parity formula is given by:

$$C = S - Ee^{-rt} + P$$

Where

C = Call PriceP = Put PriceS = Stock PriceE = Exercise/Strike Pricer = Continuously compounded interest rate '

As $\$TSLA$ does not pay a dividend, we can ignore e^{-rt} section of our equation, as it will equate to 1. When plugging in the values we have, we see that $C = \$110 - \$100 + P$. Therefore, if the put is priced at \$2 dollars, and efficient market would price the call at \$12.

This means that any call option priced under \$12 would present an arbitrage opportunity, making our answer 12.

Solution to Question 62: Differ By 2

Use Markov Chains to solve this question. Youâll notice some symmetry in each dice roll. Rolling a 1 is functionally the same as rolling a 2, 5, or 6. In the same sense, rolling a 3 is the same as rolling a 4. Using this symmetry, we can drastically decrease the number of states to find the answer to this question. Let E_0 be the starting state where we donât have any rolls yet. Let E_1 be the states of rolling a 1, 2, 5, or 6 (they all have only one following roll that gets to the goal state) and not previously rolling a number that's a difference of 2. Let E_2 be the states of rolling a 3 or 4 (both have 2 possible following goal states) and not previously rolling a number that's a difference of 2. Finally, let E_{goal} be the goal state where the two previous rolls differ by 2. Then our equations become:

$$E_0 = \frac{2}{3}E_1 + \frac{1}{3}E_2 + 1$$
$$E_1 = \frac{1}{6}E_{goal} + \frac{2}{3}E_1 + \frac{1}{6}E_2 + 1$$
$$E_2 = \frac{1}{3}E_{goal} + \frac{1}{3}E_1 + \frac{1}{3}E_2 + 1$$
$$E_{goal} = 0$$

Solving these systems of equations, we get $E_0 = \frac{17}{3}$.

Solution to Question 63: Pass the Ball

We can use Markov states to solve this question. You'll notice there is some symmetry in this problem that will allow us to solve this question with less states. The symmetry is in the placement of the other people. For example, the two people immediately next to you can be treated as the same state functionally (since they both behave the exact same), and the two people furthest from you are also functionally the same (again, since they both behave the exact same). Let P_1 be the probability you end with the ball. Let P_2 be the probability that someone immediately next to you has the ball and you are the one to end the game. Finally, let P_3 be the probability that someone furthest from you has the ball and you are the one to end the game. The equations for the states are as follows:

$$P_1 = \frac{1}{3} + \frac{2}{3}P_2$$

$$P_2 = \frac{1}{3}P_1 + \frac{1}{3}P_3$$

$$P_3 = \frac{1}{3}P_2 + \frac{1}{3}P_3$$

Note that in the last equation, if one of the two people not immediately next to you has the ball, one of the passes will be to someone that is also not next to you, which is why there is only a $\frac{1}{3}P_2$ term. Solving these equations, we see that $P_1 = \frac{5}{11}$ which is our final answer.

Solution to Question 64: Unlucky Seven II

Condition on the value of the first roll. If we roll a 1, we either lose \$1 with probability $\frac{1}{6}$ (if we roll a 6 on the second roll). Otherwise, we make a total of $2, 3, \ldots, 6$, each with probability $\frac{1}{6}$. Therefore our expected payout if we roll again is $\frac{1}{6} \cdot (-1) + \frac{2+3+4+5+6}{6} = \frac{19}{6} > 1$. This implies we should roll again if we receive a 1.

If we roll a 2, then with probability $\frac{1}{3}$ we lose \$2 (if we roll a 5 or 6). Otherwise, we receive $3, 4, 5$, or 6 with equal probability $\frac{1}{6}$. Therefore, the expected value upon rolling again is $\frac{1}{3} \cdot (-2) + \frac{3+4+5+6}{6} = \frac{7}{3} > 2$. This implies we should roll. If we roll a 3, then with probability $\frac{1}{2}$ we lose \$3 (rolling 4 or more). Otherwise, we earn $4, 5$, or 6 each with probability $\frac{1}{6}$. Therefore, our expected profit upon rolling again is $\frac{1}{2} \cdot (-3) + \frac{4+5+6}{6} = 1 < 3$. Therefore, for any value 3 or more, we should not roll again. This implies that our expected value on this game with this strategy is $\frac{\frac{19}{6} + \frac{14}{6} + 3 + 4 + 5 + 6}{6} = \frac{47}{12}$.

Solution to Question 65: 3 Heads 3 Tails II

After building a transition graph, we find can produce the following set of equations for the expcted time to absorption:

$$\begin{aligned}
\mu_H &= 1 + \frac{1}{2}\mu_T + \frac{1}{2}\mu_{HH} \\
\mu_T &= 1 + \frac{1}{2}\mu_H + \frac{1}{2}\mu_{TT} \\
\mu_{TT} &= 1 + \frac{1}{2}\mu_H \\
\mu_{HH} &= 1 + \frac{1}{2}\mu_T
\end{aligned}$$

Solving, we find $\mu_T = 6$, meaning that beginning with state T, we expect 6 more flips before 3 in a row is reached. By symmetry, $\mu_H = 6$. By law of total expectation, we expect 6 more flips before 3 in a row is reached after the first coin has been flipped. Adding the first coin flip, our final answer is 7.

Solution to Question 66: Cats and Dogs I

Label the seats $1-12$. Saying that there are at least 4 dogs in a row somewhere is equivalent to saying that at some position, reading the animals off clockwise and abbreviating dog as D and cat as C, we have the sequence CDDDD. Let A_i be the event that this happens starting from position i going clockwise, we want to find $\mathbb{P}\left[\bigcup_{i=1}^{12} A_i\right]$. Note that the A_i events are disjoint because it is impossible for more than one of them to occur since we would either have cats interfering with the dogs mid-sequence or they are completely separate sequences, which is not possible due to there only being 6 dogs. Therefore, the probability of the union is the sum of the probabilities and we get that our probability of interest is $\sum_{i=1}^{12} \mathbb{P}[A_i]$. Further note that each of the spots are exchangeable (we assigned the labels arbitrarily), so this sum is just $12\mathbb{P}[A_i]$. All that is left is to compute $\mathbb{P}[A_i]$.

We want the probability of the sequence CDDDD, which can occur in $6 \cdot 6 \cdot 5 \cdot 4 \cdot 3$ ways. The total number of ways to assign animals to those 5 spots is $12 \cdot 11 \cdot 10 \cdot 9 \cdot 8$. Therefore, $\mathbb{P}[A_1] = \dfrac{6 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8} = \dfrac{1}{44}$. Therefore, our probability of interest is $\dfrac{12}{44} = \dfrac{3}{11}$.

Solution to Question 67: Increasing Dice Order I

We can use conditional probability to solve this problem.
$P(3 \text{ strictly increasing numbers}) = P(3 \text{ different numbers}) \times P(\text{increasing order}|3 \text{ different numbers})$.
The first term is simply $\frac{6}{6} \times \frac{5}{6} \times \frac{4}{6}$ from drawing without replacement. The second term is $\frac{1}{3!} = \frac{1}{6}$, since there is only one way of permuting three distinct numbers to be strictly increasing. Hence we find the answer to be $\frac{6}{6} \times \frac{5}{6} \times \frac{4}{6} \times \frac{1}{6} = \frac{5}{54}$.

Solution to Question 68: Non-Zero Eigenvalue

We know that the trace of a matrix is the sum of all the eigenvalues repeated according to their algebraic multiplicity. We first need to find the algebraic multiplicity of this non-zero eigenvalue. The trick here is to note that each of the rows of A_n are linearly dependent. Namely, the kth row of this matrix is a multiple k times the first row. This means that the subspace $A_n x = 0$ (i.e. the null space of A_n) has geometric multiplicity $n-1$, as the other $n-1$ rows are a multiple of the first row. The dimension of the null space, however, is just the geometric multiplicity of 0 as an eigenvalue. Therefore, $\mathrm{GM}_{\lambda=0}(A_n) = n-1$.

We now use another theorem that states that for any $n \times n$ matrix A and eigenvalue of A, say λ, $1 \leq \mathrm{GM}_\lambda(A) \leq \mathrm{AM}_\lambda(A) \leq n$. As we know $\mathrm{GM}_{\lambda=0}(A_n) = n-1$, we have that $\mathrm{AM}_{\lambda=0}(A_n)$ is either $n-1$ or n. However, we can rule out n by the fact that the if the algebraic multiplicity were n, the sum of all the eigenvalues (and hence the trace of A_n) would be 0. However, one can quickly see that the trace of A_n is $1^2+2^2+\cdots+n^2 = ||x_n||^2 = \dfrac{n(n+1)(2n+1)}{6}$ by working out the multiplication. Thus, the algebraic multiplicity of $\lambda = 0$ is $n-1$, and the last eigenvalue must be

$$||x_n||^2 = \frac{n(n+1)(2n+1)}{6}$$

We can quickly see that the dominating term of $||x_n||^2$ is an n^3 term, so $k = 3$ is our answer.

Solution to Question 69: Car Crashes

Let the probability in question be p. We know that the probability there is no car crash in the hour interval is $\dfrac{1}{9}$ by complementation. This is the same as saying that there is no car crash in each of the intervals consisting of the first and last 30 minutes of the hour. By the question, we can say that the number of car crashes in disjoint intervals are independent. This is because they occur at a constant rate throughout time and the arrivals are independent. The probability of no car crash in each of those two intervals individually is $1-p$, so the probability of no car crash in both intervals is $(1-p)^2$. Thus, $(1-p)^2 = \dfrac{1}{9}$, so $p = \dfrac{2}{3}$.

Solution to Question 70: Prime First

Consider all the primes in $S: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29$. There are 10 such integers. The key idea to notice here is that for whoever selects the integer 6, that eliminates 2 and 3. Therefore, no other pairwise products of primes among the remaining prime integers are at most 30. Therefore, there would be 8 primes left, and whoever is selecting first in this new game loses, as they can only eliminate one prime factor a turn. However, this "new game" is really just the second turn our game after we eliminate 2 and 3. Therefore, Alice should go first, select the value 6 (alternatively, $12, 18$, or 24 also work, as they have the same primes in the factorization) so that she can eliminate both 2 and 3, and then Bob and Alice alternate selecting among the remaining prime integers. Then, Bob would be forced to pick some non-prime integer, which will have a prime factor among those already eliminated, so Bob will lose. Therefore, Alice should go first, and the answer is 1.

However, there is also another sneaky starting integer, which is 5. This is because regardless of what Bob selects on his first turn after Alice selects 5, Alice can force Bob to eliminate one prime factor at a time after her next turn, so this also works. Since 5 works, $5^2 = 25$ also works. Therefore,

$$100(1) + \frac{6+12+18+24+5+25}{6} = 115$$

Solution to Question 71: Decorrelation

If $\text{Cov}(X, Y-kX) = 0$, then by bilinearity, $\text{Cov}(X,Y) - k\text{Cov}(X,X) = 0$, which means that $k = \dfrac{\text{Cov}(X,Y)}{\text{Var}(X)}$. Plugging in our values, we get $k = \dfrac{1}{2}$.

Solution to Question 72: Doda Rectangle

There are 2^{12} ways to color the 12 vertices. We can break down the solution into the following three cases: (1) there are zero same-colored opposing vertices, (2) there is exactly one pair of same-colored opposing vertices, and (3) there is exactly 1 red-red opposing pair of vertices and 1 blue-blue opposing pair of vertices.

Case 1: We set colors for vertices 1-6; there are 2^6 ways to do so. Then, vertices 7-12 are set. There are 2^6 ways to have zero same-colored opposing vertices.

Case 2: We set colors for vertices 1-6; there are 2^6 ways to do so. We choose 1 of vertices 7-12 to color identically to the opposite vertex; there are $\binom{6}{1} = 6$ ways to do so. There are $6 \cdot 2^6$ ways to have exactly one pair of same-colored opposing vertices.

Case 3: We set colors for vertices 1-6; there are 2^6 ways to do so. We then choose 2 of the remaining vertices to assign as same-colored to the opposing vertex. Half of such arrangements end up with two pairs of the same color. There are $\binom{6}{2} \cdot \frac{1}{2} \cdot 2^6 = 480$ ways to have exactly one pair of same-colored opposing vertices.

Adding up all the cases, we find that the probability is $\frac{928}{2^12} = \frac{29}{128}$.

Solution to Question 73: Multinomial Expansion

Note that each term of the expansion is going to be in the form $x_1^{e_1} \ldots x_6^{e_6}$, where $e_1 + \cdots + e_6 = 18$ and $e_i \geq 0$ are integers. This is because we choose one of the x_i's from each of the 18 terms of the product, so the exponents of all the x_i together must sum to 18. Therefore, we have that we just want the number of non-negative solutions to $e_1 + \cdots + e_6 = 18$, with $e_i \geq 0$ are integers. By stars and bars, this is $\binom{23}{5}$.

Solution to Question 74: Dice Products

If one of the dice is 6, we know the product will always be divisible by 6. There are 11 rolls that involve the number 6 (5 pairs with distinct numbers Ã 2 orderings + 1 way to roll double 6). The other way to make a product divisible by 6 is to roll a number with factors of 2 and 3. Since we counted rolls with 6 already, we leave those aside. So weâll need either a 2 or 4 on one die and a 3 on the other. In this case we have 2 possibilities for the first die, 1 possibility for the second, and 2 different orderings. $2\tilde{A}1\tilde{A}2 = 4$ combinations. Adding this 4 with the previously calculated 11 and we have 15 rolls which result in a number divisible by 6. There are 36 total possibilities of 2 rolls, each being the same probability of occurring, so the probability will be $15/36 = 42.5\%$

Solution to Question 75: Tennis Deuces II

From "Tennis Deuces I", we know there are 20 ways to get to 3-3. We know we have to get to 3-3 before getting to 4-4 since any score of 4-x where x isn't a number greater than or equal to 3 means the game ended prematurely. From here, to make 4-4, either Andrew wins a point and then Beth does or vice-versa which is two different outcomes. $20 \cdot 2 = 40$ ways.

Solution to Question 76: Red and Black Urn II

The probability of the nth ball being red is just $\mathbb{E}[I_n]$, where I_n is the indicator of the event that the nth ball is red. We can use Law of Total Expectation to get that $\mathbb{E}[I_n] = \mathbb{E}[\mathbb{E}[I_n \mid X_{n-1}]]$, where X_{n-1} is the number of red balls in the urn after $n-1$ draws. The expectation on the inside is just $\dfrac{X_{n-1}}{r+b}$, as there are X_{n-1} red balls out of $r+b$ total. Therefore, the probability is $\mathbb{E}\left[\dfrac{X_{n-1}}{r+b}\right] = \dfrac{M_{n-1}}{r+b}$. With these particular values, the probability is $\dfrac{M_{10}}{20} \approx 0.64$.

Solution to Question 77: Covariance of BM

We will do this more generally for any $0 \leq s \leq t$. In other words, we want to compute $\text{Cov}(W_s, W_t)$ for any $s \leq t$. Using our standard expansion of covariance,

$$\text{Cov}(W_s, W_t) = \text{E}[W_s W_t] - \mathbb{E}[W_s]\mathbb{E}[W_t]$$

The mean of Brownian Motion at any time is 0, as $W_t \sim N(0, t)$, so the second term vanishes. The trick to compute the first term is to write $W_t = (W_t - W_s) + W_s$. We do this because of the fact that $W_t - W_s$ is independent of W_s. Therefore,

$$\mathbb{E}[W_s W_t] = \mathbb{E}[W_s((W_t - W_s) + W_s)] = \mathbb{E}[W_s(W_t - W_s)] + \mathbb{E}[W_s^2]$$

The first term above is just $\mathbb{E}[W_s]\mathbb{E}[W_t - W_s] = 0$ by the independence of W_s and $W_t - W_s$. The other term is just s. This is because the mean of W_s is 0, so $\mathbb{E}[W_s^2] = \text{Var}(W_s)$. Therefore, $\text{Cov}(W_s, W_t) = s$ for $0 \leq s \leq t$. In this case $s = 1$ and $t = 2$, so our answer is 1.

Solution to Question 78: Conditional Head Starter

Let $A(n)$ be the number of sequences of length n with no consecutive heads. Then $A(1) = 2$ and $A(2) = 3$ (only HH doesn't satisfy this). For a sequence of length n to satisfy our constraint, it either starts with T or HT. It can't start with HH, as that would violate the condition.

There are $A(n-1)$ sequences starting with T that satisfy this condition, as the first flip is T and the other $n-1$ spots must not contain any consecutive heads. There are $A(n-2)$ sequences starting with HT by the same logic with with $n-2$ remaining spots. Thus, $A(n) = A(n-1) + A(n-2)$ with $A(1) = 2$ and $A(2) = 3$. This is just the Fibonacci sequence shifted. In particularly, $A(n) = F_{n+2}$, where F_k is the kth Fibonacci number.

Given this information, all sequences that satisfy the condition are equally likely. We know that there are $A(n-2)$ sequence starting with HT (and hence H) out of the $A(n)$ total. Therefore, we get that

$$p(n) = \frac{A(n-2)}{A(n)} = \frac{F_n}{F_{n+2}}$$

In particular, $p(10) = \dfrac{F_{10}}{F_{12}} = \dfrac{55}{144}$

Solution to Question 79: Car Bidding II

Let B be the bid price. For the bid to be accepted, we know that $B > 9000$. We want to find the probability $B > 15000$, but we have this information from above, so we really want $\mathbb{P}[B > 15000 \mid B > 9000]$. By the memorylessness property, this is equivalent to asking $\mathbb{P}[B > 6000]$. Using the CDF of the exponential distribution,

$$\mathbb{P}[B > 6000] = 1 - \mathbb{P}[B \leq 6000] = 1 - \left(1 - e^{-\frac{6}{5}}\right) = e^{-\frac{6}{5}}$$

Therefore, $a = -\dfrac{6}{5}$.

Solution to Question 80: 1 or Bust

Initially, the question may look like a Markov chain, but it is not straight forward to solve it this way. For example, if you roll a 1, 2, or 3, this doesn't necessarily increase your payoff by \$1. This is due to the fact that there is still a chance you will end the game with \$0. Instead, it is easier to solve this expectation directly using an infinite geometric series. Namely, the probability we receive \$$k$ payout is to roll k values $1-3$ and then either a 4 or a 5. The probability of this is $\dfrac{1}{2^k} \cdot \dfrac{1}{3}$. Therefore,

$$\mathbb{E}(X) = \sum_{k=1}^{\infty} k \cdot \left(\frac{1}{2}\right)^k \left(\frac{1}{3}\right) = \frac{1}{3}\mathbb{E}[G] = \frac{2}{3}$$

where we evaluate the sum by recognizing the remaining term is the PMF of a Geom(1/2) random variable, so we can say the sum is just $\mathbb{E}[G]$ for $G \sim$ Geom(1/2).

Solution to Question 81: Row Your Boat

The first order of business to find the speed at which the friends travelled. Let x be this constant speed in miles per hour. When travelling upstream, they move at $x - 2$ miles per hour because the stream opposes them. When travelling downstream, the move at $x+2$ miles per hour. Therefore, we have that $3(x-2) - 5(x+2) = -32$, as they travel upstream for 3 hours and downstream for 5 hours and end up 26 miles below their initial spot. Solving for x here yields $x = 8$. Therefore, as they travel upstream going back to the campsite the next morning, they will be travelling at 6 miles per hour going back up. Therefore, it takes the friends $\dfrac{32}{6} = \dfrac{16}{3}$ hours to get back to their campsite. This is 5 hours and 20 minutes. This means they must have started at 1:40 PM. Therefore, our answer is 1340.

Solution to Question 82: Ancient Births

Since B died 129 years after C was born and at least one of B or C was alive for 100 years, this means there were 29 years where neither of them were alive. This means that there were 29 years after the death of C before B was born. Therefore, B was born in 1 B.C., so our answer is -1.

Solution to Question 83: Windless Mile

Thinking about this as a rate, let x be the (in miles/min) that the wind blows and r be the rate (in miles/min) that the man bikes at. We know that $r+x = 1/3$ and $r - x = 1/4$ from the question above. Therefore, we get that $r = 7/24$ by solving this system, meaning it would take him 24/7 minutes to bike a mile windless.

Solution to Question 84: Number Concatenate

Given an integer m, the number of digits in m is $\lfloor \log_{10}(m) \rfloor + 1$. In this case, we have that our number of digits is

$$\lfloor 1000 \log_{10}(2) \rfloor + \lfloor 1000 \log_{10}(5) \rfloor + 2$$

We know that $\lfloor 1000 \log_{10}(2) \rfloor + \lfloor 1000 \log_{10}(5) \rfloor \leq 1000 \log_{10}(2) + 1000 \log_{10}(5) = 1000 \log_{10}(10) = 1000$. However, we also know that it is at least 999 because

$$\lfloor 1000 \log_{10}(2) \rfloor + \lfloor 1000 \log_{10}(5) \rfloor > (1000 \log_{10}(2) - 1) + (1000 \log_{10}(5) - 1) = 1000 \log_{10}(10) - 2 = 998$$

The next largest integer would be 999. Therefore, x must therefore have either 1001 or 1002 digits.

We get equality of $\lfloor x \rfloor + \lfloor y \rfloor = x + y$ when x and y are both integers. However, $1000 \log_{10}(2)$ nor $1000 \log_{10}(5)$ are integers, so we must have that our answer is 1001.

Solution to Question 85: 3 Heads 3 Tails I

The probability that Anna wins is $\frac{1}{2}$ by symmetry. Further, the expected total number of tosses for Anna to win is equal to the expected total number of tosses for Brenda to win by symmetry. By the law of total expectation, our desired answer is simply double the expected total number of tosses for Anna to win. Anna wins in 3 total tosses with probability $\frac{1}{8}$. Anna wins in 4 total tosses with probability $\frac{3}{16}$ (there are 3 arrangements of 3 heads and 1 tail such that the last flip is heads). Anna wins in 5 total tosses with probability $\frac{6}{32}$ (there are 6 arrangements of 3 heads and 2 tails such that the last flip is tails). Putting it all together, we find the expected number of tosses for Anna to win is

$$\frac{1}{8} \cdot 3 + \frac{3}{16} \cdot 4 + \frac{6}{32} \cdot 5 = \frac{66}{32}$$

Multiplying by 2 gives us $\frac{33}{8}$.

Solution to Question 86: Multinomial Sum

Looking at an individual multinomial coefficient, it counts the number of ways to put 7 distinct balls into 4 distinct boxes with a certain amount in each box. Therefore, the sum over all such multinomial coefficients is just the number of ways to put 7 distinct balls into 4 distinct boxes, which can be done in $4^7 = 16384$ ways.

Solution to Question 87: Put-Call Arbitrage

From put-call parity, we know that $C - P = S - K$ must hold. We can see that $C - P = 4 - 3 = 1$ and $S - K = 10 - 4 = 6$. We can see that $1 < 6$, so there clearly is an arbitrage. When we see an inequality like this, we will long the undervalued assets and short the overvalued assets. Here, the undervalued assets are $C - P$ and the overvalued assets are $S - K$. We long $C - P$, meaning that we will long 1 unit of C and short 1 unit of P. We will then short 1 unit of S and long 4 units of the bond. This is due to the fact that bonds pay 1 at expiry, so $K = 4$ corresponds to 4 units of the bond.

We then have # Stock + # Call + # Put + # Bonds $= -1 + 1 - 1 + 4 = 3$

Solution to Question 88: High-Low Guess

This is a simple binary search problem. On the first turn, you should select 500, and in each consecutive turn, based on whether your friend says higher or lower, you guess the midpoint of the remaining region. This eliminates half of the "search space" at each turn, so the number of values you can search in k turns is 2^k, meaning that n is the smallest integer such that $2^n \geq 1000$, which is $n = 10$.

Solution to Question 89: Ten Ten

For each roll, there are two possible cases. The first is that he doesn't roll a 10, which occurs with probability $\frac{9}{10}$. In this case, given he doesn't roll a 10, the values in which he rolls are uniformly distributed over the set $\{1, 2, \dots, 9\}$, which has mean value 5. This term is contributing $\frac{9}{10}(5+\mathbb{E}[X])$ monetary value. Otherwise, he rolls a 10 and profits 0. To find the stopping point, we want to find the value at which Adam's expected value on the game of rolling another time decreases from what it is presently. Therefore, to find the stopping point, we want to solve $\mathbb{E}[X] = \frac{9}{10}(5 + \mathbb{E}[X])$, which can be solved to get $\mathbb{E}[X] = 45$.

Solution to Question 90: Voter Mayhem I

Let $P(n, m)$ be the probability of interest with n votes from Candidate A and m votes for Candidate B. Since the votes are thrown in the box randomly, the probability any given vote draw is for Candidate A is $\frac{n}{m+n}$, while it is $\frac{m}{m+n}$ for Candidate B. Conditioning on which candidate has the last vote, we get that

$$P(n, m) = P(n-1, m) \cdot \frac{n}{m+n} + P(n, m-1) \cdot \frac{m}{m+n}$$

This is because of the fact that the probability that A is always ahead in all n draws is the same as the probability that A is always ahead if they received $n-1$ votes and B received m votes, which is $P(n-1, m)$. By the same argument, we get that the probability A is always ahead with the last vote being for B is the same as if A is always ahead given A had n votes and B had $m-1$ votes. Our goal is now find $P(n, m)$.

A good warm-up is to consider an edge case of when A has $n > 1$ votes and B has $m = 1$ votes. A would always be ahead exactly when the singular B vote is not in the first two draws. If the B vote is in the first two draws, then B would have at least as many votes as A at some point in the first two draws. Otherwise, if the first two draws are for A, B can never reach more than 1 vote. The probability of this is $\frac{n}{n+1} \cdot \frac{n-1}{n} = \frac{n-1}{n+1}$. This suggests to us to consider $\frac{n-m}{n+m}$ for a general m.

To prove this, we induct on the value of $n + m$. If $n + m = 1$, this means $n = 1$ and $m = 0$. In this case, $P(1, 0) = \frac{1-0}{1+0} = 1$, which is accurate. More generally, suppose this holds for when $n+m=k$ for all $1 \leq k \leq r$. We now need to show it holds for $n+m=r+1$. Using our recurrence,

$$P(n,m) = \frac{n}{m+n} \cdot P(n-1,m) + \frac{m}{m+n} \cdot P(n,m-1) = \frac{n}{m+n} \cdot \frac{n-1-m}{n-1+m} + \frac{m}{m+n} \cdot \frac{n-(m-1)}{n+(m-1)} = \frac{n-m}{n+m}$$

We can use our recurrence relation here since if $n+m=r+1$, $n+(m-1) = (n-1)+m = r$, so the induction hypothesis applies.

For our particular case, the answer is $\frac{100-80}{100+80} = \frac{1}{9}$

Solution to Question 91: Say Your Color

The pattern here is that for the first three sounds, 0 means that the vowel is pronounced and 1 means that the consonant is pronounced. This means that YELLOW is 001.

Solution to Question 92: Basic Gamma

The premium of an option comes from the fact that they can expire in-the-money and have some intrinsic value at expiration. If a call is deep out-of-the-money, then it is unlikely for the call to expire in-the-money and hence the gamma is 0.

We can also think about this in terms of the payoff structure. We can *take derivatives* of the payoff for a call and see that gamma looks like a delta-spike, hence deep out-of-the-money calls having a gamma of 0.

Solution to Question 93: Bivariate Covariance

If $Z_1, Z_2 \sim N(0,1)$ IID, we know that the joint transformation that would yield (X,Y) is $X = Z_1$, $Y = \rho Z_1 + \sqrt{1-\rho^2} Z_2$. Therefore, $\text{Cov}(X, Y^2) = \mathbb{E}[XY^2] - \mathbb{E}[X]\mathbb{E}[Y^2]$. As we know that $X \sim N(0,1)$ marginally, $\mathbb{E}[X] = 0$. Therefore, we have to compute the first term. Substituting in, we get

$$\mathbb{E}[XY^2] = \mathbb{E}[Z_1(\rho Z_1 + \sqrt{1-\rho^2} Z_2)^2] = \mathbb{E}[Z_1(Z_1^2 + 2\rho\sqrt{1-\rho^2} Z_1 Z_2 + (1-\rho^2) Z_2^2)]$$

By distributing through the Z_1 and applying linearity of expectation, we get that the above is

$$\mathbb{E}[Z_1^3] + 2\rho\sqrt{1-\rho^2}\mathbb{E}[Z_1^2 Z_2] + (1-\rho^2)\mathbb{E}[Z_1 Z_2^2]$$

Using the independence of Z_1 and Z_2 and the fact that all odd moments of the standard normal distribution are 0, all of the above cancels and the answer is 0.

Solution to Question 94: Replacement Orbs

Let X_1 represent the number of draws needed to go from 1 blue to 2 blue orbs in the box. Similarly, let X_2 be the amount of draws needed to go from 2 blue to 3 blue orbs in the box. Then $T = X_1 + X_2$ gives us the total amount of draws needed to go from our current state to 3 blue orbs. By linearity of expectation, $\mathbb{E}[T] = \mathbb{E}[X_1] + \mathbb{E}[X_2]$. Note that $X_1 \sim \text{Geom}(2/3)$, as there is a $\frac{2}{3}$ probability on each draw that the orb will be red (and hence replaced by a blue after). Similarly, $X_2 \sim \text{Geom}(1/3)$. The means of these two are $\frac{3}{2}$ and 3, respectively, so the answer is $\mathbb{E}[T] = \frac{3}{2} + 3 = \frac{9}{2}$.

Solution to Question 95: Illegible Dice

There are 36 possible face combinations of two six-sided dice rolls. In order for each of the twelve outcomes to be equally likely, each must be able to occur exactly three times. In order for 1 to occur three times, there must be three 0s on the illegible die. Furthermore, in order for 12 to occur three times, there must be three 6s on the illegible die. Hence, we have solved the sides of the illegible die and the sum of its sides is 18.

Solution to Question 96: Sharpe Marbles

Siblings Alice and Bob play a game with marbles. Each player has one red and one blue marble and shows one marble to the other uniformly at random. If both show blue, Alice wins \$1. If both show red, Alice wins \$3. Else, Bob wins \$2. Note that the winnings come from their mother, not the other player. Let A_r and B_r define the ratio between the expected return and variance of Alice's and Bob's payoffs, respectively. What is $B_r - A_r$?

We start by solving for the expected payoffs and their variances:

$$E[A] = \frac{1}{4}(1) + \frac{1}{4}(3) + \frac{1}{2}(0) = 1$$

$$E[B] = \frac{1}{2}(2) + \frac{1}{2}(0) = 1$$

$$V[A] = \frac{1}{4}(1-1)^2 + \frac{1}{4}(3-1)^2 + \frac{1}{2}(0-1)^2 = \frac{3}{2}$$

$$V[B] = \frac{1}{2}(2-1)^2 + \frac{1}{2}(0-1)^2 = 1$$

Now, we can calculate A_r and B_r:

$$A_r = 1 \div \frac{3}{2} = \frac{2}{3}$$
$$B_r = 1 \div 1 = 1$$
$$B_r - A_r = \frac{1}{3}$$

This conclusion shows how expected return is not the only metric that should be used when calculating the practical payoff of an investment. Even though the two players have the same expected return, Alice is taking on additional risk for her position. Ceteris paribus, Bob's position is favored over Alice's because he achieves higher expected return per unit of positional risk he takes on.

Solution to Question 97: Dollar Cent Switch

Let y be the number of cents and x be the number of dollars on the check. Then the man's check is $x + 0.01y$. The amount the main was paid is $y + 0.01x$. The amount he had that was exactly twice as much as the check was $y+(0.01x-0.05)$. Therefore, $2x + 0.02y = y + (0.01x - 0.05)$.

There are two cases to consider here. In the first case, we have that $y < 50$, in which the $0.02y$ term doesn't overflow into another dollar. In this case, we would have that $2x = y$ and $0.02y = 0.01x - 0.05$. If this was true, then $0.04x = 0.01x - 0.05$, which would means that $x < 0$, which is impossible. Therefore, we have that $y \geq 50$, in which $y = 2x + 1$, as we overflow into another dollar. Additionally, we would have that $0.02y - 1 = 0.01x - 0.05$. Substituting in here,

$$0.02(2x + 1) - 1 = 0.01x - 0.05 \iff 0.03x = 0.99 \iff x = 31$$

Then we have that $y = 2 \cdot 31 + 1 = 63$. Therefore, his check was for 31.63.

Solution to Question 98: Questionable Values

There must be exact three times as many easy questions as medium so that the probability of a medium is 1/4. Thus, we can group the questions up into groups of 3 easy and 1 medium question, which sum to \$15. We need to find the minimum number of groups, say n, such that $15n$ is a perfect square. This would occur for the first time exactly when $n = 15$, so his question bank is worth $15^2 = 225$.

Solution to Question 99: Poisson Review II

Let $X \sim \text{Poisson}(7)$. In hours, the total service time is $\dfrac{1}{6} \cdot X$. We wish to compute $\mathbb{E}\left[\dfrac{1}{6}X\right] + \text{Var}\left[\dfrac{1}{6}X\right] = \frac{1}{6}\mathbb{E}[X] + \frac{1}{36}\text{Var}[X] = \frac{49}{36}$

Solution to Question 100: Cylindrical Intersection

We can model this as two cylinders $x^2+y^2 = 1$ and $y^2+z^2 = 1$. Suppose that we fix some y. We then want to find the set of points (x, z), such that $x^2 \leq 1 - y^2$ and $z^2 \leq 1 - y^2$. Namely, this just because $-\sqrt{1 - y^2} \leq x, z \leq \sqrt{1 - y^2}$. This point set is just a square of side length $2\sqrt{1 - y^2}$, which has area $4(1 - y^2)$. Then, we can just integrate the cross sections from $y = -1$ to 1, as those are the bounds of y in our intersected region. Therefore, the answer is

$$\int_{-1}^{1} 4(1 - y^2)dy = \frac{16}{3}$$

Solution to Question 101: Word Shift II

There are 3 B and 4 O in the word above. There are $\dbinom{11}{7}$ ways to pick the locations of these 7 letters in our anagram. However, there are 5 anagrams of $BBBOOOO$ that have at least 2 of the B's before all of the O's. Namely, you can move one B around among the 4 O's, yielding 5 total letters to place. You just select one of those spots to be B and those yield the 5 anagrams. Afterwards, the other 4 letters $LAHU$ can be arranged in the remaining 4 blanks completely at will in 4! ways. Therefore, the answer is

$$5 \cdot 4! \cdot \binom{11}{7} = 5 \cdot \frac{11!}{7!} = 39600$$

Solution to Question 102: Graph Search I

When rewording it to think about it as a 10−sided die, and we are trying to figure out how many rolls it will take to see all 10 sides at least once, it is easy to see that this is a classic coupon collector's problem.

The time until the first result appears is 1 as it takes one step to start somewhere. After that, the random time until a second (different) result appears is geometrically distributed with parameter of success $\frac{9}{10}$, hence a mean $\frac{10}{9}$ (as the mean of a geometrically distributed random variable is the inverse of its parameter). After that, the random time until a third (different) result appears is geometrically distributed with parameter of success $\frac{8}{10}$, hence a mean of $\frac{10}{8}$. And so on, until the random time of appearance of the last and tenth node, which is geometrically distributed with parameter of success $\frac{1}{10}$, hence with mean $\frac{10}{1}$.

We now know the mean time to see the first through last numbers pop up, so our total expectation will be summing all of the individual expectations up.

$$\sum_{k=1}^{10} \frac{10}{k} \approx 29.29$$

This yields our solution of 29.

Solution to Question 103: Black or Yellow

The first pick has a $\frac{1}{2}$ probability of being right with a payoff of \$4. The second pick has a $\frac{2}{3}$ probability of being right with a payoff of \$3, since you will always pick the majority choice. The third pick has two cases: with probability $\frac{1}{3}$, you chose a ball that was the last of its color in the previous turn, and thus know the color of the current ball, and with probability $\frac{2}{3}$, you chose a ball that was of the majority in the previous turn such that now there is one of each and thus the probability of being right is $\frac{1}{2}$. Both cases have a payoff of \$2. And finally, by elimination you know the fourth ball with a payoff of \$1. Thus, the total payoff of the game is:

$$\frac{1}{2} \times 4 + \frac{2}{3} \times 3 + \frac{1}{3} \times 2 + \frac{2}{3} \times \frac{1}{2} \times 2 + \frac{1}{1} \times 1 = \frac{19}{3}$$

Solution to Question 104: Options Gamma

We can use put-call parity and the result from Options Delta to obtain the relationship between the gamma of a call and put at the same strike (assuming Black-Scholes dynamics).

From Options Delta, we have $\Delta_c - \Delta_p = 1$. We can take another derivative with respect to the underlying and see that $\Gamma_c - \Gamma_p = 0 \Rightarrow \Gamma_c = \Gamma_p$.

This gives us another important result from options theory. The gamma of a call and a put at the same strike are the same.

Solution to Question 105: Complex Exponential

Recall that $i = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = e^{\frac{\pi}{2}i}$. This means that

$$i^i = \left(e^{\frac{\pi}{2}i}\right)^i = e^{\frac{\pi}{2}i^2} = e^{-\frac{\pi}{2}}$$

This means that $c = \frac{1}{2}$.

Solution to Question 106: Stone Ripple

We know that $A(r) = \pi r^2$ as a function of r and that $r' = 2$. After 10 seconds, the radius of the circle is $2 \cdot 10 = 20$ feet. Lastly, we find that $A' = 2\pi r r'$, so plugging in our values, we get that the rate of increase is $2\pi(20)(2) = 80\pi$. Therefore, $k = 80$.

Solution to Question 107: Random Angle II

Repeating the same process, we have that $\theta > \frac{\pi}{3}$ if and only if $\tan(\theta) > \sqrt{3}$. Therefore, we want the probability that $\mathbb{P}[A > B\sqrt{3}]$. Plotting this in the plane yields a triangle of side lengths 1 and $\frac{1}{\sqrt{3}}$, meaning that the area of the triangle is $\frac{1}{2\sqrt{3}}$. We are allowed to take ratios of areas since they are uniform distributions. Thus, the area of the entire square is 1, so the probability of this is just $\frac{1}{2\sqrt{3}}$. Therefore, $abc = 6$.

Solution to Question 108: Compound Interest I

The answer will be $100(1+0.01)^{100}$ by simple interest formula. Estimating that requires some skill. Noting the fact that $(1+1/n)^n \approx e$ for large n, we know it is about $100e \approx 271$. However, we must adjust down a little bit, so one may guess it is \$270. 270 is indeed correct, though it is on the boundary between 270 and 271.

Solution to Question 109: Negative Correlated Sum

We have that $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X,Y) = 9 + 16 + 2(-3/8)\sqrt{9}\sqrt{16} = 16$, so the standard deviation of $X+Y$ is $\sqrt{16} = 4$.

Solution to Question 110: Orthogonal Cosine

We know that $\cos(75) = \dfrac{x \cdot y}{||x||||y||}$ gives us a formula for the angle between two vectors x and y here. The cosine of the angle between Ax and Ay is

$$\frac{(Ax) \cdot (Ay)}{||Ax||||Ay||}$$

Orthogonal matrices applies to vectors preserve length i.e. the linear transformation $T(x) = Ax$ is an isometry. Therefore, $||Ax|| = ||x||$ and $||Ay|| = ||y||$. Additionally, we can write $(Ax) \cdot (Ay)$ as $(Ax)^T(Ay) = x^T A^T Ay$. However, as A is orthogonal, $A^T A = I_n$, so $x^T A^T Ay = x^T(I_n y) = x^T y = x \cdot y$ Therefore, the cosine of the angle between Ax and Ay is the same as between x and y, so the answer is also 75.

Solution to Question 111: 112 Appearance

Let μ represent the mean number of rolled needed. We are going to apply law of total expectation here to compute μ based on our first roll. If our first roll is 2, we are done. If our first roll is 1, then we only need one more 1 until we are done. We will call the expected number of rolls to finish starting with a 1 in our sequence μ_1. Otherwise, we just are at our current state again. Together, these imply that

$$\mu = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot (1 + \mu_1) + \frac{2}{3} \cdot (1 + \mu)$$

To compute μ_1, note that rolling either a 1 or 2 will make the game finish in 1 turn. Otherwise, we are back to the "nothing" state. Therefore,

$$\mu_1 = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot (1 + \mu) = 1 + \frac{2}{3}\mu$$

Substituting this into the first equation, we get that

$$\mu = 1 + \frac{1}{6} \cdot \left(1 + \frac{2}{3}\mu\right) + \frac{2}{3}\mu$$

Rearranging this yields $\frac{2}{9}\mu = \frac{7}{6}$, so $\mu = \frac{21}{4}$.

Solution to Question 112: Field Imperfection

The measurement of the length is given by $300 + X$, where X is the error, and the measurement of the width is given by $200 + Y$, where Y is the error. The area then is $(300+X)(200+Y)$, so we want $\mathbb{E}[(300+X)(200+Y)] = \mathbb{E}[60000 + 300Y + 200X + XY]$. We have that $\mathbb{E}[X] = 0$ and $\mathbb{E}[Y] = 5$ from properties of uniform random variables. By the linearity of expectation and independence of

X and Y, the above is just $60000 + 300\mathbb{E}[Y] + 200\mathbb{E}[X] + \mathbb{E}[X]\mathbb{E}[Y]$. The last two terms vanish as $\mathbb{E}[X] = 0$. The second term is 1500, so the expected area is 61500.

Solution to Question 113: Brussels Sprouts

We can use the Law of Total probability to solve this problem. Let X be the total number of Brussels sprouts you will eat and Y be the value of the dice roll. By the Law of Total Probability:

$$E[X] = P(Y = \{1,2\}) \times E[X|Y = \{1,2\}] + P(Y = \{3,4,5,6\}) \times E[X|Y = \{3,4,5,6\}]$$

$$E[X|Y = \{1,2\}] = 1.5 + E[X]$$

$$E[X|Y = \{3,4,5,6\}] = 4.5$$

$$P(Y = \{1,2\}) = \frac{1}{3}$$

$$P(Y = \{3,4,5,6\}) = \frac{2}{3}$$

Substituting in values, we find that:

$$E[X] = \frac{1}{3}(1.5 + E[X]) + \frac{2}{3}(4.5) \Rightarrow E[X] = \frac{21}{4}$$

Solution to Question 114: Coloring Components II

Let C_n be the number of connected components when we have n squares in a line. We will derive a recurrence relation for $\mathbb{E}[C_n]$.

Suppose that we want to find $\mathbb{E}[C_n]$, the expected number of connected components with n squres in a row. We can find this by conditioning on the $n-1$st square. We want to find the probability that the squares match in color. This probability is just $\frac{3}{4} \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{1}{4} = \frac{5}{8}$ because of the fact that they are either both black (the first term) or both white (the second term). If it matches, then we have $\mathbb{E}[C_{n-1}]$ connected components, as we don't add in any new ones if it matches. If it differs, then we have $1 + \mathbb{E}[C_{n-1}]$ connected components, as the differing color will add in a new component. Therefore, by Law of Total Expectation, $\mathbb{E}[C_n] = \frac{5}{8} \cdot \mathbb{E}[C_{n-1}] + \frac{3}{8} \cdot (1 + \mathbb{E}[C_{n-1}]) = \mathbb{E}[C_{n-1}] + \frac{3}{8}$. This recurrence along with the initial condition that $\mathbb{E}[C_1] = 1$ (as we have 1 component), yields the solution $\mathbb{E}[C_n] = 1 + \frac{3}{8}(n-1)$. In particular, $n = 25$, so $\mathbb{E}[C_{25}] = 10$.

Solution to Question 115: Multiple Divisors II

$$20 = 2^2 \cdot 5$$

, so this means that whatever factor we select needs to has an exponent of at least 2 in its prime factorization for 2s and also needs to be divisible by 5. Note that $20! = 20 \cdot 19 \cdot 18 \cdot \cdots \cdot 2 \cdot 1 = 2^{18} \cdot 5^4 \cdot N$ by noting that we can extract a power of 2 from every even integer, an extra power of 2 from $4, 8, 12, 16$, and 20, a third power of 2 from 8 and 16, and then a final power of 2 from 16. Similarly, we extract a power of 5 from $5, 10, 15$, and 20. The number N is irrelevant here, as it will has its own prime factorization that does not include any powers of 2 nor 5.

Since we select the divisor uniformly at random, the term $2^a 5^b$ will also be uniformly at random in $0 \leq a \leq 18$ and $0 \leq b \leq 4$. The probability that $a \geq 2$ is $\frac{17}{19}$, as it is uniformly distributed over 19 integers. The probability $b \geq 1$ is $\frac{4}{5}$ by similar logic. As these are independent, the final probability is $\frac{17}{19} \cdot \frac{4}{5} = \frac{68}{95}$.

Solution to Question 116: Prime Subset

To ensure all of the elements are coprime, the easiest way to make a candidate set would be the collection of all prime numbers at most 30. However, we also want to include 1, as 1 is coprime to every positive integer. Therefore, a candidate set would be

$$\{1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$$

This set has sum 130. However, we can do better than this by noting that we could replace any prime integers in the set by a combination of their factors that would have a larger sum. For example, you may remove 3 and 5 to replace it with 15, increasing the sum to 137. The question then becomes which integers can we do this for and not violate our condition. Let's start with taking out the 2. This is natural since 2 is our smallest integer besides 1 in our subset. The closest we can get to 30 with elements currently in the subset is 28, as $2^2 \cdot 7 = 28$. 29 is already in our subset, and 30 can't be added in presently, as 3 and 5 divide 30. Therefore, we should replace 2 and 7 with 28, increasing our sum by 19 to 149. Our remaining set is

$$\{1, 3, 5, 11, 13, 17, 19, 23, 28, 29\}$$

Afterwards, we note that $3^3 = 27$, and 27 is still coprime to everything our in our set. Similarly, $5^2 = 25$ would also be coprime to everything else, so we make these substitutions as well. This yields our final set of

$$\Omega = \{1, 11, 13, 17, 19, 23, 25, 27, 28, 29\}$$

The elements of this set sum to 193.

Solution to Question 117: Red and Black Urn I

Let X_n be the number of red balls in the urn after n drawings. Then $R_n = \mathbb{E}[X_n]$. We can derive a recurrence for R_n. In particular,

$$R_{n+1} = \mathbb{E}[X_{n+1}] = \mathbb{E}[\mathbb{E}[X_{n+1} \mid X_n]] = \mathbb{E}\left[X_n \cdot \frac{X_n}{r+b} + \left(1 - \frac{X_n}{r+b}\right)(X_n+1)\right] = \left(1 - \frac{1}{r+b}\right)R_n + 1$$

We have that $R_0 = r$, as there are r balls at the start that are red.

Note that

$$R_n - (r+b) = \left(1 - \frac{1}{r+b}\right)(R_{n-1} - (r+b)) = \left(1 - \frac{1}{r+b}\right)^2 (R_{n-2} - (r+b)) = \cdots = \left(1 - \frac{1}{r+b}\right)^n (R_0 - (r$$

Plugging in $R_0 = r$ and rearranging, we get that

$$R_n = r + b\left(1 - \left(1 - \frac{1}{r+b}\right)^n\right)$$

Plugging in the specific values, we get that our answer is about 12.82.

Solution to Question 118: Eigenspace Intersection

Let x be a vector belonging to both E_{λ_1} and E_{λ_2}. This means that $Ax = \lambda_1 x$ and $Ax = \lambda_2 x$. Equating both of these, we get that $\lambda_1 x = \lambda_2 x$. Equivalently, this means $(\lambda_1 - \lambda_2)x = 0$. Since $\lambda_1 \neq \lambda_2$, the only way this is possible is if $x = 0 \in \mathbb{R}^n$. Therefore, the intersection of these two eigenspace is just $\{0\}$, so our answer is 1.

Solution to Question 119: Light Switch

The key here is to condition on the color of the light at the start of the interval. In limit, the proportion of the time that the light is yellow is $\frac{4}{40+4+40} = \frac{1}{21}$, while the light is green and red for a limiting proportion of $\frac{10}{21}$ each. Thus, at the start of the interval, there is probability $\frac{1}{21}$ it is yellow, in which you are guaranteed to see a color switch. This is because the light is only yellow for 4 seconds at a time. There is probability $\frac{10}{21}$ it is green, which means the start of the observation period must be in the last 4 seconds of it being green to observe a color switch. This occurs with probability $\frac{1}{10}$, as the total length of the period where the light is green is 40 seconds. The same calculation applies for red too, so our final probability is

$$\frac{1}{21} \cdot 1 + \frac{10}{21} \cdot \frac{1}{10} \cdot 2 = \frac{1}{7}$$

Solution to Question 120: 60-40 Split

The trick here is to condition on whether or not the $60-$sided die is larger than 40 or in the range $1, 2, \ldots, 40$. In the case where it is strictly larger than 40, the $60-$sided die wins with probability 1. This event occurs with probability $\frac{1}{3}$. In the event that it is in the range $1, 2, \ldots, 40$, then it is like we are rolling two identical dice.

Therefore, we just need to find the probability that the $60-$sided die would be the one that is strictly larger. Conditioning on the fact that we are in the range $1, 2, \ldots, 40$, each of the two dice is equally likely to be strictly larger than the other. Therefore, we just need to subtract out the probability that they are equal and then divide by 2. The probability they are equal is $\frac{1}{40}$. This is because if we fix the first die arbitrarily, there is 1 of 40 equally likely values the second die that is equal to the first. Therefore, there is a $\frac{39}{40}$ chance they are not equal. If they are not equal, it is equally likely for either of the two to be larger, so the probability that the larger one is the $60-$sided die is $\frac{39}{80}$. This means the total probability is

$$\frac{1}{3} \cdot 1 + \frac{2}{3} \cdot \frac{39}{80} = \frac{79}{120}$$

Solution to Question 121: Company Purchase I

To calculate how much you would pay, you simply divide your cash flows by your targeted yield.

So $\frac{\$100}{.16} = \$625$

Solution to Question 122: Double Data Trouble II

When we duplicate the data, we are just replacing the matrix X with the matrix $X' = \begin{bmatrix} X \\ X \end{bmatrix}$, so $X'^T X' = 2X^T X$. Therefore, $\sigma^2(2X^T X)^{-1} = \frac{1}{2} \cdot \sigma^2(X^T X)^{-1}$, which proves our statement.

Solution to Question 123: Hours of Labor

Let Y_n be the number of hours that employees have off in a year when there are n employees. Then $\mathbb{E}[Y_n] = 24n\mathbb{E}[T_n]$, where T_n is the number of days that the employees have off in a year. The $24n$ comes from the fact that there are 24 hours in a day and that there are n employees each spending that amount of time off. We can calculate T_n by using indicators.

Let $X_1, X_2, \ldots, X_{365}$ be the indicators of the event that the ith day, $1 \leq i \leq 365$, has at least one of the n people born on it. Then $T_n = X_1 + \cdots + X_{365}$ gives the total days off in the year. Then $\mathbb{E}[T_n] = 365\mathbb{E}[X_1]$ by linearity of expectation and the fact that each day is equally likely to have someone born on it.

$\mathbb{E}[X_1]$ is the probability that at least 1 person has a birthday on day 1. The complement of this is that all n people are both on the other 364 days of the year. The probability all of them do is $\left(\frac{364}{365}\right)^n$. Therefore,

$$\mathbb{E}[T_n] = 365\left(1 - \left(\frac{364}{365}\right)^n\right)$$

This means that $\mathbb{E}[Y_n] = 24n \cdot 365 \cdot \left(1 - \left(\frac{364}{365}\right)^n\right)$ by substitution. As $n \to \infty$, the interior term tends to 1, while the exterior goes to ∞. Therefore, the answer is infinite, meaning we enter -1.

Solution to Question 124: Overlapping Subsets

For an element $x \in A \cap B$, we know that $x \in A$ and $x \in B$. Since A and B are independently selected, the events $\{x \in A\}$ and $\{x \in B\}$ are independent as well. As the subsets are selected uniformly at random, each element has probability $\frac{1}{2}$ of being in the subset or not being in the subset. Therefore, each of the 20 elements is in $A \cap B$ with probability $\frac{1}{4}$, independent between elements.

Therefore, $|A \cap B|$ just counts the number of successes (which are inclusions into the intersection) in 20 independent trials with probability $\frac{1}{4}$, so $N \sim \text{Binom}\left(20, \frac{1}{4}\right)$. For $X \sim \text{Binom}(n, p)$, $\mathbb{E}[X] = np$ and $\text{Var}(X) = np(1-p)$, so the ratio in question here is just $1 - p$. As we know the value of p, our answer is just $1 - \frac{1}{4} = \frac{3}{4}$.

Solution to Question 125: Error and Residual

Note that SSR + SSE = SST, so SST = 200. Therefore, $R^2 = 1 - \frac{SSE}{SST} = 1 - \frac{120}{200} = \frac{2}{5}$.

Solution to Question 126: Prime Sum

The prime integers at most 20 are $2, 3, 5, 7, 11, 13, 17$, and 19, which yields 8 total integers. There are $8 \cdot 7 = 56$ ways to pick 2 integers ordered for the sum. Our sum is even exactly when we don't select 2. Therefore, there are $7 \cdot 6 = 42$ ways to pick two prime integers that aren't 2. Therefore, our probability is

$$\frac{42}{56} = \frac{3}{4}$$

Solution to Question 127: Always Profit II

Something to notice about this situation is that being long on the stock is better than betting the stock goes up against your friend because in the case the stock goes down, youâll still retain 50% of your investment in the stock. Thus we should look to go long on the stock and bet the stock goes down against your friend. The profit (including returned bets) equations are as follows:
Stock goes up: $A - B$
Stock goes down: $B - 0.5A$
Making these two equations equal, we get $A = 4B/3$. Thus A and B are equal to \$4 and \$3 respectively which allows us to always profit \$1.

Solution to Question 128: Bowl of Cherries V

Let's first approach this problem by considering similar-colored cherries as distinguishable. Treating our bowls as sets, we have:

$$\begin{aligned} A &= \{r_1, r_2, p_1, p_2, p_3, p_4, p_5\}, \\ B &= \{r_3, r_4, r_5, r_6, r_7, r_8, p_6, p_7, p_8\}. \end{aligned}$$

Let's find the probability that r_1 is eaten last (we will denote this event R_1). In order for r_1 to be eaten last, we need to eat 15 cherries that are not r_1. Note that there are 8 cherries to choose from in bowl A until the 10th round, which is when bowl B is empty. Thus,

$$\begin{aligned} \mathbb{P}(R_1) &= \left(\frac{7}{8}\right)^9 \left(\frac{6}{7} \cdot \frac{5}{6} \cdot \dots \cdot \frac{1}{2}\right) \\ &= \left(\frac{7}{8}\right)^9 \cdot \frac{1}{7} \end{aligned}$$

The probability that r_3 is eaten last (we will denote this event with R_3) is trickier to compute. Let S_k denote the event that r_3 is the k-th cherry transferred from B to A. Since each arrangement of the 9 cherries in bowl B occur with equal probability, $\mathbb{P}(S_k) = \frac{1}{9}$ for all $k \in \{1, \dots, 9\}$. Note that if S_k, then there are $9 - k$ cherries remaining in bowl B once r_3 is in bowl A. And once r_3 is in bowl A, we have a very similar problem to finding $\mathbb{P}(R_1)$.

$$\begin{aligned} \mathbb{P}(R_3|S_1) &= \left(\frac{7}{8}\right)^8 \left(\frac{7}{8} \cdot \frac{6}{7} \cdot \dots \cdot \frac{1}{2}\right) \\ &= \left(\frac{7}{8}\right)^8 \cdot \frac{1}{8} \\ \mathbb{P}(R_3|S_2) &= \left(\frac{7}{8}\right)^7 \cdot \frac{1}{8} \\ &\vdots \\ \mathbb{P}(R_3|S_9) &= \left(\frac{7}{8}\right)^0 \cdot \frac{1}{8} \\ &= \frac{1}{8} \end{aligned}$$

We can now compute $\mathbb{P}(R_3)$ using the law of total probability.

$$\begin{aligned}\mathbb{P}(R_3) &= \sum_{k=1}^{9} \mathbb{P}(R_3 \cap S_k) \\ &= \sum_{k=1}^{9} \mathbb{P}(R_3|S_k)\mathbb{P}(S_k) \\ &= \frac{1}{8}\sum_{k=1}^{9} \mathbb{P}(R_3|S_k) \\ &= \frac{1}{8\cdot 9}\left(\sum_{k=1}^{9}\left(\frac{7}{8}\right)^{k-1}\right)\end{aligned}$$

The summation can be computed as follows:

$$\begin{aligned}\sum_{k=1}^{9}\left(\frac{7}{8}\right)^{k-1} &= \frac{1\left((1-\left(\frac{7}{8}\right)^{9}\right)}{1-\frac{7}{8}} \\ &= 8\left(1-\left(\frac{7}{8}\right)^{9}\right)\end{aligned}$$

Substituting, we find

$$\mathbb{P}(R_3) = \frac{1}{9}\left(1-\left(\frac{7}{8}\right)^{9}\right)$$

We define R_2 in a similar way to R_1, and we define $R_4, \ldots, R_8$ in a similar way to R_3. By symmetry,

$$\begin{aligned}&\mathbb{P}(R_1) = \mathbb{P}(R_2), \quad \text{and} \\ &\mathbb{P}(R_3) = \mathbb{P}(R_4) = \cdots = \mathbb{P}(R_8).\end{aligned}$$

$R_1, R_2, \ldots, R_8$ are mutually exclusive events, meaning that we are free to use countable additivity. So, the probability of eating a red cherry last is

$$\begin{aligned}\mathbb{P}\left(\bigcup_{k=1}^{8} R_k\right) &= \sum_{k=1}^{8}\mathbb{P}\left(R_k\right) \\ &= \frac{2}{7}\left(\frac{7}{8}\right)^{9} + \frac{2}{3}\left(1-\left(\frac{7}{8}\right)^{9}\right) \\ &= \frac{27789631}{50331648}\end{aligned}$$

Solution to Question 129: Birthday Off

Let $X_1, X_2, \ldots, X_N$ be the indicators of the event that the ith person, $1 \leq i \leq N$, has a distinct birthday distinct from the other $N-1$ people. Then

$T = X_1 + \cdots + X_N$ gives the total number of distinct birthdays in the year. Then $b(N) = \mathbb{E}[T] = N\mathbb{E}[X_1]$ by linearity of expectation and the fact that each person is equally likely to have a distinct birthday from everyone else.

$\mathbb{E}[X_1]$ is the probability that person 1 has a distinct birthday from everyone else. Fix the birthday of person 1. Then the other $N-1$ people must have a birthday on the other 364 days of the year, so the probability all of them do is $\left(\frac{364}{365}\right)^{N-1}$. Therefore,

$$b(N) = \mathbb{E}[T] = N \cdot \left(\frac{364}{365}\right)^{N-1}$$

.

The question implies that there are two integer values of n that yield the same value of $b(n)$. We can note that $b(n)$ is increasing for a while and then decreasing. Therefore, the two values of n must be consecutive. Thus, we just need to find a value n' such that $b(n') = b(n'+1)$. Our answer would then be $n' + (n'+1) = 2n' + 1$. The equation $b(n') = b(n'+1)$ means

$$n' \cdot \left(\frac{364}{365}\right)^{n'-1} = (n'+1)\left(\frac{364}{365}\right)^{n'} \iff n' = \frac{364}{365}n' + \frac{364}{365} \iff n' = 364$$

Therefore, our answer is $2 \cdot 364 + 1 = 729$.

Solution to Question 130: Car Line

Let v_A, v_B, and v_C be the speeds of Alice, Bob, and Carter, respectively. The idea here is to consider the relative speeds of the cars to one another. Alice starts a distance d behind Bob, so $\frac{d}{v_A - v_B} = 7$. Alice starts a distance $3d$ behind Carter, so $\frac{3d}{v_A - v_C} = 12$, as it takes Alice a total of 12 minutes from the instant to pass Carter. We want to find $t = \frac{2d}{v_B - v_C}$, the time it takes for Bob to catch up to Carter.

We have that $d = 7(v_A - v_B) = 4(v_A - v_C) = \frac{t}{2}(v_B - v_C)$ by rearranging the equations above. However, note that

$$v_B - v_C = (v_A - v_C) - (v_A - v_B) = \frac{d}{4} - \frac{d}{7} = \frac{3}{28}d$$

Therefore, we have that $d = \frac{t}{2} \cdot \frac{3d}{28} = \frac{3dt}{56}$, so $t = \frac{56}{3}$. In particular, this is 6 minutes and 40 seconds after Alice passes Carter (Alice passes at 12 minutes). Therefore, our answer is $6 \cdot 60 + 40 = 400$.

Solution to Question 131: Comparing Flips II

Both sequences contain the subsequence HT, so imagine writing down an infinite number of toss outcomes and scanning for HT subsequences. In order for HTH to occur first, an H must not precede HT (else HHT occurs) and an H must succeed it (to create HTH). Hence, the probability HTH wins is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ given there is an HT. In order for HHT to occur first, an H must precede HT, which happens with probability $\frac{1}{2}$ given there is an HT. Note that these are the only two outcomes that can occur and thus define our sample space $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$. The probability that your friend wins is $\frac{1}{2} \div \frac{3}{4} = \frac{2}{3}$ and the probability that you win is $\frac{1}{4} \div \frac{3}{4} = \frac{1}{3}$. In other words, HHT is twice as likely to occur before HTH.

Solution to Question 132: Colorful Bracelet

There are three cases to consider here. The first is that all three red beads are together. In this case, we get exactly 1 distinguishable bracelet, as the others are just rotations. The second case is that we have 2 distinct regions i.e. 2 beads are together separated from the last one. This means that blue beads are at both ends of the 2 consecutive red bead region. Therefore, you get 2 distinct bracelets by ordering the last red and blue bead. The last case is that all of the red bead are separated from one another, in which case we also get 1 bead that alternates in color. Adding these up yields 4 bracelets.

Solution to Question 133: Options Rho

Think about this intuitively. Interest rates can be considered to be the risk-free rate. If there is a positive risk-free rate, we would expect an asset, call it S, to also increase by this risk free rate. In other words, $S = S_0 e^{rT}$. This would cause S to increase as r increases.

If the final payoff of a call option is $\max(S_T - K, 0)$, then S_T would increase due to the statement mentioned above. This would then cause the final payoff to increase, which means that in general, the price of a European call option also increases.

Solution to Question 134: Car Question

Let E_{ij} represent the expected number of minutes before the cars are in spaces 3 and 4 when the cars are in spaces $i < j$. Clearly $E_{34} = 0$, as we already have reached our goal. We want E_{12}. The number of turns needed for a car to move forward to the next spot is $N \sim \text{Geom}(1/2)$. The first step is that we need the car in space 2 to move to space 3. This takes 2 minutes on average, so $E_{12} = 2 + E_{13}$.

Now, from this state, there are three possibilities that occur with equal probability: the car in space 1 moves before the car in space 3, the car in space 3 moves before the car in space 1, or the cars both move at the same time. We will use Law of Total Expectation to condition on which of the cars moves first from this state. You can prove these are equally probable by computing $\mathbb{P}[X = Y]$, where $X, Y \sim \text{Geom}(1/2)$ IID and using the exchangeability of the random variables. Furthermore, we use the result of First Flip to say $\mathbb{E}[X \mid X < Y] = \mathbb{E}[X \mid X = Y] = \frac{4}{3}$. Taking these for granted, let's compute the conditional expectation in each case.

Space 1 Moves Before Space 3: In this case, we reach a state where the cars are in spaces 2 and 3. By the first fact above, it takes an average of $\frac{4}{3}$ minutes for the first car to move conditioned on this. Then, the car in space 3 needs to move to space 4, taking an average of 2 minutes. Lastly, the car in space 2 needs to move to space 3, taking an average of 2 minutes as well. The expected time in this case to reach the end is $\frac{4}{3} + 2 + 2 = \frac{16}{3}$.

Space 3 Moves Before Space 2: In this case, we reach a state where the cars are in spaces 1 and 4. By the first fact above, it takes an average of $\frac{4}{3}$ minutes for the car in space 3 to move conditioned on this. Then, the car in space 1 needs to move to space 3, which means it must move from space 1 to 2 and then 2 to 3. This takes an average of 2 minutes per space, meaning the expected time to reach the end in this case is $\frac{4}{3} + 4 = \frac{16}{3}$.

Both Move Simultaneously: In this case, we reach a state where the cars are in spaces 2 and 4. It again takes an average of $\frac{4}{3}$ minutes for the cars to move conditioned on this. Then, all that needs to occur is a movement of the car in space 2 to space 3, which takes 2 minutes on average. Therefore, the expected time to reach the end in this case is $\frac{4}{3} + 2 = \frac{10}{3}$.

By the Law of Total Expectation, $E_{13} = \frac{1}{3}\left(\frac{16}{3} + \frac{16}{3} + \frac{10}{3}\right) = \frac{14}{3}$. Therefore, by our initial equation $E_{12} = 2 + E_{13} = \frac{20}{3}$.

Solution to Question 135: 77 Multiple I

Since $77000{\cdot}77{\cdot}1000$ and $7700 = 77{\cdot}100$ are multiples of 77, $69300 = 77000-7700$ is as well. We also know that $770 = 77 \cdot 10$ is a multiple of 77, so $69300 + 770 = 70070$ is also a multiple of 77. This is the smallest since if we subtract 77, we are below 70000.

Solution to Question 136: Basic Die Game VI

Let v_k be the value of the game if you accept a roll of k or more and re-roll otherwise. Our optimal strategy will be in this form because we can view each round of the game as an independent trial. Therefore, our strategy should be the same between trials. Then

$$\begin{aligned}
&v_1 = (1/6)(1+2+3+4+5+6) && \Rightarrow \quad v_1 = 7/2\\
&v_2 = (1/6)\,(v_2-1) + (1/6)(2+3+4+5+6) && \Rightarrow \quad v_2 = 19/5\\
&v_3 = (2/6)\,(v_3-1) + (1/6)(3+4+5+6) && \Rightarrow \quad v_3 = 4\\
&v_4 = (3/6)\,(v_4-1) + (1/6)(4+5+6) && \Rightarrow \quad v_4 = 4\\
&v_5 = (4/6)\,(v_5-1) + (1/6)(5+6) && \Rightarrow \quad v_5 = 7/2\\
&v_6 = (5/6)\,(v_6-1) + (1/6)(6) && \Rightarrow \quad v_6 = 1
\end{aligned}$$

All of these are derived from the fact that if we obtain a value at least our minimum stopping value, we just stop immediately. Otherwise, we just go again and it is the same game but we lose 1 in payout from the cost of the roll. Therefore, this means that our optimal EV is 4.

Solution to Question 137: Alternating Sum

Write this as $(100^2 - 99^2) + (98^2 - 97^2) + \cdots + (2^2 - 1^2)$. We use the identity $a^2 - b^2 = (a+b)(a-b)$ for all the terms. In particular, since $b = a - 1$ in this scenario, we have that $a^2 - (a-1)^2 = 2a - 1$ from the identity above. Therefore, we can write each of these terms as

$$199 + 195 + \cdots + 7 + 3 = \sum_{k=1}^{50}(4k-1) = 4 \cdot \frac{50(51)}{2} - 50 = 5050$$

Solution to Question 138: All Equal

We are given that x is the mean. Thus,

$$x = \frac{100 + 40 + 200 + 50 + 90 + 60 + x}{7} \Rightarrow x = 90$$

Solution to Question 139: Spacious Uniform Values II

In Spacious Uniform Values I, we found that the probability that no two points are within a distance x of one another is $(1-100x)^{101}$. In other words, the probability that the minimum distance between two values is at least x is that. Now, recall the identity for non-negative real-valued continuous random variables that $\mathbb{E}[X] = \int_0^\infty \mathbb{P}[X \geq x]dx$. We apply this to M, the minimum spacing between the points. We have that $\mathbb{E}[M] = \int_0^\infty \mathbb{P}[M \geq x]dx$. The maximum value of the minimum possible spacing between the points is $\frac{1}{100}$. If no point was within a distance of something larger than $\frac{1}{100}$, the total length of the interval formed would be larger than 1, contradicting that these are only supported on $(0, 1)$. Therefore, our integral upper bound is $\frac{1}{100}$. This yields

$$\mathbb{E}[M] = \int_0^{\frac{1}{100}} (1 - 100x)^{101}\, dx$$

Let $u = 1 - 100x$ so that $du = -100dx$. It is simple to verify the bounds of our integral are now 0 and 1. This yields the new integral $\frac{1}{100}\int_0^1 u^{101}du$, which is evaluated to be $\frac{1}{10200}$.

Solution to Question 140: Digit Sum

Imagine each number as as having six digits, from 000000 to 999999 (the sum of the digits of 1 million is 1, so we can add this back later). The average value of each digit is $\frac{9}{2}$, and thus the average sum of each number is:

$$6 \times \frac{9}{2} = 27$$

There are one million numbers from 0 to 999999, so the sum of all of their digits is:

$$27 \times 1000000 = 27000000$$

Adding back the sum of the digits in 1000000, 1, we find our answer to be 27000001.

Solution to Question 141: Simple Delta Hedge I

The Δ of the overall portfolio is $100 \cdot 0.33 = 33$. The underlying has $\Delta = 1$. To remain delta-neutral, we need to short 33 units of the underlying.

Solution to Question 142: Leap Frog

We want to maximize p_k. We are going to explicitly solve for p_k. We know that the frog jumps 1 or 2 steps at each time. Therefore, for position k, we can only reach it from positions $k-1$ or $k-2$. Namely, by Law of Total Probability, $p_k = \frac{1}{2}p_{k-1} + \frac{1}{2}p_{k-2}$. The $\frac{1}{2}$ represents the probability of selecting 1 or 2 steps when at position $k-1$ and $k-2$, respectively. This is a homogeneous second-order recurrence relation with constant coefficients. We can see this by rearranging to get $-p_k + \frac{1}{2}p_{k-1} + \frac{1}{2}p_{k-2} = 0$. The characteristic polynomial is $-r^2 + \frac{1}{2}r + \frac{1}{2} = 0$. By multiplying both sides by -2, this is equivalent to $2r^2 - r - 1 = 0$. By using the quadratic equation, the roots are $r = 1, -\frac{1}{2}$. Therefore, our solution is in the form $p_k = c_0 + c_1\left(-\frac{1}{2}\right)^k$. We need some initial conditions on this. Namely, it is very easy to calculate p_0 and p_1. We have that $p_0 = 1$ as we start at 0. Additionally, $p_1 = \frac{1}{2}$ since we must take 1 step starting from 0 to reach it. Otherwise, we do not hit it. We get the system of equations $c_0 + c_1 = 1$ and $c_0 - \frac{1}{2}c_1 = \frac{1}{2}$. It is simple enough to verify that $c_0 = \frac{2}{3}$ and $c_1 = \frac{1}{3}$ solve this system.

Plugging this in $p_k = \frac{2}{3} + \frac{1}{3}\left(-\frac{1}{2}\right)^k$. Note that $\left(-\frac{1}{2}\right)^k$ is positive for even k and negative for odd k. Therefore, to maximize this probability, we want to maximize how positive $\left(-\frac{1}{2}\right)^k$ is for $k > 0$. Namely, we just want k to be as small as possible and even, as evenness makes it positive and k being the smallest even one makes the term as large as possible over the positive integers. The smallest positive integer is $k = 2$, so $k = 2$ will maximize p_k. Namely, $p_2 = \frac{3}{4}$, and this is our solution.

Solution to Question 143: Stack Double

The trick here is to really work backwards. Note that if the player is the ith to flip the coin, his money will be doubled $7 - i$ times after he flips. Therefore, after obtaining his heads and paying out the sum to each of the other players, the ith player to flip must have $\frac{1.28}{2^{7-i}}$ in the bank. Furthermore, note that there is always $\$1.28 \cdot 7 = \$8.96$ in the game at all turns.

Starting with player A, this means player A must have 2 cents in the bank after he flips. This means that $x_1 - (8.96 - x_1) = 0.02$, as x_1 is what he had before the flip and $8.96 - x_1$ is what he pays out. Solving this yields $x_1 = 4.49$. For player B, we are going to calculate what he has right before he pays out everyone. We didn't need to worry about this when calculating x_1 as there was nobody before player A.

If p_2 is what player B has right before paying out, then $p_2 - (8.96 - p_2) = 0.04$, meaning $p_2 = 4.50$. Since player B has his money doubled once already, this means $x_2 = 2.25$.

More generally, if p_k is what the kth person to flip had in their bank right before paying out, $p_k - (8.96 - p_k) = 1.28 \cdot 2^{-(7-k)}$, meaning that

$$p_k = \frac{1.28 \cdot 2^{-(7-k)} + 8.96}{2}$$

However, this person already had their money doubled $k - 1$ times from the $k - 1$ people that came before them, so this implies that

$$x_k = \frac{1.28 \cdot 2^{-(7-k)} + 8.96}{2^k}$$

Plugging in $k = 1, 2, \ldots, 7$ yields that

$$x_1 = 4.49, x_2 = 2.25, x_3 = 1.13, x_4 = 0.57, x_5 = 0.29, x_6 = 0.15, x_7 = 0.08$$

Adding the squared values up and multiplying by 10000 yields the result of 269374.

Solution to Question 144: 2 Below I

Suppose you were planning to select a value n. Note that $n + 3$ is strictly better than n, as $n + 3$ will beat all integers n would beat, as well as the integer $n + 2$.

Therefore, whenever you can select n, you should select $n+3$. This means that your strategy should be to select $98, 99$, or 100 with some probabilities.

By the symmetry of the game, your friend should also select those values with the same probabilities. In particular, if you select $98-100$ with equal probability, no matter what probabilities your friend selects $98, 99$, and 100 with, the expected payout for each player is 0 by symmetry. This is because we see that 98 is beat by 99, 99 is beat by 100, and 100 is beat by 98, so each of the three outcomes is dominated by one other outcome.

Furthermore, if you select a non-uniform distribution on $98, 99, 100$ for your values, there exists a strategy your friend can select that yields positive expected payout for them. The probabilities of selecting each of the values for you would be p_1, p_2, and $1-p_1-p_2$. For a numerical demonstration, say $p_1 = 1/5$ and $p_2 = 1/2$. Then your friend should always select the value that maximizes their probability of winning. In this case, they should select 100, as the expected payout would be

$$(-1) \cdot \frac{1}{5} + (1) \cdot \frac{1}{2} + (0) \cdot \frac{3}{10} > 0$$

Namely, if x is the value assigned to probability $\max\{p_1, p_2, 1-p_1-p_2\}$, then your friend should always select the value that beats x. Therefore, to eliminate this opportunity, you should select a uniform distribution among $98-100$.

This means $X \sim \text{DiscreteUnif}(\{98, 99, 100\})$, so $\mathbb{E}[X] = 99$ and $\text{Var}(X) = \dfrac{(98-99)^2 + (99-99)^2 + (100-99)^2}{3} = \dfrac{2}{3}$.

Solution to Question 145: Private Documents

For each group of 3 people, there must be a lock none of them can access. Otherwise, a group of 3 would be able to unlock all of them. Therefore, there must be $\binom{7}{3} = 35$ locks minimum, as that is the number of subsets of 3 people that can be made. This means $n = 35$. Then, each lock has 4 keys, as every lock has a subset of 3 that can't access it, so there are $35 \cdot 4 = 140$ total keys, meaning that each person must receive 20 keys. Therefore, we have that $m = 20$. Our answer is $10 \cdot 20 + 35 = 235$.

Solution to Question 146: 7 Multiple

Intuitively, it is clear that you should select to go second. This is because you have no chance of winning on the first roll, but every roll after the first roll, there is a 1/6 probability that you win the game, as exactly 1 of the 6 possible values on the will make the sum a multiple of 7.

Let's compute the probability that you, the second player to roll, wins. Call this probability p. By conditioning on your first roll, there is a 5/6 probability you do not roll a value resulting in a sum divisible by 7 on the first turn. To come back to you, you would need your friend to also not roll a value resulting in a sum divisible by 7, which occurs with probability 5/6 too. In this case, your probability of winning when it comes back to you is p. Alternatively, you do roll the value that results in a sum divisible by 7, occurring with probability 1/6. Therefore, we have the equation

$$p = \frac{5}{6} \cdot \frac{5}{6}p + \frac{1}{6} \iff p = \frac{6}{11}$$

Solution to Question 147: Fancy Factorial

We know that $1 < n < 10$. This is because if $n = 1$, then this implies $10! = 6!$, which is clearly untrue. In addition, $n < 10$, as $6! > 1$. We can write out $\frac{10!}{6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{6!} = 10 \cdot 9 \cdot 8 \cdot 7$. With this, we know that $n \geq 7$, as 7 can't be reduced into any lower factors and it must be in the factorial of n.

Let's write out a factorization of $10 \cdot 9 \cdot 8 \cdot 7$. We can write $10 = 2 \cdot 5$. In addition, $9 \cdot 8 = 72 = 1 \cdot 3 \cdot 4 \cdot 6$. Therefore, $10 \cdot 9 \cdot 8 \cdot 7 = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 7!$, so $n = 7$.

Solution to Question 148: Dice Profits

Let us find the expected profit when we have n total rolls i.e. $n-1$ re-rolls. Our profit after n rolls would be $P_n = \max\{X_1, \dots, X_n\} - (n-1)$, as we are paid the maximum of the n rolls and we pay $n-1$ for the re-rolls. For convenience, let $M_n = \max\{X_1, \dots, X_n\}$. We can use the property for non-negative integer-valued random variables that $\mathbb{E}[X] = \sum_{k=1}^{\infty} \mathbb{P}[X \geq k]$. As the largest possible value is 20, we can say that $\mathbb{E}[M_n] = \sum_{k=1}^{20} \mathbb{P}[M_n \geq k]$.

To evaluate $\mathbb{P}[M_n \geq k]$, it is easier to evaluate via the complement, as the maximum being less than a certain value means all of them are less than that value, and then we can use independence. Namely, as M_n is integer-valued, $\mathbb{P}[M_n \geq k] = 1 - \mathbb{P}[M_n < k] = 1 - \mathbb{P}[M_n \leq k-1]$. M_n being at most $k-1$ means all of $X_1, \ldots, X_n$ are at most $k-1$. The probability this occurs for each roll is $\frac{k-1}{20}$. Therefore, $\mathbb{P}[M_n \geq k] = 1 - \left(\frac{k-1}{20}\right)^n$. Plugging this in,

$$\mathbb{E}[M_n] = \sum_{k=1}^{20} \left(1 - \left(\frac{k-1}{20}\right)^n\right) = 20 - \frac{1}{20^n} \sum_{k=0}^{19} k^n$$

The summation comes from an index shift by 1. Now, we have that

$$\mathbb{E}[P_n] = \mathbb{E}[M_n] - (n-1) = (21-n) - \frac{1}{20^n} \sum_{k=0}^{19} k^n$$

Our objective is to maximize this value. We can note that $\mathbb{E}[M_n - M_{n-1}]$, the increase in our expected value of consecutive roll numbers, decreases as n increases. This is because the maximum grows with smaller and smaller probabilities each time. Therefore, if we can find where $\mathbb{E}[P_n]$ starts to decrease, we have found our maximum, as it will never rise after.

The first few values by direct substitution and evaluating using our known summation formulas are $\mathbb{E}[P_1] = \frac{21}{2}, \mathbb{E}[P_2] = \frac{513}{40}, \mathbb{E}[P_3] = \frac{1079}{80}, \mathbb{E}[P_4] = \frac{1078667}{80000}$. Note that $\mathbb{E}[P_4] < \mathbb{E}[P_3]$, as $1079000 > 1078667$. Therefore, we should have 3 total rolls (so 2 rerolls). This yields expected profit of $\frac{1079}{80}$.

Solution to Question 149: Socks and Shelves

This is a classic conditional probability question. A sure fire way to do this problem is to use Bayes Theorem. Applying Bayes for

$$\mathbb{P}[\text{Top Shelf}|RR] =$$

$$\frac{\mathbb{P}[RR|\text{Top Shelf}] \cdot \mathbb{P}[\text{Top Shelf}]}{\mathbb{P}[RR]} = \frac{0.4 \cdot 0.4 \cdot 0.5}{0.4 \cdot 0.4 \cdot 0.5 + 0.7 \cdot 0.7 \cdot 0.5} = \frac{16}{65}$$

Solution to Question 150: Non-Consecutive Sequence

We are going to combine a lot of different results in this question, specifically from Fibonacci Limit I and II. Let $A(n)$ be the number of sequences of length

n with no consecutive heads. Then $A(1) = 2$ and $A(2) = 3$ (only HH doesn't satisfy this). For a sequence of length n to satisfy our constraint, it either starts with T or HT. It can't start with HH, as that would violate the condition.

There are $A(n-1)$ sequences starting with T that satisfy this condition, as the first flip is T and the other $n-1$ spots must not contain any consecutive heads. There are $A(n-2)$ sequences starting with HT by the same logic with with $n-2$ remaining spots. Thus, $A(n) = A(n-1) + A(n-2)$ with $A(1) = 2$ and $A(2) = 3$. This is just the Fibonacci sequence shifted. For the purposes of this question, the exact shift will not matter. All that matters is the amount of shift of the numerator relative to the denominator.

We can now derive a recurrence relation for $E(n)$. If the first flip is T, which occurs with probability $\dfrac{A(n-1)}{A(n)}$ conditional on there being no consecutive heads ($A(n-1)$ of the $A(n)$ total sequences start with T by the previous paragraph), the expected number of heads is $E(n-1)$, as the first flip is T and we count the number of heads in the remaining $n-1$ spots.

Otherwise, if the first flip is H, which occurs with probability $\dfrac{A(n-2)}{A(n)}$, then the expected number of heads is $1 + E(n-2)$, as we obtain 1 head from the first flip, the second flip must be a tails, and then we count the expected number of heads in the other $n-2$ flips. We also get that $E(1) = 1/2$ and $E(2) = 2/3$ by listing out the valid sequences and counting the number of heads appearing in them. Therefore, we have the following recurrence for $E(n)$:

$$E(n) = \frac{A(n-1)}{A(n)}E(n-1) + \frac{A(n-2)}{A(n)}\left(1 + E(n-2)\right)$$

Note that this is a non-homogenous and non-constant coefficient second order recurrence, so it is very unlikely a nice closed form exists. However, we can get asymptotics on $E(n)$. Suppose that $E(n) \sim cn$ as $n \to \infty$. Our goal is to find c.

At this point, we use Fibonacci Limit I and II. For large n we know that $\dfrac{A(n)}{A(n-1)} \to \phi$, the golden ratio $\phi = \dfrac{1+\sqrt{5}}{2}$, so $\dfrac{A(n-1)}{A(n)} \to \dfrac{1}{\phi}$. Similarly, $\dfrac{A(n-2)}{A(n)} \to \dfrac{1}{\phi+1}$. Therefore, in limit, we need to find c such that

$$cn = \frac{1}{\phi}c(n-1) + \frac{1}{\phi+1}(1+c(n-2)) \iff c\left(\frac{1}{\phi} + \frac{2}{\phi+1}\right) = \frac{1}{\phi+1} \iff c = \frac{1}{\phi+2} = \frac{2}{5+\sqrt{5}}$$

Thus, the answer is $5 \cdot 5 \cdot 2 = 50$.

Solution to Question 151: Straddle Gamma

A straddle is a derivative that is long a call option and long a put option at the same strike K. From Black-Scholes, we know that at the same strike K, the put and call have the same Γ. So, the gamma of a straddle is $\Gamma_C + \Gamma_P = 2\Gamma_C = 2(0.03) = 0.06$.

Solution to Question 152: Power to the Matrix

We first need to find the eigenvalues of A. The characteristic polynomial is $p(\lambda) = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A)$. It is easy to see that $\mathrm{tr}(A) = 6$ and $\det(A) = 5$, so we need to solve $\lambda^2 - 6\lambda + 5 = 0$. This easily factors to $(\lambda - 5)(\lambda - 1) = 0$, so $\lambda = 1, 5$ are the eigenvalues.

Next, we want the eigenvectors corresponding to each eigenvalue. We do this by solving $(A - \lambda I_2)x = 0$. For $\lambda = 1$, this becomes

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution set to this is $x = t \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ for $t \in \mathbb{R}$. Our eigenvector is therefore $e_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. ,

For $\lambda = 5$, this equation becomes

$$\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution set to this is $x = t \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ for $t \in \mathbb{R}$. Our eigenvector is therefore $e_5 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

We now need to write v in terms of our eigenvector basis. However, it is fairly clear to see that $v = e_1 + 2e_5$. By the fact that e_1 and e_5 are eigenvalues of A, we have that

$$A^{10}v = A^{10}(e_1 + 2e_5) = A^{10}e_1 + 2A^{10}e_5 = 1^{10} \cdot e_1 + 2 \cdot 5^{10} e_5 = \begin{bmatrix} 2 \cdot 5^{10} + 1 \\ 4 \cdot 5^{10} - 2 \end{bmatrix}$$

This means our answer is $5 \cdot 10 + 1 + 4 + 2 + 2 = 59$.

Solution to Question 153: Nearest Circular Neighbor

Fix the location of X_1. Let D be the length of the arc (in radians) between X_1 and the nearest point. We will convert to degrees at the end. We compute $\mathbb{E}[D] = \int_0^\pi \mathbb{P}[D \geq x]dx$. The maximal distance is π since we look in both directions. The event $\{D \geq x\}$ means that the other 19 points lie outside a region of x radians in either direction. This means that there are $2(\pi - x)$ radians that are valid for the other $n-1$ points to lie in. Therefore, the probability all of the other 19 points lie outside this region is

$$\left(\frac{2(\pi - x)}{2\pi}\right)^{19} = \left(1 - \frac{x}{\pi}\right)^{19}$$

Therefore, our expected distance is $\mathbb{E}[D] = \int_0^\pi \left(1 - \frac{x}{\pi}\right)^{19} dx = \pi \int_0^1 (1 - u)^{19} du = \frac{\pi}{20}$ Therefore, this is $\frac{180}{\pi} \cdot \frac{\pi}{20} = 9$ degrees.

Solution to Question 154: Solo or Pair

The probability of winning in game 1 is $1 - \left(\frac{5}{6}\right)^4$ by complementation, as to lose, you must roll each of the other 5 values per roll. The probability of winning in game 2 is $1 - \left(\frac{35}{36}\right)^{24}$, as to lose, you must roll any of the other 35 outcomes of the two dice per turn. We need to determine which of these is larger, which is equivalent to determine which of $\left(\frac{5}{6}\right)^4$ and $\left(\frac{35}{36}\right)^{24}$ is smaller. We know that $\frac{35^2}{36^2} = \frac{1225}{1296} \approx \frac{1225}{1300} = \frac{49}{52}$. Then, squaring this yields $\frac{2401}{2704} \approx \frac{8}{9}$. Therefore, $\left(\frac{35}{36}\right)^8 \approx \frac{64}{81} \approx \frac{4}{5}$. Now, we just need to compare $\left(\frac{5}{6}\right)^4$ to $\frac{4^3}{5^3} = \frac{64}{125}$. However, this is easy to see, as $\frac{5^4}{6^4} = \frac{625}{1296} < \frac{1}{2} < \frac{64}{125}$. Therefore, we can conclude, at least approximately, that $\left(\frac{5}{6}\right)^4$ is smaller, so Game 1 gives you better odds. When plugging into a calculator, this is indeed confirmed.

Solution to Question 155: Statistical Test Review VIII

Let $\hat{\mu}$ denote the sample mean. Our test statistic is defined as follows:

$$\begin{aligned} z &= \frac{\hat{\mu}-\mu}{\sigma/\sqrt{n}} \\ &= \frac{\sqrt{25}(165-160)}{30} \\ &= \frac{5}{6} \end{aligned}$$

The desired probability is simply $\mathbb{P}\left(Z \geq \frac{5}{6}\right) = 1 - \mathbb{P}\left(Z \leq \frac{5}{6}\right)$ for $Z \sim \mathcal{N}(0,1)$. With a calculator, we find this value to be 0.2023.

Solution to Question 156: Close Dice II

Let E_i be the expected time for this to happen given that our current roll is i and E be the expected value of the event we are interested in. Then

$$E = 1 + \frac{E_1 + E_2 + E_3 + E_4 + E_5 + E_6}{6}$$

by Law of Total Expectation. By the symmetry of the fair die, $E_1 = E_6$, $E_2 = E_5$ and $E_3 = E_4$. Note that $E_2 \neq E_3$ for the same reason as in Close Dice I. Using the substitution yields $E = 1 + \dfrac{E_1 + E_2 + E_3}{3}$.

It remains to calculate equations for E_1, E_2, and E_3. For E_1, we get the equation $E_1 = 1 + \dfrac{1}{6}E_1 + \dfrac{1}{6}E_2 + \dfrac{1}{3}E_3$. This is because if we roll a 1 or 2, we are done. If we roll a 3 or 4, they are same the EV, so we combine per above. If we roll a 5 or 6, we just get E_2 and E_1 respectively. By similar logic, we have $E_2 = 1 + \dfrac{1}{6}E_1 + \dfrac{1}{6}E_2 + \dfrac{1}{6}E_3$ and $E_3 = 1 + \dfrac{1}{3}E_1 + \dfrac{1}{6}E_2$. Solving these linear equations yields $E_1 = \dfrac{288}{115}, E_2 = \dfrac{246}{115}$, and $E_3 = \dfrac{252}{115}$. Plugging these back into the equation for E, $E = 1 + \dfrac{262}{115} = \dfrac{377}{115}$.

Solution to Question 157: Magic Doors

When John is inside the room, the probability of John selecting any one of the five doors is $\frac{1}{5}$. If John selects one of the doors that leads back to the room, he has the same probability of picking any one of the five doors. So, we can write the following relationship for random variable X, which represents the total distance traveled before reaching freedom:

$$\begin{aligned}\mathbb{E}[X] &= \frac{1}{5}\left(\mathbb{E}[X]+3\right) + \frac{1}{5}\left(\mathbb{E}[X]+4\right) + \frac{1}{5}\left(\mathbb{E}[X]+1\right) + \frac{1}{5}(2) + \frac{1}{5}(5) \\ &= \frac{3}{5}\mathbb{E}[X] + \frac{15}{5} \\ \Rightarrow \mathbb{E}[X] &= \frac{5}{2}\cdot\frac{15}{5} = \frac{15}{2}\end{aligned}$$

Solution to Question 158: Birthday Twins II

In order for the probability of at least two people having the same birthday to be greater than $\frac{1}{2}$, then the probability of no one having the same birthday must be less than $\frac{1}{2}$. Let there be n people in the class. The probability that there are no birthday collisions is $\frac{365}{365} \times \frac{364}{365} \times ... \times \frac{365-n+1}{365} < \frac{1}{2}$. The smallest such n is 23, which can be found via calculator.

Solution to Question 159: Hitting MGF

Consider the exponential martingale $Z_t = e^{\theta W_t - \theta^2 t/2}$. If you have not shown this is a martingale before, it is a good exercise to do so. We can apply the Optional Stopping Theorem to Z_t with T_a and obtain

$$\mathbb{E}[Z_{T_a}] = \mathbb{E}[Z_0] = e^0 = 1$$

Note that when T_a occurs, W_{T_a} is equally likely to be either a or $-a$, since they are equidistant from the origin. Therefore, we need to take a conditional expectation based on what W_{T_4} is. In particular, we obtain

$$\frac{1}{2}\mathbb{E}\left[e^{a\theta-\theta^2 T_a/2}\right] + \frac{1}{2}\mathbb{E}\left[e^{-a\theta-\theta^2 T_a/2}\right] = 1$$

by conditioning on $W_{T_a} = \pm a$. We can multiply by 2 on both sides and extract the leading term of each expectation to get that

$$\left(e^{a\theta}+e^{-a\theta}\right)\mathbb{E}\left[e^{-\theta^2 T_a/2}\right] = 2 \iff \mathbb{E}\left[e^{-\theta^2 T_a/2}\right] = 1/\cosh(a\theta)$$

We are looking for $\mathbb{E}[e^{-\lambda T_a}]$. This can be remedied by re-parameterizing and letting $\theta = \sqrt{2\lambda}$ so that we get our desired quantity on the LHS. In particular, we get that $\mathbb{E}\left[e^{-\lambda T_a}\right] = 1/\cosh(a\sqrt{2\lambda})$.

Plugging in $\lambda = 4$ and $a = \ln(2)$, our answer is $\dfrac{2}{e^{4\ln(2)}+e^{-4\ln(2)}} = \dfrac{2}{16+1/16} = \dfrac{32}{257}$.

Solution to Question 160: Machine Variance

Let $X_1, \ldots, X_{25}$ be the number of pizzas produced by each of the workers per day. Then $T = X_1 + \cdots + X_{25}$ gives the total number of pizzas produced per day by the shop. We have that $\text{Var}(T) = 25 \cdot \text{Var}(X_1)$ because of the fact that each of the people are independent and have the same variance. Thus, $\sigma_T = \sqrt{\text{Var}(T)} = 5 \cdot \sigma_{X_1} = 5 \cdot 60 = 300$.

Solution to Question 161: Paired Pumpkins II

Let a, b, c, d, and e be the weight of each pumpkin from lightest to heaviest. If we sum all the pairwise weights, we know we will have 4 sets of a, b, c, d, and e. Thus

$$4 \cdot (a + b + c + d + e) = 21 + \cdots + 30 = 255$$

or

$$a + b + c + d + e = \frac{255}{4}$$

You could also create equations for all the pairs to find the weights of each pumpkin but the question only asks for the sum of the weights and the above method is a clever way of reaching the answer.

Solution to Question 162: Squared GBM

The SDE that S_t satisfies is $dS_t = S_t(dt + dW_t)$ by plugging in $\mu = \sigma = 1$. Use Ito's Lemma to compute dS_t^2 by applying it to $f(x) = x^2$. This yields that

$$df(S_t) = dS_t^2 = 2S_t dS_t + \frac{1}{2} \cdot 2d[S,S]_t$$

The quadratic variation of S_t is simply just

$$d[S,S]_t = S_t^2(dt + dW_t)(dt + dW_t) = S_t^2 dt$$

Plugging dS_t and $d[S,S]_t$ in, we get that

$$dS_t^2 = 2S_t \cdot S_t(dt + dW_t) + S_t^2 dt = S_t^2(3dt + 2dW_t)$$

Therefore, we see that S_t^2 satisfies the same SDE as S_t but with $\mu' = 3$ and $\sigma' = 2$.

For a GBM S_t with drift μ and volatility σ, $\mathbb{E}[S_t] = S_0 e^{\mu t}$ We have that $S_0^2 = 1$ as well. Furthermore, as S_t^2 is also a GBM with $\mu' = 3$ and $\sigma' = 2$, plugging these in, we quickly see that

$$\mathbb{E}[S_2^2] = 1 \cdot e^{3 \cdot 2} = e^6$$

Therefore, $k = 6$.

Solution to Question 163: Statistical Test Review IV

Let p denote the probability that a customer chooses plan A. We define

$$H_0 : p = \frac{1}{3}$$
$$H_a : p > \frac{1}{3}$$

For a Bernoulli random variable X with parameter p, recall that $\mu = p$ and $\sigma^2 = p(1-p)$. Since $n = 1000$, we have a sufficiently large sample size and may assume standard normally distributed $\hat{p}$, which we define as our sample proportion: $\hat{p} = 0.4$. Our test statistic is

$$\begin{aligned} Z &= \frac{\hat{p} - \mu}{\sigma/\sqrt{n}} \\ &= \frac{\sqrt{1000}(0.4 - \frac{1}{3})}{\sqrt{\frac{1}{3} \cdot \frac{2}{3}}} \\ &\approx 4.47 \\ \Rightarrow \mathbb{P}(Z > 4.47) &\approx 3.4 \times 10^{-6} \end{aligned}$$

This p-value is much smaller than typical significance levels for α such as $\alpha = 0.05$ or $\alpha = 0.01$. Since the p-value is less than α, we may reject H_0, which tells us that there is enough evidence to suggest that customers have a preference for plan A. We should answer 1.

Solution to Question 164: Poisson Review IV

The average number of imperfections per 3 square yards is $3 \cdot 4 = 12$. Let $X \sim \text{Poisson}(12)$. Then, $\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X = 0) = 1 - \frac{12^0 e^{-12}}{1}$. Our probability is $1 - \frac{1}{e^{12}}$, so our answer is $-1 + 12 = 11$.

Solution to Question 165: Throwing Darts I

The probability that the dart lands in the central ring is the ratio of the area of the central ring to the area of the entire board, since the dart lands uniformly at random on the board. The area of the entire board is 9π and the area of the central ring is $4\pi - 1\pi$. Hence, the probability that the dart lands in the central ring is:

$$\frac{4\pi - \pi}{9\pi} = \frac{1}{3}$$

Solution to Question 166: 10 Die Sum

The most efficient approach here would be generating functions. However, we are going to solve via casework instead. The possibilities of values (not ordered in any way currently) that result in a sum of 10 are

$$(1,6,3),(1,5,4),(2,6,2),(2,5,3),(2,4,4),(3,4,3)$$

Three of this combinations have all distinct values. Namely, these are $(1,6,3),(1,5,4)$, and $(2,5,3)$. For these sets of outcomes, there are $3! = 6$ ways they can be assigned to the three dice. These yields $3 \cdot 6 = 18$ possibilities.

The other three outcomes have two distinct values. For these, there are only 3 ways they can each be assigned to the dice, as you only select the die that shows the value that differs from the other two. Then, the other two dice are fixed. These yields $3 \cdot 3 = 9$ possibilities.

Combining all of this, this yields $18 + 9 = 27$ possibilities that result in a sum of 10. We divide by $6^3 = 216$ to get the probability, yielding a final answer of $\frac{1}{8}$.

Solution to Question 167: Eight Dice

Let each die roll take its minimum value of 1 for a total sum of 8. We have 4 "dice dots" left to distribute to the 8 dice in order for the sum to be 12. This is analogous to the number of ways that 4 non-unique balls can be placed into 8 distinct boxes, which is:

$$\binom{8+4-1}{8-1} = \binom{11}{7} = 330$$

Solution to Question 168: ABC Sum

There are $10 \cdot 9 \cdot 8 = 720$ such numbers that can be formed, as we have 10 options for a, 9 for b, and 8 for c. Then, the trick is here that we need to note the average value of each digit is 4.5. This means the average value of the numbers we generate is

$$\frac{9}{2} \cdot 10^{-1} + \frac{9}{2} \cdot 10^{-2} + \cdots = \frac{9}{2} \sum_{k=1}^{\infty} \frac{1}{10^k} = \frac{1}{2}$$

Therefore, the sum must be $\frac{1}{2} \cdot 720 = 360$.

Solution to Question 169: Modified Even Coins

To obtain an even amount of heads, we have two options. If the fair coin shows H, then we must show an odd amount of heads on the other $n-1$ coins to obtain an even total. Alternatively, if the fair coin shows T, then we must show an even amount of heads on the other $n-1$ coins to obtain. If p is the probability of an even amount of heads on the other $n-1$ non-fair coins, then by Law of Total Probability, our probability of interest is $\frac{1}{2} \cdot p + \frac{1}{2} \cdot (1-p) = \frac{1}{2}$. This is because the complement of an even amount of heads is an odd amount, and these probabilities are weighted equally in the computation, so they cancel out.

Solution to Question 170: Peaky Poisson

We will solve this for general λ. Plugging in the Poisson PMF and taking the ratio yields that we want to find the largest k such that

$$\frac{\lambda^k e^{-\lambda}/k!}{\lambda^{k-1}e^{-\lambda}/(k-1)!} \geq 1$$

After cancellation, this yields $\frac{\lambda}{k} \geq 1$, or that $k \leq \lambda$. Since λ is not necessarily an integer, we have that $\lfloor \lambda \rfloor$ will be the maximum possible value in the support of X satisfying that inequality. In this case, $\lfloor 13.4 \rfloor = 13$.

Solution to Question 171: Dice Order III

An analytical approach is possible, but a more visual, and perhaps more elegant, approach will be provided. Imagine a $6 \times 6 \times 6$ cube where each axis records the value of a die such that each of the 216 voxels represents a possible, equally probable outcome. For example the voxel $(3,5,1)$ represents rolling 3 on the first die, 5 on the second die, and 1 on the third die. The set of outcomes where the minimum value is 6 is simply (6,6,6). The set of outcomes where the minimum is exactly 5 is the sub-cube from (5,5,5) to (6,6,6) minus the (6,6,6) voxel; in total, there are $2^3 - 1^3 = 7$ possibilities. The set of outcomes where the minimum is exactly 4 is the sub-cube from (4,4,4) to (6,6,6) minus the (5,5,5) to (6,6,6) sub-cube; in total, there are $3^3 - 2^3 = 7$ possibilities. The pattern is becoming clear. For the minimum to be exactly x of n d-sided dice, the number of possibilities is:

$$(d-x+1)^n - (d-x)^n$$

Solving for each event's probability, we find the expected value of the minimum of three dice rolls to be:

$$\frac{1}{216}(6) + \frac{7}{216}(5) + \frac{19}{216}(4) + \frac{37}{216}(3) + \frac{61}{216}(2) + \frac{91}{216}(1) = \frac{49}{24}$$

Solution to Question 172: Random Angle I

We know that $\tan(\theta) = \frac{A}{B}$. If $\theta > \frac{\pi}{4}$, then $\tan(\theta) > 1$, so $\frac{A}{B} > 1$, or $A > B$. Thus, we want $\mathbb{P}[A > B]$, which is just $\frac{1}{2}$ as A and B are IID and hence exchangeable.

Solution to Question 173: Butterfly Payoff

A butterfly spread is one where you long the wings and sell 2 units of the body. Here, we long a call (or put) at $K = 35$ and $K = 40$, and sell 2 calls (or puts) at $K = 37.5$. From the payoff diagram, we can see that our payoff is maximized when $S_T = 37.5$.

Solution to Question 174: 60 Heads

The number of heads that appear is $X \sim \text{Binom}(100, 1/2)$. The mean and variance of X are, respectively, $100 \cdot 1/2 = 50$ and $100 \cdot 1/2 \cdot (1 - 1/2) = 25$. Therefore, as X is the sum of 100 IID Bernoulli(1/2) random variables, we can approximate X well by $M \sim N(50, 25)$ via Central Limit Theorem. Thus, we want to calculate

$$\mathbb{P}[M \geq 60] = \mathbb{P}\left[\frac{M - 50}{5} \geq \frac{60 - 50}{5}\right] = \mathbb{P}\left[\frac{M - 50}{5} \geq 2\right]$$

The LHS of the last term is now standard normal, so our approximation is $1 - \Phi(2) = \Phi(-2) \approx 0.02275$

Solution to Question 175: Specific Partition

Arrange the elements of S in the below fashion:

1	2	3	4	5	6	7	8	9	10	11
12	13	14	15	16	17	18	19	20	21	22

Any two adjacent elements either vertically or horizontally will now satisfy our criterion. We can view problem now as finding the number of ways to tile the grid below with 1×2 tiles that can be placed either vertically or horizontally.

We derive a recurrence relation for this. Let t_n be the number of ways to tile a $2 \times n$ grid with 1×2 tiles. Consider t_{n+1} and the relationship between it and t_n. In this new grid, if we place a tile on the $(n + 1)$st column vertically, there are t_n remaining ways to tile it. If we place the tile horizontally, then we must place another horizontally below it and there are t_{n-1} valid ways to tile it in this case. Therefore, we have that $t_{n+1} = t_n + t_{n-1}$. Furthermore, we can quickly see that $t_1 = 1$ and $t_2 = 2$. Therefore, we can see that $t_n = F_{n+1}$, the $(n+1)$st Fibonacci number, as this is just the Fibonacci sequence shifted up 1. In particular, $n = 11$ here, so $t_{11} = F_{12} = 144$.

Solution to Question 176: Triangle of Primes

Suppose we have a triangle that satisfies the conditions presented in the problem. Denote the vertices as a, b, c, the values of a, b, c correspond to their labels. Let $a > b > c$. Note that it must be the case that $a - b = p_1$, $b - c = p_2$, and $a - c = p_3$, where p_1, p_2, p_3 are primes. Notice that $p_3 = p_1 + p_2$. Hence, either $p_1 = 2$ or $p_2 = 2$. Now, we use casework.

If the primes are 2, 3, 5, then the smallest number c can be between 1 and 15. If the primes are 2, 5, 7, then c can be between 1 and 13. If the primes are 2, 11, 13, then c can be between 1 and 7. Finally, if the primes are 2, 17, 19, then the $c = 1$. Hence, there are a total of 36 possible values for c. Depending on whether $p_1 = 2$ or $p_2 = 2$, there are two possible ways to assign a and b per each value of c. Our final answer is therefore $2 \cdot 36 = 72$.

Solution to Question 177: Cylindrical Cone

The volume of the cylinder is $\pi r^2 h$. The volume of the cone that is inside this cylinder is $\dfrac{1}{3}\pi r^2 h$. Therefore, since we select uniformly at random from the cylinder, the probability it lies in the cone is $\dfrac{\frac{1}{3}\pi r^2 h}{\pi r^2 h} = \dfrac{1}{3}$

Solution to Question 178: Russian Roulette IV

If you spin the barrel, the probability that you survive is $\frac{4}{6}$. If you don't spin the barrel, the probability that you survive is $\frac{3}{4}$. To understand this case, let us define the chambers 1-6 and further define chambers 5 and 6 to hold the two consecutive bullets, without loss of generality. Because our friend survives, we know that he shot either chambers 1, 2, 3, or 4. Of these 4 equal possibilities, only one of these leads to our loss (if he shot the fourth chamber, then the following is a bullet); hence, the probability of survival in this case is $\frac{3}{4}$ and the difference between these probabilities is:

$$\frac{3}{4} - \frac{2}{3} = \frac{1}{12}$$

Solution to Question 179: Confused Ant II

This seems like a very complicated problem on the surface but you can actually use Markov Chains to solve this problem. There is a lot of symmetry in cubes to help us decrease the number of states involved in this question. Letâs let your starting state be E_{00}. No matter which way the ant goes, by symmetry itâll basically always be the same position. That position being 1 side length away from the starting vertex. Lets denote the expected number of moves to get back to the initial vertex as E_1. Thus $E_{00} = E_1 + 1$. Let E_2 be the expected number of moves from being in the state of 2 side lengths away from the starting vertex. Finally, let E_3 be the expected moves from being at the opposite vertex of the starting vertex. The equations for all these states are:

$$\begin{aligned} E_{00} &= E_1 + 1 \\ E_1 &= \frac{2}{3}E_2 + \frac{1}{3}E_0 + 1 \\ E_2 &= \frac{2}{3}E_1 + \frac{1}{3}E_3 + 1 \\ E_3 &= E_2 + 1 \end{aligned}$$

We know that $E_0 = 0$. Solving all of these equations, we get $E_{00} = 8$. Thus it takes 8 steps for the ant to start at one vertex of the cube and return to it.

Solution to Question 180: Perfect Square

When you choose two integers uniformly at random in $\{1, 2, \ldots, n\}$, the probability that the sum is i is

$$P(i) = \frac{n - |n + 1 - i|}{n^2} = \frac{1}{n^2}\begin{cases} i - 1 & 2 \le i \le n + 1 \\ 2n + 1 - i & n + 1 \le i \le 2n \\ 0 & \text{else} \end{cases}$$

So you get

$$P(sq) = \sum_{k=2}^{\lfloor\sqrt{n}\rfloor} \frac{k^2 - 1}{n^2} + \sum_{k=\lfloor\sqrt{n}\rfloor+1}^{\lfloor\sqrt{2n}\rfloor} \frac{2n + 1 - k^2}{n^2}$$

And doing the sums gives

$$P(sq) = \frac{\lfloor\sqrt{n}\rfloor\left(2\lfloor\sqrt{n}\rfloor^2 + 3\lfloor\sqrt{n}\rfloor - 6n - 5\right)}{3n^2} - \frac{\lfloor\sqrt{2n}\rfloor\left(2\lfloor\sqrt{2n}\rfloor^2 + 3\lfloor\sqrt{2n}\rfloor - 12n - 5\right)}{6n^2}.$$

Letting $\epsilon = \sqrt{n} - \lfloor\sqrt{n}\rfloor$ and $\delta = \sqrt{2n} - \lfloor\sqrt{2n}\rfloor$, where $0 \le \epsilon, \delta < 1$, we have

$$P(sq) = \frac{4(\sqrt{2} - 1)}{3\sqrt{n}} + O\left(\frac{1}{n^{3/2}}\right).$$

So the limit exists and is equal to $(4/3)(\sqrt{2} - 1)$.

Solution to Question 181: Crazy Covariance

By definition, we know that $\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$. We only know the distribution of $Y \mid X = x$. Therefore, we will need to apply LOTUS on any terms involving Y.

Starting with $\mathbb{E}[Y]$, $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y \mid X]]$. One can show from the definition of the lognormal distribution that for $R \sim \mathrm{LogNorm}(\mu, \sigma^2)$, $\mathbb{E}[R] = e^{\mu + \frac{1}{2}\sigma^2}$. In this case, $\mu = 0$ and $\sigma^2 = X$. Therefore, $\mathbb{E}[Y \mid X] = e^{\frac{X}{2}}$. This implies that $\mathbb{E}[Y] = \mathbb{E}\left[e^{\frac{X}{2}}\right]$. This is just the MGF of X evaluated at $\theta = \frac{1}{2}$. As we know $X \sim \mathrm{Exp}(1)$, the MGF of X is $M_X(\theta) = \frac{1}{1-\theta}$. Plugging in $\theta = \frac{1}{2}$ yields $\mathbb{E}[Y] = 2$. We already know $\mathbb{E}[X] = 1$ by the known properties of exponential distributions.

To calculate $\mathbb{E}[XY]$, we need to apply LOTUS once again. Thus,

$$\mathbb{E}[XY] = \mathbb{E}[\mathbb{E}[XY \mid X]] = \mathbb{E}[X\mathbb{E}[Y \mid X]] = \mathbb{E}\left[Xe^{\frac{X}{2}}\right]$$

The last expression above is just the first derivative of the MGF of X evaluated at $\theta = \frac{1}{2}$. Taking the derivative of the MGF of X, we get $M_X'(\theta) = \frac{1}{(1-\theta)^2}$. Letting $\theta = \frac{1}{2}$ yields $\mathbb{E}[XY] = 4$. This means $\mathrm{Cov}(X,Y) = 4 - 2 \cdot 1 = 2$.

Solution to Question 182: First Right

Let x be the expected value of the game if you go first. With probability $\frac{1}{2}$, you will receive \$30. With probability $\frac{1}{4}$ (you land tails and your opponent lands heads), you will lose \$30. Then, with probability $\frac{1}{4}$, the game resets and your expected gain is x. This yields the equation

$$x = 0.5(30) + 0.25(-30) + 0.25(x)$$

When solved, this yields $x = 10$, so the right is worth at most \$10.

Solution to Question 183: Clock Angle I

At 9:45 PM, we know the minutes hand is at the 9. The hours hand is $\frac{3}{4}$ of the way between the 9 and the 10, as $\frac{3}{4}$ of an hour has elapsed. Therefore, the angle between them is just $\frac{3}{4}$ of the angular distance from 9 to 10 on the clock. Namely, as there are 12 equally spaced numerical values and the total degrees of a circle is 360, there are 30 degrees between each numerical value. Therefore, our answer is $\frac{3}{4} \cdot 30 = 22.5$.

Solution to Question 184: Greedy Bob

Let I of mean 0.5 and variance 0.25 be the indicator variable that denotes if Bob gets a head. Then, $Y = IX$, where I and X are independent. Thus we can write:

$$\begin{aligned} V[Y] = V[IX] &= (V[I] + E[I]^2)(V[X] + E[X]^2) - E[I]^2E[X]^2 \\ &= (0.25 + 0.25)(2 + 144) - 0.25 \times 144 = 37 \end{aligned}$$

Solution to Question 185: Bond Question II

First thing to realize is that our payments are quarterly, meaning we will have 12 payments, and a quarterly interest rate of 1.25

The first step we want to take is calculating our cash flows. In order to do this, we need to calculate how much money we lose to inflation each year we are payed out. To start, we calculate how much we are payed out per period, which is our face value * our coupon rate. In this case $1000 \cdot 2\% = 20$. We then want to calculate how much value we are giving up to inflation. This is given by the formula:

$$\sum_{n=1}^{12} \frac{20}{1.0125^n}$$

Here 20 is our coupon payment, 1.0125 represents our quarterly inflation rate of 1.25

The second step is calculating our final face value payment, which is given by the face value and then dividing by the inflation rate to the power of however many years we were losing money to inflation. In this case, it will be $1000/1.0125^{12}$, which equals 861.51.

Finally, add up the cash flows and the final face value to get our answer of 1083.09.

Solution to Question 186: Gamma Review III

We are going to evaluate this for general a, b, and k. The condition for existence of this expectation is that $a + k > 0$. One can verify this by looking at the definition of the Gamma integral. This is clearly satisfied here.

In this more general case, we have $\mathbb{E}[X^k] = \displaystyle\int_0^\infty \frac{x^{(a+k)-1}e^{-\frac{x}{b}}}{b^a\Gamma(a)}dx$ by combining the Gamma PDF with the x^k term obtained from LOTUS. This integral now seems to have the same form as the Gamma(a, b) PDF but instead with $a + k$ instead of a. Therefore, we will want to normalize to a Gamma$(a + k, b)$ PDF. We need to normalize so that we get $a + k$ everywhere we have a.

The denominator must become $b^{a+k}\Gamma(a + k)$, so multiplying and dividing by this, we get

$$\frac{b^{a+k}\Gamma(a+k)}{b^a\Gamma(a)} \int_0^\infty \frac{x^{(a+k)-1}e^{-\frac{x}{b}}}{b^{a+k}\Gamma(a+k)}dx$$

The integrand is just the PDF of a Gamma$(a+k, b)$ distribution over the entire support, so it is 1. Therefore, we get that

$$\mathbb{E}[X^k] = \frac{b^k\Gamma(a+k)}{\Gamma(a)}$$

Plugging in our specific values of $a = 2, b = 4, k = 6$, we get that

$$\mathbb{E}[X^6] = \frac{4^6 \cdot \Gamma(8)}{\Gamma(2)} = 4^6 \cdot 7! = 2^{12} \cdot 7!$$

This means our answer is $2 + 12 + 7 = 21$.

Solution to Question 187: Spacious Uniform Values I

Consider the following discrete scenario: Imagine having a deck of cards of size m and marking N of them. Find the probability that the marked cards are all at least a distance d apart. If you deal out the cards one-by-one from the top of the deck, each time you obtain a marked card, deal out d cards from the bottom of the deck and put them on top of your marked card you just dealt. All of the marked cards are at least a distance d apart precisely when none of the marked cards were in the bottom $(N-1)d$ cards. If they were, then they would be within d distance of the previous marked card dealt out, as they would lie in the region that we put between two marked cards, which is supposed to have none if the marked cards are to truly be at least d cards apart.

Let $m \to \infty$ in the above example and the intuition for this problem holds. Instead of throwing out d cards, we are going to throw out an interval of length d at each point selected. Thus, we want to find the probability that none of the "marked values" (points) are within the interval of total length $(N-1)d$ that was thrown away. This probability is $1-(N-1)d$ for each random variable. By independence and there being N such random variables (as we have N such points here), the probability is $(1-(N-1)d)^N$. In this case, $N = 101$ and $d = \dfrac{1}{1000}$, so this evaluates to $\left(\dfrac{9}{10}\right)^{101}$. By the form of our solution, we want $9 + 10 + 101 = 120$, which is the answer.

Solution to Question 188: Better in Red V

Given that we have new information entering this problem (both rolls of the selected mini cube yielded a red face shown), we need to update the probability that the chosen cube was a corner. Thus, we have to use Bayes Theorem. Let C be the event of picking a corner cube and R_2 be the event that the chosen cube shows a red face twice when rolled twice. Then we have:

$$\mathbb{P}[C \mid R_2] = \frac{\mathbb{P}[R_2 \mid C] \cdot \mathbb{P}[C]}{\mathbb{P}[R_2]}$$

The probability we roll a red face on a corner cube is $\frac{3}{6}$. Thus the probability we roll a red face twice on a corner cube is $(\frac{3}{6})^2 = \frac{1}{4}$. Thus $\mathbb{P}[R_2 \mid C] = \frac{1}{4}$.

Since there are 8 total corner cubes out of a total of 27, $\mathbb{P}[C] = \frac{8}{27}$.

Finally, the probability we roll a red face twice is $\frac{1}{4} \cdot \frac{8}{27}$ from corner pieces, $(\frac{1}{3})^2 \cdot \frac{12}{27}$ from edge pieces (two red faces), and $(\frac{1}{6})^2 \cdot \frac{6}{27}$ from center side pieces (one red face).

Putting this all together we get:

$$\mathbb{P}[C \mid R_2] = \frac{\mathbb{P}[R_2 \mid C] \cdot \mathbb{P}[C]}{\mathbb{P}[R_2]} = \frac{\frac{1}{4} \cdot \frac{8}{27}}{\frac{1}{4} \cdot \frac{8}{27} + \frac{1}{9} \cdot \frac{12}{27} + \frac{1}{36} \cdot \frac{6}{27}} = \frac{4}{7}$$

Solution to Question 189: Checkmate

Let m denote the probability that Andy wins a game against Michael, and let a denote the probability that Andy wins a game against Aaron. Since Aaron is better than Michael, we assume $a < m \Rightarrow m - a > 0$. In order to receive a prize, Andy can either win the first two games, or he can lose the first game and with the next two games. For option 1, the probability that Andy receives a prize can be written as

$$\mathbb{P}(\text{Andy receives prize} \,|\, \text{option 1}) = m \cdot a + (1-m) \cdot a \cdot m$$

And for option 2, we have

$$\mathbb{P}(\text{Andy receives prize} \,|\, \text{option 2}) = a \cdot m + (1-a) \cdot m \cdot a$$

Note the following:

$$\begin{aligned} &\mathbb{P}(\text{Andy receives prize} \,|\, \text{option 2}) - \mathbb{P}(\text{Andy receives prize} \,|\, \text{option 1}) \\ &= [(1-a) - (1-m)]\, am \\ &= (m-a)am > 0 \end{aligned}$$

Therefore, we conclude that $\mathbb{P}(\text{Andy receives prize} \,|\, \text{option 2})$ is greater than $\mathbb{P}(\text{Andy receives prize} \,|\, \text{option 1})$. Andy should pick option 2.

Solution to Question 190: Bubbly Sort

There are 99 comparisons to be made in Bubble Sort. Namely, between positions i and $i+1$ for $1 \leq i \leq 99$. At each iteration, you decide to either swap or skip the pair. Each list of swaps or skips generates a unique original permutation that would be sorted after one iteration of the algorithm. For example, with $n = 3$, the sorted list would be $(1, 2, 3)$. Consider the two example cases below:

Case 1 - Swap Swap: We want to work backwards here, so first we unswap positions 2 and 3, yielding $(1, 3, 2)$. Afterwards, we unswap positions 1 and 2, yielding $(3, 1, 2)$.

Case 2 - Swap Skip: We skip switching positions 2 and 3. However, we swap positions 1 and 2, yielding $(2, 1, 3)$.

Generally, each of the $2^{100-1} = 2^{99}$ sequences of swap and skip will lead to a unique initial configuration that yields a sorted list after 1 iteration. Therefore, the solution is $\dfrac{2^{99}}{100!}$, meaning the answer to our question is $2 + 99 \cdot 100 = 9902$.

Solution to Question 191: Game Arbitrage I

2

teams are guaranteed to make it out of the group. So, if we long every contract, we expect a payout of 2. However, if we add the time-0 price of the 4 teams, we get $1.97 < 2$. So, there is an arbitrage opportunity.

Here, we long the undervalued item and short the overvalued item. 2 corresponds to the bank account. Since interest rates are 0, we do not need to worry about discounting. We long 1 unit of every team and borrow 2 bonds. This gives us the following:

$$0.84 + 0.73 + 0.25 + 0.15 - 2 = -0.03$$

So, we get an initial credit of 3 cents.

Solution to Question 192: Duplicating Data

Correlation is not a function of sample size, and thus correlation increases by 0.

Solution to Question 193: Minimal Variance

Computing $\text{Var}(Y)$ is quite easy, as X_1 and X_2 are independent. Thus, we can just sum the variances of each term. Namely,

$$\text{Var}(Y) = \text{Var}(cX_1 + (1-c)X_2) = c^2\text{Var}(X_1) + (1-c)^2\text{Var}(X_2) = 25c^2 + 100(1-c)^2$$

We just need to maximize the RHS as a function of c. Taking the derivative in c, we get that the maximizing c satisfies

$$50c - 200(1-c) = 0 \iff c = \frac{4}{5}$$

Solution to Question 194: Hit Or Miss

In order to hit the origin, the particle must arrive at $(1, 1)$. To do so, the particle can (1) move 3 left and 3 down, (2) move 2 left, 2 down, and 1 diagonal, (3) move 1 left, 1 down, and 2 diagonal, or (4) move down 3 consecutive diagonals. The probability that the particle arrives at (1, 1) is then

$$\frac{1}{3^6} \cdot \binom{6}{3} + \frac{1}{3^5} \cdot \binom{5}{2, 2, 1} + \frac{1}{3^4} \cdot \binom{4}{1, 1, 2} + \frac{1}{3^3} = \frac{245}{3^6}$$

Finally, there is a $\frac{1}{3}$ chance that the particle reaches the origin from $(1, 1)$, as it has to move diagonally from $(1, 1)$, so our answer is

$$\frac{245}{3^6} \cdot \frac{1}{3} = \frac{245}{2187}$$

Solution to Question 195: Dollar Bills

This is a classic systems of equations problem. Let A be the number of \$10 bills you deposited and B be the number of \$20 bills you deposited. Since you deposited 150 bills, you know that $A + B = 150$. Another equation you can make is the "value" equation. Since the total value you deposited is \$2150, you know that the value from the \$10 bills is $10A$ and $20B$ from the \$20 bills. Thus $10A + 20B = 2150$. When we solve these two equations, we get $B = 65$.

Solution to Question 196: Half Cycle

First, we need to find the expected number of $k-$cycles in our permutation for $k > n$. Then, we can sum those up from $n + 1$ to $2n$ to get $\mathbb{E}[C_n]$.

First, let's fix some $n + 1 \leq k \leq 2n$. Let X_i be the indicator of whether or not the value i belongs to a cycle of length k. Then we have that

$$T = \frac{\sum_{i=1}^{2n} X_i}{k}$$

gives the total number of $k-$cycles, as we need to remove rotations (k such rotations per $k-$cycle). Using linearity of expectation, we have that

$$\mathbb{E}[T] = \frac{1}{k}\sum_{i=1}^{2n}\mathbb{E}[X_i] = \frac{2n}{k}\mathbb{E}[X_1]$$

$\mathbb{E}[X_1]$ is just the probability 1 belongs to a $k-$cycle. There are $(2n)!$ permutations of $\{1, 2, \ldots, 2n\}$. There are $\binom{2n-1}{k-1}$ ways to pick $k - 1$ more elements to belong to the $k-$cycle. Now that we have selected the elements, there are $(k - 1)!$ ways to arrange those elements. Then, there are $(2n - k)!$ ways to arrange the other $2n - k$ elements. This yields that our probability that 1 belongs to a cycle is

$$\frac{\binom{2n-1}{k-1}(k-1)!(2n-k)!}{(2n)!} = \frac{1}{2n}$$

Therefore, $\mathbb{E}[T] = \frac{1}{k}$ after substituting in.

The above tells us that the expected number of $k-$cycles for $1 \leq k \leq 2n$ is $\frac{1}{k}$. Therefore, $\mathbb{E}[C_n]$ is just the sum of the terms above from $n + 1$ to $2n$.

Namely,

$$\mathbb{E}[C_n] = \sum_{k=n+1}^{2n} \frac{1}{k} = \sum_{k=n+1}^{2n} \frac{1}{\frac{k}{n}} \cdot \frac{1}{n}$$

This is starting to look like a Riemann Integral. Let $x = \frac{k}{n}$ so that $dx = \frac{1}{n}$ (this is a discrete sum currently, so dx is just change between terms). The lower bound of our integral is $x_L = \frac{n+1}{n} \to 1$, while the upper bound is $x_U = \frac{2n}{n} = 2$. As $n \to \infty$, this sum converges to the Riemann Integral

$$\int_1^2 \frac{1}{x} dx = \ln(2)$$

Therefore, $q = 2$.

Solution to Question 197: Limiting Values II

The size of the sample space is reduced from 36 to 30 since 7 appeared six times in the initial sample space. The resulting probabilities associated with every roll:

$$2 \Rightarrow \frac{1}{30}$$
$$3 \Rightarrow \frac{2}{30}$$
$$4 \Rightarrow \frac{3}{30}$$
$$5 \Rightarrow \frac{4}{30}$$
$$6 \Rightarrow \frac{5}{30}$$
$$8 \Rightarrow \frac{5}{30}$$
$$9 \Rightarrow \frac{4}{30}$$
$$10 \Rightarrow \frac{3}{30}$$
$$11 \Rightarrow \frac{2}{30}$$
$$12 \Rightarrow \frac{1}{30}$$

The resulting expected value is:

$$\frac{1}{30}(2)+\frac{2}{30}(3)+\frac{3}{30}(4)+\frac{4}{30}(5)+\frac{5}{30}(6)+\frac{5}{30}(8)+\frac{4}{30}(9)+\frac{3}{30}(10)+\frac{2}{30}(11)+\frac{1}{30}(12)=7$$

Intuitively, this makes sense since 7 is the mean of the face values and the probabilities of the values are symmetric around 7.

Solution to Question 198: Graph Search II

When rewording it to think about it as a 11−sided die, and we are trying to figure out how many rolls it will take to see all 11 sides at least once, it is easy to see that this is a variation of the coupon collector's problem.

The time until the first result appears is 1 as it takes one step to start somewhere. After that, once again we are guaranteed to visit another node in 1 step, so we add another one. The random time until a third (different) result appears is geometrically distributed with parameter of success $\frac{9}{10}$, hence a mean $\frac{10}{9}$ (as the mean of a geometrically distributed random variable is the inverse of its parameter). After that, the random time until a fourth (different) result appears is geometrically distributed with parameter of success $\frac{8}{10}$, hence a mean of $\frac{10}{8}$. And so on, until the random time of appearance of the last and tenth node, which is geometrically distributed with parameter of success $\frac{1}{10}$, hence with mean $\frac{10}{1}$.

We now know the mean time to see the first through last numbers pop up, so our total expectation will be summing all of the individual expectations up.

$$\sum_{k=1}^{10}\frac{10}{k}+1\approx 30.29$$

This yields our solution of 30.

Solution to Question 199: Lucky Genie

Suppose we bet x units on even and y on odd. Then our total bet is $4x+6y$. Our expected profit from this game would be $0.5\cdot 6x+0.5\cdot 9y=3x+4.5y=\frac{3}{4}\cdot(4x+6y)$. Therefore, our expected value in return is $\frac{3}{4}$ of our bet, so we should not play this game with an expected value of 0.

Solution to Question 200: Penny Stack

We want to make the product as large as possible. Therefore, if we view the number of coins in each pile as a side length, we want to maximize our value as much as possible. A cube maximizes the volume here, which is the product of the side lengths, so we want to get to a cube or as close to it as possible. Let's start with 10 stacks of 10, as $100 = 10 \cdot 10$. If we divide each stack of 10 into two stacks of 5, then we get a larger product, as each stack of 10 now contributes $5 \cdot 5 = 25$ to our product instead of 10. Therefore, we now have 5 stacks of 20. However, as $5 = 3 + 2$ and $3 \cdot 2 = 6 > 5$, we should divide them up again. Now, we are going to try as many stacks of 3 as possible. This would yields 33 stacks of 3 and a lone penny. However, this is not optimal, as removing one of our stacks of 3 and putting it with the lone penny yields $4 \cdot 3^{32} > 3 \cdot 3^{32} = 3^{33}$. As we can't divide 4 into anything more optimal, our solution is $4 \cdot 3^{32}$, so the answer to the answer is $4 \cdot 3 \cdot 32 = 384$.

Solution to Question 201: Perfect Seating I

Seat the youngest person at the table in any spot. There are 6! ways to arrange the remaining 6 people that have not been seated at the table. Of those arrangements, exactly 2 of them are in increasing order of age. Namely, these occur when they are increasing in age clockwise or counter-clockwise. Therefore, 2 of these 6! equally likely permutations are in increasing order of age, so our result is $\frac{2}{6!} = \frac{1}{360}$.

Solution to Question 202: Straddle Price

For at-the-money calls and puts, we can approximate the option price with the following formula:

$$V = \sqrt{\frac{T-t}{2\pi}}\sigma S$$

A straddle is long a call and put, so we just have to plug everything into the equation and multiply by 2. Plugging everything in, we get:

$$V = 2 * \sqrt{\frac{.36}{2\pi}}(0.4)(20) = 3.83$$

Solution to Question 203: No Arithmetic

Since the sequence must be strictly increasing, we can assign a, b appropriately after selecting two integers between 6 and 29 inclusive. We can also exclude 6 and 20, since those values would create an arithmetic sequence of 4 numbers.

There are a total of $\binom{29-6+1-2}{2} = 231$ ways to do so. Then, we need to consider other ways of forming arithmetic sequences of 4 numbers. Specifically, $(a, b) \neq (7, 9), (12, 21), (16, 28)$. Our final answer is 228.

Solution to Question 204: Uniform Order II

By the definition of conditional probability, this is the same as $\dfrac{\mathbb{P}[X_1 > X_{10}, X_{20} > X_{10}]}{\mathbb{P}[X_{10} < X_{20}]}$. Since these random variables are exchangeable, we have that $\mathbb{P}[X_{10} < X_{20}]$ is just $\dfrac{1}{2}$, as either of the two random variables are equally likely to be larger than one another. For the numerator, that probability is really the same as the probability that X_{10} is the smallest of X_1, X_{10}, X_{20}. Since each of the three random variables are equally likely to be the smallest by exchangeability, the numerator is just $\dfrac{1}{3}$. Therefore, the probability in question is just $\dfrac{\frac{1}{3}}{\frac{1}{2}} = \dfrac{2}{3}$.

Solution to Question 205: Valuable Hearts

The sum of all the values of the cards in the deck is $5 \cdot \sum_{i=1}^{13} i$, as each suit contribute $1 + 2 + \cdots + 13$ to the total expect hearts, which contributes $2 \cdot (1 + 2 + \cdots + 13)$. There are 52 total cards in the deck, so the expected value of a random card is

$$\frac{5 \cdot \frac{13 \cdot 14}{2}}{52} = \frac{35}{4}$$

Solution to Question 206: Birthday Guessing

Let $D \in \{1, 2, 4, 5, 7, 8\}$ be the day of the month of Alice's birthday. If Alice's birthday is on a unique day, then Charlie will know Alice's birthday immediately. Considering that Bob is sure that Charlie does not know Alice's birthday, you must infer that the day Charlie was told is not 2 or 7, and thus the month is not June or December. Now Charlie knows that the month must be either March or September. He immediately figures out Alice's birthday, which means that the day must be unique in the March and September set of dates. In other words, Alice's birthday cannot be March 5 or September 5, but instead be one of March 4, March 8, or September 1. Among these three possibilities, March 4 and March 8 have the same month. If the month Bob had was March, then he would not have been able to figure out Alice's birthday. Thus, Alice's birthday must be September 1.

Solution to Question 207: Put Fly I

The put-fly is irrelevant here. A put-fly and a call-fly have the exact same payoff. The only difference is that one uses puts while the other uses calls. By no-arbitrage pricing, they must have the same payoff. A butterfly longs 1 unit of the wings and shorts 2 units of the body. So, we have the following:

$$V_0 = 8.1 - 2(4.2) + 1.4 = 1.1$$

We can also use put-call parity to obtain the same value.

Solution to Question 208: Make Your Martingale IV

The objective here is to calculate dX_t via Ito's Formula. The process a_t will cancel out the dt portion. Namely, $dX_t = 10W_t^9 dW_t - a_t dt + \frac{1}{2} \cdot 90W_t^8 dt$, as the integral portion has 0 second derivative. In particular we can write this as $dX_t = 10W_t^9 dW_t + (45W_t^8 - a_t)dt$, so $a_t = 45W_t^8$. This means our answer is $45 \cdot 8 = 360$.

Solution to Question 209: Coloring Components III

If we were to count overlaps, we could simply set up an indicator variable over every window of 5 squares and applying Linearity of Expectation. This is simply 16 windows of probability $1/32$ each, for an expected value of $1/2$.

To account for overlaps, let's analyze how many times we want to count consecutive chains of black squares of certain lengths. For example, if we have exactly 6 black squares in a row, we want to count this as 1 connected 5-component, but if there are exactly 10 black squares in a row, we want to count this as 2. In general, for exactly n black squares in a row, we want to count this as $\lfloor n/5 \rfloor$ connected 5-components. This can be achieved by repeating our scheme for counting overlaps for multiples of 5 but subtracting the expected number of results we would get from considering overlapping chains of length 1 greater. Denoting C_n as the expected number of chains of n consecutive black squares while counting overlaps, our answer is

$$C_5 - C_6 + C_{10} - C_{11} + C_{15} - C_{16} + C_{20}$$

$$= \left(16 \cdot \frac{1}{2^5}\right) - \left(15 \cdot \frac{1}{2^6}\right) + \left(11 \cdot \frac{1}{2^{10}}\right) - \left(10 \cdot \frac{1}{2^{11}}\right) + \left(6 \cdot \frac{1}{2^{15}}\right) - \left(5 \cdot \frac{1}{2^{16}}\right) + \left(1 \cdot \frac{1}{2^{20}}\right) = \frac{284785}{1048576}.$$

We get the term $C_k = \dfrac{21-k}{2^k}$ from the fact that there are $21-k$ spots for a length k chain to start at and the probability of it starting at any given spot is $\dfrac{1}{2^k}$. Our answer is $284785 + 1048576 = 1333361$.

Solution to Question 210: Unfriendly

The easiest way to compute this is to calculate the total number of ways to invite people and then remove those that invite both. There are $\dbinom{8}{5} = 56$ total ways to select 5 people to invite. To count the number of ways with both of the friends that refuse to attend together, fix both of them among the 5 selected. We then need to select 3 more from the remaining 6, so this yields $\dbinom{6}{3} = 20$ combinations. Therefore, we have $56 - 20 = 36$ ways to invite the people.

Solution to Question 211: Unfair Roulette

Consider Option 2 first. The probability that Miriam survives in this case is $1/2$, since there are 3 loaded chambers out of 6 total chambers. Now, let's consider Option 1. Let B denote the event that Gabe survives the first shot, or in other words, the event that the first chamber is empty. Let A denote the event that Miriam survives the next shot, or in other words, the event that the second chamber is empty. If Miriam chooses Option 1, then her chance of survival is conditioned on the fact that Gabe survived the first trigger-pull. We therefore wish to determine:

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

The probability of the first two chambers being empty is $1/3$, and the probability of the first chamber being empty is $2/3$. To visualize this, consider all the possible arrangements of the bullets within the cylinder, where E denotes an empty chamber and N denotes a non-empty chamber.

$$NENEEE$$

$$ENENEE$$

$$EENENE$$

$$EEENEN$$

$$NEEENE$$

$$ENEEEN$$

Simplifying, we find:

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{1/3}{2/3} = \frac{1}{2}$$

Option 1 gives Miriam the same survival chance as Option 2, so our answer is 3.

Solution to Question 212: Dividing Nuggets

We wish to find $\mathbb{P}(A|B)$, where A is the event that all four students are satisfied, and B is the event that Mr. Garrison's division of the nuggets gives each student at least 5. First, note the following:

$$\begin{aligned}\mathbb{P}(A|B) &= \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \\ &= \frac{\mathbb{P}(A)}{\mathbb{P}(B)}\end{aligned}$$

Since the sample space Ω is the same for the probabilities in the numerator and the denominator, and since we are told that each partition has an equal chance of occurring, our problem can be simplified into the following two sub-problems: (1) how many ways can 50 nuggets be divided into four distinguishable groups so that there are at least 6 nuggets in the first three groups and at least 18 nuggets in the fourth group, and (2) how many ways can 50 nuggets be divided into four distinguishable groups so that there are at least 5 nuggets in each group.

Let's begin with the first sub-problem. We can utilize stars and bars. Since we have 50 total nuggets and 4 buckets, we end up with 3 bars. Let's assign 6 nuggets into each of the first three groups and 18 nuggets into the fourth group. This leaves us with $50 - 3 \cdot 6 - 18 = 14$ nuggets left to split into groups. There are $\binom{17}{3}$ ways to order our 14 stars and 3 bars.

Similarly, for the second sub-problem, we can assign 5 nuggets into each of the four groups. This leaves us with 30 chicken nuggets (stars) left to be partitioned. There are $\binom{33}{3}$ ways to order our 30 stars and 3 bars.

Our answer is:

$$\frac{\binom{17}{3}}{\binom{33}{3}}$$

Finally, $17 + 2 \cdot 3 + 33 + 2 \cdot 3 = 62$.

Solution to Question 213: Covered Calls II

In order to solve, we know that our profit is going to maximize right before our calls our exercised. No matter what we have $(10.50 + 9.50) \cdot 100 = \$2000$

in profit from selling both of our calls, now we just need to see how much extra our shares will add to that profit. Right up to \$240, we will maintain ownership of all 200 shares, with all 200 gaining \$10 in value, for a total profit of $2000 + 200 \cdot 10 = \$4000$. We then need to check our next peak at \$250. This time we will maintain ownership of 100 shares, with these 100 gaining \$20 in value each, for a total profit of $2000 + 100 \cdot 20 = \$4000$.

At both peaks, we have the same max profit, so pick either and our answer is \$4000

Solution to Question 214: Counting Nash Equilibria

Let's first simplify the valuation of the chest, since we are interested in Nash equilibria we are interested in a set of strategies such that a **single** person cannot increase their *expected value* from the chest when everyone uses the Nash strategy. Therefore, we can simply think of the chest containing \$10 since this is the expected number of heads. Now we begin with the observation that if one person submits a \$10 bid, then all 10 players at best will have a 0 EV strategy. It is not hard to see the person bidding \$10 will be 0 EV since the chest is \$10 in expectation. As for the remaining 9 people, if any of them bid less than \$10, they don't win or lose anything since the chest goes to the \$10 bidder and their bid is returned. If they bid exactly \$10 then again this is 0 EV, and if they bid any more than \$10 their EV is negative so at best the remaining 9 people can do a 0 EV strategy.

However, it might be in the interest of a person bidding \$10 to decrease their bid. Consider a strategy where all 9 people will bid \$9, what should the last person do to maximize their EV? Bidding \$10 here is 0 EV as discussed before, but bidding \$9 is actually positive EV since there is a $\frac{1}{10}$ probability their \$9 bid wins the chest worth \$10 for netting them a \$1 profit. This motivates our first statement.

Claim: All people bidding \$9 is a Nash Equilibrium.

Proof: The expectation of this strategy for a given person is \$0.1 as shown above. A given person cannot exceed this in expectation since bidding any higher will result in a zero or negative expectation and bidding any lower will result in zero expectation since they will never win the auction due to the other people bidding \$9.

Now, we begin to generalize. Let's consider counting the number of Nash Equilibrium by the number of people who are bidding \$10. If there are only 0 people bidding \$10, then the only Nash Equilibrium that can exist is when all bid \$9, if there was someone who was not bidding \$9 then they could increase their expectation by bidding \$9. If there is only 1 person bidding \$10, then no such Nash equilibrium can exist since this person can increase their expectation by bidding \$9.

If there are only 2 people bidding \$10, then we claim this is a Nash equilibrium because everyone's strategy has an expected value of 0 and no one can gain by deviating. In fact, this logic holds as long as there are *at least* 2 people bidding \$10. Therefore, to count the total number of strategies let us reframe the problem. Let's denote a strategy as a 10-d tuple so $S = \{b_1, b_2, ..., b_{10}\}$ where b_i denotes the $i-$th person's bid meaning $b_i \in \{\$0, \$1, ..., \$10\}$. To count the number of Nash Equilibria which correspond to at least 2 people bidding \$10 we count the complement. That is, the number of Nash Equilibria with at least 2 people bidding ten $= \underbrace{11^{10}}_{\text{all strategies}} - \underbrace{10^{10}}_{\text{no ten bids}} - \underbrace{\binom{10}{1}10^9}_{\text{one ten bids}} = 5937424601.$
Lastly, we must add one in the case of all bids being \$9 giving us our final answer of 5937424602.

Solution to Question 215: Difference of Four

In order for the difference to be exactly four, the values must either have a maximum of 6 and a minimum of 2, or a maximum of 5 and a minimum of 1. Let us consider these cases separately. For the first case, two of the numbers are 6 and 2. The third number can be 2, 3, 4, 5 or 6. Thus, the unordered outcomes are 226, 236, 246, 256, and 266. The first outcome has 3 orderings, the second has six orderings, the third has six orderings, the fourth has six orderings, and the fifth has three orderings. Each ordering has a probability $\frac{1}{216}$ of occurring, so the total probability of this case is $\frac{3+6+6+6+3}{216} = \frac{24}{216}$. For the second case, two of the numbers are 5 and 1. The third number can be 1, 2, 3, 4, or 5. Thus, the unordered outcomes are 115, 125, 135, 145, and 155. The first outcome has three orderings, the second outcome has six orderings, the third outcome has six orderings, the fourth outcome has six orderings, and the fifth outcome has three orderings. Each ordering has a probability $\frac{1}{216}$ of occurring, so the total probability of this case is $\frac{3+6+6+6+3}{216} = \frac{24}{216}$. In total, the probability of yielding a difference between the highest and lowest values of exactly four is:

$$\frac{24+24}{216} = \frac{2}{9}$$

Solution to Question 216: Colorful Line

This is a classic anagrams problem, where we have three distinct groups (corresponding to the three colors) and every string corresponds to an arrangement of red, green, and blue. There are 19 total balls, so if all were distinct, then there are 19! ways to arrange them. However, this overcounts by considering every ball distinct, where balls of the sample color are not distinct. Therefore, we must divide out those arrangements. There are 9! arrangements of the 9 red balls that aren't distinct. Similarly, there are 7! and 3! ways to arrange for the other colors, so the answer is

$$\frac{19!}{9!7!3!} = 11085360$$

Solution to Question 217: 4 Before 2

If S represents the random sum of two dice, we note that $\mathbb{P}[S=2]+\mathbb{P}[S=4] < 1$ and these events are disjoint, so we can say that the probability Jed obtains a sum of 4 before a sum of 2 is

$$\frac{\mathbb{P}[S=4]}{\mathbb{P}[S=4]+\mathbb{P}[S=2]}$$

There are 3 combinations yielding a sum of 4, while there is only one combination yielding a sum of 2, so the answer is $\frac{3}{3+1} = \frac{3}{4}$.

Solution to Question 218: Prime Janitors

Looking at this scenario a bit longer, youâll see that the opening/closing of doors are dependent on their factors. For example, door 2 starts opened and is closed by the first janitor. Door 3 is going to be closed by the second janitor. However, door 6 is closed by the first janitor, but reopened by the second janitor. In general, door k will be closed at the end of the day if it has an odd number of distinct prime factors. This is all of the integers that are prime integers themselves, powers of prime integers (both have 1 factor), or integers that have 3 distinct prime factors.

The primes that are at most 100 are: $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73$ (25 integers)

The powers of primes that are at most 100 are: $4, 8, 16, 32, 64, 9, 27, 81, 25, 49$ (10 integers)

Integers that are at most 100 with 3 prime factors are:

$$\begin{aligned}
2 \cdot 3 \cdot 5 &= 30 \\
2 \cdot 3 \cdot 7 &= 42 \\
2 \cdot 3 \cdot 11 &= 66 \\
2 \cdot 3 \cdot 13 &= 78 \\
2 \cdot 5 \cdot 7 &= 70 \\
2^2 \cdot 3 \cdot 5 &= 60 \\
2^2 \cdot 3 \cdot 7 &= 84 \\
2 \cdot 3^2 \cdot 5 &= 90
\end{aligned}$$

There are 8 integers above. This means that there are $25 + 10 + 8 = 43$ closed doors, meaning that $100 - 43 = 57$ are still open.

Solution to Question 219: Ant in a Hurry

Open up and flatten the cube to a two-dimensional surface to see that the optimal path is a straight line between the two points. The path is the hypotenuse of a triangle with side lengths 1 and 2, and thus is of length $\sqrt{5}$.

Solution to Question 220: Better in Red I

Label the faces of each cube $1 - 6$, and then let I_i be the indicator that side i of the cube that is drawn is colored red. Then $T = I_1 + \cdots + I_6$ gives the total number of red sides of the cube. By linearity of expectation and the fact that this is the cube so all the sides are exchangeable, $\mathbb{E}[T] = 6\mathbb{E}[I_1]$. $\mathbb{E}[I_1]$ is just the probability that side 1 is colored red. We know that 10^2 of the cubes will have side 1 colored red, as each side is colored red on one face of the big cube, which is 10^2 little cubes. Therefore, the probability side 1 is red is $\frac{10^2}{10^3} = \frac{1}{10}$. Therefore, $\mathbb{E}[T] = \frac{3}{5}$ by substituting back in.

Solution to Question 221: Resell Painting

If your bid is lower than the value of the painting, you wonât lose anything, so no harm no foul. However, let's say that you do win the painting. This means that you bid higher than the actual value. Let x be your bid. Given you won the bid, we know that the the value is at most x. Therefore, the conditional distribution of the value of the painting is $\text{Unif}(0, x)$. On average, the value of the painting will be $\frac{x}{2}$. This means we will be able to sell it, on average, for $\frac{1.5 \cdot x}{2} = \frac{3}{4}x$. However, this is less than x, the amount we paid for the painting. So we shouldnât bid on this painting, thus the answer is \$0.

Solution to Question 222: Pick Your Opponent

Intuitively, to win the tournament, you must win your second game, as you have to win 2 in a row. Therefore, you should play the easier opponent in the middle, meaning you should select ABA. To verify this, let x and y be your probabilities of being Alice and Bob, respectively. We know that $x < y$ by the question.

If you select ABA, your probability of winning the tournament is $xy + xy - x^2y = 2xy - x^2y$ by inclusion-exclusion. If you select BAB, your probability of winning the tournament is $xy + xy - y^2x = 2xy - y^2x$. However, as $x < y$, we have that $y^2x > x^2y$, so your probability of winning with BAB is smaller, as you subtract a larger number.

Solution to Question 223: Bull Call Spread I

If we look at the payoff diagram of a bull call spread, we have $V_T = 0, S_T \in [0, 5]$, $V_T = S_T - 5, S_T \in [5, 10]$ and $V_T = 10, S_T \in [5, \infty)$.

So, in the worst case, we will gain 0 and in the best case, we will gain 5. In general, a bull call spread can cap the upside, but can also limit the downside loss. This option contract is the best when an investor expects a moderate increase in the stock (i.e it stays in the range of K_1 and K_2).

Solution to Question 224: Red Card Deal

Initially, it seems as if you can gain a small advantage in this game by waiting for the first moment when the red cards in the remaining deck outnumber the black. However, this may never happen. The question is now if that benefit outweighs the risk. The answer is no, and we can show this now.

Suppose you elect to use some strategy, say S, to this game. Now, apply S to the game where instead of the next card being red, the last card in the deck is red. The probability of you winning is the same in each game, as each of the remaining unturned cards has the same probability of being red. However, the probability of winning in the second game, regardless of your strategy, is $1/2$, as we always ask for the last card to be red. Therefore, it doesn't matter what strategy you use for this game, as it will always be probability $1/2$ of winning.

Solution to Question 225: Eigenshift

The eigenvalues of $A-3I_n$ are 2 and 4, as the $-3I_n$ decreases both eigenvalues by 3. Then, $(A-3I_n)^{-1}$ has eigenvalues $\frac{1}{2}$ and $\frac{1}{4}$, as the inverse has eigenvalues that are the reciprocal of the original eigenvalues. Lastly, $(A-3I_n)^{-3}$ would have eigenvalues $\frac{1}{2^3}=\frac{1}{8}$ and $\frac{1}{4^3}=\frac{1}{64}$, as we would cube the eigenvalues. Adding these up, we get that our answer is $\frac{1}{8}+\frac{1}{64}=\frac{9}{64}$.

Solution to Question 226: Statistical Test Review III

We define

$$H_0 : \mu = 132$$
$$H_a : \mu < 132$$

Here, $n = 40$, which is large enough for us to let $\mu_{\text{sample}} = 122$ be a point estimator of μ that is approximately normally distributed. We can then define our test statistic as

$$Z = \frac{\mu_{\text{sample}} - \mu}{\sigma/\sqrt{40}}$$

Under the null hypothesis, we expect $Z \sim \mathcal{N}(0,1)$. We reject H_0 if $Z < z_{0.01} = -2.326$.

$$\begin{aligned} Z &= \frac{\sqrt{40}(122-132)}{25} \\ &\approx -2.53 < -2.326 \end{aligned}$$

We may reject the null hypothesis, which means we can accuse QuantHomies of paying wages that are below the industry standard. We should respond with 1.

Solution to Question 227: Miscalculated Fibonacci

You would calculate that $G_{101} = G_{100}+F_{99} = 1+(F_{100}+F_{99}) = 1+F_{101}$. Then, you would calculate that $G_{102} = G_{101}+G_{100} = 2+(F_{101}+F_{100}) = 2+F_{102}$. More generally, we see that the error is going to accumulate just like the Fibonacci sequence itself! This is because the error in each of the two previous values is summed up. In particular, the error terms start as 1 and 1 rather than 0 and 1. Therefore, we have that the error term is F_{n+1}. Plugging in $n = 10$, as we are calculating 10 terms out above 100, we get that our error is going to be $F_{11} = 89$.

Solution to Question 228: Make Your Martingale III

By direct computation, we have that

$$\mathbb{E}[M_{n+1}^p \mid M_n^p] = \mathbb{E}\left[M_n^p \cdot \frac{1}{2^p} e^{\frac{\lambda p}{2} X_{n+1}} \mid M_n^p\right] = \frac{M_n^p}{2^p}\mathbb{E}\left[e^{\frac{\lambda p}{2} X_{n+1}} \mid M_n^p\right]$$

We know that X_{n+1} is independent of M_n^p, as M_n is only a function of $X_1, \ldots, X_n$. Therefore, $\mathbb{E}\left[e^{\frac{\lambda p}{2} X_{n+1}} \mid M_n^p\right] = \mathbb{E}\left[e^{\frac{\lambda p}{2} X_{n+1}}\right] = M\left(\frac{\lambda p}{2}\right)$, where $M(\theta) = \dfrac{\lambda}{\lambda - \theta}$ is the MGF of an $\text{Exp}(\lambda)$ random variable. Substituting in, $\mathbb{E}\left[e^{\frac{\lambda p}{2} X_{n+1}}\right] = \dfrac{\lambda}{\lambda - \frac{\lambda p}{2}} = \dfrac{2}{2-p}$. This means that $\mathbb{E}[M_{n+1}^p \mid M_n^p] = M_n^p \cdot \dfrac{2^{1-p}}{2-p}$. We have a sub-martingale whenever $\dfrac{2^{1-p}}{2-p} > 1$, which means $2 - p < 2^{1-p}$.

Note that $0 < p < 2$ because of the assumption $p > 0$ and that our step where we plug into the MGF of the exponential distribution is otherwise invalid if $p > 2$. This is since the MGF is only defined for $\theta < \lambda$, so as $\theta = \dfrac{\lambda p}{2}$ here, for $p \geq 2$, this is at least λ, which is not in our domain of definition. Thus, we restrict ourselves to $(0, 2)$. In particular, we have equality at $p = 0, 1$. As $2^{1-p} > 2-p$ for $p < 0$, we must have that $2^{1-p} < 2-p$ on $(0, 1)$ and $2^{1-p} > 2-p$ for $1 < p < 2$. This means that $1 < p < 2$ yields a sub-martingale, so $a + b = 3$.

Solution to Question 229: Exact 5 II

Let N be the amount of trials needed to obtain the first 6 and T be the number of times that 5 appears in total. By Law of Total Expectation,

$$\mathbb{E}[T] = \mathbb{E}[\mathbb{E}[T \mid N]]$$

$\mathbb{E}[T \mid N]$ is fairly easily computed by some slight tricks. Note that if N is the first time 6 appears, then there are $N - 1$ trials before that where 6 does not appear. Given that 6 does not appear, each of the other 5 values are equally-likely to show up in each of the $N - 1$ spots. Therefore, $\mathbb{E}[T \mid N] = \dfrac{N-1}{5}$. We know that $N \sim \text{Geom}\left(\frac{1}{6}\right)$, as N is a waiting time to see the value 6. Combining all of this,

$$\mathbb{E}[T] = \mathbb{E}\left[\frac{N-1}{5}\right] = \frac{\mathbb{E}[N] - 1}{5} = 1$$

Solution to Question 230: Snowman Surface

The total surface area of the snowman is $4\pi(8^2 + 11^2 + 14^2) = 1524\pi$. Half of the 11 inch sphere and all of the 14 inch sphere are at most 39 inches off the ground. This comes from the fact that the diameter is 28 inches for the 14 inch snowball, and then we add 11 inches as the radius of the 11 inch snowball. The surface area of the portion closer to the ground is $4\pi \cdot 14^2 + 2\pi \cdot 11^2 = 1026\pi$. Therefore, the answer in question is $\frac{1026\pi}{1524\pi} = \frac{171}{254}$.

Solution to Question 231: Three Repeat II

We have 4 forms that the sequence can be in, which are $HHH---$, $-HHH--$, $--HHH-$, and $---HHH$. Note that the first and second are respectively reflection of the fourth and third sequences, so we can just do the first two sequences and then multiply by 2. For the first sequence, the 4th flip must be tails, but the other two can be either parity, so this yields 4 possibilities. For the second outcome, the first and second dashes must be T, but the last can be either, so this yields 2 outcomes. Therefore, we have a total of $2(4 + 2) = 12$ sample points that have a single run of three heads, so the probability is $\frac{12}{64} = \frac{3}{16}$.

Solution to Question 232: Card Shuffling

Each shuffle can be represented as a permutation in S_{14}, the symmetric group on 14 integers. This is because each of the cards is now permuted to some other spot in the deck, meaning that the shuffle operation can be represented as a permutation of 14 numbers

In two-row notation, this can be written as:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 \\ 14 & 12 & 10 & 8 & 6 & 4 & 2 & 1 & 3 & 5 \\ 7 & 9 & 11 & 13 \end{pmatrix}$$

which, when expressed in cycle notation, would be:

$$\begin{pmatrix} 1 & 14 & 13 & 11 & 7 & 2 & 12 & 9 & 3 & 10 \\ 5 & 6 & 4 & 8 \end{pmatrix}$$

This cycle is a disjoint one, as nothing maps back before the end. Thus, this permutation has order 14. This means 14 shuffles are required to get the cards back to their original arrangement.

Solution to Question 233: Red Tower

Let T_n be the number of $n-$tall towers that can be made using these blocks. We know that the last block in the tower is either $1-$tall or $2-$tall. We condition on the two cases. If the last block is $1-$tall, then there are T_{n-1} valid towers stemming from this case. If the last block is $2-$tall, there are T_{n-2} valid towers stemming from this case. Therefore, we have the recurrence relation $T_n = T_{n-1} + T_{n-2}$.

We need 2 initial conditions, as this is a second-order recurrence. We can clearly see that $T_1 = 1$, as only a $1-$tall block works. Additionally, we see $T_2 = 2$, as we either have a $2-$tall block or two $1-$tall blocks stacked. Now, we see that this is just the Fibonacci Sequence shifted up 1 term, as the Fibonacci sequence typically starts off with $T_1 = T_2 = 1$. Therefore, $T_n = F_{n+1}$. In particular, $T_{12} = F_{13} = 233$ by using the recurrence.

Solution to Question 234: Compound Game

Let T be the total payout and X_1 be your first roll. Since our potential future outcomes depend on whether or not X_1 is even or odd, we should condition on the parity of X_1. Namely, $\mathbb{E}[T] = \mathbb{E}[\mathbb{E}[T \mid X_1]]$. If X_1 is odd, we are done, and our expected payout is $\dfrac{1+3+5}{3} = 3$. If X_1 is even, our expected payout from the first round is 4, and then we get to flip a fair coin. With probability $\dfrac{1}{2}$, we get to roll again, so in the even case, our expected payout is $4 + \dfrac{1}{2}\mathbb{E}[T]$, as with probability $\dfrac{1}{2}$ we go again. Therefore, by LOTE,

$$\mathbb{E}[T] = \frac{1}{2} \cdot 3 + \frac{1}{2}\left(4 + \frac{1}{2}\mathbb{E}[T]\right)$$

Solving for $\mathbb{E}[T]$ in the above yields $\mathbb{E}[T] = \dfrac{14}{3}$.

Solution to Question 235: Poisson Review I

Let $X \sim \text{Poisson}(7)$. We wish to compute $\mathbb{P}(X \geq 4) = 1 - \mathbb{P}(X \leq 3)$. Recall that the pmf of a Poisson random variable is $\mathbb{P}[X = x] = \frac{\lambda^x}{x!}e^{-\lambda}$.

$$\begin{aligned} 1 - \mathbb{P}(X \leq 3) &= 1 - \sum_{x=0}^{3} \frac{\lambda^x}{x!}e^{-\lambda} \\ &\approx 0.918 \end{aligned}$$

Solution to Question 236: Cube Pack

In each dimension, we can only fit 6 of the $3 \times 3 \times 3$ cubes. This is because a 7th would give us a dimension of length $21 > 20$, which is not contained inside the cube. Therefore, the answer is just $6^3 = 216$.

Solution to Question 237: Regional Manager III

Recall that the sample size for an upper-tail α-level test is $n = \frac{(z_\alpha + z_\beta)^2\sigma^2}{(\mu_0 - \mu_a)^2}$. Because $\alpha = \beta = 0.05$, then $z_{0.05} = z_\alpha = z_\beta = 1.645$. Thus:

$$n = \frac{(z_\alpha + z_\beta)^2\sigma^2}{(\mu_0 - \mu_a)^2} = \frac{(1.645 + 1.645)^2 \times 6}{(15 - 16)^2} \approx 64.9$$

The regional sales manager will require $n = 65$ observations in order to satisfy his accuracy targets of $\alpha = \beta = 0.05$.

Solution to Question 238: Binomial Maximizer

We will solve this for general n in $\text{Binom}(n, p)$ and $0 \leq k \leq n$. Let $f(p) = \binom{n}{k} p^k (1-p)^{n-k}$. We want to maximize $f(p)$ in p. By the product and chain rules,

$$\begin{aligned} f'(p) &= \binom{n}{k} \left[kp^{k-1}(1-p)^{n-k} + (n-k)p^k(1-p)^{n-k-1} \right] \\ &= \binom{n}{k} p^{k-1}(1-p)^{n-k-1} \left(k(1-p) - (n-k)p \right) \end{aligned}$$

Now, we just need to set the term inside the parentheses equal to 0, as the terms outside are not 0. This is since $0 < p < 1$. Therefore, $k(1-p) - (n-k)p = k - kp - np + kp = k - np = 0$. This means $p = \frac{k}{n}$ maximizes $f(p)$. One can check this is the maximizer by direct substitution, but this is not necessary. In this case, $k = 8$ and $n = 12$, so $p = \frac{8}{12} = \frac{2}{3}$.

Solution to Question 239: Five Below

Let random variable X denote the minimum number rolled after you repeatedly rolls a die until you get a 5. Notice that $\mathbb{P}(X = 6) = 0$, since a 5 will always be rolled. By the definition of expectation, we have:

$$\mathbb{E}[X] = \sum_{x=1}^{5} x \cdot \mathbb{P}(X = x).$$

Let's compute $\mathbb{P}(X = 1)$. In other words, we want to find the probability that there exists at least one 1 before a 5 occurs. Notice that there are only two possibilities (we don't care what the numbers between 1 and 5 actually are): (1) a 1 appears before a 5, or (2) a 5 appears before a 1. By symmetry and the law of total probability, we conclude that $\mathbb{P}(X = 1) = 1/2$.

Next, let's compute $\mathbb{P}(X = 2)$. In order for 2 to be the minimum value, a 1 cannot be rolled before the first 5, and a 2 must be rolled before the first 5. Consider an arbitrary sequence of rolled values of infinite length that contains at least one each of 1, 2, and 5. There are a couple possibilities. Just to name a few: (1) 1 appears before 2 which appears before 5, and (2) 2 appears before 5 which appears before 1, and (3) 2 appears before 1 which appears before 5. In total, there are $3! = 6$ ways to order $\{1, 2, 5\}$. Only when we have $\{2, 5, 1\}$ will we achieve $X = 2$. By symmetry and the law of total probability, for an arbitrary sequence of infinite length, the probability that a 2 appears before a 5 which appears before a 1 is $1/6$.

Similarly, for $\mathbb{P}(X = 3)$, we need to find orderings of $\{1, 2, 3, 5\}$ such that 3 appears before 5 which appears before 1 and 2 in any order. Therefore, $\mathbb{P}(X = 3) = 2!/4! = 1/12$. Repeating this logic for $\mathbb{P}(X = 4)$ and $\mathbb{P}(X = 5)$, we find $\mathbb{P}(X = 4) = 3!/5! = 1/20$ and $\mathbb{P}(X = 5) = 4!/5! = 1/5$. Putting it all together, we find:

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{x=1}^{5} x \cdot \mathbb{P}(X = x) \\
&= \frac{1}{2} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{12} + 4 \cdot \frac{1}{20} + 5 \cdot \frac{1}{5} \\
&= \boxed{\frac{137}{60}}.
\end{aligned}$$

Solution to Question 240: Close Dice I

Let E_i be the expected time for this to happen given that our current roll is i and E be the expected value of the event we are interested in. Then

$$E = 1 + \frac{E_1 + E_2 + E_3 + E_4 + E_5 + E_6}{6}$$

by Law of Total Expectation. By the symmetry of the fair die, $E_1 = E_6$, $E_2 = E_5$ and $E_3 = E_4$. Note that $E_2 \neq E_3$ because intuitively E_1 and E_6 should be larger since there is only one possible value we can roll in the next roll that will give a difference of 1. For E_3 the valid values are 2 and 4, whereas for E_2 it would be 1 and 3. Using the substitution yields $E = 1 + \dfrac{E_1 + E_2 + E_3}{3}$.

It remains to calculate equations for E_1, E_2, and E_3. We know $E_1 = \dfrac{1}{6} \cdot 1 + \dfrac{1+E_1}{6} + \dfrac{1+E_3}{6} + \cdots + \dfrac{1+E_6}{6} = 1 + \dfrac{1}{3}E_1 + \dfrac{1}{6}E_2 + \dfrac{1}{3}E_3$ because of our substitutions with $E_i = E_{7-i}$. Namely, we get $E_1 + 1$ as the expected value when we roll a 1 or 6. We get $E_3 + 1$ as the expected value when we roll 3 or 4. Lastly, we get $E_2 + 1$ as the expected value when we roll 5. If we roll a 2, then it takes 1 roll.

Using this same logic,

$$E_2 = 1 + \frac{1}{6}E_1 + \frac{1}{3}E_2 + \frac{1}{6}E_3$$

$$E_3 = 1 + \frac{1}{3}E_1 + \frac{1}{6}E_2 + \frac{1}{6}E_3$$

To solve for E_1, E_2, and E_3, we now have three equations with three unknowns. One can check that the solution is $E_1 = \dfrac{70}{17}, E_2 = \dfrac{58}{17}$, and $E_3 = \dfrac{60}{17}$, so plugging this into our equation for E, the actual expected value we are interested in,

$$E = 1 + \frac{188}{51} = \frac{239}{51}$$

Solution to Question 241: Student Appointment

We can think of this as an anagrams problem. Label the students $1 - 5$. Each scheduling then corresponds to some anagram of 111222333444555. Say that student 1 is the student unable to make the first time slot. There are $\dbinom{15}{3,3,3,3,3} = \dfrac{15!}{(3!)^5}$ total arrangements of these characters. However, we need to exclude those that start with 1. The probability a given string starts with 1 is $\dfrac{1}{5}$, as there are equal amounts of each of the 5 numbers in the strong. Therefore, $\dfrac{4}{5}$ do not start with 1. This means $\dfrac{4}{5}$ of all of the schedulings are valid, so our answer is

$$\frac{4}{5} \cdot \frac{15!}{(3!)^5} = 134,534,400$$

Solution to Question 242: Mean Babysitter

There are $\binom{10}{4} = 210$ ways to pick the 4 children that will receive food. Once the 4 children are picked, let $x_1, \ldots, x_4$ represent the amount of units of food that each of the children get. We want to find the number of non-negative integer solutions to $x_1 + x_2 + x_3 + x_4 = 12$ with each $x_i \geq 1$. This is equivalent to saying that we fix one unit of food per child before we distribute the rest and then distribute the other 8 units of food with no restrictions. Namely, the equation prior has the same number of solutions as the equation $x_1 + x_2 + x_3 + x_4 = 8$ with each $x_i \geq 0$ an integer. There are $\binom{11}{3} = 165$ solutions to this equation by stars and bars. Therefore, the babysitter has $210 \cdot 165 = 34650$ ways to distribute the food units.

Solution to Question 243: Big Mod II

Note that $15 \equiv -2 \bmod 17$, so we can say that $15^{2021} \bmod 17 = (-2)^{2021} \bmod 17$. Since $(-2)^4 = 16$, this is desirable, as $16 \equiv -1 \bmod 17$, so we can write the above as

$$\left[(-2)^4\right]^{505} \cdot (-2) \bmod 17 = (-1)^{505} \cdot (-2) \bmod 17 = 2$$

Solution to Question 244: Extra Coin

This can be solved with symmetry. Let us compare the number of heads in Alice's first n coins and Bob's n coins. There are three possible outcomes. Let E_1 be the event that Alice has strictly more heads than Bob; E_2 be the event that Alice and Bob have the same number of heads; E_3 be the event that Alice has fewer heads than Bob. By symmetry, $P(E_1) = P(E_3) = x$ and $P(E_2) = y$. Since $\sum_{\omega \in \Omega} P(\omega) = 1$, $2x + y = 1$. For E_1, Alice will always have more heads than Bob, regardless of Alice's final coin result. Furthermore, for E_3, Alice will never have more heads than Bob, regardless of Alice's final coin result. Thus, only event E_2 depends on Alice's $n+1$th coin flip, in which Alice will win half the time (with a result of heads). Thus, Alice's total probability of winning is: $x + 0.5y = x + (1 - 2x) = \frac{1}{2}$.

Solution to Question 245: Dice-Coin Paradigm

The easier term to calculate is the probability that do not observe any 1's, and then subtract this probability from 1. We condition on the number of flips of the coin needed to see the first tails. Let O be the event that we don't see a 1 in the game and T_k be the event that we see the first tails on the kth flip, $k \geq 1$. We have that $\mathbb{P}[O] = \sum_{k=1}^{\infty} \mathbb{P}[O \mid T_k]\mathbb{P}[T_k]$. We have that $\mathbb{P}[T_k]$ is the probability that the first $k-1$ flips are heads and the kth is tails. The probability of this is $\frac{1}{2^k}$. Then, $\mathbb{P}[O \mid T_k]$ is the event that there are no 1s in the k rolls, which is $\left(\frac{5}{6}\right)^k$. Therefore, we get that our sum is

$$\sum_{k=1}^{\infty}\left(\frac{5}{12}\right)^k = \frac{5}{12}\sum_{k=1}^{\infty}\left(\frac{5}{12}\right)^{k-1}$$

This sum evaluates to $\frac{\frac{5}{12}}{1-\frac{5}{12}} = \frac{5}{7}$. Therefore, the probability we see a 1 somewhere is $1-\frac{5}{7} = \frac{2}{7}$.

Solution to Question 246: Alternating Sum

Write this as $(100^2-99^2)+(98^2-97^2)+\cdots+(2^2-1^2)$. We use the identity $a^2-b^2=(a+b)(a-b)$ for all the terms. In particular, since $b=a-1$ in this scenario, we have that $a^2-(a-1)^2=2a-1$ from the identity above. Therefore, we can write each of these terms as

$$199+195+\cdots+7+3=\sum_{k=1}^{50}(4k-1)=4\cdot\frac{50(51)}{2}-50=5050$$

Solution to Question 247: Random Scale

The 200 in front doesn't matter, as subtracting 200 uniformly from all the weights won't change the ordering. Therefore, we can consider this question as if it were weights $1-6$ grams. Fix the weight of 6 grams on one side. We know that the sum of all of the weights is 21 grams, so the side with 11 or more grams would be heavier. There are $\binom{5}{2}=10$ ways to pick a pair of weights to be with the 6 gram weight. Of those, all of them but $(1,2)$ and $(1,3)$ sum to at least 5 grams. Therefore, the probability must be $\frac{10-2}{10}=\frac{4}{5}$, as 8 of the 10 possible combinations make that side heavier.

Solution to Question 248: Arbitrage Detective II

First you must sell Dollars for Euros, $\$1000\cdot .85=\$850$.

You then exchange your Euros for Pounds, $\frac{\$850}{1.5}\approx\$566.667$

And finally, Pounds for Dollars, $\$566.667 \cdot 1.8 = \$1020$

Which yields \$20 in profit.

Solution to Question 249: Paper Draw

Since there are 4 pieces of paper labelled 2 in the second urn and only 1 piece labelled 2 in the first, the probability we selected the second urn is $\frac{4}{5}$. This is directly by Bayes' rule. Given we selected this urn, we would expect to see the value 2 appear $\frac{1}{2} \cdot 40 = 20$ times, as there is probability $\frac{1}{2}$ per selection it appears in this urn. If we selected the other urn, which occurs with probability $\frac{1}{5}$, then we would expect to see the value 2 to appear $\frac{1}{8} \cdot 40 = 5$ times. Therefore, by the Law of Total Expectation, the expected number of times we see the value 2 is

$$\frac{4}{5} \cdot 20 + \frac{1}{5} \cdot 5 = 17$$

Solution to Question 250: Wandering Ant II

Let $N_{(a,b)}$ be the number of steps starting from (a, b) needed to reach the square boundary. We want $\mathbb{E}[N_{(0,0)}]$. We use Law of Total Expectation here to condition on the first step and use the symmetry of our random walk to simplify this problem greatly.

Starting from the origin, the ant moves to $(\pm 1, 0)$ or $(0, \pm 1)$ all with equal probability. However, by the symmetry of our square and our walk, $\mathbb{E}[N_{(0,1)}] = \mathbb{E}[N_{(0,-1)}] = \mathbb{E}[N_{(1,0)}] = \mathbb{E}[N_{(-1,0)}]$. Therefore, by Law of Total Expectation,

$$\begin{aligned}\mathbb{E}[N_{(0,0)}] &= \frac{1}{4}\left[(1 + \mathbb{E}[N_{(0,1)}]) + (1 + \mathbb{E}[N_{(0,-1)}]) + (1 + \mathbb{E}[N_{(1,0)}]) + (1 + \mathbb{E}[N_{(-1,0)}])\right] \\ &= 1 + \mathbb{E}[N_{(1,0)}]\end{aligned}$$

The additional 1 added to all of these represents the fact that we take 1 step to get from the origin to one of these spots.

From $(1, 0)$, we can reach the boundary in one more step if we move right. If we move up or down, we are going to reach $(1, 1)$ or $(1, -1)$. The expected time to reach the boundary from each of these positions is equal by the symmetry of our random walk. If we move left, we are back at the origin. Therefore, by Law of Total Expectation:

$$\begin{aligned}\mathbb{E}[N_{(1,0)}] &= \frac{1}{4}\big(1 + (1 + \mathbb{E}[N_{(1,1)}]) + (1 + \mathbb{E}[N_{(1,-1)}]) + (1 + \mathbb{E}[N_{(0,0)}]\big) \\ &= 1 + \frac{1}{2}\mathbb{E}[N_{(1,1)}] + \frac{1}{4}\mathbb{E}[N_{(0,0)}]\end{aligned}$$

From $(1,1)$, we can reach the boundary in one more step if we move right or up, which occurs with probability $\frac{1}{2}$. Otherwise, we end up at $(1,0)$ or $(0,1)$, and we already know by symmetry that the expected time to hit the boundary from these positions are equal. Therefore, by LOTE, $\mathbb{E}[N_{(1,1)}] = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \big(1 + \mathbb{E}[N_{(1,0)}]\big) = 1 + \frac{1}{2}\mathbb{E}[N_{(1,0)}]$ by doing the substitution.

Substituting the expressions for $\mathbb{E}[N_{(1,1)}]$ and $\mathbb{E}[N_{(0,0)}]$ into the expression for $\mathbb{E}[N_{(1,0)}]$, we find that $\mathbb{E}[N_{(1,0)}] = 1 + \frac{1}{2}\left(1 + \frac{1}{2}\mathbb{E}[N_{(1,0)}]\right) + \frac{1}{4}(1 + \mathbb{E}[N_{(1,0)}])$. We solve and see that $\mathbb{E}[N_{(1,0)}] = \frac{7}{2}$. Substituting this into the expression for $\mathbb{E}[N_{(0,0)}]$, we get $\mathbb{E}[N_{(0,0)}] = \frac{9}{2}$.

Solution to Question 251: In Order

We know that $O_1 = 0.3$ and $O_4 = 0.9$. This means that the remaining two order statistics, as we are looking at IID Unif$(0,1)$ random variables, should partition the interval $(0.3, 0.9)$ into 3 equal length parts. As the length of this interval is 0.6, the length of each of the parts should be 0.2 on average. Therefore, as the third order statistic would have 2 of these pieces before it, $\mathbb{E}[O_3 \mid O_4 = 0.9, O_1 = 0.3] = 0.3 + 2 \cdot 0.2 = 0.7$.

Solution to Question 252: Dominated Turtle

If X_n is the position of Tort and Y_n is the position of Bort (both at time n), imagine plotting (X_n, Y_n) in the plane. Essentially, we would want the number of paths staying strictly below the $X_n = Y_n$ line. Let $N^*_{10,(a,b)\to(c,d)}$ be the number of paths where Tort is below Bort the entire time starting at (a,b) and ending at (c,d) after 10 steps. Let $N_{10,(a,b)\to(c,d)}$ be the number of paths from (a,b) to (c,d) in 10 steps. Lastly, let $N^c_{10,(a,b)\to(c,d)}$ be the number of paths where Tort does catch up to Bort at some point in the 10 steps that start at (a, b) and end at (c, d). The question here is asking for $\frac{N^*_{10,(0,4)\to(0,4)}}{2^{20}}$, as each path is equally likely and there are 2^{20} total paths that can be taken (2^{10} for each of the turtles).

A quick observation is that $N^*_{10,(0,4)\to(0,4)} = N_{10,(0,4)\to(0,4)} - N^c_{10,(0,4)\to(0,4)}$. This is really just the complementary rule. Computing $N_{10,(0,4)\to(0,4)}$ is not particularly difficult, and we will do so closer to the end. However, there is no clear way to compute $N^c_{10,(0,4)\to(0,4)}$ right now, so we should focus on that first.

If some path is part of $N_{10,(0,4)\to(0,4)}$, that means that Tort must have caught up to Bort at some point. In other words, the $X_n = Y_n$ line was crossed at some point. Therefore, the idea is to reflect the path up until that meeting point across the $X_n = Y_n$ line and then keep the rest of the path untouched. In particular, the point $(0, 4)$ gets reflected to $(4, 0)$. Therefore, there is a one-to-one correspondence between paths that cross the $X_n = Y_n$ line that are counted for $N^c_{10,(0,4)\to(0,4)}$ and unconstrained paths (since once we do the reflection, there are no issues with crossing) from $(4, 0)$ to $(0, 4)$. This implies that

$$N^c_{10,(0,4)\to(0,4)} = N_{10,(4,0)\to(0,4)}$$

Substituting this into our first expression, we get that

$$N^*_{10,(0,4)\to(0,4)} = N_{10,(0,4)\to(0,4)} - N_{10,(4,0)\to(0,4)}$$

All that is left is to calculate $N_{10,(a,b)\to(c,d)}$. For this, we just need to multiply together the number of paths from a to c in 10 steps (this corresponds to Tort) and number of paths from b to d in 10 steps (this corresponds to Bort).

To get from a to c in 10 steps, if we move right x times, we must move left $10 - x$ times. Therefore, our position would be $a + x - (10 - x)$. We need this to equal c, so we must solve for x in $a + x - (10 - x) = c$, which yields $x = \frac{10 + c - a}{2}$. This implies that $N_{10,(a,b)\to(c,d)} = \binom{10}{x_1}\binom{10}{x_2}$, where $x_1 = \frac{10 + c - a}{2}$ and $x_2 = \frac{10 + d - b}{2}$.

Therefore, $N_{10,(0,4)\to(0,4)} = \binom{10}{5}^2$ and $N_{10,(4,0)\to(0,4)} = \binom{10}{3}^2$. This implies our final probability is

$$\frac{\binom{10}{5}^2 - \binom{10}{3}^2}{2^{20}}$$

Extracting the values, $a = c = 10$, $b = 5$, $d = 3$, $p = 2$, and $r = 20$. This means $a^2 + b^2 + c^2 + d^2 + p^2 + r^2 = 638$.

Solution to Question 253: Flip Again

We know that if we get 2 or more heads then we keep our result. Otherwise, we flip all of them again for \$1. The probability we re-flip is just the probability we obtain 0 or 1 heads from our first flipping process. As the number of heads in the four flips is $X_1 \sim \text{Binom}\left(4, \frac{1}{2}\right)$ distributed, we are looking for $\mathbb{P}[X_1 = 0] + \mathbb{P}[X_1 = 1]$ as our probability to flip again. Plugging in, this comes out to $\frac{5}{16}$. The expected number of heads obtained is 2 as $\mathbb{E}[X_1] = 4 \cdot \frac{1}{2} = 2$.

Therefore, if G is the expected payout of our game, by Law of Total Expectation

$$\mathbb{E}[G] = \mathbb{E}[G \mid X_1 \leq 1]\mathbb{P}[X_1 \leq 1] + \mathbb{E}[G \mid X_1 > 1]\mathbb{P}[X_1 > 1]$$

As we know that $\mathbb{P}[X_1 \leq 1] = \frac{5}{16}$, $\mathbb{P}[X_1 > 1] = 1 - \frac{5}{16} = \frac{11}{16}$. Now, we need to find the expected payout in each case. If $X_1 \leq 1$, we re-flip, in which case the expected number of heads on the second flip is 2. However, note that we pay \$1 to flip again, so $\mathbb{E}[G \mid X_1 \leq 1] = 2 - 1 = 1$.

If $X_1 > 1$, then we know that $X_1 = 2, 3$, or 4. In this case, we are not flipping again, so we just keep the payout equal to the number of heads obtained. To find this, we need to find the conditional distribution of $X_1 \mid X_1 > 1$. Namely, by Bayes' Rule, we have that

$$\mathbb{P}[X_1 = 2 \mid X_1 > 1] = \frac{\mathbb{P}[X_1 = 2, X_1 > 1]}{\mathbb{P}[X_1 > 1]} = \frac{6}{11}$$

Similarly, we can show $\mathbb{P}[X_1 = 3 \mid X_1 > 1] = \frac{4}{11}$ and $\mathbb{P}[X_1 = 4 \mid X_1 > 1] = \frac{1}{11}$. Therefore, $\mathbb{E}[G \mid X_1 > 1] = 2 \cdot \frac{6}{11} + 3 \cdot \frac{4}{11} + 4 \cdot \frac{1}{11} = \frac{28}{11}$. With all of these values, we substitute back in and find $\mathbb{E}[G] = \frac{5}{16} \cdot 1 + \frac{28}{11} \cdot \frac{11}{16} = \frac{33}{16}$.

Solution to Question 254: Lily Pads II

Each lily pad must cover $\frac{20480}{10}$ square feet to cover the entirety of the pond. Since the size of each pad is 2^N after N days of growth, then:

$$2^N = \frac{20480}{10} \Rightarrow N = 11$$

Solution to Question 255: Prime Pair

The numbers on the sides would be $2, 3, 5, 7, 11$, and 13. Therefore, the sum of the two upfaces has to be between 4 and 26, inclusive. The prime integers in this interval are $5, 7, 11, 13, 17, 19$, and 23. We now need to determine how each prime can be obtained from these dice. One thing to note is that 2 must be one of the rolls, as all other values on the die are primes larger than 2, which must be odd. Therefore, the outcomes are just primes p such that $p-2$ is also a prime and $p-2 \leq 13$. The values of p where this holds true is $p = 5, 7$, and 13. Each of these have two permutations of the die outcomes that yield that sum. Therefore, 6 such outcomes of the $6^2 = 36$ yield a prime sum, so our answer is

$$\frac{6}{36} = \frac{1}{6}$$

Solution to Question 256: Make Your Martingale II

Using Ito's Formula on $f(t, w) = w^3 - ctw$, we have that

$$dX_t = f_t(t, w)dt + f_w(t, w)dW_t + \frac{1}{2}f_{ww}(t, w)dt$$

The last term comes from the quadratic variation of W_t. This is a martingale exactly when there is no dt term i.e. when $f_t(t, w) = -\frac{1}{2}f_{ww}(t, w)$. We can quickly calculate that $f_t(t, w) = -cw$ and that $f_{ww}(t, w) = 6w$. Therefore, $-cw = -\frac{1}{2} \cdot 6w = -3w$, which means $c = 3$.

Solution to Question 257: Geometrical Progression

We can guess and check starting from $n = 2$. Above, we showed that $n = 2$ gives one underneath a square. We can iterate this process checking numbers up until a certain point.

$121 = 3^0 + 3^1 + 3^2 + 3^3 + 3^4 = 11^2$

A fact: there are only two square numbers which are sum of consecutive powers of an integers. These are 121 and 400.

Solution to Question 258: Coefficient Sum

We can utilize the hockey stick identity several times here.

$$\begin{aligned}12\binom{3}{3}+11\binom{4}{3}+\cdots+\binom{14}{3} &= \left(\binom{3}{3}+\binom{4}{3}+\cdots+\binom{13}{3}\right)+\\ &\quad\left(\binom{3}{3}+\binom{4}{3}+\cdots+\binom{12}{3}\right)+\\ &\qquad\vdots\ +\\ &\quad\binom{3}{3}\\ &= \binom{15}{4}+\binom{14}{4}+\cdots+\binom{5}{4}+\binom{4}{4}.\end{aligned}$$

Applying the hockey stick identity once again, we find

$$\binom{15}{4}+\binom{14}{4}+\cdots+\binom{5}{4}+\binom{4}{4}=\binom{16}{5}=4368.$$

Solution to Question 259: Normal Activities

As X and Y are mean 0 normal random variables, $Y-4X$ is also a mean 0 normal random variable. Therefore, $\mathbb{P}[Y>4X]=\mathbb{P}[Y-4X>0]$ is asking the probability that a normal random variable is greater than its mean. This is $\frac{1}{2}$ by the symmetry of the normal distribution about its mean.

Solution to Question 260: Hatching Eggs I

Let X denote the number of eggs laid. $X \sim \text{Pois}(6)$. Let Y denote the number of egg hatches. Note that the value of Y depends on the number of eggs hatched, X. For example, if $X=x$, then, letting $E_1, E_2, \ldots, E_n \overset{\text{iid}}{\sim} \text{Bernoulli}(0.3)$ be indicator variables for each egg, we have:

$$Y=\sum_{i=1}^{x} E_i.$$

It follows from the linearity of expectation that

$$\mathbb{E}[Y|X=x]=0.3x$$

Our last step is to use the law of total expectation.

$$\begin{aligned}\mathbb{E}[Y] &= \mathbb{E}[\mathbb{E}[Y|X]] \\ &= \mathbb{E}[0.3X] \\ &= \frac{3}{10} \cdot 6 \\ &= \frac{9}{5}\end{aligned}$$

Solution to Question 261: Sum Exceedance II

We are going to use the result of Sum Exceedance I that states that for any $0 < x < 1$, if $f(x) = \mathbb{E}[N_x]$, with $N_x = \min\{n : X_1 + \cdots + X_n > x\}$, $f(x) = e^x$. We want to find $f(2)$. We should condition on X_1, as that will tell us how much further of a sum we need to exceed. Namely, $f(2) = \mathbb{E}[N_2] = \mathbb{E}[\mathbb{E}[N_2 \mid X_1]]$. Now, given X_1, we have $2 - X_1$ left to exceed starting from the next turn and we used 1 turn, so $\mathbb{E}[N_2 \mid X_1] = 1 + N_{2-X_1}$. Therefore, $\mathbb{E}[N_2] = 1 + \mathbb{E}[N_{2-X_1}]$ by plugging back into the above. In other words, $f(2) = 1 + f(2 - X_1)$. Now, we need to condition on $X_1 = x_1$ and integrate. Namely, this means that $f(2) = \displaystyle\int_0^1 (1 + f(2 - x_1))\, dx_1 = 1 + \int_0^1 f(2 - x_1) dx_1$ By the u-substitution $u = 2 - x_1$, we get that $f(2) = 1 + \displaystyle\int_1^2 f(u) du$.

More generally, for $1 < x < 2$, by replacing 2 with x, note, that the bounds of the integral would now become $x - 1$ to x in this case, so

$$f(x) = 1 + \int_{x-1}^{x} f(u) du$$

Taking the derivative, we have that $f'(x) = f(x) - f(x - 1)$. However, as $1 < x < 2$, $0 < x - 1 < 1$, in which we know f in the interval $(0, 1)$. Therefore, using the result stated at the beginning, $f'(x) = f(x) - e^{x-1}$, as $f(x) = e^x$ in $(0, 1)$. Rearranging this differential equation, we have $f'(x) - f(x) = -e^{x-1}$. Our initial condition is that $f(1) = e$, as this makes it continuous with the part of f on $(0, 1)$.

This here is a first order linear differential equation that can be solved by integrating factors. Namely, the integrating factor here is $\mu(t) = e^{\int (-1) dx} = e^{-x}$. Multiplying by this on both sides yields $e^{-x} f'(x) - e^{-x} f(x) = -e^{-1}$. The LHS is just $(e^{-x} f(x))'$. Therefore, integrating both sides, $e^{-x} f(x) = \displaystyle\int -e^{-1} dx = -xe^{-1} + C$. Multiplying by e^x on both sides, $f(x) = -xe^{x-1} + Ce^x$. We know that $f(1) = e$, so $e = -1 + Ce$, so $C = 1 + e^{-1}$.

Therefore, $f(x) = -xe^{x-1} + \left(1 + e^{-1}\right) e^x$. Plugging in $x = 2$ yields $f(2) = e^2 - e$. Thus, $a = 1$ and $b = -1$, so $a + b = 0$.

Solution to Question 262: 0DTE Option

We know that the Δ for OTM options near-expiry is close to 0 and so is the Γ. The charm represents the decay in Δ. The -0.04 represents that the delta is already very close to 0 as if nothing were to happen in the next hour, then the Δ would decrease by 0.04 to 0. This means that we can approximate the $\Delta = 0.04$ as we are dealing with an extremely short expiry.

Since the Γ is approximately 0, we are moving further away from the strike and so the option price should decrease by 0.04. This gives a price of 0.26.

Solution to Question 263: Optimal Card Pick

Among all full houses, selecting AAA confirms that no other full house will defeat our hand. Therefore, we now want to select our paired rank as to minimize the probability of losing. In this case, we want to minimize the probability of four of a kind or a straight flush. The probability of a four of a kind is the same regardless of which two kicker cards we select, so this doesn't give us any information. However, we do want to minimize the probability of a straight flush. This means we want to select the rank that is involved in the most number of straights.

There are fewer straight that involve Aces since we already selected 3 Aces. Therefore, any rank possessing a straight involving Aces is eliminated. We see that $2 - 5$ and $10 - K$ are all eliminated with this logic. The remaining ranks would be $6 - 9$, and one can check that there are equal amounts of straights involving each of these values and not Aces. Since no other full house can defeat our hand, we are indifferent to each of these 4 hands. This means we would pick any of $AAA66, AAA77, AAA88$, or $AAA99$ to be our hand. The value of each hand is, respectively, $54, 56, 58$, and 60, so the sum of these values is 228.

Solution to Question 264: Sum and Difference

For simplicity, we can restrict ourself to the case that $Y \geq X$ and then multiply by 2 in the end to account for the other case. When doing this, we can drop the absolute value. This makes our condition $X+Y \geq 2(Y-X)$, which is $Y \leq 3X$. However, we also know that $Y \geq X$. Plotting this in the plane, we can see that the region $\{X \leq Y \leq 3X\}$ in $[0,1]^2$ is a triangle that takes up all the space in the upper half except for the triangle with vertices $(0,0), (0,1)$, and $\left(\frac{1}{3}, 1\right)$. The area of this triangle is $\frac{1}{6}$ by the triangle area formula, so the area of our shaded region is $\frac{1}{3}$. Multiplying by 2 to account for the case when $Y \leq X$, our answer for the probability is $\frac{2}{3}$.

Solution to Question 265: Double Data Trouble IV

Recall that $R^2 = 1 - \frac{SSE}{SST}$. With each of the data points doubled, the new SSE is 4 times as large as it was before. This is because we are summing squared errors and we have doubled each of the points. Therefore, the errors (in magnitude) are doubled. Once we square these, the SSE would then be 4 times as large. For this same reason, SST also is quadrupled. Accordingly, the ratio $\frac{SSE}{SST}$ is unchanged, meaning that R^2 is unchanged from before.

Solution to Question 266: Dog Days

Let T be the number of days until another bad day and X_1 be the outcome of the first day since today. Then $\mathbb{E}_B[T] = \mathbb{E}_B[T \mid B]\mathbb{P}_B[B] + \mathbb{E}_B[T \mid G]\mathbb{P}_B[G]$. The subscript B is to denote that we start on a bad day. We have that $\mathbb{P}_B[B] = \frac{7}{10}$, so $\mathbb{P}_B[G] = \frac{3}{10}$. Then, $\mathbb{E}_B[T \mid G] = 1 + \mathbb{E}_G[T]$, as we take one step and then find the expected number of days starting from a good day until a bad day. Then, $\mathbb{E}_B[T \mid B] = 1$, as it takes just 1 day to get another bad day if the first day after is bad. Therefore,

$$\mathbb{E}_B[T] = (1 + \mathbb{E}_G[T]) \cdot \frac{3}{10} + 1 \cdot \frac{7}{10} = 1 + \frac{3}{10} \cdot \mathbb{E}_G[T]$$

Note that the distribution of the number of days until a bad day starting from a good day is Geom$\left(\frac{2}{5}\right)$, as there is a $\frac{2}{5}$ probability each day of going to a bad day, so $\mathbb{E}_G[T] = \frac{5}{2}$. This implies that $\mathbb{E}_B[T] = 1 + \frac{3}{10} \cdot \frac{5}{2} = \frac{7}{4}$.

Solution to Question 267: Balanced Beans IV

For this problem, we will first start off with the method used to solve this specific question and then show give a generalized form for any n beans.

As with any other "Balanced Beans" problem, we need to split the 90 beans into three groups of 30. The worst case scenario is that the balance is not balanced. Here is where it gets difficult: How do we go about checking the remaining 60 beans? A popular method is to subdivide each group of beans into groups of powers of 3.

With 30 beans in each group, we can divide them each up into a group of 27 beans and a group of 3 beans. Then, just like in "Balanced Beans II", we rotate the groups of 27 beans around. The original groups were 27_A, 3_A, 27_B, 3_B, 27_C, 3_C. After the rotation, we have 27_C, 3_A, 27_A, 3_B, 27_B, 3_C. If the orientation of the balance doesn't change, then we know that the abnormal bean is a part of either 3_A or 3_B. If the orientation did change, depending on how they changed, we can deduce which group of 27 has the abnormal bean. The worse case is that the bean is a part of the 27 group which from "Balanced Beans III", would take an extra 3 weighings. This makes our total 5 weighings.

You may be wondering why we split the groups of 30 into 27 and 3. This is because using the rotation method and having groups of 3^n allow us to not only identify whether the abnormal ball is heavier or lighter than the others but also be left with a group of 3^n which we have proved in "Balanced Beans III" to take n weighings. Lets take another example. Say we starting with 120 beans. We then split them into 3 groups of 40 and within the 40, we can split them into groups of $27, 9, 3$ and 1. No matter which group the abnormal bean is, it will take the same number of weighings as we start by rotating the largest groups and go onto the smallest. 27 will take one more weighing than 9 beans and thus the rotation of 9 beans comes after the rotation of 27.

The generalization of this strategy becomes ceiling($\log_3(2n+3)$) where n is the amount of beans we have.

Solution to Question 268: Sample Size for Z

As a rule of thumb, a sample size of at least 30 is appropriate. Furthermore, the difference in Z and T test results converge to 0 as the sample size approaches infinity.

Solution to Question 269: Terminating Sum

The prime factorization of 100 is $100 = 2^2 \cdot 5^2$. Therefore, whatever denominator we have must be in the form $k = 2^a 5^b$ for it to terminate in finite length. If we select a denominator that is not in this form, we end up with a decimal that will repeat forever, as it is not cleanly divisible into 10. Therefore, our sum can really be written as

$$\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\left(\frac{1}{2^a 5^b}\right)^3 = \sum_{a=0}^{\infty}\frac{1}{8^a}\sum_{b=0}^{\infty}\frac{1}{125^b} = \frac{1}{1-\frac{1}{8}} \cdot \frac{1}{1-\frac{1}{125}} = \frac{8}{7}\cdot\frac{125}{124} = \frac{250}{217}$$

Solution to Question 270: Shoe Manufacturing

We are testing the null hypothesis $H_0 : p = \frac{1}{3}$ against the alternative hypothesis $H_a : p > \frac{1}{3}$. The sample proportion $\hat{p}$ is $\frac{400}{1000} = 0.4$ for $n = 1000$. The Z statistic can be calculated as:

$$Z = \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} = \frac{0.40 - 0.33}{\sqrt{\frac{0.33(1-0.33)}{1000}}} \approx 4.47$$

Solution to Question 271: Uniform Equilibrium I

There can't be such a strategy. Suppose that player 1 picks $0 < a < 1$ and player 2 picks $0 < b < 1$. Without loss of generality, say $a < b$. Player 1 could then do better by selecting $a/2$ or Player 2 could do better by picking $\dfrac{1+b}{2}$. If they are at the endpoints, then player 2 can do better by choosing anything that isn't 1. Therefore, there is no such equilibria.

Solution to Question 272: Good Grid I

Note that $a \le 0$, as $\min\{0, b\} \le 0$. Furthermore, we also see that $b \ge 0$, as $\max\{0, a\} \ge 0$. Therefore, we just need to find the probability $a \le 0$ and $b \ge 0$. There are $21^2 = 441$ total ways a and b can be selected. Of those, we have 11 choices for each a and b, as $a \le 0$ and $b \ge 0$, so our probability is just $\dfrac{11^2}{21^2} = \dfrac{121}{441}$.

Solution to Question 273: Beer Barrel I

We can solve this problem by following these steps in order:

1. Perform 14 transactions of filling and emptying the 7-quart measure, wasting 98 quarts and leaving 22 quarts in the barrel.

2. Fill the 7-quart measure, then transfer 5 quarts to the 5-quart measure, leaving 2 quarts in the 7-quart.

3. Empty the 5-quart measure, then transfer 2 quarts from the 7-quart to the 5-quart.

4. Fill the 7-quart measure, then fill up the 5-quart from the 7-quart, leaving 4 quarts in the 7-quart.

5. Empty the 5-quart measure, then transfer 4 quarts from the 7-quart to the 5-quart.

6. Fill the 7-quart measure again, then fill up the 5-quart from the 7-quart, leaving 6 quarts in the 7-quart.

7. Empty the 5-quart measure, then fill the 5-quart from the 7-quart, leaving 1 quart in the 7-quart.

8. Empty the 5-quart measure, leaving 1 quart in the 7-quart.

9. Draw off the remaining 1 quart from the barrel into the 5-quart measure, completing the task in 14 more transactions, totaling 42 transactions with the initial 28.

Solution to Question 274: Picking Candy

Your optimal strategy is to put one good candy in one box, and the remainder of the candy in the other box. The probability that you receive a good candy is $\frac{1}{2} \times 1 + \frac{1}{2} \times \frac{4}{9} = \frac{13}{18}$.

Solution to Question 275: No Adjacent Evens

There are $4! \cdot 3! = 144$ ways to arrange around the digits on the blanks once we select the locations of where the even and odd integers are located. Therefore, we need to count the number of ways to arrange the even and odd integers. In particular, we can count directly how many spot options there are for the even integers. These are

$$135, 136, 137, 146, 147, 157, 246, 247, 257, 357$$

Therefore, there are 10 ways we can select the spots, so there are $144 \cdot 10 = 1440$ total ways to arrange the integers on the blanks.

Solution to Question 276: The Ten Cards

Let A denote the person who plays first, and B denote the person who plays second. Let O denote a card turned up, and X denote a card turned down. There are 3 ways A can win.

For the third card: A turns down the 3rd from either end. This leaves: $00X0000000$. Whatever happens next, A can always leave one of the following:

$$000X000$$
$$00X00X0X0$$
$$0X00X000$$

The order does not matter.

In the first case, A copies in one triplet what B does in the other triplet, until he gets the last card.

In the second case, A similarly copies B until he gets the last card.

In the third case, whatever B does, A can leave:

$$0X0$$
$$0X0X0X0$$
$$00X00$$

and again the win is apparent.

Second Card: A turns down the 2nd from either end.

This leaves:

$$0X00000000$$

Solution to Question 277: Multiple Likely Coin

We are going to solve this more generally for $k > 1$ times as likely. We are given that if there we select a coin at random and we receive 2 heads, then it is k times as likely that we picked Coin 2 than Coin 1. Thus, if we let C_2 denote the event we selected Coin 2, C_1 denote the event we selected Coin 1, and H denote the event we flipped 2 heads from the two flips, $\mathbb{P}[C_2 \mid H] = k\mathbb{P}[C_1 \mid H]$ Writing this with conditional probability formula on both sides, we get the equality

$$\frac{\mathbb{P}[C_2 H]}{\mathbb{P}[H]} = \frac{\mathbb{P}[C_1 H]}{\mathbb{P}[H]}$$

Cancellation of denominators gives $\mathbb{P}[C_2 H] = k\mathbb{P}[C_1 H]$. Rewriting with conditional probability again on both sides yields

$$\mathbb{P}[H \mid C_2]\mathbb{P}[C_2] = k\mathbb{P}[H \mid C_1]\mathbb{P}[C_1]$$

However, since the two coins are selected at random, $\mathbb{P}[C_1] = \mathbb{P}[C_2]$. Thus, we have $\mathbb{P}[H \mid C_2] = k\mathbb{P}[H \mid C_1]$. This is now is a very workable expression. Now that each flip is independent, so $\mathbb{P}[H \mid C_2]$ is just the probability of a heads on Coin 2 squared, which is just p_2^2. Similarly, since $p_1 + p_2 = 1$, $p_1 = 1 - p_2$. Thus, the probability of 2 heads on Coin 1 is given by $(1 - p_2)^2$. Therefore, we now have the equality $p_2^2 = k(1 - p_2)^2 \implies p_2^2 = kp_2^2 - 2kp_2 + k$. Rearranging to get 0 on a side, we get $(k-1)p_2^2 - 2kp_2 + k = 0$. This is a polynomial in p_2. We use the quadratic formula to solve:

$$p_2 = \frac{-(-2k) \pm \sqrt{(-2k)^2 - 4(k-1)(k)}}{2(k-1)} = \frac{2k \pm \sqrt{4k}}{2(k-1)}$$

Notice that we must use the $-$ branch of the square root here, as $2k > 2(k-1)$, so $\frac{2k}{2(k-1)} > 1$, and a probability can not be larger than 1. Thus, we use the $-$ branch as our valid branch to solve for p_2, and we get that $p_2 = \frac{k - \sqrt{k}}{k-1}$. Plugging in $k = 4$, we obtain $p_2 = \frac{2}{3}$.

Solution to Question 278: Pairwise Digit Sums II

We will start with the largest number. Start in the left-most position with 9. Then, the second digit from left should be 8. We can't have replicate digits, as otherwise, we could pair any other digit in the number with either one of the two repeated digits and get the same sum. Then, the next digit is 7, as we still have all unique digit sums. Therefore, our number is in the form $987ab$, with a and b still to be determined. a can't be 6, as $6 + 9 = 7 + 8$. However, with $a = 5$, we still maintain all unique digit sums. Thus, our form is $9875b$. If $b = 4$, then $4 + 9 = 8 + 5$. If $b = 3$, then $3 + 9 = 5 + 7$. However, with $b = 2$, we get all unique digit sums. Therefore, $y = 98752$.

For the smallest, the smallest first digit is 1. Then, the smallest second digit is 0. The smallest third digit is 2. Thus, our number is in the form $102cd$. If

$c = 3$, $1+2 = 3+0$, but for $c = 4$, we still maintain unique sums. Therefore, our form is $1024d$. If $d = 5$, then $5+0 = 4+1$. If $d = 6$, then $6+0 = 4+2$. However, for $d = 7$, we maintain unique sums, so $x = 10247$. Therefore, $y - x = 88505$.

Solution to Question 279: Make Your Martingale V

We can quickly compute $dY_t = tdW_t$ quite simply. First, let $f(t, y) = ty$. We can see that $f_t(t, y) = y$, $f_y(t, y) = t$, and $f_{yy}(t, y) = 0$. Therefore, by Ito's Formula, $dZ_t = Y_t dt + t^2 dW_t$. Therefore, if $a_t = \int_0^t Y_s ds$, Z_t would be a martingale. This is because $da_t = Y_t dt$, so it would cancel out the $Y_t dt$ that currently exists in dZ_t. Therefore, this means $k = 1$.

Solution to Question 280: Longest Rope II

Let X be the distance from the LHS that the rope is cut. Then we know that $X \sim \text{Unif}(0, 1)$ and $L = \max\{X, 1 - X\}$ is the length of the longer piece, as the division at point X yields two pieces of length X and $1 - X$. We know that $L \geq 0.5$, as at least one of the two pieces must be longer than $1/2$ in length. However, we see that the event $\{L = c\}$ corresponds to $X = c$ or $X = 1 - c$, so we get that $L \sim \text{Unif}(0.5, 1)$, meaning that $\text{Var}(L) = \frac{(0.5)^2}{12} = \frac{1}{48}$.

Solution to Question 281: Straddle Arbitrage I

We can replicate a straddle at strike K by going long 1 unit of the call and 1 unit of the put. So, we should have the following equality:

$$V_0 = P_0 + C_0$$

When plugging in the prices, we see this equality doesn't hold. We have $5.4 \stackrel{?}{=} 4.2 + 1.4 = 5.6$

To take the arbitrage, we long the undervalued item and short the overvalued item. Here, we long the straddle and short the vanilla call and put. The bond and underlying stock are irrelevant.

Stock + # Call + # Put + # Bonds + # Straddle $= 0 - 1 - 1 + 0 + 1 = -1$

Solution to Question 282: Poisson Review V

Since there are an average of 4 imperfections per square yard, there are on average 40 imperfections per 10 square yards. Let $X \sim \text{Poisson}(40)$. We wish to compute $\mathbb{E}[10X]+\text{Var}(10X) = 10\mathbb{E}[X]+100\text{Var}(X)$. Plugging in 40 for both $\mathbb{E}[X]$ and $\text{Var}(X)$, we find our answer to be 4400.

Solution to Question 283: Sock Drawer I

Let there be b blue socks and g green socks. Then the probability of the first sock being blue is $\frac{b}{b+g}$; and if the first sock is blue, the probability of the second sock being blue is $\frac{b-1}{b+g-1}$. The probability that both are blue is given to be $\frac{1}{2}$, or $\frac{b}{b+g} \cdot \frac{b-1}{b+g-1} = \frac{1}{2}$

One could start with $g = 1$ and try successive values of r, then go to $g = 2$, and so on, to find that $g = 1$ and $b = 3$ satisfy the equation. Thus, there are a total of 4 socks at minimum.

Solution to Question 284: Thank You, Quant!

We first use Poisson thinning to note that the inflow of clients sent a thank you card from each firm is also a Poisson process. For the two respective firms, the rates are $6 \cdot \dfrac{1}{6} = 1$ and $10 \cdot \dfrac{1}{5} = 2$. Therefore, the time between clients being sent thank you cards for each firm follow IID Exp(1) and Exp(2) distributions, respectively. The time between any clients being sent thank you cards is $\min\{X_1, X_2\}$, where $X_i \sim \text{Exp}(i)$, as whichever firm has a shorter interarrival time of clients being sent thank you cards will represent the time between clients (from either firm) being sent cards.

Then, we can note that for any $t > 0$, from the known facts about the exponential random variable,

$$\mathbb{P}[\min\{X_1, X_2\} > t] = \mathbb{P}[X_1 > t]\mathbb{P}[X_2 > t] = e^{-t} \cdot e^{-2t} = e^{-3t}$$

Therefore, we have that $\min\{X_1, X_2\} \sim \text{Exp}(3)$. In particular, this means the mean of this is $\dfrac{1}{3}$, which is our answer.

Solution to Question 285: Fixed Point Limit I

The events that each of the values is a fixed point are independent, as we can choose a function f from the set of all functions, so as there are n possible values in S_n and each has probability $1/n$ of being fixed (exactly one of the n values would make it fixed), we have that $F_n \sim \text{Binom}(n, 1/n)$. By Poisson Limit Theorem, we have that $F_n \implies \text{Poisson}(1)$, so $\mathbb{P}[F_n = 5] = \frac{1}{5!}e^{-5}$, so our answer is $1/120$.

Solution to Question 286: Movie Arrivals

The easiest approach here is to note that we should condition on $N = n$. We can see that $N \sim \text{Geom}\left(\frac{1}{3}\right)$, as we are counting the number of people that arrive until a woman does, so $\mathbb{P}[N = n] = \frac{2^{n-1}}{3^n}$ by plugging into the PMF. Then, given $N = n$, all of the first n people must arrive more than 1 minute apart from one another. The interarrival times of the Poisson process are Exp(1) distributed, so the probability any given interarrival time is at least 1 is e^{-1}. Therefore, for this to occur for all n people, our probability is e^{-n}. By law of total probability, the probability of our event is

$$\sum_{n=1}^{\infty} \frac{2^{n-1}}{(3e)^n} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{2}{3e}\right)^n = \frac{1}{2}\cdot\frac{\frac{2}{3e}}{1-\frac{2}{3e}} = \frac{1}{3e-2}$$

Thus, $abc = 1 \cdot 2 \cdot 3 = 6$.

Solution to Question 287: Deviation Probability

Given that $\sigma^2 = 4$, we know the standard deviation (σ) is equal to 2. Thus, we are trying to find the area under the normal probability distribution that is greater than 2 standard deviations away from the mean.

We can use a cumulative distribution calculator to see that $\mathbb{P}(X < 54) = \Phi(2) \approx 0.977$.

Thus $\mathbb{P}(X > 54) = 1 - \Phi(2) \approx 0.023$.

Solution to Question 288: Greedy Pirates

In the 2-pirate case, the most senior pirate, denoted pirate 2, will distribute all of the gold to himself since he will always get at least 50

Solution to Question 289: Exponential Maximum Asymptotic

Let us compute this probability directly. $\mathbb{P}[M_n \le \ln(n)] = \mathbb{P}[\max\{X_1, \dots, X_n\} \le \ln(n)]$ by definition. Then, the maximum being $\le \ln(n)$ means each individual random variable is at most $\ln(n)$. This means the above is just $\mathbb{P}[X_1 \le \ln(n), \dots, X_n \le \ln(n)]$. By independence, we can split this up to obtain $\mathbb{P}[X_1 \le \ln(n)] \dots \mathbb{P}[X_n \le \ln(n)]$. Since all of the random variables are identically distributed, we can just take one of them and raise it to the nth power, so this is $\mathbb{P}[X_1 \le \ln(n)]^n$.

$\mathbb{P}[X_1 \le \ln(n)] = 1 - e^{-\ln(n)} = 1 - \frac{1}{n}$ by the CDF of the Exp(1) distribution. Therefore, $\mathbb{P}[M_n \le \ln(n)] = \left(1 - \frac{1}{n}\right)^n$. Recall from Calculus that for any real x, $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$. This is just the exponential limit with $x = -1$, so our limit is $e^{-1} = \frac{1}{e}$. Therefore, $b = 1$.

Solution to Question 290: Segment Traversal

In total, there are n line segments when the process is done. This means that there are $\binom{n}{2}$ pairs of line segments that could possibly intersect. However, not all of these have equal probabilities of intersecting. In particular, each of the n points selected represents the intersection of two line segments on the circumference of the circle, meaning that those two segments can't intersect inside the circle. Therefore, we must subtract n possible pairs of line segments that could intersect, as those n pairs can't intersect each other inside the circle. This yields $\binom{n}{2} - n$ possible pairs of line segments that can intersection. By linearity of expectation and the exchangeability of the remaining pairs, we just need to now find the probability of intersection between any two line segments to find the total number of expected intersections.

Our question can be formalized as the following: Suppose that we have X_1, X_2, X_3, and X_4 selected uniformly at random from the circumference of a circle. Draw a chord between X_1 and X_2 and a chord between X_3 and X_4. Find the probability that the chords intersect. Since the circle is rotationally invariant, we can just lock in X_1 at $(1, 0)$. Then, we can consider all possible orderings of X_2, X_3, and X_4 in terms of distance (radians CCW) from the origin. By drawing out the different permutations, you can see that there is an intersection between the segments if X_2 is between X_3 and X_4 (or vice versa). This corresponds to 2 of the $3! = 6$ possible orderings of the random variables, so the probability is 1/3.

The above yields that the expected number of intersections is $\frac{\binom{n}{2} - n}{3} = \frac{n(n-3)}{6}$. In particular, $n = 12$ here yields 18 intersections on average.

Solution to Question 291: Expecting HTH

Let x be the expected value of the number of tosses to see HTH at the start of the game. There is a $\frac{1}{2}$ probability the toss is a T and you start over with a new expected value of $x + 1$. There is a $\frac{1}{4}$ probability you get HH and then start over with a new expected value of x. There is a $\frac{1}{8}$ probability you get an HTT and then start over with a new expected value of $x+3$. There is a $\frac{1}{8}$ probability that you get HTH and end the game. Thus:

$$x = \frac{1}{2}(x + 1) + \frac{1}{4}(x) + \frac{1}{8}(x + 3) + \frac{1}{8}(3)$$

Solving for x, we find that $x = 10$.

Solution to Question 292: Mental Black Scholes

We can use the approximation to price both puts and calls when interest rates are low and the option is at-the-money (put-call parity).

$$c(t) \approx \sigma S \sqrt{\frac{T - t}{2\pi}}$$

We can use the approximation of $\frac{1}{\sqrt{2\pi}} \approx 0.4$

This then gives us $c(t) \approx .4 \cdot 100 \cdot 0.4 \cdot \sqrt{0.25} = 8$

Note, we need to divide implied volatility by 100 as in Black-Scholes, it's reported as a percentage.

Solution to Question 293: Zero Volatility Returns

With $\sigma = 0$, $\frac{dS_t}{S_t}$ is now non-random. Namely, it is just μdt. Therefore, the ratio here grows at a constant rate $\mu = 1$ that is non-random. Therefore, by approaching this as a standard differential equation and integrating on $(0, 2)$, $\ln(S_2/S_0) = 2$, so $S_2/S_0 = e^2$, meaning $a = 2$.

Solution to Question 294: Coin Toss Game I

Its worth noting that it doesn't matter what each player flips in their first two flips. They'll each have a $\frac{1}{2}$ chance to get the other flip they need after each of their first flips and assuming they have the coin. Thus, starting on the third flip, Alice will have a $\frac{1}{2}$ chance to win the game right there. If she doesn't get the other flip she needed on the third flip, Alice needs Bob to flip what he flipped previously. If that happens, we start back at a state similar to the third flip. Given this, we can model this situation as a singular state equation. Say P is the probability that Alice wins in the state of the third flip (both Alice and Bob need one specific roll). Then Alice will win half the time and a fourth of the time we repeat the same state. So

$$P = \frac{1}{2} + \frac{1}{4} \cdot P$$

Thus $P = \frac{2}{3}$ which is the probability Alice wins.

Solution to Question 295: The Last Airbender

The earth card is absent from our sample space and thus does not affect your probability of winning. Imagine ordering the remaining three cards in the order that you flip them- there are a total of 3! possible permutations. Of these permutations, favorable outcomes are those where the first two cards are air then water or water then air. Thus, the probability that you win is:

$$\frac{2}{3!} = \frac{1}{3}$$

Solution to Question 296: Observing Cars

The hour interval can be broken down into two disjoint thirty-minute interval such that the probability p of observing any accident is uniform. The probability that we do not observe an accident within a thirty-minute interval is $1 - p$. Furthermore, the probability that we do not observe any accidents within two independent thirty-minute intervals (or an hour) can be written as:

$$(1-p)^2 = 1 - \frac{3}{4} \Rightarrow p = 1/2$$

Solution to Question 297: Writing to Recruiters

This is a classic indicator variable problem. Let E_k be the indicator variable for each k letter where it is 1 if the letter goes to the right firm and 0 if it doesnât. Then $T = \sum_{i=1}^{N} E_i$ gives the total number of letters that go to the correct firm. By linearity of expectation and the fact that the random variables are exchangeable, $\mathbb{E}[T] = N\mathbb{E}[E_1]$. The probability for each of these letters to go to their correct firm is $\frac{1}{N}$. Thus,

$$\mathbb{E}[T] = N \cdot \frac{1}{N} = 1$$

You can double check this is the case for small values N, such as 2 and 3.

Solution to Question 298: Chess Tournament III

For ease in explanation, let x_1 denote the highest-rated player, and let x_2 denote the second-highest rated player. In order for x_1 and x_2 to meet in the final round, x_1 and x_2 must be in two different sub-brackets. This occurs with probability $\frac{2^{n-1}}{2^n-1}$. Then, x_1 must win all $n-1$ of their sub-bracket games, which occurs with probability p^{n-1}. Additionally, x_2 must win all $n-1$ of their sub-bracket games, which similarly occurs with probability p^{n-1}. Finally, x_2 must triumph over x_1 in the final round, which occurs with probability $1-p$. Putting it all together, we find

$$\mathbb{P}(x_2 \text{ beats } x_1 \text{ in final}) = \frac{2^{n-1} \cdot p^{2n-2} \cdot (1-p)}{2^n - 1}$$

Plugging in the desired values for n and p, we conclude

$$\begin{aligned}\mathbb{P}(x_2 \text{ beats } x_1 \text{ in final}) &= \frac{8 \cdot \left(\frac{3}{4}\right)^6 \cdot \frac{1}{4}}{15} \\ &= \frac{2}{15} \cdot \left(\frac{3}{4}\right)^6 = \frac{243}{10240}\end{aligned}$$

Solution to Question 299: Egg Drop II

In Egg Drop I, the strategy was to keep the worst case constant by scaling how high we jump at each interval based on how far up it is in our tower. In other words, we make larger jumps at lower floors in the tower so that we can test every floor afterwards in a given range. Namely, with at most d drops and 2 eggs, we could test $\frac{d(d+1)}{2}$ floors completely. The key here is to note that when we drop one egg and it breaks, we really get reduced down to the two egg case. Therefore, instead of making jumps that decrease linearly (which is what we do in the two egg case), we are going to make jumps that decrease by the maximum increment of the two egg case. This is so that we are sufficiently able to search everything in a given range in the two egg case afterwards.

Putting this into math, we want to find a first cutoff point such that if the egg breaks at floor X on the first drop, we can examine the floors below it in $9-1=8$ trials. We can examine up to $\frac{8\cdot 9}{2}=36$ floors in 8 trials by Egg Drop I, so we should drop our first egg at floor 37. If it doesn't break, then we need to increment such that if it breaks on our next trial, we can examine the remaining space between 37 and the next floor in $9-2=7$ drops. We can explore up to $\frac{8\cdot 7}{2}=28$ floors with 7 drops, so we should place our next egg at floor 66 such that we can explore floors $38-65$ (28 floors) in the remaining 7 drops.

Continuing this pattern, when we have k drops left, we are going to increment by $\frac{k(k+1)}{2}+1$ floors. This means that the next egg after would be placed at floor $66+21+1=88$, the egg after at $88+15+1=104$, and so on. The remaining floors to drop eggs at would be $115, 122, 126$, and 128. If it breaks at 128 but not at 126, we try 127 to verify the answer. If it doesn't break at 128, we try 129 instead and see if it breaks. If it breaks, we found our answer. If it doesn't break, then if $n=130$, we would know that the egg breaks at floor 130, as it must break on one of the floors in our building. We can't explore any higher than 130 due to the fact that if it breaks at some point above 130, we would have to search between 129 and the breaking point, which takes strictly more than one egg drop. Therefore, $n=130$ is our answer.

Solution to Question 300: Bowl of Cherries III

We suppose the cherry transfer has already happened. Let R denote the event that the randomly picked cherry from bowl B is red. Let A denote the event that the randomly picked cherry originated from bowl A. Then, A^c denotes the event that the randomly picked cherry originated from bowl B.

We are given the following from the problem statement:

$$\mathbb{P}(G|A)=\frac{1}{4}$$
$$\mathbb{P}(G|A^c)=\frac{3}{4}$$
$$\mathbb{P}(A)=\frac{1}{3}$$
$$\mathbb{P}(A^c)=\frac{2}{3}$$

By the Law of Total Probability (since A, A^c together form a partition of Ω),

$$\begin{aligned}
\mathbb{P}(G) &= \mathbb{P}(G \cap A) + \mathbb{P}(G \cap A^c) \\
&= \mathbb{P}(G|A)\mathbb{P}(A) + \mathbb{P}(G|A^c)\mathbb{P}(A^c) \\
&= \frac{1}{4} \cdot \frac{1}{3} + \frac{3}{4}\frac{2}{3} \\
&= \frac{7}{12}
\end{aligned}$$

Solution to Question 301: Put Option Price Estimate

To give the best estimate, we need to find the best lower bound. We can do this by building a step function that subreplicates the put option for all values of S_T. Mathematically, the put option can be written as $\max(15 - S_T, 0)$. We can bound this on the lower side with a binary put of $K = 12.5$. However, we need to have 2.5 units. The binary put and vanilla put will have the same payout at $S_T = 12.5$.

We can continue this process. We can then long 2.5 units of the $K = 10$ strike binary put. Now, the binary put and vanilla put will agree in payoffs at $S_T = 12.5$ and $S_T = 10$. Repeating this, we can see that we need to long 2.5 units of every binary put with strike $K < 15$. This gives us the tightest lower bound of:

$$L_0 = 2.5(0.64 + 0.24 + 0.31 + 0.20) = 3.925$$

This is the best lower-bound estimate of a vanilla put.

Solution to Question 302: Median Roll

By the symmetry of the dice, the median value being 3 and 4 are equally likely. Similarly, 2 and 5 are equally likely, as are 1 and 6. As the mean value of each median pair is $\frac{7}{2}$, this implies that the median is also $\frac{7}{2}$.

Solution to Question 303: You Got Mail

The probability that the mail makes it to Mailbox 1 is $\frac{1}{3}$. There is also a $\frac{2}{3}$ chance the mail isn't read given it is in Mailbox 1 by the question. Therefore, the probability of the mail being in Mailbox 1 and not noticing it is $\frac{2}{9}$.

The two ways you didn't read the mail is that it ended up in one of the other mailboxes or it ended up in Mailbox 1 but you didn't notice it. We calculated the latter already, and the former is just $\frac{2}{3}$, so the answer is $\frac{\frac{2}{9}}{\frac{2}{9}+\frac{2}{3}} = \frac{1}{4}$

Solution to Question 304: Bowl of Cherries I

The only way for Amy to have no cherries left is if the last cherry drawn from the bowl is red. We're therefore left with a simple ordering problem: how many ways can Amy order m indistinguishable red cherries and n indistinguishable purple cherries such that the last cherry is red?

We take one red cherry and force it to be drawn last. Then, we have $m-1$ red cherries and n purple cherries to arrange however we would like. Correcting for overcounting (since red cherries are indistinguishable from each other and purple cherries are indistinguishable from each other), we find that there are

$$\frac{(m+n-1)!}{(m-1)!\,n!}$$

possible orderings.

There are a total of

$$\frac{(m+n)!}{m!\,n!}$$

possible orderings of the m red and n purple cherries without the restriction.

Therefore, the probability that the bowl is empty (the last cherry draw from the bowl is red) is

$$\frac{\frac{(m+n-1)!}{(m-1)!\,n!}}{\frac{(m+n)!}{m!\,n!}} = \frac{m}{m+n} = \frac{5}{13}$$

Solution to Question 305: Bear and Bull Market

Consider pairs of consecutive days. The first day, the price drops to $\frac{99}{100}$. Then, the price rises to $\frac{101}{100}$. Therefore, every 2 days, the price is multiplied by a factor of

$$\frac{99}{100} \cdot \frac{101}{100} = \frac{9999}{10000} < 1$$

Thus, in the long run, the price converges to 0, as it is exponentially decreasing.

Solution to Question 306: Counting Up

When Jay guesses 2, Kay can only pick the values 3 to 10. Regardless of what Kay says, Jay should then select 11. In this pattern, Jay will select all values in the form $9k + 2$ for k an integer. The largest value in this form less than 200 is 191. Once Jay says 191, Kay must pick a value 192 to 199. Regardless of what Kay says, the value will be within 8 of 200, so Jay can say 200 on the next turn. The reason to think of this is because 200 is also in the form $9k + 2$.

Solution to Question 307: Simulation Scheme

Let N be the number of coin flips needed to simulate the event. With probability $\frac{3}{4}$, we determine whether or not the event happened on the first trial. If we obtain TT, which occurs with probability $\frac{1}{4}$, then we restart our trial process, but the number of flips we have done increases by 2. Therefore, with probability $\frac{3}{4}$, it takes 2 flips to simulate, and with probability $\frac{1}{4}$, it takes $2 + \mathbb{E}[N]$ flips to simulate, as we have performed 2 flips and we end up exactly where we started with the next trial. Thus, we have that $\mathbb{E}[N] = \frac{3}{4} \cdot 2 + \frac{1}{4}(2 + \mathbb{E}[N])$. Solving for $\mathbb{E}[N]$ yields $\mathbb{E}[N] = \frac{8}{3}$

Solution to Question 308: Poor Odds

In the case that Angelina loses all her money after winning exactly once, Angelina must have played exactly 11 games. Moreover, Angelina must have won within the first three rounds; there are three possible ways for Angelina to satisfy this condition. Hence, our answer is $(0.1)(0.9)^{10} \cdot 3 \approx 0.105$.

Solution to Question 309: Colosseum Fight I

It turns out that all strategies are equally good. Suppose that each gladiator is given x balls if his strength is x. Then, we vertically stack all of the balls into into some random order. When two gladiators fight, the winner is going to be the one whose ball is highest vertically in the stack between the two. If the two strengths are x and y, the probability of x having the highest ball in the stack between the two is $\frac{x}{x+y}$. Afterwards, the winner obtains the balls corresponding to the losing gladiator. The balls in the updated stack are still uniformly distributed, so we get independence between trials. However, Bob wins the tournament precisely when the ball at the top of the stack belongs to him. As the sum of all of Alice's power is 10 and Bob's power is 30, this occurs with probability $\frac{3}{4}$. Note that this result is independent of the strategy, as we are never fixing a way for the gladiators to fight against one another.

Solution to Question 310: Restack Rings

Try to do this with smaller n. If there's only one ring, it only takes 1 move. With two rings, it takes 3 moves. With three rings, it takes 7 moves. You will see, and can formally prove, that it will take $2^n - 1$ moves to recreate the tower for any n. Plugging in $n = 10$, we arrive at the answer 1023.

Solution to Question 311: Lily Pads I

The 8 lily pads can be thought of as a single lily pad that is three days old, since $2^3 = 8$. Thus, it will take $10 - 3 = 7$ days for the surface of the pond to be covered.

Solution to Question 312: Say 50!

Notice here that if you say 39, its functionally checkmate because you can always compliment whatever your opponent adds to make 50. Lets say they say 1 and bring the total to 40. Then you can say 10 and win. No matter what your opponent says, you can always compliment them to add 11 between both of your turns. Now we can work backwards. If we want to say 39, saying 28 is again, functionally checkmate because you can compliment whatever your friend says to make 39. Go back another 11 and we get 17. One more time and we get 6. Thus you should say the number 6 first and then compliment your friend to keep adding 11 to this 6 till it becomes 50.

Solution to Question 313: Basic Dice Game III

Your payoff is dependent on the first roll, so we can use the Law of Total Expectation. Let x be the fair value of this game. There is a $\frac{1}{2}$ probability that you roll a 1, 2, or 3 (on average 2). The other $\frac{1}{2}$ of the time, you roll a 4, 5, or 6 (on average 5), add this to your total, and essentially restart the game with an expected value of x. Thus, we can write:

$$x = \frac{1}{2} \times (2) + \frac{1}{2} \times (x + 5)$$

$$x = 7$$

Solution to Question 314: Coin Flipping Competition III

We know that $T, G, P \sim \text{Geom}\left(\frac{1}{2}\right)$ IID, as they are looking for the distribution of the first heads. As these are independent, we can multiply the individual PMFs to get the joint PMF, so the joint PMF is $\mathbb{P}[T = t, G = g, P = p] = \left(\frac{1}{2}\right)^t \left(\frac{1}{2}\right)^g \left(\frac{1}{2}\right)^p$ for $t, g, p = 1, 2, \ldots$.

We now need to get a region of summation for this probability. Let's let t be free, so we sum t from 1 to ∞. Then, we know $G > T$, so we sum over $g = t+1$ to ∞. After that, we know $P > G$, so we sum inner most from $p = g + 1$ to ∞. Therefore, our sum is $\sum_{t=1}^{\infty} \sum_{g=t+1}^{\infty} \sum_{p=g+1}^{\infty} \left(\frac{1}{2}\right)^t \left(\frac{1}{2}\right)^g \left(\frac{1}{2}\right)^p$. As the inner most summation only concerns p, we ignore the rest for now. $\sum_{p=g+1}^{\infty} \frac{1}{2^p} = \frac{\frac{1}{2^{g+1}}}{1-\frac{1}{2}} = \frac{1}{2^g}$. Now, our summation is $\sum_{t=1}^{\infty} \sum_{g=t+1}^{\infty} \frac{1}{2^t} \cdot \frac{1}{2^{2g}} = \sum_{t=1}^{\infty} \sum_{g=t+1}^{\infty} \frac{1}{2^t} \cdot \frac{1}{4^g}$. Ignoring the first term, as our sum only concerns g, $\sum_{g=t+1}^{\infty} \frac{1}{4^g} = \frac{\frac{1}{4^{t+1}}}{1-\frac{1}{4}} = \frac{4}{3} \cdot \frac{1}{4^{t+1}}$.

Now, our final summation is $\frac{1}{3} \sum_{t=1}^{\infty} \frac{1}{2^t} \cdot \frac{1}{4^t} = \frac{1}{3} \sum_{t=1}^{\infty} \frac{1}{8^t}$ after shoving the constants to the front. The last sum is simply $\frac{\frac{1}{8}}{1-\frac{1}{8}} = \frac{1}{7}$, so our solution is $\frac{1}{21}$.

Solution to Question 315: Power Grid

Consider the light bulbs in a matrix arrangement as follows:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

There are clearly $2^9 = 512$ equally-likely outcomes of the light bulbs being activated or not. Now, we need to count the outcomes that satisfy our event.

The key here is to consider light bulb 5. If it is powered on, then none of $2, 4, 6$, and 8 can be powered on. However, $1, 3, 7$, and 9 are free to be on or off. Therefore, there are $2^4 = 16$ outcomes in this case. The other case is more complicated.

Suppose that light bulb 5 is off now. We now have 4 sub-cases to consider corresponding to the different combinations of when light bulbs 4 and 6 are powered or not.

Case 1 - Both On: In this case, $1, 3, 7$, and 9 all must be off. However, 2 and 8 are free to be on or off, so there are $2^2 = 4$ outcomes in this case.

Case 2 - One On: To simplify, we will consider when 4 is on and 6 is off. Then, we can just multiply by 2 to account for the other case. Since 4 is on, 1 and 7 must be off. For 2 and 3, either exactly one is on or neither is on. This accounts for 3 cases. This holds similarly for light bulbs 8 and 9, so there are $2 \cdot 3 \cdot 3 = 18$ combinations in this case.

Case 3 - Both Off: If both 4 and 6 are off, then light bulbs $1-3$ can either be all off (1 case), have exactly one on (3 cases), or have 1 and 3 on with 2 off (1 case). This yields 5 total cases. This similarly holds for lightbulbs $7-9$, so there are $5^2 = 25$ combinations for this case.

Adding all of these up, we get that there are $16 + 4 + 18 + 25 = 63$ total combinations that are favorable, so our answer is $\dfrac{63}{512}$.

Solution to Question 316: Standing Table

Lets fix the first leg at a random spot on the circumference. The second leg is expected to be $\frac{1}{4}^{\text{th}}$ of the circumference away from the initial leg (draw out the circle, you will see that the average spot for the second leg is half way between where the first leg is and the opposite point of the first leg). Finally, for the table to stand up, the center of the table (center of mass) needs to be within the triangle that is formed by connecting the three legs. This leaves us with $\frac{1}{4}^{\text{th}}$ of the circumference to place the final leg which allows for the table to stand. Thus the answer is $\frac{1}{4}$.

Solution to Question 317: Largest Ball

We can actually solve this problem for a more generalized form. Let k be the largest ball when we choose n balls from a total of N in the urn. Then we have to choose $n-1$ balls from $k-1$ balls. Going through all possible values of k (n to N), and using linearity of expectation, we get the following equation:

$$\mathbb{E} = \frac{\sum_{k=n}^{N} k \binom{k-1}{n-1}}{\binom{N}{n}} = \frac{\sum_{k=n}^{N} n \binom{k}{n}}{\binom{N}{n}} = \frac{n}{\binom{N}{n}} \left[\binom{N}{n} + \binom{n+1}{n} + \cdots + \binom{N}{n} \right]$$

The term in the parentheses is just $\binom{N+1}{n+1}$ by the hockey stick identity. Therefore, our answer is

$$\frac{n}{\binom{N}{n}} \binom{N+1}{n+1} = \frac{n(N+1)}{(n+1)}$$

Plugging in 10 for n and 20 for N, we get a final answer of $\frac{210}{11}$.

Solution to Question 318: Doubling Bacterium

The population doubles every 10 minutes, so it would double 4 times from 55 to 95 minutes. Therefore, there would be $200 \cdot 2^4 = 3200$ bacteria after 95 minutes.

Solution to Question 319: Rolls in a Row

We can use Markov Chains for this problem. Let E_0 be the expectation state of having no 6s in a row. Let E_1 be the expectation state of having one 6 in a row (so 6 is the most recent roll). Finally, let E_2 be the expectation state of having two 6s in a row (our goal). Then our equations become:

$$E_0 = \frac{1}{6}(E_1 + 1) + \frac{5}{6}(E_0 + 1) = \frac{1}{6}E_1 + \frac{5}{6}E_0 + 1$$
$$E_1 = \frac{1}{6}(E_2 + 1) + \frac{5}{6}(E_0 + 1) = \frac{1}{6}E_2 + \frac{5}{6}E_0 + 1$$
$$E_2 = 0$$

Solving these equations, we get $E_0 = 42$. Thus it takes 42 rolls on average to get two 6s in a row.

Solution to Question 320: Fibonacci Limit II

We can write $\dfrac{F_{n+2}}{F_n} = \dfrac{F_{n+1}}{F_n} + 1$. Using the result of Fibonacci Limit I, the first term converges to the golden ratio $\dfrac{1+\sqrt{5}}{2}$, so the limit here is

$$\frac{3+\sqrt{5}}{2}$$

The answer is $3 \cdot 5 \cdot 2 = 30$.

Solution to Question 321: Sum Leak I

We are going to prove a more general version for $\mathbb{E}\left[\dfrac{S_m}{S_n}\right]$ with $m \leq n$. We have that

$$\mathbb{E}\left[\frac{S_m}{S_n}\right] = \mathbb{E}\left[\frac{X_1 + \cdots + X_m}{X_1 + \cdots + X_n}\right] = \sum_{i=1}^{m} \mathbb{E}\left[\frac{X_i}{X_1 + \ldots X_n}\right]$$

As the X_i's are IID, they are exchangeable, so $\mathbb{E}\left[\dfrac{X_1}{X_1 + \cdots + X_n}\right] = \cdots = \mathbb{E}\left[\dfrac{X_n}{X_1 + \cdots + X_n}\right]$. However, the sum of all of the expectations above must be 1, as they would add to $\dfrac{S_n}{S_n} = 1$. Thus, each one is $\dfrac{1}{n}$. As there are m of these terms in the sum, we get $\mathbb{E}\left[\dfrac{S_m}{S_n}\right] = \dfrac{m}{n}$. Our specific case is $m = 20$ and $n = 40$, so our answer is $\dfrac{1}{2}$.

Solution to Question 322: Adult Concert

Let x denote the total number of people at the concert before the bus arrives. $x \equiv 0 \mod 12$. Then, $x + 50 \equiv 0 \mod 25$. Since $50 \equiv 0 \mod 25$, we can rewrite the previous expression as $x \equiv 0 \mod 25$. The minimum possible value of x satisfying these two conditions is $x = 300$. $x \cdot \frac{5}{12} = 125$.

Solution to Question 323: American Call Arbitrage

American and European calls on non-dividend paying stocks should have the same value. The rationale is follows: if you can exercise the American option and gain $S - K$ anytime, it must be worth at least the European call for there to be no arbitrage.

We will short the European call and long the American call. If the call expires in-the-money, then both options have value $S - K$ and we obtain 0, but we keep our 0.06. If the calls expire out-of-the-money, then both options have value 0 and we once against keep our 0.06.

Solution to Question 324: Straddle Output

The expected value of the straddle is $\mathbb{E}[|S - K|] = \mathbb{E}[|S|]$, as $K = 0$. We are going to generalize this calculation for when $S \sim N(0, \sigma^2)$.

$$\begin{aligned}
E[|S|] &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} |x| \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \\
&= \frac{2}{\sqrt{2\pi\sigma^2}} \int_0^{\infty} x \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \\
&= \sqrt{\frac{2}{\pi\sigma^2}} \left(-\sigma^2 \exp\left(-\frac{x^2}{2\sigma^2}\right)\right)\Bigg|_0^{\infty} \\
&= \sqrt{\frac{2}{\pi}}\sigma,
\end{aligned}$$

From line 1 to line 2, we use symmetry of the integrand about 0. Then, from line 2 to line 3, we apply a $u-$substitution to obtain that indefinite integral. In particular, $\sigma = 1$ here, so our answer is $v = \sqrt{\frac{2}{\pi}}$, meaning that $v^2 = \frac{2}{\pi}$, so $a = 2$.

Solution to Question 325: Paired Values I

To get an odd sum, we have 1 odd term and 2 even terms (3 odd terms is impossible). This means our arrangement is in the form

$$OO\ EE\ EO$$

There are $3 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = (3!)^2 = 36$ ways to directly assign the values to the blanks. There are 2 ways to arrange the EO term (OE vs. EO). Then, there are $3! = 6$ ways to arrange the groups around, yielding $72 \cdot 6 = 432$ total arrangements. Thus, the probability is $\frac{432}{6!} = \frac{3}{5}$.

Solution to Question 326: Uniform Product I

As of right now, we have three sources of randomness coming from X, Y, and Z. Therefore, we should try to remove one of those sources by conditioning. Let's condition on $Z = z$. By law of total probability, we get

$$\mathbb{P}[X > YZ] = \int_0^1 \mathbb{P}[X > YZ \mid Z = z] f_Z(z) dz$$

Where $f_Z(z) = 1$ on $(0, 1)$ is the PDF of Z. Now, we see that $\mathbb{P}[X > YZ \mid Z = z] = \mathbb{P}[X > zY]$. As $0 < z < 1$, the region $X > zY$ is equivalent to saying $Y < \frac{X}{z}$. As $0 < z < 1$, the slope of this line is at least 1. Therefore, the region above this line is a triangle of sides length 1 and z, so the area is $\frac{z}{2}$. This means the probability we are below this line (and hence in the complement of the triangular region) is $1 - \frac{z}{2}$.

This means our final probability is

$$\int_0^1 \left(1 - \frac{z}{2}\right) dz = \left(z - \frac{z^2}{4}\right)\Big|_0^1 = \frac{3}{4}$$

Solution to Question 327: Circular Hop

We will use the fact that for a (non-circular) symmetric random walk starting at 0 with boundaries a and $-b$, for $a, b > 0$ being integers, the expected time to hit a boundary is ab. This is a fairly well-known fact and is a good exercise to prove if you have never done so before. Imagine we flattened out the circle into a line. In this case, to hit 0 from 1, we either need to move 1 unit up or 99 units down (this would get us to 0 by hitting spot 99 and circling back to 0). The answer is immediate from this fact. Namely, if we treat 1 as our starting point i.e. 0, we can treat the boundaries as 1 and -99, so our answer is just 99 by the lemma above.

Solution to Question 328: Skater Boy

We know that we select an angle uniformly at random and then we move 1 unit in that direction. Thus, we have that after n movements, if $\theta_1, \dots, \theta_n$ are the angles selected, $X = \sum_{i=1}^n \cos(\theta_i)$ and $Y = \sum_{i=1}^n \sin(\theta_i)$, where each of the θ_i are IID $\text{Unif}(0, 2\pi)$ random variables. Thus, the expected squared distance from the origin is just

$$D^2 = X^2 + Y^2 = \left(\sum_{i=1}^{n} \cos(\theta_i)\right)^2 + \left(\sum_{i=1}^{n} \sin(\theta_i)\right)^2$$

Next, we can expand out each of the summations to get that the above is $\sum_{i=1}^{n} \cos^2(\theta_i) + \sum_{i \neq j} \cos(\theta_i)\cos(\theta_j) + \sum_{i=1}^{n} \sin^2(\theta_i) + \sum_{i \neq j} \sin(\theta_i)\sin(\theta_j)$. Grouping the terms together nicely, we get that the above is

$$\sum_{i=1}^{n} \left(\cos^2(\theta_i) + \sin^2(\theta_i)\right) + \sum_{i \neq j} \left(\cos(\theta_i)\cos(\theta_j) + \sin(\theta_i)\sin(\theta_j)\right)$$

The first sum evaluates to just n by trigonometric identities. The second has an interior equal to $\cos(\theta_i - \theta_j)$. Thus, we have that $D^2 = n + \sum_{i \neq j} \cos(\theta_i - \theta_j)$. Therefore, the expectation $\mathbb{E}[D^2] = n + \sum_{i \neq j} \mathbb{E}[\cos(\theta_i - \theta_j)]$. Computing this expectation is quite simple. The joint PDF of θ_i and θ_j is given by $f(\theta_i, \theta_j) = \frac{1}{4\pi^2} I_{(0,2\pi)}(\theta_i) I_{(0,2\pi)}(\theta_j)$, so this expectation is given by $\int_0^{2\pi} \int_0^{2\pi} \frac{\cos(\theta_i - \theta_j)}{4\pi^2} d\theta_i d\theta_j$. The first integral evaluates to $\int_0^{2\pi} -\frac{1}{4\pi^2} \left(\sin(2\pi - \theta_j) - \sin(-\theta_j)\right) d\theta_j = 0$, as $\sin(-\theta_j) = \sin(2\pi - \theta_j)$ by periodicity. Thus, the expectation is just n. Alternatively, you could see that since θ ranges over $(0, 2\pi)$, the mean of the cosine and sine on those intervals is 0, so that can go quicker.

Our specific case here is $n = 16$, so 16 is our answer.

Solution to Question 329: Short Wood

Let X and Y be the two distances from the LHS where the wood is cut (in meters). Both of these are IID Unif$(0, 1)$ random variables. Since X and Y are IID, they are exchangeable. Thus, for simplicity, assume $Y > X$. We will multiply by 2 at the end to account for when $X > Y$. The lengths of the three pieces would then by given by $X, Y - X$, and $1 - Y$. Thus, we want $\mathbb{P}[\min(X, Y - X, 1 - Y) < 0.05] = 1 - \mathbb{P}[\min(X, Y - X, 1 - Y) > 0.05]$.

By properties of the minimum, this is the same as $1 - \mathbb{P}[X > 0.05, Y > X + 0.05, Y < 0.95]$. The last two terms are obtained by rearranging the inequalities. Drawing the region where all three of those conditions are true out in the plane yields a triangular region. Since the slope is one, the two legs are of the same length. Note that $X > 0.05$ and $Y < 0.95$. Thus, we have that the endpoint of where we leave our region of interest is $X = 0.9$, as then $Y = 0.95$.

Thus, the length of each leg is $\frac{17}{20}$, so the area of the triangle is just $\frac{1}{2}\left(\frac{17}{20}\right)^2 = \frac{289}{800}$. Thus, we have that since X and Y are exchangeable, we multiply the above probability by 2, so we get $\frac{289}{400}$, which means the probability is $\frac{111}{400}$ by complementation.

Solution to Question 330: Brownian Bridge

We will solve this more generally for $0 < s < t < 1$. Write $X_s = W_s - sW_1 = \left(W_s - \frac{s}{t}W_t\right) + \frac{s}{t}(W_t - tW_1) = \left(W_s - \frac{s}{t}W_1\right) + \frac{s}{t}X_t$. For fixed times, both of the terms are normally distributed. Furthermore,

$$\mathrm{Cov}\left(W_s - \frac{s}{t}W_t, X_t\right) = \mathrm{Cov}\left(W_s - \frac{s}{t}W_t, W_t - tW_1\right) = \mathrm{Cov}(W_s, W_t) - t\mathrm{Cov}(W_s, W_1) - \frac{s}{t}\mathrm{Cov}(W_t, W_t) + s\mathrm{Cov}(W_t, W_1)$$
$$= s - st - s + st = 0$$

Therefore, we have that $W_s - \frac{s}{t}W_t$ and X_t are independent, as they are uncorrelated normals. Thus, $\mathbb{E}\left[W_s - \frac{s}{t}W_t \mid X_t\right] = \mathbb{E}[W_s] - \frac{s}{t}\mathbb{E}[W_t] = 0$. Using this, we can find that

$$\mathbb{E}[X_s \mid X_t] = \mathbb{E}\left[\left(W_s - \frac{s}{t}W_1\right) + \frac{s}{t}X_t \mid X_t\right] = \frac{s}{t}X_t$$

We obtain this from the fact that the first term vanishes by the above and that X_t is known in the second term. In particular, we have that $s = 1/2, t = 3/4$, and $X_t = 3$, so our answer is $\frac{2}{3} \cdot 3 = 2$.

Solution to Question 331: Sheep Sharing

Let A, J, C be the number of sheep each of Alfred, John, and Charles got respectively. We then have the following system:

$$A = J + \frac{20J}{100}$$
$$A = C + \frac{25J}{100}$$
$$J = 3600$$

Solving the system, we get $C = 3456$

Solution to Question 332: Green Ball Draw

Note that sampling without replacement is an exchangeable process. Therefore, this probability is the same as if we were to draw a blue on the first draw and want the probability the second is green. If the first ball is blue, then there are 3 green and 6 blue balls in the urn left. Therefore, the probability the second is green is $\frac{3}{9} = \frac{1}{3}$.

Solution to Question 333: Exercise Gamma

Gamma is the largest with the lowest strike. We can think about this in terms of value of the underlying. A $1 move on a $10 stock is a lot more valuable than a $1 move on a $1000 stock.

Solution to Question 334: 2D Paths I

Your character will take a total of 12 steps, regardless of the path it takes. Of these 12 steps, 6 must be up, and the other 6 must be right. In other words, of the 12 steps, you choose 6 of them to be up, and the rest will be filled in as right. Thus, the total number of paths you character can take is:

$$\binom{12}{6} = \frac{12!}{6! \times 6!} = 924$$

Solution to Question 335: Keg of Wine

Let x denote the capacity of the jug. Then, the proportion of wine in the keg after Tuesday can be denoted as:

$$\begin{aligned} \frac{10 - x - \left(\frac{10-x}{10}\right) x}{10} &= \frac{1}{2} \\ 20 - 2x - \frac{10x - x^2}{5} &= 10 \\ \Rightarrow x &= 10 - 5\sqrt{2} \end{aligned}$$

This means that $a + b + c = 17$.

Solution to Question 336: Light Bulb

Label the switches 1, 2, 3, and 4. Because there are four possible answers, you need at least two binary choices to differentiate between the results, one of which is entering the room to see if the light is on or off. The second will be if the bulb is warm. Our strategy will be to turn on switches 1 and 3 for 20 minutes, then turn off switch 3 and turn on switch 2. Thus, there are four possibilities from the two binary choices based on the characteristics of the bulb when we go into the room once:

$$\text{On and warm} \Rightarrow 1$$
$$\text{On and cold} \Rightarrow 2$$
$$\text{Off and warm} \Rightarrow 3$$
$$\text{Off and cold} \Rightarrow 4$$

Thus, we only need to enter the room once.

Solution to Question 337: Counting Odds

The sequence of numbers within our range that are divisible by three are $3, 6, ..., 2022$. Every other term in this sequence will be even since adding an odd number to x flips whether or not it is odd or even. Thus, the new sequence in consideration is $3, 9, ..., 2019$. Note that the difference between each term is six, which is intuitive since we are skipping every other multiple of three. Adding 3 to each term, the sequence becomes $6, 12, ..., 2022$. Dividing each term by 6, the sequence becomes $1, 2, ..., 337$. Hence, there are 337 odd numbers from 1 to 2023.

Solution to Question 338: Balanced Beans II

The first step is similar to the other "Balanced Beans" problem and its to split the beans into 3 groups of four. Put one group on one side and another group on the other side, leaving one group unweighted. The best case scenario is if the scale is balanced on both sides which means the abnormal bean is in the unweighted group. In this case, you can take two beans from that group and weigh them on either side. If the scale is balanced, you know the abnormal bean is one of the other two beans in the 3^{rd} group. Thus you can pick one of these two beans and replace either side with that bean. If the scale continues to stay balanced, you know the last bean (never touched the scale) is the abnormal one. Otherwise, its the new bean you put on the scale. Its a similar approach if the scales are not balanced after you selected two beans from the 3^{rd} group. You replace one side with a bean you know is normal and if it continues to remain unbalanced, the bean that stayed on the scale is abnormal. Otherwise its the bean you replaced with a normal bean.

Now lets cover the more difficult situation where the two groups of four aren't balanced. The key realization with this problem is to move around subsets of each group, specifically subsets of three beans. Lets call the two groups of four on the scale Group 1 (left side) and Group 2 (right side) while the unweighted group is Group 3. We take three beans from Group 1 and move them to Group 2, take three beans from Group 2 and move them to Group 3, and three beans from Group 3 and move them to Group 1. Now the best case scenario is the balance doesn't change its orientation (if the left side was lighter before the moving of groups of three, it continues to stay lighter). This is the best case scenario because we know the abnormal bean is either the bean that didn't move from from Group 1 or the bean that didn't move from Group 2. We can easily deduce which one is the abnormal one by weighing one of these beans against a known normal bean and deduce if the bean weighed is the normal or abnormal bean.

Lets say the scale does change orientation after the 2$^{\text{nd}}$ weighing. If the scale flips orientations (still unbalanced but not the same side being lighter than the other), then we know the abnormal bean was in the group of three beans we moved from Group 1 to Group 2. We would also know if this bean is lighter or heavier than the others. If the left side (Group 1) weighed more than the right (Group 2), and it flipped to the right weighing more, then we know the abnormal bean weighs more than the others. Same thing if its lighter but we'd notice the left side going from being lighter to heavier. Weigh any two beans from this group of three that's currently in Group 2 to deduce the abnormal bean. Lets go back to the case where the scales were originally unbalanced but after the 2nd weighing, the becomes balanced. This means that the abnormal bean is in the group of three we moved from Group 2 to Group 3. We would also know if this bean is heavier or lighter than the others as we could remember if the right side (Group 2) was up or down after the second weighing. Now weigh any two beans from this group of three to deduce the abnormal bean. The total number of 3 weightings needed to guarantee you know which bean is the abnormal one.

Solution to Question 339: Reciprocal SDE

Let $Y_t = \dfrac{1}{X_t}$. We can write the SDE in the question as $dY_t = Y_t(2dt - dW_t)$. Note that $X_t = \dfrac{1}{Y_t}$ from definition. Therefore, let $f(y) = \dfrac{1}{y}$. We have that $f'(y) = -\dfrac{1}{y^2}$ and $f''(y) = \dfrac{2}{y^3}$.

From Ito's Formula, we get that $dX_t = df(Y_t) = f'(Y_t)dY_t + \dfrac{1}{2}f''(Y_t)d[Y,Y]_t$, where $d[Y,Y]_t$ is the quadratic variation of Y_t. We already know dY_t from the SDE above. We get that

$$d[Y,Y]_t = (Y_t(2dt - dW_t))^2 = Y_t^2\left(4d[t,t]_t - 4d[t,W]_t + d[W,W]_t\right) = Y_t^2 dt$$

by the fact that $d[t,t]_t = d[t,W]_t = 0$.

Plugging everything in,

$$dX_t = -\frac{1}{Y_t^2} \cdot Y_t(2dt - dW_t) + \frac{1}{Y_t^3} \cdot Y_t^2 dt = \frac{1}{Y_t}(-dt + dW_t) = X_t(-dt + dW_t)$$

This yields that our answer is $-1 \cdot 1 = -1$.

Solution to Question 340: Marble Mischief

Let B and G represent the event that the absolute last marble in our $1200-$long sequence is blue or green, respectively. Let R be the event in question. By Law of Total Probability,

$$\mathbb{P}[R] = \mathbb{P}[R \mid G]\mathbb{P}[G] + \mathbb{P}[R \mid B]\mathbb{P}[B]$$

$\mathbb{P}[G] = \frac{1}{2}$ and $\mathbb{P}[B] = \frac{1}{3}$ by the exchangeability of the draws. Then, for the event to occur conditioned on green being the absolute last marble in the first case, we need blue to be the last among red and blue marbles. This would occur with probability $\frac{400}{600} = \frac{2}{3}$, as there are 600 marbles of those colors and 400 are blue.

Similarly, among red and green, the probability the last is green is $\frac{600}{800} = \frac{3}{4}$. Therefore,

$$\mathbb{P}[R] = \frac{2}{3} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{3} = \frac{7}{12}$$

Solution to Question 341: Picture Day

There are two ways for the tallest two students to be arranged. Let us focus on the remaining eight students and concern ourselves with the four vacant positions to the left of the center. These four positions can be filled with any of the combination of the eight students, since each combination has one possible arrangement that follows the height invariant. The remaining four students that are not chosen must go to the other side of the line, of which there is only one arrangement to satisfy the height invariant. Thus, there are a total of $\binom{8}{4}$ possible arrangements of the eight non-center students. In total, because there are two possible arrangements of the two middle students, the total number of ways the students can line up is:

$$\binom{2}{1} \times \binom{8}{4} = 140$$

Solution to Question 342: Full Solutions

Since 3 and 7 are clearly relatively prime, there must exist a solution to this equation. We note that $3(-2) + 7(1) = 1$, so if we multiply by 10000 on both sides, $3(-20000) + 7(10000) = 10000$, which means $(x, y) = (-20000, 10000)$ is a solution to this equation. By the latter hint, we can write all solutions to this equation in the form

$$(-20000 + 7n, 10000 - 3n)$$

We now need to reduce this equation so that the integer in front of $-3n$ is as small as possible. We note that $9999 = 3 \cdot 3333$, so we note that if $m = n - 3333$, we can rewrite the form of solutions as

$$(-20000 + 7(m + 3333), 10000 - 3(m + 3333)) = (3331 + 7m, 1 - 3m)$$

This is the general form where b is as small as possible but possible. In particular $ab = 3331$.

Solution to Question 343: Arbitrage Detective IV

We can see here, we have quite an unusual options chain we can take advantage of. As our chains are near inverses of each other, assume the underlying is selling at \$175. Usually, on a chain, when the strike of our call option increases, the call option's value approaches full value, meaning the difference between our prices should be close to the difference in our strike prices. In this example, as our call option increases in strike price, we would expect the prices to get increasingly farther away. In this case, the prices actually start converging, showing our chain is improperly formed.

The same is true on the put side, we should see the difference in price increase as the strike price decreases, but again, the opposite is true.

In order to take advantage of this, we can use an iron condor strategy. Usually an iron condor is used to take advantage of a neutral market, but in this case, we just care about the large mispricings in the chain. We use a put credit and a call credit spread, netting \$12 - \$9 of credit on both sides, with a max loss of \$5. So $3 \cdot 2 - 5 = \$1$. And as all contracts control 100 shares, we get $\$1 \cdot 100 = \$100$.

Solution to Question 344: Trinomial Call Pricing I

First, note that the probabilities are given rather than calculating any risk-neutral probabilities. This is due to the fact that there are infinite probabilities that can be used (i.e the system of equations that needs to be solved has infinite solutions). To provide an exact answer, the probabilities must be provided for a trinomial model, when compared to the binomial model.

We can see that the stock price has final values $10.4, 8, 4$. Of these, the option only has value when the stock price is 10.4. More specifically, the call option will have value $S_T - K = 10.4 - 8 = 2.4$. The time-0 price is simply the expected value at time-1, which is just $0.2 * 2.4 = 0.48$.

Solution to Question 345: Russian Roulette I

After the cylinder is initially spun, the position of the bullet is fixed. Thus, you win if the bullet is in chambers 1, 3, or 5, and lose if the bullet is in chambers 2, 4, or 6. The probability of winning is $\frac{1}{2}$.

Solution to Question 346: Big Smalls

Let Z_x be Bob's gain if he selects the value x. Furthermore, let Y be the randomly generated number. We can ignore the case where $Y = x$, as the payout there is 0 for both people. By Law of Total Expectation, we have

$$\mathbb{E}[Z_x] = \mathbb{E}[Z_x \mid x > Y]\mathbb{P}[x > Y] + \mathbb{E}[Z_x \mid x < Y]\mathbb{P}[x < Y]$$

We can see that $\mathbb{P}[x > Y] = (x-1)/100$ and $\mathbb{P}[x < Y] = (100-x)/100$, as there are $x-1$ values in $\{1, \ldots, x-1\}$ and $100-x$ numbers in $\{x+1, \ldots, 100\}$

To compute $\mathbb{E}[Z_x \mid x > Y]$, we can see that if $Y < x$, it is uniform on $\{1, \ldots, x-1\}$, so the expected amount Bob pays is $x/2$.

To compute $\mathbb{E}[Z_x \mid x < Y]$, we just simply note that Bob receives his number as the payout, which is x. Combining all of the above,

$$\begin{aligned} \mathbb{E}[Z_x] &= \left(\frac{x-1}{100}\right)\left(-\frac{x}{2}\right) + \left(\frac{100-x}{100}\right)(x) \\ &= \frac{201x - 3x^2}{200}. \end{aligned}$$

The derivative of this function if $\frac{1}{200}(201-6x)$, so this derivative equals 0 precisely when $x^* = \frac{201}{6} = \frac{67}{2}$. Since the parabola is symmetric about the axis of symmetry, which is at the $x-$value of the maximum (33.5 in this case), we conclude that guessing 33 and 34 give equal payouts. Namely, $\mathbb{E}[Z_{33}] = \mathbb{E}[Z_{34}] = \frac{1683}{100}$

Solution to Question 347: Beta Difference

Since S_{324} and T_{324} are both comprised of 324 IID Beta$(1,2)$ random variables, they are also IID. By the Central Limit Theorem, they are also approximately normally distributed, as they are defined as the sum of a large number of IID random variables. All that remains is to find the mean and variance. We do this for S_{324}, but as they are IID, these apply to T_{324} too.

Namely, by Linearity of Expectation and the fact the X_i are IID, we have that $\mathbb{E}[S_{324}] = 324\mathbb{E}[X_1] = 324 \cdot \frac{1}{3} = 108$. Furthermore, as the X_i are IID, $\text{Var}(S_{324}) = 324\text{Var}(X_1) = 324 \cdot \frac{1}{18} = 18$. Therefore, S_{324} and T_{324} are each well-approximated by independent $N(108, 18)$ random variables. As they individually are well-approximated by independent normal random variables, their difference is also well-approximated by a normal random variable. Namely, $S_{324} - T_{324}$ is approximately distributed as $N(108-108, 18+18)$, which is $N(0, 36)$.

With all of this in mind, $\mathbb{P}[S_{324} - T_{324} > 10] = \mathbb{P}\left[\frac{(S_{324} - T_{324}) - 0}{6} > \frac{10}{6}\right]$. The LHS of the interior is now approximately $Z \sim N(0,1)$ distributed, so the approximate probability in question is $\mathbb{P}\left[Z > \frac{5}{3}\right] = \mathbb{P}\left[Z \leq -\frac{5}{3}\right] = \Phi\left(-\frac{5}{3}\right)$. We use symmetry in switching the direction and sign. Therefore, $a = -\frac{5}{3}$.

Solution to Question 348: Dice Upon Dice

To find the expected total S, it may help to start by first finding $\mathbb{E}[M]$, the average number of dice that we will be rolling in the last step. To do this, we should condition on N, the number of dice we roll in the intermediary step. If X_i represents the outcome of a fair die roll, then $\mathbb{E}[M] = \mathbb{E}\left[\sum_{i=1}^{N} X_i\right]$, as we roll

N dice. To calculate this expected value, use Law of Total Expectation of N to get

$$\mathbb{E}[M] = \mathbb{E}[\mathbb{E}[M \mid N]] = \mathbb{E}[\mathbb{E}[X_1 + \cdots + X_N \mid N]] = \mathbb{E}\left[\frac{7}{2}N\right] = \mathbb{E}[N]\mathbb{E}[X_1] = \left(\frac{7}{2}\right)^2 = \frac{49}{4}$$

Now, we know that $S = \sum_{i=1}^{M} X_i$, so by the exact same process above, $\mathbb{E}[S] = \mathbb{E}[M]\mathbb{E}[X_1] = \left(\frac{7}{2}\right)^3 = \frac{343}{8}$.

Solution to Question 349: Probability of Unfair Coin II

Let D be the event that we select the double-headed coin and H be the event that the coin showed heads. We want $\mathbb{P}[D \mid H]$. By Bayes' Rule, we have that $\mathbb{P}[D \mid H] = \frac{\mathbb{P}[H \mid D]\mathbb{P}[D]}{\mathbb{P}[H]}$. For the denominator, we should condition on the ways a head could be obtained. There are three types of coins: the fair coins, the double-headed coin, and the double-tailed coin. Let F be the event of selecting a fair coin and T be the event of selecting the double-tailed coin. We have that $\mathbb{P}[H \mid T] = 0$ since this coin has no heads on it. We know $\mathbb{P}[H \mid F] = \frac{1}{2}$ since it is fair and $\mathbb{P}[F] = \frac{98}{100}$ since 98 of the coins are fair. Lastly, $\mathbb{P}[H \mid D] = 1$ since both sides are heads and $\mathbb{P}[D] = \frac{1}{100}$ as there is only 1 of them. Combining all of this,

$$\mathbb{P}[D \mid H] = \frac{\frac{1}{100} \cdot 1}{\frac{1}{100} \cdot 1 + \frac{98}{100} \cdot \frac{1}{2} + 0} = \frac{1}{50}$$

Solution to Question 350: Voter Mayhem II

Let $Q(n, m)$ be the probability of interest with n votes from Candidate A and m votes for Candidate B. Since the votes are thrown in the box randomly, the probability any given vote draw is for Candidate A is $\frac{n}{m+n}$, while it is $\frac{m}{m+n}$ for Candidate B. Conditioning on which candidate has the last vote, we get that

$$Q(n, m) = Q(n-1, m) \cdot \frac{n}{m+n} + Q(n, m-1) \cdot \frac{m}{m+n}$$

This is because of the fact that the probability that A is never behind in all n draws is the same as the probability that A is not behind if they received $n-1$ votes and B received m votes, which is $Q(n-1, m)$. By the same argument, we get that the probability A is never behind with the last vote being for B is the same as if A is never behind given A had n votes and B had $m-1$ votes. Our goal is now find $Q(n,m)$.

A good warm-up is to consider an edge case of when A has $n>1$ votes and B has $m=1$ votes. A would never be behind exactly when the singular B vote is not the first one drawn. The probability of this is $\frac{n}{n+1}$. This suggests to us to consider $\frac{n}{n+m}$ for a general m.

To prove this, we induct on the value of $n+m$. If $n+m=1$, this means $n=1$ and $m=0$. In this case, $Q(1,0)=\frac{1}{1+0}=1$, which is accurate. More generally, suppose this holds for when $n+m=k$ for all $1\le k\le r$. We now need to show it holds for $n+m=r+1$. Using our recurrence,

$$Q(n,m)=\frac{n}{m+n}\cdot Q(n-1,m)+\frac{m}{m+n}\cdot Q(n,m-1)=\frac{n}{m+n}\cdot\frac{n-1}{n-1+m}+\frac{m}{m+n}\cdot\frac{n}{n+(m-1)}=\frac{n}{n+m}$$

We can use our recurrence relation here since if $n+m=r+1$, $n+(m-1)=(n-1)+m=r$, so the induction hypothesis applies.

For our particular case, the answer is $\frac{100}{100+80}=\frac{5}{9}$

Solution to Question 351: Matrix Editor

Let I be the number of steps needed to obtain a singular matrix starting from the identity matrix. This value depends on which value in our matrix is edited first, so let's apply Law of Total Expectation to what value is edited first.

If we select one of the diagonal elements, then it is easy to verify we obtain a singular matrix. Otherwise, if we select one of the off-diagonal elements, we end up with either $\begin{bmatrix}1&1\\0&1\end{bmatrix}$ or $\begin{bmatrix}1&0\\1&1\end{bmatrix}$. However, a property of determinants is that for any square matrix A, $\det(A)=\det(A^T)$. Additionally, we select the elements uniformly at random, so these matrix will have the same expected time until they reach a singular matrix. Let O be the expected time to reach a singular matrix starting from $\begin{bmatrix}1&1\\0&1\end{bmatrix}$. Then by LOTE,

$$\mathbb{E}[T]=\frac{1}{2}\cdot 1+\frac{1}{2}\left(1+\mathbb{E}[O]\right)=1+\frac{1}{2}\mathbb{E}[O]$$

Now, we need to evaluate $\mathbb{E}[O]$. It is easy to verify that if we select any element besides the top right corner, we end up with a singular matrix in the next iteration. Therefore, with probability $\frac{3}{4}$, it takes 1 step. If we select the top right corner, we are back at the identity matrix. Therefore,

$$\mathbb{E}[O] = \frac{3}{4} \cdot 1 + \frac{1}{4}\left(1 + \mathbb{E}[I]\right) = 1 + \frac{1}{4}\mathbb{E}[I]$$

Plugging in the equation for $\mathbb{E}[O]$ into the one for $\mathbb{E}[I]$ and simplifying, $\mathbb{E}[I] = \frac{12}{7}$.

Solution to Question 352: Fish Capture

$$\frac{1}{4}$$

of the captured fish were tagged. Therefore, this implies that you tagged roughly $\frac{1}{4}$ of the fish population in the pond. This means there are roughly $50 \cdot 4 = 200$ fish in the pond.

Solution to Question 353: Delayed Ruin

We are going to use the result of Voter Mayhem I (often called the "Ballot Theorem"). For the gambler to ruin in exactly $n+2k$ rounds, the gambler must win exactly k of the rounds and lose $n+k$ of the rounds. The probability that the gambler obtains exactly k wins in $n+2k$ rounds of the game is

$$\binom{n+2k}{k} p^k (1-p)^{n+k}$$

Now, consider looking from the last round back to the start. The number of losses must always be strictly leading compared to the number of wins. If the number of wins/losses equalize beforehand, the gambler would be bankrupt at an earlier state than $n+2k$, which we don't want to happen. Therefore, given that $n+2k$ rounds are played, the probability this is the position at which the gambler goes bankrupt is just the probability that the number of losses always is strictly ahead of the number of wins (when counted from the point of ruin), which is

$$\frac{(n+k)-k}{n+2k} = \frac{n}{n+2k}$$

Therefore, the answer is

$$\binom{n+2k}{k} p^k (1-p)^{n+k} \cdot \frac{n}{n+2k}$$

Plugging in the respective values, the answer is approximately 0.1084.

Solution to Question 354: Defining Standard Deviation

$$\hat{\sigma} = \sqrt{\frac{\sum_i (x_i - \bar{x})^2}{N-1}} = \sqrt{\frac{\sum_{i=1}^{5} (i-3)^2}{5-1}} = \sqrt{\frac{5}{2}} \approx 1.58$$

Solution to Question 355: Bacterial Survival II

Let x be the probability that the bacterial population will die out. There are 4 total possibilities for the first bacterium: either it dies, stays the same, splits into two, or splits into three. In the first case, the bacterial population dies with probability 1. In the second case, the bacterial population dies with probability x, since the bacterial population dies when the bacterium that stays the same dies, which happens with probability x. In the third case, the bacterial population dies with probability x^2, as both bacteria that are formed must die in order for the population to die. In the fourth and final case, the bacterial population dies with probability x^3, as all three bacteria that are formed must die in order for the population to die. Thus, we can write

$$x = \frac{1}{4} \times 1 + \frac{1}{4}x + \frac{1}{4}x^2 + \frac{1}{4}x^3 = \sqrt{2} - 1 \approx 0.41$$

Therefore, our answer is $2 + 1 = 3$.

Solution to Question 356: Flash Drive Finders

To make computation easier for ourselves, we'll consider the amounts in \$5000 units. Gabe's flashdrive is worth 100 units, and each flash drive finder costs 1 unit per week.

Suppose Gabe hires n flash drive finders. Then, the probability that none of the flash drive finders find the flash drive is $\left(\frac{1}{10}\right)^n$. Gabe's expected payoff is then $\left(1 - \left(\frac{1}{10}\right)^n\right) 100 - n$. Our problem becomes the following:

$$\begin{aligned} n_{\text{optimal}} &= \arg\max_{n \in \mathbb{N}} \left\{ \left(1 - \left(\frac{1}{10}\right)^n\right) 100 - n \right\} \\ &= \arg\min_{n \in \mathbb{N}} \left\{ 100 \left(\frac{1}{10}\right)^n + n \right\} \end{aligned}$$

When $n = 1$, $100\left(\frac{1}{10}\right)^n + n = 11$. When $n = 2$, $100\left(\frac{1}{10}\right)^n + n = 3$. When $n \geq 3$, the n term alone will be greater than the value of $100\left(\frac{1}{10}\right)^n$ for $n = 2$, so 2 is our final answer.

Solution to Question 357: Central Containment

Fix the first point arbitrarily. The second point can also lie anywhere, but notice that to form a triangle which contains the center, the third point must lie on the portion which has equivalent size as the length of the arc between the first two points, reflected over the center. Thus, think about the position of the second point. The size of that portion could be anywhere between 0 and π. On average, it is $\frac{\pi}{2}$, and hence the answer is

$$\frac{\frac{\pi}{2}}{2\pi} = \frac{1}{4}$$

Solution to Question 358: Ascending Stairs

Let $f(x)$ be the number of ways to ascend to step x. You can only step to x from either step $x-1$ or $x-2$. In other words, $f(x) = f(x-1) + f(x-2)$. We know that $f(1) = 1$ and $f(2) = 2$, so we can solve for $f(5) = 8$ recursively.

Solution to Question 359: RPS Galore

Note that if Jack plays S, Jane must have played either R or P to get a distinct winner. Therefore, from the 6 times Jack played S, they would have 4 wins from the times that Jane played P and 2 losses from the two R. From the last 4 games, Jane played S 4 times, so they would 3 losses to the three R from Jack and 1 win from the time that they played P. Therefore, Jack has 7 total wins.

Solution to Question 360: Defining Mean

E[Z] = E[3X- 2Y] = 3E[X] - 2E[Y] = 6-2=4

Solution to Question 361: Determination I

We can interpret R^2 as the proportion of total variability in Y captured by the model on the RHS. As the total variability in Y is constant and including in more factors includes the information about variability captured by each of the factors, we know that the new R^2 must be at least 0.05. We can show 0.05 is attainable by letting $X_2 = X_1$ in an edge case. In that case, the information captured by the new regression is precisely the same as the old regression.

Solution to Question 362: High Roller

If X and Y represent the individual die outcomes, then XY is our payout. By independence of the dice rolls, $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] = (3.5)^2 = \frac{49}{4}$. If we are to re-roll, then we know the expected value of our second roll would be $\frac{49}{4}$. Therefore, our strategy is to re-roll if our product is 8 or less. This is because 8 would be the first integer that is at least 4 less than the mean product of the two dice, $\frac{49}{4}$.

The probability we obtain a product of 8 or less is $\frac{6+4+2+2+1+1}{36} = \frac{4}{9}$. We obtain these terms from looking at the ways to obtain a product at most 8 by fixing the value of the first die. For example, if you have a 1 on the first die, regardless of your other die value, you will have a product at most 8, yielding 6 combinations to our total. Therefore, we re-roll with probability $\frac{4}{9}$, in which our expected value is $\frac{49}{4} - 4 = \frac{33}{4}$ for the re-roll. Otherwise, we keep the value of the die. If we keep the value, by drawing out a table that has all the die rolls on the axes and the products at the intersection of rows and columns, one can find that the expected value of the die roll given that it is more than 8 is $\frac{92}{5}$. Therefore, our expected value on the game is $\frac{4}{9} \cdot \frac{33}{4} + \frac{5}{9} \cdot \frac{92}{5} = \frac{125}{9}$.

Solution to Question 363: 2D Paths II

The character has two sub-paths: from $(0,0)$ to $(2,3)$ and from $(2,3)$ to $(6,6)$. The total number of paths from $(0,0)$ to $(2,3)$ is $\binom{5}{2}$. The total number of paths from $(2,3)$ to $(6,6)$ is $\binom{7}{4}$. Thus, the total number of paths to $(6,6)$ such that your character can pass through $(2,3)$ is:

$$\binom{5}{2} \times \binom{7}{4} = 10 \times 35 = 350$$

Solution to Question 364: Card Turner

Let T be the total duration of the game. We are going to generalize this to n pairs of cards. Then $T = T_1 + T_2 + \cdots + T_n$, where T_i is the amount of draws needed to go from $i-1$ pairs drawn to i pairs drawn. By linearity of expectation, $\mathbb{E}[T] = \sum_{i=1}^{n} \mathbb{E}[T_i]$. We need to find $\mathbb{E}[T_i]$ now. If $i-1$ pairs are already drawn, then there are $2n - 2(i-1) = 2(n-i+1)$ cards left in the deck. For each drawing, fix the first card. The probability the second card matches that first card is $\frac{1}{2(n-i+1)-1}$, as there are $2(n-i+1)-1$ cards left after the first of the pair is drawn, and only 1 of the cards is the same value as first card. Therefore, as this is independent between trials, $T_i \sim \text{Geom}\left(\frac{1}{2(n-i+1)-1}\right)$, meaning $\mathbb{E}[T_i] = 2(n-i+1)-1$. Therefore,

$$\mathbb{E}[T] = \sum_{i=1}^{n} 2(n-i+1) - 1 = \sum_{i=1}^{n}(2i-1) = n^2$$

In particular, $n = 10$ here, so the answer is 100.

Solution to Question 365: Stacked Derivative

The trick here is to use logarithmic differentiation. In other words, let $y = f(x) = (\ln(x))^x$. Then $\ln(y) = x\ln(\ln(x))$. Using the product and chain rules, we have that

$$\frac{y'}{y} = \ln(\ln(x)) + x\cdot\frac{1}{x\ln(x)} \iff y' = y\left[\ln(\ln(x)) + \frac{1}{\ln(x)}\right] = (\ln(x))^x\left[\ln(\ln(x)) + \frac{1}{\ln(x)}\right]$$

Plugging in $x = e$, the above expression evaluates to 1.

Solution to Question 366: Make a Market I

We want no arbitrage pricing, so our pricing should be the same as a replicated portfolio. In other words, the price of $\text{Price}_{A+B} = \text{Price}_A + \text{Price}_B$. There are two perspectives we can take: a market-maker and a market-taking perspective. The market-maker perspective is more important in the context of trading interviews, so we will focus on this.

When we consider $A+B$, we are considering a long position of A and a long position of B. So, if we want to set our bid (and not be taken out by other market-makers), we should set our bid to be the $\text{Bid}_A + \text{Bid}_B$ and ask to be $\text{Ask}_A + \text{Ask}_B$. This gives us an ask of 17 and a bid of 14 and so $Y^2 - X^2 = 17^2 - 14^2 = 93$.

Solution to Question 367: 9 Sum I

Our number is in the form $x_1x_2x_3x_4x_5x_6$, where each x_i is a digit. The only restriction we have is that $x_1 \geq 1$ so it is a valid 6−digit integer. To get a sum of 9, we are essentially looking for the number of integer solutions to $x_1 + x_2 + \cdots + x_6 = 9$ with $x_2, \ldots, x_6 \geq 0$, $x_1 \geq 1$. Letting $x_1' = x_1 - 1$, we can see $x_1 = x_1' + 1$ and the restriction is that $x_1' \geq 0$, so we can say we want the number of non-negative integer solutions to $x_1' + x_2 + \cdots + x_6 = 8$. This is a classic stars-and-bars question that solves to $\binom{8+6-1}{6-1} = \binom{13}{5} = 1287$

Solution to Question 368: Overlapping Segments

There is a sneaky trick here. We only care about which endpoint is paired with the lowest of the four endpoints (in terms of value). If it is the second smallest, then we get no overlap. Otherwise, we have overlap. As it is equally likely for the remaining three random variables to be ordered in any way with respect to the last three spots, the probability is just $\frac{2}{3}$.

Solution to Question 369: Digit Match

Label the digits $1 - 11$, with digit 1 being the leftmost digit. Let X_i be the indicator of the event that the ith digit is an 8. Then

$$T = X_1 + \cdots + X_{11}$$

gives the total number of digits that are 8. By linearity, we have that $\mathbb{E}[T] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_{11}]$. Since our digit is uniformly at random selected, we have 9 equally-likely options for the first digit and 10 equally-likely options for the remaining 10 digits. Therefore, $\mathbb{E}[X_1] = \frac{1}{9}$ and $\mathbb{E}[X_k] = \frac{1}{10}$ for $2 \leq k \leq 11$. Therefore, $\mathbb{E}[T] = \frac{1}{10} \cdot 10 + \frac{1}{9} = \frac{10}{9}$.

Solution to Question 370: Make Your Martingale I

Our goal is to compute $\mathbb{E}[M_{n+1} \mid M_n]$ as a function of c. Afterwards, we must find the value of c so that the conditional expectation is M_n. Doing this, $\mathbb{E}[M_{n+1} \mid M_n] = \mathbb{E}\left[M_n e^{X_{n_1}+c} \mid M_n\right] = M_n e^c \mathbb{E}[e^{X_{n+1}} \mid M_n]$ by taking out constants and what is known. Note that X_{n+1} is independent of M_n, so $\mathbb{E}[e^{X_{n+1}} \mid M_n] = \mathbb{E}[e^{X_{n+1}}]$. The computation of this is quite simple as X_{n+1} only takes two values, so

$$\mathbb{E}[e^{X_{n+1}}] = e \cdot \frac{1}{2} + e^{-1} \cdot \frac{1}{2}$$

Therefore, $\mathbb{E}[M_{n+1} \mid M_n] = M_n e^c \cdot \frac{e + e^{-1}}{2}$.

To make M_n a martingale, we need all the constants to simplify to 1. Namely, this means $e^c = \dfrac{2}{e+e^{-1}}$. Therefore, $c = \log\left(\dfrac{2}{e+e^{-1}}\right) = -\log\left(\dfrac{1}{2}e + \dfrac{1}{2}e^{-1}\right)$. For our question, the answer is $a+b+c+d = \dfrac{1}{2} + \dfrac{1}{2} + 1 - 1 = 1$.

Solution to Question 371: Regional Manager I

He is testing the null hypothesis $H_0 : \mu = 15$ against the alternative hypothesis $H_a : \mu > 15$. Because n is sufficiently large, he can utilize the Z statistic. We are given that $\mu = 15, \bar{y} = 17, s^2 = 9, andn = 36$ which can be used in the z-score formula:

$$z = \frac{\bar{y} - \mu}{\frac{s}{\sqrt{n}}} = \frac{17 - 15}{\frac{3}{\sqrt{36}}} = 4$$

Solution to Question 372: 2 In A Row

There are three cases to consider: $HHT-$, $THHT$, and $-THH$, where the $-$ can be either H or T. The probability of the first and last occurring are $p^2(1-p)$ each, as we obtain two heads and a tail in those respective orders. The probability of the second case is $p^2(1-p)^2$, as you must obtain two heads and two tails in that order. Therefore, if $f(p)$ represent our probability of this event as a function of p,

$$f(p) = 2p^2(1-p) + p^2(1-p)^2 = p^4 - 4p^3 + 3p^2$$

To maximize $f(p)$, we take the derivative with respect to p and set it equal to 0. Namely,

$$f'(p) = 4p^3 - 12p^2 + 6p = 2p(2p^2 - 6p + 3) = 0 \iff p = \frac{6 \pm \sqrt{12}}{4}$$

Since $\dfrac{6+\sqrt{12}}{4} > 1$, we take the negative square root on this, and our answer is $p^* = \dfrac{3-\sqrt{3}}{2}$ after simplification. Thus, $a+b+c = 8$.

Solution to Question 373: Missing Million II

We do not locate the door with \$1 million exactly when the wheel does not tell us any information both times. This occurs with probability $2/5$ per trial, so the probability you don't get any information about the location of the money is $\dfrac{2}{5} \cdot \dfrac{2}{5} = \dfrac{4}{25}$. This means that with probability $\dfrac{21}{25}$, you are guaranteed to know where the money is located via the wheel. Otherwise, with probability

$\frac{4}{25}$, you guess uniformly at random among the doors and locate the money with probability $\frac{1}{3}$. This implies the total probability of finding the money is

$$\frac{21}{25} + \frac{1}{3} \cdot \frac{4}{25} = \frac{67}{75}$$

Solution to Question 374: Fishy Bear

Let C represent the event that the 5th fish is caught and F represent the event that the bear is fishing on the 5th turn. By the Law of Total Probability,

$$\mathbb{P}[C] = \mathbb{P}[C \mid F]\mathbb{P}[F] + \mathbb{P}[C \mid F^c]\mathbb{P}[F^c]$$

$\mathbb{P}[F]$ is just the probability that the bear has caught at most 2 fish in the first 4 turns. This probability is just $\frac{\binom{4}{0}+\binom{4}{1}+\binom{4}{2}}{2^4} = \frac{11}{16}$ Therefore, $\mathbb{P}[F^c] = 1 - \frac{11}{16} = \frac{5}{16}$.

Then, we know that $\mathbb{P}[C \mid F] = \frac{1}{2}$ by the question. Additionally, $\mathbb{P}[C \mid F^c] = 0$, as if the bear isn't fishing at the 5th turn, it can't be caught. Therefore, $\mathbb{P}[C] = \frac{11}{16} \cdot \frac{1}{2} + 0 = \frac{11}{32}$.

Solution to Question 375: Coin Flipping Competition I

We know that if T and G represent the number of flips needed for Ty and Guy, respectively, to obtain their first heads, $T, G \sim \text{Geom}\left(\frac{1}{2}\right)$ IID. We are looking for $\mathbb{P}[T \geq 4G]$. The easiest way to do this is to condition on $G = g$ and use Law of Total Probability. Therefore,

$$\mathbb{P}[T \geq 4G] = \sum_{g=1}^{\infty} \mathbb{P}[T \geq 4G \mid G = g]\mathbb{P}[G = g]$$

By the definition of the PMF of G, $\mathbb{P}[G = g] = \frac{1}{2^g}$. Now, $\mathbb{P}[T \geq 4G \mid G = g] = \mathbb{P}[T \geq 4g]$. To evaluate this probability, let's think about what this event says. This means that it takes Ty at least $4g$ flips to obtain his first heads. This is equivalent to saying that all of the first $4g - 1$ flips are tails, and this occurs with probability $\frac{1}{2^{4g-1}}$. This is exactly the probability of interest there.

All that is left is to compute the sum. $\mathbb{P}[T \geq 4G] = \sum_{g=1}^{\infty} \frac{1}{2^{4g-1}} \cdot \frac{1}{2^g} = 2\sum_{g=1}^{\infty} \left(\frac{1}{32}\right)^g$. This is a geometric series with ratio $\frac{1}{32}$ starting at the first term, so this evaluates to $2 \cdot \frac{\frac{1}{32}}{1-\frac{1}{32}} = \frac{2}{31}$.

Solution to Question 376: Basic Dice Game IV

We know that the value we need to not roll depends on the value we roll first. This implores us to use Law of Total Expectation and condition on the first value that we roll. Namely, if X_1 is our first roll and T is our total, $\mathbb{E}[T] = \mathbb{E}[\mathbb{E}[T \mid X_1]] = \sum_{i=1}^{6} \mathbb{E}[T \mid X_1 = i]\mathbb{P}[X_1 = i]$.

We know $\mathbb{P}[X_1 = i] = \frac{1}{6}$ for all $1 \leq i \leq 6$, so $\mathbb{E}[T] = \frac{1}{6}\sum_{i=1}^{6} \mathbb{E}[T \mid X_1 = i]$

To compute $\mathbb{E}[T \mid X_1 = i]$, we first know that we start with i in the bank from our first roll. Therefore, $\mathbb{E}[T \mid X_1 = i] = i + \mathbb{E}[T' \mid X_1 = i]$, where T' is the money that is made after the first roll.

We know that with probability $\frac{1}{6}$, we roll the value i again, and our bank is now i larger than before and we are at the same game as before. Otherwise, with probability $\frac{5}{6}$, we cash out next round, as we don't roll the value i. If we are given that we don't roll i on the second die, then our expected value would be $\frac{21-i}{5}$, as the sum of the other faces is $21 - i$ and each of those 5 would be equally likely if it is not the value i. Therefore, by Law of Total Expectation,

$$\mathbb{E}[T' \mid X_1 = i] = \frac{1}{6}\left(i + \mathbb{E}[T' \mid X_1 = i]\right) + \frac{5}{6} \cdot \frac{21-i}{5}$$

Solving this, we see that i cancels and we obtain that $\mathbb{E}[T' \mid X_1 = i] = \frac{21}{5}$ for each i. Adding back the first roll, $\mathbb{E}[T \mid X_1 = i] = i + \frac{21}{5}$.

Therefore, as $\mathbb{E}[T]$ is just the average of $\mathbb{E}[T \mid X_1 = i]$ for each i, so $\mathbb{E}[T] = \frac{1}{6}\sum_{i=1}^{6}\left(\frac{21}{5} + i\right) = \frac{21}{5} + \frac{7}{2} = \frac{77}{10}$.

Solution to Question 377: Expected Chord Length

Let θ and ϕ be IID Unif$(0, 2\pi)$. Picking the two angles is equivalent to picking the two points uniform on the circumference. The measure of the angle between θ and ϕ is $\theta - \phi$. To get the length of the line segment connecting the two points, we can take a slight shortcut. The length here only depends on the distance between our points we select in terms of angle. In other words, regardless of where we end up having our points located, we can just rotate the circle so that one of them is located at $(1, 0)$. Therefore, we can arbitrarily fix $\theta_1 = 0$. Thus, the length between $\theta_1 = 0$ and the point at $\theta_2 = \theta$ is $\sqrt{(1-\cos(\theta))^2 + \sin^2(\theta)} = \sqrt{2}\sqrt{1-\cos(\theta)}$. This means that finding the expected length of the chord is equivalent to finding $\sqrt{2}\mathbb{E}[\sqrt{1-\cos(\theta)}]$, where $\theta \sim$ Unif$(0, 2\pi)$.

Now, $\sqrt{2}\mathbb{E}[\sqrt{1-\cos\theta}] = \dfrac{\sqrt{2}}{2\pi}\displaystyle\int_0^{2\pi} \sqrt{1-\cos\theta}d\theta$. Now, the best approach is to conjugate the interior by multiplying and dividing by $\sqrt{1+\cos\theta}$ so that we get $\sqrt{1-\cos^2\theta} = |\sin\theta|$. Doing this, we get $\dfrac{\sqrt{2}}{2\pi}\displaystyle\int_0^{2\pi} \frac{|\sin\theta|}{\sqrt{1+\cos\theta}}d\theta$.

Note that the region on $(0, \pi)$ and $(\pi, 2\pi)$, our integrand is symmetric, so we can just evaluate over one interval and double it. This means our new integral is $\dfrac{\sqrt{2}}{\pi}\displaystyle\int_0^{\pi} \frac{\sin\theta}{\sqrt{1+\cos\theta}}d\theta$. Let $u = 1 + \cos\theta$. Then $du = -\sin\theta d\theta$. Our bounds would respectively become 2 and 0, but the negative from the du flips them back. Therefore, our new integral is $\dfrac{\sqrt{2}}{\pi}\displaystyle\int_0^{2} u^{-\frac{1}{2}}du$. Evaluating this, we get $\dfrac{2\sqrt{2}}{\pi} \cdot \sqrt{u}\Big|_0^2 = \dfrac{4}{\pi}$.

Solution to Question 378: Rabbit Hop II

Using the same idea as Rabbit Hop I, we are looking for subsets of $\{1, 2, \ldots, 9\}$ that have odd size. This is because we have to select an odd amount of integers $1 - 9$ such that when we move to step 10, the total number of moves is even. In other words, we need to find the number of odd-sized subsets of a set of size 9.

We can match up odd and even subsets by matching a subset $A \subseteq \{1, 2, \ldots, 9\}$ with odd cardinality to A^c, which would have even cardinality. This is since if $|A| = k$, where k is odd, $9 - k$ must be even, so we have a one-to-one matching of odd and even subsets, so there must be equal amounts of odd and even subsets. Thus, there are $\dfrac{1}{2} \cdot 2^9 = 256$ such paths.

Solution to Question 379: Primitive Preference

The percentage of boys in this society is dependent on nature, not on couples' preferences. Every newborn child has an equal probability of being either a boy or a girl, regardless of the gender of any other children. Thus, the percentage of boys in this society approaches 50%.

Solution to Question 380: Statistical Test Review V

The population variance must be known, though one could make an argument that it can be estimated accurately by the sample variance when $n \geq 30$. The sample must of course be a random sample of the population. The distribution of the sample mean must be approximately normal, which suggests $n \geq 30$ by the Central Limit Theorem. However, if it is given that the population distribution is normal, then it is not necessarily the case that $n \geq 30$ must be satisfied to proceed with the Z test. Therefore, our answer is 123.

Solution to Question 381: Arbitrage Detective III

Once again, we can exploit an improperly priced options chain. As our time to expiration increases, the cost of our option must also increase in turn. For the same underlying, at the same strike, our contract with the closer expiry will always be worth less than its same strike counterpart with a further expiration date.

Using this knowledge, we can see that the \$165 dollar strike is mispriced. In order to capitalize, we simply need to sell the front month at \$165, giving us a credit of \$21, and then buy the back month at \$20. Giving us a net credit of \$1. Using this strategy, we lock in \$1 of profit per share.

This is then multiplied by 100, giving us our final total of \$100.

Solution to Question 382: Lollipop Mix

LOLLIPOP consists of 3 L, 2 O, 2 P, and 1 I. There are a total of $\frac{8!}{3!2!2!} = 1680$ total distinguishable permutations of LOLLIPOP. We find the probability that a random permutation starts and ends with the same letter. Let this event be S.

We condition on the first letter of our permutation. Namely, $\mathbb{P}[S] = \mathbb{P}[S \mid L]\mathbb{P}[L] + \mathbb{P}[S \mid O]\mathbb{P}[O] + \mathbb{P}[S \mid P]\mathbb{P}[P] + \mathbb{P}[S \mid I]\mathbb{P}[I]$. The last term is 0, as if we start with I, we can't end with I, as there is only one I total. The middle two terms are the same, as there are 2 of each letter in LOLLIPOP. We do the calculations for just O. Namely, $\mathbb{P}[O] = \frac{2}{8}$, as each letter is equally likely to appear first. Then, given we start with O, the probability we end with O is $\frac{1}{7}$, as only one of the 7 remaining letters is O. Thus, each of the two middle terms is $\frac{1}{28}$

Lastly, $\mathbb{P}[L] = \frac{3}{8}$ and $\mathbb{P}[S \mid L] = \frac{2}{7}$ by the same logic above. Thus, this first term is $\frac{3}{28}$. Adding these all up, $\mathbb{P}[S] = \frac{5}{28}$, so the total number of permutations that start and end with the same letter is $\frac{5}{28} \cdot 1680 = 300$.

Solution to Question 383: 4 and 5 Sum

Since 4 and 5 are relatively prime, in theory, it should be all of them, as we know that the Diophantine equation $4x + 5y = d$ has a solution $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ if and only if $\gcd(4, 5) = 1 \mid d$. This is not required for this problem, as we can intuitively see that $4 \equiv -1 \bmod 5$, so some combination of 4 and 5 should yield each integer. All multiplies of 5 can be formed by just writing it as a sum of some amount of 5s. Then, we can swap out some of those 5s for 4s to get the values between. This only works if there are sufficiently many 5s to swap out. This is a possibility when there are < 5 5s in our sum, so we know that every integer ≥ 16 is possible to be written in this form. We can now manually determine the integers that can't be written in this form are $1, 2, 3, 6, 7$, and 11. Therefore, 6 of 1000 integers can't be written in this form, so 994 can.

Solution to Question 384: Strictly Better

Let S be Simon's number and J be Jimmy's number. By Law of Total Probability, we have that $\mathbb{P}[S > J] = \sum_{k=1}^{1000} \mathbb{P}[S > J \mid J = k]\mathbb{P}[J = k]$. We have that $\mathbb{P}[J = k] = \frac{1}{1000}$ for each k since we choose the number completely at random. If $J = k$, then that means Simon must have a value in the set $\{k+1, \ldots, 3000\}$, which contains $3000-(k+1)+1 = 3000-k$ elements. Therefore, the probability

Simon picks a value greater than Jimmy's k is $1-\frac{k}{3000}$. Therefore, we just need to evaluate $\sum_{k=1}^{1000}\frac{1}{1000}\left(1-\frac{k}{3000}\right)=1-\frac{1}{1000\cdot 3000}\cdot\frac{1000(1001)}{2}=1-\frac{1001}{6000}=\frac{4999}{6000}$.

Solution to Question 385: Jellybean Jar I

To get two red jellybeans at the start, the child has $6\cdot 5=30$ options of red jellybean. Then, for the two blues, the child has $10\cdot 9=90$ options. Thus, there are 2700 ways for the child to obtain this sequence. The total number of ways to draw 4 jellybeans without replacement is $16\cdot 15\cdot 14\cdot 13$. Therefore, our probability is $\frac{2700}{16\cdot 15\cdot 14\cdot 13}=\frac{45}{728}$.

Solution to Question 386: Playlist

By exchangeability, we can just consider the first 10 songs in the playlist as being all Rae Sremmurd songs or DaBaby songs (since there are 6 Rae Sremmurd and 4 DaBaby). Consider 10 blanks. If all of the Rae Sremmurd songs come before the 2nd DaBaby song, the last 3 blanks must all be DaBaby songs. In the first 7 spots, there are $P(7,6)=\frac{7!}{1!}=7!$ ways to permute the 6 DaBaby songs to the first 7 blanks. In the remaining 4 blanks, they must be DaBaby songs, so there are 4! ways to arrange the 4 DaBaby songs in the remaining blanks. There are 10! total ways to arrange the order of the songs, so our answer is $\frac{7!4!}{10!}=\frac{24}{10\cdot 8\cdot 9}=\frac{1}{30}$.

Solution to Question 387: Missing 7

To compute this probability, we need to find the number of assignments where 7 is missing. Then, we need to find the number of assignments where 7 is missing and nothing else is.

The number of assignments where 7 is missing is fairly easy. Namely, each of the 7 people can select any integer in $\{1,\ldots,6\}$, so there are 6^7 ways they can select numbers. From how we have set this up, we are assigning numbers to the people. Therefore, to compute the numerator, note that to have the value 7 be the only one missing, exactly 2 people must pick one value and the other 5 people must pick the other 5 values.

There are $\binom{7}{2}$ ways to pick the two people that select the same value. Then, there are 6 ways to pick the value they both select. Lastly, there are 5! ways to permute the 5 people to the other 5 values. This means our total probability is

$$\frac{\binom{7}{2}6!}{6^7} = \frac{35}{648}$$

Solution to Question 388: Random Particles

One key feature to note is the collision property. Before a collision, two particles are going in opposite directions towards each other. After a collision, the two particles are still going in opposite directions, but now away from each other. Thus, you can imagine switching the labels of the two particles after the collision and its as if the collision never occurred. Furthermore, there is no difference with regards to which direction a particle moves in due to symmetry. A particle at the x-th meter will fall off of the line in x seconds, on average. If it were traveling in the other direction, you can set x to $1-x$. Hence, the problem is asking for the expected value of the maximum of 1000 independent and identically distributed random variables that are distributed uniformly between 0 and 1. The expected value of the minimum for a uniform distribution of n samples on $[0,1]$ is $\frac{n}{n+1}$, or $\frac{1000}{1001}$ for this problem.

Solution to Question 389: Shuffled Deck

If a card starts in position x, after 1 shuffle, the card will now be in position y, where $2x \equiv y \pmod{53}$. Note that we never obtain the value 0 as a position, as 2 and 53 are relatively prime, so we can say that $1 \leq y \leq 52$. Therefore, after n shuffles, the card in position x at the beginning is now in position w, where $2^n x \equiv w \pmod{53}$. By the same logic, $1 \leq w \leq 52$. Therefore, we need to find the smallest n such that $2^n x \equiv x \pmod{53}$.

As 53 is prime, we can invert x on both sides and say that it is the smallest n such that $2^n \equiv 1 \pmod{53}$. By Fermat's Little Theorem, we know that $2^{52} \equiv 2 \pmod{53}$, as 53 is prime. Inverting a 2 on both sides yields that $n = 52$ works as a solution. One can also verify this is the minimum number of shuffles needed by noting that in the group $(\mathbb{Z}/p\mathbb{Z}, \cdot)$, where p is a prime, the order of all non-identity elements is $p-1$. This is a theorem from Abstract Algebra.

Solution to Question 390: Water(melon) Weight

There is 1 pound of non-water in the watermelon. After drying, we know that this pound consists of 2% of the total weight of the watermelon. Hence, the watermelon must weight 50 pounds.

Solution to Question 391: Dice Order I

A brute force approach is possible, but a more analytical approach will be provided here. In order for the maximum to be exactly four, the maximum must be less than or equal to four, but not less than or equal to three. In order words:

$$P(\max = 4) = P(\max \leq 4) - P(\max \leq 3)$$

$$= \frac{4}{6} \times \frac{4}{6} - \frac{3}{6} \times \frac{3}{6}$$

$$= \frac{7}{36}$$

Solution to Question 392: Basic Eigenvalues

We can obtain the characteristic function by finding the solutions λ that solve $\det(A - \lambda I) = 0$

We have $A - \lambda I = \begin{bmatrix} -5-\lambda & 2 \\ -7 & 4-\lambda \end{bmatrix}$

This gives the characteristic function $(-5-\lambda)(4-\lambda)+14 = \lambda^2+\lambda-6 = 0$. We can factor this into $(\lambda+3)(\lambda-2) = 0$, giving $\lambda = -3$, $\lambda = 2$. Plugging the values in, we get $(-3)^2 + 2^2 = 13$.

Solution to Question 393: Digit Multiplier

For the product to end in those four digits, we already know that none of the four integers can be even. Otherwise, the product would be even. We also know that none of the integers can be 5, as then the product of the integers will end in a 5 as well. Therefore, this is equivalent to just asking the probability that each of the digits we select is in the subset $\{1, 3, 7, 9\}$. The probability of this for each digit is $\frac{4}{9}$, so the probability of interest is $\frac{4^4}{9^4} = \frac{256}{6561}$.

Solution to Question 394: Parameter Picker

The easiest thing to first look at is the support. Note that our support of Y is an interval of 0.06 length, so this implies that $Y = 0.06X + b$, as the support of X is normally an interval of length 1. Then, note that it starts at -0.02, so $b = -0.02$, as this shifts the support to our desired interval of $[-0.02, 0.04]$. In other words, we have that $Y = \frac{3X-1}{50}$. We have that $\mathbb{E}[Y] = \frac{3\mathbb{E}[X]-1}{50} = \frac{1}{400}$, so $\mathbb{E}[X] = \frac{3}{8}$. Then, we have that $\text{Var}(Y) = \frac{9}{2500}\text{Var}(X) = \frac{3}{32000}$, so $\text{Var}(X) = \frac{5}{192}$. Plugging in the expressions for $\mathbb{E}[X]$ and $\text{Var}(X)$ in terms of a and b, $\frac{a}{a+b} = \frac{3}{8}$ and $\frac{ab}{(a+b)^2(a+b+1)} = \frac{5}{192}$. Note that $\frac{ab}{(a+b)^2} = \left(\frac{a}{a+b}\right)\left(1 - \frac{a}{a+b}\right) = \mathbb{E}[X](1-\mathbb{E}[X]) = \frac{3}{8} \cdot \frac{5}{8} = \frac{15}{64}$. Thus, $\frac{15}{64} \cdot \frac{1}{a+b+1} = \frac{5}{192}$. In other words, $a+b+1 = 9$, so $a+b = 8$. Therefore, $\frac{a}{a+b} = \frac{a}{8} = \frac{3}{8}$, so $a = 3$. Plugging back in, since $a+b = 8$, $b = 8-a = 5$. Therefore, $a-b = 3-5 = -2$.

Solution to Question 395: Triangle Walk

It is easier to compute the probability that Joey never travels beyond 1 unit from his starting point. Let N denote the total number of turns that Joey has taken without violating the condition. Note that Joey can only ever be at two states (let x_0 denote Joey's original starting point, and let x_1 denote any neighboring point that is exactly 1 unit away) without violating the condition. Let X denote the state that Joey is in. Then, we have

$$\begin{aligned}
\mathbb{P}(X = x_0, T = 0) &= 1 \\
\mathbb{P}(X = x_1, T = 0) &= 0 \\
\mathbb{P}(X = x_0, T = 1) &= 0 \\
\mathbb{P}(X = x_1, T = 1) &= 1
\end{aligned}$$

We wish to compute $\mathbb{P}(X = x_0, T = 0) + \mathbb{P}(X = x_1, T = 0)$. Note the following recurrence relation:

$$\begin{aligned}
\mathbb{P}(X = x_0, T = t) &= \frac{1}{6}\mathbb{P}(X = x_1, T = t-1) \\
\mathbb{P}(X = x_1, T = t) &= \frac{1}{3}\mathbb{P}(X = x_1, T = t-1) + \mathbb{P}(X = x_0, T = t-1)
\end{aligned}$$

Following this recurrence relation, we find:

$$\begin{aligned}
\mathbb{P}(X = x_0, T = 0) &= 1 & \mathbb{P}(X = x_1, T = 0) &= 0 \\
\mathbb{P}(X = x_0, T = 1) &= 0 & \mathbb{P}(X = x_1, T = 1) &= 1 \\
\mathbb{P}(X = x_0, T = 2) &= \frac{1}{6} & \mathbb{P}(X = x_1, T = 2) &= \frac{1}{3} \\
\mathbb{P}(X = x_0, T = 3) &= \frac{1}{18} & \mathbb{P}(X = x_1, T = 3) &= \frac{5}{18} \\
\mathbb{P}(X = x_0, T = 4) &= \frac{5}{108} & \mathbb{P}(X = x_1, T = 4) &= \frac{4}{27} \\
\mathbb{P}(X = x_0, T = 5) &= \frac{2}{81} & \mathbb{P}(X = x_1, T = 5) &= \frac{31}{324}
\end{aligned}$$

Finally,

$$1 - (\mathbb{P}(X = x_0, T = 5) + \mathbb{P}(X = x_1, T = 5)) = 1 - \frac{13}{108} = \frac{95}{108}$$

Solution to Question 396: Thorough Frog

For lily pad 50 to be the truly last lily pad the frog hasn't visited, the frog must first get to lily pad 49 or 51. Each one is equally likely to be the first neighbor of 50 that the frog hops to by the symmetry of the frog hopping. WLOG, say the frog hops to lily pad 49 first. Now, we can treat 49 as our new 0, and we can consider a rightwards hop as $+1$ and a leftwards hop as -1. This is now asking the probability the frog (now a "random walker") hits -98 before hitting 1. We only need to hit -98 since we just need to visit the other neighbor of 50, which is 51. This lily pad is 98 hops in the other direction. Therefore, since this is a simple symmetric random walk, the probability this occurs is just

$$\frac{1}{1+98} = \frac{1}{99}$$

Solution to Question 397: Disc EV

For any point (x, y) in our circle, we know that $(-x, y)$ has the same probability density, as we are uniform over the circle. Note also that our circle is symmetric about reflections over the axes. The effect of operation $(x, y) \mapsto (-x, y)$ is a reflection of our unit disc across the $y-$axis. This means that the joint distribution of (X, Y) is the same as the joint distribution of $(-X, Y)$. In particular, this means $\mathbb{E}[XY] = \mathbb{E}[(-X)Y] = -\mathbb{E}[XY]$. This means that $\mathbb{E}[XY] = 0$.

Solution to Question 398: Bond Practice IV

$n = 2 \times 8.0 = 16; r = \frac{0.03}{2} = 0.01500$; $C = \frac{0.06 \times 1{,}000}{2} = 30$; $P =$ price of bond:

$$P = \left(\frac{30}{0.01500}\right)\left(\frac{(1+0.01500)^{16}-1}{(1+0.01500)^{16}}\right) + \frac{1,000}{(1+0.01500)^{16}}$$
$$P = (2,000)\left(\frac{1.26899-1}{1.26899}\right) + \frac{1,000}{1.26899}$$
$$P = 423.9379 + 788.0310$$
$$P = 1,211.97$$

Solution to Question 399: Rain Chance I

The probability it doesn't rain on the two days are 0.6 and 0.3, respectively. Therefore, with independence, the probability it doesn't rain on either day is

$0.6 \cdot 0.3 = 0.18$.

Solution to Question 400: Prime Product

For the product of the three integers to be prime, we note that two of the integers must be -1 and 1. Otherwise, we would end up with two factors that are not one multiplied together, and this is no longer a prime number. The difference between these integers is 2. We should have -3 as our last integer, as we want the product to be positive. We can now verify that $-3 \cdot -1 \cdot 1 = 3$, which is prime. The above tells us $a = -3$ and $k = 2$, so our answer is $9 + 4 = 13$.

Solution to Question 401: Covered Calls I

When in covered call, we can only lose money when our underlying goes down in value. In this case, we have a \$10.50 credit from selling our call, which will offset the losses from our stock going down. So we can lose up to \$10.50 on our shares without going in the red overall, so our breakeven point is simply $230 - 10.5 = 219.5$ If the share price falls below \$219.50, the credit we gained will not be covering our losses on the underlying any longer.

Solution to Question 402: Increasing Dice Order II

If a collection of values are strictly increasing, then they must be unique.

$$\mathbb{P}(\text{strictly increasing}) = \mathbb{P}(\text{increasing} \,|\, \text{unique}) \cdot \mathbb{P}(\text{unique})$$

For any set of 5 unique integer values between 1 and 10 inclusive, there exists only 1 ordering out of a total of 5! possible orderings such that the values are strictly increasing.

$$\mathbb{P}(\text{strictly increasing} \,|\, \text{unique}) = \frac{1}{5!}$$

Finally, let's compute $\mathbb{P}(\text{unique})$.

$$\mathbb{P}(\text{unique}) = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{10^5}$$

Putting it all together, we get:

$$\begin{aligned}\mathbb{P}(\text{strictly increasing}) &= \frac{1}{5!} \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{10^5} \\ &= \frac{252}{10^5}\end{aligned}$$

The value of x is 252.

Solution to Question 403: Series Moment

We know the classic series representation of $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, so we apply this here. Thus, $\mathbb{E}\left[\frac{1}{1-X}\right] = \sum_{k=0}^{\infty} \mathbb{E}[X^k]$. Now, we just have to find the kth moment of X.

We can write $M(\theta)$ using a Taylor Series expansion of $e^{\frac{\theta}{2}}$ to get

$$M(\theta) = 2\sum_{k=0}^{\infty} \frac{\theta^k}{2^k k!} - 1 = 1 + \sum_{k=1}^{\infty} \frac{\theta^k}{2^{k-1} k!}$$

This yields that $\mathbb{E}[X^k] = \frac{1}{2^{k-1}}$, for $k \geq 1$, as the moment is the coefficient of $\frac{\theta^k}{k!}$. Plugging this into our original expression, we get

$$1 + \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = 1 + 2 = 3$$

Solution to Question 404: Particle Reach V

From Particle Reach I, we know that the probability that the particle ever reaches position 1 is strictly less than 1 for $p < 1/2$. Therefore, there is positive probability that the particle never reaches 1 i.e. that infinitely many steps are needed to reach position 1. This implies that the expected number of steps needed is ∞, so the answer is -1 here.

Solution to Question 405: Nested Root

Let x be the value of this expression. Then $x = \sqrt{6+x}$ from the nesting demonstrated here. Squaring both sides, we see that $x^2 = x+6$, so $x^2 - x - 6 = 0$ This factors to $(x-3)(x+2) = 0$, so $x = 3, -2$. However, as the value of this expression must be positive, we can conclude that $x = 3$.

Solution to Question 406: Seating Drama

We know that, from left to right, we have $XY524$, where X and Y are each one of either 1 or 3. Since 1 doesn't want to sit next to 5, $Y = 3$, so person 3 is to the left of person 5.

Solution to Question 407: Cube Shadow

There are 2 distinct n-sided shadow polygons you can create by rotating the die. Starting the light pointing directly at any face, you can see only one face, and your shadow result is a square. Rotate around any one of the 3 axes, and you can see two faces, but you only get a rectangle. Rotate along any two axes at once, and you'll be able to see 3 faces of the die. When viewing 3 faces, there are 9 distinct edges you see. The 3 in the middle all being shared by exactly 2 faces, and the 6 on the exterior, (only touching a single face) which make up the edges of the shadow. This shadow is a 6-sided polygon, a hexagon, making our answer 6.

Solution to Question 408: Covariance Review II

The correlation coefficient ρ is defined as

$$\begin{aligned}
\rho &= \frac{\mathrm{Cov}(X_1, X_2)}{\sigma_1 \sigma_2}, \text{ where} \\
\sigma_1^2 &= \mathrm{Var}(X_1) \text{ and} \\
\sigma_2^2 &= \mathrm{Var}(X_2).
\end{aligned}$$

The correlation coefficient must satisfy $-1 \leq \rho \leq 1$.

$$\begin{aligned}
-1 \leq &\frac{\mathrm{Cov}(X_1, X_2)}{\sigma_1 \sigma_2} \leq 1 \\
-1 \leq &\frac{\mathrm{Cov}(X_1, X_2)}{12} \leq 1 \\
-12 \leq &\mathrm{Cov}(X_1, X_2) \leq 12
\end{aligned}$$

The maximum possible value for $\mathrm{Cov}(X_1, X_2)$ is 12.

Solution to Question 409: Equal Money

Let b and g be the amount of money that boys and girls started with, respectively. Then, if each boy gives s_1 to each girl and each girl gives s_2 to each boy, then we want $b - 3s_1 + 3s_2 = g - 9s_2 + 9s_1$. This means that $12(s_2 - s_1) = g - b$. Since 12 needs to cleanly divide the number of pennies and girls start with more money, we could let $g - b = 0.12$. The smallest way to obtain this without anyone having negative money is if $g = 0.24$ and $b = 0.12$. Then, since both genders need to give money to one another, let $s_2 = 0.02$ and $s_1 = 0.01$. We would thus have that everyone has 0.15 at the end.

Solution to Question 410: Rainbow Selection

Having the 5 more red balls drawn than yellow does not affect our expectation on the number of balls drawn for any of the other colors in the first 70 draws. You can see this by considering a combined "red and yellow" color. Then, in this case, we would expect each of the other 5 colors to be drawn with probability $\frac{1}{7}$ on the next draw, as the drawing process is exchangeable.

Now, if r and y represent the probabilities of a red and yellow ball, respectively, on the next draw, then we know that $r + y = \frac{2}{7}$, as that is the remaining probability left after giving the other 5 colors $\frac{1}{7}$ probability each. However, note that we selected 5 more red balls than yellow in the first 70 draws. Therefore, there are 5 fewer red balls left in the urn than yellow. This implies that $r = y - \frac{5}{70} = y - \frac{1}{14}$, as there are 70 balls left and 5 fewer of them are red as opposed to yellow.

Therefore, we get $2y - \frac{1}{14} = \frac{2}{7}$, so $y = \frac{5}{28}$.

Solution to Question 411: Circular Cut

The arc containing $(1, 0)$ contains two pieces: the clockwise and counter-clockwise piece. By symmetry, the lengths of these two pieces are equal in expectation. Therefore, let L be the length of the clockwise piece. $2L$ is the length of our entire arc containing $(1, 0)$. Using the identity $\mathbb{E}[L] = \int_0^{2\pi} \mathbb{P}[L \geq x]dx$, we can compute the expected length in a clean fashion. Namely, $\mathbb{P}[L \geq x]$ is just the probability that there are no points in $(0, x]$. As we have 3 points, this probability is simply $\left(1 - \frac{x}{2\pi}\right)^3$, as each is independently not in that arc with probability $1 - \frac{x}{2\pi}$. Therefore, we have that

$$\mathbb{E}[L] = \int_0^{2\pi} \left(1 - \frac{x}{2\pi}\right)^3 dx = \frac{\pi}{2}$$

This means that the expected length of the full arc containing $(1, 0)$ is π, meaning $q = 1$.

Solution to Question 412: Trading Cards

We want to find $\mathbb{E}\left[\binom{N}{4}\right]$, where $N \sim \text{Poisson}(6)$. This is because there are $\binom{n}{4}$ ways to pick the cards when there are n cards in the box. Thus, using LOTUS, $\mathbb{E}\left[\binom{N}{4}\right] = \sum_{n=0}^{\infty} \binom{n}{4} \cdot \mathbb{P}[X = n]$. However, we know that for $n < 4$, the binomial coefficient is 0, so we can really start this summation at 4, so this is $\sum_{n=4}^{\infty} \frac{n!}{4!(n-4)!} \cdot \frac{6^n}{n!} e^{-6} = \frac{e^{-6}}{4!} \sum_{n=4}^{\infty} \frac{6^n}{(n-4)!}$.

The trick here is to try to get a $n-4$ in the exponent of the 6^n and then index shift our summation to be from $n = 0$ to ∞. To do this, we take out 6^4 from the exponent to get $\frac{6^4 e^{-6}}{4!} \sum_{n=4}^{\infty} \frac{6^{n-4}}{(n-4)!}$. Now, we index shift back by 4 to let $v = n - 4$ so that the sum ranges from $v = 0$ to ∞ and we obtain $\frac{6^4 e^{-6}}{4!} \sum_{v=0}^{\infty} \frac{6^v}{v!}$. This last summation is just e^6, so our answer is $\frac{6^4}{4!} = 54$.

Solution to Question 413: Maximal Variance

For the variance to be a maximum, we need to optimize in two ways. Namely, we have to maximize the spread from the mean as well as the randomness about that mean. To maximize the distance of all points away from the mean, we should have a discrete random variable. This is because a continuous random variable has density throughout an interval, which makes it more compact about the mean. We can't have just one value in our discrete random variable, as then the variance would be 0. Therefore, with 2 values, we should aim to separate them from the mean the furthest. This can be done by taking the values -1 and 1, as those are the endpoints of our allowed interval. Then, to maximize how "random" our distribution is, we should set $p = \frac{1}{2}$. This would maximize something called the entropy. Note that with $p = \frac{1}{2}$, the mean is 0.

Since the mean is 0, all that needs to be computed is $\mathbb{E}[X^2]$, where X is as above. however, X^2 is just 1 with probability 1, so $\mathbb{E}[X^2] = 1$. Thus, $\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 1$.

Solution to Question 414: Increasing Uniform Chain I

Let $f(x)$ be the expected largest number given that the current largest is x. We want $f(0)$, as we are selecting uniform values on $(0, 1)$, so our initial largest value is 0. We don't choose a starting value lower than 0 since we can't generate values lower than 0.

With probability x, the next value is smaller than x, and our largest value will be x. Otherwise, if our next value is $y > x$, y is uniformly distributed on $(x, 1)$, so the expected maximum in that case is $f(y)$. We integrate over all $x < y < 1$ since we are applying Law of Total Probability. Written as an equation, this is

$$f(x) = x^2 + \int_x^1 f(y)dy$$

Taking the derivative to convert into a differential equation, we get that $f'(x) = 2x - f(x)$. Equivalently, $f'(x) + f(x) = 2x$. An initial condition for this differential equation is $f(1) = 1$, as we can't go any higher than 1.

This is a linear first order differential equation, so we can use the method of integrating factors here. In particular, the integrating factor is $\mu(x) = e^{\int 1dx} = e^x$. Multiplying this on both sides, $(e^x f(x))' = 2xe^x$, meaning $e^x f(x) = \int 2xe^x dx = 2e^x(x - 1) + C$. This means that $f(x) = 2x - 2 + Ce^{-x}$. Using our initial condition $f(1) = 1$, this yields $C = e$, so $f(x) = 2(x - 1) + e^{1-x}$. In particular, $f(0) = e - 2$, so our answer is $1 - 2 = -1$.

For a sanity check, we can note that the answer to this question and the answer to Decreasing Uniform Chain sum to 1. This makes perfect sense. If $X_i \sim \text{Unif}(0, 1)$, then $1 - X_i \sim \text{Unif}(0, 1)$, so there is a symmetry between the two problems.

Solution to Question 415: Repetitious Game I

If Audrey gets tails on her first toss (which has probability 1/2), then she must start over again. If Audrey gets heads on her first toss, then she can either get HH or HT after her second toss (each with probability 1/4). If Audrey gets HT, then she must start over again. Otherwise, she is finished with the game.

Let X denote the number of tosses before Audrey finishes the game. From our reasoning above, we can write:

$$\mathbb{E}[X] = \frac{1}{2}(\mathbb{E}[X] + 1) + \frac{1}{4}(\mathbb{E}[X] + 2) + \frac{1}{4} \cdot 2$$

We can now solve for $\mathbb{E}[X]$.

$$\mathbb{E}[X] = \frac{6}{4} \cdot 4 = 6$$

Solution to Question 416: Presidential Options

We can see that Pennsylvania (Trump) will pay 1.00 in every outcome where the 2020 election (Trump) pays 1.00 due to the condition mentioned above. When looking at the initial costs, we can see that the payoff of Pennsylvania (Trump) dominates the payoff of 2020 election (Trump). This gives an arbitrage opportunity as this contract shouldn't be worth less than the US election. We will long the undervalued contract, and short the overvalued contract. In other words, we can long 1 unit of Pennsylvania (Trump) and short 1 unit of 2020 election (Trump) and receive a credit of $0.16 - 0.17 = -\$0.01$.

Here, regardless of the outcome, we can see that we cannot lose money. If Trump wins the election, we will keep our 1 cent as the final value of our portfolio will be $1 - 1 = 0$. If Trump only wins Pennsylvania, we have a value of $1 - 0 = 1$. Finally, if Trump loses Pennsylvania, we have a value of $0 - 0 = 0$. In all cases, we get to (at least) keep our 1 cent credit.

Solution to Question 417: Reflip

Once you flip all of the coins for the first time, you would only want to reflip the tails because flipping the heads again risks you losing the value of the heads that are already present. At the end of two flips, the probability a coin did not show tails in either flip is just $\frac{1}{2^2} = \frac{1}{4}$ because of the independence of the outcomes. Therefore, the probability of obtaining a heads before the end is $\frac{3}{4}$. Since there are 8 coins, the expected number of heads (and hence expected value of the game) is $\frac{3}{4} \cdot 8 = 6$.

Solution to Question 418: Breakeven Price I

The payoff of the call option at expiry is $\max(S_T - 4, 0)$. For a non-negative payout, we need $S_T = 4$. However, since we paid \$3, we need S_T to be \$3 higher to make money. Another way to think about it is:

$$\max(S_T - 4, 0) - 3 = 0$$

Solving for S_T, we get $S_T = 7$.

Solution to Question 419: Put Call Futures Parity

Our put call parity formula is given by

$$\text{Future's Price} - \text{Call Price} + \text{Put Price} - \text{Strike Price} = 0$$

Solving for our Put Price, we get Put Price $= -110 + 3 + 120 = 13$

Solution to Question 420: Unique IDs

From the information given, we know that E has to be the smallest integer so it has to be 0. For $ABCD$, we can pick 4 of the remaining numbers. Since they are all distinct, there will be only one ordering for these 4 numbers to match our criteria. The same can be said about the 5 remaining numbers $FGHIJ$. Thus the answer is $\binom{9}{4} = 126$.

Solution to Question 421: Casino War

This can be solved with symmetry. There are three distinct outcomes that define the sample space. Let E_1 be the event that your number is larger than the dealer's; E_2 be the event that your number is equal to the dealer's; E_3 be the event that your number is smaller than the dealer's. By symmetry, $P(E_1) = P(E_3)$, so we only need to find $P(E_2)$. We know that when we are dealt a card, there are 51 cards that the dealer could have, of which only 3 could be of the same number, so $P(E_2) = \frac{3}{51}$. Since $\sum_{\omega \in \Omega} P(\omega) = 1$, we can solve for $P(E_1)$:

$$P(E_1) + P(E_2) + P(E_3) = 12P(E_1) + P(E_2) = 12P(E_1) + \frac{3}{51} = 1P(E_1) = \frac{8}{17}$$

.

Solution to Question 422: 100 Factorial Digits

First, we can see that $\log_{10}(100) = 2$, so to get the number of digits, we just have to add 1. Thus, we want to find $\log_{10}(100!) = \log(1) + \log(2) + \cdots + \log(100)$ and then add 1. We can approximate this as the integral of $\log_{10}(x)$ on $[1, 100]$.

Integrating $\int_1^{100} \log(x)\ dx \approx 157$. Adding 1, we get 158.

Solution to Question 423: Conditional Uniform

Drawing this out in the plane, the region that is bound by the axes and $x+y=1$ is a triangle with side lengths 1. The region of interest for our event is the region that is below $y = x - 1/2$. Shading in this region, we see that it is also a triangle. The intersection point is $(3/4, 1/4)$, so the length of each side of the triangle of our region of interest is $\frac{\sqrt{2}}{4}$. Since X and Y are IID Unif$(0,1)$, their conditional joint distribution is uniform over the bigger triangle bound by $x+y=1$ and the axes. Thus, we can take ratios of areas, and determine that the area of the big triangle is $1/2$, while the area of our smaller region of interest is $\frac{1}{16}$, meaning our probability is $\frac{1}{8}$.

Solution to Question 424: Six Card Sum

Let $X_1, X_2, \ldots, X_6$ denote the amount of money won on turn $1, 2, \ldots, 6$, respectively. By linearity of expectation, the total amount of money won is simply $\mathbb{E}[X_1] + \mathbb{E}[X_2] + \ldots + \mathbb{E}[X_6]$. Clearly, $\mathbb{E}[X_1] = \frac{1}{2}$, so let's begin our discussion with the second turn.

After Jamie makes a guess from round 1, regardless of the guess and the true card, Jamie will have information for the correct 3 card-2 card split of the remaining 5 cards. Under the optimal strategy, Jamie should guess the value of the card that appears 3 times in the remaining 5 cards. Hence, $\mathbb{E}[X_2] = \frac{3}{5}$.

The expected value of round 3 depends on the previous two guesses; specifically, there is either a 3-1 split or a 2-2 split. Once again, Jamie should know the true split and guess accordingly. It is not difficult to conclude that the probability of a 3-1 split is $\frac{2}{5}$, since there are 20 total orderings of 3 aces and 3 jacks, and $\frac{4}{1} \cdot 2$ of those orderings begin with ace-ace or jack=jack. That means that there is a $\frac{3}{5}$ chance that there is a 2-2 split. By the law of total expectation, we have $\mathbb{E}[X_3] = \frac{3}{5} \cdot \frac{1}{2} + \frac{2}{5} \cdot \frac{3}{4} = \frac{3}{5}$.

The expected value of round 4 can be computed similarly. There is either a 2-1 split or a 3-0 split. The 3-0 split occurs with probability $\frac{1}{10}$, while the 2-1 split occurs with probability $\frac{9}{10}$. By linearity of expectation, we find $\mathbb{E}[X_4] = \frac{1}{10} \cdot 1 + \frac{9}{10} \cdot \frac{2}{3} = \frac{7}{10}$.

The expected value of round 5 can be conditioned on a 2-0 split which occurs with probability $\frac{2}{5}$ or a 1-1 split which occurs with probability $\frac{3}{5}$. By linearity of expectation, we find $\mathbb{E}[X_5] = \frac{2}{5} \cdot 1 + \frac{3}{5} \cdot \dfrac{1}{2} = \frac{7}{10}$.

By round 6, there is only one card remaining, so $\mathbb{E}[X_6] = 1$. Putting it all together, our expected value is $\frac{1}{2} + \frac{3}{5} + \frac{3}{5} + \frac{7}{10} + \frac{7}{10} + 1 = \frac{5+6+6+7+7+10}{10} = \frac{41}{10}$.

Solution to Question 425: Same Heads

To solve this question, we first need to find the probability that each player gets each number of heads. We can use the binomial formula for this.

$$\mathbb{P}[0 \text{ heads}] = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

$$\mathbb{P}(1 \text{ head}) = \binom{4}{1}\left(\frac{1}{2}\right)^4 = \frac{1}{4}$$

$$\mathbb{P}(2 \text{ heads}) = \binom{4}{2}\left(\frac{1}{2}\right)^4 = \frac{3}{8}$$

$$\mathbb{P}(3 \text{ heads}) = \binom{4}{3}\left(\frac{1}{2}\right)^4 = \frac{1}{4}$$

$$\mathbb{P}(4 \text{ heads}) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

Since the probability of rolling a certain amount of heads is independent for the two people, we can just square all these probabilities to find the probability that two people get the same number of heads.

$$\mathbb{P}(0 \text{ heads for both}) = \left(\frac{1}{16}\right)^2 = \frac{1}{256}$$

$$\mathbb{P}(1 \text{ head for both}) = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

$$\mathbb{P}(2 \text{ heads for both}) = \left(\frac{3}{8}\right)^2 = \frac{9}{64}$$

$$\mathbb{P}(3 \text{ heads for both}) = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

$$\mathbb{P}(4 \text{ heads for both}) = \left(\frac{1}{16}\right)^2 = \frac{1}{256}$$

Summing up these probabilities, we get

$$\mathbb{P}(\text{same \# of heads}) = 2 \cdot \frac{1}{256} + 2 \cdot \frac{1}{16} + \frac{9}{64} = \frac{35}{128}$$

Solution to Question 426: No Heads

Let H be the number of heads obtained. We know that give $N = n$, we flip the coin n times. The probability that none of the coins appear heads in n flips is $\left(\frac{1}{2}\right)^n$. Therefore, this is $\mathbb{P}[H = 0 \mid N = n]$.

We want $\mathbb{P}[H = 0] = \sum_{n=0}^{\infty} \mathbb{P}[H = 0 \mid N = n]\mathbb{P}[N = n]$ by Law of Total Probability. We know $\mathbb{P}[N = n] = \frac{2^n}{n!}e^{-2}$ by our known distribution of N. We just calculated the other term above, so $\mathbb{P}[H = 0] = \sum_{n=0}^{\infty} \frac{1}{2^n} \cdot \frac{2^n}{n!} e^{-2} = e^{-2} \sum_{n=0}^{\infty} \frac{1}{n!}$. This remaining sum is just the Taylor expansion of e, so $\mathbb{P}[H = 0] = e^{-1}$. This means $a = -1$.

Solution to Question 427: Taylor Sum

Using Taylor Series, we can write $e^{x^2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}$. Therefore,

$$f(x) = (1+x+2x^2+3x^3) \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} + x \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} + 2x^2 \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} + 3x^3 \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}$$

Multiplying in the powers of x, we have that

$$f(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} + \sum_{k=0}^{\infty} \frac{x^{2k+1}}{k!} + 2\sum_{k=0}^{\infty} \frac{x^{2k+2}}{k!} + 3\sum_{k=0}^{\infty} \frac{x^{2k+3}}{k!}$$

To find the 7th derivative at 0, we need to sum up $c_7 \cdot 7!$ in each sum, where c_7 is the coefficient of x^7. However, we quickly see that the first and third sums do not have any such term, as all of their terms have even powers. Therefore, those can immediately be eliminated. Looking at the second sum, we see that $c_7 = \dfrac{1}{3!} = \dfrac{1}{6}$, as we get an exponent of 7 with $k = 3$. Therefore, we get a contribution of $\dfrac{7!}{3!}$ from that term. From the last sum, we see that $k = 2$ yields an exponent of 7, so the coefficient there is $\dfrac{3}{2!} = \dfrac{3}{2}$, so that sum adds a total of $\dfrac{3}{2} \cdot 7!$. Adding these two terms up, we get a sum of 8400, which we can conclude to be $f^{(7)}(0)$.

Solution to Question 428: Visible Number of Heads

We decompose $X = X_1 + X_2 + ... + X_{20}$ where we define:

$$X_i = \begin{cases} 1 & \text{if person } i \text{ is taller than everyone before him,} \\ 0 & \text{otherwise.} \end{cases}$$

We then claim that $E[X_i] = \frac{1}{i}$, since among the first i people, they are all equally likely to be the tallest, so the probability that the $i-$th person is tallest is $\frac{1}{i}$. Then, it follows from Linearity of Expectation that

$$\mathbb{E}[X] = \sum_{i=1}^{20} \frac{1}{i} \approx 3.6$$

For large n (number of people), a good approximation is $\log(n)$.

Solution to Question 429: Infinite Exponents

Let $y = x^{x^{\cdots}} = 2$. Then,

$$x^y = x^{x^{\cdots}} x^2 = 2x = \sqrt{2}$$

Solution to Question 430: Oil Transport

The key realization in this question is how to place the storage ports to maximize the amount of oil you are carrying per oil lost. This realization leads to the idea of having multiples of 1000 gallons at any storage port. If you don't, you aren't maximizing the efficiency metric above. Where do we place the first storage port to hold 2000 gallons of oil? Well, we need to leave $\frac{2000}{3}$ gallons every time we arrive there from Port A. This means that the distance should be $1000 - \frac{2000}{3} = \frac{1000}{3}$ miles away from Port A. If we only have this storage port, we will bring $1000 - \frac{2000}{3} = \frac{1000}{3}$ gallons twice to Port B $= \frac{2000}{3} \approx 667$ gallons. But we can do better by having another storage port that is a multiple of 1000 gallons, that being the port that will have 1000 gallons. From the first storage port, we can go 500 miles and drop off 500 gallons to the second storage port twice. This means this second storage port should be located 500 miles away from the first storage port $= 500 + \frac{1000}{3} = \frac{2500}{3}$ miles away from Port A. Now we have 1000 gallons of oil that is $1000 - \frac{2500}{3} = \frac{500}{3}$ miles away from Port B. We can carry these 1000 gallons all at once and lose $\frac{500}{3}$ gallons of oil which leaves us with $1000 - \frac{500}{3} = \frac{2500}{3} \approx 833$ gallons of oil at Port B.

Solution to Question 431: ITM Expiration

If the put expires out-of-the-money, this means that the call must expire in-the-money. The probability that the put expires out-of-the-money is $1 - 0.42 = 0.58$ as the put must expire either ITM or OTM. Then, the probability that the call expires in-the-money is also 0.58.

Solution to Question 432: Least Multiple of 15

In order for a number to be a multiple of 15, it must be perfectly divisible by both 3 and 5. In order for the number to be divisible by 5, it must end in either a 0 or 5- thus, our number must end in a 0. In order for the number to be divisible by 3, the sum of its digits must be a multiple of 3- thus, we will add three 1's digits to the number. Hence, the smallest positive multiple of 15 whose digits consist of 1's and 0's only is 1110.

Solution to Question 433: Russian Roulette III

If you don't spin the barrel, the probability that you survive is $\frac{3}{5}$, since one of the safe chambers has already been pulled by your friend. If you do spin the barrel, the probability that you survive is $\frac{4}{6}$, since any of the 6 chambers is possible, 2 of which are lethal. Thus, the difference in probability is:

$$\frac{4}{6} - \frac{3}{5} = \frac{1}{15}$$

Solution to Question 434: Marketing Claims

We are testing the null hypothesis $H_0 : p = 0.25$ against the alternative hypothesis $H_a : p < 0.25$. We are given $n = 80$, $p = 0.25$, and $\alpha = 0.05$, and are looking for $\hat{p}$. The z statistic when $\alpha = 0.05$ is -1.645, which can be substituted into the appropriate formula:

$$z = \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} = \frac{\hat{p} - 0.25}{\sqrt{\frac{0.25(1-0.25)}{80}}} = -1.645 \Rightarrow \hat{p} \approx 0.17$$

Solution to Question 435: Deriving Log-Price Dynamics

We can use Ito's Lemma on $f(x) = \ln(x)$. We have $f'(x) = \frac{1}{x}$ and $f''(x) = -\frac{1}{x^2}$.

Ito's Lemma says that $df = f'(S_t)dS_t + \frac{1}{2}f''(S_t)(dS_t)^2$. Plugging in the dynamics from above into dS_t, the derivatives, and using the fact that $(dS_t)^2 = 16S_t^2 dt$, we get the following.

$$\begin{aligned} df &= \frac{1}{S_t}(3S_t dt + 4S_t dW_t) + \frac{1}{2}\frac{-1}{S_t^2}16S_t^2 dt \\ &= 3dt + 4dW_t - 8dt \\ &= -5dt + 4dW_t \end{aligned}$$

So, we have $a = -5$ and $b = 4$. Plugging the values in, we get $(-5)^2 + 4^2 = 41$.

Solution to Question 436: Equal Flip Timer

By symmetry, we know that $\mathbb{P}[B] = \mathbb{P}[C]$, as they are both flipping coins with the same heads probability. Therefore, as we want all three terms to be equal, they all should have probability $\frac{1}{3}$, as they sum to 1. In other words, we need to find a value p such that the probability both of them obtain their first heads at the same time is $\frac{1}{3}$.

The probability it takes exactly k flips for each of them would be $p(1-p)^{k-1}$, as they get $k-1$ tails and 1 head. Thus, the probability both get it in exactly k flips is $p^2(1-p)^{2(k-1)}$. Summing this up from $k = 1$ to ∞ yields

$$\sum_{k=1}^{\infty} p^2(1-p)^{2(k-1)} = p^2 \sum_{k=1}^{\infty} \left((1-p)^2\right)^{k-1} = \frac{p^2}{1-(1-p)^2}$$

Setting this expression above equal to $\frac{1}{3}$,

$$\frac{p^2}{1-(1-p)^2} = \frac{1}{3}$$

The only solution in $(0,1)$ is $p = \frac{1}{2}$.

Solution to Question 437: Poisoned Kegs I

Each patient can serve as a binary indicator for each keg. Hence, only 1 month is needed to determine the poisoned keg. As an example, suppose our kegs are labeled $K_1, K_2, \ldots, K_8$. Then, we could have servant 1 try K_1, K_3, K_5, K_7, servant 2 try K_2, K_3, K_6, K_7, and servant 3 try K_4, K_5, K_6, K_7. Each possible configuration of how the 3 servants die corresponds to one possible poisoned keg.

Solution to Question 438: Russian Dolls

If we label the dolls $1-7$, with 7 being the largest and 1 being smallest, note that every subset of $\{1, 2, \ldots, 7\}$ of size at least 2 creates a valid nesting by arranging them with the largest element of the set on the outside, second largest element inside the first largest, etc. There is 1 subset of $\{1, 2, \ldots, 7\}$ of size 0 and $\binom{7}{1} = 7$ subsets of size 1. There are $2^7 = 128$ total subsets, so there are $128 - 8 = 120$ orderings.

Solution to Question 439: Option Arbitrage

For this to be an arbitrage, there must be a 0 probability that we lose money. In other words, our constructed portfolio must have a non-negative payoff everywhere **and** we must pay at most 0 to enter the position.

For this to occur, we see that we must construct a butterfly spread. To do this, we buy 1 unit of the 1000-strike call, short 2 units of the 1010-strike call, and buy 1 unit of the 1020-strike call. The cost to enter this spread is -0.25. In other words, we are paid 25 cents to enter a spread, in which there is a non-negative payoff everywhere. We are guaranteed to *at least* collect the 25 cents.

Solution to Question 440: The Picking Hat

We want to determine an optimal threshold x, where, if AJ's draw is at least x, then he will not play another round. Notice that, if AJ draws again, without taking into consideration the \$1 penalty, then AJ's expected payoff is the same as the previous round. So, we can write the following expression, where A denotes the total winnings.

$$\begin{aligned}\mathbb{E}[A] &= -1 + \sum_{i=x}^{100} \frac{i}{100} + \sum_{i=1}^{x-1} \frac{\mathbb{E}[A]}{100} \\ &= -1 + \frac{(100-x+1)(100+x)}{200} + \frac{x-1}{100}\mathbb{E}[A]\end{aligned}$$

Let's isolate $\mathbb{E}[A]$.

$$\mathbb{E}[A] = \frac{(101-x)(100+x)-200}{2(101-x)}$$

We now wish to determine

$$x_{\text{optimal}} = \arg\max_x \left\{ \frac{(101-x)(100+x)-200}{2(101-x)} \right\}.$$

Let's take the derivative with respect to x.

$$\begin{aligned}\frac{d}{dx}\mathbb{E}[A] &= \frac{2(101-x)(-2x+1)+2((101-x)(100+x)-200)}{4(101-x)^2} \\ &= \frac{10001-202x+x^2}{4(101-x)^2}\end{aligned}$$

Setting $\frac{d}{dx}\mathbb{E}[A] = 0$ and solving for x, we find

$$\begin{aligned}10001 - 202x_{\text{optimal}} + x_{\text{optimal}}^2 &= 0 \\ x_{\text{optimal}} &= \frac{202 \pm \sqrt{800}}{2} \\ &= 101 \pm 10\sqrt{2}\end{aligned}$$

Since $x \leq 100$, $x_{\text{optimal}} = 101 - 10\sqrt{2} \approx 86.9$. Our optimal integer x is then either 86 or 87. Plugging both into $\mathbb{E}[A]$, we find $\mathbb{E}[A]$ is maximized when $x = 87$ and has value $\frac{1209}{14}$.

Solution to Question 441: Proper Tables

Attach the first leg of the table at an arbitrary position. Since the density is uniform, we do not need to worry about anything. Now, we know that the other legs can't be within 90 degrees of the first leg. Therefore, we can imagine shading out a region spanning 90 degrees in each direction emanating from the first leg location. This shades out half of the table. From this point on, it may be helpful to get a piece of paper and draw all of this out to convince yourself of the arguments below.

Now, we need to ensure that the remaining two legs are not within 90 degrees of one another. However, this probability is dependent upon the location of the second leg. This is because wherever we place the second leg, there is going to be some overlap potentially in the region that is shaded out because of it. How much overlap there is depends on how far away the first leg is from the second. We know that the valid range to put the second leg is 90 to 270 degrees (all orientations are assumed to be CCW) away from the first. By the symmetry of our region and our uniform distribution of the points, as the region that is available to place the third point in is equal in length (and hence probability) if we put the second at $180+x$ versus $180-x$, we can just consider the region 90 to 180 degrees away from the first point and double it to get the total probability.

If we place our second leg at $90 + x$ degrees, $0 < x < 90$, then there are $90 - x$ degrees remaining at which we can place the third leg. Therefore, by using continuous version of Law of Total Probability the probability we are interested in is $\int_0^{90} 2 \cdot \frac{1}{360} \cdot \frac{90-x}{360} dx$. Recall that the 2 is because we double to account for the other region of 180 to 270 degrees away CCW. Therefore, $\frac{2}{360^2} \int_0^{90} 90 - x dx = \frac{1}{16}$, which is our solution.

Solution to Question 442: Optimal Ball Draw

There is a $\frac{2}{5}$ chance that Rajiv draws the first white ball on the first draw. There is a $\frac{3}{5} \cdot \frac{2}{4} = \frac{3}{10}$ chance that Rajiv draws the first white ball on the second draw. There is a $\frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} = \frac{1}{5}$ chance that Rajiv draws the first white ball on the third draw. And finally, there is a $\frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} = \frac{1}{10}$ chance that Rajiv draws the first white ball on the fourth draw.

If Rajiv draws a white ball on the first draw, then if he were to keep going until the last white ball is drawn, the expected value of his gain/loss would be $\frac{1}{4} \cdot 1$ (HTTT) $+ \frac{1}{4} \cdot 0$ (THTT) $+ \frac{1}{4} \cdot -1$ (TTHT) $- \frac{1}{4} \cdot -2$ (TTTH) $= -\frac{1}{2}$. We conclude that if Rajiv draws his first white ball on the first draw, the he should stop immediately and receive the payoff of +\$1.

If Rajiv draws a white ball on the second draw, then if he were to keep going until the last white ball is drawn, the expected value of his gain/loss would be $\frac{1}{3} \cdot 1$ (HTT) $+ \frac{1}{3} \cdot 0$ (THT) $+ \frac{1}{3} \cdot -1$ (TTH) $= 0$. So, if Rajiv draws his first white ball on the second draw, his payoff will be $0 + 0 = +\$0$ regardless of whether he keeps drawing until the last head is drawn.

If Rajiv draws a white ball on the third draw, then if he were to keep going until the last white ball is drawn, the expected value of his gain/loss would be $\frac{1}{2} \cdot 1$ (HT) $+ \frac{1}{2} \cdot 0$ (TH) $= 0$. So, if Rajiv draws his first white ball on the third draw, he should keep drawing until he gets the second white ball, with total payoff $-1 + \frac{1}{2} = -\$\frac{1}{2}$

And of course, if Rajiv gets 3 blacks in a row for his first 3 draw, he should finish drawing the last two white balls, which gives him a payoff of $-3+2 = -\$1$.

In total, his expected profit is $\frac{2}{5} - \frac{1}{5} \cdot \frac{1}{2} - \frac{1}{10} = \frac{1}{5}$.

Note that, while the above solution is more methodical, one could simply write out all $\frac{5!}{2!3!} = 10$ permutations of 2 white and 3 black balls. It would then be much faster to visualize potential strategies.

Solution to Question 443: Particle Reach VII

We generalize this for $p \geq 1/2$. Let T_1 be the number of steps needed to move from position 0 to 1. We want $\mathbb{E}[T_1]$. We use Law of Total Expectation to condition on what happens at the first step. If the particle moves right at the first step, which occurs with probability p, then $T_1 = 1$, as the particle has hit 1. Otherwise, with probability $1-p$, the particle moves left. The expected number of steps in this would be $1+2\mathbb{E}[T_1]$, as the number of steps needed to move from -1 to 0 and 0 to 1 are the same by the Markov Property. This means that

$$\mathbb{E}[T_1] = p \cdot 1 + (1-p) \cdot (1 + 2\mathbb{E}[T_1])$$

Rearranging and solving yields $\mathbb{E}[T_1] = \dfrac{1}{2p-1}$. With $p = 1/2$, this yields ∞. Therefore, the answer is -1. Note that with $p = 1/2$, the random walk is symmetric. This shows that 1 (or really any position) is a null recurrent state, as the probability of reaching it in finite time is 1 but the expected number of steps needed to reach the state is infinite.

Solution to Question 444: Dice Roll Intuition

Consider the transition graph for both cases, where we consider three states: (A) no specific numbers, (B) the first specific number, and (C) 2 specific numbers in a row. There is a slight difference in the two transition graphs; namely, there is a $\frac{2}{3}$ probability of returning to state A when we work with case 1, whereas there is a $\frac{5}{6}$ probability of returning to state A when considering case 2. Since we want to maximize the number of rolls it takes for us to finish the game, we would prefer case 2.

Solution to Question 445: Odd Stars

We can write an odd integer x as $x = 2k + 1$ for some non-negative integer k. Thus, let $x_i = 2k_i + 1$ for non-negative integers k_i. Then $(2k_1 + 1) + \cdots + (2k_6 + 1) = 96$ for $k_i \geq 0$ integers. Rearranging, this is the same as $k_1 + \cdots + k_6 = 45$ with $k_i \geq 0$ are integers. There are $\binom{50}{5}$ solutions to this by stars and bars, so our answer is $50 \cdot 5 = 250$.

Solution to Question 446: Dart Distance

Let R be the radial distance from the center. We can compute $\mathbb{E}[R]$ using our classic trick of $\mathbb{E}[R] = \displaystyle\int_0^\infty \mathbb{P}[R \geq r]dr$. We only need to integrate this on $[0, 1]$ since the dart can't land more than a distance 1 from the center. The event $\{R \geq r\}$ means that the dart lands in the "donut" region that is outside of the circle of radius r centered at the origin. The total area of the dartboard is π, and the area of the circular region the dart can't land in is πr^2, so the probability that the dart lands outside the circle of radius r centered at the origin is $\dfrac{\pi - \pi r^2}{\pi} = 1 - r^2$. Therefore, we have that

$$\mathbb{E}[R] = \int_0^1 1 - r^2 dr = \frac{2}{3}$$

Solution to Question 447: Put Theta

Theta is maximized (absolutely) at-the-money, in this case, when $S = K$. Since we buy a put, we will be paying theta and thus have negative theta. If the price increases, we go away from the strike (maximal absolute theta). This means that our theta will decrease absolutely. We pay theta, and thus our new theta will approach 0. This means that our theta increases.

Solution to Question 448: Coin Toss Game II

There are several states to this situation so we can solve this question using Markov Chains. Let $P_{x,y}$ be the probability Alice wins at the state where Alice has flipped x heads and Bob has flipped y heads and it is Alice's turn. We then arrive at these following equations:

$$P_{0,0} = \frac{1}{4} \cdot P_{0,0} + \frac{1}{4} \cdot P_{1,0} + \frac{1}{4} \cdot P_{0,1} + \frac{1}{4} \cdot P_{1,1}$$

$$P_{1,0} = \frac{1}{2} + \frac{1}{4} \cdot P_{1,0} + \frac{1}{4} \cdot P_{1,1}$$

$$P_{0,1} = \frac{1}{4} \cdot P_{0,1} + \frac{1}{4} \cdot P_{1,1}$$

$$P_{1,1} = \frac{1}{2} + \frac{1}{4} \cdot P_{1,1}$$

Solving these equations yields $P_{0,0} = \frac{16}{27}$.

Solution to Question 449: Bakery Boxes

Given that he chooses the value k, we want to find the number of non-negative integer solutions to the equation $x_1 + \cdots + x_k = k$. This is because we can treat each commodity he puts in the gift basket as a separate "item", and we want the number of distinct combinations of different quantities of those items. By Stars and Bars, this is $\binom{k+k-1}{k-1} = \binom{2k-1}{k-1}$. Therefore, since each value of k is equally likely to be selected, the answer will just be $\frac{1}{4}\left[\binom{1}{0} + \binom{3}{1} + \binom{5}{2} + \binom{7}{3}\right] = \frac{49}{4}$.

Solution to Question 450: Tic Tac Toe

Embed the $4 \times 4 \times 4$ cube in the larger $6 \times 6 \times 6$ cube. Every winning path of the $4 \times 4 \times 4$ cube corresponds to ending at two unique ending squares on the surface of the $6 \times 6 \times 6$ cube when extending those paths out to the larger cube. To visualize this, try this with a standard 2D 3×3 tic tac toe grid embedded in a 5×5 grid and extend out the winning paths. Therefore, for each path, we can put it in two-to-one correspondence with one of the $6^3 - 4^3 = 152$ cubes on the surface of the larger cube. It is a two-to-one correspondence since each path ends at two unique points on the surface of the larger cube. Therefore, the total number of paths must be $\frac{152}{2} = 76$.

Solution to Question 451: Combinatorial Electrician

There are $\binom{12}{4} = 495$ ways to pick a subset of 4 light bulbs to be in our circuit. Therefore, label the circuits $1 - 495$ and let $X_1, \ldots, X_{495}$ be the indicators that circuit i is a complete $4-$circuit. Then $T = X_1 + \cdots + X_{495}$ gives the total number of $k-$circuits. By the exchangeability of the circuits (no circuit is more likely to be complete than any other) and linearity of expectation, $\mathbb{E}[T] = 495\mathbb{E}[X_1]$. $\mathbb{E}[X_1]$ is just the probability that circuit 1 is $4-$complete. We know that between any 2 pairs of light bulbs, there must be an active wire, and there are 4 light bulbs here, so there are $\binom{4}{2} = 6$ active wires for this circuit to be complete. The probability of all 6 being active is $\frac{1}{2^6} = \frac{1}{64}$. Therefore, $\mathbb{E}[T] = \frac{495}{64}$.

Solution to Question 452: Collecting Toys I

Choosing a distinct toy on each new box follows a geometric distribution. For the first distinct toy, we know that the first box we open will be a new toy. For the second distinct toy, there is a $\frac{4}{5}$ probability of receiving a new toy; thus, it should take $\frac{5}{4}$ boxes on average to receive a new toy. This logic follows for the third $(\frac{5}{3})$, fourth $(\frac{5}{2})$, and fifth $(\frac{5}{1})$ toy. Hence, the total number of boxes we opened on average to collect all five toys is:

$$1 + \frac{5}{4} + \frac{5}{3} + \frac{5}{2} + \frac{5}{1} = \frac{137}{12}$$

Solution to Question 453: Word Shift I

There are 3 B and 4 O in the word above. There are $\binom{11}{7}$ ways to pick the locations of these 7 letters in our anagram. However, there is only one anagram of $BBBOOOO$ that has all of the B's before all of the O's. Namely, it is the one listed there. Afterwards, the other 4 letters $LAHU$ can be arranged in the remaining 4 blanks completely at will in 4! ways. Therefore, the answer is

$$4! \cdot \binom{11}{7} = \frac{11!}{7!} = 7920$$

Solution to Question 454: Soccer Practice

Among the 98 remaining penalty kicks, you must score exactly 65, which occurs $\binom{98}{65}$ ways. Let us now find the probability of each one of these outcomes occurring. The denominator of the probability must be 99!, which is the product of the total number of kicks at each time step ($\frac{1}{2} \times \frac{x}{3} \times \frac{y}{4} \times ... = \frac{z}{99!}$). Similarly, the numerator must be $65! \times 33!$, since there must be exactly 65 successful kicks and exactly 33 failed kicks. Thus, the probability of any such arrangement of is $\frac{65!33!}{99!}$, and the probability that you score exactly 66 penalty kicks is:

$$\binom{98}{65} \times \frac{65! \times 33!}{99!} = \frac{1}{99}$$

Solution to Question 455: Leg Count

Let x be the number of cows. Then $2x$ is the number of chickens and $2(2x) = 4x$ is the number of spiders. Therefore, with x cows, there are $4x + 2(2x) + 8(4x) = 40x$ legs. Therefore, $40x = 440$, so $x = 11$. The number of spiders then is $4 \cdot 11 = 44$.

Solution to Question 456: Sum Exceedance III

By Sum Exceedance I, we know that the expected number of uniforms to exceed 1 is e. Namely, we have that $S_{N_1} = \sum_{i=1}^{N_1} X_i$ gives the value of the sum after it exceeds 1. As N_1 is a stopping time, we can say $\mathbb{E}[S_{N_1}] = \mathbb{E}[N_1]\mathbb{E}[X_i] = \frac{e}{2}$. Therefore, $c = \frac{1}{2}$.

Solution to Question 457: Identical Hands

This is going to boil down to a combinatorics problem. First let us determine the denominator, which is total combinations of cards that our dealer can give out. For our first player, he as 12 cards and he is going to choose 3 of them to give, yielding our first term as $\binom{12}{3}$. Similarly for the second player, there are 9 cards, and he will be giving out 3 of them, so our denominator will then be multiplied by $\binom{9}{3}$. This pattern continues, yielding our denominator as $\binom{12}{3} \cdot \binom{9}{3} \cdot \binom{6}{3} \cdot \binom{3}{3}$.

For our numerator, there are 4 ways to choose the first king for the first player, 3 ways to choose the second king for the second player, 2 for the third, and 1 for the last player. This pattern is similar for the queens and jacks, leaving our final equation as:

$$\frac{4! \cdot 4! \cdot 4!}{\binom{12}{3} \cdot \binom{9}{3} \cdot \binom{6}{3} \cdot \binom{3}{3}} = \frac{72}{1925}$$

Solution to Question 458: Returning Books

Let X be the number of hours the system takes to register the return action, and let I be an indicator variable that denotes if the book is returned to another library; I has mean 0.3 and variance $0.3(1-0.3) = 0.21$. We can rewrite X as $X = 2 + 3I$, as it takes two hours if it's returned correctly (i.e. $I = 0$) and five if not. Thus, we can solve for $V[X]$:

$$V[X] = V[2 + 3I] = 9 \times V[I] = 1.89$$

Solution to Question 459: Slippery Ladder I

We can view the ladder as the hypotenuse of a triangle whose sites are the ground and the wall. Using Pythagorean Theorem, we find that

$$30^2 + y^2 = 50^2 \iff y = 40$$

Therefore, the tip of the ladder is 40 feet above ground at that time. We now take the derivative of both sides of $x^2 + y^2 = z^2$. We know that $x' = 3$, as the base length is increasing as the ladder slips down the wall. Furthermore, $z' = 0$, as the ladder doesn't change in length. We want y'. The derivative is $2xx' + 2yy' = 2zz'$. However, the RHS is 0 by the fact $z' = 0$. Therefore, we can just plug in and solve for y'.

$$2(30)(4) + 2(40)y' = 0 \iff y' = -3$$

This means that our answer is 3, as the ladder slides down the wall at a rate of 3 feet per minute.

Solution to Question 460: 3 Card Straight

Since each deck has 52 cards, there are $\binom{416}{3}$ ways to pick the three cards with order irrelevant. Now, to count the total number of straights, we should include all types of straights first and then remove straight flushes, as it is difficult to count it directly.

The straight can be in the form $A23, 234, \ldots, QKA$, which is 12 options. Then, there are 32 replicates of each rank in the deck, as you get 4 replicates per deck. This means that there are 32 options per rank in the deck, so there are $12 \cdot 32^3$ straights. To remove straight flushes, there are 12 ways to pick the straight, 4 ways to pick the suit, and now 8 cards of each rank and suit in the deck (as we have 8 replicate decks), so there are $12 \cdot 4 \cdot 8^3$ straight flushes. This means that the total number of straights is $12 \cdot 32^3 - 12 \cdot 4 \cdot 8^3 = 368640$

Putting this all together, our probability is $\dfrac{368640}{\dbinom{416}{3}} = \dfrac{768}{24817}$

Solution to Question 461: Crossing Pairs

Suppose the pairs of points are picked sequentially, then for our first point pick any point along the circle. As for the possible partner of the first point, we can either choose to pick its left or right neighbor, or the point across from it. Any other choice will result in one point trapped inside the formed chord forcing an intersection.

First, the probability that the first point is matched with a left/right neighbor is $\frac{2}{5}$. Applying the same logic now to the remaining four points, we get that the probability these pairs do not intersect is $\frac{2}{3}$.

In the other $\frac{1}{5}$ chance that the first point is paired with the point directly across, we need the remaining four points not to intersect the chord. The only way for this to happen is if the pairs on each side of the chord are paired with each other. This has probability of $\frac{1}{3}$.

Putting these together, we find that the probability of no intersection is $\frac{2}{5}(\frac{2}{3}) + \frac{1}{5}(\frac{1}{3}) = \frac{1}{3}$.

Solution to Question 462: Minecraft Mine

We have that $T = \displaystyle\sum_{i=1}^{N} X_i$, where X_i is the time spent on the ith selection of tunnel. This is because Steve has to do N trials to escape, since N is the number of trials needed before he gets out. We just have to take the average of the three times spent going down the three paths to get $\mathbb{E}[X_i]$, as they are equally likely. Thus,

$$\mathbb{E}[X_i] = \frac{2+4+8}{3} = \frac{14}{3}$$

This directly implies that $B_n = \dfrac{14}{3}n$ by linearity of expectation.

To find A_n, if we have that $N = n$, that means on the first $n-1$ runs of the tunnel, we have that Steve took either path 1 or 2 instead of 3, and then on the nth run, he chooses path 3. We thus have that

$$\mathbb{E}[T \mid N = n] = \mathbb{E}\left[\sum_{i=1}^{n-1} X_i + X_n \;\middle|\; X_i \neq 8, 1 \leq i \leq n-1, X_n = 8\right]$$

These conditional expectations are equivalent because we have that $X_i = 8$ corresponds to selecting path 3. Thus, simplifying the conditional expectation, we have that this is equivalent to $\sum_{i=1}^{n-1} \mathbb{E}[X_i \mid X_i \neq 8] + 8$ by linearity. The expectation in the sum is just the conditional expectation of X_i given that it is either 2 or 4. Since these occur with equal probability, the conditional expectation inside the expectation is just 3. Since we sum it $n-1$ times, we have that $A_n = \mathbb{E}[T \mid N = n] = 3n + 5$. This means that $\frac{A_n}{B_n} = \frac{9n+15}{14n}$. The limit of this function is just $\frac{9}{14}$, as it is the coefficients of n divided.

Solution to Question 463: Pick Your Urn Wisely

Lets calculate the expected payout if you switch urns versus if you stay with your current urn. For notational purposes, let urn 1 be the urn that started with 4 \$1 chips and urn 2 be the one that started with 3. Define the random variable X as the urn that you selected your first chip from. We can say that $\mathbb{P}[X = 1] = \frac{4}{3}\mathbb{P}[X = 2]$ as urn 1 started with 4 \$1 chips. Therefore, as those 2 values must sum to 1, $\mathbb{P}[X = 1] = \frac{4}{7}$ and $\mathbb{P}[X = 2] = \frac{3}{7}$.

Let P_0 be the payout if we stay with the same urn. If we don't switch, then $\mathbb{E}[P_0] = \mathbb{E}[P_0 \mid X = 1]\mathbb{P}[X = 1] + \mathbb{E}[P_0 \mid X = 2]\mathbb{P}[X = 2]$. If $X = 1$, then after drawing one \$1 chip, there are 6 \$10 and 3 \$1 chips left, so $\mathbb{E}[P_0 \mid X = 1] = \frac{63}{9} = 7$. Similarly, there would be 7 \$10 and 2 \$1 chips after the first draw if $X = 2$, so $\mathbb{E}[P_0 \mid X = 2] = \frac{72}{9} = 8$. Plugging these values in yields

$$\mathbb{E}[P_0] = 7 \cdot \frac{4}{7} + 8 \cdot \frac{3}{7} = \frac{52}{7}$$

Let P_1 be your profit if you switch. Similarly, we have $\mathbb{E}[P_1] = \mathbb{E}[P_1 \mid X = 1]\mathbb{P}[X = 1] + \mathbb{E}[P_1 \mid X = 2]\mathbb{P}[X = 2]$. If $X = 1$, then by switching, we pick from an untouched urn 2, so $\mathbb{E}[P_1 \mid X = 1] = \frac{73}{10} = 7.3$. Similarly, if $X = 2$, then we pick from an untouched urn 1, so $\mathbb{E}[P_1 \mid X = 2] = \frac{64}{10} = 6.4$. Therefore,

$$\mathbb{E}[P_1] = 6.4 \cdot \frac{3}{7} + 7.3 \cdot \frac{4}{7} = \frac{242}{35} < \frac{245}{35} = 7$$

Therefore, you should not switch, and your expected payout is $\frac{52}{7}$.

Solution to Question 464: Minimum Variance Portfolio

We are going to solve this with more general values the means and variances.

Let R_A and R_B be the returns of stock A and B, respectively. We have that if you purchase k shares of stock B, then the return of your portfolio is given by $R_A + kR_B$. We have that

$$\text{Var}(R_A + kR_B) = \text{Var}(R_A) + \text{Var}(kR_B) + 2\text{Cov}(R_A, kR_B)$$

By properties of variance and covariance, the previous expression is equal to $\sigma_A^2 + k^2\sigma_B^2 + 2k\text{Cov}(R_A, R_B)$.

From the definition of correlation, $\text{Cov}(R_A, R_B) = \rho\sigma_A\sigma_B$. Thus, we have that the variance (as a function of k) is $k^2\sigma_B^2 + 2\rho\sigma_A\sigma_b k + \sigma_A^2$. Note that this is a quadratic function in k, so to find the minimum k, we need to just use the handy formula $-\frac{b}{2a}$ to find the minimum. This means that $k = -\frac{2\rho\sigma_A\sigma_B}{2\sigma_B^2} = -\rho\frac{\sigma_A}{\sigma_B}$ is our minimum. Note that if $\rho > 0$, then this implies we should short B.

Plugging in our specific values of $\sigma_A = 2, \sigma_B = 3$, and $\rho = -1/2$, we get that $k = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$

Solution to Question 465: Matching Socks I

By Pigeon Hole Principle, you will need to have 4 socks to guarantee that at least two have the same color. This is because in the worst-case scenario, the first 3 socks you grab are each of a different color- the fourth sock must pair one of the first 3 socks.

Solution to Question 466: Coin Runs

The first flip will always start a new run but every flip after that has a $\frac{1}{2}$ chance of starting a new run. With 99 flips, each having $\frac{1}{2}$ chance of starting a new run, we get $99 \cdot \frac{1}{2} = 49.5$. We add one more for the first flip and our final answer is $49.5 + 1 = 50.5$ runs.

Solution to Question 467: Company Purchase II

To calculate, we use the following equation:

Company Value $= \frac{\text{Cash Flow}}{\text{Discount Rate - Cash Flow Growth Rate}}$ where Cash Flow Growth Rate ¡ Discount Rate.

So, Company Value $= \frac{100}{10\%-5\%} = \frac{100}{.05} = \$2000$

Solution to Question 468: Die Multiple I

We condition on the parity of the first roll. Let N be the number of rolls needed for this and E be the event that the first roll is even. Then

$$\mathbb{E}[N] = \mathbb{E}[N \mid E]\mathbb{P}[E] + \mathbb{E}[N \mid E^c]\mathbb{P}[E^c]$$

As the die is fair, $\mathbb{P}[E] = \mathbb{P}[E^c] = \frac{1}{2}$. If the first roll is even, we are done in one roll, so $\mathbb{E}[N \mid E] = 1$. If the first roll is odd, then we just need to roll another odd value to obtain an even sum. There is probability $\frac{1}{2}$ per trial of rolling an odd value, so the distribution of the number of rolls needed to see another odd value is Geom(0.5), which has mean 2. Therefore, $\mathbb{E}[N \mid E^c] = 3$, as we must account for the 1 roll at the beginning. This means our total answer is $\mathbb{E}[N] = \frac{1+3}{2} = 2$.

Solution to Question 469: Positive 25

Let x_1, x_2, and x_3 be the three integers. This question is asking the number of solutions to the equation $x_1 + x_2 + x_3 = 25$, where $x_1, x_2, x_3 \geq 1$ are integers. We can solve this by shifting this back to a problem whose solutions just need to be non-negative, as we know how to solve that with stars and bars. Since the "objects" (i.e. the individual 1's here) are indistinguishable, distribute each of the x_i's 1 to start with. Then, we have 22 objects (i.e. 1's) remaining and the arrangements are not restricted, so this is equivalent to the number of solutions to $x_1 + x_2 + x_3 = 22$, where $x_1, x_2, x_3 \geq 0$. There are $\binom{22+3-1}{3-1} = \binom{24}{2} =$ 276

Solution to Question 470: Swift Betting

We need to calculate our expected profit as a function of x. If we guess x, then the probability Juliana stops before $234x$ is x. In this case, we pay Juliana $-2x$. Therefore, with probability $1-x$, we are paid x. Therefore, the expected profit when guessing x is $x(-2x)+(1-x)x = x-3x^2$. Taking the derivative, we have that the maximizer is when $1-6x = 0$, so $x = \frac{1}{6}$.

Solution to Question 471: Real Roots

Recall there there are no real solutions to this equation when $B^2 - 4C < 0$. In other words, we would need $C > \frac{B^2}{4}$. The joint PDF of B and C is $f_{B,C}(b,c) = be^{-b^2/2} \cdot 2e^{-2c} I_{(0,\infty)}(b) I_{(0,\infty)}(c)$ by independence of B and C. Now, finding this probability, we get

$$\mathbb{P}[C > B^2/4] = \int_0^\infty \int_{b^2/4}^\infty be^{-b^2/2} \cdot 2e^{-2c} dc db = \int_0^\infty be^{-b^2/2} \cdot e^{-b^2/2} db = \int_0^\infty be^{-b^2} db = \frac{1}{2}$$

The first equality comes from just bounding this region in the plane, the second is just the survival function of the Exponential distribution, and the last equality comes from $u-$substitution with $u = b^2$.

Solution to Question 472: Contracts and Pricing III

Removing all the extra useless information, we know $X \sim \text{Exp}(\beta)$ and the fair contract is worth \$0.7 thousand dollars, meaning the expected gain should be 0.

Assuming Kevin bought the contract, he will only exercise his right to purchase Justin's tuition refund if the actual value of the tuition refund is worth more than 0.5 thousand dollars. Otherwise, Kevin will simply take the loss in the price paid for the contract. Kevin's gain/loss, G may be expressed as follows:

$$G = \begin{cases} -0.7 + X - 0.5 & X > 0.5 \\ -0.7 & X \leq 0.5 \end{cases}$$

The expected gain can then be computed via the law of total expectation.

$$\mathbb{E}[G] = \mathbb{E}[G|X > 0.5]\mathbb{P}(X > 0.5) + \mathbb{E}[G|X \leq 0.5]\mathbb{P}(X \leq 0.5)$$

Note that

$$\begin{aligned}
\mathbb{E}[G|X > 0.5] &= -1.2 + \int_{0.5}^{\infty} \frac{x}{\beta} e^{-x/\beta}\, dx \\
&= -1.2 + \left[-xe^{-x/\beta}\right]_{0.5}^{\infty} + \int_{0.5}^{\infty} e^{-x/\beta}\, dx \\
&= -1.2 - \left[e^{-x/\beta}(\beta + x)\right]_{0.5}^{\infty} \\
&= -1.2 + e^{-0.5/\beta}(\beta + 0.5), \\
\mathbb{P}(X > 0.5) &= \int_{0.5}^{\infty} \frac{1}{\beta} e^{-x/\beta}\, dx \\
&= e^{-0.5/\beta}
\end{aligned}$$

So, our expected gain is

$$\begin{aligned}
\mathbb{E}[G] &= \mathbb{E}[G|X > 0.5]\mathbb{P}(X > 0.5) + \mathbb{E}[G|X < 0.5] \\
&= \left(-1.2 + e^{-0.5/\beta}(\beta + 0.5)\right) e^{-0.5/\beta} - 0.7\left(1 - e^{-0.5/\beta}\right) \\
&= e^{-1/\beta}(\beta + 0.5) - 0.5e^{-0.5/\beta} - 0.7 \\
&= 0
\end{aligned}$$

Solving the above equation for β with the help of a computer, we find $\beta \approx 1.5$.

Solution to Question 473: Cooked Steaks

Let A, B, C, and D be the 4 steaks and $A_1, A_2, B_1, B_2, C_1, C_2, D_1$, and D_2 be the sides of all the steaks. The most basic way to do it would be pick three steaks, say A, B, and C, and cook A_1, B_1, and C_1 in the first two minutes, A_2, B_2, and C_2 in the second period of two minutes, and then take 4 minutes to cook D, yielding a total of 8 minutes. This is inefficient because of the treatment of D.

Note that there must be 8 total "cookings", so the idea here is that we can start with A_1, B_1, and C_1 cooked, taking two minutes. Afterwards, we want to get part of D cooked, so now cook A_2, B_2, and D_1, taking two minutes minutes. All that is left now is to cook C_2 and D_2, taking the last two minutes. In this way, we aren't wasting more time with D being the only one on the grill. This yields a total of 6 minutes, which is optimal because you can only cook 6 sides in 4 minutes, and there are 8 sides total.

Solution to Question 474: Sharing Resources

Each person is rationed $\frac{8}{3}$ bushels of wheat, including Charlie. Of this, Alice contributes $5 - \frac{8}{3} = \frac{7}{3}$ and Bob contributes $3 - \frac{8}{3} = \frac{1}{3}$ to Charlie. Because Charlie receives $\frac{8}{3}$ bushels of wheat in exchange for \$8, each $\frac{1}{3}$ bushels costs \$1. In other words, Alice receives \$7 and Bob receives \$1. Alice receives \$6 more than Bob.

Solution to Question 475: Basic Dice Game I

The fair value of the second roll is $\frac{1+2+3+4+5+6}{6} = 3.5$. Thus, you should only opt to re-roll if your first roll was below this, since you know that the second roll will do better on average. In other words, you will not re-roll if you obtain a 4, 5, or 6, which happens $\frac{1}{2}$ of the time with an expectation of 5. Hence, the fair value of this game is:

$$\frac{1}{2} \times 5 + \frac{1}{2} \times 3.5 = 4.25$$

Solution to Question 476: Shootout

With $n = 2$, clearly the pirates can only point at one another. With $n = 3$, there are also only 2 options for the pirates to point at. Therefore, $n \geq 4$.

Note that there are $\binom{n}{2}$ pairs of pirates when we have n pirates total. With $n = 4$, there would be 6 pairs of pirates but 8 cannons that need to be pointed. Therefore, by Pigeonhole Principle, at least 2 pirates will be pointing at one another. If $n = 5$, then there would be 10 pairs of pirates and 10 cannons that need to be pointed. Therefore, the way we can match is that pirate i, $1 \leq i \leq 5$, points at pirates $(i + 1) \bmod 5$ and $(i + 2) \bmod 5$. This satisfies our criterion, so $n = 5$ is our answer.

Solution to Question 477: Captive Marbles

If the bowls are not perfectly balanced, then we know that one of the bowls must have a larger than 0.5 probability of you drawing a white ball while the other has less than a 0.5 probability. Taking this to the extreme, the best case is that one of the bowls has probability 1 of you surviving and the other has probability 0.5 of you surviving, which yields a 75% chance of survival. The closest we can get to this is by putting 1 white marble in one of the urns and then the other 99 marbles in the other urn. This gives you probability 1 of survival in one urn and $\frac{49}{99}$ probability in the other. This is as close as we can get to the theoretical optimum, so the probability of survival here is

$$\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{49}{99} = \frac{74}{99}$$

Solution to Question 478: Random Triangle

The triangle is formed by drawing chords between each pair of points. As the points are IID, the three side lengths are exchangeable. This means that the expected perimeter is just obtained by tripling the expected length of one of the sides of the triangle. In other words, if P is the perimeter and S_{AB} is the length of the chord between points A and B, $\mathbb{E}[P] = 3\mathbb{E}[S_{AB}]$. This question now boils down to finding the expected length of a chord drawn in a circle.

Let θ and ϕ be IID Unif$(0, 2\pi)$. Picking the two angles is equivalent to picking the two points uniform on the circumference. The measure of the angle between θ and ϕ is $\theta - \phi$. To get the length of the line segment connecting the two points, we can take a slight shortcut. The length here only depends on the distance between our points we select in terms of angle. In other words, regardless of where we end up having our points located, we can just rotate the circle so that one of them is located at $(1, 0)$. Therefore, we can arbitrarily fix $\theta_1 = 0$. Thus, the length between $\theta_1 = 0$ and the point at $\theta_2 = \theta$ is $\sqrt{(1-\cos(\theta))^2 + \sin^2(\theta)} = \sqrt{2}\sqrt{1-\cos(\theta)}$. This means that finding the expected length of the chord is equivalent to finding $\sqrt{2}\mathbb{E}[\sqrt{1-\cos(\theta)}]$, where $\theta \sim \text{Unif}(0, 2\pi)$.

Now, $\sqrt{2}\mathbb{E}[\sqrt{1-\cos\theta}] = \frac{\sqrt{2}}{2\pi}\int_0^{2\pi} \sqrt{1-\cos\theta}\,d\theta$. Now, the best approach is to conjugate the interior by multiplying and dividing by $\sqrt{1+\cos\theta}$ so that we get $\sqrt{1-\cos^2\theta} = |\sin\theta|$. Doing this, we get $\frac{\sqrt{2}}{2\pi}\int_0^{2\pi} \frac{|\sin\theta|}{\sqrt{1+\cos\theta}}d\theta$.

Note that the region on $(0, \pi)$ and $(\pi, 2\pi)$, our integrand is symmetric, so we can just evaluate over one interval and double it. This means our new integral is $\frac{\sqrt{2}}{\pi}\int_0^{\pi} \frac{\sin\theta}{\sqrt{1+\cos\theta}}d\theta$. Let $u = 1 + \cos\theta$. Then $du = -\sin\theta d\theta$. Our bounds would respectively become 2 and 0, but the negative from the du flips them back. Therefore, our new integral is $\frac{\sqrt{2}}{\pi}\int_0^{2} u^{-\frac{1}{2}}du$. Evaluating this, we get $\frac{2\sqrt{2}}{\pi} \cdot \sqrt{u}\Big|_0^2 = \frac{4}{\pi}$.

Thus, the expected perimeter of the triangle is $3 \times \frac{4}{\pi} = \frac{12}{\pi}$ and $a = 12$.

Solution to Question 479: Odd Coefficients

We can write out all the binomial coefficients as follows:

$$\begin{array}{cccccc}
\binom{1}{1} & \binom{3}{1} & \binom{5}{1} & \binom{7}{1} & \cdots & \binom{999}{1} \\
& \binom{3}{3} & \binom{5}{3} & \binom{7}{3} & \cdots & \binom{999}{3} \\
& & \binom{5}{5} & \binom{7}{5} & \cdots & \binom{999}{5} \\
& & & \binom{7}{7} & \cdots & \binom{999}{7} \\
& & & & \vdots & \vdots
\end{array}$$

We only write out the entries for $n \geq k$, as the binomial coefficients are 0 otherwise. Looking at the columns of this matrix, we see that the kth column of the matrix counts the number of odd subsets of $\{1, 2, \ldots, 2k-1\}$. Exactly half all subsets of any finite set are odd-sized. There are 2^n subsets of a set of size n. Therefore, exactly 2^{n-1} of them are odd. This means that the kth column sums to $2^{2k-1-1} = 4^{k-1}$. We can verify this holds for the first 4 columns here. There are 500 columns in this matrix, as the last corresponds to $999 = 2 \cdot 500 - 1$. Therefore, the sum of all of these is given by:

$$\sum_{k=1}^{500} 4^{k-1} = \sum_{k=0}^{499} 4^k = \frac{4^{500} - 1}{3}$$

Therefore, our answer is $4 \cdot 500 \cdot 3 \cdot 1 = 6000$.

Solution to Question 480: Exact 5 I

There is a slick trick here that simplifies this problem greatly. At each roll, we only care if the roll appearing is a 5 or 6, as the other values are not counted. Given that we have either a 5 or 6 rolled, it is equally-likely to be each one, as the original die is fair. Thus, we really only are looking for the probability of the sequence 55556 conditioned on when we have a fair die that rolls only 5 or 6. This idea incorporates the fact that we don't consider the other values. Based on the idea that each of the two values in this conditioned die are equally-likely, the answer is just $\frac{1}{2^5} = \frac{1}{32}$.

Solution to Question 481: Poker Hands IV

There are a total of $\binom{52}{5}$ total hand combinations. To count the number of hands that contain a single pair, we need to look at the pair itself and the possible other cards. The pair has 13 possible face values and $\binom{4}{2}$ possible suit values. The other cards have $\binom{12}{3}$ possible face values and 4 possible suit values for each card. Thus, the probability that you have a single pair is:

$$\frac{13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3}{\binom{52}{5}} = \frac{352}{833}$$

Solution to Question 482: Hasty Horseman

Suppose the army moves at speed 1 and the horseman moves at speed $s > 1$. Assume the amount of time it takes for the horseman to get from the back to the front is t_1 and then the time it takes for him to get to the back from the front is t_2. The distance travelled by the rider is $s(t_1 + t_2)$. The distance travelled by the army is $t_1 + t_2$, so $t_1 + t_2 = 50$. Therefore, we just need to find s.

Relative to the crowd, the horseman moves at a speed of $s - 1$ in the first leg and a speed of $s + 1$ in the second leg when returning. Therefore, $t_1(s - 1) = t_2(s + 1) = 50$, as the horseman had to go from the back to the front in both cases relative to the line. This means that $t_1 = \dfrac{50}{s-1}$ and $t_2 = \dfrac{50}{s+1}$, so $50 = \dfrac{50}{s-1} + \dfrac{50}{s+1}$. Multiplying by $\dfrac{s^2-1}{50}$ on both sides and rearranging yields that s satisfies $s^2 - 2s - 1 = 0$, so

$$s = \frac{2 \pm \sqrt{2^2+4}}{2} = 1 \pm \sqrt{2}$$

Since $s > 1$, we have that $s = 1 + \sqrt{2}$, so the distance that the horseman travels is $50(1 + \sqrt{2})$. Therefore, our answer is $50 \cdot 2 \cdot 1 = 100$.

Solution to Question 483: Balanced Beans III

Since we know the abnormal bean is heavier than the others, at each weighing, we can divide our prospective beans by a factor of 3. In this problem, we first make 3 groups of 6 beans and weigh 2 of them on the balance. If they are not balanced, we know the heavier side has the abnormal bean. Otherwise, if the balances are level, we know the abnormal bean is in the group that wasn't weighed. We do this again (split the 6 potential beans into 3 groups of 2) and find the group of 2 in which the abnormal bean is a part of. Then we can simply weigh one bean on either side to find the heavier one. Thus it takes a total of 3 weighings.

If we were to generalize this for any n beans where one is known to be heavier than the others, then the answer is simply ceiling($\log_3(n)$).

Solution to Question 484: Coin Pair IV

Let T be the total number of coin flips needed and X be the total number of heads that appear on the first flipping of the coins. Then $\mathbb{E}[T] = \mathbb{E}[\mathbb{E}[T \mid X]] = \sum_{k=0}^{4} \mathbb{E}[T \mid X = k]\mathbb{P}[X = k]$ by Law of Total Expectation. $X \sim \text{Binom}\left(4, \frac{1}{2}\right)$ because X counts the number of heads appearing in 4 flips of a fair coin. To set our notation, let e_i represent the additional number of flips needed to stop our process once we have i heads.

If $X = 4$, then obviously we don't flip any coins again, so $\mathbb{E}[T \mid X = 4] = 4$. This would mean $e_4 = 0$. Similarly, if $X = 3$, then we aren't able to flip just one tails, so $\mathbb{E}[T \mid X = 3] = 4$. This would mean $e_3 = 0$.

If $X = 2$, then we are going to continually flip the two tail coins again until we don't obtain TT. We have that $\mathbb{E}[T \mid X = 2] = 4 + e_2$, as it takes 4 flips at the beginning and then need e_2 additional flips to stop. We have that

$$e_2 = \frac{1}{4} \cdot (2 + e_4) + \frac{1}{2} \cdot (2 + e_3) + \frac{1}{4} \cdot (2 + e_2)$$

This stems from the fact that we flip 2 coins, and we get $0, 1$, or 2 heads with those prescribed probabilities fries those two coins. Solving for e_2 here by plugging in $e_3 = e_4 = 0$ is $e_2 = \frac{8}{3}$, so $\mathbb{E}[T \mid X = 2] = 4 + \frac{8}{3} = \frac{20}{3}$.

Iterating this logic, if $X = 1$, then we flip two tails until they don't appear TT. We know that $\mathbb{E}[T \mid X = 1] = 4 + e_1$ by the same logic as above. By doing the same conditioning argument, we have that

$$e_1 = \frac{1}{4} \cdot (2 + e_3) + \frac{1}{2} \cdot (2 + e_2) + \frac{1}{4} \cdot (2 + e_1)$$

Plugging in $e_3 = 0$ and $e_2 = \frac{8}{3}$ yields that $e_1 = \frac{40}{9}$. Therefore, $\mathbb{E}[T \mid X = 1] = \frac{40}{9} + 4 = \frac{76}{9}$.

Lastly, if $X = 0$, we have that $\mathbb{E}[T \mid X = 0] = 4 + e_0$. By doing the same conditioning argument once again, we get that

$$e_0 = \frac{1}{4} \cdot (2 + e_2) + \frac{1}{2} \cdot (2 + e_1) + \frac{1}{4} \cdot (2 + e_0)$$

Solving for e_0 by plugging in $e_2 = \frac{8}{3}$ and $e_1 = \frac{40}{9}$ yields $e_0 = \frac{176}{27}$. We can now compute $\mathbb{E}[T \mid X = 0] = \frac{176}{27} + 4 = \frac{284}{27}$.

Plugging the values and the PMF of X into our expression from the beginning, we get that

$$\mathbb{E}[T] = \frac{1}{16} \cdot 4 + \frac{4}{16} \cdot 4 + \frac{6}{16} \cdot \frac{20}{3} + \frac{4}{16} \cdot \frac{76}{9} + \frac{1}{16} \cdot \frac{284}{27} = \frac{176}{27}$$

Solution to Question 485: Second Moment

Using the relation $\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$, we have $25 = \mathbb{E}[X^2] - 3^2$, so $\mathbb{E}[X^2] = 34$.

Solution to Question 486: Binary Call

The payoff of a binary call and binary put is a step function. Combining them together yields a line at $y = 1$. So, a binary call and a binary put add up to a bond (a constant is the same as a bond). Since we are dealing with time-0 prices, we need to use the discount factor instead of 1.

$$P_0 + C_0 = Z_0$$

We can then find the time-0 price of the binary call.
$C_0 = Z_0 - P_0 = 0.9 - 0.36 = 0.54$

Solution to Question 487: Overlapping Data

Let us first compute $\text{Cov}(U, V)$. We know that $\text{Cov}(U, V) = \text{Cov}(3X_1 + \cdots + 3X_{50} + X_{51} + \cdots + X_{100}, X_{51} + \cdots + X_{100} + 3X_{101} + \cdots + 3X_{150})$. Since the X_i random variables are IID, this means that the only time the covariances are not 0 is when we are looking at the pairs of random variables that are in both U and V. Namely, these are $X_{51}, \ldots, X_{100}$. Therefore, $\text{Cov}(U, V) = \text{Cov}(X_{51}, X_{51}) + \cdots + \text{Cov}(X_{100}, X_{100}) = \text{Var}(X_{51}) + \cdots + \text{Var}(X_{100}) = 50$ by the information in the question.

To get $\text{Corr}(U, V)$, we need the variances of both U and V. By the symmetry of U and V (they have the same overlapping random variables and there are factors of 3 for the same number of non-overlapping random variables), $\text{Var}(U) = \text{Var}(V)$. Therefore,

$$\text{Var}(U) = \text{Var}(3X_1 + \cdots + 3X_{50} + X_{51} + \cdots + X_{100}) = 3^2\text{Var}(X_1) + \cdots + 3^2\text{Var}(X_{50}) + \text{Var}(X_{51}) + \cdots + \text{Var}(X_{100}) = 500$$

.

Lastly, $\text{Corr}(U, V) = \dfrac{\text{Cov}(U, V)}{\sqrt{\text{Var}(U)\text{Var}(V)}} = \dfrac{50}{500} = \dfrac{1}{10}$.

Solution to Question 488: Limiting Product

We know that $\log(Y_n) = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} \log(X_k)$. As $-\log(X_k) \sim \text{Unif}(0, 1)$, which has mean 1, we can quickly see $\mathbb{E}[\log(X_k)] = -1$, which is finite. Therefore, the Strong Law of Large Numbers applies. We thus get that

$$\log(Y_n) = \frac{\displaystyle\sum_{k=1}^{n} \log(X_k)}{n} \to \mathbb{E}[\log(X_1)] = -1$$

where the convergence above is almost surely. Therefore, this means Y_n converges to e^{-1} almost surely, so $a = -1$.

Solution to Question 489: Generational Wealth II

It is known that if $X_1, \ldots, X_n \sim \text{Unif}(0, 1)$ IID, then if $M_n = \max\{X_1, \ldots, X_n\}$, $\mathbb{E}[M_n] = \dfrac{n}{n+1} = 1 - \dfrac{1}{n+1}$. In fact, one can show that $M_n \sim \text{Beta}(n, 1)$, but we don't need this level of depth for this question. Namely, we want to find n that maximizes our expected payout. If we generate n numbers total, that means we have $n - 1$ re-generations of numbers. The expected max would be $1 - \dfrac{1}{n+1}$, and we would have to pay $\dfrac{n-1}{20}$ to get those re-generations. Thus, our payout as a function of the total number of generations that we do is $f(n) = 1 - \dfrac{1}{n+1} - \dfrac{n-1}{20}$. We want maximize in n, so treat $f(n)$ as continuous and take the derivative in n. Then, we want to set this equal to 0 to find the maximizer.

This means $f'(n) = \frac{1}{(n+1)^2} - \frac{1}{20} = 0$, which means $20 = (n+1)^2$. This means $n = \sqrt{20} - 1$. As this is not an integer, we should test the two integers closest to this value to see what the expected value is. As $4 < \sqrt{20} < 5$, we should test $n = 3$ and $n = 4$ to see what the expected value is. One can quickly see by direct substitution that $f(3) = f(4) = \frac{13}{20}$, so our answer is $\frac{13}{20}$.

Solution to Question 490: Basic Die Game VII

Intuitively, given that the player does not roll a 6, the conditional expectation of their die roll is 3. The sum of the rest of the faces of the die is 15, which are distributed over 5 faces. Therefore, the player should keep any value at least 3 and re-roll any value below 3. Let e_3 be the expected payout of this strategy. We have that

$$e_3 = \frac{1}{3} \cdot e_3 + \frac{1}{2} \cdot 4$$

This is since with probability 1/3, the person rolls 1 or 2, in which the game essentially restarts. With probability 1/2, the person rolls $3, 4$, or 5, which has conditional expectation 4. Solving for this, we get that $e_3 = 3$, which is our optimal strategy. One can also show that if you re-roll anything below 4, you get the same expected payout. This makes intuitive sense because of the fact that at 3, you are indifferent to keeping or rolling again.

Solution to Question 491: Needy Friends

Let x be the amount of money that the friend allocates and n be the number of people in the present week that needed money. Then $\frac{x}{n}$ was the payout for each of them. We can translate what the man said into math. The first statement says that

$$\frac{x}{n-5} = \frac{x}{n} + 2$$

This is because there are $n-5$ people that need money compared to the previous week. The second statement says that

$$\frac{x}{n+4} = \frac{x}{n} - 1$$

Therefore, the goal is to solve these two equations simultaneously. In the first equation, multiplying by $n(n-5)$ on both sides yields $nx = x(n-5) + 2n(n-5)$, which simplifies to $5x = 2n(n-5) = 2n^2 - 10n$. Multiplying by $n(n+4)$ on both sides in the second equation, $nx = x(n+4) - n(n+4)$, which simplifies to $4x = n(n+4) = n^2 + 4n$. Subtracting twice of the second equation from the first equation yields that $-3x = -18n$, which means that $\frac{x}{n} = 6$.

Solution to Question 492: Deja Vu

We solve this using the classic trick $\mathbb{E}[X] = \sum_{k=1}^{\infty} \mathbb{P}[X \geq k]$ for non-negative integer-valued random variables X. Let X be the number of distinct faces showing up. We now need to find $\mathbb{P}[X \geq k]$ for $k = 1, 2, \ldots, 6$. We know for $k \geq 7$ this is 0 since there are only 6 sides on the die.

$\mathbb{P}[X \geq 1] = 1$, as you have to roll at least one distinct side to be able to repeat it. Now, for $\mathbb{P}[X \geq 2]$, we need to roll any value besides our first face before the first face again. This is determined within one roll, and the probability of this is $\frac{5}{6}$, as there are 5 values that haven't been selected yet. For $\mathbb{P}[X \geq 3]$, we need to obtain 2 distinct faces, which occurs with probability $\frac{5}{6}$, and then we need to roll another distinct face on the next turn, which occurs with probability $\frac{2}{3}$. Therefore, $\mathbb{P}[X \geq 3] = \frac{5}{6} \cdot \frac{2}{3} = \frac{5}{9}$.

Continuing this pattern, $\mathbb{P}[X \geq 4] = \frac{5}{9} \cdot \frac{1}{2} = \frac{5}{18}$, $\mathbb{P}[X \geq 5] = \frac{5}{18} \cdot \frac{1}{3} = \frac{5}{54}$, and $\mathbb{P}[X \geq 6] = \frac{5}{54} \cdot \frac{1}{6} = \frac{5}{324}$. Therefore, $\mathbb{E}[X] = \sum_{k=1}^{6} \mathbb{P}[X \geq k] = 1 + \frac{5}{6} + \cdots + \frac{5}{324} = \frac{899}{324}$.

Solution to Question 493: Game Show III

If you have not done the problem "Game Show I", please do so before this question. From "Game Show I", we already know the best choice is to always switch doors. Now the problem is to calculate the EV of the game.

$\frac{9}{10}^{\text{ths}}$ of the time, you pick a door with a goat initially. Given the strategy of always switching after the three doors with goats behind them are revealed, we are left with 6 doors to switch to, one of which has the cash prize. Thus the probability we switch to the cash prize door is $\frac{9}{10} \cdot \frac{1}{6} = 0.15$.

The other $\frac{1}{10}^{\text{th}}$ of the time, we initially pick the cash prize door and switch to a losing door. Thus the EV of this game is $0.15 \cdot \$1000 = \$150$.

Solution to Question 494: Connected Origin

Let A be the event that the origin belongs to a connected component. Suppose A holds. Then for any length n, there must be a path γ of length n that passes through the origin. For each path γ, let A_γ be the event that path γ is present. Since our path is of length n, $\mathbb{P}[A_\gamma] = p^n$, where p is the probability that each individual component of the path is present.

Now, by sub-additivity and the fact that the collection of all infinite components passing through the origin is a subset of the length n components passing through the origin (and then the rest of the path is irrelevant after that),

$$\mathbb{P}[A] \leq \mathbb{P}\left[\bigcup_\gamma A_\gamma\right] \leq \sum_\gamma \mathbb{P}[A_\gamma] = \sum_\gamma p^n$$

How many paths γ are there of length n passing through the origin? An easy upper bound is to note that the initial direction has 4 choices, and then after that, there are at most 3 other directions that the path can go at each step, as it can't go backwards to the spot it was before. Therefore, an upper bound on the number of such paths is $4 \cdot 3^{n-1}$. Therefore $\mathbb{P}[A] \leq 4 \cdot 3^{n-1} \cdot p^n = \dfrac{4}{3} \cdot (3p)^n$. For this probability to tend to 0, we need $3p < 1$, so $p < \dfrac{1}{3}$. Therefore, our interval is $\left(0, \dfrac{1}{3}\right)$, so $a = \dfrac{1}{3}$.

Solution to Question 495: Integral Variance V

As this integrand is a deterministic function $f(s)$, the integral is normally distributed with mean 0 and variance $\displaystyle\int_0^t (f(s))^2 ds$. In this case, $f(s) = s$, so the variance is $\displaystyle\int_0^t s^2 ds = \frac{t^3}{3}$. This implies $k = \dfrac{1}{3}$.

Solution to Question 496: Sum Covariance

Since X, Y, and Z are uncorrelated, $\text{Var}(U) = \text{Var}(X) + \text{Var}(Y) = 100$ and $\text{Var}(V) = \text{Var}(X) + \text{Var}(Z) = 289$. Furthermore, $\text{Cov}(U, V) = \text{Cov}(X + Y, X + Z) = \text{Cov}(X, X) = \text{Var}(X) = 64$ by bilinearity of covariance and the lack of correlation between the other random variables. Therefore, by the correlation formula,

$$\mathrm{Corr}(U, V) = \frac{\mathrm{Cov}(U, V)}{\sigma_U \sigma_V} = \frac{64}{\sqrt{100 \cdot 289}} = \frac{64}{\sqrt{10^2 \cdot 17^2}} = \frac{32}{85}$$

Solution to Question 497: Coin Priors

The prior distribution being Beta$(10, 10)$ means that we can treat ourselves starting with 10 heads and 10 tails on the coin as "prior information" when calculating the distribution of p. Then, the information provided says that in the next 80 flips, we obtain 50 heads, meaning we also obtain 30 tails. This means that after 100 total flips, we have 60 heads and 40 tails. Therefore, our posterior distribution is Beta$(60, 40)$. The mean of a Beta(a, b) distribution is $\frac{a}{a+b}$, so the mean of our posterior distribution is $\frac{60}{100} = \frac{3}{5}$.

Solution to Question 498: Doubly Blue

Let BB represent the event of drawing two blues and B represent the event of the first ball being blue. Let U_1 represent the two blue urn, U_2 represent the two red urn, and then U_3 represent the mixed urn. We want

$$\mathbb{P}[BB \mid B] = \frac{\mathbb{P}[BB]}{\mathbb{P}[B]}$$

The equality above comes from the fact that $BB \subseteq B$. To calculate each, we just condition on the urn we are in. Namely,

$$\mathbb{P}[B] = \mathbb{P}[B \mid U_1]\mathbb{P}[U_1] + \mathbb{P}[B \mid U_2]\mathbb{P}[U_2] + \mathbb{P}[B \mid U_3]\mathbb{P}[U_3]$$

Since the original urn is selected uniformly at random, $\mathbb{P}[U_1] = \mathbb{P}[U_2] = \mathbb{P}[U_3] = \frac{1}{3}$. Furthermore, we have that $\mathbb{P}[B \mid U_1] = 1, \mathbb{P}[B \mid U_2] = 0$, and $\mathbb{P}[B \mid U_3] = \frac{1}{2}$ by the proportions of blue balls in each urn. Since we replace the ball between trials, to get $\mathbb{P}[BB \mid U_i]$, we just have to square $\mathbb{P}[B \mid U_i]$. This means that

$$\mathbb{P}[B] = \frac{1 + 1/2 + 0}{3} = \frac{1}{2}$$

and

$$\mathbb{P}[BB] = \frac{1 + 1/4 + 0}{3} = \frac{5}{12}$$

Therefore, the answer is $\mathbb{P}[BB \mid B] = \frac{5/12}{1/2} = \frac{5}{6}$.

Solution to Question 499: Builders

Suppose 600 total units need to be built to fully create the pipe. It follows then that Alice builds at 5 units per minute and Bob builds at 6 units per minute (indeed, their respective completion times are 120 and 100 minutes). After 40 minutes, Alice and Bob together would have completed $40 \cdot (5 + 6) = 440$ of the 600 units. After Charlie joins, they complete the remaining 160 units in 10 minutes, meaning that they were building at a rate of 16 units per minute with all three working. As they were at 11 per minute before Charlie joined, this means Charlie builds at 5 units per minute, the same rate as Alice, so Charlie also would take 120 minutes to build the pipe on his own.

Solution to Question 500: Valid Expressions

Consider all sequences in the form $1-2-3-4-5-6-7-8-9$. We know that each $-$ can either be a $+, *$, or nothing. That gives 3 options for each of the 8 dashes that we listed, so there are 3^8 possible sequences by the multiplication rule.

Solution to Question 501: Hidden Code

First, let's find how many strings there are that have no two consecutive characters the same, disregarding the other constraint. There are 5 options for the first character and then 4 options for each of the other 4, as the only character disallowed at each step is the one used in the previous spot. Therefore, there are $5 \cdot 4^4 = 1280$ such sequences. We now need to exclude all of sequences that have $3, 4$, or 5 of a given character that also satisfy the condition that no two consecutive characters are the same.

There are no sequences included in the 1280 above that have 4 or 5 of the same character, as there will always be two of the same that touch. For the case of 3 of the same character, the sequence would be in the form $XYXZX$, where X, Y, and Z are any of the letters in our set with $X \neq Y$ and $X \neq Z$. There are 5 options for X and 4 options for each of Y and Z, so there are $5 \cdot 4^2 = 80$ sequences that we have to remove. Therefore, our total number of valid sequences is $1280 - 80 = 1200$.

Solution to Question 502: First Pair

The first card dealt out will determine the rank of the pair. It can be any card as all cards have the same number of duplicates of the rank. Therefore, let the first card be arbitrary. There are 3 cards of that rank remaining of 51 cards in the deck, so the probability of a pair is $\frac{3}{51} = \frac{1}{17}$.

Solution to Question 503: Single Double Sum

Let p_i be the probability of rolling i on this die. We know that to get sum of 2, we must roll 1 twice. Therefore, we must have that $p_2 = p_1^2$. Similarly, to get a sum of 3, we must roll a 1 and a 2 in some order, so $p_3 = 2p_1p_2$. To get a sum of 4, we either roll 1 and 3 in some order or roll 2 consecutive 2s. Therefore, $p_4 = 2p_1p_3 + p_2^2$. To get a sum of 5, we must either roll a 1 and a 4 OR a 2 and a 3 in some order. Therefore, $p_5 = 2p_1p_4 + 2p_2p_3$. Lastly, to get a sum of 6, we must obtain either 2 consecutive 3s, a 2 and a 4 in some order, or a 1 and a 5 in some order. Therefore, $p_6 = p_3^2 + 2p_2p_4 + 2p_1p_5$.

Substituting, as $p_2 = p_1^2$, $p_3 = 2p_1^3$. Continuing this pattern of substitution, $p_4 = 5p_1^4, p_5 = 14p_1^5$, and $p_6 = 42p_1^6$. We also know that $p_1 + \cdots + p_6 = 1$, as this accounts for all possible outcomes. Therefore, the equation p_1 satisfies is

$$42p_1^6 + 14p_1^5 + 5p_1^4 + 2p_1^3 + p_1^2 + p_1 = 1$$

Therefore, we have that $\sum_{k=1}^{6} c_k = 65$.

Solution to Question 504: Taxman

Notice that we can only take one prime off the board since all the primes are divisible only by 1 and themselves. Thus for our first pick, lets choose 7 (the largest prime on the board). The taxman then takes 1 and the board becomes 2, 3, 4, 5, 6, 8, 9, and 10. We can't choose 2, 3, or 5 since they are prime and don't have any more factors. We should never choose 4 because we then can't take 8 later. In this case, the best choice to take is 9 since it is one of the larger numbers and only has one factor left (3). Now the board becomes 2, 4, 5, 6, 8, and 10. Out of this set, 6 is the only number with only one factor so lets take it. Now the taxman takes 2. The board is now 4, 5, 8, and 10. Now it doesn't matter what order we take 8 and 10, the taxman will get 4 and 5. Thus the numbers we have are 6, 7, 8, 9, and 10 which gives you a score of 40 and the taxman has 1, 2, 3, 4, 5 which gives them a score of 15. You win with a score of 40!

Solution to Question 505: Digit Halving

Let $x =$ ab be our integer. Then $x = 10a+b$. In addition, we know that $ab = \frac{x}{2}$ by our property, so $ab = \frac{10a+b}{2}$. Solving for b, $2ab = 10a+b$, which means that $b(2a-1) = 10a$, so $b = \frac{10a}{2a-1} = 5+\frac{5}{2a-1}$. We know that b is an integer, so this implies $\frac{5}{2a-1}$ is an integer. The only divisors of 5 are 1 and 5, so $2a-1$ is either 1 or 5. If it is 1, then this implies $b = 10$, which is impossible as b must be a singular digit. Therefore, $2a-1 = 5$, so $a = 3$. Plugging this in, $b = 6$. Therefore, the integer is 36.

Solution to Question 506: Throwing Darts II

We first want to calculate the probability of one dart landing in the inner ring. Because the dart lands uniformly at random on the board, we can calculate the probability as the ratio of the area of the inner ring to the area of the entire board. The area of the entire board is 9π and the area of the inner ring is 1π. Thus, the probability that the dart lands in the inner ring is:

$$\frac{\pi}{9\pi} = \frac{1}{9}$$

We now want to calculate the probability that at least one of our three darts will land in this inner circle. To do this, we will calculate the probability of not getting any darts in the inner ring, which is:

$$\left(\frac{8}{9}\right)^3 = \frac{512}{729}$$

Finally, we use the complement rule, $1 - P(A^c) = P(A)$ to come to the answer:

$$1 - \frac{512}{729} = \frac{217}{729}$$

Solution to Question 507: Variance Product

$\text{Var}(XY) = \mathbb{E}[X^2Y^2] - (\mathbb{E}[XY])^2$. By independence, $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] = -12$. The first term can be obtained by the same idea that $\mathbb{E}[X^2Y^2] = \mathbb{E}[X^2]\mathbb{E}[Y^2] = 20 \cdot 45 = 900$ by independence. Thus, $\text{Var}(XY) = 900 - (-12)^2 = 756$.

Solution to Question 508: Turducken Hunt

Suppose that Mordecai shoots first at position x. Then with probability x he wins, while with probability $1-x$, Rigby wins. Alternatively, if Rigby shoots first at position x, then with probability x^2 he wins, while with probability $1-x^2$, Mordecai wins. Therefore, the optimal strategy for each is to minimize the other person's chance of winning. For Mordecai, this function that he needs to minimize is $\max\{x^2, 1-x\}$. For Rigby, this function is $\min\{x, 1-x^2\}$.

However, note that x^2 is monotonically increasing and $1-x$ is monotonically decreasing. Similarly, x is monotonically increasing and $1-x^2$ is monotonically decreasing. Therefore, for each function, there is a unique point at which the two respective functions are equal and that will be the minimum.

For Mordecai, the equation that must be solved is $x^2 = 1-x$. For Rigby, the equation that needs to be solved is $x = 1-x^2$. However, by rearranging both equations, we can see that the optimal x for both is when $x+x^2 = 1$. This implies that they should both shoot at the same time. Solving this quadratic equation yields the optimal $x^* = \dfrac{\sqrt{5}-1}{2}$.

Rigby's equation is the chance of Mordecai winning, so as $x^* = 1-(x^*)^2$, we have that Mordecai's chance of winning is just $\dfrac{\sqrt{5}-1}{2}$.

Solution to Question 509: Temperature Conversion

Let F denote temperature in °F and C denote temperature in °C. Recall that the conversion formula between the units is:

$$C = \frac{5}{9}(F-32)$$

Thus:

$$E[C] = \frac{5}{9} \times E[F] - \frac{5}{9} \times 32 = \frac{5}{9} \times 80 - \frac{5}{9} \times 32 \approx 26.67^\circ\text{C}$$

$$V[C] = (\frac{5}{9})^2 \times V[F] = (\frac{5}{9})^2 \times 25 \approx 7.72^\circ\text{C}$$

$$E[C] - V[C] \approx 26.67 - 7.72 = 18.95$$

Solution to Question 510: Absolutely Brownian

We are going to solve this for general $T_a = \inf\{t > 0 : |W_t| > a\}$ for any $a > 0$. It is a well-known fact that $Z_t = f(t, W_t)$ defines a martingale, where $f(t, w) = w^4 - 6w^2t + 3t^2$. By using the Optional Stopping Theorem, we have that

$$\mathbb{E}[Z_{T_4}] = \mathbb{E}[W_{T_a}^4 - 6W_{T_a}^2 T_a + 3T_a^2] = \mathbb{E}[Z_0] = 0$$

Since $W_{T_a} = \pm a$ with equal probability, we know that $W_{T_a}^4 = a^4$ with probability 1. Similarly, $W_{T_a}^2 = a^2$ with probability 1. Therefore, by linearity, we have that $a^4 - 6a^2\mathbb{E}[T_a] + 3\mathbb{E}[T_a^2] = 0$. From a more basic martingale argument on $W_t^2 - t$, one can easily prove that $\mathbb{E}[T_a] = a^2$. Therefore, we have that $\mathbb{E}[T_a^2] = \frac{5}{3}a^4$ by rearrangement.

To find the variance, we just use our classic relationship $\text{Var}(T_a) = \mathbb{E}[T_a^2] - (\mathbb{E}[T_a])^2 = \frac{5}{3}a^4 - a^4 = \frac{2}{3}a^4$. In our case, $a = 4$, so the answer is $\frac{512}{3}$.

Solution to Question 511: Incomplete Deck

Let n be the number of cards in the deck. Let a, b, and c be some integers such that these equations hold true:

$$n - 1 = 3a$$

$$n - 3 = 5b$$

$$n - 3 = 4c$$

We can rearrange the first equation to become:

$$n - 3 = 3a - 2$$

These three equations means that

$$n - 3 = \text{LCM}(3a - 2, 4, 5)$$

or

$$n - 3 = \text{LCM}(3a - 2, 20)$$

We know that the multiples of 20 are 20, 40, 60, etc but since we know we have less cards than a standard deck, we should only care about 20 and 40. Out of these two, 40 is the number that is also divisible by $3a - 2$ when a is 14. Since

$$\text{LCM}(3a - 2, 4, 5) = 40$$

we get $n - 3 = 40$ which means $n = 43$.

Solution to Question 512: Birth Paradox

Let B^* denote a boy born on a Friday, and let B denote a boy not born on a Friday. Additionally, let G denote a girl. The two ways we can obtain 2 boys is if they are both born on a Friday, or if only one of them is born on a Friday. We know at least one is a boy born on a Friday, so BB is not possible. The total ways to obtain at least one boy born on a Friday is to have the two cases above OR have one boy born on a Friday and one girl (irrelevant which day she is born on). Thus, we want $\dfrac{\mathbb{P}[B^*B^* \cup B^*B]}{\mathbb{P}[B^*B^* \cup B^*B \cup B^*G]}$. I am disregarding order of birth here, which will account for later. These are all disjoint events, so by the Kolmogorov Axioms on the probability measure, we can separate out each of the individual events and take the sum of their probabilities. We can assume independence upon gender from birth to birth, equally likely probabilities of birth on any given day of the week, and independence between day of the week of birth and gender. We can find easily find $\mathbb{P}[B^*] = \dfrac{1}{2} \cdot \dfrac{1}{7} = \dfrac{1}{14}$ and $\mathbb{P}[B] = \dfrac{1}{2} \cdot \dfrac{6}{7} = \dfrac{6}{14}$. Thus, since there is independence from birth to birth, $\mathbb{P}[B^*B^*] = \dfrac{1}{14} \cdot \dfrac{1}{14} = \dfrac{1}{196}$ and $\mathbb{P}[B^*B] = 2 \cdot \dfrac{1}{14} \cdot \dfrac{6}{14} = \dfrac{12}{196}$. Note the factor of 2 out front, as we can arrange the location of the boy born on a Friday to either be the first born or the second born, so there are 2 arrangements. Lastly, $\mathbb{P}[B^*G] = 2 \cdot \dfrac{1}{14} \cdot \dfrac{1}{2} = \dfrac{14}{196}$. Factor of 2 out front for same reason as above. Thus, we have our final probability as

$$\frac{\mathbb{P}[B^*B^* \cup B^*B]}{\mathbb{P}[B^*B^* \cup B^*B \cup B^*G]} = \frac{\mathbb{P}[B^*B^*] + \mathbb{P}[B^*B]}{\mathbb{P}[B^*B^*] + \mathbb{P}[B^*B] + \mathbb{P}[B^*G]} = \frac{\frac{1}{196} + \frac{12}{196}}{\frac{1}{196} + \frac{12}{196} + \frac{14}{196}} = \frac{13}{27}$$

How do we reconcile this increase due to something seemingly irrelevant? It is much easier to have a boy born on a Friday if you have two boys than one boy. So if we have the information that a boy is born on a Friday, it is more likely that there are two boys.

Solution to Question 513: Probability Discussion

First, let's model this mathematically. Let G and S be the number of minutes after 4 PM that Gabe and the student show up, respectively. Then we have that $G \sim \text{Unif}(30, 60)$ and $S \sim \text{Unif}(0, 60)$. The event that the meeting occurs is the just the event that $|G - S| \leq 10$, as they must differ by up to 10 minutes.

We should draw out the region in the plane to see what we are working with. If you put G on the x-axis and S on the y-axis, we have a tall rectangle. When drawing the two lines corresponding to $|G - S| \leq 10$, you will notice that it is easier to calculate the probability of the complement, as those are easier regions to work with. You will see that there is one trapezoid and one triangle. It is easy enough to use some basic algebra to note that the slope is 1, so the line will go as far vertically as it does horizontally. You will find that the base of the trapezoid is 30, and the two heights are 20 and 50, so its area is 1050. The two sides of the triangle are 20 each, so the area is 200. Thus, the probability of the complement is $\frac{25}{36}$. Thus, the probability in the question is $\frac{11}{36}$

Solution to Question 514: Minimal Flipping

Let X_i be the number of flips person i performs until they obtain their first tails. The statement that none flip more than 2 times is the same as $\max(X_1, X_2, X_3) \leq 2$.

Thus, we want

$$\mathbb{P}[\max(X_1, X_2, X_3) \leq 2] = \mathbb{P}[X_1 \leq 2]\mathbb{P}[X_2 \leq 2]\mathbb{P}[X_3 \leq 2] = (\mathbb{P}[X_1 \leq 2])^3$$

by the fact the X_i random variables are IID. We have that $X_1 \sim \text{Geom}(0.75)$, as there is a 75% chance of obtaining a tails on any flip, so $\mathbb{P}[X_1 \leq 2] = \mathbb{P}[X_1 = 1] + \mathbb{P}[X_1 = 2] = \frac{3}{4} + \frac{3}{4} \cdot \frac{1}{4} = \frac{15}{16}$, so the probability is $\left(\frac{15}{16}\right)^3$

Solution to Question 515: Balanced Beans I

We put 3 beans on each side of the balance scale. If the scale is balanced, then we simply compare the final 2 beans. Else, say the left side is heavier, we randomly pick 2 of the 3 beans to compare. If the scale is balanced, then the remaining bean is heavier. Else, the heavier side contains the heavier bean. Hence, we only need to use the balance scale twice.

Solution to Question 516: Dinner at Dorsia

Let quant x arrive X minutes after 8:00pm and quant y arrive Y minutes after 8:00pm. The two quants meet if and only if $|Y - X| \leq 10$. The area covered by this constraint in the sample space $X, Y \in [0, 60]$ is equal to $60^2 - 50^2 = 1100$. Since both X and Y are uniformly distributed, the probability that the two quants meet is $\frac{1100}{3600} = \frac{11}{36}$.

Solution to Question 517: Unknown Starter

For simplicity, we can assume the values of the cards are $1, 2$, and 3, as n is arbitrary and is just a shifting factor. If you make a decision at the first or third card, you have no choice on what the card will be, so the expected value will be 2 in each of those cases. Therefore, to do better, we should consider devising a strategy at the second card drawn.

Suppose the two cards you have drawn, in order, are a and b. The possible values for $a - b$ are ± 2 or ± 1. We break up into cases based on the values of $a - b$.

Case 1: If $a - b = \pm 2$, then you know the largest and smallest card among the three, so you just select the larger card in this case and you're done.

Case 2: If $a - b = 1$, which occurs with the starting sequence 21 and 32, we would have 3 and 1 as our last card. The expected value of the second card in this case is $3/2$ (either 1 or 2 with equal probability), whereas the expected value of the last card is 2 (3 or 1 with equal probability), so we should select the third card in this case.

Case 3: If $a - b = -1$, which occurs with the starting sequence 12 or 23, we would again have 3 and 1 as our last card. The expected value of the last card is the same still, but the expected value of the second card is now $5/2$, which is larger than 2, so we should stay at the second card.

In summary, for each of the permutations of $1, 2, 3$ under this strategy, we select the following cards:

$$213 : 3, 231 : 3, 321 : 1, 123 : 2, 132 : 3, 312 : 2$$

Therefore, the expected value of the card is

$$\frac{3 + 3 + 1 + 2 + 3 + 2}{6} = \frac{8}{3}$$

This was the case where $n = 1$, so to shift with $n, n + 1$, and $n + 2$, the answer would be $n + 4/3$. This means $c = 4/3$.

Solution to Question 518: Family Ties

Let x be the age of the 10s digit child and y be the age of the 1s digit child. We know that the age of the grandmother is $10x + y$. Therefore, the sum of all of their ages is $11x+2y$. Thus, $11x+2y = 98$. We know that each of x and y need to be an integer between 1 and 9, inclusive of both, as the age of each child is a digit. Since $11x + 2y = 98$, we need x to be large, as it has a significantly larger coefficient.

Consider $x = 7, 8, 9$. For $x = 7$, $2y = 21$, which means y is larger than 10. Therefore, this does not work. If $x = 8$, then $2y = 10$, in which case $y = 5$. This is a possibility. If $x = 9$, then $2y = -1$, meaning y is negative (and also not an integer), so this is impossible. Therefore, $x = 8$ and $y = 5$, so the age of the grandmother is 85.

Solution to Question 519: Plane Partition

Let L_n be the maximum number of regions with n lines. Note that $L_1 = 2$ by the above and $L_2 = 4$, as we want to draw two non-parallel lines. The key observation is here that the $(n+1)$st line that is added should intersect each of the first n lines, and this yields $n+1$ distinct regions. We can see this in the small cases of $n = 1, 2$, and 3. More rigorously, the $(n + 1)$st line splits k of the old regions exactly when it intersects the existing lines in $k-1$ places. As two lines can only intersect in at most one point, then this implies $L_{n+1} \leq L_n + (n+1)$, as we get $k \leq n+1$ given that the new line can intersect the old lines in at most n places.

However, it is also possible to place the line so that it is non-parallel to every other existing line and it doesn't intersect at any existing intersection points. Therefore, $L_{n+1} \geq L_n + (n+1)$ by the same logic. This means we derived the recurrence $L_{n+1} = L_n + (n+1)$, which simply has the solution

$$L_n = \frac{n(n+1)}{2} + 1$$

Plugging in $n = 10$, we see that our answer is 56.

Solution to Question 520: Conditional Expectation I

We first need to compute c.

$$\begin{aligned}\iint_{\mathbb{R}^2} f_{X_1,X_2}(x_1,x_2)dx_1\,dx_2 &= 1\\ \int_0^1\int_0^{x_2} c(1-x_2)\,dx_1\,dx_2 &= 1\\ \int_0^1 cx_2(1-x_2)\,dx_2 &= 1\\ c\left[\frac{x_2^2}{2}-\frac{x_3^3}{3}\right]_0^1 &= 1\\ \frac{c}{6} &= 1\\ c &= 6\end{aligned}$$

Next, we need to determine the conditional pdf $f_{X_1|X_2}(x_1|x_2)$. Recall

$$f_{X_1|X_2}(x_1|x_2) = \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_2}(x_2)}.$$

We must compute the marginal pdf of X_2.

$$\begin{aligned}\int_0^{x_2} 6(1-x_1)\,dx_1 &= 6x_2(1-x_2)\\ f_{X_2}(x_2) &= \begin{cases}6x_2(1-x_2) & 0\le x_2\le 1\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

Plugging in, we find

$$\begin{aligned}f_{X_1|X_2}(x_1|x_2) &= \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_2}(x_2)}\\ &= \begin{cases}\frac{1}{x_2} & 0\le x_1\le x_2\\ 0 & \text{otherwise}\end{cases}.\end{aligned}$$

The last step is to apply the law of total expectation.

$$\begin{aligned}
\mathbb{E}[X_1] &= \mathbb{E}\left[\mathbb{E}[X_1|X_2 = x_2]\right] \\
&= \int_0^1 \mathbb{E}[X_1|X_2 = x_2] f_{X_2}(x_2)\, dx_2 \\
&= \int_0^1 \left[\int_0^{x_2} \frac{1}{x_2}\, dx_1\right] 6x_2(1-x_2)\, dx_2 \\
&= \int_0^1 \frac{x_2}{2} 6x_2(1-x_2)\, dx_2 \\
&= \frac{1}{4}
\end{aligned}$$

Solution to Question 521: Racecar Speed

Be careful with this problem, we can't simply take the average of the speeds of the laps because the racecar spends different amounts of time around each lap. Since we don't know the length of the racetrack, we have to create a variable for how long it takes to go around the racetrack. Say x is the amount of time it takes to go around the first lap in seconds. This means that the second lap will take $\frac{x}{1.5}$ seconds to go around and the third lap will take $\frac{x}{2}$ seconds to go around. That means the average of the speeds (in mph) is

$$\frac{(100 \cdot x + 150 \cdot \frac{x}{1.5} + 200 \cdot \frac{x}{2})}{x + \frac{x}{1.5} + \frac{x}{2}} = \frac{1800}{13}$$

.

Solution to Question 522: Ordering at Chipotle

For each topping, you have the binary option to either get it or not. Thus, there are $2^{10} - 1$ different bowl combinations (-1 for excluding the null set).

Solution to Question 523: Likely Targets I

The key here is to note that since the targets are of extremely small radius, we can essentially treat them as points. The approximate probability we hit target A would be approximately $f(x_A)\varepsilon$, where $f(x_A)$ is the probability density at point A. A similar statement holds for B. Since the two targets are disjoint, our goal is to maximize the sum of the probability densities. Our probability density here is dependent on what μ we select. Therefore, as a function of μ, we need to maximize

$$f(\mu) = \frac{1}{2\sqrt{2\pi}}\left(e^{-\frac{(-1-\mu)^2}{8}} + e^{-\frac{(3-\mu)^2}{8}}\right)$$

The two interior terms are just the density of a $N(\mu, 4)$ distribution at -1 and 3. To do this, we take the derivative and set it equal to 0. In particular,

$$f'(\mu) = \frac{1}{2\sqrt{2\pi}}\left[\frac{-1-\mu}{4}e^{-\frac{(-1-\mu)^2}{8}} + \frac{3-\mu}{4}e^{-\frac{(3-\mu)^2}{8}}\right] = 0$$

In particular, note that if $\mu = 1$, then the two terms inside are the same but of opposite sign by symmetry, so $\mu = 1$ is a critical point. One can verify that this is indeed a maxima, so $\mu = 1$ is our answer. This intuitively makes sense, as the symmetry of the normal distribution should imply that the maxima would be symmetric in both points for our target.

Solution to Question 524: Bowl of Cherries IV

Let A denote the event that the cherry that Jenny eats originates from bowl A. Let R denote the event that the cherry Jenny eats is red. After a cherry is transferred from A to B, there are 10 total cherries, one of which originates bowl A. Therefore,

$$\begin{aligned}\mathbb{P}(A) &= \frac{1}{10}\\ \Rightarrow \mathbb{P}(A^c) &= \frac{9}{10}.\end{aligned}$$

We wish to find $\mathbb{P}(R)$. By applying the law of total probability, we can rewrite this value as

$$\begin{aligned}\mathbb{P}(R) &= \mathbb{P}(R \cap A) + \mathbb{P}(R \cap A^c)\\ &= \mathbb{P}(R|A)\mathbb{P}(A) + \mathbb{P}(R|B)\mathbb{P}(B).\end{aligned}$$

If we are given that the cherry originates from bowl A, then the probability that it is red is simply $\frac{4}{9}$. Similarly, if we are given that the cherry originates from bowl B, then the probability that it is red is simply $\frac{6}{9} = \frac{2}{3}$. Putting it all together, we conclude

$$\begin{aligned}\mathbb{P}(R) &= \mathbb{P}(R|A)\mathbb{P}(A) + \mathbb{P}(R|B)\mathbb{P}(B)\\ &= \frac{4}{9}\cdot\frac{1}{10} + \frac{6}{9}\cdot\frac{9}{10}\\ &= \frac{29}{45}\end{aligned}$$

Solution to Question 525: Particle Reach VI

We generalize this for $p \geq 1/2$. Let T_1 be the number of steps needed to move from position 0 to 1. We want $\mathbb{E}[T_1]$. We use Law of Total Expectation to condition on what happens at the first step. If the particle moves right at the first step, which occurs with probability p, then $T_1 = 1$, as the particle has hit 1. Otherwise, with probability $1-p$, the particle moves left. The expected number of steps in this would be $1+2\mathbb{E}[T_1]$, as the number of steps needed to move from -1 to 0 and 0 to 1 are the same by the Markov Property. This means that

$$\mathbb{E}[T_1] = p \cdot 1 + (1-p) \cdot (1 + 2\mathbb{E}[T_1])$$

Rearranging and solving yields $\mathbb{E}[T_1] = \frac{1}{2p-1}$. With $p = 2/3$, this yields an answer of 3.

Solution to Question 526: Carded Pair

To observe either 2 kings or 1 king and 1 ace, we need to condition on which of king or ace appears first in the deck. If king appears first, then we need to find the number of cards until we observe either a king or an ace. If ace appears first, then we need to find the number of cards until we observe a king. For the first king/ace to be drawn, it is equally likely to be either of them. From here on out, K is king and A is ace.

First, we need to find the expected number of cards until the first K or A appears. We can use the First Ace methodology to use the dividers as K or A, yielding 8 dividers. These 8 dividers divide the other $52 - 8 = 44$ cards into 9 equally-sized regions in expectation, so there are $44/9$ cards on average before the first K or A. Then, we need to add in 1 card for actually picking that K or A.

Now, with probability $1/2$, we selected K. In this case, we need to find the expected number of cards until either a K or A appears in the remaining deck. There are now 7 dividers (4 A and 3 K) and $52 - 44/9 - 1 - 7 = 352/9$ cards left in the deck on average. Since there are 7 dividers, there are now 8 equally-sized regions in expectation, so the expected number of cards before observing the next K or A is $352/72$. Then, we must add in 1 for selecting the K or A, yielding a conditional expectation of $424/72$ in this case.

With probability $1/2$, we selected A. In this case, we need to find the expected number of cards until a K appears. There are now 4 dividers (corresponding to the 4 K) and $52 - 44/9 - 1 - 4 = 379/9$ cards left in the deck on average. The 4 dividers split up our remaining deck into 5 equally-sized regions in expectation, so there are $379/45$ cards before the first K on average. Then, we need to add in 1 more back for selecting the K, so our conditional expectation here is $424/45$.

Putting it all together, our final answer is

$$\frac{44}{9} + 1 + \frac{1}{2} \cdot \frac{424}{45} + \frac{1}{2} \cdot \frac{424}{72} = \frac{1219}{90} \approx 13.54$$

Solution to Question 527: Deriving Put-Call Parity II

We know that the time-T payout of a call-option is $\max(S_T - K, 0)$. The put-option has time-T payout of $\max(K - S_T, 0)$. Using the underlying, the bond, and the put, we can see that we have

$$\max(S_T - K, 0) = S_T - K + \max(K - S_T, 0)$$

In other words, a call option is equivalent to going long 1 unit of the underlying, borrowing K units of the bond, and long 1 unit of the put. This then gives us $V_0 = 8 - 10e^{-.02} + 3.2 = 1.40$

Intuitively, this also makes sense. The current price of the underlying is 8 while the strike is 10. So, our puts are in-the-money while the calls are out-of-the-money. In other words, the puts should have more value than the calls (which we indeed see).

This is also known as put-call parity, where we can replicate a call-option with a forward $(S_T - K)$ and a put option at strike K. This is a very important concept in options theory and forms the basis of *no arbitrage*.

Solution to Question 528: Probability of Unfair Coin I

We can use Bayes' theorem to solve this problem. Let U be the probability that the coin is unfair, and H be the probability that the ten tosses turn up heads. Intuitively, $P(H|U) = 1$ and $P(A) = \frac{1}{1000}$ as given in the problem. With these, we can solve for $P(U|H)$:

$$P(U|H) = \frac{P(H|U)P(U)}{P(H)} = \frac{P(H|U)P(U)}{P(H|U)P(U) + P(H|U^c)P(U^c)} = \frac{\frac{1}{1000} \times 1}{\frac{1}{1000} \times 1 + \frac{999}{1000} \times \frac{1}{1024}} = \frac{1024}{2023} \approx 0.5$$

.

Solution to Question 529: Dollar Draw

The average value of each bill selected is $\frac{100+3\cdot 20+4\cdot 10}{8} = 25$. Since draws are exchangeable, the expected payout of each draw is the exact same, so the expected total of the bills we select is $25 \cdot 7 = 175$.

Alternatively, we can compute this by considering the bill we don't select, and then subtract this result from 200. With probability $\frac{1}{2}$, we leave a \$10 bill. With probability $\frac{3}{8}$, we leave a \$20 bill. With probability $\frac{1}{8}$, we leave the \$100 bill. Therefore, the expected value of the bill we leave is $\frac{1}{2}\cdot 10+\frac{3}{8}\cdot 20+\frac{1}{8}\cdot 100 = 25$. This means the expected sum of the bill we do select is $200 - 25 = 175$.

Solution to Question 530: Ace Distribution

To distribute 52 cards to 4 players equally, there are a total of $\frac{52!}{13!\times 13!\times 13!\times 13!}$ possibilities. There are a total of 4! ways to distribute the 4 aces to the 4 players. There are a total of $\frac{48!}{12!\times 12!\times 12!\times 12!}$ ways to distribute the remaining 48 cards. The final probability is the number of favorable outcomes over the total number of outcomes, or

$$4! \times \frac{48!}{12!\times 12!\times 12!\times 12!} \div \frac{52!}{13!\times 13!\times 13!\times 13!} \approx .1055$$

Solution to Question 531: Green Light

Fix the 4th green ball in the 9th spot. This means that there are 3 green balls in the first 8 spots. There are $\binom{8}{3}$ ways to pick the locations of those in the first 8 spots. There are $\binom{15}{4}$ ways to pick the locations of the 4 green balls with no restrictions, so the probability in question is just $\frac{\binom{8}{3}}{\binom{15}{4}} = \frac{8}{195}$

Solution to Question 532: Team Winners

There are a total of $\binom{32}{2} = 496$ games played in this tournament, meaning that there are 2^{496} total outcomes of the tournament. We want the number of outcomes where each team wins a distinct amount of games. However, this fixes the tournament outcome completely, as each team must have a distinct integer $0, 1, \ldots, 31$ wins, and note that $0 + 1 + \cdots + 31 = \frac{31 \cdot 32}{2} = 496$. Therefore, we have already fixed the outcomes of the tournament, so we really just need to assign labels to which team obtains which amount of wins. There are 32! ways to do this, so our answer is $\frac{32!}{2^{496}}$. This means $a = 32, b = 496$, so $a + b = 528$.

Solution to Question 533: Partial Derivatives

The trick here is to realize that only the counts matter and not the order of the derivatives. This implies we should take a stars and bars approach. Let n_i be the order of partial derivative given to x_i, where $1 \leq i \leq 10$. This means that counting 5th order partial derivatives is equivalent to counting the number of non-negative integer solutions to $n_1 + \cdots + n_{10} = 5$, as each unique solution to this corresponds to the partial derivative $\frac{\partial f}{\partial x_1^{n_1} \ldots \partial x_{10}^{n_{10}}}$. There are $\binom{14}{9} = 2002$ non-negative integer solutions to the this equation by stars and bars.

Solution to Question 534: Spherical Coodinates

We know that our point is uniformly at random selected from this sphere. Let R be the random radius (distance from the origin) of this point. We know that $R^2 = X_1^2 + \cdots + X_{10}^2$. Since this sphere is clearly symmetric about the origin and all of the coordinates share the same marginal distributions, they are exchangeable and hence has the same expectation. Therefore, $\mathbb{E}[R^2] = 10\mathbb{E}[X_1^2]$.

We know that $\text{Var}(X_1) = \mathbb{E}[X_1^2] - (\mathbb{E}[X_1])^2$. However, as our sphere is again symmetric about the origin, the mean of X_1 is 0, so we just need to find the second moment of X_1 and we are done. This boils down to, by the above, finding the expected squared distance from the center.

Let's compute $\mathbb{P}[R \leq r]$. The volume of a $n-$dimensional sphere of radius r is a constant $C_n \cdot r^n$. Therefore, as we select uniformly at random from this sphere, the probability of the event $\{R \leq r\}$ just means that our point belongs to the sub-sphere of radius r from the big sphere of radius 12. The probability of this is $\frac{C_{10} r^{10}}{C_{10} \cdot 12^{10}} = \frac{r^{10}}{12^{10}}$. Therefore, the probability density of R (found by taking the derivative with respect to r of the CDF) is $f_R(r) = \frac{10r^9}{12^{10}} I_{(0,12)}(r)$. The indicator comes from the fact that the radius must be somewhere between

0 and 12. Obtaining $\mathbb{E}[R^2]$ is now straightforward using LOTUS. We have that $\mathbb{E}[R^2] = \frac{1}{12^{10}} \int_0^{12} 10r^{11} dr = \frac{10}{12^{11}} 12^{12} = 120.$

Plugging this back into our original equation, we have that $120 = 10\mathbb{E}[X_1^2]$, meaning $\mathrm{Var}(X_1) = \mathbb{E}[X_1^2] = 12$.

Solution to Question 535: English and History Majors

We are able to obtain the proportion of students majoring in just English by subtracting out the intersection proportion from the English proportion. This means 15% of students major in only English. Similarly, 30% of students major in just History. Adding all the single major students and the double major students, we get 60% of students major in either English or History. Therefore, 40% do not major in either of the subjects.

Solution to Question 536: Limiting Random Variable I

Define $Z_n = X_{4n-3}^2 + X_{4n-2}^2 X_{4n-1} X_{4n}$. Then $Y_n = \frac{Z_1 + \cdots + Z_n}{n}$. by substituting in. Therefore, as the Z_i random variables are IID, $\mathbb{E}[Y_n] = \frac{1}{n} \cdot n\mathbb{E}[Z_1] = \mathbb{E}[Z_1]$. By substituting in the definition of Z_1, $\mathbb{E}[Z_1] = \mathbb{E}[X_1^2 + X_2^2 X_3 X_4]$.

By linearity of expectation and independence, $\mathbb{E}[Z_1] = \mathbb{E}[X_1^2] + \mathbb{E}[X_2^2]\mathbb{E}[X_3]\mathbb{E}[X_4]$. We can quickly compute $\mathbb{E}[X_1^2] = \mathrm{Var}(X_1) + (\mathbb{E}[X_1])^2 = 25 + 20 = 45$, so $\mathbb{E}[Z_1] = 45 + 45 \cdot 5 \cdot 5 = 1170$, so $\mathbb{E}[Y_n] = 1170$ for all n, meaning 1170 is our answer.

Solution to Question 537: Confident Double Heads

Let p_n be the probability that given we observe n consecutive heads, our coin is the double-headed coin. Let H_n be the event that n consecutive heads are observed and D be the event that we selected the double-headed coin. We want $p_n = \mathbb{P}[D \mid H_n] = \frac{\mathbb{P}[D \cap H_n]}{\mathbb{P}[H_n]}$ by the definition of conditional probability. On the numerator, we can use the definition of conditional probability again to receive $\mathbb{P}[D \mid H_n] = \mathbb{P}[H_n \mid D]\mathbb{P}[D]$. The probability of selecting the double-headed coin is $\frac{1}{20}$. Given we select the double-headed coin, we will flip any amount of heads consecutively with probability 1, so $\mathbb{P}[H_n \mid D] = 1$.

On the denominator, we need to condition on whether or not we receive the double-headed coin. Thus,

$$\mathbb{P}[H_n] = \mathbb{P}[H_n \mid D]\mathbb{P}[D] + \mathbb{P}[H_n \mid D^c]\mathbb{P}[D^c]$$

We already calculated the first term in the numerator. For the second term, we know $\mathbb{P}[D^c] = 1 - \mathbb{P}[D] = \frac{19}{20}$. Then, $\mathbb{P}[H_n \mid D^c] = \frac{1}{2^n}$, as we flip a heads on a fair coin with probability $\frac{1}{2}$ in each turn. Therefore, $\mathbb{P}[H_n] = \frac{1}{20} + \frac{19}{20 \cdot 2^n}$. Therefore, $p_n = \frac{\frac{1}{20}}{\frac{1}{20} + \frac{19}{20 \cdot 2^n}} = \frac{1}{1 + \frac{19}{2^n}}$. Our goal is to find the smallest n such that $p_n > \frac{19}{20}$.

This means that $\frac{1}{1 + \frac{19}{2^n}} > \frac{19}{20}$. Multiplying by the denominator on both sides yields $1 > \frac{19}{20} + \frac{361}{20 \cdot 2^n}$. Multiplying by $20 \cdot 2^n$ on both sides yields $2^n > 361$, which occurs for $n \geq 9$. This means 9 times is the minimum number we need.

Solution to Question 538: Sequence Probability

Every sequence of n coin flips has a $\left(\frac{1}{2}\right)^n$ chance of happening (assuming a fair coin). This is since there are 2^n possible sequences of length n. For 4 flips, this comes out to $\frac{1}{16}$.

Solution to Question 539: Conditional 8 Sum

There are 5 outcomes of the dice that show a sum of 8. These are $(6, 2), (5, 3), (4, 4), (3, 5)$, and $(2, 6)$. Of those, two of them are $(6, 2)$ and $(2, 6)$. Therefore, our answer is simply $\frac{2}{5}$.

Solution to Question 540: Doubly 5 II

We are going to compute the complement of seeing 4 and 6 before two 5s. The trick here is that no other rolls matter besides $4-6$. Therefore, conditioned on being in $4-6$, each of the three values appears with probability $1/3$. We only need to consider the orderings of the rolls $4, 6, 5$, and 5. There are three cases where we get 4 and 6 before two 5s.

Case 1 - 5 on First Roll: In this case, we need the next two rolls to be 4/6 and then the other of 4/6 that wasn't first. The probability of this case is $\frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{9}$, as there is a $1/3$ chance 5 is first, then $2/3$ chance of rolling 4/6 before another 5, and then $1/2$ chance of rolling the other of 4/6 before a 5.

Case 2 - 4/6 Comes First and Second: In this case, the probability is just $\frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$, as it is the same as Case 1 after we roll the 5 in Case 1.

Case 3 - 4/6 Comes First and 5 Comes Second: In this case, the probability is $\frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{6}$. There is a $2/3$ probability 4/6 comes first, then a $1/2$ probability that 5 comes before the other of 4/6, and $1/2$ probability that the other of 4/6 comes before the other 5.

Adding these up, we obtain a probability of $\frac{11}{18}$ that both 4 and 6 appear before two 5s. Therefore, our answer is $1 - \frac{11}{18} = \frac{7}{18}$. This makes intuitive sense, as there are different permutations of 4 and 6 that they can appear in, whereas two 5s only have one permutation.

Solution to Question 541: High or Die

How likely it is to obtain a value that is at least as large as the first value depends on the first value itself. Therefore, we should condition on X_1, the value of the first roll. This means that $\mathbb{E}[N] = \mathbb{E}[\mathbb{E}[N \mid X_1]]$. Given X_1, there are $7 - X_1$ values on the die at least X_1, so the probability of obtaining a value at least X_1 any given roll would be $\frac{7 - X_1}{6}$. Therefore, the expected number of rolls needed to do so is $\frac{6}{7-X_1}$. Plugging this in and using LOTUS,

$$\mathbb{E}[N] = \mathbb{E}\left[\frac{6}{7-X_1}\right] = \sum_{i=1}^{6} \frac{6}{7-i} \cdot \frac{1}{6} = \frac{49}{20}$$

Solution to Question 542: High or Die

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$$\mathbb{E}[N] = \mathbb{E}\left[\frac{6}{7-X_1}\right] = \sum_{i=1}^{6} \frac{6}{7-i} \cdot \frac{1}{6} = \frac{49}{20}$$

Solution to Question 543: Squared Matrix

By multiplying the first row of A by the first column of A, we get that $1+2x = 9$ by equating entries, meaning $x = 4$. Afterwards, by multiplying the second row of A by the by the second column of A, we get that $2x + 4 + 3y = 21$. Since $x = 4$ from before, we have that $3y = 9$ after rearrangement, yielding $y = 3$. Therefore, $xy = 12$.

Solution to Question 544: Leaked IP

Let T be the event the culprit is a trader and S be the event that the witness says that the person leaking IP is a trader. We want $\mathbb{P}[T \mid S]$. We know that $\mathbb{P}[T] = 2/3$ from the question. Also from the question, we can conclude $\mathbb{P}[S \mid T] = 2/3$, as the witness is correct 2/3 of the time, so given the culprit was a trader, 2/3 of the time, the witness would also say it is a trader. This also means $\mathbb{P}[S \mid T^c] = 1/3$, as the witness is incorrect 1/3 of the time. Therefore, we have that

$$\mathbb{P}[T \mid S] = \frac{\mathbb{P}[S \mid T]\mathbb{P}[T]}{\mathbb{P}[S \mid T]\mathbb{P}[T] + \mathbb{P}[S \mid T^c]\mathbb{P}[T^c]} = \frac{2/3 \cdot 2/3}{2/3 \cdot 2/3 + 1/3 \cdot 1/3} = \frac{4}{5}$$

Solution to Question 545: Jane's Children

The sample space of two children given at least one child is a daughter is $\Omega = \{(b,g),(g,b),(g,g)\}$, where (b,g) means that the older child is a boy and the younger child is a girl. Thus, the probability that Jane has two daughters is $\frac{1}{3}$.

Solution to Question 546: Tricky Bob I

Call Bob's success probability p_2. Let's write out the expected value as a function of p_1 and p_2. The probability of HH is p_1p_2. The probability of TT is $(1-p_1)(1-p_2)$. The probability of TH or HT is $p_1(1-p_2)+p_2(1-p_1) = p_1+p_2-2p_1p_2$. From Bob's perspective, the respective profits for each of these outcomes are $6, 4$, and -5. Therefore, the expected value is $6p_1p_2+4(1-p_1)(1-p_2)-5(p_1+p_2-2p_1p_2) = 6p_1p_2+4-4p_1-4p_2+4p_1p_2-5p_1-5p_2+10p_1p_2 = 20p_1p_2-9p_2-9p_1+4$. Factoring, this becomes $(20p_1-9)p_2+(4-9p_1)$.

Now, assuming that $p_1 \neq \dfrac{9}{20}$ (such that we can rearrange and divide), we see that $p_2 > \dfrac{9p_1-4}{20p_1-9}$. The RHS is 0 when $p_1 = \dfrac{4}{9}$, as the numerator vanishes. In addition, let's find when the RHS is at least 1. This implies $9p_1-4 \geq 20p_1-9$, which means that $p_1 \leq \dfrac{5}{11}$. One can verify that the expected value in this case (by plugging into the expected value equation previously) is p_2-1. However, as a probability is at most one 1, unless $p_2 = 1$, Bob has a strictly negative expected value. Similarly, by plugging in $p_1 = \dfrac{4}{9}$, one can find the expected value to be $-p_2$. Therefore, unless $p_2 = 0$, Bob once again has a strictly negative expected value. Therefore, $a = \dfrac{4}{9}$ and $b = \dfrac{5}{11}$, so $b-a = \dfrac{1}{99}$.

Solution to Question 547: Equicorrelated

We are going to consider $\mathrm{Var}(\overline{X})$ here, where $\overline{X} = \dfrac{X_1+\cdots+X_7}{7}$. By plugging this in and using properties of variance, we see that

$$\mathrm{Var}(\overline{X}) = \frac{1}{49}\mathrm{Var}(X_1+\cdots+X_7) = \frac{1}{49}\left[\sum_{i=1}^{7}\mathrm{Var}(X_i) + \sum_{i\neq j}\mathrm{Cov}(X_i,X_j)\right]$$

Since we know the mean and variance of each of the random variables, we know that the first sum is just 7. Similarly, we know that $\mathrm{Cov}(X_i,X_j) = \rho(1)(1) = \rho$ and that there are $7\cdot 6 = 42$ terms in that sum. Therefore, the second sum is just 42ρ. Thus,

$$\mathrm{Var}(\overline{X}) = \frac{7+42\rho}{49}$$

Our condition is that $\text{Var}(\overline{X}) \geq 0$, as the variance of any random variable must be non-negative. Thus, we can disregard the denominator and find that

$$7 + 42\rho \geq 0 \iff \rho \geq -\frac{1}{6}$$

Therefore, our answer is $-\frac{1}{6}$.

Solution to Question 548: Stamp Sum

Note that $21, 42, 63$, and 84 are the smallest values that can be expressed as change that are equivalent to $1, 2, 3$, and 4 modulo 5, respectively. Afterwards, if we add some amount of 5 value stamps to these, we are able to obtain stamps of any number equivalent to those in modulo 5. Therefore, the largest value we can't create would is something 4 modulo 5, as that is the highest starting point. Namely, the largest integer that can't be created here is 79, as anything that is 4 modulo 5 and at least 84 can be created, so 79 is our answer.

Solution to Question 549: Poker Hands V

There are a total of $\binom{52}{5}$ total hand combinations. To count the number of hands that contain a three-of-a-kind, we need to look at the three-of-a-kind itself and the possible other cards. The three-of-a-kind has 13 possible face values and $\binom{4}{3}$ possible suit values. The other cards have $\binom{12}{2}$ possible face values and 4 possible suit values for each card. Thus, the probability that you have a three-of-a-kind is:

$$\frac{13 \cdot \binom{4}{3} \cdot \binom{12}{2} \cdot 4^2}{\binom{52}{5}} = \frac{88}{4165}$$

Solution to Question 550: Binary String

Let S be the event the string starts with 1 and O be the event it ends with 00. We want $|S \cup O| = |S| + |O| - |SO|$. There are 2^7 strings starting with 1, as we fix the first spot of the string and the other 7 spots have 2 options each. Similarly, $|O| = 2^6$ and $|SO| = 2^5$ by the same logic. We have that $|S \cup O| = 128 + 64 - 32 = 160$.

Solution to Question 551: Kiddie Pool

This is a rates problem so let's answer it as such. For every minute, the first hose fills up $\frac{1}{20}^{\text{th}}$ of the pool, the second hose fills up $\frac{1}{10}^{\text{th}}$, and the drain removes

$\frac{1}{15}^{\text{th}}$ of the pool. Thus the combined rate is $\frac{1}{20} + \frac{1}{10} - \frac{1}{15} = \frac{1}{12}$. With the two hoses and the drain open, we fill up $\frac{1}{12}^{\text{th}}$ of the pool every minute. Thus it takes 12 minutes to completely fill up the pool.

Solution to Question 552: Good Grid II

Our sample space here is the collection of all pairs of intervals. Note that we have two cases here for each of the two intervals we form. First, we could have that our interval is a singular point, which occurs if $a = b$ or $c = d$. There are $2n+1$ ways for this to occur. Alternatively, we can have that $a \neq b$ or $c \neq d$. In this case, for every 2 distinct values we select from S_n, exactly one arrangement of them will make a valid interval. Therefore, there are $\binom{2n+1}{2}$ ways to pick two distinct points from S_n. Adding these two cases together, there are

$$\frac{(2n+1)(2n)}{2} + (2n+1) = \binom{2n+2}{2}$$

ways to form an interval, so there are $\binom{2n+2}{2}^2$ ways to pick the two intervals.

Let a $|$ represent an endpoint of an interval. There are $2n+2$ locations a $|$ can be put, as we can put them in any spot from before $-n$ to after n, which is $2n+2$ spots. For example, with $n = 1$, $|-1|01$ represents $a = b = -1$, whereas $|-10|1$ represents $a = -1$ and $b = 0$. Additionally, we see that a proper nesting is really just a way to select 4 distinct spots from the $2n+2$, as we can't have the endpoints of I_1 be equal to either of the endpoints of I_2. Therefore, there are $\binom{2n+2}{4}$ ways to pick the intervals such that they don't overlap. Thus, the probability is

$$p(n) = \frac{\binom{2n+2}{4}}{\binom{2n+2}{2}^2} = \frac{1}{6} \cdot \frac{2n(2n-1)}{(2n+2)(2n+1)}$$

As $n \to \infty$, the term with n in it tends to 1, so $\lim_{n\to\infty} p(n) = p = \dfrac{1}{6}$.

Therefore,

$$\frac{p - p(10)}{p} = \frac{\frac{1}{6} - \frac{1}{6} \cdot \frac{20\cdot 19}{22\cdot 21}}{\frac{1}{6}} = \frac{41}{231}$$

Solution to Question 553: Mathematical Birthday

The man must have lived in the 1800s or 1900s. Starting with the first condition, the only combination of a and b that gets close to this range is $5^4 + 6^4 = 1921$. Therefore, the man must have supposedly been $5^2+6^2 = 61$ in 1921. This would imply the man was born in 1860. We now need to attempt to verify the other conditions. To be $2m$ years old in the year $2m^2$, we need to find m satisfying $1860 + 2m = 2m^2$, so $930 = 31 \cdot 30 = m(m-1)$, so $m = 31$ works here. This would be consistent with the first piece of information. Lastly, we must find an n such that $1860+3n = 3n^4$, which means that $620 = 5\cdot 124 = 5(5^3-1) = n^4-n = n(n^3-1)$. Thus, $n = 5$, Therefore, we get that $abmn = 5 \cdot 6 \cdot 5 \cdot 31 = 4650$.

Solution to Question 554: Swapping X and Y

Consider the formula to calculate β in a simple linear regression, regressing Y onto X: $\beta = \frac{\text{Cov}(X,Y)}{Var(X)}$. The denominator corresponds to the variance of the independent variable. If we swap Y and X, the denominator becomes $\text{Var}(Y)$. From looking at the data, we can see that the variance of Y is larger than the variance of X. So, the denominator will be bigger in the case of the second regression, decreasing the value of β.

Solution to Question 555: Geometric PGF

By LOTUS, we have that

$$p(z) = \mathbb{E}[z^X] = \sum_{k=1}^{\infty} z^k \mathbb{P}[X=k] = \sum_{k=1}^{\infty} p(1-p)^{k-1} \cdot z^k = pz \sum_{k=1}^{\infty} (z(1-p))^{k-1} = \frac{pz}{1-(1-p)z}$$

The last sum converges since $|(1-p)z| \leq 1$. Plugging in our desired values, we obtain our answer of $\dfrac{2/9}{1-1/3\cdot 1/3} = \dfrac{1}{4}$.

Solution to Question 556: Fibonacci Sum

Define S as the sum in equation. We can expand out the first few terms of the sum as

$$\sum_{k=0}^{\infty} \frac{F_k}{10^{k+1}} = \frac{F_0}{10} + \frac{F_1}{10^2} + \frac{F_2}{10^3} + \frac{F_3}{10^4} + \cdots = \frac{F_0}{10} + \frac{F_1}{10^2} + \left(\frac{F_0}{10^3} + \frac{F_1}{10^3}\right) + \left(\frac{F_1}{10^4} + \frac{F_2}{10^4}\right) + \ldots$$

We used the definition of the Fibonacci sequence in the second equality. Now, we can regroup the terms as

$$S = \frac{F_0}{10} + \frac{F_1}{100} + \left(\frac{F_0}{10^3} + \frac{F_1}{10^4} + \dots\right) + \left(\frac{F_1}{10^3} + \frac{F_2}{10^4} + \dots\right)$$

As $F_0 = 0$, we can add in a $\frac{F_0}{10^2}$ term to the beginning of the second parenthesized group without changing the value. Note, the first parenthesized statement is just $\frac{1}{10^2}S$, as all the exponents are shifted by 2. Similarly, the second parenthesized statement is just $\frac{1}{10}S$. Therefore, we have that

$$S = \frac{0}{10} + \frac{1}{100} + \frac{1}{100}S + \frac{1}{10}S \iff \frac{89}{100}S = \frac{1}{100} \iff S = \frac{1}{89}$$

Solution to Question 557: Integral Variance III

As W_s^2 is square-integrable, adapted, and continuous almost surely, $\int_0^t W_s^2 dW_s$ is a martingale with mean 0. By Ito Isometry, we have that

$$\mathbb{E}\left[\left(\int_0^t W_s^2 dW_s\right)^2\right] = \int_0^t \mathbb{E}[(W_s^2)^2]ds = \int_0^t \mathbb{E}[W_s^4]ds$$

As $W_s \sim N(0,s)$, $W_s = \sqrt{s}Z$, where $Z \sim N(0,1)$. In particular, since $\mathbb{E}[Z^4] = 3$, $\mathbb{E}[W_s^4] = \mathbb{E}[(\sqrt{s}Z)^4] = 3s^2$. Now, we evaluate evaluate that integral to simply be $\int_0^t 3s^2 ds = t^3$. As we already know the integral is mean 0, t^3 is also the variance, so $k = 1$.

Solution to Question 558: 9 Appearance

The first thing is to count the number of times 9 appears in $1-99$. Then, we can multiply this by 10, as there are 10 starting digits $0-9$ that can go in front of this. A starting digit of 0 here really means that the number is less than 100. Afterwards, we can add in 100 9s afterwards for the 900s to account for the starting digit of 9.

9 occurs in each of $9, 19, \dots, 89$ once, yielding 9 9s. Afterwards, $90-99$ has 11 9s, as 99 counts as 2. Therefore, in $1-99$, the digit 9 occurs 20 times. Thus, the answer is $10 \cdot 20 + 100 = 300$.

Solution to Question 559: Shiny Pennies

We can use complimentary probability; we'll begin by compute the probability that the third shiny penny appears within the first 4 draws. There are two cases: (1) the third shiny penny appears on the third draw, and (2) the third shiny penny appears on the fourth draw. This is now a very simple ordering problem. There are $\binom{7}{3} = 35$ total orderings. There is 1 ordering that satisfies case 1, and 3 orderings that satisfy case 2 (there is exactly 1 dull penny that appears in the first 4 draws, and the last of the first 4 draws must be a shiny penny). Our answer is $\frac{31}{35}$.

Solution to Question 560: Three Riflemen

We are looking for the distribution of the first success of a process with probability 1/2 of success per trial. The distribution of the number of trials needed is $N \sim \text{Geom}(1/2)$. Therefore, $\mathbb{P}[N = k] = \dfrac{1}{2^k}$ for $k \geq 1$. If A wins, then the shot is on one of trials $1, 4, 7, \ldots$, as it must cycle back to them. Therefore, the probability of interest is

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{3k+1} = \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{8^k} = \frac{1}{2} \cdot \frac{1}{1-\frac{1}{8}} = \frac{4}{7}$$

Another way to quickly see this is that we could call the probability of person A winning p. Every outcome where person B has the opportunity to win is the same as person A except scaled by a factor of half since person B must ensure person A misses right before them, so the probability for person B is $\dfrac{1}{2}p$. By the same logic, the probability person C wins is $\dfrac{1}{4}p$. Adding all of these up yields

$$p\left(1 + \frac{1}{2} + \frac{1}{4}\right) = 1 \iff p = \frac{4}{7}$$

Solution to Question 561: Call Option Change

An at-the-money call option has $\Delta = 0.5$. We know that Γ is the sensitivity of Δ to the price change of the underlying. If the underlying moves by \$1, then we would expect the call option to increase by 0.5. However, now, the delta has also increased by gamma ($\Delta = 0.5 + 0.03 = 0.53$). In other words, $\Delta = 0.53$. When the underlying moves another dollar up, then we need to increase the call option by this new delta. This leaves the following approximation of the new price of the option.

$$C = 2.3 + 0.5 + 0.53 = 3.33$$

Solution to Question 562: 9 For 1

Let X_i be the digit in the 10^i spot in X, with X_0 being in the ones spot. For each digit X_i, it is equally likely whether or not it is a 1 or 9. This is because of the fact that we select k uniformly at random, so there as are equally many $8-$digit integers with k 1s as k 9s, we can view them as just flipped around. Therefore, $\mathbb{E}[X_i] = \dfrac{1+9}{2} = 5$. We can write $X = 10^7 \cdot X_7 + \cdots + 10^1 \cdot X_1 + X_0$. Therefore,

$$\mathbb{E}[X] = 10^7 \cdot \mathbb{E}[X_8] + \cdots + \mathbb{E}[X_1] = 5\left(10^8 + \cdots + 10 + 1\right) = 5 \cdot 11111111 = 55555555$$

Solution to Question 563: Lognormal I

Let $Z \sim N(0,1)$. Then we can say that $X = e^Z$. We want to find $\mathbb{E}[X] = \mathbb{E}[e^Z] = M_Z(1)$, where $M_Z(\theta)$ is the MGF of Z. The MGF of $Z \sim N(0,1)$ is $M_Z(\theta) = e^{\frac{1}{2}\theta^2}$. Therefore, $M_Z(1) = e^{\frac{1}{2}}$, so our answer is $\ln(e^{\frac{1}{2}}) = \dfrac{1}{2}$.

Solution to Question 564: Coin-Die Oddity

The number of dice rolled is dependent upon the number of heads that we see. Therefore, we should condition on the number of heads obtained. We see $0, 1$, and 2 heads with respective probabilities $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{1}{4}$.

If we obtain no heads, then the sum is 0, which is even. Therefore, our probability in this case is 0. If we obtain 1 head, then 3 of the 6 equally-likely values are odd, so the probability is $\frac{1}{2}$ in this case. If we obtain 2 heads, then we roll 2 dice. The possible values are $3, 5, 7, 9$, and 11. The number of combinations resulting in those sums is $2+4+6+4+2 = 18$. There are $6^2 = 36$ total outcomes, so the probability of an odd value in this case is $\frac{1}{2}$ as well. Therefore, by the Law of Total Probability, the probability of an odd value is

$$\frac{1}{4} \cdot 0 + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = \frac{3}{8}$$

Solution to Question 565: Dollar Break

The most clear choice is that we should have 4 pennies. This is so that we can't create any other coin using these 4 pennies. Next, let's look at quarters. We only want to have 1 quarter because if we have more than one, we will be able to create change potentially by 50 cents with 2 quarters. We want to ensure that doesn't happen. Next, we want to load in as many dimes as possible. We can put in 9 dimes and still have no change for a dollar since the closest we can get is 99 cents (7 dimes, 1 quarter, 4 pennies). We can't add in any nickels, as then we would be able to make a dollar with 1 quarter, 7 dimes, and a nickel. Therefore, our maximal value is $0.25 + 9 \cdot 0.10 + 4 \cdot 0.01 = 1.19$.

Solution to Question 566: Multiple Divisors I

The prime factorization of 10^{99} is $2^{99} \cdot 5^{99} = (2^{80} \cdot 5^{80}) \cdot 2^{19} \cdot 5^{19}$. The term in parentheses is 10^{80}. As we know that there are 100 options of exponent for our divisor for each of 2 and 5 (can choose an exponent between 0 and 99, inclusive of both), there must be $100^2 = 10000$ divisors total that we can select from. Of those, we need those with the exponent for both 2 and 5 to be at least 80. By the previous rewriting of our expression, this just means that we must choose an exponent for both 2 and 5 that are between 80 and 99, inclusive of both. This gives us 20 options for each, implying that there are $20^2 = 400$ total divisors satisfying this. Putting it all together, our probability must therefore be $\frac{400}{10000} = \frac{1}{25}$.

Solution to Question 567: Infinite Maturity

Quite simply, a European option can only be exercised at expiration. Given our call option has infinite maturity (meaning it will never expire), you can never exercise it, leading to it being

Note: There is another line of thinking here. Although you are never able to realize its value, this option still technically does hold value (as shown through the Black-Scholes model), the problem is you are never able to practically extract that value. So in theory, the option keeps it initial value, but practically you are unable to extract it.

Solution to Question 568: The Big Three

To obtain a sum of 18 from three dice, we need to have at least 3 sixes among the 5 dice. Therefore, we really are just asking for the probability of $3, 4$, or 5 sixes among 5 die rolls. The probability of 5 sixes is just $\frac{1}{6^5}$, as it is just $\frac{1}{6}$ probability for each die. The probability of 4 sixes is $5 \cdot \frac{5}{6} \cdot \frac{1}{6^4}$, as we have 5 ways to select the die that isn't a 6, a $\frac{5}{6}$ probability that the select die is not a 6, and then $\frac{1}{6}$ probability for each of the other 4 dice to be a 6. Lastly, the probability that we have exactly 3 sixes is $\binom{5}{2} \cdot \left(\frac{5}{6}\right)^2 \cdot \frac{1}{6^3}$. This is because there are $\binom{5}{2} = 10$ ways to pick the two dice that aren't a 6, $\frac{5}{6}$ probability for each of them to not be a 6, and $\frac{1}{6}$ probability for each of the remaining three dice to be a 6. Adding all of these up yields

$$\frac{1}{6^5} + \frac{25}{6^5} + \frac{250}{6^5} = \frac{276}{6^5} = \frac{23}{648}$$

Solution to Question 569: 3 Ace

There are $48 \cdot 13 = 624$ hands that consist of all 4 suits of some rank. This is because there are 13 ways to pick the rank in which we have all 4 suit. Then, there are 48 ways to pick the last card. We just need to divide this by the total number of hands with at least 3 cards of some rank. We know 624 hands have exactly 4 of some rank. Therefore, we can just count the number of hands with exactly 3 cards of some rank. There are 13 ways to pick the rank that we have 3 of. There are 4 ways to pick the suit that is excluded from our selection. In addition, there are $\binom{48}{2}$ ways to pick two remaining cards for our hand that are not of the desired rank. Therefore, our answer is $\frac{624}{624 + 4 \cdot 13\binom{48}{2}} = \frac{1}{95}$.

Solution to Question 570: 20-30 Die Split II

Let W be the event Alice wins. Then we can use Law of Total Probability to condition on whether or not Alice's roll is larger than 20. Let A be the value Alice rolls. Furthermore, let B_1 and B_2 be the values that Bob rolls on the first and second roll. We have that

$$\mathbb{P}[W] = \mathbb{P}[W \mid A > 20]\mathbb{P}[A > 20] + \mathbb{P}[W \mid A \leq 20]\mathbb{P}[A \leq 20]$$

It is clear to see that $\mathbb{P}[A > 20] = \frac{1}{3}$, as this accounts for 10 of 30 values. Furthermore, $\mathbb{P}[W \mid A > 20] = 1$, as this is strictly larger than any value Bob can roll.

For the other case, we compute

$$\mathbb{P}[W \mid A \leq 20] = \sum_{a=1}^{20} \mathbb{P}[\max\{B_1, B_2\} < a]\mathbb{P}[A = a \mid A \leq 20] = \sum_{a=1}^{20} \mathbb{P}[B_1 < a]^2\mathbb{P}[A = a \mid A \leq 20]$$

This is the event of interest as we want the Alice to have a strictly higher value than Bob. We simplify the maximum statement by noting that B_1 and B_2 are IID and that $\max\{B_1, B_2\} < a$ is equivalent to saying that $B_1 < a, B_2 < a$.

We know $\mathbb{P}[A = a \mid A \leq 20] = \frac{1}{20}$, as we know $A \leq 20$, so there are 20 equally-likely values. Furthermore, we have that $\mathbb{P}[B_1 < a] = \frac{a-1}{20}$, as we want B_1 to take some value between 1 and $a-1$, inclusive of both. Therefore,

$$\mathbb{P}[W \mid A \leq 20] = \frac{1}{20}\sum_{a=1}^{20}\left(\frac{a-1}{20}\right)^2 = \frac{247}{800}$$

Combining these, we have that

$$\mathbb{P}[W] = \frac{1}{3} + \frac{2}{3} \cdot \frac{247}{800} = \frac{647}{1200}$$

Solution to Question 571: Measuring Water

Two pints may be measured in fourteen transactions as show in the table below. The first column represents the amount of water in the 7 pint vessel, whereas the second column represents the amount of water in the 11 pint vessel.

7 11

1: 7 0

2: 0 7

3: 7 7

4: 3 11

5: 3 0

6: 0 3

7: 7 3

8: 0 10

9: 7 10

10: 6 11

11: 6 0

12: 0 6

13: 7 6

14: 2 11

Solution to Question 572: Rook on a Chessboard

Consider the case of 2 x 2 chessboard. For you to win, the rook must be on the top-right square or the bottom-left square. Player 1 has no choice but to move the rook onto one of those squares, and you will always win.

We can expand this to bigger chessboards by ensuring that you always move the rook to a position that is diagonal to the top-left square. This gives a diagonal line from the bottom-right square to the top-left square of *safe* squares. If you move the rook to one of these squares, you are guaranteed to win. Thus, however Player 1 moves, you will move the rook to the corresponding square on the diagonal.

Solution to Question 573: Deviating Sums

Let X be the number chosen from the first set and Y be the number chosen from the second set. Since X and Y are independent, we know that

$$Var(X+Y) = Var(X) + Var(Y) = 2 \cdot Var(X)$$

To find $Var(X)$, we can use the equation $Var(X) = E(X^2) - E(X)^2$.

$$E(X^2) = \frac{0^2+1^2+2^2+3^2+4^2}{5} = 6$$

$$E(X)^2 = (\frac{0+1+2+3+4}{5})^2 = 2^2 = 4$$

Thus $Var(X) = 6 - 4 = 2$ and $Var(X+Y) = 2 \cdot 2 = 4$. Since we are finding the standard deviation, we have to take the square root of the variance. Thus the answer is 2.

Solution to Question 574: Tennis Deuces I

Both players need to win exactly 3 points against each other to obtain a deuce. This means that out of a length of 6 points, 3 come from Person 1 and 3 come from Person 2. Thus we can choose 3 positions out of the total 6 points for Person 1 to score which gives us $\binom{6}{3}$ which yields 20 ways.

Solution to Question 575: Proper Statement

We solve this for more general n. The trick here is to refer back to the idea of Catalan numbers being proper parenthesizations of just (). There are C_n such parenthesizations of length $2n$. Then, for each pair of parentheses, we have the option to select it as either parentheses or curly braces. Therefore, there are 2^n such ways we can select each pair being either parentheses or curly braces. Thus, $P_n = C_n \cdot 2^n$. In particular, $P_5 = C_5 \cdot 2^5 = \dfrac{1}{6} \cdot \dbinom{10}{5} \cdot 2^5 = 1344$.

Solution to Question 576: Lead Count

Fix the first flip arbitrarily since it is equally likely to be heads or tails. Without loss of generality, say it is heads. Then we would need at least 5 heads in the next 9 flips to be leading, as we need strictly more than 5 heads total in the 10 flips and we know the first is heads. The trick here to compute this probability is to note that $\dbinom{9}{i} = \dbinom{9}{9-i}$ for each $0 \leq i \leq 9$. Therefore, we have that

$$\binom{9}{0} + \cdots + \binom{9}{4} = \binom{9}{5} + \cdots + \binom{9}{9}$$

As each sequence of heads and tails is equally likely and we note that there and the same number of sequences with at least 5 and strictly less than 5 heads in the 9 flips, we see that our answer is $\dfrac{1}{2}$.

Solution to Question 577: 18 Sides

Suppose there are n sides labeled 6, and $18 - n$ sides labeled 3. Then, the expected value of a roll can be written as $\frac{6n}{18} + \frac{3(18-n)}{18} = 4$. Solving for n, we find

$$\begin{aligned} \frac{6n}{18} + \frac{3(18-n)}{18} &= 4 \\ \frac{n}{6} + 3 &= 4 \\ n &= 6 \end{aligned}$$

Solution to Question 578: Defining Regression

The covariance between the X and Y is zero, and thus a is 0. The y-intercept of the regression is thus $E[Y]$, so b is 8. In conclusion, $a + b = 8$.

Solution to Question 579: Binomial Contract Pricing I

First, we need to figure out the risk-neutral probabilities. We solve the following equation to determine these probabilities.

$$2q = (1-q) * 0.5 = 1$$

This gives us $q = 1/3$. We can then find the final potential payoffs, we can either double twice, half twice, or double, then half. This gives us final underlying values of either $20, 5, 1.25$. We can square these to get the contract values. We can then use the risk-neutral probabilities to back-propagate to determine the initial value.

For example, for the first up-move, when the underlying has value $S_1 = 10$, we either will have $S_2 = 20$ or $S_2 = 5$. We have these contract values and can then calculate the expectation of the contract for this time and value. This gives us:

$$E_1 = \frac{1}{3}(400) + \frac{2}{3}(25) = 150$$

This can be repeated for the other values to get $E_0 = 56.25$.

Solution to Question 580: ATM Implied Volatility

We can use the following approximation to calculate the price of a vanilla European option.

$$V = \sqrt{\frac{T-t}{2\pi}}\sigma S$$

To calculate the implied volatility, we can plug in the values above. $T - t = 3/12$, $S = 12$, and $V = 2.3$. Solving for σ, we obtain $\sigma = .961$.

Solution to Question 581: Perfect Correlation II

Suppose that X and Y have respective variances σ_X^2 and σ_Y^2. Then $\text{Cov}(X+Y, X-Y) = \text{Cov}(X,X) - \text{Cov}(Y,Y) = \sigma_X^2 - \sigma_Y^2$. Similarly, we know that $\text{Var}(X+Y) = \sigma_X^2 + \sigma_Y^2 + 2\sigma_X\sigma_Y = (\sigma_X + \sigma_Y)^2$, so $\sigma_{X+Y} = \sigma_X + \sigma_Y$. By a similar argument, we get $\text{Var}(X-Y) = \sigma_X^2 + \sigma_Y^2 - 2\sigma_X\sigma_Y = (\sigma_X - \sigma_Y)^2$, so $\text{Var}(X-Y) = |\sigma_X - \sigma_Y|$. Note that the absolute values are needed here since we can't have a negative variance.

Therefore, we have that $\rho(X+Y, X-Y) = \dfrac{\sigma_X^2 - \sigma_Y^2}{(\sigma_X + \sigma_Y)|\sigma_X - \sigma_Y|} = \dfrac{\sigma_X - \sigma_Y}{|\sigma_X - \sigma_Y|} = 1$. We get the 1 from the condition that $\sigma_X > \sigma_Y$, so the numerator is positive.

Another way to see this is that if X and Y are perfectly correlated, then $Y = aX + b$ for some constants a and b. Since $\sigma_X > \sigma_Y$, it must be the case that $|a| < 1$. Therefore, $X + Y = X(1+a) + b$ and $X - Y = X(1-a) + b$. As $|a| < 1$, both of the constants $1 + a$ and $1 - a$ are positive. Therefore, as these are both linear transformation of X with the same signs, they must have correlation 1.

Solution to Question 582: Bus Wait I

As you show up at a uniformly random time throughout the day, the time until the next bus is just uniformly distributed between 0 and 10 minutes. Therefore, the answer is just 5, as that is the mean of this random variable.

Solution to Question 583: 100 Factorial Digits

First, we can see that $\log_{10}(100) = 2$, so to get the number of digits, we just have to add 1. Thus, we want to find $\log_{10}(100!) = \log(1) + \log(2) + \cdots + \log(100)$ and then add 1. We can approximate this as the integral of $\log_{10}(x)$ on $[1, 100]$.

Integrating $\int_1^{100} \log(x)\, dx \approx 157$. Adding 1, we get 158.

Solution to Question 584: Missing Product

Note that since a, b, c, and d don't necessarily have to be integers, prime factorization matching may not solve the question entirely. One thing to quickly notice is that the integers 9 and 10, as well as 27 and 30 are in our set. Therefore, this implies two of the values, say a and b, satisfy, $a = \frac{9}{10}b$. One may now take the guess that $a = 9$ and $b = 10$. However, this doesn't work, as one of the other integers would have to be $c = 1$, and there would be too many missing values remaining. The next natural guess would be $a = \frac{9}{2}$ and $b = 5$, as you can obtain the values 9 and 10 from setting another integer $c = 2$. Running with this, we see that $ac = 9$ and $bc = 10$. To find d, the values left that we haven't matched yet are $12, 27$, and 30. However, we see that $12 = 2 \cdot 6, 27 = \frac{9}{2} \cdot 6$, and $30 = 5 \cdot 6$, which implies $d = 6$. The last value should be $ab = \frac{9}{2} \cdot 5 = \frac{45}{2}$

Solution to Question 585: Modified RNG

Setting this up as a conditional distributions question, let X be Jimmy's number. Then $X \sim \text{Unif}(0, 1)$ and $Y \mid X = x \sim \text{Unif}(x, 1)$. We want to find $\mathbb{E}[Y]$ and $\text{Var}(Y)$. We start with $\mathbb{E}[Y]$ first. Since we know the conditional distribution of Y, we should apply Law of Total Expectation to get $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y \mid X]]$. $\mathbb{E}[Y \mid X] = \frac{X+1}{2}$ as $Y \mid X \sim \text{Unif}(X, 1)$. Thus, $\mathbb{E}[Y] = \frac{1}{2}(\mathbb{E}[X] + 1) = \frac{3}{4}$ since $\mathbb{E}[X] = \frac{1}{2}$.

Now, for $\text{Var}(Y)$, we want to use Law of Total Variance, so $\text{Var}(Y) = \text{Var}(\mathbb{E}[Y \mid X]) + \mathbb{E}[\text{Var}(Y \mid X)]$. We have $\mathbb{E}[Y \mid X] = \frac{X+1}{2}$ from the previous part. Furthermore, we have that $\text{Var}(Y \mid X) = \frac{(1-X)^2}{12}$ by the known properties of uniform random variables. Therefore, $\text{Var}(Y) = \text{Var}\left(\frac{1+X}{2}\right) + \frac{1}{12}\mathbb{E}\left[(1-X)^2\right]$.

Let's start with $\text{Var}\left(\frac{1+X}{2}\right)$. We know constant shifts don't change the variance, so this can be reduced to $\text{Var}\left(\frac{X}{2}\right)$. Constants inside variance need to be squared to move outside, so this becomes $\frac{1}{4}\text{Var}(X)$. It is known $\text{Var}(X) = \frac{1}{12}$, so this first term becomes $\frac{1}{48}$.

For the second term, $\mathbb{E}[(1-X)^2] = \int_0^1 (1-x)^2 dx = \int_0^1 x^2 dx = \frac{1}{3}$ by making the $u-$substitution $u = 1 - x$. Therefore, the second term is $\frac{1}{36}$. Added together, $\text{Var}(Y) = \frac{7}{144}$. Dividing $\mathbb{E}[Y]$ by $\text{Var}(Y)$, we get that the ratio is $\frac{108}{7}$.

Solution to Question 586: Hatching Eggs II

Let X denote the number of eggs laid. $X \sim \text{Pois}(6)$. Let Y denote the number of egg hatches. Note that the value of Y depends on the number of eggs hatched, X. For example, if $X = x$, then, letting $E_1, E_2, \ldots, E_n \overset{\text{iid}}{\sim} \text{Bernoulli}(0.3)$ be indicator variables for each egg, we have:

$$Y = \sum_{i=1}^{x} E_i.$$

We are interested in $\text{Var}[Y]$. We can utilize the law of total variance.

$$\text{Var}[Y] = \mathbb{E}[\text{Var}[Y|X]] + \text{Var}[\mathbb{E}[Y|X]]$$

Note that

$$\begin{aligned} \text{Var}[Y|X] &= 0.3X \\ \mathbb{E}[Y|X] &= X(0.3)(0.7) \\ &= 0.21X \end{aligned}$$

Plugging these values in, we find

$$\begin{aligned} \text{Var}[Y] &= \mathbb{E}[\text{Var}[Y|X]] + \text{Var}[\mathbb{E}[Y|X]] \\ &= 0.3^2\text{Var}[X] + 0.21\mathbb{E}[X] \end{aligned}$$

Recall that $\text{Var}[X] = \lambda = 6$ and $\mathbb{E}[X] = \lambda = 6$. Our final answer is

$$\begin{aligned} \text{Var}[Y] &= 0.3^2 \cdot 6 + 0.21 \cdot 6 \\ &= 1.8 \end{aligned}$$

Solution to Question 587: Egg Drop I

Suppose that we have a strategy with a maximum number of N drops. For the first drop of the first egg, we can try the N-th floor. If the egg breaks, we can start to try the second egg from the first floor and increase the floor number by one until the second egg breaks. At most, there are $N-1$ floors to tests. If the first egg dropped from the N-th floor does not break, then we have $N-1$ drops left. This time we can only increase the floor number by $N-1$ for the first egg since the second egg can only cover $N-2$ floors if the first egg breaks. If the egg dropped from the $2N-1$-th floor does not break, then we have $N-2$ drops left. Therefore, we can only increase the floor number by $N-2$ for the first egg since the second egg can only cover $N-3$ floors if the first egg breaks. Using such logic, we can see that the number of floors that these two eggs can cover with a maximum of N drops is

$$N + (N-1) + \ldots 1 \geq 100 \frac{N(N+1)}{2} \geq 100 N = 14$$

Solution to Question 588: Casino Combo

The die has the most restrictive values (only 1-6), so roll the die and let it land on one of the sides arbitrarily. The probability that we choose a card of that rank from the deck is $\frac{1}{13}$. The probability the roulette wheel lands on that value is $\frac{1}{38}$. Therefore, the probability, by independence, is $\frac{1}{13 \cdot 38} = \frac{1}{494}$.

Solution to Question 589: Price an Option II

To find the time-0 price, we need to create a replicating portfolio. We can see that the payoff $\max(3S_T, 15)$ is the same as $15 + 3\max(S_T - 5, 0)$. This replicating portfolio is the same as being long 3 call-options with strike $K = 5$, but now shifted vertically by 15. Constants can be treated as n units of a bond, paying 1 at maturity. So, to replicate the final payoff, we have 15 bonds and 3 call options.

So, we have time-0 price of $V_0 = 15(0.9) + 3(4.2) = 26.1$. Note: we need to discount the bonds back to the present value of 0.9.

Solution to Question 590: Fair Load

Emma can obtain an odd sum in two ways: The fair die is even and the loaded is odd OR The fair die is odd and the loaded is even.

Case 1: The fair die is even with probability $\frac{1}{2}$. The probability that the loaded die is odd is $\frac{1+3+5}{1+2+3+4+5+6} = \frac{3}{7}$

Case 2: The fair die is odd with probability $\frac{1}{2}$. The probability that the loaded die is even is $\frac{2+4+6}{21} = \frac{4}{7}$

Putting it together, we have that the probability of an odd sum is $\frac{1}{2} \cdot \frac{3}{7} + \frac{1}{2} \cdot \frac{4}{7} = \frac{1}{2}$

Solution to Question 591: Exercise Vega

Vega is proportional to the strike price when assuming Black-Scholes dynamics. If you can exercise at a larger strike, then the option is more valuable and all things else, should be more sensitive to volatility changes.

Solution to Question 592: OLS Review III

It can be shown that $s^2 = \frac{\text{SSE}}{n-2}$ is an unbiased estimator for σ^2 in the case where there are 2 β parameters. First, let's determine the least-squares estimators for the β parameters. We know that

$$\begin{aligned}\hat{\beta}_1 &= \frac{S_{xy}}{S_{xx}},\\ \hat{\beta}_0 &= \bar{y} - \hat{\beta}_1\bar{x}.\end{aligned}$$

For our data, we easily find $S_{xy} = \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = 7$, and $S_{xx} = \sum_{i=1}^{n}(x_i - \bar{x})^2 = 10$. Hence $\hat{\beta}_1 = \frac{7}{10}$, and $\hat{\beta}_0 = 1 - 0 = 1$. We can now compute the sum of squared estimate of errors (SSE) with the following formula:

$$\begin{aligned}\text{SSE} &= S_{yy} - \hat{\beta}_1 S_{xy}\\ &= 6 - \frac{7}{10}\cdot 7 = \frac{11}{10}\\ \Rightarrow \frac{\text{SSE}}{n-2} &= \frac{11/10}{5-2} = \frac{11}{30}\end{aligned}$$

Solution to Question 593: Friendly Competition

Let p be the probability player 1 wins. If the first flip is tails, then player 1 loses automatically. If we go HT, then we are back where we started and player 1 has probability p of winning. This occurs with probability $\frac{2}{3}\cdot\frac{1}{3} = \frac{2}{9}$ by independence of the coins. If we go HH, then player 1 wins. This occurs with probability $\frac{2}{3}\cdot\frac{2}{3} = \frac{4}{9}$ Therefore, by Law of Total Probability, $p = \frac{2}{9}p + \frac{4}{9}$. Solving this for p yields $p = \frac{4}{7}$.

Solution to Question 594: Weird Die

You can obtain a sum of 7 by rolling a permutation of $(1, 6), (2, 5)$, or $(3, 4)$. The probability of a permutation of $(1, 6)$ is $2 \cdot 1/6 \cdot 1/6 = 1/18$. The probability of a permutation of $(2, 5)$ is $2 \cdot 1/6 \cdot (1/5 - x)$. The probability of a permutation of $(3, 4)$ is $(1/10 + 2x) \cdot (1/5 - x)$. Adding all of these and expanding, if $p(x)$

represents the probability of obtaining a sum of 7 in 2 rolls with x as the input for our die,

$$p(x) = -4x^2 + \frac{4}{15}x + \frac{73}{450}$$

We can use the formula $x^* = -\frac{b}{2a}$ to find the $x-$coordinate of the maximum. In particular, $x^* = -\frac{4/15}{-8} = \frac{1}{30}$.

Solution to Question 595: Digit Difference

We are allowed to select d_9 from $\{8, 9\}$, d_0 from $\{0, 1\}$, and d_i from $\{i-1, i, i+1\}$ for all $i \in \{1, 2, \ldots, 8\}$. The average we wish to compute is the expected value of $\sum_{i=0}^{9} d_i \cdot 10^i$, where each digit d_i is selected independently per a uniform distribution over its set of possible values. $\mathbb{E}[d_i] = i$ for all $1 \le i \le 8$, as it is uniform over $\{i-1, i, i+1\}$. For $i = 0$ and 9, the means are 0.5 and 8.5, respectively. By linearity of expectation,

$$\mathbb{E}\left[\sum_{i=0}^{9} d_i \cdot 10^i\right] = \sum_{i=0}^{9} \mathbb{E}[d_i]\cdot 10^i = 8.5\cdot 10^9 + \sum_{i=1}^{8} i\cdot 10^i + 0.5\cdot 10^0 = 9376543210.5 = \frac{18753086421}{2}.$$

Solution to Question 596: Safe Cracking

Let us look at the first two constraints while thinking about the possibilities of each of the three digits. The units digit must be an even number that is not six, and thus has four possibilities. The tens and hundreds place both have nine possibilities. Now onto the third constraint to determine the total number of codes that satisfy all three. In order for one of the digits to appear, there are four cases to consider: first two digits are the same and the third is different; the first and third digits are the same and the second is different, the second and third digits are the same and the first is different; all three digits are the same. We will consider each one separately.

Case 1: The first two digits are either the same odd number or the same even number. If they are both odd, then the third digit can be any of its four choices. If they are both even, then the third digit can be any of the three choices such that it is a different even number. The number of total possibilities is $5 \times 4 + 4 \times 3 = 32$.

Case 2: The first and third digits can be any one of four choices, limited by the options that the third digit has. The second digit can be any one of the eight choices such that it is a different number. The number of total possibilities is $4 \times 8 = 32$.

Case 3: Note that this case is symmetrical to the previous case. The second and third digits can be any one of four choices, limited by the options that the third digit has. The first digit can be any one of the eight choices such that it is a different number. The number of total possibilities is $4 \times 8 = 32$.

Case 4: The three digits can be any one of four choices, limited by the options that the third digit has. The number of total possibilities is 4.

In conclusion, the total number of three digit entries satisfy these three requirements is $32 + 32 + 32 + 4 = 100$.

Solution to Question 597: Relatively Prime Coins

If X heads are flipped out of a total of n coin tosses, then $Y = n - X$ tails are flipped. Let's substitute this expression in for Y, simplify, and apply the linearity of expectation.

$$\begin{aligned}\mathbb{E}[XY] &= \mathbb{E}[X(n-X)] \\ &= n \cdot \mathbb{E}[X] - \mathbb{E}[X^2]\end{aligned}$$

Note that $X \sim \text{Binom}(n, 0.5)$. Recall that for some $Z \sim \text{Binom}(n, p)$, $\mathbb{E}[Z] = np$ and $\text{Var}(Z) = npq$, where $q = 1 - p$. So,

$$\begin{aligned}\mathbb{E}[X] &= \frac{n}{2} \\ \text{Var}[X] &= \frac{n}{4}\end{aligned}$$

We can compute $\mathbb{E}[X^2]$ as follows:

$$\begin{aligned}\text{Var}[X] &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= \frac{n}{4} \\ \Rightarrow \mathbb{E}[X^2] &= \frac{n}{4} + \left(\frac{n}{2}\right)^2\end{aligned}$$

Returning to our original expression of interest,

$$\begin{aligned}\mathbb{E}[XY] &= \mathbb{E}[X(n-X)] \\ &= n \cdot \mathbb{E}[X] - \mathbb{E}[X^2] \\ &= \frac{n^2}{2} - \frac{n^2}{4} - \frac{n}{4} \\ &= \frac{n^2 - n}{4}.\end{aligned}$$

Our solution is therefore $1 - 1 + 4 = 4$.

Solution to Question 598: Oil Profits I

In the event that there is oil (which occurs with probability p), the company makes $\$1,050,000$ in profit. If there is no oil, which occurs with probability $1 - p$, the company loses $\$350,000$. Therefore, we need to find the smallest p such that

$$1050000p - 350000(1 - p) \geq 0$$

The smallest such p can be solved to be $1/4$.

Solution to Question 599: Double Data Trouble III

From Double Data Trouble II, we know that the variance of our estimate is reduced by $\frac{1}{2}$, so the standard error is reduced by $\frac{1}{\sqrt{2}}$. As $t_0 = \frac{\hat{\beta}_1}{s\left\{\hat{\beta}_1\right\}}$, we know that the estimate $\hat{\beta}_1'$ is the same from Double Data Trouble I, and that the standard error in the denominator is $\frac{1}{\sqrt{2}}$ as large for $\hat{\beta}_1'$, we get that $t_1 = \sqrt{2}t_0$, meaning that $t_1^2 = 2t_0^2$.

Solution to Question 600: Asset Dynamics

When we have an asset following dynamics $dX_t = a\ dt + b\ dW_t$, where $a, b \in \mathbb{R}$, we have the following properties.

$$\mu_T = X_0 + aT$$
$$\sigma_T^2 = b^2 T$$

We can also obtain these results intuitively without using the fact that a standard Brownian Motion follows $W_t \sim \mathcal{N}(0, t)$. The Brownian motion component is the randomness contributing part, with mean 0. Thus, we can expect our time T value to increase in direction by $a\ dt$ with an initial position of X_0. We can then use properties of a normal distribution to obtain the variance of our time T position (as the dW_t term contributes solely to the randomness of our time T position. We can see that $b\,\mathcal{N}(0, t) = \mathcal{N}(0, b^2 t)$.

Solution to Question 601: Precise N Sum

For $X_1 + X_2 + X_3 = N$ to occur, we would need $X_1 + X_2 \leq N$ and $X_3 = N - X_1 - X_2$ to occur simultaneously. This is because all of the values we select are non-negative. Let $A_N = \{X_1 + X_2 \leq N\}$ and $B_N = \{X_3 = N - X_1 - X_2\}$. Then we want

$$\mathbb{P}[A_N \cap B_N] = \mathbb{P}[B_N \mid A_N]\mathbb{P}[A_N]$$

$\mathbb{P}[B_N \mid A_N]$ asks for the probability $X_3 = N - X_1 - X_2$ given $X_1 + X_2 \leq N$. This probability is just $\dfrac{1}{N+1}$ since X_3 is discrete uniform over $\{0, 1, \ldots, N\}$ and the requested value is in the support since we know $0 \leq X_1 + X_2 \leq N$, meaning $N - X_1 - X_2$ is too.

For $\mathbb{P}[A_N]$, the trick here is to treat X_1 and X_2 as continuous uniform on $[0, N]$. At the end of the day, we only need an approximation, although this probability can be found explicitly. As X_1 and X_2 are both continuous and symmetric about their means, which are both $\dfrac{N}{2}$, $X_1 + X_2$ will be symmetric about its mean, which is N. Therefore, the probability in the continuous case is $\dfrac{1}{2}$, meaning the probability converges to that in the discrete case. Therefore, $p_N \approx \dfrac{1}{2(N+1)}$. Therefore, using the known rules of limits of rational functions, $Np_N = \dfrac{1}{2} \cdot \dfrac{N}{N+1} \to \dfrac{1}{2}$.

Solution to Question 602: Maximum Volatility

We can approach this problem in one of two ways, either intuitively or through the Black-Scholes equation.

First we'll look through the lens of the Black-Scholes equation. The Black-Scholes equation for the price of a European call option: $\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC = 0$

In these equations:

- C represents the price of the European call option.

- S is the current stock price.

- t is the time to expiration.

- r is the risk-free interest rate.

- σ is the volatility of the underlying stock.

Using this information, and plugging in an infinity for σ, we can simplify our equation to $\infty - rC = 0$, meaning the price of our option must tend to infinity as well.

Thinking about this more intuitively, under normal conditions, the greater the volatility (ν), the more the seller demands in order to protect against big moves against his position. When this volatility is tending to infinity, this means the seller needs near infinite protection in order to justify selling the call.

Solution to Question 603: Option Dice II

First, we need to see how many combinations of dice rolls land us in profit. Listed in increasing order they are $(1,1),(1,2),(2,1),(1,3),(3,1)$. With these combinations, a roll of $(1,1)$ yields a profit of 3 units $(4-1=3)$ with a $\frac{1}{36}$ chance in doing so, a roll of $(1,2)$ or $(2,1)$ yields a profit of 2 units $(4-2=2)$ with a $\frac{2}{36}$ chance in doing so, and a roll of $(1,3)$ or $(3,1)$ yields a profit of 1 units $(4-3=1)$ with a $\frac{2}{36}$ chance in doing so.

Putting this all together we have $\frac{1}{36}\cdot 3+\frac{2}{36}\cdot 2+\frac{2}{36}\cdot 1=\frac{9}{36}$, showing our contract should be priced at $1/4$.

Solution to Question 604: 1900 Age

Let x be the age at which he died. Then $29x$ was the year he was born. The year he died then was thus $29x+x=30x$. Since the man was alive in the year 1900, the death must have been after 1900. Since the year he died must have been divisible by 30, the next multiple of 30 after 1900 is 1920. This would imply $x=64$, which is consistent with the information we have. The year he would have been born in was $29\cdot 64=1856$, and he lived 64 years to pass away in 1920. Therefore, in 1900, the man turned 44 years old.

Solution to Question 605: Uniformly Correlated

Note that we can write $U = 1 - V$. This will help us simplify some of the calculations. Thus,

$$\text{Cov}(U, V) = \text{Cov}(1 - V, V) = -\text{Var}(V) = -\text{Var}(X^2 + Y^2)$$

We need to calculate this variance, which requires some algebra on the definition of variance. We have that $\text{Var}(X^2 + Y^2) = \mathbb{E}[(X^2 + Y^2)^2] - (\mathbb{E}[X^2 + Y^2])^2$. We have that $\mathbb{E}[X^2 + Y^2] = \mathbb{E}[X^2] + \mathbb{E}[Y^2]$. We have that $\mathbb{E}[X^2] = \text{Var}(X) + (\mathbb{E}[X])^2 = \frac{1}{3}$ from the formulas for the variance and expectation of a Unif$(0, 1)$. The same holds for $\mathbb{E}[Y^2]$. The harder term is the first one. We have that

$$\mathbb{E}[(X^2 + Y^2)^2] = \mathbb{E}[X^4 + Y^4 + 2X^2Y^2] = \mathbb{E}[X^4] + \mathbb{E}[Y^4] + 2\mathbb{E}[X^2]\mathbb{E}[Y^2]$$

A fairly easy to show fact is that for a Unif$(0, 1)$, the kth moment is $\frac{1}{k+1}$. You can see this by noting that if $R \sim \text{Unif}(0, 1)$, $\mathbb{E}[R^k] = \int_0^1 r^k dr = \frac{1}{k+1}$ by LOTUS. Thus, each of the first two terms in this are $\frac{1}{5}$. Thus, we have that $\text{Var}(X^2 + Y^2) = \left(\frac{1}{5} + \frac{1}{5} + 2\left(\frac{1}{3}\right)^2\right) - \left(\frac{1}{3} + \frac{1}{3}\right)^2 = \frac{28}{45} - \frac{20}{45} = \frac{8}{45}$. Thus, $\text{Cov}(U, V) = -\frac{8}{45}$

Solution to Question 606: Random Bivariate

We know that $Y = M + S(\rho Z_1 + \sqrt{1 - \rho^2} Z_2)$, where Z_1 and Z_2 are IID $N(0, 1)$. This is the definition of the transform that yields a bivariate normal pair. We have then that $\text{Var}(Y) = \text{Var}(M + S(\rho Z_1 + \sqrt{1 - \rho^2} Z_2) = \text{Var}(M) + \text{Var}(S(\rho Z_1 + \sqrt{1 - \rho^2} Z_2))$. This is because the two terms are independent.

$\text{Var}(M) = 1$ by known formulas of the Exp(1) distribution. By the Law of Total Variance,

$$\text{Var}(S(\rho Z_1 + \sqrt{1 - \rho^2} Z_2)) = \mathbb{E}[\text{Var}(S(\rho Z_1 + \sqrt{1 - \rho^2} Z_2) \mid S)] + \text{Var}(\mathbb{E}[S(\rho Z_1 + \sqrt{1 - \rho^2} Z_2) \mid S])$$

The second term is just 0 because $\mathbb{E}[S(\rho Z_1 + \sqrt{1 - \rho^2} Z_2) \mid S] = S\mathbb{E}[\rho Z_1 + \sqrt{1 - \rho^2} Z_2] = 0$ by the fact $Z_1, Z_2 \sim N(0, 1)$. The first term is $\mathbb{E}[S^2]$, as

$$\text{Var}(S(\rho Z_1 + \sqrt{1 - \rho^2} Z_2) \mid S) = S^2(\text{Var}(\rho Z_1) + \text{Var}(\sqrt{1 - \rho^2} Z_2)) = S^2$$

Thus, $\mathbb{E}[S^2] = 2$ by known formulas, so our answer is $\text{Var}(Y) = 3$.

Solution to Question 607: Rebuy Gambler

The easier quantity to compute is the probability that Alice loses the game, which is the complement here. For Alice to lose the game, she first needs to go bankrupt in the initially, and then bankrupt a second time. Using the Gambler's Ruin paradigm for equal-odds rounds, the probability she initially goes bankrupt is $\frac{10}{25} = \frac{2}{5}$. Afterwards, Bob would have \$25 and Alice would have \$15 upon rebuying. The probability Alice goes bankrupt here is $\frac{25}{40} = \frac{5}{8}$. Combining these, the probability Alice loses is

$$\frac{2}{5} \cdot \frac{5}{8} = \frac{1}{4}$$

Therefore, Bob loses with probability $\frac{3}{4}$. This makes sense, as this is the same result as if Alice had \$30 bankroll initially and Bob had \$10 initially. Alice essentially does have \$30, as she can rebuy for \$15 midway.

Solution to Question 608: Find the Triangle

Let's look at triangles that have consecutive sides. For example, $3, 4, 5$. Afterwards, we can calculate the height of the triangle. We can do this with Heron's formula.

$$\begin{aligned} A &= 0.25 * \sqrt{(a+b+c)} \\ \frac{1}{2}bh &= 0.25 * \sqrt{(a+b+c)} \\ h &= \frac{0.5}{b}\sqrt{(a+b+c)} \end{aligned}$$

Guessing and checking for various consecutive sides, we obtain a triangle with sides $13, 14, 15$, making 14 the base, the height 12, and the area 84.

Solution to Question 609: Rectangles on Chess Board

Think about what rectangles represent. Rectangles represent two pairs of start and end points (up/down and left/right). When we analyze the problem in this aspect, the problem boils down to choosing a start and end edge out of the lateral and horizontal edge. There are 9 lateral edges to choose from for the lateral start and ends of the rectangle and 9 horizontal edges to choose from for the horizontal start and ends of the rectangle. Thus the problem becomes

$$\binom{9}{2}^2 = 36^2 = 1296$$

Solution to Question 610: Lowest Target

There are 12 total targets and 4 columns. Label the columns $1-4$. Then we are just counting the number of anagrams that can be made with 2 1s, 3 2s, 3 3s, and 4 4s. This is because we can identify a unique breaking order by the column we shoot in at each step. The number of anagrams is just given by the multinomial coefficient $\binom{12}{2,3,3,4} = 277200$

Solution to Question 611: Squares on a Chess Board

Lets start by considering the largest squares we can make. The largest is an 8×8 square and there's only one of these on the board. How about 7×7 squares? Well, by decreasing the dimensions of the squares by one, we allowed for one more movement horizontally and vertically of the square so we have 4 7×7 squares. If we continue this, you will see the pattern that there are $(9-n)^2$ $n \times n$ squares on the board. Thus the answer boils down to $\displaystyle\sum_{n=1}^{8} n^2 = 204$.

Solution to Question 612: Peeled Dice

The value on the second roll is going to depend on the value peeled off from the first roll. Therefore, we condition on the value peeled off on the first roll. You roll each value on the die with equal probability, so if T denotes the total of the two rolls, $\mathbb{E}[T] = \mathbb{E}[\mathbb{E}[T \mid X_1]] = \frac{1}{6}\sum_{i=1}^{6} \mathbb{E}[T \mid X_1 = i]$.

$\mathbb{E}[T \mid X_1 = i]$ is now what we need to compute. We know that i is already contributed to our total from the first roll. Then, i is peeled off the die, so that now the remaining values sum to $21-i$ and there are 5 equally-likely values. Therefore, the expected value of the second roll given the first is i is $\dfrac{21-i}{5}$ Thus, $\mathbb{E}[T \mid X_1 = i] = \dfrac{21-i}{5} + i = \dfrac{21+4i}{5}$.

Plugging this in,

$$\begin{aligned}\mathbb{E}[T] &= \frac{1}{6}\sum_{i=1}^{6}\frac{21+4i}{5}\\ &= \frac{7}{10}\sum_{i=1}^{6}1 + \frac{2}{15}\sum_{i=1}^{6} i\\ &= \frac{42}{10} + \frac{2}{15}\cdot\frac{6(7)}{2} = \frac{21}{5} + \frac{14}{5} = 7\end{aligned}$$

Another way to think about this is that the expected value that you remove from the first roll is 3.5, so the expected sum of remaining values is 17.5. As there are 5 equally-likely values on the die, the expected value of the second roll is still 3.5, so adding these two up gives 7 as well.

Solution to Question 613: Ranged Stars and Bars

We are going to solve this problem more generally. Namely, we want to find the number of non-negative integer solutions to

$$n+1 \le x_1 + \cdots + x_n \le 2n$$

We know by stars and bars that for a fixed k, the number of non-negative integer solutions to $x_1 + \cdots + x_n = k$ is $\binom{n+k-1}{n-1}$. Therefore, the number of non-negative integer solutions to $x_1 + \cdots + x_n \le 2n$ is just the sum of the number of non-negative integer solutions to $x_1 + \cdots + x_n = k$ for $0 \le k \le 2n$, so this yields

$$\binom{n-1}{n-1} + \cdots + \binom{2n-1}{n-1} + \binom{2n}{n-1} + \cdots + \binom{3n-1}{n-1}$$

Using the hockey stick identity $\sum_{i=k-1}^{r-1}\binom{i}{k-1} = \binom{r}{k}$, $1 \le k < r$, with $k = n$ and $r = 3n$, we get that the above is just $\binom{3n}{n}$. We do similar to find that the number of non-negative integer solutions to $x_1 + \cdots + x_n \le n$ is

$$\binom{n-1}{n-1} + \cdots + \binom{2n-1}{n-1} = \binom{2n}{n}$$

Therefore, the number of non-negative integer solutions to $n+1 \le x_1+\cdots+x_n \le 2n$ is just the difference of the above values, as $x_1 + \cdots + x_n \le 2n$ includes solutions that are also $\le n$. Therefore, the number solutions to this is

$$\binom{2n}{n-1} + \cdots + \binom{3n-1}{n-1} = \binom{3n}{n} - \binom{2n}{n}$$

Plugging in $n = 5$ yields that our answer is $\binom{15}{5} - \binom{10}{5} = 2751$

Solution to Question 614: Car Crash

We can solve this question with linearity of expectation. Let us define an indicator variable Z_i as follows:

$$Z_i = \begin{cases} 1 & \text{if there are no slower cars in front of } X_i \\ 0 & \text{otherwise} \end{cases}$$

Then, the total number of clusters K is simply $K = \sum_{i=1}^{N} Z_i$. By linearity of expectation, it follows that $\mathbb{E}[K] = \sum_{i=1}^{N} \mathbb{E}[Z_i]$.

Consider Z_N, the indicator variable corresponding to the car at the front of the pack. No matter what, $Z_N = 1$ since there are no other cars in front of it. Next, let's consider Z_i. In this case, there are $N - i$ cars in front of car X_i. Including car X_i, there are $N - i + 1$ total cars. The probability that X_i is the slowest car among the total $N - i + 1$ cars is simply $\frac{1}{N-i+1}$. Hence, $\mathbb{E}[Z_i] = \frac{1}{N-i+1}$. Plugging this in, we find

$$\begin{aligned} \mathbb{E}[K] &= \sum_{i=1}^{N} \mathbb{E}[Z_i] \\ &= \sum_{i=1}^{N} \frac{1}{N-i+1} \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} \end{aligned}$$

When $N = 10$, we compute that

$$\begin{aligned} \mathbb{E}[K] &= \sum_{i=1}^{10} \frac{1}{i} \\ &= \frac{7381}{2520} \end{aligned}$$

Solution to Question 615: Inheritance Split

Let x be the amount Abe receives. Then $100 - x$ is what Ben receives. The line there says that $\dfrac{100 - x}{4} - \dfrac{x}{3} = 11$. Solving for x here algebraically yields $x = 24$.

Solution to Question 616: Two Puts

We want to find the best super-replication. Since $35 > 33$, the $K = 35$ put should always be greater than the $K = 33$ put. So, a reasonable price estimate is 4.5. However, we can do better. In fact, we can divide the strikes by each other and find a more accurate super-replication.

In fact, we can go long K_2/K_1 units of the $K = 35$ put. This will give a closer super-replication and an estimate. We have: $\frac{33}{35}(4.5) = 4.24$. Note, this super-replication does not hold when $S_T < 0$. However, a stock price cannot go negative (though futures can!) and we do not need to worry about this.

Solution to Question 617: 9 Sum II

We can verify quickly that 100000 does not have digits that sum to 9. Therefore, we are looking at numbers at most 5 digits. The trick to put everything on the same footing is that we append 0s to the left of numbers less than 5 digits until they reach 5 digits. For example, 81 becomes 00081 and 144 becomes 00144. Therefore, our integer is in the form $x_1x_2x_3x_4x_5$, where each $0 \leq x_i \leq 9$, as we can have less than $5-$digit integers. The number of integers whose digits sum to 9 is just the number of non-negative integer solutions to $x_1+x_2+x_3+x_4+x_5 = 9$. This is a classic stars-and-bars problem and evaluates to $\binom{9+5-1}{5-1} = \binom{13}{4} = 715$.

Solution to Question 618: Paired Values II

We don't prove this here, but each distinct pairing of the 6 integers creates a unique sum. There are 6! ways to permute the digits to the 6 spots. However, the first two, second two, and last two can be exchanged within each block to give the same sum, as order of multiplication doesn't matter, so we divide by $2^3 = 8$. As well, we can change around the order of the blocks and get the same sum, so there are 3! ways to arrange the blocks. Thus, the answer is $\frac{6!}{3!2^3} = 15$.

Solution to Question 619: Variance of Uniform

We can quickly see that the mean of X is $\frac{1}{2}$, as X is uniformly distributed over the interval $(0, 1)$, so the mean is right in the center of the interval due to the uniform distribution of probability density. We use LOTUS to compute $\mathbb{E}[X^2]$. Namely, the PDF of X is $f(x) = I_{(0,1)}(x)$, so

$$\mathbb{E}[X^2] = \int_{\mathbb{R}} x^2 I_{(0,1)}(x)dx = \int_0^1 x^2 dx = \frac{1}{3}$$

Therefore, using the formula $\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$

Solution to Question 620: Colorful Socks II

Let the first sock be arbitrarily drawn. We can do this since there are the same amount of socks of each color in the drawer. To obtain 2 socks of matching color, we either need the first and second socks to match or we need the third sock to match one of the first two differing color socks.

For the first case, the probability that we obtain a matching color sock on the second draw is $\frac{1}{19}$, as we drew out 1 sock and only 1 of the remaining 19 is of that same color. After that, we are out of socks of that color, so it is irrelevant what we pick next. Alternatively, suppose the first and second socks differ in color. This occurs with probability $\frac{18}{19}$ by complementation. To get a pair, the third sock must match one of the first 2 colors of socks drawn. This occurs with probability $\frac{2}{18}$, as there are 2 socks of the same color as the first two pairs drawn out of 18 total. Therefore, the probability of this case is $\frac{2}{19}$. Adding the two cases together, our probability is $\frac{3}{19}$.

Solution to Question 621: Sticky Delta

A sticky delta model means that the implied volatility is a function purely of the deltas. In other words, we need to find the Δ of the $K = 6$ option and plug it into the function to find the implied volatility. We know that Γ is the derivative of Δ and thus can use a first order approximation. Since we are dealing with $S_0 = 4$ and $K = 4$, the second order term does not matter as Γ is maximized (approximately) for at-the-money options. We also know that ATM options have 0.5 delta.

We can then find the new delta with $\Delta_1 = 0.5 - 0.03(2) = 0.44$. We subtract since the strike is increasing, and we are becoming further out-of-the-money. Out-of-the-money options have a $\Delta < 0.5$. We then plug this into the function: $f(0.44) = (.44 - .5)^2 + .3 = .31$

Solution to Question 622: Pizza Munching

Let T be the amount of time that Garrett takes to eat his slice of pizza. We know that $T \sim \text{Unif}(1, 5)$ from the question. We know he has spent at least 2 minutes eating his slice, and we want the probability he finishes in the next minute, so we want $\mathbb{P}[T \leq 3 \mid T \geq 2]$. By the definition of conditional probability, this is

$$\frac{\mathbb{P}[2 \leq T \leq 3]}{\mathbb{P}[T \geq 2]} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Solution to Question 623: Soccer Bets

The best way to tackle this question is to look at the outcomes and make equations for the net amount given each case. For example, if Team Quant wins, your net value is $2A - B - C$, as you receive $2A$ profit, and then lose the B you bet on Team Guide and the C you bet on a tie. Repeating this logic, you obtain the following equations:

$$\begin{aligned} \text{If Team Quant wins: } & 2A - B - C \\ \text{If Team Guide wins: } & 3B - A - C \\ \text{If tie: } & 10C - A - B \end{aligned}$$

Since we want our profit to be the same irrespective of the outcome, we need to make all these equations equal to each other. Solving these systems of equations, we get $A = 44$, $B = 33$, $C = 12$ as the smallest even bets you can make. To double check, when we plug these values into the three equations we made, we seem to profit 43 no matter the scenario. Thus, the answer is 43.

Solution to Question 624: Dollar Cent Switch II

Let y be the number of cents that he had and x be the number of dollars he had upon entry. Upon arrival, he had $x + 0.01y$ in his pocket. He spent half of his money and had x cents and $\frac{y}{2}$ dollars upon leaving. This means that $\frac{1}{2} \cdot (x + 0.01y) = \frac{y}{2} + 0.01x$. We can immediately note that y is even as he has exactly half as many cents.

There are two cases to consider here, which correspond to if x is even or odd. In the first case, we would get that $\frac{x}{2} = \frac{y}{2}$, meaning $x = y$. However, we would also get that $0.005y = 0.01x$, meaning $y = 2x$. The only possible case where these happen simultaneously is if $x = y = 0$, but his pocket is not empty, so this can't be correct. Therefore, x must be odd. In this case, we have $\frac{x-1}{2}$ dollars and $0.50 + 0.005y$ cents beforehand.

In this case, we would get that $\frac{x-1}{2} = \frac{y}{2}$, meaning $y = x - 1$. We would also get that $0.50 + 0.005y = 0.01x$. Substituting in $y = x - 1$, this yields that $0.495 = 0.005x$, meaning $x = 99$. Therefore $y = 99 - 1 = 98$, so he had 99.98 upon entering.

Solution to Question 625: Product Over Sum

Let X and Y be the outcomes of the two rolls. Then we want to find when $X + Y \geq XY$. Solving for Y, we see that $Y \leq \frac{X}{X-1}$. If $X = 1$, then this holds for any Y. If $X = 2$, then this holds for $Y = 1, 2$. However, for $X > 2$, this only holds when $Y = 1$ by plugging into the inequality. Therefore, the condition is just equivalent to when is there at least one 1 rolled or we get the outcome $(2, 2)$. The probability of this is $\frac{12}{36} = \frac{1}{3}$ by drawing out the table or using Inclusion-Exclusion and adding in the case of $(2, 2)$.

Solution to Question 626: Numerical Triangle

The sum of all the sides, including the vertices, must be 51. The sum of the integers $1 - 9$ is 45. Therefore, a sum of 6 must be on the three vertices so that they are double-counted and the sum turns out to be 51. The only way to do this is fix $1, 2$, and 3 on the vertices, as these are the only 3 integers summing to 6.

Let A be the side with vertices 1 and 2, B be the side with vertices 1 and 3, and C be the side with vertices 2 and 3. We have the integers $4, 5, 6, 7, 8$, and 9 left. We have to assign sums of $14, 13$, and 12, respectively, to sides A, B, and C.

We can get a sum of 14 by either using 6/8 or 5/9. This gives us two options for side A. Once we pick what is assigned to A, our remaining sides are fixed. Namely, if we assign 6/8 to A. Then we must assign 4/9 to side B and 5/7 to side C. Similarly, if we assign 5/9 to side A, then we must assign 6/7 to side B and 4/8 to side C. Therefore, there are only 2 arrangements of the triangle's values that are distinct.

All that is left is reflections of the triangle to evaluate distinctness. There are 3 reflection symmetries of an equilateral triangle corresponding to axes of symmetry from the three vertices. Therefore, we have 3 reflections that create distinct triangles. This means we have a total of $2 \cdot 3 = 6$ distinct answers.

Solution to Question 627: Prime or Not

No, 1027 is not prime because it is factorable.

$$1027 = 1000 + 27 = 10^3 + 3^3 = (10+3)(10\hat{A} + 3\hat{A} - 10 \times 3) = 13 \times 79$$

Solution to Question 628: Bond Practice V

$n = 2 \times 12.0 = 24; r = \frac{0.06}{2} = 0.03000$; $C = \frac{0.06 \times 1,000}{2} = 30$; $P =$ price of bond:

$$P = \left(\frac{30}{0.03000}\right)\left(\frac{(1+0.03000)^{24} - 1}{(1+0.03000)^{24}}\right) + \frac{1,000}{(1+0.03000)^{24}}$$
$$P = (1,000)\left(\frac{2.03279 - 1}{2.03279}\right) + \frac{1,000}{2.03279}$$
$$P = 508.0663 + 491.9337$$
$$P = 1,000.00$$

Solution to Question 629: Pizza Passcode

Let x_i be the number of times that digit i, $0 \leq i \leq 9$, appears in the code. Then $x_0 + x_1 + \cdots + x_9 = 7$ and each $x_i \geq 0$ is an integer. Furthermore, each non-negative integer solution to this equation yields a unique code. The first one in the question corresponds to $x_1 = x_2 = 2$, $x_3 = x_4 = x_5 = 1$. Therefore, we just need to count the number of non-negative integer solutions to this, which is just

$$\binom{10+7-1}{10-1} = \binom{16}{9} = 11440$$

by stars and bars.

Solution to Question 630: Pharmaceutics I

The level of significance is also the probability that the test statistic is in the rejection region given the null hypothesis is true.

$$\alpha = P(\text{rejecting } H_0 \text{ when } H_0 \text{ is true}) = P(x \leq 12 \mid p = 0.8)$$

$$= \sum_{i=0}^{12} \binom{20}{i} \times 0.8^i \times 0.2^{12-i} \approx 0.032$$

Solution to Question 631: Factorial Zeros

We get a zero when we have a power of 10. As there are more integers divisible by 2 than by 5 in the first 100 integers, we really just need to count the exponent of 5 in the prime factorization of 100!. We get one power of 5 from each of the integers $5, 10, 15, \ldots, 100$, yielding an exponent of 20. However, we also need to account for the terms that are divisible by 5 multiple times. In this case, those would be divisible by 25, so these have an extra power of 5 that has not been stripped yet, so the integers $25, 50, 75$, and 100 have an extra power of 5 to strip, yielding 5^{24} in our prime factorization. Therefore, our answer is 24.

Solution to Question 632: Sum Leak II

We are now going to prove a more general version of $\mathbb{E}\left[\frac{S_m}{S_n}\right]$ with $m > n$. Now, we would have

$$\mathbb{E}\left[\frac{S_m}{S_n}\right] = \mathbb{E}\left[\frac{S_n + X_{n+1} + \cdots + X_m}{S_n}\right] = 1 + \sum_{i=n+1}^{m} \mathbb{E}\left[\frac{X_i}{S_n}\right]$$

Note that X_i is not in the denominator term for $i = m+1$ to n, as the denominator only goes up to n. Therefore, the X_i are independent of S_n for $i > n$ and there are $m - n$ of these terms. Therefore, this expectation is $1 + (m - n)\mathbb{E}[X_1]\mathbb{E}\left[\frac{1}{S_n}\right]$. Plugging in our specific values yields $1 + (40 - 20) \cdot 2 \cdot \frac{1}{10} = 5$.

Solution to Question 633: Marble Runs

Let us define random variables $X_1, \ldots, X_{100}$ such that

$$X_i = \begin{cases} 1 & \text{if the } i\text{-th draw is the beginning of a new run} \\ 0 & \text{otherwise} \end{cases}$$

In the case of $RBBRRRBRR$, for example, we would have $X_1 = 1, X_2 = 1, X_4 = 1, X_7 = 1, X_8 = 1$. Note that, for a sequence of 100 red and blue marbles drawn in any order, it is always the case that $X_1 = 1$. Let Z represent the total number of runs.

$$Z = \sum_{i=1}^{100} X_i$$

By the linearity of expectation,

$$\begin{aligned}\mathbb{E}\left[Z\right] &= \mathbb{E}\left[\sum_{i=1}^{100} X_i\right] \\ &= \sum_{i=1}^{100} \mathbb{E}\left[X_i\right] \\ &= 1 + \sum_{i=2}^{100} \mathbb{E}\left[X_i\right]\end{aligned}$$

Now let's consider $\mathbb{E}[X_i] = \mathbb{P}(X_i = 1)$ for $i \geq 2$.

$$\begin{aligned}\mathbb{P}(X_i = 1) &= \mathbb{P}(((i-1)\text{-th draw } R, i\text{-th draw } B) \cup ((i-1)\text{-th draw } B, i\text{-th draw } R)) \\ &= 2 \cdot \mathbb{P}((i-1)\text{-th draw } R, i\text{-th draw } B) \\ &= 2 \cdot \mathbb{P}(i\text{-th draw } B \mid (i-1)\text{-th draw } R) \cdot \mathbb{P}((i-1)\text{-th draw } R)\end{aligned}$$

The probability that the $(i-1)$-th draw is red is simply $\frac{50}{100}$. The probability that the i-th draw is blue given that the previous draw is red is simply $\frac{50}{99}$. Plugging this in, we find

$$\begin{aligned}\mathbb{P}(X_i = 1) &= 2 \cdot \frac{50}{100} \cdot \frac{50}{99} \\ &= \frac{50}{99} \\ &= \mathbb{E}[X_i]\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}\left[Z\right] &= 1 + \sum_{i=2}^{100} \frac{50}{99} \\ &= 51\end{aligned}$$

Solution to Question 634: Elliptical Area

Let E and C be the areas of the ellipse and circle, respectively. We are looking for $\mathbb{P}[E < C]$. We have that $C = \dfrac{\pi}{4} A^2$ by the formula for the area of a circle (note that the radius of the circle is $A/2$).

We want $\mathbb{P}[C > E] = \mathbb{P}\left[\dfrac{\pi}{4} A^2 > \pi AB\right] = \mathbb{P}[A > 4B]$ by substituting in the definitions and simplifying. We now have a region in the plane we can integrate the joint density of A and B over to obtain this probability. The joint density of A and B is easily obtained by multiplying the individual densities together, as they are independent, so $f(a,b) = e^{-(a+b)}I_{(0,\infty)}(a)I_{(0,\infty)}(b)$. Thus, we have that

$$\mathbb{P}[A > 4B] = \int_0^\infty \int_{4b}^\infty e^{-(a+b)}dadb = \int_0^\infty e^{-5b}db = \frac{1}{5}$$

Solution to Question 635: Generous Banker

Consider this game on the finite set $\{1, 2, \ldots, N\}$. We can quickly see that in this finite state game, the banker selecting uniformly at random is not optimal, as to maximize the expected payout, you would select N as your value and your expected payout on the game would be N.

What we want to do is find a distribution on the positive integers so that the expected payout from the banker is constant regardless of the value that he selects. This would be because of the fact that no matter which probability distribution you select, you would now have constant payout on the game, so you can't do any better by selecting another distribution. If N is the value the banker selects, to have constant expected payout per value, we would need a distribution with PMF in the form $\mathbb{P}[N = n] = \dfrac{C}{n^2}$ for positive integers n.

To find C, we note that $\sum_{n=1}^{\infty} \mathbb{P}[N = n] = 1$, so $C \cdot \sum_{n=1}^{\infty} \dfrac{1}{n^2} = 1$. This sum here is well-known to evaluate to $\dfrac{\pi^2}{6}$, so $C = \dfrac{6}{\pi^2}$. This PMF is actually known as the Zeta(2) distribution. The expected payout is therefore $\dfrac{6}{\pi^2}$ for you, as any probability distribution you assign is just going to be weighting of $\dfrac{6}{\pi^2}$ for each value conditional on selecting some value. Therefore, the answer is 12 to the question of interest.

Solution to Question 636: Arbitrage Detective IV

We can see here, we have quite an unusual options chain we can take advantage of. As our chains are near inverses of each other, assume the underlying is selling at \$175. Usually, on a chain, when the strike of our call option increases, the call option's value approaches full value, meaning the difference between our prices should be close to the difference in our strike prices. In this example, as our call option increases in strike price, we would expect the prices to get increasingly farther away. In this case, the prices actually start converging, showing our chain is improperly formed.

The same is true on the put side, we should see the difference in price increase as the strike price decreases, but again, the opposite is true.

In order to take advantage of this, we can use an iron condor strategy. Usually an iron condor is used to take advantage of a neutral market, but in this case, we just care about the large mispricings in the chain. We use a put credit and a call credit spread, netting \$12 - \$9 of credit on both sides, with a max loss of \$5. So $3 \cdot 2 - 5 = \$1$. And as all contracts control 100 shares, we get $\$1 \cdot 100 = \$100$.

Solution to Question 637: Ferry Stops

Let be the amount of people we started with. After the first round, there are $\frac{1}{4}x + 7$ people on the ferry. After the second stop, there are

$$\frac{1}{4}\left(\frac{1}{4}x + 7\right) + 7 = \frac{1}{16}x + \frac{35}{4}$$

After the last stop, there are

$$\frac{1}{4}\left(\frac{1}{16}x + \frac{35}{4}\right) + 7 = \frac{1}{64}x + \frac{147}{16} = \frac{x + 588}{64}$$

people on the ferry. We must find the smallest integer x such that $x + 588$ is divisible by 64. Note that $10 \cdot 64 = 640$, which is the smallest integer larger than 588 that is divisible by 64. Thus, there are $640 - 588 = 52$ people on the ferry at the start, meaning there are 10 people on at the end.

Solution to Question 638: Racecar Driver

Let p be the probability that he runs a given stoplight. Then $p(1-p) = \frac{9}{100}$ by interpreting the question into math. We can quickly see that $p = \frac{1}{10}$, as we know the value is less than half. Therefore, the probability that the fourth light is the first he runs is $p(1-p)^3 = \frac{729}{10000}$.

Solution to Question 639: Diner Dash

There are 5^4 ways for the friends to choose their diner locations. For the numerator, there are 5 ways for the first friend to choose a location, 4 ways for the second friend, 3 for the third, and 2 for the fourth, for a total of 120 ways. Our answer is $\frac{120}{5^4} = \frac{24}{125}$.

Solution to Question 640: Investment Arbitrage

We can hedge our position in investing in the stock by betting on it doing down at the casino. Let p be the profit you make. Suppose you purchase s stocks and purchase b units in the casino to bet on the stock going down. Then $p = 100s - 100b$ if the stock goes up, as you make 100 per stock if it goes up and lose 100 in the casino since you bet on it going down. If it goes down, then $p = 100b - 50s$. Equating these, we see that $100s - 100b = 100b - 50s$, which means $s = \frac{4b}{3}$. Therefore, to ensure both are integers, we can let $b = 3$ and $a = 4$, which is the minimal position with integer units, yields a profit of 100 guaranteed.

Solution to Question 641: Binomial Pricing I

First, we need to calculate the risk-neutral probabilities. We cannot assume that the stock has a 50% chance of increasing by 100% or decreasing by 50%. We setup the following equation:

$$2q + (1 - q).5 = 1$$

This gives $q = 1/3$. We can now setup our one-period binomial pricing model. We know that the stock price can be either $S_1 = 20$ or $S_1 = 5$, depending on if we increase by 100% or decrease by 50%. If we end at $S_1 = 20$, then we have an option payoff of $C_1 = 15$. Otherwise, we have an option payoff of $C_1 = 0$. We know the risk-neutral probabilities, so we can now calculate the fair price of our call option.

$$E(C_0) = \frac{1}{3}(15) + \frac{2}{3}(0) = 5$$

Solution to Question 642: 30 Side Difference

Let $Z = |X - Y|$. We are going to compute the PMF of Z directly. We know that there are a total of $30^2 = 900$ possible outcomes for the rolls. Now, for a fixed k, we want to find the number of outcomes satisfying $\{Z = k\}$.

If the absolute difference between the rolls is exactly k, then the outcomes are in the form $(i, i + k)$ for some appropriate i. What is the range of i we can have? Namely, we know that $i \geq 1$, as that is the minimal possible value of the die. The upper bound is $i = 30 - k$, as that would correspond to the pair $(30 - k, 30)$. We can permute the values to either of the 2 dice, so there are

$2(30-k)$ outcomes corresponding to $\{Z=k\}$ Therefore,

$$\mathbb{E}[Z] = \sum_{k=0}^{29} k \cdot \frac{2(30-k)}{900} = \frac{1}{450}\sum_{k=1}^{29} 30k - k^2 = \frac{1}{450}\left(\frac{30(29)(30)}{2} - \frac{29(30)(59)}{6}\right) = \frac{899}{90}$$

Solution to Question 643: Head-Tail Product

If X heads are flipped out of a total of n coin tosses, then $Y = n - X$ tails are flipped. Let's substitute this expression in for Y, simplify, and apply the linearity of expectation.

$$\begin{aligned}\mathbb{E}[XY] &= \mathbb{E}[X(n-X)] \\ &= n \cdot \mathbb{E}[X] - \mathbb{E}[X^2]\end{aligned}$$

Note that $X \sim \text{Binom}(n, 0.5)$. Recall that for some $Z \sim \text{Binom}(n,p)$, $\mathbb{E}[Z] = np$ and $\text{Var}(Z) = npq$, where $q = 1-p$. So,

$$\begin{aligned}\mathbb{E}[X] &= \frac{n}{2} \\ \text{Var}[X] &= \frac{n}{4}\end{aligned}$$

We can compute $\mathbb{E}[X^2]$ as follows:

$$\begin{aligned}\text{Var}[X] &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= \frac{n}{4} \\ \Rightarrow \mathbb{E}[X^2] &= \frac{n}{4} + \left(\frac{n}{2}\right)^2\end{aligned}$$

Returning to our original expression of interest,

$$\begin{aligned}\mathbb{E}[XY] &= \mathbb{E}[X(n-X)] \\ &= n \cdot \mathbb{E}[X] - \mathbb{E}[X^2] \\ &= \frac{n^2}{2} - \frac{n^2}{4} - \frac{n}{4} \\ &= \frac{n^2 - n}{4}.\end{aligned}$$

Our solution is therefore $1 - 1 + 4 = 4$.

Solution to Question 644: Fixed Point Variance

Note that $Y = 1000 - X$, as $X + Y = 1000$ always. Therefore, $\text{Var}(X - Y) = \text{Var}(1000 + 2X) = 4\text{Var}(X)$. It now remains to compute $\text{Var}(X)$. Note that $X = X_1 + \cdots + X_{1000}$, where X_i is the indicator of whether or not i is fixed.

Then $\mathbb{E}[X] = 1000\mathbb{E}[X_1]$ by exchangeability. $\mathbb{E}[X_1]$ is just the probability 1 is fixed, which is $\frac{1}{1000}$, as it is equally likely to end in any of the 1000 spots.

To compute $\mathbb{E}[X^2]$, we expand out the sum squared. Namely,

$$\mathbb{E}[X^2] = \mathbb{E}[(X_1 + \cdots + X_{1000})^2] = \sum_{i=1}^{1000} \mathbb{E}[X_i^2] + \sum_{i \neq j} \mathbb{E}[X_i X_j]$$

The first term is where $i = j$ and the second term is where $i \neq j$. Note that $X_i^2 = X_i$, as X_i only takes the values 0 and 1, so $\mathbb{E}[X_i^2] = \frac{1}{1000}$.

For the second sum, as the X_i random variables are exchangeable, for any $i \neq j$, $\mathbb{E}[X_i X_j] = \mathbb{E}[X_1 X_2]$. This expectation is just the probability both 1 and 2 are fixed. This probability is just $\frac{1}{1000} \cdot \frac{1}{999}$, as once 1 is fixed, 2 goes into spot 2 with probability $\frac{1}{999}$. Therefore, $\mathbb{E}[X_1 X_2] = \frac{1}{1000} \cdot \frac{1}{999}$.

There are 1000 terms in the first sum, while there are $1000^2 - 1000 = 1000(1000 - 1) = 1000 \cdot 999$ terms in the second sum. Therefore, we have that

$$\mathbb{E}[X^2] = 1000 \cdot \frac{1}{1000} + 1000 \cdot 999 \cdot \frac{1}{1000} \cdot \frac{1}{999} = 2$$

We thus have that $\mathbb{E}[X^2] = 2 - 1^2 = 1$, so $\text{Var}(X - Y) = 4(1) = 4$.

Solution to Question 645: Going Extinct

The number of years before extinction follows a geometric distribution where $p = 0.1$. The expected number of years before extinction is thus $\frac{1}{p} = 10$. This can also be solved for analytically. Let x be the expected number of years before extinction. There is a $\frac{1}{10}$ probability that the species will only last one year, and a $\frac{9}{10}$ probability that the species will extend its expected number of years by one. Hence:

$$x = \frac{1}{10}(1) + \frac{9}{10}(x + 1) \Rightarrow x = 10$$

Solution to Question 646: Cubic Sum

We can write $441 = 147 \cdot 3$, $588 = 147 \cdot 4$, and $735 = 147 \cdot 5$, so $441^3 + 588^3 + 735^3 = 147^3(3^3 + 4^3 + 5^3) = 147^3 \cdot 216 = (6 \cdot 147)^3 = 882^3$. Therefore, $x = 882$.

Solution to Question 647: 2D Paths IV

From 2D Grids I, we know that there are a total of $\binom{8}{3} = 56$ total paths from $(0,0)$ to $(5,3)$ with no restrictions. Now, we want to remove those that don't satisfy our restriction.

Note that if the $y = x$ line is crossed, which runs from $(0,0)$ to $(3,3)$ inside the grid, then we will touch the diagonal that runs from $(-1,0)$ to $(2,3)$. Now, for each path that does touch this new diagonal, reflect the part that comes before the first touch across that diagonal. In specific, we now that have $(0,0)$ reflects to $(-1,1)$ and each path that touches this new diagonal corresponds to a unique path from $(-1,1)$ to $(5,3)$. There are $\binom{8}{2} = 28$ such paths, so our total number of paths that don't cross our original diagonal is $56 - 28 = 28$.

Solution to Question 648: Uniform Product II

Plot the line $XY > 0.5$ within the unit square. You'll see its a curved line from point $(0.5,1)$ to $(1,0.5)$. We can find the area under this curve and take its compliment to find the answer to this question (from $0.5 \le x \le 1$). Thus

$$\begin{aligned}\mathbb{P}[XY > 0.5] &= \int_{0.5}^{1}\left(1 - \frac{1}{2x}\right)dx \\ &= \frac{1}{2}\left(1 + \ln\left(\frac{1}{2}\right)\right) \approx 0.15\end{aligned}$$

Solution to Question 649: Positively Normal

The key idea here is that (X,Y) is radially symmetric about the origin. In other words, the density at each point of the circumference of the circle of radius $r > 0$ centered at the origin is constant. One can see this from the fact that the $f_{X,Y}(x,y) = f_X(x)f_Y(y) = \frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}$, so the density at a point (x,y) depends on it's distance from the origin. This problem boils down to finding the angle swept our by our region of interest compared to the total region.

The region $\{Y > 0\}$ in the plane is the upper-half plane, which spans π radians. The region $Y > \sqrt{3}X$ covers all of the 2nd quadrant and also $\frac{\pi}{6}$ radians in the 1st quadrant, as the line $y = \sqrt{3}x$ makes an angle $\frac{\pi}{3}$ with the positive $x-$axis. Our region of interest is $\frac{\pi}{6}+\frac{\pi}{2}=\frac{2\pi}{3}$ radians of the total π radians, so our answer is just

$$\frac{\frac{2\pi}{3}}{\pi}=\frac{2}{3}$$

Solution to Question 650: 7 Multiple

Intuitively, it is clear that you should select to go second. This is because you have no chance of winning on the first roll, but every roll after the first roll, there is a $1/6$ probability that you win the game, as exactly 1 of the 6 possible values on the will make the sum a multiple of 7.

Let's compute the probability that you, the second player to roll, wins. Call this probability p. By conditioning on your first roll, there is a $5/6$ probability you do not roll a value resulting in a sum divisible by 7 on the first turn. To come back to you, you would need your friend to also not roll a value resulting in a sum divisible by 7, which occurs with probability $5/6$ too. In this case, your probability of winning when it comes back to you is p. Alternatively, you do roll the value that results in a sum divisible by 7, occurring with probability $1/6$. Therefore, we have the equation

$$p=\frac{5}{6}\cdot\frac{5}{6}p+\frac{1}{6} \iff p=\frac{6}{11}$$

Solution to Question 651: Geometric Distribution

We can interpret the information as saying that it takes us at least 5 coin flips to obtain our first heads. We want to find the probability that it takes up no more than 8 coin flips to obtain our first heads. The "at least 5 flips" portion means that the first 4 flips were all tails. Therefore, from 4 tails, find the probability the first head occurs on or before the 8th flip. Given the memorylessness of the geometric distribution, this is really asking the probability we obtain a heads in the next 4 flips.

The complement is easy to compute, which is the event that we don't obtain a heads in the next 4 flips. The probability of all tails for the next 4 flips is $\frac{3^4}{4^4}=\frac{81}{256}$. Therefore, the probability we do get a heads in the next 4 flips is $1-\frac{81}{256}=\frac{175}{256}$.

Solution to Question 652: Defeating Dragons

Since there are changing states to this situation, we need to employ a Markov Chain to answer this question. Each state is differentiated by the amount of heads the dragon has. Let P_x be the probability you defeat the dragon when the dragon is at x heads. We already know that $P_0 = 1$ and $P_5 = 0$, thus we need to make equations for every other state in between. Also, its important to notice you are either going to a state with two less, the same amount, or one more head. Since the probability of going down two heads is the same as one more head, the probability of each scenario happening is $\frac{1}{2}$ as if you repeat the same state, you'll still have to either go down two heads or up one head again. Also, when the dragon is at one head, you will either stay at the same number or increase the number of heads by one, so $P_1 = P_2$. Every other state equation is as follows:

$$P_2 = \frac{1}{2} \cdot P_0 + \frac{1}{2} \cdot P_3$$

$$P_3 = \frac{1}{2} \cdot P_1 + \frac{1}{2} \cdot P_4$$

$$P_4 = \frac{1}{2} \cdot P_2 + \frac{1}{3} \cdot P_5$$

Solving all these equations, we obtain the answer of $P_3 = \frac{3}{5}$.

Solution to Question 653: Binary Strangle

We can replicate a binary strangle as a binary call with strike $K = 20$ and a binary put with strike $K = 15$. We don't have a binary put, so we must replicate the $K = 15$ binary put with a $K = 15$ binary call. We know that $P_0 = Z_0 - C_0 = 0.9 - 0.73 = 0.17$.

Combining it all, we obtain $V_0 = C_0 + P_0 = 0.13 + 0.17 = 0.30$

Solution to Question 654: Adjacent Birthdays

Let N be the number of people needed for this to occur. Then $\mathbb{E}[N] = \sum_{k=1}^{\infty} \mathbb{P}[N \geq k]$ by our alternative form of expectation. The event $\{N \geq k\}$ here means that none of the first $k - 1$ people in the room share adjacent birthdays. Each birthday removes 3 possible days for the remaining people to have. Therefore, the ith person entering the room has $365-3(i-1)=368-3i$ possible days for their birthday to not have it overlap with any other birthday. This means that

$$\mathbb{P}[N \geq k] = \frac{365}{365} \cdot \frac{362}{365} \cdot \frac{359}{365} \cdot \cdots \cdot \frac{368-3k}{365} = \frac{\prod_{i=1}^{k}(368-3i)}{365^k}$$

Now, we sum over the support of N. There are 365 possible birthdays, and each person entering removes up to 3 days. Therefore, we know that $N \leq 122$ with probability 1, as with 122 people, there must be a pair that is adjacent. This is because if none were adjacent, the 122 people would block out $122 \cdot 3 = 366 > 365$ days, which is not possible. Thus, two must overlap. Therefore, we can stop our sum at 122. Our final expectation is

$$\mathbb{E}[N] = \sum_{k=1}^{122} \frac{\prod_{i=1}^{k}(368-3i)}{365^k} \approx 13.5$$

Solution to Question 655: Comparing Flips I

Both HHT and HTT begin with heads, so we can ignore any pattern with leading tails. Consider the case where our sequence begins with heads. Given that the first flip is heads, each of the following four cases occurs with probability $\frac{1}{4}$: HHH, HHT, HTH, HTT. In the case of HHH, then Audrey will see the pattern HHT first with probability 1, since it is impossible for her to roll two tails without first seeing HHT. In the case of HHT, HHT appears first. In the case of HTT, HTT appears first. And finally, in the case of HTH, we are returned to our original state of our sequence of interest beginning with heads. Since HHT, HTT, and HHH occur with equal probability, and since HHT and HHH result in HHT appearing first, the probability that HHT appears first is $\frac{2}{3}$.

Solution to Question 656: Plane Boarding

This can be solved with symmetry. Consider your seat and the drunk passenger's seat. Either your seat is taken before the drunk passenger's seat, or the drunk passenger's seat is taken before your seat. Because there is nothing special about your seat or the drunk passenger's seat, each passenger that has to make a random choice of seats will have an equal probability of filling either your seat or the drunk passenger's seat, assuming both are vacant. Hence, the probability is $\frac{1}{2}$.

Solution to Question 657: Circular Delete

At the first step, you are keeping all integers n with $n \equiv 0 \bmod 2$. Afterwards, you are keeping all integers with $n \equiv 0 \bmod 4$. More generally, at step k, you are keeping all integers $n \equiv 0 \bmod 2^k$. Therefore, we just need to find the largest k such that $2^k \leq 2000$. 2^k will thus be our last integer. This is $k = 10$, which corresponds to $2^{10} = 1024$.

Solution to Question 658: Multiplicative Mix

We know that $bc = 15 = 3 \cdot 5$. Since 21 is not divisible by 5, this means $b = 3$ and $c = 5$. Since $c = 5$, this means $d = 9$. Since $b = 3$, this means $a = 7$. Adding the squares up, we get that our answer is $3^2 + 5^2 + 7^2 + 9^2 = 164$.

Solution to Question 659: Triangular Change

We can write $P(a, b, c) = a + b + c$ to be the perimeter of the triangle at a given time. Taking the derivative in time, we see

$$\frac{d(a+b+c)}{dt} = \frac{da}{dt} + \frac{db}{dt} + \frac{dc}{dt} = 3 - 2 - 2 = -1$$

Solution to Question 660: Pocket Aces

P(two aces) = P(first ace) $\times P(\text{second ace}|\text{first ace}) = \frac{4}{52} \times \frac{3}{51} = \frac{1}{221}$

Solution to Question 661: Ping Pong Tournament II

In order for competitor 49 to survive the first 10 rounds, either (1) competitor 49 does not appear until the 11th round, or (2) competitor 49 plays their first round within the first 10 rounds, and competitor 50 does not play within the first 10 rounds.

Consider case 1. Note that there are a total of 39 rounds between rounds 11 and 49, inclusive. If we want competitor 49 to play their first round after the 10th round, then competitor 49 must be at position 12 or later (39 possible positions) in a random ordering of the 50 competitors. The probability of case 1 occuring is then $\dfrac{39}{50}$.

Next, consider case 2. In this case, in a random ordering of 50 competitors, competitor 49 must be among the first 11 competitors, and competitor 50 must be among the last 39 competitors. This occurs with probability $\frac{\binom{11}{1}\binom{39}{1}48!}{50!} = \frac{429}{2450}$. By countable additivity, we find the answer to be $\frac{39}{50} + \frac{429}{2450} = \frac{234}{245}$.

Solution to Question 662: Square Cross

Let this event be called E. To compute $\mathbb{P}[E]$, we condition on the radius $R = r$. Namely, we have that

$$\mathbb{P}[E] = \int_0^{10} \mathbb{P}[E \mid R = r] f_R(r) dr$$

where $f_R(r) = \frac{1}{10} I_{(0,10)}(r)$ is the PDF of R. Given that $R = r$, our center must stay in the square that is a distance r away from each side. In particular, the side length of this square region where we can select our center is $20 - 2r$. Thus, the probability that it lies within this region is

$$\frac{(20-2r)^2}{20^2} = \frac{(10-r)^2}{10^2}$$

This means our probabiility is

$$\mathbb{P}[E] = \int_0^{10} \frac{(10-r)^2}{10^3} dr = \frac{1}{10^3} \cdot \frac{10^3}{3} = \frac{1}{3}$$

Solution to Question 663: Boy Chairs

Since there are 3 boys and 3 girls and all 3 children are of the same gender, it is equally likely for all 3 of the chairs to be occupied by boys and by girls. Therefore, the expected value is just $\frac{1}{2} \cdot 3 + \frac{1}{2} \cdot 0 = \frac{3}{2}$, as it is equally likely for either 3 or 0 boys to occupy the chairs.

Solution to Question 664: Leading Sum

To find a, we need to have some polynomial $P(n) = an^{1001}$ for some a such that the limit of this sum divided by $P(n)$ is 1 as $n \to \infty$. Using this fact, we can write this as

$$1 = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} k^{1000}}{P(n)} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} k^{1000}}{an^{1001}}$$

Distributing the n through and multiplying by a, we get that

$$a = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^{1000} \frac{1}{n} = \int_0^1 x^{1000} dx = \frac{1}{1001}$$

Solution to Question 665: Cheater

Let K be the event that Gabe knew the answer to the question, C be the event he gets a question right, and G be the event he guessed on a question. We know that $\mathbb{P}[K] = 0.6$, $\mathbb{P}[G] = \mathbb{P}[K^c] = 1 - \mathbb{P}[K] = 0.4$, $\mathbb{P}[C \mid G] = 0.2$, as he selects completely at random, and it is reasonable to assume that $\mathbb{P}[C \mid K] = 1$, as if he knows the answer, he will get it correct. We want to find $\mathbb{P}[K \mid C]$. By definition of conditional probability, this is $\mathbb{P}[K \mid C] = \dfrac{\mathbb{P}[KC]}{\mathbb{P}[C]}$. On the top, we apply conditional probability again to write this as

$$\mathbb{P}[KC] = \mathbb{P}[C \mid K]\mathbb{P}[K]$$

and on the bottom, we apply the Law of Total Probability to get

$$\mathbb{P}[C] = \mathbb{P}[C \mid K]\mathbb{P}[K] + \mathbb{P}[C \mid G]\mathbb{P}[G]$$

Thus, we know that $\mathbb{P}[K \mid C] = \dfrac{\mathbb{P}[C \mid K]\mathbb{P}[K]}{\mathbb{P}[C \mid K]\mathbb{P}[K] + \mathbb{P}[C \mid G]\mathbb{P}[G]} = \dfrac{1 \cdot 0.6}{1 \cdot 0.6 + 0.2 \cdot 0.4} = \dfrac{15}{17}$

Solution to Question 666: Statistical Test Review I

Let us define

$$\begin{aligned} H_0 &: p = 0.5 \\ H_a &: p \neq 0.5 \end{aligned}$$

Our rejection region is $|x - 18| \geq 4$. Recall, by definition, that α is the probability that the test statistic, X, is within the rejection region when H_0 is true. Hence,

$$\begin{aligned} \alpha &= \sum_{x=0}^{14} \binom{36}{x} (0.5)^x (0.5^{36-x}) + \sum_{x=22}^{36} \binom{36}{x} (0.5)^x (0.5)^{36-x} \\ &\approx 0.243 \end{aligned}$$

Solution to Question 667: Expecting Heads

We always expect one head between the first and last coins. The three middle coins follow a Binom$(3, 0.5)$, where the expectation is $\dfrac{3}{2}$. Thus, the expected number of heads is $\frac{5}{2}$.

Solution to Question 668: Signed Correlation

Correlation is a measure of linear association between X and Y that is unaffected by scaling and shifting. Thus, we have that $\mathrm{Corr}(aX + b, cY + d) = \mathrm{sign}(ac)\mathrm{Corr}(X, Y)$. In this case, $\mathrm{sign}(ac) = -1$, so our answer is $-1 \cdot -0.4 = 0.4$.

Solution to Question 669: Leftwards Frog

The frog must hop 7 times to visit all of the lily pads, so there are 7! ways it can move. Consider the set $S = \{2, 3, 4, 5, 6, 7, 8\}$. These are the unvisited vertices of the frog. Some arrangement where the frog makes exactly one leftwards hop is exactly the same as asking how many ways we can partition S into two subsets, say A and B, where the frog hops from 1 to all of the lily pads in A, ordered from left to right and then all of the lily pads in B left to right. For example, let $A = \{2, 3, 6, 7\}$ and $B = \{4, 5, 8\}$. The frog would then hop in the order $2 \to 3 \to 6 \to 7 \to 4 \to 5 \to 8$. However, we have to be careful here, as if either B is empty or the entire set $\{2, 3, 4, 5, 6, 7, 8\}$, no left hop occurs. However, there are other cases as well. Namely, to not left hop, the set A must contain elements such that $\max(A) < \min(B)$. This occurs exactly when A contains the starting sequence of the elements. This would mean we exclude $A = \{2\}, \{2, 3\}, \{2, 3, 4\}, \ldots, \{2, 3, 4, 5, 6, 7\}$ (we already excluded the full set). This adds 6 additional sets we must exclude. Therefore, Therefore, we must remove 8 elements from the $2^7 = 128$ possibilities of how we can create the subsets. Therefore, our answer is

$$\frac{2^7 - 8}{7!} = \frac{1}{42}$$

Solution to Question 670: Tigers Love Sheep

In the 1-tiger case, the tiger will eat the sheep since it does not need to worry about being eaten after. In the 2-tiger case, either tiger knows that if they eat the sheep, then they will be eaten by the other tiger once they become a sheep. Thus, neither tiger will eat the sheep. In the 3-tiger case, the sheep will be eaten since each tiger will realize that once it eats the sheep and turns into a sheep, there will be 2 tigers left, and neither tiger will eat in the 2-tiger case. In the 4-tiger case, each tiger understands that if it eats the sheep, then it will be eaten after turning into a sheep, since they know that the sheep is eaten in the 3-tiger case. Following this logic, we can naturally show that if the number of tigers is even, the sheep will not be eaten. If the number is odd, the sheep will be eaten.

Solution to Question 671: Sphere Slicer

There will be $\binom{6}{2} = 15$ total edges on the sphere. Label these edges $1 - 15$, and let X_i be the indicator of the event that the plane intersects the ith edge. Then $T = X_1 + \cdots + X_{15}$ gives the total number of edges that are intersected by the plane.

Suppose that we have two points on our sphere. We want to find the probability that the segment between them is intersected by the plane. The key here is that we draw a vector from the origin to the two points that we consider. The angle between those two vectors will be somewhere in $(0, \pi)$. The cosine of the angle between the two points is the dot product of their vectors, as they are already magnitude 1. Therefore, if the two points are labelled 1 and 2 and their corresponding vectors are v_1 and v_2, we have that the angle θ between them satisfies $\cos\theta = v_1 \cdot v_2$. In other words, $\theta = \arccos(v_1 \cdot v_2)$. As the angle with the origin of our normal direction is uniform on $(0, \pi)$ we have that the probability of intersection is just $\frac{\arccos(v_1 \cdot v_2)}{\pi}$. Our final task then is to compute this for every pair of points.

The pairs of points that we have here are nice in the sense that many are orthogonal to one another. In particular, there are only three pairs of points that are not orthogonal, which are $(1, 0, 0)$ with $(-1, 0, 0)$, $(0, 1, 0)$ with $(0, -1, 0)$, and $(0, 0, 1)$ with $(0, 0, -1)$. The dot product of all of these pairs of points is -1, so $\arccos(-1) = \pi$. For the other 12 segments, we have $\arccos(0) = \frac{\pi}{2}$. Therefore, we get that

$$\mathbb{E}[T] = 12 \cdot \frac{\frac{\pi}{2}}{\pi} + 3 \cdot \frac{\pi}{\pi} = 9$$

Solution to Question 672: Derivative Difference

Note that the original differential equation implies that $f'(x) - f''(x) = 3(x+1)^2$, $f''(x) - f^{(3)}(x) = 6(x+1)$, $f^{(3)}(x) - f^{(4)}(x) = 6$, and $f^{(k)}(x) - f^{(k+1)}(x) = 0$ for all $k \geq 4$. This last expression implies that $f^{(4)}(x) = 0$ for all x. Therefore, we just have to work backwards by plugging in x.

Namely, $f^{(3)}(9) = 6$. Then, $f''(9) = 6 + 6(9+1) = 66$. Afterwards, $f'(9) = 66 + 3(9+1)^2 = 366$. Lastly,

$$f(9) = 366 + (9+1)^3 = 1366$$

Solution to Question 673: Primes and Composites

Let's split up all the outcomes into prime and composite.

$$\text{Primes: } 2, 3, 5, 7 \text{Composites: } 4, 6, 8, 9, 10$$

The EV of this game comes out to $(2+3+5+7-2-3-4-4.5-5)/9 = -\frac{1}{6}$. Since the EV is negative, the answer is 0.

Solution to Question 674: DisCard Game

No matter how the cards are permuted, you and the dealer will always have the same number of cards in your respective piles because each pair of discarded cards has one black and one red card, so an equal number of red and black cards are always discarded. In other words, there are an equal number of red and black cards that are not discarded and in your piles. Thus, the number of red cards left for you and the number of black cards left for the dealer are always the same and the casino always wins. You should pay \$0 to play this game.

Solution to Question 675: Bacterial Survival I

If the colony with one cell goes extinct with probability $\frac{1}{3}$, then for a colony starting with three cells to go extinct, you can view this as saying the independent lineages of each of the three cells must go extinct. The probability for each that it goes extinct at some point is $\frac{1}{3}$, so for all three to go extinct, as they are independent, the probability is $\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{27}$.

Solution to Question 676: Modified Deck Pair

If a matching pair has already been removed, then there are 38 cards remaining. There are 9 sets of 4 identically-numbered cards remaining, and 1 set of 2 identically-numbered cards remaining. The desired probability is simply

$$\frac{9\binom{4}{2}+1}{\binom{38}{2}} = \frac{55}{703}.$$

Solution to Question 677: Double Data Trouble VI

From Double Data Trouble V, we know that the variance of our estimate is the same, so the standard error is also the same. As

$$t_0 = \frac{\hat{\beta}_1}{s\left\{\hat{\beta}_1\right\}}$$

we know that the estimate $\hat{\beta}_1'$ is the same from Double Data Trouble I, and that the standard error in the denominator is same for $\hat{\beta}_1'$. Therefore, we get that $t_1 = t_0$, meaning that $t_1^2 = t_0^2$.

Solution to Question 678: Optimal Bidders I

It is known that given $X_1, \ldots, X_n \sim \text{Unif}(0,1)$ IID and $M_n = \max\{X_1, \ldots, X_n\}$, $\mathbb{E}[M_n] = \frac{n}{n+1} = 1 - \frac{1}{n+1}$. This also extends now to $\text{Unif}(500, 1000)$, as this is just a scaling and shifting of a $\text{Unif}(0,1)$. Namely, if $X \sim \text{Unif}(0,1)$, $500X + 500 \sim \text{Unif}(500, 1000)$. Therefore, if M_n' is the maximum of n IID $\text{Unif}(500, 1000)$ random variables and M_n is as above, $\mathbb{E}[M_n'] = 500\mathbb{E}[M_n] + 500 = 1000 - \frac{500}{n+1}$. The cost of obtaining n people is $5n$. Therefore, our profit, which we can denote as P_n, is $P_n = M_n' - 5n$, so $\mathbb{E}[P_n] = \mathbb{E}[M_n'] - 5n = 1000 - \frac{500}{n+1} - 5n$. We want to find n that maximizes this. Calling this function $f(n)$, we can treat n as continuous and find the value of n that maximizes it by taking the derivative and setting it equal to 0.

This means $f'(n) = \frac{500}{(n+1)^2} - 5 = 0$. Rearranging yields $(n+1)^2 = 100$, so as n must be positive, $n = 9$ bidders gives us the maximal expected profit. Plugging this in, $\mathbb{E}[P_9] = 905$, which is our solution.

Solution to Question 679: Egg Drop II

In Egg Drop I, the strategy was to keep the worst case constant by scaling how high we jump at each interval based on how far up it is in our tower. In other words, we make larger jumps at lower floors in the tower so that we can test every floor afterwards in a given range. Namely, with at most d drops and 2 eggs, we could test $\frac{d(d+1)}{2}$ floors completely. The key here is to note that when we drop one egg and it breaks, we really get reduced down to the two egg case. Therefore, instead of making jumps that decrease linearly (which is what we do in the two egg case), we are going to make jumps that decrease by the maximum increment of the two egg case. This is so that we are sufficiently able to search everything in a given range in the two egg case afterwards.

Putting this into math, we want to find a first cutoff point such that if the egg breaks at floor X on the first drop, we can examine the floors below it in $9 - 1 = 8$ trials. We can examine up to $\frac{8 \cdot 9}{2} = 36$ floors in 8 trials by Egg Drop I, so we should drop our first egg at floor 37. If it doesn't break, then we need to increment such that if it breaks on our next trial, we can examine the remaining space between 37 and the next floor in $9 - 2 = 7$ drops. We can explore up to $\frac{8 \cdot 7}{2} = 28$ floors with 7 drops, so we should place our next egg at floor 66 such that we can explore floors $38 - 65$ (28 floors) in the remaining 7 drops.

Continuing this pattern, when we have k drops left, we are going to increment by $\frac{k(k+1)}{2} + 1$ floors. This means that the next egg after would be placed at floor $66 + 21 + 1 = 88$, the egg after at $88 + 15 + 1 = 104$, and so on. The remaining floors to drop eggs at would be $115, 122, 126$, and 128. If it breaks at 128 but not at 126, we try 127 to verify the answer. If it doesn't break at 128, we try 129 instead and see if it breaks. If it breaks, we found our answer. If it doesn't break, then if $n = 130$, we would know that the egg breaks at floor 130, as it must break on one of the floors in our building. We can't explore any higher than 130 due to the fact that if it breaks at some point above 130, we would have to search between 129 and the breaking point, which takes strictly more than one egg drop. Therefore, $n = 130$ is our answer.

Solution to Question 680: Big or Small Deck?

The idea here is that the deck of size 13 has a significantly larger probability of having no aces in it. Conditional on there being an ace in the deck of 13, the deck of 13 is faster on average. However, the probability of no ace is so much larger in the 13 than it is in the 39 that the deck of 39 is the better selection.

Solution to Question 681: Double Data Trouble V

When we double the data, we are just replacing the matrix X with the matrix $X' = 2X$. However, we also have to replace σ with 2σ. Accordingly, $X'^T X' =$

$4X^TX$ by simple matrix multiplication. Then Therefore, $(2\sigma)^2(4X^TX)^{-1} = \frac{1}{4} \cdot 4 \cdot \sigma^2(X^TX)^{-1} = \sigma^2(X^TX)^{-1}$, which proves our statement.

Solution to Question 682: Basic Die Game V

The expected value of a fair die roll is 3.5. The cost of the re-roll is \$1. Therefore, the expected payout if Alice decides to roll again is 2.5. This means Alice should accept any value at least 3 on her first roll and roll again otherwise. Namely, if P is Alice's payout and X_1 is the value of the first die roll, then

$$\mathbb{E}[P] = \mathbb{E}[P \mid X_1 \leq 2]\mathbb{P}[X_1 \leq 2] + \mathbb{E}[P \mid X_1 > 2]\mathbb{P}[X_1 > 2]$$

The two probabilities are $1/3$ and $2/3$, respectively. Then, $\mathbb{E}[P \mid X_1 \leq 2] = 2.5$ by our discussion above about the expected payout upon rolling again. $\mathbb{E}[P \mid X_1 > 2] = \frac{3+4+5+6}{4} = 4.5$, as each of the values > 2 are equally-likely. Therefore,

$$\mathbb{E}[P] = \frac{1}{3} \cdot \frac{5}{2} + \frac{2}{3} \cdot \frac{9}{2} = \frac{23}{6}$$

Solution to Question 683: Pairwise Digit Sums I

The first observation is that all the digits in this number must be unique. If there was a repeated digit, say the integer was in the form $aabcde$, then you could write $a + b$ twice as two different sums with the two a. Let's attempt to create the smallest $6-$digit number with distinct sums. We note that the digits $0, 1,$ and 2 are all safe to use. We can't add 3, since $3 + 0 = 2 + 1$, so 4 is the first value that can't be created with pairwise sums of existing integers. Then, 7 is the first value after $0, 1, 2, 4$ that can't be created with pairwise sums. Then, lastly, 12 would be the first value after $0, 1, 2, 4, 7$ that can't be created using pairwise sums. However, 12 is not a digit, so this is impossible.

However, in the process, we have shown a $5-$digit integer does exist. Namely, 74210 is such an integer by our construction above. There are many others, but we just need existence.

Solution to Question 684: Rainbow Trains

Let us denote the colors 1, 2, and 3 such that a correct rainbow ordering of the cars is 123. There are 6! ways to order the three cars, and of those orders, only three preserve the rainbow order of the cars: 123, 231, and 312. Thus, the probability that the conductor can rearrange the order of the cars into rainbow order is:

$$\frac{3}{6!} = \frac{1}{2}$$

Solution to Question 685: Big Bubble I

We have that $A(r) = \pi r^2$ and $C(r) = 2\pi r$ represent the area and circumference of a circle as a function of the radius. We know that r', the rate of increase of the radius, is constant. Therefore, $A' = 2\pi r r'$ and $C' = 2\pi r'$. Setting these equal to each other,

$$2\pi r r' = 2\pi r' \iff r = 1$$

Solution to Question 686: German Tanks

We need to find N such that the expected value of the $X_{(6)}$, the maximum among 6 distinct uniform random draws from $\{1, 2, \ldots, N\}$ is 120. Namely, note that this is equivalent to the "First Ace" problem, but instead you're looking at the expected value of the 6th instead of the first. Intuitively, if 120 is the 6th order statistic, then in expectation, there are 5 other tanks before it that should partition $\{1, 2, \ldots, 120\}$ into 6 equal length parts. This means that besides all of tanks that are the order statistics themselves (since we have discrete values), there are 114 other serial numbers. The spacings between these tanks should be equal in expectation, so the average distance between them should be $\frac{114}{6} = 19$. Therefore, as 120 is the 6th order statistic and we know each order statistic on average has a spacing of 19, the maximum tank number should just be $120 + 19 = 139$.

Solution to Question 687: Same Side

The first flip will dictate what side the rest of the flips must be. Therefore, this flip is arbitrary. Now, once flipped, the other three flips match that first result each with probability $\frac{1}{2}$, so our answer is $\frac{1}{2^3} = \frac{1}{8}$.

Solution to Question 688: Prime Hunter

This is a simple binomial distribution, with our successes happening at a $\frac{1}{2}$ probability, i.e. we roll a $2, 3$, or 5, and our failures happening $\frac{1}{2}$ of the time as a well. Thus, if S is the number of successful rolls, $S \sim \text{Binom}(6, 0.5)$.

We then get

$$\mathbb{P}[S = 3] = \binom{6}{3} \cdot \left(\frac{1}{2}\right)^6 = \frac{20}{64} = \frac{5}{16}$$

Solution to Question 689: Sum Standard Deviation

Using the variance of sum formula, we know $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X,Y)$. Plugging in our given values, $4^2 = 1^2 + 3^2 + 2\rho(3)(1)$, so $6\rho = 6$, meaning that $\rho = 1$.

Solution to Question 690: Building Blocks

There are $\binom{3}{2}$ ways to select the two colors of interest for building the tower. There are $\binom{21}{5}$ total combinations of blocks able to be chosen to construct the tower. Finally, there are $\binom{14}{5}$ combinations to select the blocks for the tower. However, this overcounts as you are also counting towers that are composed of just one color. Thus we need to subtract $2 \cdot \binom{7}{5}$ ways of this happening. Combining all of these, we get a final probability of

$$\binom{3}{2} \cdot \frac{\binom{14}{5} - 2 \cdot \binom{7}{5}}{\binom{21}{5}} = \frac{5880}{20349}$$

.

Solution to Question 691: Existent MGF

We know that $M(\theta) = \mathbb{E}[e^{\theta X}]$ for all θ such that this expectation exists. Thus, $M(1) = \mathbb{E}[e^X] = 7$. Consider the random variable $Y = e^X$. Then

$$\text{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = \mathbb{E}[e^{2X}] - (\mathbb{E}[e^X])^2 = M(2) - (M(1))^2$$

By definition, $\text{Var}(Y) \geq 0$, so $M(2) - 7^2 \geq 0$, meaning $M(2) \geq 49$.

By applying the same logic now to $Z = e^{2X}$, we have that $\text{Var}(Z) = M(4) - (M(2))^2 \geq M(4) - 49^2 \geq 0$, so $M(4) \geq 49^2 = 2401$.

Alternatively, one can apply Jensen's Inequality to $Y = e^X$ and $f(y) = y^4$. As y^4 is convex on $(0, \infty)$, we have that $(\mathbb{E}[Y])^4 = (M(1))^4 = 7^4 = 2401 \leq \mathbb{E}[Y^4] = M(4)$.

Solution to Question 692: Cube Slice

Consider the center cube. To form the 6 sides of it, at least 6 cuts need to be made. However, the most basic strategy possible in cutting works: Just cut the top 4 times (2 times vertically and 2 times horizontally) and then 2 times on the side to create the layering of the cubes. Therefore 6 is the minimum.

Solution to Question 693: Fireball Thrower

We have that M and L are independent, as they come from collections of random variables that are independent of each other. The PDF of M is given by

$$f(x) = 2e^{-x}(1 - e^{-x})I_{(0,\infty)}(x)$$

by using the order statistics formula. Similarly, the PDF of L is

$$g(y) = 4e^{-2y}(1 - e^{-2y})I_{(0,\infty)}(y)$$

Thus, the joint PDF of M and L is $8e^{-(x+2y)}(1-e^{-x})(1-e^{-2y})I_{(0,\infty)}(x)I_{(0,\infty)}(y)$.

The remainder is just integrating over the region. The double integral is $\int_0^\infty \int_{2y}^\infty 8e^{-(x+2y)}(1 - e^{-x})(1 - e^{-2y})dxdy$. We can now evaluate this as

$$\int_0^\infty 8e^{-2y}(1-e^{-2y}) \int_{2y}^\infty e^{-x} - e^{-2x} dxdy = \int_0^\infty 4e^{-4y}(2-e^{-2y})(1-e^{-2y})dy = \frac{1}{2}$$

Solution to Question 694: Laser Game

If we pick r, the probability that the dart is within a radius r of the center is $\frac{r^2}{100}$, as that is the ratio of the area of the circle of radius r to the circle of radius 10. This means the probability it is outside that circle is $1 - \frac{r^2}{100}$. Thus, our expected profit when we select r is $p(r) = \frac{r^2}{100} \cdot -(10 - r) + \left(1 - \frac{r^2}{100}\right) \cdot r = -\frac{r^2}{10} + r = -r\left(\frac{r}{10} - 1\right)$. The maximum of a downwards facing parabola is halfway between the zeroes. The zeroes here are $r = 0, 10$. Therefore, the maximum value is at $r = 5$.

Solution to Question 695: Contracts and Pricing II

Let G denote Steve's gain after purchasing the contract and rationally deciding to either purchase or forgo the purchase of Amy's house 6 months in the future. Let Amy's house's current value be denoted as v. Let the price of the contract be denoted as p.

$$G = \begin{cases} -p + (C - 1)v & \text{if } C \geq 1 \\ -p & \text{otherwise} \end{cases}$$

$C \geq 1$ with probability 0.6. By the law of total expectation,

$$\begin{aligned}
\mathbb{E}[G] &= \mathbb{E}[\mathbb{E}[G|C]] \\
\mathbb{E}[G] &= \mathbb{E}[G|C \geq 1]\mathbb{P}[G \geq 1] + \mathbb{E}[G|C < 1]\mathbb{P}[G < 1] \\
&= (-p + 0.3v) \cdot 0.6 - p \cdot 0.4 \\
&= -p + 0.18v
\end{aligned}$$

Note that a fair price for the contract assumes Steve does not lose or gain anything.

$$p = 0.18v$$

Steve should pay no more than 18% of Amy's house's current value.

Solution to Question 696: Bowl of Cherries VI

We must select 3 cherries for bowl A, 5 cherries for bowl B, 4 cherries for bowl C, and 4 cherries for bowl D. Then, we must account for the possible orderings of the bowls, since we must treat each bowl as indistinguishable from the rest. This occurs with probability

$$a = \left(\frac{\binom{16}{4}\binom{12}{4}\binom{8}{5}}{4^{16}}\right)\left(\frac{4!}{2!}\right).$$

The probability that all four bowls contain the same number of cherries is

$$b = \left(\frac{\binom{16}{4}\binom{12}{4}\binom{8}{4}}{4^{16}}\right)\left(\frac{4!}{4!}\right).$$

We conclude

$$\begin{aligned}
\frac{a}{b} &= \frac{12 \cdot \binom{8}{5}}{\binom{8}{4}} \\
&= \frac{12 \cdot 56}{70} \\
&= \frac{48}{5}
\end{aligned}$$

Solution to Question 697: Infected Dinner I

The minimum amount of time would correspond to the most people becoming infected the fastest. This would be when each infected person talks to a healthy person at every step. After 1 minute, there would be 2 infected people. After 2 minutes, there would be 4 infected people, as each of the 2 infected people talks to someone healthy. In general, after n minutes, there would be 2^n infected people. We now just have to find the minimum n such that $2^n > 1000$, which is just $n = 10$.

Solution to Question 698: Confused Ant I

There are a total of eight vertices: one is the starting vertex denoted v_0, three are the vertices adjacent to the starting vertex denoted v_1, three are the vertices adjacent to the ending vertex denoted v_2, and one is the ending vertex denoted v_3. Let $E[v_i]$ be the expected number of edges travelled to arrive at v_3 from v_i. This problem has now been set up as calculating the expecting hitting time of a state within a Markov chain:

$$\begin{aligned}
E[V_0] &= 1 + \frac{1}{3} \times E[V_1] + \frac{1}{3} \times E[V_1] + \frac{1}{3} \times E[V_1] \\
E[V_1] &= 1 + \frac{1}{3} \times E[V_0] + \frac{1}{3} \times E[V_2] + \frac{1}{3} \times E[V_2] \\
E[V_2] &= 1 + \frac{1}{3} \times E[V_1] + \frac{1}{3} \times E[V_1] + \frac{1}{3} \times E[V_3] \\
E[V_3] &= 0
\end{aligned}$$

Substituting in $E[V_3]$ and solving, we see that $E[V_0] = 10$.

Solution to Question 699: Acceptance-Rejection Sampling

We need to find a such that it is closest to $f(x)$. In other words, a is just going to be the maximum of $f(x)$ on $[0, 1]$. We can quickly see that $f'(x) = 24x - 36x^2 = 12x(2 - 3x)$, so $f'(x) = 0$ when $x = 0, 2/3$. One can also see that $x = 0$ is a local minimum and $x = 2/3$ is a local maximum, so $f(x)$ is maximized at $x = 2/3$. This value is $a = f(2/3) = 16/9$.

For any pair (X, Y) that is generated, there is probability $\frac{1}{16/9} = \frac{9}{16}$ that it is accepted. Therefore, it takes an average of $\frac{16}{9}$ generated pair per accepted pair, meaning that there are $\frac{7}{9}$ rejected pairs per accepted pair on average.

Solution to Question 700: Smallest Factorizaiton

With a little bit of work, one can show that the prime factorization of $1234567890 = 2 \cdot 3^2 \cdot 5 \cdot 3607 \cdot 3803$. Therefore, if we were to multiply 3607 by $2 \cdot 5 = 10$ and 3803 by $3^2 = 9$, we should get two factors that are really close. These must be the closest because of the fact that our two factors must be multiples of 3607 and 3803, respectively, so we want to make those multiples as close as possible. The way to do this is by multiplying 3607 by a slightly larger number than 3803 as to close the gap. The arrangement listed the above gives the two factors that are closest together, so our two numbers are $3607 \cdot 10 = 36070$ and $3803 \cdot 9 = 34227$. Therefore, their sum is $36070 + 34227 = 70297$.

Solution to Question 701: Zero Appearance

We should perform complementary counting. There are 10000 positive integers at most 10000. There are 9 single-digit integers that don't have a 0. There are 9^2 two-digit integers without 0. There are 9^3 three-digit integers that don't have 0. Lastly, there are 9^4 four-digit integers that don't have 0. Therefore, the answer is

$$10000 - (9 + 9^2 + 9^3 + 9^4) = 2620$$

Solution to Question 702: Particle Reach IX

We are going to generalize this for $p > 1/2$. Let T_1 be the number of steps needed to move from position 0 to 1. We want $\text{Var}(T_1) = \mathbb{E}[T_1^2] - (\mathbb{E}[T_1])^2$. We know $\mathbb{E}[T_1] = \dfrac{1}{2p-1}$ from Particle Reach VI. We use Law of Total Expectation to condition on what happens at the first step.

If the particle moves right at the first step, which occurs with probability p, then $T_1 = 1$, as the particle has hit 1. Therefore, in this case, $T_1^2 = 1$ as well. Otherwise, with probability $1-p$, the particle moves left. The number of steps in this would be $1 + T_0 + T_1$, where T_0 is the number of steps needed to go from -1 to 0. In particular, this means that

$$\mathbb{E}[T_1^2] = p \cdot 1 + (1-p)\mathbb{E}[(1 + T_0 + T_1)^2]$$

Note that $\mathbb{E}[T_0] = \mathbb{E}[T_1]$ and $\mathbb{E}[T_0^2] = \mathbb{E}[T_1^2]$ by the Markov Property, so after expanding out everything, substituting, and rearranging, one gets that

$$\mathbb{E}[T_1^2] = \frac{-4p^2 + 6p - 1}{(2p-1)^3}$$

Therefore, $\text{Var}(T_1) = \mathbb{E}[T_1^2] - (\mathbb{E}[T_1])^2 = \dfrac{-4p^2 + 6p - 1}{(2p-1)^3} - \dfrac{1}{(2p-1)^2} = \dfrac{4p(1-p)}{(2p-1)^3}$. With $p = 2/3$, we have that $\text{Var}(T_1) = 24$.

Solution to Question 703: Greater Than

By symmetry, an ordering of X, Y, Z is equally likely to occur as any other ordering. There are a total of 6 orderings, only one of which satisfying $X \geq Y \geq Z$, so our answer is $\frac{1}{6}$.

Solution to Question 704: Complex Circle

We can write $Z = X+Yi$, where X and Y are real-valued and satisfy $X^2+Y^2 = 1$. We know that on the real unit circle, X and Y are uncorrelated and identically distributed. Thus, we have that

$$\mathbb{E}[Z^2] = \mathbb{E}[(X+Yi)^2] = \mathbb{E}[X^2 - Y^2] + 2i\mathbb{E}[XY] = 0$$

The first term vanishes by the identical distribution of X and Y, while the second term vanishes by X and Y being uncorrelated and mean 0.

The easiest solution is to note that the boundary of the unit circle maps to itself under $f(z) = z^2$, as every point on it can be written as $e^{i\theta}$ for some θ, so $Z^2 = e^{2i\theta}$, which has image just the boundary of the unit circle. Thus, $\mathbb{E}[Z^2] = \mathbb{E}[Z] = 0$, as the circle is symmetric in both components.

Solution to Question 705: Expecting Jacks

Suppose Bill does not draw a jack on the first draw (which occurs with probability $\frac{12}{13}$). Then, he must start over again. If Bill drew a jack on the previous draw, he must draw another jack to continue; otherwise, with probability $\frac{1}{13} \times \frac{12}{13}$, he must start over. We follow this reasoning for all four jack draws:

$$\begin{aligned}\mathbb{E}[X] &= \frac{12}{13}\left(\mathbb{E}[X]+1\right) + \frac{12}{13^2}\left(\mathbb{E}[X]+2\right) + \frac{12}{13^3}\left(\mathbb{E}[X]+3\right) + \frac{12}{13^4}\left(\mathbb{E}[X]+4\right) + \frac{1}{13^4}\cdot 4 \\ &= 30940\end{aligned}$$

Solution to Question 706: Option Dice III

First we need to calculate the true value of this call option. To do this, we first need to see how many combinations of dice rolls land us in profit. Listed in increasing order they are $(4,5)$, $(5,4)$, $(4,6)$, $(6,4)$, $(5,5)$, $(5,6)$, $(6,5)$, and $(6,6)$. With these combinations, a roll of $(4,5)$ or $(5,4)$ yields a profit of 1 units $(20-19=2)$ with a $\frac{2}{36}$ chance in doing so, and a roll of $(5,5)$ yields a profit of 6 units $(25-19=6)$ with a $\frac{1}{36}$ chance in doing so, a roll of $(5,6)$ or $(6,5)$ yields a profit of 11 units $(30-19=11)$ with a $\frac{2}{36}$ chance in doing so, and a roll of $(6,6)$ yields a profit of 17 units $(36-19=17)$ with a $\frac{1}{36}$ chance in doing so. Lastly, $(4,6)$ and $(6,4)$ yield profit of 5 units $(24-19=5)$ with probability $\frac{2}{36}$.

Putting this all together we have $\frac{2}{36}\cdot 1+\frac{2}{36}\cdot 5+\frac{1}{36}\cdot 6+\frac{2}{36}\cdot 11+\frac{1}{36}\cdot 17 = 19/12$, showing our contract should be priced at $19/12$.

To make a 2−wide market around this value, we buy 1 unit cheaper than the theoretical value of 19/12 and sell 1 unit more expensive than 19/12, which comes out to a market of $\frac{7}{12}$ @ $\frac{31}{12}$, and adding them together (the same as multiplying our theo by 2) yields our final answer of $\frac{19}{6}$.

Solution to Question 707: Mod Mondays

Every seventh day is a Monday. There will be $mod(100, 7) = 2$ days remaining after the last Monday, and thus 100 days after Monday is Wednesday (4).

Solution to Question 708: Bolt Variance II

The manufacturer is testing $H_0 : \sigma_1^2 = \sigma_2^2$ against $H_a : \sigma_1^2 > \sigma_2^2$. The appropriate test statistic is:

$$F = \frac{s_1^2}{s_2^2} = \frac{0.0003}{0.0001} = 3$$

Solution to Question 709: Uniform Summation

By directly applying LOTUS, $\mathbb{E}\left[\frac{1}{1-U}\right] = \int_0^{0.5} \frac{1}{1-x} \cdot 2dx = -2\ln(1-x)\Big|_0^{0.5} = -2\ln(0.5) = 2\ln(2) = \ln(4)$.

On the other hand, $\mathbb{E}\left[\frac{1}{1-U}\right] = \mathbb{E}\left[\sum_{k=0}^{\infty} U^k\right] = \sum_{k=0}^{\infty} \mathbb{E}[U^k]$.

We calculate $\mathbb{E}[U^k]$ by LOTUS as well. Namely, $\mathbb{E}[U^k] = \int_0^{0.5} 2x^k dx = \frac{2x^{k+1}}{k+1}\Big|_0^{0.5} = \frac{1}{2^k(k+1)}$. This exactly matches with the terms in the summation in the question. Therefore, the sum is $\ln(4)$, meaning $a = 4$.

Solution to Question 710: Sock Drawer II

Consider the number of ways for Sandy to pick out a matching pair on Monday and fail to do so on Tuesday and Wednesday. This is equal to the number of ways for her to pick out a matching pair on Wednesday and fail to do so on Monday and Tuesday, as any order of these three pairs of socks is equally likely. This is simply 5 matching pairs to choose from on Monday, $\binom{8}{2} - 4$ ways to choose an unmatched pair on Tuesday, and $\binom{6}{2} - 2$ ways to choose an unmatched pair on Wednesday. The total number of ways to choose pairs for the three days is $\binom{10}{2,2,2,4} = \frac{10!}{2!^3 \cdot 4!}$. This works out to

$$\frac{5 \cdot 24 \cdot 13}{45 \cdot 28 \cdot 15} = \frac{26}{315}$$

If you failed to notice the symmetry during an interview, it is possible to obtain the same answer through careful casework in conditioning on the results of each day.

Solution to Question 711: 5 Pairwise Sum

We can impose an ordering with $a \leq b \leq c \leq d \leq e$. We then know that $a+b=5$ and $d+e=22$, as the smallest two integers must add up to the smallest sum and the two largest integers must add up to the largest sum. Additionally, we have that every single of the a, b, c, d, and e appears in 4 of the 10 pairwise sums, as each integer is added up to every other integer but itself. Therefore, if we sum up all the values in the list, the result is $4 \cdot (a+b+c+d+e)$. Adding these integers up yields $5+11+11+13+13+14+16+19+22+22=146$, so $4 \cdot (a+b+c+d+e) = 146$. However, we know $a+b=5$ and $d+e=22$, so we can get that

$$4 \cdot (5+c+22) = 146 \iff c = 9.5$$

The trick now is to note that by the ordering we have, $a+c$ must be the second smallest sum possible after $a+b$, so we have that $a+c=11$, meaning $a=1.5$. Since we know $a+b=5$, we get that $b=3.5$. By similar logic, we know that the second largest sum after $d+e$ is $c+e$, so we know that $22=c+e$, meaning that $e=12.5$. Lastly, since $d+e=22$ as well, we know $d=c=9.5$. Putting it all together, we obtained the solution

$$(a, b, c, d, e) = (1.5, 3.5, 9.5, 9.5, 12.5)$$

Therefore, $a^2+b^2+c^2+d^2+e^2 = \frac{3^2+7^2+19^2+19^2+25^2}{2^2} = \frac{1405}{4}$

Solution to Question 712: Finite Coin Equalizer

We are going to use the result of Voter Mayhem (often called the "Ballot Theorem"). We can rephrase the question as follows: What is the probability that exactly n heads and n tails occur in $2n$ flips AND there are always strictly more heads than tails until the last flip. Let H be the event that n heads and n tails occur in $2n$ flips and M be the event that there are always more heads and tails until the last flip. We have that $\mathbb{P}[H \cap M] = \mathbb{P}[M \mid H]\mathbb{P}[H]$. $\mathbb{P}[H]$ is quite easy to compute, as we can use a binomial random variable PMF. In particular, if $X \sim \text{Binom}(2n, p)$,

$$\mathbb{P}[H] = \mathbb{P}[X = n] = \binom{2n}{n} p^n (1-p)^n$$

For $\mathbb{P}[M \mid H]$, we know that one of the two people must be ahead the entire time and the last equalizes. We can condition on the parity of the last coin. If it is tails, which occurs with probability $1-p$, then the probability is $P_{n,n-1}$, where the heads must be ahead the entire time. If it is tails, which occurs with probability p, then the probability is $P_{n,n-1}$ as well, where the tails must be ahead the entire time. Therefore, adding the two cases up, the probability is $\mathbb{P}[M \mid H] = P_{n,n-1} = \frac{1}{2n-1}$. Therefore

$$\mathbb{P}[M \cap H] = \frac{\binom{2n}{n} p^n (1-p)^n}{2n-1}$$

Evaluating this with $p = 1/3$ and $n = 5$, we get the answer of approximately 0.0152.

Solution to Question 713: Bus Wait II

Let G be the event of refilling on gas and T be the time it takes for the bus to arrive. We are going to condition on showing up on a turn when the bus refills for gas vs. when it doesn't. On average, 1 in every 10 trips will have a gas refill. In this case, the trip length is 70 minutes.

Therefore, in 9 of every 10 trips, the bus doesn't refill. Therefore, on average, in every 160 minutes, 70 of those will be when the bus is on a refill trip. Therefore, the probability of you appearing during a refill trip is $\frac{7}{16}$. The expected wait of that trip would be $\frac{1}{2} \cdot 70 = 35$. Then, with probability $\frac{9}{16}$ the trip is regular and the expected wait is 5 minutes as per the previous part of this question. Therefore, by Law of Total Expectation, the total wait time is

$$35 \cdot \frac{7}{16} + 5 \cdot \frac{9}{16} = \frac{145}{8}$$

Solution to Question 714: Circular Slice II

From Circular Slice I, we know that the probability that the two regions are disjoint is $\frac{1}{6}$. Therefore, the probability they are not disjoint (i.e. intersect at least once) is $\frac{5}{6}$. Now, we need to compute the probability that they intersect at exactly one part. We know the probability they are disjoint is $\frac{1}{6}$. Thus, we just need to find the probability that they intersect at possible parts (the endpoints of 0 radians and θ_1 radians). Just as in Circular Slice I, we will treat these as $\text{Unif}(0,1)$ random variables instead of $\text{Unif}(0,2\pi)$ random variables by scaling them down.

For there to be an intersection at the left end, we need $\alpha < \theta_1$. This is because then the start of the blue region will still be in the red region. To have an intersection at the right endpoint i.e. at proportion 1 of the circle, we need $\alpha + \theta_2 > 1$. This is because this arc needs to wrap back around to the point $(1,0)$. Therefore, to find the probability of 2 intersections, we are looking for $\mathbb{P}[\theta_1 > \alpha, \theta_2 > 1-\alpha]$. Similar to the previous question, we will need to condition on the value of α to obtain this probability. This means that

$$\mathbb{P}[\theta_1 > \alpha, \theta_2 > 1-\alpha] = \int_0^1 \mathbb{P}[\theta_1 > \alpha, \theta_2 > 1-\alpha \mid \alpha = x] f_\alpha(x)dx$$

Evaluating the interior probability, this becomes $\mathbb{P}[\theta_1 > x, \theta_2 > 1-x] = \mathbb{P}[\theta_1 > x]\mathbb{P}[\theta_2 > 1-x] = x(1-x)$. Therefore, the integral is $\int_0^1 x(1-x)dx = \frac{1}{6}$.

Combining all of the above, we now see the probability of exactly one purple region is $1 - \frac{1}{6} - \frac{1}{6} = \frac{2}{3}$, so by the formula for conditional probability, given that there was at least one purple region, the probability there is exactly one is $\frac{\frac{2}{3}}{\frac{5}{6}} = \frac{4}{5}$.

Solution to Question 715: Convergent intervals

The point that the intervals converge to will become arbitrarily close to the center of our most recent interval. If our current interval is of length x, the distance of the next interval's center to the current interval's center is distributed $U(0, x/4)$. Recalling that the variance of a continuous uniform random variable is $\frac{(b-a)^2}{12}$, we can consider the point of convergence as an infinite sum of uniformly distributed random variables.

$$\text{Var}\left(\sum_{n=0}^{\infty} X_i\right) = \sum_{n=0}^{\infty} \text{Var}(X_i) = \sum_{n=0}^{\infty} \frac{(1/4 \cdot (1/2)^n)^2}{12} = \frac{1}{36}$$

Solution to Question 716: Parking Rush

Let E_1, E_2, E_3 denote the times at which the 3 executives leave. Let $L_1, L_2, \ldots, L_9$ denote the times at which the 9 latecomers arrive. Note that since $E_1, E_2, E_3, L_1, L_2, \ldots, L_9 \overset{\text{iid}}{\sim}$ $\text{Unif}([12, 15])$, by symmetry, any ordering of $E_1, E_2, E_3, L_1, L_2, \ldots, L_9$ by increasing time of day occurs with equal probability.

Suppose we are ordering 3 Es and 9 Ls. Consider any arbitrary ordering of the 3 Es and 9 Ls. If the ordering allows all spots to be filled by 3:00 PM, then we can assign Andy to 3 of the 9 Ls. So, the probability that Andy gets a parking spot conditioned on the event that all spots are filled by 3:00 PM is $\frac{1}{3}$.

In the other case, the probability should intuitively be less than $\frac{1}{3}$, because not all parking spots can be filled. For example, consider the following ordering: $LLLLLLLLLEEE$. No matter which L we assign Andy to, there is no chance that he will find an empty parking spot.

Now that we've gained a bit more intuition, we can now properly divide our problem up into 4 cases: (A) there are 3 empty parking spots at 3:00 PM, (B) there are 2 empty parking spots at 3:00 PM, (C) there is 1 empty parking spot at 3:00 PM, and (D) there are no empty parking spots at 3:00 PM. Note that there are a total of $\binom{12}{3} = 220$ possible orderings of 3 Es and 9 Ls. Let S denote the event that Andy finds a parking spot.

Case A: There are 3 empty parking spots at 3:00 PM. The only way for there to be 3 empty parking spots at 3:00 PM is with the following ordering: $LLLLLLLLLEEE$. This ordering occurs with probability $\frac{1}{220}$. Then,

$$\begin{aligned}
\mathbb{P}(S \cap A) &= \mathbb{P}(S \mid A)\,\mathbb{P}(A) \\
&= 0 \cdot \frac{1}{220} \\
&= 0
\end{aligned}$$

Case B: There are 2 empty parking spots at 3:00 PM. One way for this to occur is if the last two letters are both E, of which there are $\binom{10}{9} = 10$ possible orderings. But this includes the case where the last three letters are all E, so we must subtract 1. In addition, $LLLLLLLLEELE$, $LLLLLLLLEEEL$ work. In this case, there is exactly one L that Andy may be assigned to such that he is able to park. Hence,

$$\begin{aligned}\mathbb{P}(S \cap B) &= \mathbb{P}(S \mid B)\,\mathbb{P}(B) \\ &= \frac{1}{9} \cdot \frac{11}{220} \\ &= \frac{1}{180}\end{aligned}$$

Case C: There is 1 empty parking spot at 3:00 PM. One way for this to occur is if the last letter is an E, which occurs in $\binom{11}{2} = 55$ orderings. But this includes the case where two or more of the final letters are E, as well as the case $LLLLLLLLEELE$, so we must subtract $10 + 1 = 11$. In addition, $LLLLLLLEEELL$, $LLLLLLLEELEL$, as well as any ordering ending with EEL except for the ordering ending with $EEEL$ (there are $\binom{9}{8} - 1 = 8$ such orderings) work. If one parking spot is vacant at 3:00 PM, then there are 2 Ls that Andy may be assigned to such that Andy has a parking spot.

$$\mathbb{P}(S \cap C) = \mathbb{P}(S, |, C)\,\mathbb{P}(C) \quad = \frac{2}{9} \cdot \frac{54}{220} \;= \frac{3}{55}$$

Case D: There are no empty parking spots at 3:00 PM. There are $220 - 54 - 11 - 1 = 154$ orderings remaining. As previously discussed, if no parking spots are empty at 3:00 PM, then there are 3 Ls that Andy may be assigned to such that Andy has a parking spot.

$$\mathbb{P}(S \cap D) = \mathbb{P}(S, |, D)\,\mathbb{P}(D) \quad = \frac{3}{9} \cdot \frac{154}{220} \;= \frac{7}{30}$$

Now, we may employ the law of total probability.

$$\mathbb{P}(S) = \mathbb{P}(S \cap A) + \mathbb{P}(S \cap B) + \mathbb{P}(S \cap C) + \mathbb{P}(S \cap D) \quad = 0 + \frac{1}{180} + \frac{3}{55} + \frac{7}{30} \;= \frac{581}{1980}$$

The complement of the union of these events is:

$$1 - \frac{581}{1980} = \frac{1399}{1980}$$

Solution to Question 717: Perfect Correlation III

Suppose that X and Y have respective variances σ_X^2 and σ_Y^2. Then $\mathrm{Cov}(X + Y, X - Y) = \mathrm{Cov}(X, X) - \mathrm{Cov}(Y, Y) = \sigma_X^2 - \sigma_Y^2$. Similarly, we know that $\mathrm{Var}(X + Y) = \sigma_X^2 + \sigma_Y^2 - 2\sigma_X\sigma_Y = (\sigma_X - \sigma_Y)^2$, so $\sigma_{X+Y} = |\sigma_X - \sigma_Y|$. Note that the absolute values are needed here since we can't have a negative variance. By a similar argument, we get $\mathrm{Var}(X - Y) = \sigma_X^2 + \sigma_Y^2 + 2\sigma_X\sigma_Y = (\sigma_X + \sigma_Y)^2$, so $\mathrm{Var}(X - Y) = \sigma_X + \sigma_Y$.

Therefore, we have that $\rho(X+Y, X-Y) = \dfrac{\sigma_X^2 - \sigma_Y^2}{(\sigma_X + \sigma_Y)|\sigma_X - \sigma_Y|} = \dfrac{\sigma_X - \sigma_Y}{|\sigma_X - \sigma_Y|} = -1$. We get the -1 from the condition that $\sigma_Y > \sigma_X$, so the numerator is negative.

Another way to see this is that if X and Y are perfectly correlated, then $Y = aX + b$ for some constants a and b. Since $\sigma_Y > \sigma_X$, it must be the case that $|a| > 1$. Therefore, $X + Y = X(1 + a) + b$ and $X - Y = X(1 - a) + b$. As $|a| > 1$, one of the constants between $1 + a$ and $1 - a$ is positive and one is negative. Therefore, as these are both linear transformation of X with opposing signs, they must have correlation -1. Note that this argument does not depend at all on whether or not the correlation of X and Y is -1 or 1.

Solution to Question 718: 100 Lights

The i-th bulb is switched by person j if j is a factor of i. Thus, the i-th bulb will remain on if it has an odd number of factors. We call a number that has an odd number of factors a perfect square, and there are 10 perfect squares from 1 to 100.

Solution to Question 719: Careful Coin Question

Fix some subset A of the sample space that is non-empty and not the entire sample space. Suppose our question is "Is our sequence $\omega \in A$?" If $|A|$ denotes the number of sequences in A, then the probability we guess correctly is

$$\frac{|A|}{2^{100}} \cdot \frac{1}{|A|} + \left(1 - \frac{|A|}{2^{100}}\right) \cdot \frac{1}{2^{100} - |A|} = \frac{2}{2^{100}} = \frac{2}{2^{100}} = 2^{-99}$$

Regardless of which subset we restrict to, we get a 2^{-99} probability of guessing correctly. Therefore, our answer is $2 \cdot 99 = 198$.

Solution to Question 720: Price an Option III

To price this asset, we need to find a time-T replication, and by no-arbitrage, we can find the time-0 price using the portfolio of T_0 prices.

We see that we can replicate this asset by going long 15 units of the bond, shorting 1 unit of the underlying, and going long 2 units of the $K = 15$ call. In other words:

$$|x - 15| = 15 - x + 2\max(S_T - 15, 0)$$

Using the time-0 prices, we get $15(0.9) - 7 + 2(0.7) = 7.9$.

Upon closer inspection, one can see that this is the same as a straddle with strike $K = 15$. Typically, this can be created by going long a call and put at the same strike K. Here, we do not have access to the put, and instead, we are replicating a put with the underlying and a bond (this is a forward contract).

Solution to Question 721: Poisoned Kegs II

The idea here is that we can number of kegs in binary and 0−index them. In other words, label the kegs 0 to $n-1$. With 10 servants, we can have up to 10 digits in our binary expansion of $n-1$. Similarly, we can label the people $0-9$ and write keg label $i = a_{0i} + a_{1i} \cdot 2^1 + \cdots + a_{9i} \cdot 2^9$, where each of $a_{0i}, a_{1i}, \ldots, a_{9i} = 0, 1$. We then give wine from urn i to all servants whose indices are in the subset

$$S_i = \{0 \leq k \leq 9 : a_{ki} = 1\}$$

For example, if $i = 17 = 0000010001_2$, we would give wine from keg 17 to servants 0 and 4. Afterwards, the sequence of servants that die lets us uniquely see which keg is poisoned. For example, if servants 0 and 4 die, this means that keg 17 is poisoned. We essentially are creating a one-to-one mapping between the kegs and binary sequences of length 10, so this means that $n = 2^{10} = 1024$ is the maximum number of kegs we can test. We can't test any more than this because there are only 2^{10} possible subsets of the servants, meaning that we would need two subsets of servants to test two different bottles if $n > 2^{10}$, which wouldn't allow us to uniquely identify it.

Solution to Question 722: The Last Roll

This is because 6 can reach 1000 from a current sum of anything between $994 - 999$. Conversely, a 1 only works if the current sum is 999. Having a smaller value on the last roll reduces the possibilities of the previous sums before exceeding 1000.

Solution to Question 723: Basic Delta Hedging

An important fact to remember is that the Δ of an at-the-money call option is 0.5 (derived from Black-Scholes). This means that for every 1 move in the underlying, the call option will move by 0.50. We want an overall portfolio Δ of 0. Since Δ of a portfolio is the Δ of its parts, the 10 calls have an overall Δ of 5. To obtain an overall portfolio Δ of 0, we need to obtain an asset of $\Delta = -5$. This is obtained by shorting 5 units of the underlying as the underlying is $\Delta = 1$.

Solution to Question 724: Conditioned Heads

We can interpret this information as saying that we have obtained 5 heads in the first 6 flips of our fair coin. We want the probability that in 9 flips we obtain exactly 6 heads. This is equivalent to saying that we want the probability that we observe one head in the next 3 flips of our fair coin. The probability of this, using the Binomial PMF or by calculating the combinatorics, is $\binom{3}{1}\frac{1}{2^3} = \frac{3}{8}$.

Solution to Question 725: Discrete Walker

This is just a standard symmetric random walk problem. The probability that 10 is hit before 0 starting at 6 is just $\frac{6}{10}$ by considering the ratio of the distance from the lower boundary (0) to our starting position (6) to the entire length of our region (10).

Solution to Question 726: Cone Combo

From geometry, we know that the volume of a cone with height h and base radius r is $\frac{1}{3}\pi r^2 h$. Therefore, as $r = 1$ and $h = 2$ here, the volume of this cone is $\frac{2\pi}{3}$. We also know that the radius decreases linearly with height from the bottom. In other words, we know that the radius as a function of the height x from the base, say $r(x)$, is linear. This means it is in the form $r(x) = c_0 + c_1 x$. Since $r(0) = 1$ (we are at the base) and $r(2) = 0$ (we are at the tip), we get that $r(x) = 1 - \frac{x}{2}$ by solving the resultant linear system.

Consider the complement of our event, which is that we are above a height 1 from the base. This region is another conic region. However, the base radius is now $1 - \frac{1}{2} = \frac{1}{2}$ and our height is 1, so the volume of the conic region that is complementary to our region of interest is $\frac{1}{3} \cdot \pi \left(\frac{1}{2}\right)^2 \cdot 1 = \frac{\pi}{12}$. Therefore, as our selection of the point is uniform throughout the volume of the cone, the probability that we select a point above height 1 from the base is $\frac{\frac{1}{12}}{\frac{2}{3}} = \frac{1}{8}$. Therefore, the probability of the event we are interested in is $\frac{7}{8}$.

Solution to Question 727: Theater Seating

There are two possible choices of seats where the boys can sit (either positions 1, 3, 5, 7, 9 or 2, 4, 6, 8, 10); the girls must choose the other. For each choice, there are 5! possible permutations of the boys and 5! permutations of the girls. Thus, the total number of seating arrangements is:

$$2 \times (5!)^2 = 28800$$

Solution to Question 728: Ant Collision II

Imagine that whenever two ants collide they change bodies. This is a reasonable assumption because of the fact in each collision, both ants will move in the direction of the other ant after the collision. Therefore, it is as if no collision happened at all in this paradigm. With this, we can see that every one of Alice's ants will collide with every one of Bob's ants. This means that there are $40 \cdot 80 = 3200$ total ant collisions.

Solution to Question 729: Pair Die Sum

There is symmetry in this game, as the dice are identical besides their color. Therefore, if X and Y are the sums of the upfaces of the blue and red dice, respectively, then we want $\mathbb{P}[X > Y]$. Note that by symmetry, $\mathbb{P}[X > Y] = \mathbb{P}[X < Y]$. The only missing case here to be exhaustive is $\mathbb{P}[X = Y]$. Therefore, we have that

$$\mathbb{P}[X > Y] = \frac{1 - \mathbb{P}[X = Y]}{2}$$

by substituting and rearranging in $\mathbb{P}[X > Y] + \mathbb{P}[X = Y] + \mathbb{P}[X < Y] = 1$. It now remains to evaluate $\mathbb{P}[X = Y]$. This is just a simple case of needing to count up all of the possible sums that they could equal. In particular, $\mathbb{P}[X = Y] = \sum_{k=2}^{12} \mathbb{P}[X = k]\mathbb{P}[Y = k]$ by the independence of X and Y. Using the PMF of the sum of two dice, we evaluate this to be $\dfrac{1^2 + 2^2 + \cdots + 6^2 + \cdots + 2^2 + 1^2}{36^2} = \dfrac{146}{1296}$. Therefore,

$$\mathbb{P}[X > Y] = \frac{1 - \frac{146}{1296}}{2} = \frac{575}{1296}$$

Solution to Question 730: 20-30 Die Split III

Let W be the event Alice wins. First, we want to find $\mathbb{P}[W]$. Let A and B be value of Alice's roll and Bob's first roll, respectively. The key here is to condition on whether or not $A \leq 20$. Namely, this means

$$\mathbb{P}[W] = \mathbb{P}[W \mid A \leq 20]\mathbb{P}[A \leq 20] + \mathbb{P}[W \mid A > 20]\mathbb{P}[A > 20]$$

We can quickly see that $\mathbb{P}[A \leq 20] = \frac{2}{3}$, as this accounts for 20 of 30 equally-likely values. If $A > 20$, then clearly Alice wins regardless of what Bob obtains, so $\mathbb{P}[W \mid A > 20] = 1$.

If $A \leq 20$, then we are really rolling 2 independent 20−sided fair dice. Namely, to compute $\mathbb{P}[W \mid A \leq 20]$, we see that the probability of a tie is $\frac{1}{20}$, as there are 20 values they can agree upon out of $20^2 = 400$ outcomes. Therefore, the probability the two rolls are different is $\frac{19}{20}$. Because of the fact that the two dice are symmetric, $\mathbb{P}[A > B] = \frac{1}{2} \cdot \frac{19}{20} = \frac{19}{40}$, as when the two rolls are not the same, it is equally-likely whether $A > B$ or $A < B$.

Combining these all together, we see that $\mathbb{P}[W] = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot \frac{19}{40} = \frac{39}{60}$. Now, Bob is going to roll again whenever $\mathbb{P}[W \mid B = b] > \mathbb{P}[W] = \frac{39}{60}$. This is because he wants to minimize Alice's chance of winning. It is easy to see that $\mathbb{P}[W \mid B = b] = 1 - \frac{b}{30}$, as Alice just needs to roll a value of at least $b + 1$.

We just need to find the maximum value of b, say b^*, such that $\mathbb{P}[W \mid B = b] = 1 - \frac{b}{30} > \frac{39}{60}$. Solving this inequality, we get that $b \leq 10.5$, so $b^* = 10$ is the maximum value that Bob rolls again. Let W_1 be the event that Alice wins in this new game where Bob can roll again. We have that

$$\mathbb{P}[W_1] = \mathbb{P}[W] + \frac{10}{20}\left(\mathbb{P}[W] - \mathbb{P}[W \mid B \leq 10]\right)$$

We get this because of the fact that Bob rolls again with probability $\frac{1}{2}$, which is when he rolls at most 10. Then, the probability Alice wins is going to change by $\mathbb{P}[W] - \mathbb{P}[W \mid B \leq 10]$.

To calculate $\mathbb{P}[W \mid B \leq 10]$, this is simply just average all the cases where $B = b$ for $1 \leq b \leq 10$, as when $B \leq 10$, the value is uniformly distributed on the first 10 positive integers. Therefore,

$$\mathbb{P}[W \mid B \leq 10] = \frac{1}{10}\sum_{b=1}^{10} 1 - \frac{b}{30} = \frac{49}{60}$$

Therefore, we get that

$$\mathbb{P}[W] = \frac{39}{60} + \frac{1}{2}\left(\frac{39}{60} - \frac{49}{60}\right) = \frac{17}{30}$$

Solution to Question 731: Pricing Put Spread

We can plot the payout and see that the maximum payout is 4 when $S_T < 20$. So, the best upper-bound at time-T is 4. By no arbitrage, this relationship must hold to time $t = 0$. Since this is a constant, this acts like a bond and we should apply the discount factor. This gives us our answer of $V_0 = 4 * 0.9 = 3.6$.

This then gives us another no-arbitrage relationship for puts of different strikes:

$$P(K_2) - P(K_1) \leq (K_2 - K_1)B$$

where $K_2 > K_1$ and B is the discount factor.

Solution to Question 732: Gambling Addiction

We use the fact that for non-negative integer-valued random variables, $\mathbb{E}[X] = \sum_{k=1}^{\infty} \mathbb{P}[X \geq k]$. Let's compute $\mathbb{P}[W \geq k]$ for a fixed k. This is equivalent to saying that Gabe loses each of his first $k-1$ games.

On turn k, $k \geq 1$, there is a probability of $\frac{1}{k+2}$ of winning (we select uniformly at random from $k+2$ items, and only 1 will be selected). Thus, round k, there is a probability of $1 - \frac{1}{k+2} = \frac{k+1}{k+2}$ of not winning. Therefore, the probability Gabe loses each of his first $k-1$ rounds is

$$\mathbb{P}[W \geq k] = \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \cdots \cdot \frac{k-1}{k} \cdot \frac{k}{k+1} = \frac{2}{k+1}$$

Plugging this into our expectation formula, we see that

$$\mathbb{E}[W] = \sum_{k=1}^{\infty} \mathbb{P}[W \geq k] = \sum_{k=1}^{\infty} \frac{2}{k+1} = \infty$$

We know this sum diverges as it is asymptotic to the harmonic sum. Therefore, our answer to this question is -1.

Solution to Question 733: Clock Angle II

We know that both hands will be very close to each other but should note that the hour hand will be slightly further clockwise than the minute hand because its been 15 minutes into the 3^{rd} hour. In fact, it is precisely one-fourth of the way between the 3 and 4. Since there are 12 total space between numbers of the clock, each space has to occupy 30°. Thus, $\frac{1}{4} \cdot 30° = 7.5°$.

Solution to Question 734: Particle Reach VIII

Let T be the total number of steps needed to go from position 0 to 9. We can write $T = T_1 + \cdots + T_7$, where T_i is the number of steps needed to move from position $i - 1$ to position i. By the Markov Property and Linearity of Expectation, $\mathbb{E}[T] = 7\mathbb{E}[T_1]$. $\mathbb{E}[T_1] = 3$ by the results of Particle Reach VI, so the answer is 21.

Solution to Question 735: Squid Game II

Let's start with person 1. We know that since they are leading, they must get everything right in order to be the first one across so

$$\mathbb{P}[1 \text{ win}] = \frac{1}{2^{10}}$$

Next, the only way person 2 can win is if person 1 is eliminated before reaching the end, and person 2 gets the remaining tiles correct. So, suppose person 1 is eliminated at tile $k \in$ where $1 \leq k \leq 10$. The probability of person 1 making to tile k is

$$\underbrace{\left(\frac{1}{2}\right)^{k-1}}_{\text{k-1 correct}} * \underbrace{\frac{1}{2}}_{\text{last incorrect}}$$

so then the probability of person 2 winning is getting all remaining ones correct so we have $\mathbb{P}[2 \text{ win}|1 \text{ ends at k}] = (\frac{1}{2})^{10-k}$.

Applying conditional probability, we observe that

$$\mathbb{P}[2 \text{ win}] = \sum_{k=1}^{10} \mathbb{P}[2 \text{ win}|1 \text{ ends at k}]\mathbb{P}[1 \text{ ends at k}] = \sum_{k=1}^{10} \left(\frac{1}{2}\right)^{10-k} \left(\frac{1}{2}\right)^{k} = 10{*}\left(\frac{1}{2}\right)^{10}$$

Starting to generalize, we consider $\mathbb{P}[3 \text{ win}]$ and like before, 3 can win only if both person 1 and 2 are eliminated before reaching the end **and** person 3 gets the remaining correct. Analyzing as before, let's suppose person 1 makes it to tile i and person 2 makes it to tile j with $i < j$ and $i, j \in$ where $1 \leq i, j, \leq 10$.

It follows from the previous argument that $\mathbb{P}[\text{1 ends at i}] = \left(\frac{1}{2}\right)^i$ and $\mathbb{P}[\text{2 ends at j}|\text{1 ends at i}] = (\frac{1}{2})^{j-i}$. Now, we see that

$$P[\text{3 win}] = \sum_{i=1}^{10} \sum_{j=i+1}^{10} \mathbb{P}[\text{3 win}|\text{2 ends at j}] P[\text{2 ends at j}|\text{1 ends at i}] P[\text{1 ends at i}] = \binom{10}{2} (\frac{1}{2})^{10}$$

The core idea here is that if we want to determine the $\mathbb{P}[\text{i win}]$ this is equivalent to figuring how many ways you can arrange the $i-1$ people before i to fail. Each of these outcomes has equal probability mass of $(\frac{1}{2})^{10}$.

Thus, we have that $\mathbb{P}[\text{i win}] = \binom{10}{i-1}(\frac{1}{2})^{10}$ and we know that $\binom{10}{i-1}$ is maximized when $i - 1 = 5 \implies$ person 6 has the highest probability of winning with a probability of

$$\binom{10}{6-1}\left(\frac{1}{2}\right)^{10} = \frac{252}{1024} = \frac{63}{256}$$

Solution to Question 736: Limiting Random Variable II

Define $Z_n = X_{4n-3}^2 + X_{4n-2}^2 X_{4n-1} X_{4n}$. Then $Y_n = \dfrac{Z_1 + \cdots + Z_n}{n}$. by substituting in. Therefore, as the Z_i random variables are IID, $\text{Var}(Y_n) = \dfrac{1}{n^2} \cdot n\text{Var}(Z_1) = \dfrac{\text{Var}(Z_1)}{n}$. We showed that $\mathbb{E}[Z_1] < \infty$ in the first part of this question. As $\text{Var}(Z_1) = \mathbb{E}[(X_1^2 + X_2^2 X_3 X_4)^2] - (\mathbb{E}[Z_1])^2$, these quantities here are all related to the first 4 moments of the X_i random variables, which we know are all finite. Therefore, $\text{Var}(Z_1) < \infty$. This lastly implies that $\lim_{n\to\infty} \text{Var}(Y_n) = 0$, as it decays like a constant divided by n.

Solution to Question 737: Unique-ish Solution

By the rank-nullity theorem, we know that $\text{rank}(A) + \text{null}(A) = n$. We know $\mathcal{R}(A) \neq \mathbb{R}^3$, as $Ax = e_1$ does not have a solution. However, since $Ax = e_2$ does have a solution, we can conclude that $\mathcal{N}(A) = \{0\}$. This means that $n = \text{rank}(A)$ and that $n \leq 3$ from the fact that $\text{rank}(A) \leq \min\{m, n\}$ for a $m \times n$ matrix. However, we also know that $n \geq 2$ from the question. If $\text{rank}(A) = 3$, then $\mathcal{R}(A) = \mathbb{R}^3$, which is a contradiction to what we previous stated, so this means $n = \text{rank}(A) = 2$.

Solution to Question 738: Digit Multiplication II

The prime factorization of 96 is $3 \cdot 32 = 3 \cdot 2^5$. Therefore, we need to have 5 twos and a three in our number accounted for. For the largest possible value, we should arrange the values in descending order left to right, as this would give the largest weight to digits of largest value. Therefore, our largest value is 322222. For the smallest value, note that $2^3 = 8$ and $2 \cdot 3 = 6$. Therefore, we condense down 4 of the twos and the three into an 8 and 6. We can't condense more, as otherwise those digits will be larger than 10, so the smallest value that can be made from 8, 2, and 6 is clearly 268. Thus, our answer is $322222 - 268 = 321954$.

Solution to Question 739: Intersecting Intervals

Observe that it is the relative order of the selected points that is relevant. We can think of this as valid permutations of five A_i and five B_i, where A_i and B_i represent the left endpoint and right endpoint of the ith interval, respectively. Note that for a permutation to be valid, B_i must come after A_i. Of the 10! permutations, only $\frac{1}{2^5}$ of these satisfy this requirement, as any pair A_i, B_i is in the correct order with probability $\frac{1}{2}$ independent of the rest. For an intersection to exist, it is necessary and sufficient that all A_i come before all B_i. There are $(5!)^2$ permutations for which this holds, all of which are valid. Hence, our probability is

$$\frac{\text{Valid permutations with non-empty intersection}}{\text{Valid permutations}} = \frac{2^5 \cdot (5!)^2}{10!} = \frac{8}{63}$$

Our desired answer is $8 + 63 = 71$.

Solution to Question 740: Stranded at Sea

Let p be the probability that you do not see a boat in 10 minutes. Then the probability of not seeing a boat within one hour is:

$$p^6 = 1 - .737856 \Rightarrow p = 0.8$$

The probability that you will see a boat within the next 10 minutes is:

$$1 - p = 0.2$$

Solution to Question 741: Regional Manager II

We are given that $\mu = 15, \bar{y} = 17, s^2 = 9, n = 36$, and $\alpha = 0.05$. Because n is sufficiently large, he can utilize the Z statistic. The rejection region is:

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\bar{x} - 15}{3/\sqrt{36}} > 1.645 \Rightarrow \bar{x} > 15.82$$

The probability of a Type II error, also known as β, is:

$$P(\bar{x} \leq 15.82 \mid H_0 = 16) = P(z \leq \frac{15.82 - 16}{3/\sqrt{36}}) = P(z \leq -0.36) \approx 0.3594$$

Solution to Question 742: Random Determinant

A first observation is that A and A^T have the same elements on the diagonal, so the diagonal elements are 0. Therefore, if $A = [A_{ij}]_{i,j=1,2}$, then $\det(A - A^T) = 0 - (A_{12} - A_{21})(A_{21} - A_{12}) = (A_{12} - A_{21})^2$. Thus, all we are really looking for is $\mathbb{E}[(A_{12} - A_{21})^2]$, where $A_{12}, A_{21} \sim \text{Bernoulli}(0.5)$. $(A_{12} - A_{21})^2 = 1$ exactly when $A_{12} \neq A_{21}$, as the difference will be either 1 or -1. This occurs with probability $\frac{1}{2}$. Otherwise, if they are equal, then $(A_{12} - A_{21})^2 = 0$, also occurring with probability $\frac{1}{2}$. Therefore, $\mathbb{E}[\det(A - A^T)] = \mathbb{E}[(A_{12} - A_{21})^2] = \frac{0+1}{2} = \frac{1}{2}$.

Solution to Question 743: Triangular Selection II

The complement of this event is that Matt and Aaron select disjoint triangles. Let Aaron's triangle be fixed arbitrarily. This consists of 3 of the 6 points in our space. Of the $\binom{6}{3} = 20$ ways that Matt can select his triangle, exactly 1 has no vertices in common with Aaron's triangle. Namely, he has to pick the 3 vertices Aaron did not pick. Therefore, the answer is just $1 - \frac{1}{20} = \frac{19}{20}$.

Solution to Question 744: Colosseum Fight II

In this game, instead of giving the strength x gladiator x balls, we will give him a light bulb whose lifetime is $\text{Exp}(1/x)$ distributed. We select this parameter so the mean is x. When two gladiators fight, we turn on their light bulbs, and the one goes out first is the loser. Once a gladiator wins, we immediately turn off his light bulb and then turn it on again when he is chosen for his next fight. The memorylessness property says that the excess life of this bulb (i.e. how long the bulb has left of life) is still exponential with the same parameter, so this is a valid way to encapsulate the constancy of strength after a fight.

One can show fairly easily that if $X_1 \sim \text{Exp}(\lambda_1)$ and $X_2 \sim \text{Exp}(\lambda_2)$, then $\mathbb{P}[X_1 < X_2] = \frac{\lambda_1}{\lambda_1 + \lambda_2}$. We apply this with $\lambda_1 = 1/x$ and $\lambda_2 = 1/y$ i.e. X_1 and

X_2 respectively represent the light bulbs of gladiators with strengths x and y. We thus get that

$$\mathbb{P}[X_1 < X_2] = \frac{1/x}{1/x + 1/y} = \frac{y}{x+y}$$

which represents the probability the strength y gladiator wins, exactly like we wanted!

During the tournament, Bob and Alice have one light bulb active at all times. The winner is exactly who has the greater total lighting time (the sum of all time intervals where a gladiator for their team has their light bulb active). Since the exponentials are memoryless, the order of selection is irrelevant. Therefore, there is no optimal strategy.

Solution to Question 745: Double Dice Payoff

First we need to find the probability of every outcome. Double 6s occur with probability $(\frac{1}{6})^2 = \frac{1}{36}$. We can count the number of 6 and non-6 combos there are and youâll see there are 10 (1-5 on the first die and 6 on the other with two different orderings). This gives that event a probability of $\frac{10}{36}$. Since the other event is just a recurrence of these two events, we can normalize the probabilities of the two events we care about. We do this by finding the probability of our target event and divide it with the probability of the non-recurrent events. So the probability of a double 6s in this case can be updated to $\frac{\frac{1}{36}}{(\frac{1}{36} + \frac{10}{36})} = \frac{1}{11}$. This gives the probability the game ends in a 6 and non 6 of $\frac{10}{11}$. Finally, we can solve for x. The profit equation here is $100 \cdot \frac{1}{11} - x \cdot \frac{10}{11}$ which has to be greater than or equal to 0. Thus $\frac{100}{11} \geq \frac{10x}{11}$. The maximum value of x is thus \$10.

Solution to Question 746: Always Profit I

Something to notice about this situation is that being long on the stock is better than betting the stock goes up against your friend because in the case the stock goes down, youâll still retain 50% of your investment in the stock. Thus we should look to go long on the stock and bet the stock goes down against your friend. The profit equations are as follows: if the stock goes up, you make $A - B$; if the stock goes down, you make $B - 0.5A$. Making these two equations equal, we get $A = 4B/3$. Thus A and B are equal to \$4 and \$3 respectively which allows us to always profit \$1.

Solution to Question 747: Covariance Review IV

We will present two solutions: one using just the basic definition of covariance, and another using properties of covariance of linear combinations of random variables.

Recall

$$\mathrm{Cov}(Y_1, Y_2) = \mathbb{E}[Y_1 Y_2] - \mathbb{E}[Y_1]\mathbb{E}[Y_2]$$

and

$$\rho_{Y_1,Y_2} = \frac{\mathrm{Cov}(Y_1, Y_2)}{\sigma_{Y_1}\sigma_{Y_2}}.$$

Then,

$$\begin{aligned}
\mathrm{Cov}(Y_1, Y_2) &= \mathbb{E}[(1+2X_1)(3-4X_2)] - \mathbb{E}[1+2X_1]\mathbb{E}[3-4X_2] \\
&= \mathbb{E}[3] + 6\mathbb{E}[X_1] - 4\mathbb{E}[X_2] - 8\mathbb{E}[X_1 X_2] - (1 + 2\mathbb{E}[X_1)(3 - 4\mathbb{E}[X_2]) \\
&= 3 + 6\mathbb{E}[X_1] - 4\mathbb{E}[X_2] - 8\mathbb{E}[X_1X_2] - 3 - 6\mathbb{E}[X_1] + 4\mathbb{E}[X_2] + 8\mathbb{E}[X_1]\mathbb{E}[X_2] - 8\mathbb{E}[X_1X_2] \\
&= 8\mathbb{E}[X_1]\mathbb{E}[X_2] \\
&= -8\mathrm{Cov}(X_1, X_2)
\end{aligned}$$

Additionally,

$$\begin{aligned}
\mathrm{Var}[Y_1] &= \mathrm{Var}[1+2X_1] \\
&= 4\mathrm{Var}[X_1] \\
\Rightarrow \sigma_{Y_1} &= 2\sigma_{X_1} \\
\mathrm{Var}[Y_2] &= \mathrm{Var}[3-4X_2] \\
&= 16\mathrm{Var}[X_2] \\
\Rightarrow \sigma_{Y_1} &= 4\sigma_{X_2}
\end{aligned}$$

We conclude,

$$\begin{aligned}
\rho_{Y_1,Y_2} &= \frac{\mathrm{Cov}(Y_1, Y_2)}{\sigma_{Y_1}\sigma_{Y_2}} \\
&= \frac{-8\mathrm{Cov}(X_1, X_2)}{8\sigma_{X_1}\sigma_{X_2}} \\
&= -\rho_{X_1,X_2} \\
&= -0.2.
\end{aligned}$$

Another possible solution utilizes the following two properties of covariance:

$$\mathrm{Cov}\left(\sum_{i=1}^{m} a_i X_i, \sum_{j=1}^{n} b_j Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \mathrm{Cov}(X_i, Y_j),$$
$$\mathrm{Cov}(c, X) = 0.$$

Substituting appropriately, we find

$$\mathrm{Cov}\left(1 + 2X_1, 3 - 4X_2\right) = -8\mathrm{Cov}\left(X_1, X_2\right).$$

We proceed from here in a similar manner as in the first solution.

Solution to Question 748: Option Dice I

The only time that our contract will be in the money is when we roll two 6's. When we do roll two 6's, and obtain a product of 36, and will be profiting 6 units. $(36 - 30 = 6)$. Putting this together, we have a $\frac{1}{36}$ chance of of profiting 6 units, so our option should be priced at $6 \cdot \frac{1}{36} = \frac{1}{6}$

Solution to Question 749: Soccer Jerseys

We want to find a value c so that $\mathbb{P}[S + M \leq c] = 0.975$. $M \sim N(70, 20^2)$ and $S \sim N(80, 15^2)$ represent the prices of Mane and Salah jerseys, respectively. Since these are independent, $M + S \sim N(150, 25^2)$ by adding the parameters. Then, standardizing,

$$\mathbb{P}[S + M \leq c] = \mathbb{P}\left[\frac{S + M - 150}{25} \leq \frac{c - 150}{25}\right] = 0.975$$

The LHS is now standard normal, so the LHS is $\Phi\left(\frac{c-150}{25}\right)$. Therefore, we need to find c such that $\Phi\left(\frac{c-150}{25}\right) = 0.975$. Using the inverse CDF, we get that $\frac{c-150}{25} \leq \Phi^{-1}(0.975) = 1.96$. Rearranging, $c = 150 + 25 \cdot 1.96 = 199$.

Solution to Question 750: Upface Correlation

We know that, by definition, $\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$. Let X_i be the indicator of the event that you roll a 1 on roll i, and Y_i be the indicator of the event that you roll a 5 on roll i. Thus, we have that $X = \sum_{i=1}^n X_i$ and $Y = \sum_{i=1}^n Y_i$. Plugging these in, we get that

$$\text{Cov}(X, Y) = \mathbb{E}\left[\sum_{i=1}^n X_i \sum_{i=1}^n Y_i\right] - \mathbb{E}\left[\sum_{i=1}^n X_i\right]\mathbb{E}\left[\sum_{i=1}^n Y_i\right] = \mathbb{E}\left[\sum_{i=1}^n \sum_{j=1}^n X_i Y_j\right] - \mathbb{E}\left[\sum_{i=1}^n X_i\right]\mathbb{E}\left[\sum_{i=1}^n Y_i\right]$$

Evaluating the two expectations at the end are not difficult. Note that $\mathbb{E}\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n \mathbb{E}[X_i]$ by linearity, and $\mathbb{E}[X_i] = \mathbb{P}[\text{roll a one on any given turn}] = \frac{1}{6}$. The similar logic applies to $\mathbb{E}[Y_i]$, and you also will get $\frac{1}{6}$ for that. Thus, $\sum_{i=1}^n \frac{1}{6} = \frac{n}{6}$, so the second term comes out to be $\frac{n \cdot n}{6 \cdot 6} = \frac{n^2}{36}$.

Now, for the first sum, we need to be clever. We need to break this up into two sums: $i = j$ and $i \neq j$. This yields

$$\mathbb{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} X_i Y_j\right] = \mathbb{E}\left[\sum_{i\neq j} X_i Y_j + \sum_{i=1}^{n} X_i Y_i\right] = \sum_{i\neq j}\mathbb{E}[X_i Y_j] + \sum_{i=1}^{n}\mathbb{E}[X_i Y_i]$$

Now, let's deal with these sums individually. $\mathbb{E}[X_i Y_i]$ takes the value of the probability that both X_i and Y_i occur. For both X_i and Y_i to occur, one would need to roll BOTH a one and a five on a single turn i. There is no chance of being able to do this, so $\mathbb{E}[X_i Y_i] = 0$. Thus, the entire second sum evaluates to 0. For the first expectation, it takes the value of the probability of both X_i and Y_j occurring. For both to occur, one would need to roll a 1 on roll i and a 5 on roll j. Since we know $i \neq j$, this is definitely possible to occur, and since rolls are independent, the probability of this is just $\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$. Thus, the first sum evaluates to $\sum_{i\neq j} \frac{1}{36} = \frac{n(n-1)}{36}$. We get the $n(n-1)$ term from the fact that there were n^2 terms before in the sum (n in the outer, n in the inner sum, so $n \cdot n = n^2$), and then we remove n of them from that original double sum when separating out the case where $i \neq j$ from $i = j$, so there are $n^2 - n$ terms in the first sum remaining. Thus, plugging in, we have $\text{Cov}(X, Y) = \frac{n^2 - n}{36} - \frac{n^2}{36} = -\frac{n}{36}$.

Note that there is a much quicker route to this solution: We have that $X + Y \sim \text{Binom}(n, 1/3)$, as $X + Y$ counts the number of times that either 1 or 5 appears, and at least one of these appears per roll with probability 1/3. The variance of $X + Y$ would therefore be $\frac{2n}{9}$ from the variance formula for binomials. We have that $\text{Var}(X)$ and $\text{Var}(Y)$ are both $\frac{5n}{36}$ from the variance of binomial formula. Plugging into the variance of a sum formula,

$$\frac{2n}{9} = \frac{5n}{36} + \frac{5n}{36} + 2\text{Cov}(X, Y)$$

We have that $2\text{Cov}(X, Y) = -\frac{n}{18}$ after rearranging, meaning that $\text{Cov}(X, Y) = -\frac{n}{36}$.

To find the correlation, $\sigma_X = \sigma_Y = \dfrac{\sqrt{5n}}{6}$ from the above. Therefore,

$$\rho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{-\frac{n}{36}}{\frac{5n}{36}} = -\frac{1}{5}$$

from the formula for correlation.

Solution to Question 751: Bullseye

Let's first find the probability Fred lands a bullseye within n throws. The complement is just that he misses the bullseye on each of the n throws. There is a 0.9 probability per throw that he misses the bullseye. Therefore, the probability of hitting at least once in n throws is $1 - (0.9)^n$.

We want to find an n so that $1 - (0.9)^n \geq 0.9$. Dividing and solving $n \geq \dfrac{\ln(0.1)}{\ln(0.9)} \approx 21.9$. Note that we switch the direction of the inequality since $\ln(0.9) < 0$. We need a whole integer, as there are no partial throws, so $n \geq 22$ is the minimum amount.

Solution to Question 752: Head-Tail Equality

Our first flip will either turn up to be H or T. Therefore, the difference (in absolute value) between the number of H and T will be 1 after 1 flip. Let H_n and T_n, respectively, be the number of heads and tails that occur in the first n flips. If N represents our stopping time, then $N = \min\{n > 0 : H_n = T_n\}$.

The key here is that we consider $S_n = |H_n - T_n|$, the absolute difference between the number of heads and tails in the first n flips. We can see that S_n is a symmetric random walk on the non-negative integers with a reflecting boundary at 0. This is because we either move left or right 1 with equal probability at each step when $S_n > 0$. These correspond to either the difference between heads and tails increasing or decreasing by 1 at each step. Furthermore, the fact that this is a reflecting random walk is irrelevant to this question. This is because we want to stop at the first time where we return to $S_n = 0$, when $H_n = T_n$ and don't care what happens after that. Therefore, our question simplifies to finding the expected return time to 0 in a symmetric random walk on the integers starting from 1. Mathematically, we want $\mathbb{E}[N]$, where $N = \min\{n > 0 : S_n = 0\}$ and we start at $S_0 = 0$.

There are some further simplifications that can be made. Since the random walk is shift-invariant, this is the expected time starting from 0 to hit -1 in our random walk. Since our random walk is symmetric, the expected time to hit -1 is the same as the expected time to hit 1. If R_n is a symmetric random walk on the integers (recall that we no longer care that it is on the positive integers), then we want $\mathbb{E}[N_1]$, with $N_1 = \min\{n : R_n = 1\}$.

Suppose that $\mathbb{E}[N_1] < \infty$. As we can represent $R_n = \sum_{i=1}^{n} X_i$, with each X_i being ± 1 with equal probability IID (this is the sum of left and right steps), by Wald's Identity, $\mathbb{E}[R_{N_1}] = \mathbb{E}[N_1]\mathbb{E}[X_1]$. We know that the LHS is 1 with probability 1 by the definition of N_1. By a direct calculation, we know $\mathbb{E}[X_1] = 0$. Therefore, if $\mathbb{E}[N_1]$ were finite, the LHS would be 1 and the RHS 0, leading to a contradiction. Thus, this expectation must be infinite and the answer is 0.

Solution to Question 753: Bread Slicer I

Let X be the distance from the left endpoint that Gabe cuts the piece of bread. Then $X \sim \text{Unif}(0, 8)$ by definition. The three resultant pieces are then of lengths $5, X$, and $8 - X$. We can use the condition for triangles be valid i.e. the sum of any 2 sides is longer than the remaining side to determine the valid values of X that make this a triangle.

The three inequalities say that $5 < X + (8 - X) = 8$, which is useless. Then, $X < 5 + (8 - X)$, so $X < \frac{13}{2}$ is our first condition. In addition $8 - X < 5 + X$, so that $X > \frac{3}{2}$ is our second condition. Therefore, finding a valid triangle is equivalent to finding $\mathbb{P}\left[\frac{3}{2} < X < \frac{13}{2}\right]$, which is simply just $\frac{5}{8}$ by realizing this interval is length 5 out of a total length 8 interval.

Solution to Question 754: Gamma Review I

Using the reduction property $\Gamma(a + 1) = a\Gamma(a)$, we note that $\Gamma\left(\frac{9}{4}\right) = \frac{5}{4} \cdot \Gamma\left(\frac{5}{4}\right) = \frac{5}{4} \cdot \frac{1}{4} \cdot \Gamma\left(\frac{1}{4}\right)$. Therefore, our answer is $\frac{5}{16}$.

Solution to Question 755: Mile Rate

Consider the rate at which the car moves. For the average rate to be 60 mph between the two laps, the total time taken between the two laps must be 2 minutes. Since the racecar driver drove at 30 mph in the first mile, they spent 2 minutes on the first lap. Therefore, the racecar driver would have to complete the second lap instantaneously i.e. move at an infinite speed. This is not possible, so $x = -1$.

Solution to Question 756: Simple Delta Hedge II

We first need to obtain the Δ of the put option. Delta can be interpreted probabilistically as the probability of finishing in-the-money. So, we can see that the Δ of the put is -0.75 (also can use put-call parity). We are selling 100 units of the option, so we have an overall Δ of $-0.75 * -1 * 100 = 75$. To hedge this, we need to sell 75 units of the underlying.

Solution to Question 757: Empty Boxes

$$P_n$$

is the proportion of empty bins, so $P_n = \frac{N_n}{n}$, where N_n is the number of empty bins when we do the process with n balls. Therefore, $\mathbb{E}[P_n] = \frac{1}{n}\mathbb{E}[N_n]$. This is our new goal to compute.

Label the bins $1 - n$ and let I_i be the indicator that the ith bin is empty after the process. Then $N_n = I_1 + \cdots + I_n$. By taking expected values and the exchangeability of the indicators, $\mathbb{E}[N_n] = n\mathbb{E}[I_1]$. $\mathbb{E}[I_1]$ is just the probability the first bin is empty after the process. This means that all of the balls went into the other $n - 1$ bins on every trial. The probability of this for each individual trial is $1 - \frac{1}{n}$. As each ball is independent, $\mathbb{E}[I_1] = \left(1 - \frac{1}{n}\right)^n$. Therefore, $\mathbb{E}[P_n] = \left(1 - \frac{1}{n}\right)^n$.

As $\ln(x)$ is a continuous function, we can evaluate the limit first and apply the logarithm later. $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = e^{-1}$ by the limit definition of e^x. Applying the natural log, our final answer is -1.

Solution to Question 758: Lawn Teamwork

Given the information we have, we can solve for the rate at which both men mow a lawn per minute. Mr. Rabbit mows $\frac{1}{60}$ of a lawn per minute. Mr. Turtle mows $\frac{1}{75}$ of a lawn per minute. We can use rate $\cdot$ time $=$ work done to solve for how long it takes. Let x be the number of minutes it takes for both of them to do one lawn. $(\frac{1}{60}+\frac{1}{75})\cdot x = 1$ lawn. Solving for x, we get $\frac{100}{3}$ minutes.

Solution to Question 759: Couples Reunited

Label the couples $1-9$ and let I_i be the indicator that the ith married couple is paired up together. Then $T = I_1 + \dots I_9$ gives the total number of couples that are paired up. By linearity of expectation and the exchangeability of the couples (no couple is more likely to be paired up together than any other), $\mathbb{E}[T] = 9\mathbb{E}[I_1]$. $\mathbb{E}[I_1]$ is just the probability that couple 1 is paired together. Fix the first member of the couple in some random pair. Of the 17 other people that can be this person's partner in the pair, only 1 is their married partner, so $\mathbb{E}[I_1] = \dfrac{1}{17}$. Thus, by plugging in above, $\mathbb{E}[T] = \dfrac{9}{17}$.

Solution to Question 760: Even 7

First, let's find the number of ways in which we have at least 2 dice show an even value. Since the dice are fair, we can pair each combination up (a, b, c) satisfying our condition with $(7-a, 7-b, 7-c)$. These give mirror reflections between permutations that have at least 2 even values and less than 2 even values. Therefore, half of all outcomes have at least 2 even values, so as there are $6^3 = 216$ total outcomes, $\dfrac{1}{2}\cdot 216 = 108$ of them have at least 2 even values.

For at least two even values to appear AND have a sum at most 7, the outcomes must be some permutation of $(1,2,4), (2,2,3), (2,2,1)$, or $(2,2,2)$. Respectively, there are $3!, 3, 3,$ and 1 permutation of each sequence, so this means that there are $6+3+3+1 = 13$ permutations that satisfy our conditions. Thus, the probability is $\dfrac{13}{108}$.

Solution to Question 761: Particle Reach III

We know from Particle Reach I that the probability that the particle eventually hits 1 from 0 is 1/2. Therefore, by the Markov property, this the same as the probability that the particle ever reaches $i+1$ from position i. We can then say that the probability 7 is ever reached from 0 is the product of the probabilities that 1 is reached from 0, 2 is reached from 1, and so on, until 7 is reached from 6. All of these probabilities are just 1/2, so the answer is $\frac{1}{2^7} = \frac{1}{128}$.

Solution to Question 762: All Attainable Values

The numbers $1, 2, 5$ and 6 must always be among the numbers on the die. If they weren't the sums of $2, 3, 11$, and 12 would be impossible. To make sums of 5 and 9 possible, we must include one of 3 or 4 on our die. The last value can be anything afterwards.

Let D_3 and D_4 represent the sets of dice arrangements that have 3 and 4 present, respectively. We want $|D_3 \cup D_4|$, the number of dice that contain at least one of 3 or 4. By Inclusion-Exclusion, we have that

$$|D_3 \cup D_4| = |D_3| + |D_4| - |D_3 \cap D_4|$$

$|D_3| = |D_4| = 6$, as we fix 1 side to be 3 (or 4), and then the other can be anything. Then $|D_3 \cap D_4| = 1$, as this corresponds to just the standard die with all values $1 - 6$. Therefore, there are $6 + 6 - 1 = 11$ dice satisfying this property.

Solution to Question 763: Better in Red III

Let R be the event that the last side not visible to you is red and W be the event that the other sides visible are not red. We want $\mathbb{P}[R \mid W] = \frac{\mathbb{P}[R \cap W]}{\mathbb{P}[W]}$. For $\mathbb{P}[R \cap W]$, we need the last side to be red and have that side be facing away from us. Since the orientation is random, the probability the red side is facing away from us is $\frac{1}{6}$. In addition, 6 of the 27 cubes have exactly one red side, so $\mathbb{P}[R \cap W] = \frac{1}{27}$. Then, in the denominator, we can have 5 white sides from choosing a cube that would satisfy the numerator event or just choosing the center cube. The center cube is white on all sides so there is no need to orient it. Therefore, $\mathbb{P}[W] = \frac{1}{27} + \frac{1}{27} = \frac{2}{27}$. Substituting in, we get $\mathbb{P}[R \mid W] = \frac{1}{2}$.

Solution to Question 764: Meeting on a Grid

Observe that the sum of Alan's coordinates increases by 1 each minute, and Barbara's decreases by 1. Since this sum starts at 8 for Barbara and 0 for Alan, if they meet at all, they must meet after 4 minutes, and they must do so on the diagonal $y = 4 - x$. During the first 4 minutes, both players are allowed to move in their originally permitted directions unconstrained, so there are 2^4

equally likely ways for each player to reach the diagonal and therefore $2^8 = 256$ ways for both players. There exists a bijection between the number of ways to travel from one corner to the other and the number of viable paths for Alan and Barbara to meet, as any distinct example of one can be viewed uniquely as an example of the other. Hence, we have $\binom{8}{4} = 70$ ways for Alan and Barbara to meet of the original 256, so the probability is $\dfrac{70}{256}$.

Solution to Question 765: Die To Number

A number is divisible by 8 if its last 3 digits are divisible by 8. We can therefore ignore the first 13 rolls. The last digit must be even, so the final roll must produce either $2, 4,$ or 6. At this point, we can just list all 3-digit combinations that are divisible by 8.

$$\begin{aligned}
\text{Ends in 2} &: 112, 152, 232, 312, 352, 432, 512, 552, 632 \\
\text{Ends in 4} &: 144, 224, 264, 344, 424, 464, 544, 624, 664 \\
\text{Ends in 6} &: 136, 216, 256, 336, 416, 456, 536, 616, 656
\end{aligned}$$

We conclude that our probability is $\frac{27}{6^3} = \frac{1}{8}$.

Solution to Question 766: Exponential + Uniform

The easiest way to do this is to condition on the value of U. This is because the exponential is a lot easier to deal with to find a tail probability. The CDF of V is given by $F_V(v) = 1 - e^{-v}$ for $v > 0$. Therefore, as $0 \leq U \leq 1$, $1 - U$ is as well. Therefore,

$$\mathbb{P}[U + V > 1] = \int_0^1 \mathbb{P}[U + V > 1 \mid U = u] f_U(u) du$$

where $f_U(u)$ is the PDF of U. This PDF is just 1 on $(0, 1)$, so we are fine.

$\mathbb{P}[U+V > 1 \mid U = u] = \mathbb{P}[V > 1-u] = 1-F_V(1-u) = e^{-(1-u)} = e^{u-1}$. This means our answer is $\displaystyle\int_0^1 e^{u-1} du = 1 - e^{-1}$. The answer is then $1 \cdot (-1) = -1$.

Solution to Question 767: Exact Bills I

Logically speaking, the most difficult amount to form will be very high. For example, any number from \$80 to \$90 can be harder to form by just adding \$10 to it (requires two more bill). Testing a few possibilities out, you'll see that \$99 is the most difficult amount to form as it requires 4 \$20 bills, 3 \$5 bill, and 4 \$1 bills which gives us 11 bills. However, we can't make \$100 yet so we can either another \$1 bill to our collection or a \$100 bill which brings out total to 12 bills.

Solution to Question 768: Uniform Movement

Note that $X = 3+U_1$ and $Y = 3+U_2$, where $U_1, U_2 \sim \text{Unif}(0,1)$ IID. Therefore, we can really write this as

$$\mathbb{E}[|(3+U_1) - (3+U_2)|] = \mathbb{E}[|U_1 - U_2|]$$

The trick here is that $|U_1 - U_2|$ is the length of the middle segment of two order statistics of the Unif$(0,1)$ random variable. This is because of the absolute value here denoting that we look at the length of the middle segment between the two order statistics. In expectation, each of the intervals $(0, \min\{U_1, U_2\}), (\min\{U_1, U_2\}, \max\{U_1, U_2\})$, and $(\max\{U_1, U_2\}), 1)$ should be the same length. Therefore, as their lengths sum to 1, each should be $\frac{1}{3}$ in length. In particular, this means that $\mathbb{E}[|X-Y|] = \mathbb{E}[|U_1 - U_2|] = \frac{1}{3}$.

Solution to Question 769: Unlucky Seven I

The first observation that can be made is that if we roll a $3, 4, 5$, or 6, we should not consider rolling again. This is because we can only improve our stance by rolling a number larger, but the sum is guaranteed to go over 7 in any of these cases if our first roll is any of these numbers. Therefore, we should look at just rolling a 1 or 2. If we roll a 1, we either lose \$1 with probability $\frac{1}{6}$ (if we roll a 6 on the second roll). Otherwise, we make $1, 2, \ldots, 5$, each with probability $\frac{1}{6}$. Therefore our expected payout if we roll again is $\frac{1}{6} \cdot (-1) + \frac{1+2+3+4+5}{6} = \frac{7}{3} > 1$. This implies we should roll again if we receive a 1. If we roll a 2, then with probability $\frac{1}{3}$ we lose \$2 (if we roll a 5 or 6). Otherwise, we receive $1, 2, 3$, or 4 with equal probability $\frac{1}{6}$. Therefore, the expected value upon rolling again is $\frac{1}{3} \cdot (-2) + \frac{1+2+3+4}{6} = 1 < 2$. This implies we should not roll again and keep the two. As a result, we keep all values that aren't 1, and otherwise, we roll a 1 again. This yields an expected value of $\frac{\frac{7}{3}+2+3+4+5+6}{6} = \frac{67}{18}$.

Solution to Question 770: Ramanujan's Run

First, to simplify things, let's consider 10 minutes as 1 complete unit of time. We have:

$$X_1, X_2, X_3 \overset{\text{iid}}{\sim} \text{Unif}([0,1]), \text{and}$$
$$T_1, T_2, T_3 \overset{\text{iid}}{\sim} \text{Unif}([0,1]).$$

Note that Ramanujan may either take two attempts or three attempts to reach the lecture hall. As long as Ramanujan has some time left over (for example, if $T_1 < 1$), then he will have the opportunity to run another time (that isâin our exampleâRamanujan will add X_2 to his total distance traveled regardless of the value of T_2). We wish to find the probability that Ramanujan makes it to class on time, which can be expressed as

$$\mathbb{P}(X_1 + X_2 > 1, T_1 < 1) + \mathbb{P}(X_1 + X_2 < 1, X_1 + X_2 + X_3 > 1, T_1 + T_2 < 1).$$

Note that the events $X_1 + X_2 > 1, T_1 < 1$ and $X_1 + X_2 < 1, X_1 + X_2 + X_3 > 1, T_1 + T_2 < 1$ are in fact mutually exclusive, which justifies the application of countable additivity. Now, let's simplify, making use of the fact that events only involving T_n are independent from events only involving X_n.

$$\begin{aligned}
&\mathbb{P}(\text{Ramanujan on time}) \\
&= \mathbb{P}(X_1 + X_2 > 1, T_1 < 1) + \mathbb{P}(X_1 + X_2 < 1, X_1 + X_2 + X_3 > 1, T_1 + T_2 < 1) \\
&= \mathbb{P}(X_1 + X_2 > 1)\mathbb{P}(T_1 < 1) + \mathbb{P}(X_1 + X_2 < 1, X_1 + X_2 + X_3 > 1)\mathbb{P}(T_1 + T_2 < 1) \\
&= \mathbb{P}(X_1 + X_2 > 1) + \mathbb{P}(X_1 + X_2 < 1, X_1 + X_2 + X_3 > 1)\mathbb{P}(T_1 + T_2 < 1)
\end{aligned}$$

Notice that $\mathbb{P}(T_1 + T_2 < 1) = \mathbb{P}(T_1 + T_2 > 1) = \mathbb{P}(X_1 + X_2 < 1) = \mathbb{P}(X_1 + X_2 > 1)$. We quickly find $\mathbb{P}(T_1 + T_2 < 1) = \frac{1}{2}$ by plotting the inequality on the T_1-T_2 plane. More specifically, one may compute this probability with

$$\begin{aligned}
\mathbb{P}(T_1 + T_2 < 1) &= \int_0^1 \int_0^{1-t_2} f_{T_1,T_2}(t_1, t_2)\, dt_1\, dt_2, \text{ where} \\
f_{T_1,T_2}(t_1, t_2) &= f_{T_1}(t_1) f_{T_2}(t_2) = 1 \quad \text{due to independence}
\end{aligned}$$

Substituting, we have

$$\mathbb{P}(\text{Ramanujan on time}) = \frac{1}{2} + \frac{1}{2} \cdot \mathbb{P}(X_1 + X_2 + X_3 > 1, X_1 + X_2 < 1).$$

It is much easier to compute $\mathbb{P}(X_1 + X_2 + X_3 > 1, X_1 + X_2 < 1)$ by plotting the region satisfying the inequality $X_1 + X_2 + X_3 > 1$ on the X_1-X_2-X_3 plane, where the volume of interest is strictly above the region $X_1 + X_2 < 1$ on the X_1-X_2 plane. However, one may choose to compute this probability as follows:

$$\begin{aligned}
&\mathbb{P}(X_1 + X_2 + X_3 > 1, X_1 + X_2 < 1) \\
&= \int_0^1 \int_0^{1-x_2} \int_{1-x_1-x_2}^1 f_{X_1,X_2,X_3}(x_1, x_2, x_3)\, dx_3\, dx_1\, dx_2, \text{where}
\end{aligned}$$

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = f_{X_1}(x_1) f_{X_2}(x_2) f_{X_3}(x_3) = 1 \quad \text{due to independence}$$

The value of the above integral is $\frac{1}{3}$. So,

$$\begin{aligned}\mathbb{P}(\text{Ramanujan on time}) &= \frac{1}{2} + \frac{1}{6} \\ &= \frac{2}{3}\end{aligned}$$

Solution to Question 771: Placing Dots

The first dot will always be placed on a distinct edge. The second dot has a $\frac{3}{4}$ chance of being placed on a distinct edge. The third dot has a $\frac{2}{4}$ chance of being placed on a distinct edge. Thus, the probability that the three dots all lie on distinct edges is:

$$1 \times \frac{3}{4} \times \frac{2}{4} = \frac{3}{8}$$

Solution to Question 772: Sum Exceedance I

We are going to generalize this question. For $0 < x < 1$ (this restriction will be explained later), let $m(x) = \mathbb{E}[N_x]$, where $N_x = \min\{n : X_1 + \cdots + X_n > x\}$. By the Law of Total Expectation, $\mathbb{E}[N_x] = \mathbb{E}[\mathbb{E}[N_x \mid X_1]]$. If $X_1 > x$, then it takes 1 trial and this occurs with probability $1 - x$. If $X_1 \leq x$, then we have a remaining sum of $x - X_1$ remaining to exceed. Therefore, our expectation $1 + m(x - X_1)$ in this case. We need to integrate over all values of X_1 from 0 to x by Law of Total Expectation. The expression that results is

$$m(x) = (1 - x) \cdot 1 + \int_0^x (1 + m(x - y))dy = 1 + \int_0^x m(x - y)dy$$

We can make the $u-$substitution $u = x - y$ and obtain the new integral equation $m(x) = 1 + \int_0^x m(u)du$. Assuming differentiability of $m(x)$, take the derivative on both sides and apply the fundamental theorem of calculus to obtain $m'(x) = m(x)$. The classic solution to this equation is $m(x) = Ce^x$. We now need an initial condition to obtain a particular solution. A reasonable initial condition is that $m(0) = 1$, as with probability 1, our first random variable is larger than 0, so it takes exactly 1 random variable to exceed 0. This initial condition implies $C = 1$ and $m(x) = e^x$.

Note that this equation is only valid for $0 < x < 1$. This is because for $0 < x < 1$, we can exceed x with one random variable. Otherwise, for $x > 1$, the RHS is $m(x) - m(x - 1)$. It is a good exercise to verify this. As $\ln(2) < 1$, the above is irrelevant for this question, so $m(\ln(2)) = e^{\ln(2)} = 2$.

Solution to Question 773: Limited Urns

Let W_1 denote the event that the white ball is drawn on the first draw, and W_2 denote the event that the white ball is drawn on the second draw. W_1^c and W_2^c would then represent the events of drawing a black ball on draws 1 and 2, respectively. We want to find $\mathbb{P}[W_2 \mid W_1]$. Let the event U_i be when urn i is selected. Since we select the urn uniformly random, $\mathbb{P}[U_i] = \dfrac{1}{n}$, for all $1 \leq i \leq n$. Suppose we select urn $1 \leq k \leq n$. There is 1 white ball and $2^k - 1$ black balls in it. There is independence between draws, as the ball selected is replaced after the first draw. Thus, on either draw, since we select the ball completely at random, $\mathbb{P}[W_1 \mid U_k] = \mathbb{P}[W_2 \mid U_k] = \dfrac{1}{2^k}$, and $\mathbb{P}[W_1^c \mid U_k] = \mathbb{P}[W_2^c \mid U_k] = \dfrac{2^k-1}{2^k}$. We have the conditioning on U_k, as we assumed that we are in urn k. Thus, the probability of two white balls, given we are in urn k, by independence is just $\mathbb{P}[W_1W_2 \mid U_k] = \mathbb{P}[W_1 \mid U_k]\mathbb{P}[W_2 \mid U_k] = \dfrac{1}{4^k}$. The second draw from the urn is either a white ball or a black ball, so the other case to consider is if we are in urn k, what is the probability the second ball is black? By independence:

$$\mathbb{P}[W_1W_2^c \mid U_k] = \mathbb{P}[W_1 \mid U_k]\mathbb{P}[W_2 \mid U_k] = \frac{2^k-1}{4^k} = \frac{1}{2^k} - \frac{1}{4^k}.$$

We want $\mathbb{P}[W_2 \mid W_1] = \dfrac{\mathbb{P}[W_1W_2]}{\mathbb{P}[W_1]}$ by conditional probability. On the top, we condition on all n urns, as we can get two white balls from urn 1, from urn 2, etc. By the Law of Total Probability, we have:

$$\mathbb{P}[W_1W_2] = \mathbb{P}[W_1W_2 \mid U_1]\mathbb{P}[U_1] + \mathbb{P}[W_1W_2 \mid U_2]\mathbb{P}[U_2] + \cdots + \mathbb{P}[W_1W_2 \mid U_n]\mathbb{P}[U_n]$$

We know that each $\mathbb{P}[U_i] = \dfrac{1}{n}$, as well as that for all $1 \leq i \leq n$, $\mathbb{P}[W_1W_2 \mid U_i] = \dfrac{1}{4^i}$. By substitution:

$$\begin{aligned}\mathbb{P}[W_1W_2] &= \frac{1}{4}\cdot\frac{1}{n} + \frac{1}{4^2}\cdot\frac{1}{n} + \cdots + \frac{1}{4^n}\cdot\frac{1}{n} \\ &= \frac{1}{n}\sum_{j=1}^{n}\left(\frac{1}{4}\right)^j\end{aligned}$$

On the bottom, we can have the first draw be a white ball by having the first being a white ball and the second being white OR having the second draw being black. Thus, the denominator is going to be $\mathbb{P}[W_1] = \mathbb{P}[W_1W_2] + \mathbb{P}[W_1W_2^c]$, as those two events are disjoint. The first term in this sum is the numerator. We are going to apply the Law of Total Probability to our denominator by conditioning on the urns again to get:

$$\mathbb{P}[W_1W_2^c] = \mathbb{P}[W_1W_2^c \mid U_1]\mathbb{P}[U_1] + \mathbb{P}[W_1W_2^c \mid U_2]\mathbb{P}[U_2] + \cdots + \mathbb{P}[W_1W_2^c \mid U_n]\mathbb{P}[U_n]$$

The $\mathbb{P}[U_i]$ term is the same as in the numerator, and we also have that:

$$\mathbb{P}[W_1 W_2^c \mid U_i] = \frac{2^i - 1}{4^i} = \frac{1}{2^i} - \frac{1}{4^i}, \forall 1 \leq i \leq n$$

Thus, we have:

$$\begin{aligned}
\mathbb{P}[W_1 W_2^c] &= \frac{2-1}{4} \cdot \frac{1}{n} + \frac{2^2 - 1}{4^n} \cdot \frac{1}{n} + \cdots + \frac{2^n - 1}{4^n} \cdot \frac{1}{n} \\
&= \frac{1}{n} \sum_{j=1}^{n} \frac{2^j - 1}{4^j}
\end{aligned}$$

And therefore:

$$\begin{aligned}
\mathbb{P}[W_2 \mid W_1] &= \frac{\frac{1}{n} \sum_{j=1}^{n} \left(\frac{1}{4}\right)^j}{\frac{1}{n} \sum_{j=1}^{n} \left(\frac{1}{4}\right)^j + \frac{1}{n} \sum_{j=1}^{n} \frac{2^j - 1}{4^j}} \\
&= \frac{\sum_{j=1}^{n} \left(\frac{1}{4}\right)^j}{\sum_{j=1}^{n} \left(\frac{1}{4}\right)^j + \sum_{j=1}^{n} \left(\left(\frac{1}{2}\right)^j - \left(\frac{1}{4}\right)^j\right)} \\
&= \frac{\sum_{j=1}^{n} \left(\frac{1}{4}\right)^j}{\sum_{j=1}^{n} \left(\frac{1}{2}\right)^j}
\end{aligned}$$

All that is left is to take the limit as $n \to \infty$. Note that sums are then geometric when we shift the initial index of the summation to start at 0 such that $\sum_{j=1}^{n} \left(\frac{1}{4}\right)^j = \frac{1}{4} \sum_{j=1}^{n} \left(\frac{1}{4}\right)^{j-1}$. The sum evaluates to $\frac{1}{1 - \frac{1}{4}} = \frac{4}{3}$, so with the factor of $\frac{1}{4}$ out front, the numerator evaluates to $\frac{1}{3}$. On the denominator, $\sum_{j=1}^{n} \left(\frac{1}{2}\right)^j = \frac{1}{2} \sum_{j=1}^{n} \left(\frac{1}{2}\right)^{j-1} = \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} = 1$. As the sum in the numerator approaches $\frac{1}{3}$ and the sum in the denominator approaches 1 as $n \to \infty$, $p(n) \to \frac{1}{3}$ as $n \to \infty$.

Solution to Question 774: Binary Zeroes

In base 10, we know that the number of trailing zeros for n is the highest power of 10 that divides n. Similarly, in binary, we want the highest power of 2 dividing 142!. We get 71 powers of 2 from all the even terms. Then, there are floor $\left(\frac{142}{2^2}\right) = 35$ additional powers of 2 coming from terms divisible by 4. We keep doing this to obtain that there are

$$\sum_{k=1}^{\infty} \text{floor}\left(\frac{n}{2^k}\right)$$

trailing zeroes in binary for $n!$. Note that this sum is finite since eventually $2^k > n$, so the floor will become 0. With $n = 142$, we get

$$\sum_{k=1}^{\infty} \text{floor}\left(\frac{142}{2^k}\right) = 71 + 35 + 17 + 8 + 4 + 2 + 1 = 138$$

Solution to Question 775: Base Exponent Square

We know that $\sqrt{k^k} = k^{\frac{k}{2}}$ by basic algebra. Therefore, if k is even, $\frac{k}{2}$ is an integer, so if k is even, this works. This allows 151 integers. Furthermore, if k is a perfect square, then $k = n^2$ for some integer n, so $\sqrt{k^k} = \sqrt{(n^2)^{n^2}} = \sqrt{n^{2n^2}} = n^{n^2}$, which is an integer. Therefore, odd perfect squares are not accounted for yet, so we can add those in. Namely, these are $11^2, 13^2, 15^2, 17^2$, and 19^2. This allows 5 more possibilities, so our total answer is $151 + 5 = 156$.

Solution to Question 776: Three Sides

In order to observe three heads or three tails, at least three tosses must occur but no more than five tosses can occur. The game ends with three tosses if you observe HHH or TTT, each of which occurs with probability $\frac{1}{8}$, yielding a probability of $\frac{1}{4}$. The game ends with four tosses if we begin with a permutation of HHT and end with an H, or if we begin with a permutation of TTH and end with a T. There are six total possibilities, each of which occurs with probability $\frac{1}{16}$, yielding a probability of $\frac{3}{8}$. The game ends with five tosses if we begin with a permutation of HHTT (the last toss will end the game regardless). There are six total possibilities, each of which occurs with probability $\frac{1}{16}$, yielding a probability of $\frac{3}{8}$. The expected value is thus:

$$E[X] = \frac{1}{4}(3) + \frac{3}{8}(4) + \frac{3}{8}(5) = \frac{33}{8}$$

Solution to Question 777: Party Groups II

An intuitive answer is that we discovered in Party Guests I that there are $\sum_{k=1}^{50} \frac{1}{k}$ groups on average, and since there are 50 people total, the average size of a given group is $\frac{50}{\sum_{k=1}^{50} \frac{1}{k}} \approx 11.1$. This is indeed correct. We show this using the "cycles" interpretation from the previous part of the question.

There are $n!/k$ $k-$cycles among the permutations, while there are $n! \cdot H_n$ total cycles in all of the permutations (adding up the cycles of each length), where $H_n = \sum_{k=1}^{n} \frac{1}{k}$. Therefore, the probability that a given cycle is a $k-$cycle is $\frac{n!/k}{n!H_n} = \frac{1}{kH_n}$. Therefore, the expected length of a cycle would be

$$\sum_{k=1}^{n} k \cdot \frac{1}{kH_n} = \frac{n}{H_n}$$

Solution to Question 778: Sequence Terminator

An incorrect solution goes as follows: We will have no 3s or 5s appearing before the first 1, as they reset the sequence and are not added. Therefore, our sample space is really $1, 2, 4$, and 6. On average, the 1 will appear after 4 throws of the die, so the answer is 4.

This above solution is incorrect because it more implicitly assumes that we just "ignore" 3s and 5s that appear. Therefore, this is not the correct assumption we want to make. Instead, we can reframe this problem as follows: At some point, the last 3 or 5 will appear before the first 1. As the die is memoryless (i.e. is not affected by what has already appeared), given that the first 1 appears before both the first 3 and first 5, what is the expected number of rolls needed to obtain the 1?

Let's let N represent the number of rolls needed to see a 1, and let B be the event that 1 occurs before both 3 and 5. By symmetry, $\mathbb{P}[B] = 1/3$, as each is equally likely to appear first. We now condition on the value of the first roll.

Let R_i represent the event that we roll value i on the first roll. This would imply by Law of Total Expectation that

$$\mathbb{E}[N \mid B] = \sum_{i=1}^{6} \mathbb{E}[N \mid R_i, B]\mathbb{P}[R_i \mid B]$$

For $i = 2, 4, 6$, we have that $\mathbb{P}[R_i \mid B] = \dfrac{P[B \mid R_i]\mathbb{P}[R_i]}{\mathbb{P}[B]}$. Since the event that we roll $i = 2, 4, 6$ on the first roll doesn't affect the probability of appearance of 1 before 3 and 5, we have that $\mathbb{P}[B \mid R_i] = \mathbb{P}[B]$. Clearly $\mathbb{P}[R_i] = 1/6$ based on the fact the die is fair, so $\mathbb{P}[R_i \mid B] = 1/6$ for $i = 2, 4, 6$.

We have that $\mathbb{P}[R_3 \mid B] = \mathbb{P}[R_5 \mid B] = 0$, as there is no chance that our first roll can be 3 or 5 if 1 comes before them. Thus, since the conditional probability measure is a probability measure, we have that $\mathbb{P}[R_1 \mid B] = 1 - 3 \cdot 1/6 = 1/2$.

If we roll a 1 on the first roll, then our sequence is complete and of length 1, so $\mathbb{E}[N \mid B, R_1] = 1$. Otherwise, if $i = 2, 4, 6$, then $\mathbb{E}[N \mid B, R_i] = 1 + \mathbb{E}[N \mid B]$, as we used up one roll and need to continue forward. Therefore, the equation we need to solve is

$$\mathbb{E}[N \mid B] = \frac{1}{2}\left(1 + \mathbb{E}[N \mid B]\right) + \frac{1}{2}$$

This is easily solved to yield $\mathbb{E}[N \mid B] = 2$.

You can instead view this problem as saying that whenever we roll a $1, 3$, or 5, we make it a 1 so that the 1 occurs before the 3 and 5. Therefore, the probability of rolling an odd integer on each trial is $1/2$, meaning that $N \mid B \sim \text{Geom}(1/2)$, which has mean 2. This agrees with the answer above.

Solution to Question 779: 1 Head Up

Of the $2^3 = 8$ possible outcomes, all but TTT have at least one head, so our sample space consists of $2^3 - 1 = 7$ equally-likely outcomes. Of these outcomes, 3 of them have exactly one head. Namely, these are HTT, THT, and TTH. Therefore, our probability is $\dfrac{3}{7}$.

Solution to Question 780: ATM Option Pricing

Think about this result intuitively. If $T = 1000$, then the option essentially acts like the underlying itself. This result can also be verified with Black-Scholes, taking $K = S_0$ and $t \to \infty$.

Solution to Question 781: Surface Rotation

We can write $\frac{dy}{dx} = -\frac{x}{\sqrt{r^2 - x^2}}$, so we know that

$$S_r = \int_{r^2}^{r} 2\pi f(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{r^2}^{r} 2\pi\sqrt{r^2 - x^2}\sqrt{1 + \frac{x^2}{r^2 - x^2}} dx = \int_{r^2}^{r} 2\pi r dx = 2\pi(r^2 - r^3)$$

To maximize S_r, we just take the derivative in r, obtaining that $\frac{dS_r}{dr} = 2\pi(2r - 3r^2) = 2\pi r(2 - 3r)$, which is 0 at $r = 0, 2/3$. Since we want $0 < r < 1$, we have $r^* = 2/3$ is our maximizer, yielding $S_{r^*} = \frac{8}{27}\pi$, so $q = \frac{8}{27}$.

Solution to Question 782: Golden Pancakes

Denote the sides of the six pancakes s_i where $1 \leq i \leq 6$. Without loss of generality, let s_1, s_2, and s_3 be the toasted sides and s_1 and s_2 be on the same pancake. Out of the six possible sides we could have observed, we are given to have observed one of the three toasted sides, s_1, s_2, or s_3. Of these three sides, two (s_1 and s_2) belong to a pancake that is toasted on both sides. Thus, the probability that the other side of the pancake is also toasted is $\frac{2}{3}$.

Solution to Question 783: Cube Colorer

We are going to solve this for a $k \times k \times k$ cube. In particular, there are $6k^2$ painted sides, as each side has k^2 faces painted. There are $6k^3$ total faces in the cube, as each of the k^3 cubes has 6 faces. Therefore, the probability that we obtained a painted side when we roll the selected $1 \times 1 \times 1$ cube is $\frac{6k^2}{6k^3} = \frac{1}{k}$. In this case, $k = 5$, so our answer is $\frac{1}{5}$.

Solution to Question 784: Significant Others

Recall that under the null hypothesis, the distribution of the $t-$statistic is t_{n-1}, where $t = \frac{\hat{\beta}_0}{s\left\{\hat{\beta}_0\right\}}$. The variance of $\hat{\beta}_0$ is $\text{Var}(\hat{\beta}_0) = \frac{\sigma^2}{\sum x_i^2}$ since x_i are assumed to have mean 0. σ^2 is the variance of the error term ε.

The common estimator for σ^2 is s^2, our sample variance, given by $s^2\left\{\hat{\beta}_0\right\} = \frac{1}{n-1}\sum\left(y_i - \hat{\beta}_0 x_i\right)^2 = \frac{SSR}{n-1}$. In simple linear regression, we have that $SSR = (1-\rho^2)\sum y_i^2$ (again since y_i data is mean 0). Lastly, as usual with OLS, we know that $\hat{\beta}_0 = \frac{\text{Cov}(X,Y)}{\text{Var}(X)} = \rho \cdot \frac{\sigma_Y}{\sigma_X}$. Once again, we estimate σ_X and σ_Y by $s_x = \sqrt{\frac{1}{n-1}\sum x_i^2}$ and $s_y = \sqrt{\frac{1}{n-1}\sum y_i^2}$. When we take the ratio, the coefficients out front cancel.

Plugging this all back in to our $t-$statistic, $t = \rho \cdot \frac{\sqrt{\sum y_i^2}}{\sqrt{\sum x_i^2}} \cdot \frac{1}{\sqrt{\text{Var}(\hat{\beta}_0)}} = \rho \cdot \frac{\sqrt{\sum y_i^2}}{\sqrt{\sum x_i^2}} \cdot \frac{\sqrt{\sum x_i^2}}{\sqrt{(1-\rho^2)\sum y_i^2}}\sqrt{n-1} = \frac{\rho}{\sqrt{1-\rho^2}}\sqrt{n-1}$.

Plugging in $\rho = \frac{1}{10\sqrt{3}}$, we get that the $t-$statistic is $t_n = \sqrt{\frac{n-1}{299}}$. Therefore, we must find the smallest n such that $t_n > 1.645$. We use the 95th percentile value because we are doing a one-sided test. Solving this yields $n \geq 299 \cdot (1.645)^2 + 1 = 810.1$. Since n must be an integer, $n = 811$ would the smallest value.

Solution to Question 785: Ant Collision I

Imagine that whenever two ants collide they change bodies. This is a reasonable assumption because of the fact in each collision, both ants will move in the direction of the other ant after the collision. Therefore, it is as if no collision happened at all. This means that 80 ants reach Alice and 40 ants reach Bob. Therefore, $80 - 40 = 40$ is our answer.

Solution to Question 786: Bond Practice VI

$n = 2 \times 20.0 = 40; r = \frac{0.09}{2} = 0.04500$; $C = \frac{0.06 \times 1{,}000}{2} = 30$; $P =$ price of bond:

$$P = \left(\frac{30}{0.04500}\right)\left(\frac{(1+0.04500)^{40}-1}{(1+0.04500)^{40}}\right) + \frac{1{,}000}{(1+0.04500)^{40}}$$
$$P = (666.6667)\left(\frac{5.81636-1}{5.81636}\right) + \frac{1{,}000}{5.81636}$$
$$P = 552.0475 + 171.9287$$
$$P = 723.98$$

Solution to Question 787: Breakeven Price II

Since we are writing (selling) a call option, we have payoff $-\max(S_T - 12, 0)$. We also obtain a credit of \$2.50 as a premium since we are providing insurance. We can then solve the following equation:

$$2.50 - \max(S_T - 12, 0) = 0$$

Solving this yields $S_T = 14.5$ (see Breakeven Price I)

Solution to Question 788: Covariance Review V

Note that

$$\mathrm{Var}(2X_1 - 4X_2) = 4\mathrm{Var}(X_1) + 16\mathrm{Var}(X_2) - 16\mathrm{Cov}(X_1, X_2).$$

Recall our definition of covariance:

$$\mathrm{Cov}(X_1, X_2) = \mathbb{E}[X_1X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2].$$

In order to compute the covariance we need to complete the following steps: (1) determine the value of c, (2) compute $\mathbb{E}[X_1X_2]$ from the joint pdf, and (3) determine the marginal pdfs for X_1 and X_2 in order to compute $\mathbb{E}[X_1]$ and

$\mathbb{E}[X_2]$. Let's begin with step 1.

$$
\begin{aligned}
\iint_{\mathbb{R}^2} f_{X_1,X_2}(x_1,x_2)dx_1\,dx_2 &= 1\\
\int_0^1 \int_0^{x_2} c(1-x_2)\,dx_1\,dx_2 &= 1\\
\int_0^1 cx_2(1-x_2)\,dx_2 &= 1\\
c\left[\frac{x_2^2}{2} - \frac{x_3^3}{3}\right]_0^1 &= 1\\
\frac{c}{6} &= 1\\
c &= 6
\end{aligned}
$$

Next, for step 2, we need to compute $\mathbb{E}[X_1X_2]$.

$$
\begin{aligned}
\int_0^1 \int_0^{x_2} 6x_1x_2(1-x_2)\,dx_1\,dx_2 &= \int_0^1 6x_1 \int_0^{x_2} x_2(1-x_2)\,dx_1\,dx_2\\
&= \int_0^1 x - 3x^3 + 2x^4\,dx\\
&= \left[\frac{2}{5}x^5 - \frac{3}{4}x^4 + \frac{1}{2}x^2\right]_0^1\\
&= \frac{3}{20}
\end{aligned}
$$

On to step 3. Here is $f_{X_1}(x_1)$:

$$
\begin{aligned}
\int_{\mathbb{R}} f_{X_1,X_2}(x_1,x_2)\,dx_2 &= \int_{x_1}^1 6(1-x_2)\,dx_2\\
&= \left[6x_2 - 3x_2^2\right]_{x_1}^1\\
&= 3 - 6x_1 + 3x_1^2
\end{aligned}
$$

And here is $f_{X_2}(x_2)$:

$$
\begin{aligned}
\int_{\mathbb{R}} f_{X_1,X_2}(x_1,x_2)\,dx_1 &= \int_0^{x_2} 6(1-x_2)\,dx_1\\
&= 6x_2 - 6x_2^2
\end{aligned}
$$

We can use these marginal pdfs to compute $\mathbb{E}[X_1]$, $\mathbb{E}[X_2]$, $\mathbb{E}[X_1^2]$, and $\mathbb{E}[X_2^2]$.

$$\begin{aligned}
\mathbb{E}[X_1] &= \int_0^1 x_1(3 - 6x_1 + 3x_1^2)\, dx_1 \\
&= \left[\frac{3}{4}x_1^4 - 2x_1^3 + \frac{3}{2}x_1^2\right]_0^1 \\
&= \frac{1}{4} \\
\mathbb{E}[X_2] &= \int_0^1 x_2(6x_2 - 6x_2^2)\, dx_2 \\
&= \left[-\frac{3}{2}x_2^4 + 2x^3\right]_0^1 \\
&= \frac{1}{2} \\
\mathbb{E}[X_1^2] &= \int_0^1 x_1^2(3 - 6x_1 + 3x_1^2)\, dx_1 \\
&= \left[\frac{3}{5}x_1^5 - \frac{3}{2}x_1^4 + x_1^3\right]_0^1 \\
&= \frac{1}{10} \\
\mathbb{E}[X_2^2] &= \int_0^1 x_2^2(6x_2 - 6x_2^2)\, dx_2 \\
&= \left[-\frac{6}{5}x_2^5 + \frac{3}{2}x^4\right]_0^1 \\
&= \frac{3}{10}
\end{aligned}$$

Putting it all together, we find

$$\begin{aligned}
\mathrm{Cov}(X_1, X_2) &= \mathbb{E}[X_1X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2] \\
&= \frac{3}{20} - \frac{1}{8} \\
&= \frac{1}{40}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Var}(X_1) &= \mathbb{E}[X_1^2] - (\mathbb{E}[X_1])^2 \\
&= \frac{3}{80} \\
\mathrm{Var}(X_2) &= \mathbb{E}[X_2^2] - (\mathbb{E}[X_2])^2 \\
&= \frac{1}{20}
\end{aligned}$$

Our solution is therefore

$$\begin{aligned}\mathrm{Var}(2X_1 - 4X_2) &= 4\mathrm{Var}(X_1) + 16\mathrm{Var}(X_2) - 16\mathrm{Cov}(X_1, X_2)\\ &= \frac{12}{80} + \frac{16}{20} - \frac{16}{40}\\ &= \frac{11}{20}.\end{aligned}$$

Solution to Question 789: Repeated Flipper

The game, regardless of the round that it ends on, ends when we either obtain all tails or all heads on the flips of the remaining coins. These occur with equal probability, so the answer is simply $\frac{1}{2}$.

Solution to Question 790: Lucky Chuck

This is a problem in which we need to calculate the probabilities of each event and then our net gain/loss from them. To roll 3 dice and not roll our selected number is a probability of $\frac{5}{6}^3$, in which case we lose our initial bet (a unit value we can denote as -1).

Next, we calculate the odds of only getting 1 of our dice to match our selected number, which is $\binom{3}{1} \cdot \frac{5}{6}^2 \cdot \frac{1}{6} = \frac{75}{216}$, in which case we gain 1 unit of value.

For 2 dice to match, it's $\binom{3}{2} \cdot \frac{5}{6} \cdot \frac{1}{6}^2 = \frac{15}{216}$ with a value of 2 units. And for 3 dice to match, it's $\frac{1}{6}^3 = \frac{1}{216}$ with a value of 3 units.

This makes our expected unit value of playing the game $\frac{\text{â}1\cdot125+1\cdot75+2\cdot15+3\cdot1}{216} = \frac{\text{â}17}{216}$

With a bet size of \$20 and 1 play, we are expecting to lose $20 \cdot \frac{-340}{216}$ on average, or $\frac{-85}{54}$

Solution to Question 791: Pulling Cards in Order

Instead of thinking about the situation as picking out cards one at a time, think about picking 4 cards out of the 100 all at once. Since all the numbers are unique, there is only 1 way the numbers can show up in ascending order out of 4! orderings. Thus the answer is $\frac{1}{4!} = \frac{1}{24}$.

Solution to Question 792: Doubly 5 I

We are going to break up the event based on when Jenny stops rolling. We know that she needs to roll at least 4 times, as she needs to observe both 2 5s, as well as a 4 and a 6.

Suppose that Jenny stops after n rolls. In the first $n-1$ rolls, she must have obtained exactly 2 5s. There are $\binom{n-1}{2}$ ways to pick the locations of those two fives. Then, the last roll must either be a 4 or a 6, and that value must not appear in the rolls beforehand. There are 2 options for the last roll. WLOG, let it be 6. In the other $n-3$ rolls that come beforehand, they are able to be any value $1-4$, but there must be at least one 4 in the sequence so that we have both 4 and 6 occur. There are 4^{n-3} sequences total for the other $n-3$ rolls, of which we need to exclude the ones with no 4s in them, which is 3^{n-3} sequences. Therefore, there are $4^{n-3}-3^{n-3}$ sequences for the other rolls. Lastly, there are 6^n total sequences of length n. Therefore, the probability she stops after $n \geq 4$ rolls is

$$\frac{2 \cdot \binom{n-1}{2} \cdot (4^{n-3}-3^{n-3})}{6^n}$$

To find the total probability, we just sum up over n now. This is the sum

$$\sum_{n=4}^{\infty} \frac{2 \cdot \binom{n-1}{2} \cdot (4^{n-3}-3^{n-3})}{6^n} = 2\left[\sum_{n=4}^{\infty} \binom{n-1}{3-1} \frac{1}{6^3} \cdot \left(\frac{2}{3}\right)^{n-3} - \sum_{n=4}^{\infty} \binom{n-1}{3-1} \frac{1}{6^3} \cdot \left(\frac{1}{2}\right)^{n-3}\right]$$

The terms in the parenthesis look similar to Negative Binomial distributions with $r=3$ and $p=1/3$ and $1/2$, respectively However, they are slightly off, we need the constant terms $\frac{1}{6^3}$ to be $\frac{1}{3^3}$ and $\frac{1}{2^3}$, respectively. Therefore, we just need to multiply by 2^3 and 3^3 in each sum. However, the support of a Negative Binomial starts at $r=3$, and these start at $r=4$. To remedy this, it is easy to take the complement to evaluate each sum with the complement and evaluating the PMF at 3 for each. Doing all of this,

$$2\left[\frac{1}{2^3}\sum_{n=4}^{\infty} \binom{n-1}{3-1} \frac{1}{3^3} \cdot \left(\frac{2}{3}\right)^{n-3} - \frac{1}{27} \cdot \sum_{n=4}^{\infty} \binom{n-1}{3-1} \frac{1}{2^3} \cdot \left(\frac{1}{2}\right)^{n-3}\right] = \frac{1}{4} \cdot \frac{26}{27} - \frac{2}{27} \cdot \frac{7}{8} = \frac{19}{108}$$

Solution to Question 793: Independent Children

To demonstrate independence, we must have that $\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$. We first find $\mathbb{P}[A]$ and $\mathbb{P}[B]$. For A, the probability that with n children there is at least one of each gender is easily calculated via the complement. The probability of all boys or all girls is just $\frac{2}{2^n} = \frac{1}{2^{n-1}}$, so the probability of at least one child of each gender is $\mathbb{P}[A] = 1 - \frac{1}{2^{n-1}}$. For $\mathbb{P}[B]$, we can either have 0 or 1 girls in the family of n, so there is 1 outcome where we have all boys (equivalently, 0 girls) and n outcomes where we have one girl (choose the birth location of the girl among the n), so $\mathbb{P}[B] = \frac{n+1}{2^n}$. For $\mathbb{P}[A \cap B]$, we need the probability of at most one girl and at least one boy and one girl. This is just the second case that went into calculating $\mathbb{P}[B]$, as that was exactly 1 girl and at least 1 boy, so $\mathbb{P}[A \cap B] = \frac{n}{2^n}$.

Now, we must solve $\frac{n}{2^n} = \frac{2^{n-1}-1}{2^{n-1}} \cdot \frac{n+1}{2^n}$ for n. Multiplication by 2^{2n-1} and rearranging yields that $2^{n-1} = n+1$, which can be seen to be equal at $n = 3$ by testing a few values.

Solution to Question 794: Distinct Biased Coins

The probability of winning given $H = t$ is t^6. Thus:

$$P(win) = \int_0^1 2t \times t^6 dt = \frac{1}{4}$$

Solution to Question 795: Exact Bills II

Referring to "Exact Bills I", \$99 is the most difficult amount to make using the bills at our disposal. It requires a total of 4 \$20 bills, 3 \$5 bills, and 4 \$1 bills. We also have the exact change for any transactional amount less than \$100 as long as we have those bills and of that quantity as we can just use a subset of those bills.

Solution to Question 796: Double Data Trouble I

Recall that $R^2 = 1 - \frac{SSE}{SST}$. With twice the data that is identical to the original, the new SSE is exactly twice what it was before, as we have duplicated every data point and the mean is unchanged. For this same reason, SST also doubles. Therefore, the ratio $\frac{SSE}{SST}$ is unchanged, meaning that R^2 is unchanged from before.

Solution to Question 797: Bowl Distributions

Once the white balls are distributed, there exists only one way for the black balls to be distributed such that each bowl gets 3 balls. There are $\binom{7+5-1}{5-1} = 330$ ways to do this without restrictions. However, we need to ensure that no bowl gets more than 3 white balls. Luckily for us, only one bowl can get more than 3 balls, which means we can avoid PIE. There are 5 ways to choose a bowl to assign 4 white balls to. Then, the remaining 3 balls need to be partitioned amongst the 5 bowls; there are a total of $\binom{3+5-1}{5-1} = 35$ ways to do this. Our answer is

$$330 - 5 \cdot 35 = 155$$

Solution to Question 798: Brownian Difference

By expanding out the interior, $\mathbb{E}\left[\left(\sum_{m=1}^{2^n} \Delta_{m,n}^2 - t\right)^2\right] = \mathbb{E}\left[\left(\sum_{m=1}^{2^n} \Delta_{m,n}^2\right)^2 - 2t\sum_{m=1}^{2^n} \Delta_{m,n}^2 + t^2\right]$. Applying linearity of expectation, we get that the above is

$$\mathbb{E}\left[\left(\sum_{m=1}^{2^n} \Delta_{m,n}^2\right)^2\right] - 2t\mathbb{E}\left[\sum_{m=1}^{2^n} \Delta_{m,n}^2\right] + t^2$$

We know that since $\Delta_{m,n} \sim N(0, t/2^n)$ by stationarity, $\mathbb{E}[\Delta_{m,n}^2] = \frac{t}{2^n}$. Therefore, the sum in the second term is $2^n \cdot \frac{t}{2^n} = t$. For the first sum, we expand out the square and pair up all the terms. Namely,

$$\mathbb{E}\left[\left(\sum_{m=1}^{2^n} \Delta_{m,n}^2\right)^2\right] = \mathbb{E}\left[\sum_{i=1}^{2^n}\sum_{j=1}^{2^n} \Delta_{i,n}^2\Delta_{j,n}^2\right] = \mathbb{E}\left[\sum_{i=1}^{2^n} \Delta_{i,n}^4\right] + \sum_{1\leq i\neq j\leq 2^n} \mathbb{E}[\Delta_{i,n}^2]\mathbb{E}[\Delta_{j,n}^2]$$

We get the last equality above by breaking up into where $i = j$ and $i \neq j$. This is because of the independence when $i \neq j$. For the first sum, we use the fact that if $Z \sim N(0, \sigma^2)$, then $\mathbb{E}[Z^4] = 3\sigma^4$. In this case, the first sum evaluates to $2^n \cdot 3\left(\frac{t}{2^n}\right)^2$. There are $2^n \cdot 2^n - 2^n = 2^n(2^n - 1)$ terms in the second sum. We already evaluated each term inside that second sum to be $\left(\frac{t}{2^n}\right)^2$, so we get that the combination of the two sums above yields

$$\mathbb{E}\left[\left(\sum_{m=1}^{2^n} \Delta_{m,n}^2\right)^2\right] = t^2\left(1 + \frac{1}{2^{n-1}}\right)$$

Adding up all three terms from the second line, we get that our answer is $\frac{t^2}{2^{n-1}}$. Plugging in $t = 1$ and $n = 5$ yields $\frac{1}{16}$ as our answer.

Solution to Question 799: Optimal Bidders II

It is known that given $X_1, \ldots, X_n \sim \text{Unif}(0, 1)$ IID and $S_{i,n}$ is the ith smallest, $1 \leq i \leq n$, among $X_1, \ldots, X_n$, $\mathbb{E}[S_{i,n}] = \frac{i}{n+1}$. This also extends now to Unif(500, 1000), as this is just a scaling and shifting of a Unif(0, 1). Namely, if $X \sim \text{Unif}(0, 1)$, $500X + 500 \sim \text{Unif}(500, 1000)$. Therefore, if $S'_{i,n}$ is the ith smallest among n IID Unif(500, 1000) random variables and $S_{i,n}$ is as above, $\mathbb{E}[S'_{i,n}] = 500\mathbb{E}[S_{i,n}] + 500 = 500 + \frac{500i}{n+1}$. In this case, $i = n - 1$, as we are paid out the second largest bid, so the expected payout with n people is $500 + 500\frac{n-1}{n+1} = 1000\left(1 - \frac{1}{n+1}\right)$. The cost of obtaining n people is $10n$. Therefore, our profit, which we can denote as P_n, is $P_n = S'_{i,n} - 10n$, so $\mathbb{E}[P_n] = \mathbb{E}[S'_{i,n}] - 10n = 1000\left(1 - \frac{1}{n+1}\right) - 10n$. We want to find n that maximizes this. Calling this function $f(n)$, we can treat n as continuous and find the value of n that maximizes it by taking the derivative and setting it equal to 0.

This means $f'(n) = \frac{1000}{(n+1)^2} - 10 = 0$. Rearranging yields $(n + 1)^2 = 100$, so as n must be positive, $n = 9$ bidders gives us the maximal expected profit. Plugging this in, $\mathbb{E}[P_9] = 810$, which is our solution.

Solution to Question 800: Poisoned Kegs III

Using the same idea as Poisoned Kegs II, we can let each of the 10 servants represent a binary indicator for the first 2^{10} kegs. Thus, in the first 2^{10} kegs, we will know exactly which keg is poisoned. However, now we just need to narrow it down to 10 kegsper possible death sequence. The easiest way to do this is to have each of the 2^{10} subsets of servants test 10 different kegs. Then, if some subset of servants die, the king will know that one of those 10 kegs that the subset of servants died to has the poison. Therefore, the king can test 10 times as many kegs if he only wants at least a 10% chance of identifying it. Therefore, the king can test up to $n = 2^{10} \cdot 10 = 10240$ kegs

Solution to Question 801: Integer Polygon

The prime factorization of 360 is $360 = 2^3 \cdot 3^2 \cdot 5$. Therefore, there are $(3+1)(2+1)(1+1) = 24$ total factors of 360. However, we need to remove 1 and 2 as factors, as those are not polygons. Therefore, there are $24 - 2 = 22$ values of n that are valid.

Solution to Question 802: Binary Option

We can create a replication of this. The region where $S_T \geq 24$ is simple as it is just the binary call. We can imagine the negative region as a binary put. We know that a $P_0 + C_0 = B_0$. So, we can price the binary put as $P_0 = B_0 - C_0 = 0.9 - 0.73 = 0.17$. Since this portion of the payoff is negative, we are shorting the binary put. Combining all the values, we get:

$$V_0 = C_0 - P_0 = 0.73 - 0.17 = 0.56$$

Note: this derivatives contract almost acts like a binary call, but we may have to pay if it expires OTM. This type of contract may be useful when we are looking for a cheaper contract (0.73 vs. 0.56).

Solution to Question 803: Even Blocks

Let's number the 12 slots where the blocks can be placed from 1-12. Notice that, if, say, a red block is in an odd-numbered slot, then the other red block must be in an even-numbered slot. Hence, there are $6! \cdot 6!$ ways to order the 12 blocks such that there are an even number of blocks between every identically colored pair of blocks. Since there are a total of $12!/2^6$ ways to order the blocks with no condition in place, our probability is $\frac{6! \cdot 6! \cdot 2^6}{12!} = \frac{16}{231}$.

Solution to Question 804: Planets Aligned

The planets can be aligned either on the same or opposite side of the sun. Therefore, the periods are really $20, 30$, and 45. We want the least common multiple of these, which is simply 180.

Solution to Question 805: Coin Pair III

Let T be the total number of heads and X be the total number of heads that appear on the first flipping of the coins. Then $\mathbb{E}[T] = \mathbb{E}[\mathbb{E}[T \mid X]] = \sum_{k=0}^{4} \mathbb{E}[T \mid X = k]\mathbb{P}[X = k]$ by Law of Total Expectation. $X \sim \text{Binom}\left(4, \dfrac{1}{2}\right)$ because X counts the number of heads appearing in 4 flips of a fair coin.

If $X = 4$, then obviously we don't flip any coins again, so $\mathbb{E}[T \mid X = 4] = 4$. Similarly, if $X = 3$, then we aren't able to flip just one tails, so $\mathbb{E}[T \mid X = 3] = 3$.

If $X = 2$, then we are going to continually flip the two tail coins again until we don't obtain TT. Conditioned on the fact that we don't obtain TT, 2 of the 3 equally-likely outcomes result in us having 3 heads, while 1 of the 3 results in having 4 heads, so $\mathbb{E}[T \mid X = 2] = \frac{2}{3} \cdot 3 + \frac{1}{3} \cdot 4 = \frac{10}{3}$.

Iterating this logic, if $X = 1$, then we flip two tails until they don't appear TT. With probability $\frac{2}{3}$ we end up with 2 heads, but we can iterate the process again from there on the remaining two tails and reach an expected $\frac{10}{3}$ heads. With probability $\frac{1}{3}$, we reach 3 heads and we're done. Therefore, $\mathbb{E}[T \mid X = 1] = \frac{2}{3} \cdot \frac{10}{3} + \frac{1}{3} \cdot 3 = \frac{29}{9}$.

Lastly, if $X = 0$, with probability $\frac{2}{3}$, we end up with 1 head, so we can iterate the process again to have an expected number of heads of $\frac{29}{9}$. With probability $\frac{1}{3}$, we end up with 2 heads, and we iterate the process again to get an expected number of heads of $\frac{10}{3}$. Therefore,

$$\mathbb{E}[T \mid X = 0] = \frac{2}{3} \cdot \frac{29}{9} + \frac{1}{3} \cdot \frac{10}{3} = \frac{88}{27}$$

Plugging the values and the PMF of X into our expression from the beginning, we get that

$$\mathbb{E}[T] = \frac{1}{16} \cdot 4 + \frac{4}{16} \cdot 3 + \frac{6}{16} \cdot \frac{10}{3} + \frac{4}{16} \cdot \frac{29}{9} + \frac{1}{16} \cdot \frac{88}{27} = \frac{88}{27}$$

Solution to Question 806: Too Many Primes to Count

Let X be the probability that they are tied after 10000 throws each.

There are two ways player 2 can win:

Case 1: Player 2 can be ahead after 10000 throws from each, probability $\frac{1}{2} \cdot (1\hat{a}X)$

Case 2: They can be tied at 10000 and player 2 might throw a prime number on the 10001st, probability $\frac{1}{2} \cdot X$

Thus the total probability that player 2 wins is $\frac{1}{2} \cdot (1 - X) + \frac{1}{2} \cdot X = \frac{1}{2}$

Solution to Question 807: Infected Dinner II

Consider grouping the 1000 people into 25 groups of 40 people. In 25 rounds of 39 minutes each, every person in a given bubble talks to every other person in that bubble. Additionally, each of the other bubbles are paired up.

Suppose the original sick person is in Bubble 25 and the last healthy person is in Bubble 24. In round k, we can have all the people in bubble b talk to all of the people in bubble $24 - b + k \bmod 25$, where 0 maps to 25. Therefore, we see that after k rounds, the sick people belong to people in Bubble 25 and Bubbles $1, 2, \ldots, k-1$. This is since at round 1, Bubble 25 talks with itself. Afterwards, it talks with Bubbles $1, 2, \ldots, k-1$. Therefore, Bubble 24 will be completely untouched until after the first 24 rounds, which is 936 minutes. Then, in the 937th minute, everyone in the 24th bubble talks to everyone in the 25th bubble, meaning all are infected in the first minute. Therefore, after 937 minutes, everyone is infected.

Solution to Question 808: Exponential Brownian Motion

Using Ito's lemma with $f(x) = x^n$, we have that

$$dX_t = nW_t^{n-1}dW_t + \frac{1}{2}n(n-1)W_t^{n-2}d[W,W]_t = nW_t^{n-1}dW_t + \frac{1}{2}n(n-1)W_t^{n-2}dt$$

For this to be a martingale, the dt term must vanish, which occurs precisely when $n = 0, 1$. Therefore, $0! + 1! = 2$.

Solution to Question 809: Sheep Stealing

Let n be the number of sheep in the flock. Let n_1 and n_2 be the numbers remaining after the first and second raids respectively. We have

$$409 = n_2 - \left(\frac{n_2}{5} + \frac{3}{5}\right) \Rightarrow n_2 = 512$$

Then, we have

$$512 = \left(n_1 - \frac{n_1}{4}\right) + \frac{1}{4} \Rightarrow n_1 = 683$$

Finally, we have

$$683 = n - \left(\frac{n}{3} + \frac{1}{3}\right) \Rightarrow n = 1025$$

Solution to Question 810: Make Your Martingale VI

We compute $\mathbb{E}[S_t \mid S_s]$ for any $0 \leq s \leq t$ first. Namely,

$$\mathbb{E}[S_t \mid S_s] = \mathbb{E}\left[e^{N(t)\log(c)-\lambda t} \mid S_s\right] = \mathbb{E}\left[e^{((N(t)-N(s))+N(s))\log(c)-\lambda s-\lambda(t-s)} \mid S_s\right]$$

Now, we can take out $e^{N(s)\log(c)-\lambda s} = S_s$. By the independent increments property of the Poisson Process, the remaining terms inside are independent of S_s, so the conditional expectation becomes unconditional. Namely, this means

$$\mathbb{E}[S_t \mid S_s] = S_s\mathbb{E}\left[e^{(N(t)-N(s))\log(c)-\lambda(t-s)}\right] = S_s e^{-\lambda(t-s)}M(\log(c))$$

where $M(\theta)$ is the MGF of a Poisson$(\lambda(t-s))$ random variable. We get this because of the fact that $N(t) - N(s) \sim$ Poisson$(\lambda(t-s))$. We have that $M(\theta) = e^{\lambda(t-s)(e^\theta-1)}$, so $M(\log(c)) = e^{\lambda(t-s)(c-1)}$.

Therefore, $\mathbb{E}[S_t \mid S_s] = S_s e^{\lambda(t-s)(c-2)}$. To make the constant 1, we need to make the exponent 0 regardless of s and t. This means $c = 2$.

Solution to Question 811: Bags of Fruit

The key here is to use the fact that all bags are mislabeled. For example, the bag labeled "mix" must contain either only apples or only oranges. Note how the bags labeled "apple" and "orange" are symmetric, and thus we choose a fruit from the "mix" bag. Without loss of generality, if the fruit we get is an apple, then we know that the "mix" bag is truly the "apple" bag. We are left to identify the true "mix" and "orange" labels from the mislabeled "apple" and "orange" bags. Because we know that all bags are mislabeled, the oranges must be in the "apple" bag, not the "orange" bag, and the mixed fruit are in the "orange" bag. Note we only picked one fruit from the "mix" bag.

Solution to Question 812: Minimal Shade

The total number of unit squares shaded black is cs. Meanwhile, if each row has at most $t-1$ black unit squares, then by the Pigeonhole Principle, at most $r(t-1)$ unit squares could have been shaded black. Thus, if $cs > r(t-1)$, then any valid shading configuration must yield a row with at least t black unit squares.

Conversely, if $cs \leq r(t-1)$, then there exists a shading configuration in which no rows have at least t black unit squares, which we construct as follows: (1) number the rows $0, \ldots, r-1$ and the columns $0, \ldots, c-1$; (2) in each column $i \in \{0, 1, \ldots, c-1\}$, shade the s unit squares in the rows $((si+j) \bmod r)$ for integers $j \in \{0, 1, \ldots, s-1\}$. If we follow this procedure by iterating i and j in ascending order, then for each integer $k \in \{1, 2, \ldots, cs\}$, the kth square shaded lies in row $((k-1) \bmod r)$. Therefore, row 0 has at least as many black unit squares as any other row, and row 0 has exactly $\lceil cs/r \rceil$ black unit squares per our construction. By the assumption $cs \leq r(t-1)$, $\lceil cs/r \rceil$ is at most $t-1$, so no row has more than $t-1$ black unit squares. Thus, the answer to this problem is $c = \left\lfloor \frac{r(t-1)}{s} \right\rfloor + 1$. Evaluating with our given values, we get the value 43.

Solution to Question 813: Beer Barrel II

To achieve the task in the fewest possible transactions (17 in total), follow these steps:

1. Fill the 7-quart measure.

2. Fill the 5-quart measure.

3. Empty 108 quarts from the barrel.

4. Empty the 5-quart measure into the barrel.

5. Fill the 5-quart measure from the 7-quart measure.

6. Empty the 5-quart measure into the barrel.

7. Pour 2 quarts from the 7-quart measure into the 5-quart measure.

8. Fill the 7-quart measure from the barrel.

9. Fill up the 5-quart measure from the 7-quart measure.

10. Empty the 5-quart measure into the barrel.

11. Pour 4 quarts from the 7-quart measure into the 5-quart measure.

12. Fill the 7-quart measure from the barrel.

13. Fill up the 5-quart measure from the 7-quart measure.

14. Empty the contents of the 5-quart measure.

15. Fill the 5-quart measure from the barrel.

16. Empty 5 quarts from the 5-quart measure.

17. Empty 1 quart from the barrel into the 5-quart measure.

By following these steps, you can accomplish the task using only 17 transactions, which is the minimum number required.

Solution to Question 814: Show The Face

An incorrect solution goes as follows: Since we want the first 6 to appear before the first 5, we can eliminate 5 from our sample space and look at the reduced sample space conditional on no roll being a 5. There are 5 remaining values in this sample space that are equally probable, so the number of rolls needed to see a 6 conditional on this information is Geom(1/5), which has mean 5. Therefore, the answer is 5

This above solution is incorrect because it more implicitly assumes that we just "ignore" 5s that appear. Therefore, this is not the correct assumption we want to make.

Let's let N represent the number of rolls needed to see a 6, and let B be the event that 6 occurs before 5. By symmetry, $\mathbb{P}[B] = 1/2$. We now condition on the value of the first roll. Let R_i represent the event that we roll value i on the first roll. This would imply by Law of Total Expectation that

$$\mathbb{E}[N \mid B] = \sum_{i=1}^{6} \mathbb{E}[N \mid R_i, B]\mathbb{P}[R_i \mid B]$$

For $i = 1, 2, 3, 4$, we have that $\mathbb{P}[R_i \mid B] = \dfrac{P[B \mid R_i]\mathbb{P}[R_i]}{\mathbb{P}[B]}$. Since the event that we roll $i = 1, 2, 3, 4$ on the first roll doesn't affect the probability of appearance of 6 before 5, we have that $\mathbb{P}[B \mid R_i] = \mathbb{P}[B]$. Clearly $\mathbb{P}[R_i] = 1/6$ based on the fact the die is fair, so $\mathbb{P}[R_i \mid B] = 1/6$ for $i = 1, 2, 3, 4$.

We have that $\mathbb{P}[R_5 \mid B] = 0$, as there is no chance that our first roll can be 5 is 6 comes before 5. Thus, since the conditional probability measure is a probability measure, we have that $\mathbb{P}[R_6 \mid B] = 1 - 4 \cdot 1/6 = 1/3$.

If we roll a 6 on the first roll, then we get it in 1 roll, so $\mathbb{E}[N \mid B, R_6] = 1$. Otherwise, if $i = 1, 2, 3, 4$, then $\mathbb{E}[N \mid B, R_i] = 1 + \mathbb{E}[N \mid B]$, as we used up one roll and need to continue forward. Therefore, the equation we need to solve is

$$\mathbb{E}[N \mid B] = \frac{2}{3}\left(1 + \mathbb{E}[N \mid B]\right) + \frac{1}{3}$$

This is easily solved to yield $\mathbb{E}[N \mid B] = 3$.

You can instead view this problem as saying that whenever we roll a 5 or 6, we make it a 6 so that the 6 occurs before the 5. Therefore, the probability of rolling a 5 or 6 on each trial is 1/3, meaning that $N \mid B \sim \text{Geom}(1/3)$, which has mean 3. This agrees with the answer above.

Solution to Question 815: Cyclic 4

Clearly the integer isn't 4, so it has at least 2 digits. Suppose it has exactly 2 digits. Then it is in the form $a4$ for some integer a. We would want $4(10a+4) = 40 + a$, as $40 + a$ is the representation of $4a$. This would mean that $39a = 24$ which is not possible.

Note here that the first digit of our integer must be 1, as when multiplied by 4, the first digit must now be 4. Therefore, for three digits, we can see that the form is $1a4$ for an integer a. We would need to have that $4(104 + a) = 410 + a$, so $3a = -6$. This is not possible, as our integer must be non-negative.

Next, suppose it has 4 digits. Therefore, it is in the form $1ab4$. We would need to satisfy $4(1004 + 100a + 10b) = 4100 + 10a + b$, which would imply that $390a + 39b = 84$. This is not possible, as $39 \nmid 84$.

We are starting to see a pattern here. The integer on the RHS with n non-fixed digits is $4 \cdot 10^n + 10^{n-1}$, whereas the integer on the LHS is $4 \cdot 10^n + 16$. Therefore, we need to find the smallest n such that their difference, which is $10^{n-1} - 16$, is divisible by 39. The terms on the LHS, before division, are all $39 \cdot 10^k$ for $0 \leq k \leq n - 1$. Try equating this representation to the example above.

For $n = 3$, this number is 984, which is not divisible by 39. However, for $n = 4$, the number on the RHS is $9984 = 39 \cdot 256$. Therefore, the number must be in the form $1abcd4$, as we have 4 free digits in the middle. Then, the equation we need to satisfy in the non-negative integers is

$$39000a + 3900b + 390c + 39d = 9984$$

Clearly $a = 0$, as everything must be non-negative. Dividing by 39, we see that $100b + 10c + d = 256 = 100 \cdot 2 + 10 \cdot 5 + 6$, which clearly yields $a = 0, b = 2, c = 5, d = 6$. Therefore, our answer is 102564. We can quickly verify that $410256 = 4 \cdot 102564$.

Solution to Question 816: Perfect Correlation I

Suppose that X and Y have respective variances σ_X^2 and σ_Y^2. Then $\text{Cov}(X+Y, X-Y) = \text{Cov}(X,X) - \text{Cov}(Y,Y) = \sigma_X^2 - \sigma_Y^2$. Similarly, we know that $\text{Var}(X+Y) = \sigma_X^2 + \sigma_Y^2 + 2\sigma_X\sigma_Y = (\sigma_X + \sigma_Y)^2$, so $\sigma_{X+Y} = \sigma_X + \sigma_Y$. By a similar argument, we get $\text{Var}(X-Y) = \sigma_X^2 + \sigma_Y^2 - 2\sigma_X\sigma_Y = (\sigma_X - \sigma_Y)^2$, so $\text{Var}(X-Y) = |\sigma_X - \sigma_Y|$. Note that the absolute values are needed here since we can't have a negative variance.

Therefore, we have that $\rho(X+Y, X-Y) = \dfrac{\sigma_X^2 - \sigma_Y^2}{(\sigma_X + \sigma_Y)|\sigma_X - \sigma_Y|} = \dfrac{\sigma_X - \sigma_Y}{|\sigma_X - \sigma_Y|} = -1$. We get the -1 from the condition that $\sigma_Y > \sigma_X$, so the numerator is negative.

Another way to see this is that if X and Y are perfectly correlated, then $Y = aX + b$ for some constants a and b. Since $\sigma_Y > \sigma_X$, it must be the case that $|a| > 1$. Therefore, $X + Y = X(1+a) + b$ and $X - Y = X(1-a) + b$. As $|a| > 1$, one of the constants between $1+a$ and $1-a$ is positive and one is negative. Therefore, as these are both linear transformation of X with opposing signs, they must have correlation -1.

Solution to Question 817: Defining Variance

V[Z] = V[3X-2Y] = 9V[X] + 4V[Y] = 36+16=52

Solution to Question 818: Whole Lotta Dice

No matter what the sum of any 99 of the dice, there is only one face on the 100th die that will yield a sum that's divisible by 100. For example, say the first 99 rolls add up to 3075. Only rolling a 25 on the last die will make the sum divisible by 100. Thus the answer is $\dfrac{1}{100}$.

Solution to Question 819: Fixed Point Limit II

For large n, the distribution of $F_n \approx \text{Binom}(n, 1/n)$ because each of the n values has probability $1/n$ of being mapped to itself. However, each value being a fixed point are not independent events, as knowledge of one being fixed increases the probability of another being fixed. However, for large n, the change is so small that it is nearly independent. One can make this argument rigorous to show that the (formal) conditions of the Poisson Limit Theorem apply. Since Poisson Limit Theorem applies, we can conclude that as $n \to \infty$, $F_n \implies \text{Poisson}(1)$ (this means convergence in distribution). Therefore, $\mathbb{P}[F_n = 5]$ will converge to $\dfrac{1}{5!}e^{-5}$, so our answer is $\dfrac{1}{120}$.

Solution to Question 820: Fair Bounty I

As the money is behind a random door, the number of selections needed to find the money is discrete uniform on $\{1, 2, \ldots, 7\}$. Note that it is not geometric since the success probability changes each time. The expected number of selections needed to find the money is therefore 4. For the game to be fair, this means each draw must cost $x = \dfrac{200}{4} = 50$.

Solution to Question 821: OLS Review II

We assume $\mathbb{E}[Y] = \beta_0 + \beta_1 x$. The method of least squares will allow us to find $\hat{\beta}_0, \hat{\beta}_1$, the estimators for $\mathbb{E}[Y]$. We know that

$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}},$$
$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}.$$

For our data, we easily find $S_{xy} = \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = 7$, and $S_{xx} = \sum_{i=1}^{n}(x_i - \bar{x})^2 = 10$. Hence $\hat{\beta}_1 = \frac{7}{10}$, and $\hat{\beta}_0 = 1 - 0 = 1$. The sum of the two estimators is $\dfrac{17}{10}$.

Solution to Question 822: Absolute Normal Difference

Since X and Y are independent, we have that $Y - X \sim N(0, 5)$. Therefore, if $W = Y - X \sim N(0, 5)$, we want to compute $\mathbb{E}[|W|]$. Note that $W = \sqrt{5}Z$, where $Z \sim N(0, 1)$, so we can really just compute $\sqrt{5}\mathbb{E}[|Z|]$. We can compute this via LOTUS for a general $Z \sim N(0, \sigma^2)$, as the calculation is the same:

$$
\begin{aligned}
E[|Z|] &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} |x| \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \\
&= \frac{2}{\sqrt{2\pi\sigma^2}} \int_0^{\infty} x \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \\
&= \sqrt{\frac{2}{\pi\sigma^2}} \left(-\sigma^2 \exp\left(-\frac{x^2}{2\sigma^2}\right)\right)\Bigg|_0^{\infty} \\
&= \sqrt{\frac{2}{\pi}}\sigma,
\end{aligned}
$$

From line 1 to line 2, we use symmetry of the integrand about 0. Then, from line 2 to line 3, we apply a $u-$substitution to obtain that indefinite integral. In particular, $\sigma = \sqrt{5}$ here, as $\sigma^2 = 5$, so our answer is $\sqrt{\frac{10}{\pi}}$, meaning that $bK = 0.5 \cdot 10 = 5$.

Solution to Question 823: Meek Mill

Let P be the event Meek Mill performs, L be the event he is in Philadelphia, and B be the event he is in Baltimore. We want

$$
\mathbb{P}[L \mid P] = \frac{\mathbb{P}[PL]}{\mathbb{P}[L]} = \frac{\mathbb{P}[P \mid L]\mathbb{P}[L]}{\mathbb{P}[P \mid L]\mathbb{P}[L] + \mathbb{P}[P \mid B]\mathbb{P}[B]} = \frac{0.8 \cdot 0.1}{0.8 \cdot 0.1 + 0.2 \cdot 0.8} = \frac{1}{3}
$$

All of the quantities are derived from the question itself by interpreting them in our language of events.

Solution to Question 824: Likely Targets III

The key here is to note that since the targets are of extremely small radius, we can essentially treat them as points. The approximate probability we hit target A would be approximately $f(x_A)\varepsilon$, where $f(x_A)$ is the probability density at point A. However, since B is also of small radius 2ε, the probability we hit target B is approximately $2f(x_B)\varepsilon$. Likewise, the probability we hit target C is approximately $3f(x_C)\varepsilon$. Since the targets are disjoint, our goal is to maximize the weighted sum of the probability densities. Our probability density here is dependent on what μ we select. Therefore, as a function of μ, we need to maximize

$$
f(\mu) = \frac{1}{2\sqrt{2\pi}} \left(e^{-\frac{(-1-\mu)^2}{8}} + 2e^{-\frac{(3-\mu)^2}{8}} + 3e^{-\frac{(5-\mu)^2}{8}}\right)
$$

The interior terms are just the density of a $N(\mu, 4)$ distribution at $-1, 3,$ and 5, respectively. To do this, we take the derivative and set it equal to 0. In particular,

$$f'(\mu) = \frac{1}{2\sqrt{2\pi}} \left[\frac{-1-\mu}{4} e^{-\frac{(-1-\mu)^2}{8}} + \frac{3-\mu}{2} e^{-\frac{(3-\mu)^2}{8}} + 3 \cdot \frac{5-\mu}{4} e^{-\frac{(5-\mu)^2}{8}} \right] = 0$$

Using a computer system, $f'(\mu) = 0$ for $\mu \approx 4.211$. One can verify that this is indeed a maxima, so 4.21 is our answer. This intuitively makes sense, as the increased size of the region around 3 and 5 means that we should assign more density there. Furthermore, the region around 5 is slightly larger than the region around 3, so the center should be placed closer to 5 than 3.

Solution to Question 825: Lots of LOTUS

To find $\mathbb{E}[Z]$, this is equivalent to finding $\mathbb{E}[e^{\theta X - \frac{1}{2}\theta^2}]$. Note that $\theta \in \mathbb{R}$, so this is the same as $e^{-\frac{1}{2}\theta^2}\mathbb{E}[e^{\theta X}]$. Note that $\mathbb{E}[e^{\theta X}]$ is just the MGF of X. We can find the MGF of X quickly. X has PDF $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, so

$$\begin{aligned}
\mathbb{E}[e^{\theta X}] &= \int_{\mathbb{R}} e^{\theta x} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2} + \theta x} dx \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}[(x-\theta)^2 - \theta^2]} dx \\
&= e^{\frac{1}{2}\theta^2} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\theta)^2}{2}} dx
\end{aligned}$$

The integrand inside is the PDF of a $N(\theta, 1)$ distribution, so the integral is just 1. Therefore, the MGF of X is $e^{\frac{1}{2}\theta^2}$, implying $\mathbb{E}[Z] = e^{\frac{1}{2}\theta^2} \cdot e^{-\frac{1}{2}\theta^2} = 1$.

Solution to Question 826: Minimax Box

We will want to use our alternate form of expectation for non-negative integer-valued random variables $\mathbb{E}[X] = \sum_{k=1}^{\infty} \mathbb{P}[X \geq k]$. In this case, our random variable X is the smallest value of cards in the box with card 100. The event $\{X \geq k\}$ means that all of cards valued $1, 2, \dots, k-1$ are in the other box. Each of those cards is in the other box with probability $\frac{1}{2}$, so $\mathbb{P}[X \geq k] = \frac{1}{2^{k-1}}$. The maximal smallest value is 100, which occurs in the event that 100 is the only card in the box it is placed in. Therefore,

$$\mathbb{E}[X] = \sum_{k=1}^{100} \frac{1}{2^{k-1}} = \sum_{k=0}^{99} \frac{1}{2^k} = \frac{1 - 2^{-100}}{1 - \frac{1}{2}} = 2 - 2^{-99} = 2(1 - 2^{-100})$$

This means the answer to our question is $2 \cdot 100 = 200$.

Solution to Question 827: Lock and Key Pair

Try 4 of the keys on the first lock. If any work, then we can stop early. However, the worst case scenario is that the one we don't test is the one that works, in which we would attempt all 4 keys. Similarly, of the 4 remaining unassigned keys, pick 3 of them to test. Repeat this with 3 locks and 2 locks left to yield 2 and 1 attempts, respectively. Therefore, we need to try the locks a total of $4 + 3 + 2 + 1 = 10$ times.

Solution to Question 828: Maximize Head Ratio II

The strategy here is close to optimal but needs simulations to fully reach optimality. However, this strategy makes sense intuitively because if the current ratio is 0.5 : 1, we should always try and get another head as we can always get back to a ratio of 0.5 : 1.

To put this strategy in a mathematical context, the optimal ratio will be $\dfrac{n+1}{2n+1}$ where n is a number from 0 to infinity. Thus we should always end on an odd roll. To find the expectation, we need to find a weighted sum of this series, the weight being the probability we end on $2n+1$. When we stop on $2n+1$ and $n \neq 0$, we know the last toss needs to be heads and for $k = 1, 2, ..., 2n$, the first k tosses can't include more heads than tails (we would've stopped the flips earlier in that case).

To count such sequences, we can use Dyck words (length $2n$) and the n-th Catalan number can be found by $C_n = \dfrac{1}{n+1}\binom{2n}{n}$.

Since each of the Catalan numbers have probability of $\left(\dfrac{1}{2}\right)^{2n}$ and the last flip is heads with probability of $\dfrac{1}{2}$, the expectation of the ratio becomes $\displaystyle\sum_{n=0}^{\infty} p_n \cdot \frac{n+1}{2n+1} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{4^n(2n+1)} \cdot \binom{2n}{n}$. Using the Taylor series for $\arcsin(x)$, this summation simplifies to

$$\frac{1}{2} \cdot \arcsin(1) = \frac{\pi}{4} \approx 0.785$$

Solution to Question 829: Mean Cards

The optimal strategy will be the in the form of keeping all cards at least k for some threshold k. Therefore, as a function of k, say $f(k)$, let's find the expected profit of the strategy.

The mean of all of the cards valued at least k is $\frac{100+k}{2}$, which would be our payout. Then, we need to factor in the discarding cost, so we would discard all the $k-1$ cards below value k. Therefore, $f(k) = \frac{100+k}{2} - (k-1) = 51 - \frac{k}{2}$ for $1 \leq k \leq 100$. This is a decreasing function as k increases, so the optimal strategy is just to keep all of the cards! The expected payoff from this strategy is $51 - \frac{1}{2} = \frac{101}{2}$.

Solution to Question 830: Mixing Wine

Pretend the total capacity of each glass is 24. Then, there is 8 in glass A and 6 in glass B. When we combine contents into a jug, 14 of the 48 parts of the jug is wine. Hence, our answer is $\frac{7}{24}$.

Solution to Question 831: Random Minimal Sum

We can write $S = \sum_{i=1}^{\infty} \frac{X_i}{2^i} I_{N \geq i}$, as this indicator evaluates to 0 if $i > N$ and 1 otherwise, so the summation is equivalent. Assuming valid interchange of expectation and infinite summation (this can be rigorously justified with Fubini-Tonelli Theorem), $\mathbb{E}[S] = \sum_{i=1}^{\infty} \frac{1}{2^i} \mathbb{E}[X_i I_{N \geq i}]$. What remains is to calculate that expectation.

To calculate $\mathbb{E}[X_i I_{N \geq i}]$, it is easiest to condition on the value of X_i. This is because of the fact that it is easy to evaluate the probability all of the remaining values are at least X_i from this. Therefore, $\mathbb{E}[X_i I_{N \geq i}] = \mathbb{E}[\mathbb{E}[X_i I_{N \geq i} \mid X_i]] = \mathbb{E}[X_i \mathbb{E}[I_{N \geq i} \mid X_i]]$. This expectation on the inside evaluates to the probability that given X_i, $1 > X_1 > \ldots X_{i-1} > X_i$. The probability all of the first $i-1$ random variables are at least as large as X_i is $(1-X_i)^{i-1}$, as it occurs with probability $1 - X_i$ independently for each random variable. The probability that $X_1 > X_2 > \cdots > X_{i-1}$ is $\frac{1}{(i-1)!}$, as there are $(i-1)!$ equally likely orderings of the IID values and this is one of them. Therefore, as the ordering and the restriction on the minimum value are independent, $\mathbb{E}[I_{N\geq i} \mid X_i] = \dfrac{(1-X_i)^{i-1}}{(i-1)!}$. Therefore, $\mathbb{E}[X_i I_{N\geq i}] = \mathbb{E}\left[\dfrac{X_i(1-X_i)^{i-1}}{(i-1)!}\right]$.

Using LOTUS,

$$\begin{aligned}\mathbb{E}\left[\frac{(1-X_i)^{i-1}}{(i-1)!}\right] &= \int_0^1 t \cdot \frac{(1-t)^{i-1}}{(i-1)!} \cdot 1dt \\ &= \frac{1}{(i-1)!}\int_0^1 t(1-t)^{i-1}dt \\ &= \frac{\Gamma(i)\Gamma(2)}{\Gamma(i+2)} \cdot \frac{1}{\Gamma(i)} \\ &= \frac{1}{(i+1)!}\end{aligned}$$

Note that this is because of the Beta integral $\displaystyle\int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$. Using this, we can sum the series:

$$\mathbb{E}[S] = \sum_{i=1}^{\infty} \frac{1}{2^i(i+1)!} = 2\sum_{i=1}^{\infty} \frac{1}{2^{i+1}(i+1)!} = 2\sum_{i=2}^{\infty} \frac{\left(\frac{1}{2}\right)^i}{i!}$$

This last sum is just that of $\sqrt{e}$ without the $i=0$ and $i=1$ terms. Therefore, that sum evaluates to $\sqrt{e} - 1 - \dfrac{1}{2\cdot 1!} = \sqrt{e} - \dfrac{3}{2}$. Substituting this in, $\mathbb{E}[S] = 2\sqrt{e} - 3$. This means $a = 2, c = -3, b = \frac{1}{2}$. This implies $a^2 + c^2 + 4b = 15$.

Solution to Question 832: Paying Bail

This is a Markov Chain question and you can tell by the existence of states in this question (state of given \$0, \$100, \$200, \$300, \$400, and \$500). Let E_0 be the number of bills you have to give on average given the state of having given the jailor \$0 or starting over after giving a fake bill. Then E_i for i in $\{100, 200, 300, 400, 500\}$ be the states of having given the jailor \$$i$ (all real) in a row (again, the average number of bills needed to be given). The equations become:

$$E_0 = \frac{1}{2}E_1 + \frac{1}{2}E_0 + 1$$

$$E_1 = \frac{1}{2}E_2 + \frac{1}{2}E_0 + 1$$

$$E_2 = \frac{1}{2}E_3 + \frac{1}{2}E_0 + 1$$

$$E_3 = \frac{1}{2}E_4 + \frac{1}{2}E_0 + 1$$

$$E_4 = \frac{1}{2}E_5 + \frac{1}{2}E_0 + 1$$

$$E_5 = 0$$

When you solve for E_0 (the state of expectation we are trying to find), you get 62 bills needing to be pulled out on average.

Solution to Question 833: Dual Die View

The key here is to set up a Markov chain. Let e_i be the expected number of rounds it takes given you already have i distinct faces. We want e_0. The boundary condition is that $e_6 = 0$, as we would already be done at that point. We can work backwards on this.

We would have that $e_5 = 1 + \frac{25}{36}e_5 + \frac{11}{36}e_6$. This is because if we already have 5 values, then with probability $\frac{5}{6}$ per die we roll a value that has already been rolled. This doesn't contribute to anything further, so $e_5 = \frac{36}{11}$ after rearranging.

We would have that $e_4 = 1 + \frac{16}{36}e_4 + \frac{18}{36}e_5 + \frac{2}{36}e_6$ because of the fact that if we have 4 values, with probability $\frac{4}{6}$ per die we roll a value we have already seen. We get the $\frac{2}{36}$ from the fact that the first die has 2 values that could appear that are new and only 1 new value for the last one. As the probabilities must sum to 1, the coefficient of e_5 must be $\frac{18}{36}$. Substituting in, we get that $e_4 = \frac{271}{55}$.

Continuing this pattern, we have that $e_3 = 1+\frac{9}{36}e_3+\frac{21}{36}e_4+\frac{6}{36}e_5$ by similar logic to the above. For example, to get 2 new distinct values, there are 3 values the first die can take and 2 values the second can take. Plugging in the prior values gives $e_3 = \dfrac{949}{165}$. The equations for the remaining values are

$$\begin{aligned} e_2 &= 1 + \frac{4}{36}e_2 + \frac{20}{36}e_3 + \frac{12}{36}e_4 \\ e_1 &= 1 + \frac{1}{36}e_1 + \frac{15}{36}e_2 + \frac{20}{36}e_3 \\ e_0 &= 1 + \frac{1}{6}e_1 + \frac{5}{6}e_2 \end{aligned}$$

When finishing up the solving of the system above, the final result is $e_0 = \dfrac{70219}{9240} \approx 7.6$. This is just above the 7.35 that you get if you were to divide the solution for one die (14.7) by two. This is a good sanity check.

Solution to Question 834: Position Guess

The optimal strategy is going to be in the form of guessing the minimum if the value is $\leq x$, median if the value is between x and y, and maximum if the value if $\geq y$. However, note that $y = 1 - x$ because of the fact that if $X_1, X_2, X_3 \sim \text{Unif}(0,1)$ IID, $1 - X_1, 1 - X_2, 1 - X_3 \sim \text{Unif}(0,1)$ IID as well, so the threshold for guessing the maximum would be $1 - x$.

Now, we need to find $p(x)$, the probability as a function of x of guessing correctly under this rule. There are three cases to consider, which is if the value if $\leq x$, between x and $1 - x$, and then $\geq 1 - x$. The first and third cases we can treat as the same by the same argument as above. The probability of the revealed value being at most x is x. Then, given the value is in $(0, x)$, it is uniform on that interval. Say X_1 is the value showed to us. By conditioning on $X_1 = a$,

$$\mathbb{P}[X_1 = \min\{X_1, X_2, X_3\}] = \int_0^x \mathbb{P}[X_1 = \min\{X_1, X_2, X_3\} \mid X_1 = a] f_{X_1}(a) da = \frac{1}{x}\int_0^x (1-a)^2 da = \frac{1}{3}a^2 -$$

The $(1 - a)^2$ term comes from the probability both X_2 and X_3 are larger than a. Multiplying by the probability of $X_1 \leq a$ and doubling to account for the third case, we get a term of $2a\left(\frac{1}{3}a^2 - a + 1\right)$ for our final probability.

Now, we account for the middle case, which occurs with probability $1 - 2a$. If X_1 is in this interval, it is uniform throughout it. Therefore, conditioning on $X_1 = a$ once again, we would now guess X_1 is the median, so

$$\mathbb{P}[X_1 = \text{med}\{X_1, X_2, X_3\}] = \int_x^{1-x} \mathbb{P}[X_1 = \text{med}\{X_1, X_2, X_3\} \mid X_1 = a] f_{X_1}(a) da = \frac{1}{1-2x}\int_x^{1-x} 2a(1-a)$$

The $2a(1-a)$ term comes from the fact that one of the remaining two random variables is above and the other is below a with 2 ways to order them. Multiplying by $1-2x$ to account for the probability of the median being in $(x, 1-x)$, adding it to the first term of $p(x)$ from above, and simplifying it all, we get $p(x) = 2x^3 - 4x^2 + 2x + \frac{1}{3}$.

To maximize $p(x)$, we take the derivative. Namely, $p'(x) = 6x^2 - 8x + 2$, of which the zeroes are $x^* = \frac{8 \pm \sqrt{64-48}}{12} = \frac{1}{3}, 1$. As $0 < x < \frac{1}{2}$ (since then the regions would overlap), we conclude $x^* = \frac{1}{3}$ is our maximizer, with $p(x^*) = \frac{17}{27}$.

Solution to Question 835: Child Births

If n is the number of samples we observed in our data collection and $\sum_{i=1}^{n} x_i = N$, where x_i is the value associated with the ith sample, then for a prior distribution of Gamma(a, b), our posterior mean is

$$\frac{a + \sum_{i=1}^{n} x_i}{b+n}$$

In this case, $n = 34$ and $\sum_{i=1}^{34} x_i = 100$, so we get that our posterior mean is $\frac{32+100}{10+34} = 3$.

Solution to Question 836: Broken Trading System

If we look at the Black-Scholes PDE, we notice that many of the greeks are inherently in the differential equation. We can write the PDE as the following:

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma + rS\Delta = rC$$

We can plug in all the values, and solve for C. This gives us $C = 6.67$. Note, if $r = 0$, we cannot obtain a value for the option using this method.

Solution to Question 837: Close Couples

Let I_i be the indicator that couple i sits together. Then we have that $T = \sum_{i=1}^{n} I_i$ counts the number of couples that sit together total. By linearity of expectation, $\mathbb{E}[T] = \sum_{i=1}^{n} \mathbb{E}[I_i]$. We have that $\mathbb{E}[I_i]$ is the probability couple i sits together. Fix the husband within couple i at an arbitrary seat in the table. The two people on either side of him must be women, as there is an alternating pattern at the table. Therefore, of the n spots that this man's wife can sit at, 2 are adjacent to him, and all arrangements are equally likely. Therefore, the probability of this is just $\frac{2}{n}$. Therefore, we have that $\mathbb{E}[I_i] = \frac{2}{n}$, so $\mathbb{E}[T] = \sum_{i=1}^{n} \frac{2}{n} = 2$.

Solution to Question 838: Dangerous Doubles

The four possible outcomes of this game are TT, HT, TH, and HH. TT gives you \$0, as there are no heads. HT and TH both give you \$2, as there is exactly one head. HH makes you pay \$7 by the information in the question. As all four outcomes are equally likely, our expected profit/loss on the game is $\frac{0+2+2-7}{4} = -\frac{3}{4}$.

Solution to Question 839: Fair Bounty II

Thinking of the two doors with money as aces, we want to find the expected number of doors that need to be opened, say b, until we find the first door with money. Our answer would then be $\frac{200}{b}$, as the expected cost to find the first door with money is $b \cdot \frac{200}{b} = 200$.

With the doors having money fixed, there are 5 other empty doors. The two doors with money divide up our total selection into 3 equally-sized regions in expectation. Therefore, there are, on average, $\frac{5}{3}$ doors before the first door with money. We must add one in for the selection of the door with actual money. We get $b = \frac{8}{3}$, so

$$x = 200 \cdot \frac{3}{8} = 75$$

Solution to Question 840: Non-Disjoint Subsets

There are $2^5 = 32$ ways to select each of A and B, as there are 2^5 subsets of a set of size 5. Note that each of the 5 elements are equally-likely to belong to $A \cap B, A \cap B^c, A^c \cap B$, and $A^c \cap B^c$. We count by complement here. For the two sets to be disjoint, each of the elements must belong to $A \cap B^c, A^c \cap B$, or $A^c \cap B^c$ Thus, there are 3 options per element, so there are 3^5 ways to assign the elements so that the sets A and B are disjoint. Thus, there are $2^5 \cdot 2^5 - 3^5 = 781$ ways to select A and B so that they are disjoint. There are $2^5 \cdot 2^5 = 1024$ total ways to select A and B, so the probability is $\dfrac{781}{1024}$.

Solution to Question 841: Uniform Triangle

For the triangle to not be valid, one of $X > Y + Z, Y > X + Z$, or $Z > X + Y$ must hold. Note that these three events are also mutually disjoint, as only the largest side could be larger than the sum of the other two. By the symmetry of these regions, we can just compute the probability of one of them and then multiply by 3.

To compute $\mathbb{P}[Z > X + Y]$, note that the plane $z = x + y$ inside the unit cube bounds a tetrahedron above it. Namely, this tetrahedron with vertices $(0, 0, 0), (1, 0, 1), (0, 1, 1), and (0, 0, 1)$. The volume of this tetrahedron is $dfracbh3$, with b being the area of the base and h being the height. Namely, the base is a right triangle with side lengths 1 and 1, so $b = \dfrac{1}{2}$. Clearly $h = 1$ by looking at $(0, 0, 0)$ to $(0, 0, 1)$. Therefore, the volume is $\dfrac{1}{6}$. As there are three such regions, the probability our triangle can't be formed is $\dfrac{1}{2}$, meaning with probability $\dfrac{1}{2}$ it can be formed.

Solution to Question 842: First Flip

Let X and Y respectively count the number of flips needed for Jay and John to get their first heads. We want $\mathbb{E}[X \mid X < Y]$. The distributions of X and Y are IID $\text{Geom}\left(\dfrac{1}{2}\right)$. Using the direct definition of LOTUS, we have

$$\mathbb{E}[X \mid X < Y] = \sum_{k=1}^{\infty} k \cdot \mathbb{P}[X = k \mid X < Y] = \sum_{k=1}^{\infty} k \cdot \frac{\mathbb{P}[X = k, X < Y]}{\mathbb{P}[X < Y]}$$

The numerator can be written as $\mathbb{P}[X = k, Y > k] = \mathbb{P}[X = k]\mathbb{P}[Y > k]$ by independence of X and Y. The PMF of X is $\mathbb{P}[X = k] = \frac{1}{2^k}$ for integers $k \geq 1$. Similarly, $\mathbb{P}[Y > k]$ means that it takes more than k flips for John to see his first heads, meaning all of the first k flips are tails. This also occurs with probability $\frac{1}{2^k}$. Lastly, we just need to compute $\mathbb{P}[X < Y]$.

Note that as X and Y are IID, they are exchangeable, so $\mathbb{P}[X > Y] = \mathbb{P}[X < Y]$. Furthermore, regardless of the outcome, one of the events $\{X = Y\}, \{X > Y\}$, or $\{X < Y\}$ will occur and they are disjoint. Therefore, $\mathbb{P}[X > Y] + \mathbb{P}[X < Y] + \mathbb{P}[X = Y] = 1$. Substituting in the equality above, $\mathbb{P}[X < Y] = \frac{1 - \mathbb{P}[X = Y]}{2}$. It just remains to find $\mathbb{P}[X = Y]$.

Simply enough, we just sum over all possible values of k that both X and Y can be, so $\mathbb{P}[X = Y] = \sum_{k=1}^{\infty} \mathbb{P}[X = k]\mathbb{P}[Y = k] = \sum_{k=1}^{\infty} \frac{1}{4^k} = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}$. Thus, $\mathbb{P}[X < Y] = \frac{1 - \frac{1}{3}}{2} = \frac{1}{3}$ as well.

Substituting all of this back into our original summation for $\mathbb{E}[X \mid X < Y]$, we get

$$\mathbb{E}[X \mid X < Y] = \sum_{k=1}^{\infty} k \cdot 3 \cdot \frac{1}{4^k} = \sum_{k=1}^{\infty} k \cdot \frac{3}{4} \cdot \left(\frac{1}{4}\right)^{k-1} = \frac{4}{3}$$

We evaluate the last sum by noting that the term after the k is just the PMF of a $\text{Geom}\left(\frac{3}{4}\right)$ random variable, so we actually showed that $X \mid X < Y \sim \text{Geom}\left(\frac{3}{4}\right)$, and thus the expectation is $\frac{1}{\frac{3}{4}} = \frac{4}{3}$.

Solution to Question 843: Josephus' Dilemma

This is a classic brain teaser known as the Josephus Problem. The best way to solve this problem is to start with simpler cases and see which position is the last alive. When thereâs only 1 person, Person 1 is the last to be alive (obviously). 2 people, still Person 1. 3 people, Person 3 is the last to remain alive. 4 people, Person 1 is last. If you keep doing this, youâll notice that when there are 2^n soldiers, Person 1 is the last to be alive. The greatest power of 2 that's less than or equal to 2000 is 1024. This means that 976 soldiers have to be killed and then the person that has the sword will be the one to last be alive.

After n people have been killed, the $2n + 1$ Person will have the sword. For our case, this comes out to $2 * 976 + 1 = 1953$. Thus you should be in the 1953^{rd} position of the circle to be the last person alive.
For a more general case with N people in the circle, we can express N as $2^n + r$ and the position last to survive is $2r + 1$.

Solution to Question 844: Largest Inscribed Circle

Recall from euclidean geometry that the area of a triangle can be expressed as the product of its inradius and its semiperimeter. Following routine application of the pythagorean theorem, we find that the altitude of the triangle from the base of length 24 is $4\sqrt{7}$, so the area is $48\sqrt{7}$. The semiperimeter, being half the perimeter, is 28, so we can compute the inradius as $\frac{12\sqrt{7}}{7}$. The area of the incircle is $\frac{144\pi}{7}$, so our answer is $144 + 7 = 151$.

Solution to Question 845: Dot Removal

It is equally likely to select an even or odd value to decrease the value by 1 on. Subtracting one turns the side into one of opposite parity. If we select an even value to decrease, with probability $\frac{2}{6} = \frac{1}{3}$ we roll an even value. If we select an odd value to decrease, with probability $\frac{4}{6} = \frac{2}{3}$ we roll an even value. Therefore, the total probability is

$$\frac{\frac{1}{3} + \frac{2}{3}}{2} = \frac{1}{2}$$

Solution to Question 846: Clockwise Murder

Notice that person 1 will always win if there are a total of 2^k people. This is because powers of 2 can halve without a remainder. If N is not a power of 2, then we simply need to eliminate the first $N - X$ people. The number of the person begins when there are X total people left will win. Starting with player 1, we eliminate $N - X$ people, meaning that we kill every even-numbered player until $2(N - X)$, inclusive. Our survivor is therefore numbered $2(N - X) + 1 = 2N - 2X + 1$. Hence, $a + 2b + 3c = 1$.

Solution to Question 847: Coloring Components I

Let C_n be the number of connected components when we have n squares in a line. We will derive a recurrence relation for $\mathbb{E}[C_n]$.

Suppose that we want to find $\mathbb{E}[C_n]$, the expected number of connected components with n squres in a row. We can find this by conditioning on the $n-1$st square. We know that with probability $\frac{1}{2}$ each, it matches or differs from the $n-1$st square. If it matches, then we have $\mathbb{E}[C_{n-1}]$ connected components, as we don't add in any new ones if it matches. If it differs, then we have $1+\mathbb{E}[C_{n-1}]$ connected components, as the differing color will add in a new component. Therefore, by Law of Total Expectation, $\mathbb{E}[C_n] = \frac{1}{2}\cdot\mathbb{E}[C_{n-1}]+\frac{1}{2}\cdot(1+\mathbb{E}[C_{n-1}]) = \mathbb{E}[C_{n-1}] + \frac{1}{2}$. This recurrence along with the initial condition that $\mathbb{E}[C_1] = 1$ (as we have 1 component), yields the solution $\mathbb{E}[C_n] = \frac{n+1}{2}$. In particular, $n = 25$, so $\mathbb{E}[C_{25}] = 13$.

Solution to Question 848: Bank Arbitrage

We can see that if we borrow \$100 from the bank account and use that money to buy the bond, we will gain a free \$2.08 at the end of the 1 year. This is because we will owe the bank account \$100 and some interest, but we will gain more interest from the bond. In the 1 year, we can see that our \$100 bond will grow to $V_T = 100e^{0.05(1)}$. Similarly, we can see that we will owe the bank account $V_T = 100e^{.03(1)}$. We can return the principal and interest from the bond to repay the bank account, and obtain $100e^{0.05(1)} - 100e^{.03(1)} \approx 2.08$ while doing so.

Solution to Question 849: Grid Filling I

This is not possible. There are 5 odd and 4 even numbers, and for three numbers to sum to an even number, either one or three of the numbers must be even. In other words, we cannot have any row or column with two evens. Looking at the columns, by Pigeonhole Principle, there must be at least one column with all even numbers, because there are four even numbers for three columns, and no column can have only two even numbers as noted. However, once a column is stacked with three even numbers, the last even number will necessarily cause a row to have two even numbers, and thus that row will sum to an odd. Because there are no possible combinations where the sum of each row and column is even, the probability that the sum of each row and column is even is 0.

Solution to Question 850: Midrange

The range of X_1 and X_2 is $O_2 - O_1$, as it is the larger minus the smaller. The midpoint is simply $\frac{X_1 + X_2}{2}$. Therefore, we want $\mathbb{P}\left[O_2 - O_1 > \frac{X_1 + X_2}{2}\right]$. However, the trick here is that X_1 and X_2 correspond O_1 and O_2, as each random variable must be one of the order statistics. Therefore, this probability is the same as $\mathbb{P}\left[O_2 - O_1 > \frac{O_1 + O_2}{2}\right]$. Rearranging, this is equivalent to $\mathbb{P}[O_2 > 3O_1]$. The joint PDF of the order statistics, by our order statistics formula, is given by $f(x, y) = 2e^{-(x+y)} I_{(0,\infty)}(x) I_{(x,\infty)}(y)$. Therefore, we have that our probability is given by

$$\int_0^\infty \int_{3x}^\infty 2e^{-(x+y)} dy dx = \int_0^\infty 2e^{-4x} dx = \frac{1}{2}$$

Solution to Question 851: Maximize Head Ratio I

Half of the time you will get a heads on the first flip and your ratio would be $1 : 1$. This is the best ratio you could achieve playing this game so you stop there. The other half of the time you get tails on the first flip (current ratio $0 : 1$). However, due to the law of large numbers, you know that the more you flip, the ratio of heads to total flips will get closer and closer to $0.5 : 1$. Thus the expected ratio is $0.5 * 1 + 0.5 * 0.5 = 0.75$.

Solution to Question 852: Poisson Review III

$$\mathbb{E}[50 - 2X - X^2] = 50 - 4 - \mathbb{E}[X^2]$$

. We can compute $\mathbb{E}[X^2]$ from our knowledge of the variance of X.

$$\begin{aligned} \text{Var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= \mathbb{E}[X^2] - 4 = 2 \\ \mathbb{E}[X^2] &= 6 \end{aligned}$$

Our answer is $50 - 4 - 6 = 40$.

Solution to Question 853: Picky Primes

All primes besides 2 are odd, so we get a sum that is odd precisely when we select integers that are all not 2. In other words, we select our 4 integers from the other 15 primes. There are $\binom{15}{4}$ ways to pick the 4 from the other 15 and $\binom{16}{4}$ total ways to pick 4 primes from the 16. Therefore, our probability is

$$\frac{\binom{15}{4}}{\binom{16}{4}} = \frac{3}{4}$$

Solution to Question 854: OLS Review I

Recall $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}$, and $\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}}$, where $S_{xy} = \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$ and $S_{xx} = \sum_{i=1}^{n}(x_i - \bar{x})^2$. We find $\bar{x} = 0, \bar{y} = 1.5$. Plugging in values, we find:

$$\begin{aligned} S_{xy} &= -6, \\ S_{xx} &= 10, \\ \hat{\beta}_1 &= -0.6, \\ \hat{\beta}_0 &= 1.5 \end{aligned}$$

We conclude $\mathbb{E}[Y] = \hat{y} = 1.5 - 0.6x$, so our answer is $1.5 - 0.6 = 0.9$.

Solution to Question 855: Square Billy

Notice that if Billy is on one of the diagonals of the square, then the probabilities that he ends up on a horizontal or vertical side should equal one another. At (1, 2), three moves make Billy end up on a diagonal, and one move (left) makes Billy immediately end up on a vertical side. Hence, our probability is

$$\frac{1}{4} \cdot 1 + \frac{3}{4} \cdot \frac{1}{2} = \frac{5}{8}$$

Solution to Question 856: Basic Put Delta

If a put is deep in-the-money, it essentially acts like the underlying itself. A put is essentially another way to short (or sell) a stock. Selling a stock has a delta of -1 (think about taking the derivative of the stock with respect to itself). So, a deep ITM put will also have a delta of -1.

Solution to Question 857: 29 Divide

Let $S(n)$ be the sum of the digits of an integer n. The difference between any integer n and $S(n)$ is divisible by 9 i.e. $9 \mid n - S(n)$ for any integer n. Let x be the missing digit. Then we know that as there are 9 digits and one is missing, the sum of all the digits in 2^{29} is $45 - x$. Plugging this in, $9 \mid 2^{29} - (45 - x)$.

Now, note that $2^{29} = 2^5 \cdot (2^6)^4 = 2^5 \cdot 64^4 = 2^5 \cdot (63+1)^4$ by basic rearrangement. By the Binomial Theorem, every single term in $(63+1)^4$ is divisible by 63 except the last term, which is 1. Therefore, we can write $(63+1)^4 = 63k+1$ for some integer k. Substituting this back in, we get that

$$2^{29} = 2^5 \cdot (63k+1) = 63 \cdot 2^5 \cdot k + 2^5$$

Subtracting 2^5, we have that $2^{29} - 2^5 = 63 \cdot 2^5 \cdot k$. This means that $9 \mid 2^{29} - 2^5$.

Using this fact, as we know $9 \mid 2^{29} - (45 - x)$ and $9 \mid 2^{29} - 2^5$, we know that $9 \mid 2^5 - (45 - x)$, as we just subtracted two values divisible by 9. This means that $9 \mid x - 13$. As x is a digit, it must be between 0 and 9, inclusive of both. The only value where the above holds true in that range is $x = 4$, so 4 is the missing digit.

Solution to Question 858: Options Rho II

Mathematically, a put is $\max(K - S_T, 0)$, where K is the strike. K essentially acts as a constant cashflow. Since this is a cashflow and interest rates are at play, we must discount this. As a result, the quantity $K - S_T$ becomes smaller as K decreases due to the discounting. As a result, the price of a put option decreases.

Another way to think about this is that call and put options are dependent on the forward price. The forward price increases if rates increase, and so the price of call options will increase while those of put options will decrease.

Solution to Question 859: Points on a Circle I

Denote the center of a unit circle with point O. Select a point A and a point B such that $\angle AOB \in [0, \pi]$ (this selection is always possible). Denote $\angle AOB$ with θ. Notice that $\theta \sim \text{Unif}([0, \pi])$.

Now, consider an arbitrary value of θ. We must determine the region where point C may be located such that $\triangle ABC$ is obtuse. Note that $\triangle ABC$ is obtuse if A, B, C all fall in the same semicircle. There is a region of circumference $2\pi - \theta$

where C may be placed such that A, B, C all fall in the same semicircle. So, if we are given the value of θ, then $\triangle ABC$ is obtuse with probability $\frac{2\pi-\theta}{2\pi}$. By the law of total probability, we have (written somewhat informally):

$$\begin{aligned}
\mathbb{P}(\triangle ABC \text{ obtuse}) &= \int_0^\pi \mathbb{P}(\triangle ABC \text{ obtuse} \mid \theta)\, \mathbb{P}(\theta)\ d\theta \\
&= \int_0^\pi \frac{2\pi-\theta}{2\pi} \cdot \frac{1}{\pi}\ d\theta \\
&= \int_0^\pi \frac{1}{\pi} - \frac{\theta}{2\pi^2}\ d\theta \\
&= \frac{3}{4}
\end{aligned}$$

Solution to Question 860: Longest Rope I

Let $X \in [0, 1]$ be the location of the first cut and $Y \in [0, 1]$ be the location of the second cut such that $X < Y$. Because each segment is equally likely to be the longest, the expected length of the longest piece doesn't depend on which piece we choose, so we can solve for $E[X|X \text{ is the longest}]$. This, with our earlier constraint of $X < Y$, gives us two additional constraints:

$$X > Y - X \Rightarrow Y < 2X$$

$$X > 1 - Y \Rightarrow Y > 1 - X$$

These two constraints derive from the fact that X is given to be greater than the other two segments, $Y - X$ and $1 - Y$. Graphing the constraints within our sample space, we see that the area satisfying our inequalities is the triangle A with vertices $(\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 1)$, and $(\frac{1}{3}, \frac{2}{3})$, and triangle B with vertices $(\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 1)$, and $(1, 1)$. We wish to find the probability of an outcome occurring in either areas and the expected value of X within each area.

$$\text{Area of A: } \frac{1}{2} \times \frac{1}{2} \times (\frac{1}{2} - \frac{1}{3}) = \frac{1}{24}$$

$$\text{Area of B: } \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8} = \frac{3}{24} = 3 \times \text{Area of A}$$

$$\text{Expected value of X within A: } \frac{1}{2} - \frac{1}{3}(\frac{1}{2} - \frac{1}{3}) = \frac{4}{9}$$

$$\text{Expected value of X within B: } \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} = \frac{2}{3}$$

The expected value of X within each triangle is calculated using the property of centroids. A centroid, or the center of area for a triangle, is $\frac{1}{3}$ of the way from the base to the opposite vertex, a property that is most easily seen by integration. Finally, we can solve for the expected value of X given X is the longest piece:

$$E[X|X \text{ is the longest}] = \frac{1}{4} \times \frac{4}{9} + \frac{3}{4} \times \frac{2}{3} = \frac{11}{18}$$

Solution to Question 861: Bet Size I

Use Kelly Criterion formula. The formula follows $\frac{p(b+1)-1}{b}$ where p is the probability of winning the bet and b is the profit ratio. In this case, p is 0.75 and b is 1. Thus you should bet $\frac{0.75(1+1)-1}{1} = 0.5$. Thus you should bet 50% of your bankroll on this bet.

Solution to Question 862: Splitwise

Let X be the subtotal. Then, the post-tip total is $1.6X$, and your split is $\frac{1.6X}{8} = .2X$. Hence, your split is:

$$2 \times \frac{182.30}{10} = 36.46$$

Solution to Question 863: Fresh Fruits

Let a, p, and m denote the weight of one apple, one pear, and one plum respectively.

We have:

$$p + 3a = 10m$$
$$a + 6m = p$$

The first equation represents one pear and three apples weighing the same as 10 plums. The second equation represents 1 apple and 6 plums weight the same as one pear.

We substitute for p from the second equation into the first. This gives

$$a + 6m + 3a = 10m$$

Let $a = m$, then we see that $p = 7m$.

Solution to Question 864: Integral Limit

Let $u = x^{-1}$. Then $x = u^{-1}$ and $dx = -u^{-2}du$. So this u-sub yields

$$\begin{aligned}
\lim_{n\to\infty} n\int_{\frac{1}{1+\varepsilon}}^{1} \frac{1}{1+u^{-n}}u^{-2}du &= \lim_{n\to\infty} n\int_{\frac{1}{1+\varepsilon}}^{1} \sum_{k=1}^{\infty}(-1)^{k+1}u^{kn-2}du \\
&= \sum_{k=1}^{\infty}(-1)^{k+1}\lim_{n\to\infty}\int_{\frac{1}{1+\varepsilon}}^{1} nu^{kn-2}du \\
&= \sum_{k=1}^{\infty}(-1)^{k+1}\lim_{n\to\infty}\frac{n}{kn-1}\left(1-\frac{1}{(1+\varepsilon)^{kn-1}}\right) \\
&= \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = \ln(2),
\end{aligned}$$

The last sum there is well-known to converge to $\ln(2)$. We change the order of integration, summation, and limits using the absolute convergence of this sum on $|u| < 1$. Our answer is $k = 2$.

Solution to Question 865: Mixed Set I

There are $\binom{3}{2} = 3$ ways to pick the two elements from $\{1, 2, 3\}$ that we want to include. Then, we are able to include or exclude any amount of the remaining7 elements, which can be done in $2^7 = 128$ ways. Therefore, our answer is $3 \cdot 128 = 384$.

Solution to Question 866: Colorful Apples

Note that the 4 green apples divide up the other 50 red apples into 5 regions. Since we have no other information about these regions, in expectation, they are equally-sized. Therefore, we would expect 10 apples per region. In particular, exactly one region comes after the last green apple, so we would expect 10 red apples to be left after drawing the last green apple.

Solution to Question 867: Clarence's Bread

Here, we are looking for the expected number of draws needed until Clarence grabs all 3 large loaves of bread from the bag. Since we are doing this drawing without replacement, this implies we should implement some type of "First Ace" approach. However, the issue here is that when Clarence draws a large loaf, he replaces it by another small loaf. Therefore, this isn't exactly like the paradigm above, as when you draw an Ace, it is not replaced in any way. Our new strategy is to thus consider the expected number of draws needed to see each consecutive large loaf.

First, let's find the number of loaves Clarence eats until (and including) his first selection of a large loaf. There are 3 large loaves and 7 small loaves. Therefore, we can treat the large loaves as our "dividers" and the 7 small as filler between them. The loaves split up our drawing process into 4 equally-sized components (in expectation).

Therefore, we expect $\frac{7}{4}$ small loaves before the first large loaf. Afterwards, Clarence eats one of the two small loaves that results from the large loaf and then replaces the other in the bag. Therefore, he has eaten a total of $\frac{7}{4}+1=\frac{11}{4}$ loaves thus far. Additionally, there are $\frac{7}{4}-1=\frac{3}{4}$ fewer small loaves in the bag on average.

Now, we apply the same paradigm again but on $7-\frac{3}{4}=\frac{25}{4}$ small loaves and 2 large loaves. We get 3 equally-sized regions in expectation from the two large loaves, so there should be $\frac{25}{12}$ small loaves coming before the next large loaf. Therefore, Clarence eats $\frac{25}{12}+1=\frac{37}{12}$ small loaves on average between the first and second large loaf selection. This brings our total thus far to $\frac{11}{4}+\frac{37}{12}=\frac{35}{6}$.

With one large loaf left and an (on average) $\frac{25}{4}-\frac{13}{12}=\frac{31}{6}$ small loaves left, we would expect half of those small loaves to come before the large loaf. Therefore, we expect Clarence to eat $\frac{31}{12}+1=\frac{43}{12}$ more large loaves until completion.

This yields that Clarence would need to eat an expected number of $\frac{35}{6}+\frac{43}{12}=\frac{113}{12}$ small loaves total.

Solution to Question 868: Normal LOTUS I

By multivariate LOTUS, we can set up our double integral for this expectation as

$$\mathbb{E}\left[\frac{X}{Y+1}\right]=\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{x}{y+1}\cdot xe^{-x(y+1)}I_{(0,\infty)}(x)I_{(0,\infty)}(y)dydx$$

Combining and applying the indicators, we obtain

$$\int_0^\infty \int_0^\infty \frac{x^2 e^{-x(y+1)}}{y+1} dydx$$

Note that the inner integrand looks to be somewhat in the form of a Gamma$\left(3, \frac{1}{y+1}\right)$ random variable PDF. This is because of numerator specifically. We switch the order of integration because the variable of integration for this PDF is X. Therefore, we get $\int_0^\infty \frac{1}{y+1} \int_0^\infty x^2 e^{-x(y+1)} dxdy$ after moving out the $\frac{1}{y+1}$. This integral must integrate to the normalizing constant that we would multiply and divide by to get the PDF. Since $a = 3$ and $b = \frac{1}{y+1}$ for this Gamma distribution, the normalizing constant is $\left(\frac{1}{y+1}\right)^3 \Gamma(3) = \frac{2}{(y+1)^3}$. Therefore, the inner integral evaluates to $\frac{2}{(y+1)^3}$. Using this, we get the remaining integral as

$$\int_0^\infty \frac{2}{(y+1)^4} dy$$

Some simple calculus shows that this last integral is just $\frac{2}{3}$, meaning that $\mathbb{E}\left[\frac{X}{Y+1}\right] = \frac{2}{3}$.

Solution to Question 869: Coordinate Jumper

Any path from $(0,0,0)$ to $(3,4,5)$ will consist of $3, 4$, and 5 movements in the x, y, and $z-$directions, respectively. Therefore, the number of distinct paths can really be written as an anagram of $XXXYYYYZZZZZ$, where X, Y, and Z represent movements in the x, y, and $z-$directions, respectively. There are

$$\binom{12}{3,4,5} = \frac{12!}{3!4!5!} = 27720$$

such sequences, so this is our answer.

Solution to Question 870: Good Accuracy

The total area of the dartboard is 9π, as the circles are concentric. The area of the circle of radius 1 is just π. The area of the second region is $(2^2 - 1^2)\pi = 3\pi$, as the circles are concentric. Lastly, the area of the third region is $(3^2 - 2^2)\pi = 5\pi$. Therefore, the probability of the dart landing in each of the three regions is, respectively, $\frac{1}{9}, \frac{3}{9}$, and $\frac{5}{9}$. There are $3! = 6$ orders in which we can land in the three regions. Additionally, the probability of any such particular ordering is $\frac{1 \cdot 3 \cdot 5}{9^3} = \frac{5}{243}$. Therefore, the probability of landing in the three distinct regions in 3 shots is

$$3! \cdot \frac{5}{243} = \frac{10}{81}$$

Solution to Question 871: Make a Market II

We want to quote a market on the product $A-B$. A naive answer may be to set our bid at $4-10=-6$ and our ask at $5-12=-7$. However, this is incorrect. To replicate $A-B$, we need to long 1 unit of A and short 1 unit of B. Since we are market-makers, we will quote the bid of $A-B$ to be $\text{Bid}_A - \text{Ask}_B = 4-12 = -8$ since we are willing to buy A at the bid of A and sell B at the ask of B. Using similar logic, we can quote our ask to be $\text{Ask}_A - \text{Bid}_B$ as we are now selling A and going long B. This gives us an ask of $5-10=-5$ and a market of -8 @ -5.

Plugging in our ask and bid, we get $Y^2 - X^2 = (-5)^2 - (8)^2 = -39$.

Solution to Question 872: Straddle Arbitrage II

We are under the assumption that the straddles themselves must have non-negative prices, so the answer to this question must be in the form of a straddle spread. If we construct the straddle spread, we can see that we have a negative payoff of -3 (regardless of the straddle spread we choose to create... short the first vs. short the second). This means that if we receive a credit of at least 3, our minimum payoff will be 0 and there will be no chance of losing money.

Solution to Question 873: Real Solutions

Define $f(x) = Mx^3 + Nx - c$. If there a zero in the interval $(0,1)$ AKA some $x^* \in (0,1)$ such that $f(x^*) = 0$, then we must have that $f(0)$ and $f(1)$ are of opposite signs. This is by the intermediate value theorem. One can show that $\mathbb{P}[f(x) \text{ is strictly increasing}] = 1$ by taking the derivative and noting it is always positive for $x \in (0,1)$. We have that $f(0) = -c$ and $f(1) = M + N - c$. We know that since $M, N \sim \text{Unif}(0,1)$, they are both positive values. They only take values between 0 and 1 individually.

For $0 < c < 2$, we have that $f(0) = -c < 0$ and $f(1) = M + N - c$. If we want $f(1) > 0$, then $M + N - c > 0$, so $M + N > c$. Thus, we want the value $\mathbb{P}[M+N>c]$. However, this is easier stated as $1-\mathbb{P}[M+N\le c]$. This is essentially asking for the CDF of $M+N$. Using convolution (or any other method you prefer), you get that the PDF $M+N$ is given by

$$f(z) = zI_{(0,1)}(z) + (2-z)I_{[1,2)}(z)$$

To find the CDF, we need to actually split up into cases of $0<c<1$ and $1\le c<2$, as the PDF splits there. For $0<c<1$, this is just the integral

$$\int_0^c z\,dz = \frac{c^2}{2}$$

as we are solely on the first branch of that indicator with $0<c<1$. If $1\le c<2$, we can write this integral as

$$\int_0^c f(z)dz = \int_0^1 f(z)dz + \int_1^c f(z)dz$$

The first integral evaluates to our previous expression evaluated at 1, which is just $\frac{1}{2}$, and our second becomes the integral $\int_1^c (2-z)dz = \left(2z-\frac{z^2}{2}\right)\Big|_1^c = 2c-\frac{c^2}{2}-\frac{3}{2}$. Thus, for $1\le c<2$, we have the CDF as

$$\frac{1}{2} + \left(2c-\frac{c^2}{2}-\frac{3}{2}\right) = 2c-\frac{c^2}{2}-1$$

Thus, our total CDF is given by

$$p(c) = \mathbb{P}[M+N\le c] = \frac{c^2}{2}I_{(0,1)}(c) + \left(2c-\frac{c^2}{2}-1\right)I_{[1,2)}(c)$$

This means that $a=\frac{1}{2}, b=d=0, r=-\frac{1}{2}, s=2, t=-1$. Adding up all of this, we get 1.

Solution to Question 874: Equalizer

Let p be the probability that Mike wins. If they roll the same value, which occurs with probability $\frac{1}{6}$, then Mike wins with probability 1. The probability that Mike rolls a strictly larger or smaller value than Dave are equal by symmetry of the dice. Namely, they are $\frac{1-\frac{1}{6}}{2}=\frac{5}{12}$. If Dave rolls a strictly larger value than Mike, then Mike wins with probability p. If Mike rolls a strictly larger value than Dave, then Mike does not win i.e. wins with probability 0. Therefore, by Law of Total Probability, this can be written as

$$p = \frac{1}{6}\cdot 1 + \frac{5}{12}\cdot p$$

Solving this yields $p=\frac{2}{7}$.

Solution to Question 875: Russian Roulette II

Let p be the probability that the first player, your friend, loses and $1-p$ be the probability that the second player, you, loses. We can condition the probability p on the first trigger pull. The first player loses with certainty $\frac{1}{6}$ of the time. Else, he essentially becomes the second player in the game with a conditional probability $1-p$ of losing. Thus,

$$p = \frac{1}{6} \times 1 + \frac{5}{6} \times (1-p) \Rightarrow p = \frac{6}{11}$$

Solution to Question 876: Contracts and Pricing I

We are interested in the expected gain after purchasing the contract. Specifically, if we purchase the contract, then we will only exercise our right to buy the stock at \$40 on Thursday if the value of the stock is greater than \$40. Let G denote the gain after purchasing the contract.

$$\mathbb{E}[G] = -15 + 0.2 \cdot 0.5 \cdot (50-40) + 0.2 \cdot 0.5 \cdot (x-40)$$

A fair price for the contract would assume $\mathbb{E}[G] = 0$.

$$\begin{aligned} 15 &= 1 + 0.1x - 4 \\ 18 &= 0.1x \\ x &= 180. \end{aligned}$$

Solution to Question 877: Basketball Practice

Consider $n = 3$. There are two possible outcomes: Frank makes 2 total free throws, or Frank makes 1 total free throw. Each occurs with probability $\frac{1}{2}$. Next, consider $n = 4$. If Frank makes 3 total free throws, then he must have made every free throw beginning with $n = 3$. This occurs with probability $\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$. If Frank makes 2 total free throws, then he either misses the 3rd free thrown or the 4th free throw, which occurs with probability $\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{3}$. If Frank makes 1 total free throw, then he must have missed both the 3rd and 4th free throws, which occurs with probability $\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$. We are already starting to see a pattern here, but we'll keep going for $n = 5$ just to confirm our suspicions. If Frank makes 4 total free throws, then he must have made every free throw beginning with $n = 3$, which occurs with probability $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{4}$. If Frank makes 3 total free throws, then he misses on either the 3rd, 4th, or 5th attempt, which occurs with probability $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{2}{4} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{2}{4} = \frac{1}{4}$. Similarly, if Frank makes 2 total free throws, then he makes it on either the 3rd, 4th, or 5th attempt, which occurs with probability $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{2}{4} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{2}{4} = \frac{1}{4}$. If Frank makes 1 total free throw, then he must have missed every attempt beginning with $n = 3$, which occurs with probability $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{4}$. Following this pattern, we see that the probability of achieving any number of throws on by the n-th attempt is just $\frac{1}{n-1}$. So our solution for this problem is $\frac{1}{99}$

Solution to Question 878: Game Show I

Let us look at the strategy in which you switch. With this strategy, you would only lose if you happen to pick the door with the car behind it on the first try, which happens with probability $\frac{1}{3}$. In other words, you would win whenever you happen to choose a goat on the first try, which happens with probability $\frac{2}{3}$.

Solution to Question 879: No Rock

Note that there is no point in playing paper, as paper doesn't defeat scissors nor paper. Therefore, suppose that you play rock with probability a and scissors with probability $1 - a$. Furthermore, suppose that your opponent plays paper with probability b and scissors with probability $1 - b$. If $P(a, b)$ represents your expected profit on this game with a and b fixed as above, we have that

$$P(a, b) = ab(-1) + a(1 - b)(1) + (1 - a)b(1) + (1 - a)(1 - b)(0) = a + b - 3ab$$

Keeping b fixed, taking the partial derivative, we see that $\dfrac{\partial P}{\partial a} = 1-3b$. Similarly, taking the partial derivative in b yields that $\dfrac{\partial P}{\partial b} = 1 - 3a$. Setting these both equal to 0 yields that there is an equilibrium at $a = b = \dfrac{1}{3}$. Therefore, your expected profit per round is

$$P(1/3, 1/3) = \frac{1}{3} + \frac{1}{3} - \frac{1}{3} = \frac{1}{3}$$

Solution to Question 880: Stages of Life

Let b, y, and m represent the number of years he spent in each stage of life in the order above. We want $b + y + m + 13$. We know that his age must be divisible by $3, 4$, and 5. The smallest integer such that this is possible is $60 = 3 \cdot 4 \cdot 5$. It so happens that this works, as he must have spent 15 years as a boy, 12 years as a youth, 20 years as a man, and 13 years in old age, adding up to 60.

Solution to Question 881: Digit Multiplication I

The prime factorization of 10000 is $10000 = 10^4 = 2^4 \cdot 5^4$. Therefore, our number needs to have 4 of each 2 and 5. Note that $2^3 = 8$, so we can condense three of the 2s into a single digit. Therefore, we need to make the smallest possible number using 4 of the digit 5 and one each of 2 and 8. We should arrange the digits from left to right in increasing order of value to make the number as small as possible. This is because the digits furthest to the left have the highest magnitude in terms of the final number. Therefore, the answer is 255558.

Solution to Question 882: Consecutive 1s

This question is simple enough that we can count the outcomes that fit our criteria. First are all the outcomes with 3 ones in a row. These include:

$\cdot 111xx$

$\cdot 111x1$

$\cdot x111x$

$\cdot xx111$

$\cdot 1x111$

Now for 4 ones in a row:

$\cdot 1111x$

$\cdot x1111$

Finally, thereâs only one way to get 5 ones in a row:

$\cdot 11111$

Now we need to find the probability of each of these outcomes and then we can add them up. The probability of having 3 ones and 2 non-ones is $\left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^2$. Similarly, the probability of having 4 ones and a non-one is $(\frac{1}{6})^4 \cdot (\frac{5}{6})$. Finally, all ones has a probability of $(\frac{1}{6})^5$. There are 3 possibilities with only 3 ones, 4 possibilities with 4 ones, and 1 possibility with 5 ones. Thus our answer becomes

$$3 \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^2 + 4 \cdot \left(\frac{1}{6}\right)^4 \cdot \left(\frac{5}{6}\right) + \left(\frac{1}{6}\right)^5 = \frac{1}{81}$$

Solution to Question 883: Busted 6 I

Given that we receive either a 5 or a 6, we roll 5 and 6 with equal probability 1/2. In the case we roll a 6, our payout is 0. In the case it is a 5, you cash out the sum of the previous rolls. If P is your payout, then

$$\mathbb{E}[P] = \mathbb{E}[P \mid 6]\mathbb{P}[6] + \mathbb{E}[P \mid 5]\mathbb{P}[5]$$

where 5 and 6 represent the events of rolling a 5 and 6 first, respectively. The first term vanishes, as the conditional expectation is 0. Therefore, $\mathbb{E}[P] = \frac{1}{2} \cdot \mathbb{E}[P \mid 5]$. On average, it takes 3 turns for the 5 to appear, since we are in the set with probability 1/3 per trial. This means that there are 2 roll on average before the 5. The expected payout of each of those rolls is 2.5, as we must have values $1-4$ in each of those first 2 trials. Furthermore, the outcomes are equally likely, so we have that $\mathbb{E}[P \mid 5] = 2 \cdot 2.5 = 5$. Therefore,

$$\mathbb{E}[P] = \frac{1}{2} \cdot 5 = \frac{5}{2}$$

Solution to Question 884: Coin Pair V

Let T be the total number of coin flips needed and X be the total number of heads that appear on the first flipping of the coins. Then $\mathbb{E}[T] = \mathbb{E}[\mathbb{E}[T \mid X]] = \sum_{k=0}^{4} \mathbb{E}[T \mid X = k]\mathbb{P}[X = k]$ by Law of Total Expectation. $X \sim \text{Binom}\left(4, \frac{1}{2}\right)$ because X counts the number of heads appearing in 4 flips of a fair coin. To set our notation, let e_i represent the additional number of flips needed to stop our process once we have i heads.

If $X = 4$, then obviously we don't flip any coins again, so $\mathbb{E}[T \mid X = 4] = 4$. This would mean $e_4 = 0$.

If $X = 3$, then we would be consider one heads and one tails. We should turn over the tails so that we have a guaranteed head i.e. we end up with either 3 or 4 heads after this process. Then, with equal probability, we would get a heads or tails on the remaining coin, so $e_3 = 2 + \frac{1}{2}e_3 + \frac{1}{2}e_4$. Putting in $e_4 = 0$ yields that $e_3 = 4$. This means $\mathbb{E}[T \mid X = 3] = 8$.

If $X = 2$, then we are considering 2 tails, meaning we turn one over and flip the other. This is exactly the same scenario as above, as we are guaranteed to have 3 or 4 heads after with the same probabilities, so $e_2 = 4$ as well. This means $\mathbb{E}[T \mid X = 2] = 8$.

If $X = 1$, then we consider 2 tails. We turn one over, and then we flip the other. We will end with either 2 or 3 heads after this iteration with equal probability. Therefore, $e_1 = 2 + \frac{1}{2}e_2 + \frac{1}{2}e_3$. As $e_2 = e_3 = 4$, $e_1 = 6$. This means $\mathbb{E}[T \mid X = 1] = 10$.

Lastly, if $X = 0$, we just consider 2 tails one again. We turn one over, and then we flip the other. We will end with either 1 or 2 heads after this iteration with equal probability, so $e_0 = 2 + \frac{1}{2}e_1 + \frac{1}{2}e_2$. Plugging in $e_1 = 6$ and $e_2 = 4$ yields $e_0 = 7$, so $\mathbb{E}[T \mid X = 0] = 11$.

Plugging the values and the PMF of X into our expression from the beginning, we get that

$$\mathbb{E}[T] = \frac{1}{16} \cdot 4 + \frac{4}{16} \cdot 8 + \frac{6}{16} \cdot 8 + \frac{4}{16} \cdot 10 + \frac{1}{16} \cdot 11 = \frac{135}{16}$$

Solution to Question 885: Ranged Max

Let $M = \max\{X_1, X_2, X_3, X_4\}$. We want $\mathbb{P}[2 < M \leq 3] = \mathbb{P}[M \leq 3] - \mathbb{P}[M \leq 2]$. The CDF of M is easy to derive. $\{M \leq x\}$ means that $\{X_1, X_2, X_3, X_4 \leq x\}$. By independence,

$$\mathbb{P}[M \leq x] = \mathbb{P}[X_1 \leq x]\mathbb{P}[X_2 \leq x]\mathbb{P}[X_3 \leq x]\mathbb{P}[X_4 \leq x] = (\mathbb{P}[X_1 \leq x])^4 = \frac{x^4}{4^4}$$

Therefore, $\mathbb{P}[2 < M \leq 3] = \frac{3^4 - 2^4}{4^4} = \frac{65}{256}$

Solution to Question 886: Ship Meeting

Let's call the two ships 1 and 2 in the order presented in the question. WLOG, let ship 1 arrive at $5:00$ PM and ship 2 arrive at $10:00$ PM. Clearly ship 2 is sailing at a slower speed than ship 1, as it took longer from the meeting point and they left at the same time. Let v_1 and v_2 be the velocities of the two ships and d be the total distance between the islands. From the meeting point, the distance to island B is $4v_1$, as it takes 4 hours to get to island B afterwards.

Similarly, the distance to island A is $9v_2$, as it takes 9 hours afterwards to get to island A. Therefore, $d = 4v_1 + 9v_2$.

However, the time travelled to get to that meeting point for ship 1 was $\frac{9v_2}{v_1}$, as it travels at a rate of v_1. Similarly, the time travelled to get to that meeting point for ship 2 was $\frac{4v_1}{v_2}$. Since they left at the same time, these are equal, and we get that $v_1 = \frac{3}{2}v_2$ by the above.

This means that the time travelled beforehand is $\frac{4 \cdot \frac{3}{2}v_2}{v_2} = 6$ hours. so both ships left at $7:00$ AM. Our answer is 700.

Solution to Question 887: No Marble Missing

Let B and G represent the events that there are at least one blue and one green marble, respectively, in the 4 selected. We are looking for $\mathbb{P}[B \cap G]$. These events are not disjoint, so it's easier to work the complement and use inclusion-exclusion. This namely means

$$\mathbb{P}[B \cap G] = 1 - \mathbb{P}[B^c \cup G^c] = 1 - (\mathbb{P}[B^c] + \mathbb{P}[G^c] - \mathbb{P}[B^c \cap G^c])$$

We have that $\mathbb{P}[B^c] = \mathbb{P}[G^c] = \left(\frac{5}{6}\right)^4$, as there is a $\frac{5}{6}$ chance per trial that you don't receive a blue (or green) marble. Then, $\mathbb{P}[B^c \cap G^c] = \left(\frac{4}{6}\right)^4$, as you must choose one of the other 4 colors in each of the trials. Substituting it all in, we get

$$\mathbb{P}[B \cap G] = 1 - 2 \cdot \frac{625}{1296} + \frac{256}{1296} = \frac{151}{648}$$

Solution to Question 888: Fair Match

We know that the PMFs of the value of a roll of a fair $6-$sided and $8-$sided dice are symmetric about their respective means 3.5 and 4.5. Therefore, the PMFs of the sums of independent $6-$sided and $8-$sided dice are symmetric about their means. Namely, S_m has a symmetric PMF about $3.5m$ and T_n has a symmetric PMF about $4.5n$. Therefore, if we can find values of m and n such that the means are the same, both sums will be symmetric about the means, and the condition in the question will be satisfied. This means we must find the smallest m and n satisfying $3.5m = 4.5n$. These would be the same as the smallest m and n such that $7m = 9n$ by multiplication of 2 on both sides. As 7 and 9 are relatively prime, the smallest integer solution is $m^* = 9$ and $n^* = 7$, so $m^*n^* = 63$.

Solution to Question 889: Equivariant

We have that $\mathrm{Cov}(X+Y, X-Y) = \mathrm{Var}(X) - \mathrm{Cov}(X,Y) + \mathrm{Cov}(X,Y) - \mathrm{Var}(Y)$ by bilinearity. Since we suppose that X and Y have the same variance, all terms cancel and the RHS is 0, so this means $\mathrm{Corr}(X+Y, X-Y) = 0$.

Solution to Question 890: Matching Socks II

Since we want to count the expected number of pairs that remain in the drawer, it makes sense to use indicators and linearity of expected here to count. Label the pairs of socks $1-6$ and let I_i be the indicator that pair i is still in the drawer after drawing out the pairs of socks. Then $T = I_1 + \cdots + I_6$ gives the total number of pairs of socks remaining in the drawer. By linearity of expectation and the exchangeability of the sock pairs, $\mathbb{E}[T] = 6\mathbb{E}[I_1]$.

$\mathbb{E}[I_1]$ is just the probability of the event that I_1 indicates. Namely, this is the probability that pair 1 is still in the drawer after drawing out the socks. To not pick any of pair 1 socks, we must pick 8 socks from the other 10, so there are $\binom{10}{8}$ ways to do this. There are $\binom{12}{8}$ ways to pick 8 socks in general, so the probability (and hence $\mathbb{E}[I_1]$) is $\frac{\binom{10}{8}}{\binom{12}{8}} = \frac{1}{11}$. Therefore, $\mathbb{E}[T] = \frac{6}{11}$.

Solution to Question 891: Game Show II

Note that we cannot use a frequentist approach to solve this problem, as the audience member would reveal the car $\frac{1}{3}$ of the time in repeated play, which is not the case in this instance. Instead, the audience member has just happened to open a door and revealed a goat. That is, the audience member removed one door from the sample space. The probability that you win the car if you switch is $\frac{1}{2}$.

Solution to Question 892: Colorful Cube

There are 6! colorings of the sides total. There are 8 ways to pick the corner of intersection. Then, there are $3! = 6$ ways to arrange the 3 colors among the sides that intersect at that corner. Then, there are $3! = 6$ ways to arrange the other 3 colors among the other 3 sides. Thus, our probability is $\dfrac{8 \cdot 6^2}{6!} = \dfrac{2}{5}$.

Alternatively, fix one color on any given side. The second color among blue, green, and red has a $\dfrac{4}{5}$ chance of sharing an edge with the first color (only the side opposite of the first color doesn't share an edge). Then, after that, 2 of the remaining 4 sides share one of the two vertices in common with the other two sides. Thus, the probability is $\dfrac{4}{5} \cdot \dfrac{1}{2} = \dfrac{2}{5}$

Solution to Question 893: Colorless Sides

Given that we have new information entering this problem (that we can see 5 sides are not painted), we need to update the probability that the chosen cube has a colored side. Thus, we have to use Bayes theorem. Let S be the event of the cube having exactly one colored side and B being the event that the five sides of the cube shown are blank. Then we have

$$\mathbb{P}[S \mid B] = \frac{\mathbb{P}[B \mid S] \cdot \mathbb{P}[S]}{\mathbb{P}[B]}$$

There's a $\frac{1}{6}$ chance that a one colored side cube has the painted side not showing. Thus $\mathbb{P}[B \mid S] = \frac{1}{6}$. And since there's 6 cubes that have only one painted face out of 27. Thus $\mathbb{P}[S] = \frac{6}{27} = \frac{2}{9}$. Finally, the probability we have a cube that shows five unpainted sides is $\frac{1}{6} \cdot \frac{2}{9}$ from the cubes with one painted side and $1 \cdot \frac{1}{27}$ from the center cube with no painted sides. Combining everything together we get

$$\mathbb{P}[S \mid B] = \frac{\mathbb{P}[B \mid S] \cdot \mathbb{P}[S]}{\mathbb{P}[B]} = \frac{\frac{1}{6} \cdot \frac{2}{9}}{\frac{1}{6} \cdot \frac{2}{9} + 1 \cdot \frac{1}{27}} = \frac{1}{2}$$

Solution to Question 894: Rotation Matrix Ranked

Interpreting the rotation matrix geometrically, we note that rotations do not change the length i.e. they are isometries. Therefore, the only way to have $0 \in \mathbb{R}^3$ after rotation is to start with 0. This implies that $\mathcal{N}(R_{\alpha,\beta,\gamma}) = \{0\}$ and that $\text{null}(R_{\alpha,\beta,\gamma}) = 0$. As $\text{rank}(R_{\alpha,\beta,\gamma}) + \text{null}(R_{\alpha,\beta,\gamma}) = 3$ by rank-nullity theorem, we can conclude that $\text{rank}(R_{\alpha,\beta,\gamma}) = 3$, so our answer is $3^2 + 0^2 = 9$.

Solution to Question 895: Die Rank

We can calculate $\mathbb{E}[X_1 X_2] = \mathbb{E}[X_1]\mathbb{E}[X_2] = (3.5)^2 = 12.25$ by direct calculation via independence of the rolls. Furthermore, we can calculate $\mathbb{E}[X_1^2] =$

$\frac{1^2+2^2+3^2+4^2+5^2+6^2}{6} \approx 15.17$, so we know that $b > a$. We now need to find where c lies relative to these. It's worth just thinking about this intuitively, as the calculations are quite difficult.

By inspection, we note that the distribution here is skewed towards the middle (we would need 3 6s or 3 1s to get either extreme). Note that in the sum for $\mathbb{E}[X_1^2]$, we gain much more through the squared 6 than we lose in the squared 1. This ranks it below $\mathbb{E}[X_1^2]$.

To compare to $\mathbb{E}[X_1 X_2]$, we note if either one of the dice is low, it reduces our product greatly. This reduction occurs with very high probability, whereas the median is more likely to reach at least a moderate value (say 3 or 4), so the median is skewed more right. Therefore, we get that $b > c > a$, corresponding to the answer 231.

Solution to Question 896: Circular Birthdays

There are $(7-1)! = 6! = 720$ distinct ways for the people to sit at the table with no restrictions. Only two of these seating arrangements have the people in age order: Namely, they are just when they increase CCW or CW. Therefore, our probability is

$$\frac{2}{720} = \frac{1}{360}$$

Solution to Question 897: Uniform Equilibrium II

The idea here is that Player 2 should go for the point that minimizes the maximum possible distance that Player 1 can be from the point. Having Player 2 always select 1/2 would limit Player 1 to earning at most 1/2. For player 1, they would want to select the values 0 or 1 only. If player 1 always selects 0, for example, then player 2 can do better by just lowering his value. Therefore, he should run a mixed strategy with values 0 and 1. If player 1 biases towards one of 0 or 1, player 2 can do better on average by selecting a value closer to whichever one has higher probability. Therefore, player 1 should choose equally at random between 0 and 1.

All of this yields an expected payout of 1/2 for player 1, as no matter which of 0 or 1 player 1 selects, the payout is 1/2.

Solution to Question 898: Lapping

We can think about this in terms of number of laps completed per Alice lap. For every lap Alice completes, Bob completes 2/3 of a lap. This is since it takes Bob 3/2 as long to complete a lab. Therefore, we note that at the moment Alice completes 3 laps, Bob completes his second lap, so Alice will be exactly 1 lap ahead of Bob at this instant. Alice completes 3 laps after 18 minutes, so the answer is 18.

Solution to Question 899: Coin Pair II

For each pair we flip again, we have a positive probability of obtaining 2 heads. If we are to flip every pair of coins, regardless of the initial outcome, enough times, we would end with all heads. Therefore, the answer is just 4, as with probability 1, we will obtain $HHHH$ at some point.

Solution to Question 900: Triangular Partition

We can avoid principle of inclusion-exclusion by employing a complementary approach. First, we draw an arbitrary triangle $\triangle ABC$. Consider when AB is the base of the triangle. We draw a line parallel to AB such that it splits the height of length h into two lines of lengths $\frac{3h}{4}$ and $\frac{h}{4}$. We notice that P must fall within the region between that line and AB in order for our condition to be satisfied. Repeating for bases BC and CA and subsequently length-chasing with similar triangles, we find that there is a triangle with sides of length $\frac{1}{4}AB, \frac{1}{4}BC, \frac{1}{4}CA$ in the middle of $\triangle ABC$ where P cannot fall. So, our answer is $1 - \frac{1}{16} = \frac{15}{16}$.

Solution to Question 901: Cats and Dogs II

Since this table is circular, there are $\dfrac{12!}{12} = 11!$ arrangements of the animals. As we want exactly 4 dogs in a row, this implies that we need to have the sequence $CDDDDC$ somewhere. In particular, there are 6 ways to pick the first cat. Then, there are $6 \cdot 5 \cdot 4 \cdot 3$ ways to order the 4 dogs in our sequence. Then, there are 5 ways to pick the cat on the other side. Afterwards, we still have 6 animals left over, so there are 6! ways to order the remaining animals. This implies that our probability is

$$\frac{6 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 5 \cdot 6!}{11!} = \frac{15}{77}$$

Solution to Question 902: Big Bubble II

We know that $V(r) = \frac{4}{3}\pi r^3$ and $S(r) = 4\pi r^2$ represent the volume and the surface area of the bubble as a function of radius. We know that $r' = \frac{9}{4\pi}$ is constant. Taking the derivatives of each, we see that $V' = 4\pi r^2 r'$ and $S' = 8\pi r r'$. We know at this moment that $V' = 36$, so plugging these in and solving yields

$$36 = 4\pi r^2 \cdot \frac{9}{4\pi} \iff r = 2$$

Afterwards, we have that

$$S' = 8\pi(2) \cdot \frac{9}{4\pi} = 36$$

Solution to Question 903: 9 Toss 4

The probability that the product is divisible by 4 is equal to 1 minus the probability that the product is divisible by 2 (case 1) or is odd (case 2). For case 1 to occur, there must be exactly 1 value that is divisible by 2; all others must be odd. This occurs with probability

$$\mathbb{P}(\text{case } 1) = \left(\frac{1}{2}\right)^8 \frac{1}{3}\binom{9}{1}.$$

Case 2 occurs with probability

$$\mathbb{P}(\text{case } 2) = \left(\frac{1}{2}\right)^9.$$

Our solution is

$$\begin{aligned} 1 - \mathbb{P}(\text{case } 1) - \mathbb{P}(\text{case } 2) &= 1 - \frac{3}{2^8} - \frac{1}{2^9} \\ &= \frac{2^9 - 6 - 1}{2^9} \\ &= \frac{505}{512}. \end{aligned}$$

Solution to Question 904: Exponential Ball Draw

If Alice can leave $3m + 1$ balls in the urn for an integer m for Bob to select from, she will have a winning strategy. As $100 = 3(33) + 1$, she should let Bob go first. This means $p = 2$. Once Bob selects, Alice can always leave $3x + 1$ balls for some integer x for Bob to select from. This is because if Bob selects

$2^k = (3-1)^k = 3n + (-1)^k$ balls for an integer n, given that there are already $3m+1$ balls in the urn for Bob to select from, then there would now be

$$3(m-n) + 1 - (-1)^k = 3m' + 1 + (-1)^{k+1}$$

remaining balls for an integer m'. Thus, you should pick 2^y balls, where $y \equiv k+1 \mod 2$ This would cancel out the $(-1)^{k+1}$ term at back.

All of the above implies that the strategy for Alice is to let Bob go first. If Bob selects 2^k, then Alice should select 2^{k+1}, if possible. If that isn't possible based on how many balls remain, Alice should select a smaller exponent equal in parity to 2^{k+1} i.e. $2^{k-1}, 2^{k-3}, \ldots$ that is possible to select. In our scenario here, since Bob selects 32 on his first turn, there are 68 left, so Alice should select 64 balls, as that is still possible. This means $b = 64$, so our answer is $100 \cdot 2 + 64 = 264$.

Solution to Question 905: Hide and Seek

In order for the seeker to not find anyone in the first spot, you and your friend must be in one of the four other spots. Thus, the probability that you and your friend are not caught in the first spot is:

$$\frac{4}{5} \times \frac{4}{5} = \frac{16}{25}$$

Solution to Question 906: Specific Card Pull I

In this scenario, we must realize we have only 13 cards of interest, instead of the whole deck. So by exchangeability, one of our cards of interest must appear first. Our two of hearts has a 1/13 chance to appear first, as does any other card of interest, by exchangeability. Therefore, there is a 1/13 chance that there will be no kings, queens, or jacks before the two of hearts.

Solution to Question 907: Split 100

There are a total of $\binom{100}{30}$ ways to split the 100 numbers into the two groups. In order for 6 and 66 to be in separate groups, one must be assigned to the smaller group, and the other must be assigned to the larger group. There are $2 * \binom{98}{29}$ ways to ensure that 6 and 66 are in separate groups. Putting it all together we find our answer to be $2 * \binom{98}{29} / \binom{100}{30} = \frac{14}{33}$.

Solution to Question 908: Rain Chance II

Let A and B be the events that it rains on Saturday and Sunday, respectively. We want to upper bound $\mathbb{P}[(A \cup B)^c] = 1 - \mathbb{P}[A \cup B]$. To get the maximum upper bound, we need to minimize $\mathbb{P}[A \cup B]$. The extreme case to minimize the quantity is that $A \subseteq B$ i.e. rain on Saturday always implies rain on Sunday. If $A \subseteq B$, then $\mathbb{P}[A \cup B] = \mathbb{P}[B] = 0.7$. Therefore, $\mathbb{P}[(A \cup B)^c] \leq 1 - 0.7 = 0.3$.

Solution to Question 909: Independent Zeta

We know that by LOTUS, $\mathbb{E}[X]\alpha \sum_{k=1}^{\infty} k \cdot \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$. Thus, X and Y do not have finite means, so the expression $\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$ can't be explicitly computed, as you obtain $\infty - \infty$, which is indeterminate. Therefore, the answer is 12345.

Solution to Question 910: Football Bets

We know that if Calvin doesn't accept the bet, his expected value is -10, as he loses his initial bet. If Calvin accepts the bet, then with probability p he wins 20 and with probability $1 - p$ he loses 20. Therefore, we need to find the smallest p such that $20p - 20(1 - p) \geq -10$. Solving this equation for p yields $p \geq \frac{1}{4}$, so $p = \frac{1}{4}$ is the smallest value where he should accept.

Solution to Question 911: Cat Dog Line

Let c and d, respectively, be the proportion of cats and dogs on the sidewalk. We know $c + d = 1$. Let's find d in terms of c. Note that of all the cats, $\frac{6}{7}$ are followed by a dog. Then, of those dogs, $\frac{3}{4}$ of them are followed by another dog. Of those dogs, $\frac{3}{4}$ are followed by another dog, etc. Therefore, we can write

$$d = \frac{6}{7}c + \frac{6}{7} \cdot \frac{3}{4}c + \frac{6}{7} \cdot \left(\frac{3}{4}\right)^2 c + \cdots = \frac{6c}{7} \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k = \frac{24c}{7}$$

Substituting this into our initial equation, $\frac{31c}{7} = 1$, so $c = \frac{7}{31}$. This means $d = \frac{24}{31}$.

Alternatively, you can create a Markov chain representing this scenario. Namely, if state 1 is dog and state 2 is cat, the Markov chain is

$$\begin{bmatrix} 3/4 & 1/4 \\ 6/7 & 1/7 \end{bmatrix}$$

The steady state of this Markov chain yields the same answer as above.

Solution to Question 912: Slippery Snail

For the first 13 hours, the snail moves up 4 feet and then gets blown back 2 feet, moving at a net rate of 2 feet per hour. Thus, after 13 hours, the snail is 26 feet away from the starting point. Then, in the 14th hour, the snail will reach the 30 foot mark and eat the food instantaneously, so it takes 14 hours for the snail to eat the food.

Solution to Question 913: Bull Call Spread II

Similar to pricing via replication, we can also establish bounds using replication when we cannot do a perfect replication. If we had the prices of the calls, we could calculate the exact price of the bull-call spread through exact replication. When we want to look at the best upper-bound, we want to find a replicate that super-replicates the portfolio *everywhere*. In this case, the line $y = 5$ will give the best superreplication (from the given information). Since 5 is a constant, it is essentially risk-free cashflow and we need to discount this by the discount factor.

Hence, we get obtain $0.9 * 5 = 4.5$.

Solution to Question 914: Power Digits

Only the units digit is relevant, so we can consider the first set as $\{1, 3, 5, 7, 9\}$ instead. Note that, by Euler's theorem, $x^4 \equiv 1 \mod 10$ if $\gcd(x, 10) = 1$. There are 5 numbers in $\{2004, 2005, \dots, 2023\}$ that are divisible by 4, and m may equal $1, 3, 7, 9$. Additionally, $11^x \equiv 1 \mod 10$ for any positive integer x. We must therefore add 15, since 5 values of n that are divisible by 4 have already been accounted for. Finally, we notice that $19^{2x} = 361^x \cong 1 \mod 10$ for any positive integer $2x$, so we must add 5 values of n that are divisible by 2 but not by 4 to avoid overcounting. Our probability is therefore

$$\frac{4 \cdot 5 + 15 + 5}{5 \cdot 20} = \frac{2}{5}$$

Solution to Question 915: Game Time

When a person with a \$5 bill purchases a ticket, the cashier gains a \$5 bill. However, when a person with a \$10 bill purchases a ticket, the cashier loses a \$5 bill in change. What we are looking for in this scenario is the number of arrangements in which at any point, there have been more people with a \$10 bill purchasing the ticket than the \$5 bill.

A way to think about this problem is to let someone with a \$5 bill be an (and someone with a \$10 bill be a). The equivalent formulation of the problem above is to count the number of proper parenthesizations of 19 (and and 19). This is equivalent because of the fact that a proper parenthesization can't have more closing parentheses than opening parentheses at any point throughout. For example, in the 2 opening and closing case, (()) and ()() are valid, but)(() and ())(are not valid. This problem is well-known to be solved by the Catalan numbers. In particular, the number of proper parenthesizations with n opening and closing parentheses is $C_n = \dfrac{1}{n+1}\binom{2n}{n}$.

In the above, we are only identifying people by their bill number. This means that there are $\displaystyle\binom{2n}{n}$ distinct arrangements of people in the line with n of each type. Therefore, our final probability is just $\dfrac{1}{n+1}$ by division. In particular, $n = 19$ here, so the answer is $\dfrac{1}{20}$.

Solution to Question 916: Mean Difference

We are going to generalize this to any μ_i and any number n. Using our known relationship between the second moment and variance, we should think to consider $\text{Var}(X_1 - \overline{X})$. We will also generalize to consider any X_i instead of just X_1. We know that

$$\text{Var}(X_i - \overline{X}) = \mathbb{E}[(X_i - \overline{X})^2] - (\mathbb{E}[X_i - \overline{X}])^2$$

Note that by linearity of expectation, the second term is just $(\mu_i - \overline{\mu})^2$. For $\text{Var}(X_i - \overline{X})$, we know that by the generalized formula for variance, this is

$$\text{Var}(X_i) + \text{Var}(\overline{X}) - 2\text{Cov}(X_i, \overline{X})$$

The first two terms are σ^2 and $\dfrac{\sigma^2}{n}$, respectively. This is from our known facts about sample variance. The final term we can get easily by bilinearity of Covariance.

$$\mathrm{Cov}(X_i, \overline{X}) = \frac{2}{n}\mathrm{Cov}(X_i, X_1+X_2+\cdots+X_i+\cdots+X_n) = \frac{2}{n}\left(\mathrm{Cov}(X_i, X_1) + \cdots + \mathrm{Cov}(X_i, X_i) + \cdots + \mathrm{Cov}(\right.$$

Note that for all $j \neq i$, $\mathrm{Cov}(X_i, X_j) = 0$, as the RVs are independent. The only remaining term, $\mathrm{Cov}(X_i, X_i) = \mathrm{Var}(X_i) = \sigma^2$ by definition. Thus, we have that $\mathrm{Var}(X_i - \overline{X}) = \dfrac{n-1}{n}\sigma^2$, and thus we have that $\mathbb{E}[(X_i - \overline{X})^2] = (\mu_i - \overline{\mu})^2 + \dfrac{n-1}{n}\sigma^2$.

Plugging in $\mu_1 = 1$, $\overline{\mu} = \dfrac{1+3+5+\cdots+19}{10} = 10, n = 10$, and $\sigma^2 = 100$, our final answer is 171.

Solution to Question 917: Log Comparison

Since X, Y must be between 0 and 1 (we'll treat these bounds as exclusive), it follows that $\log_2 X, \log_2 Y$ must have negative values.

In order for $\lfloor \log_2 X \rfloor = \lfloor \log_2 Y \rfloor = -1$, it is obvious that $X, Y \in [0.5, 1)$. This occurs with probability $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. In order for $\lfloor \log_2 X \rfloor = \lfloor \log_2 Y \rfloor = -2$, we need $X, Y \in [0.25, 5)$, which occurs with probability $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$. We quickly discover that the probability that $\lfloor \log_2 X \rfloor = \lfloor \log_2 Y \rfloor = -n$ is $\frac{1}{2^{2n}}$. Summing it all up, we find our answer to be $\frac{1/4}{1-1/4} = \frac{1}{3}$.

Solution to Question 918: Extrinsic Value II

The intrinsic value is the amount that you would gain if you could exercise at time-0. Here, we have strike $K = 15$, so the payout of the put is $\max(15 - S_0, 0) = \max(15 - 17, 0) = 0$. So, we have 0 intrinsic value and the option price comes entirely from the time-value.

$$I^2 + E^2 = 0^2 + 7.2^2 = 51.84$$

Solution to Question 919: Unit Fraction Representation

We can see that $\dfrac{1}{5} < \dfrac{179}{720} < \dfrac{1}{4}$, so $\dfrac{1}{5}$ is our first fraction. Then, $\dfrac{179}{720} - \dfrac{1}{5} = \dfrac{35}{720} = \dfrac{7}{144}$. At this point, we see that $\dfrac{1}{20} < \dfrac{7}{144} < \dfrac{1}{21}$, so we should next add $\dfrac{1}{21}$ to our total. Now, $\dfrac{7}{144} - \dfrac{1}{21} = \dfrac{3}{3024} = \dfrac{1}{1008}$. Therefore, the largest denominator is 1008.

Solution to Question 920: Gamma Review IV

By definition, we have that $\text{Var}(e^X) = \mathbb{E}[e^{2X}] - (\mathbb{E}[e^X])^2$, Note that the MGF of a random variable X is given by $M_X(\theta) = \mathbb{E}\left[e^{\theta X}\right]$. Therefore, we can rewrite the previous expression as $M_X(2) - (M_X(1))^2$. For our given distribution, we have that $M_X(\theta) = \left(1 - \dfrac{\theta}{3}\right)^{-8}$. This can be easily derived or also can be just taken from a table of known MGFs. Thus, $M_X(1) = \left(\dfrac{3}{2}\right)^8$, while $M_X(2) = 3^8$, so

$$\text{Var}(e^X) = 3^8 - \left(\frac{3}{2}\right)^{16}$$

Therefore, $a = 3, b = 8, c = 16$, so $abc = 384$.

Solution to Question 921: Three Consecutive Heads

Let x be the expected number of coin tosses it takes to observe three consecutive heads. On the first toss, there is a $\frac{1}{2}$ probability that you observe a tail and essentially start over with 0 consecutive heads, except that your expected number of coin tosses is $x+1$ since you wasted a toss observing a tail. There is a $\frac{1}{4}$ probability of receiving HT and essentially starting over with 0 consecutive heads, except that your expected number of coin tosses is $x+2$ since you wasted two tosses observing HT. There is a $\frac{1}{8}$ probability of receiving HHT and essentially starting over with 0 consecutive heads, except that your expected number of coin tosses is $x+3$ since you wasted three tosses observing tails HHT. Finally, there is a $\frac{1}{8}$ probability of receiving HHH and requiring three tosses. Thus, we can write this as:

$$x = \frac{1}{2}(x+1) + \frac{1}{4}(x+2) + \frac{1}{8}(x+3) + \frac{1}{8} \times 3x = 14$$

Solution to Question 922: Squid Game I

Since a given contestant is not able to distinguish the tile, we can assume that each contest guesses a tile incorrectly with $p = 0.5$. Therefore, the expected number of tiles until a person falls is $\frac{1}{p} = 2$. Therefore, we should expect that after 5 people have gone, 10 tiles will have been chosen with the last being chosen incorrectly by the 5th person meaning we should choose to be in slot 6.

Solution to Question 923: Ship Stops

Let X_i be the number of minutes that wave check i, $1 \leq i \leq 5$, stops you for. Let wave check 5 be the special one. Then $T = 12 + X_1 + \cdots + X_5$ gives the total duration of your trip. By Linearity of Expectation, $\mathbb{E}[T] = 12 + \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_4] + \mathbb{E}[X_5]$. For $X_1, \ldots, X_4$, we know they have the same distribution of 0 with probability 75% and 1 with probability 25%. Therefore, $\mathbb{E}[X_i] = 0.25 \cdot 1 + 0.75 \cdot 0 = 0.25$ for each of $1 \leq i \leq 4$. Lastly, X_5 is 0 with probability 25% and 4 with probability 75%, so $\mathbb{E}[X_5] = 0.75 \cdot 4 + 0.25 \cdot 0 = 3$. Adding all of these up, our total is $\mathbb{E}[T] = 12 + 4 \cdot 0.25 + 3 = 16$.

Solution to Question 924: Normal LOTUS II

Note that X and Y are independent and that $X \sim \text{Exp}(5)$ and Y has PDF $2yI_{(0,1)}(y)$. Thus, we have that $\mathbb{E}\left[\frac{X^2}{Y}\right] = \mathbb{E}[X^2]\mathbb{E}\left[\frac{1}{Y}\right]$. We have that $\mathbb{E}[X^2] = \text{Var}(X) + (\mathbb{E}[X])^2 = \frac{1}{25} + \frac{1}{25} = \frac{2}{25}$. We have that by LOTUS, $\mathbb{E}\left[\frac{1}{Y}\right] = \int_0^1 2dy = 2$. Thus, the expectation in question is equal to $\frac{4}{25}$.

Solution to Question 925: 3rd Head

The distribution of flips needed is $N \sim \text{NegBinom}(3, 1/4)$, which has mean $\mathbb{E}[N] = \frac{3}{1/4} = 12$.

Solution to Question 926: Different Variance

We know that $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$. Replacing Y with $-Y$ yields the formula $\text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y) - 2\text{Cov}(X, Y)$. Therefore, considering $\text{Var}(X + Y) - \text{Var}(X - Y)$, we get that it reduces to $4\text{Cov}(X, Y)$. Therefore, as we know $\text{Var}(X + Y) = 18$ and $\text{Var}(X - Y) = 10$, $4\text{Cov}(X, Y) = 8$, meaning $\text{Cov}(X, Y) = 2$.

Solution to Question 927: Last Light Bulb Standing

We are going to use a generalization of the memorylessness property for exponential random variables called the strong memorylessness property, which states that if A and B are independent non-negative random variables that are almost surely finite and $X \sim \text{Exp}(\lambda)$, then $\mathbb{P}[X > A + B \mid X > A] = \mathbb{P}[X > B]$. Note that this is not just the memorylessness property itself, as that deals with deterministic (non-random) times. Proving this is very similar to how you prove the standard memorylessness property. Assuming this and the standard property of exponentials that if $X_1 \sim \text{Exp}(\lambda_1)$ and $X_2 \sim \text{Exp}(\lambda_2)$, $\mathbb{P}[X_1 > X_2] = \dfrac{\lambda_2}{\lambda_1 + \lambda_2}$, we can complete this question fairly easily. For notation, let $S_n = \sum_{i=1}^{n} T_i$. Namely, we have that

$$\mathbb{P}[X > S_{10}] = \mathbb{P}[X > S_{10} \mid X > S_9]\mathbb{P}[X > S_9] = \mathbb{P}[X > S_9 + T_{10} \mid X > S_9]\mathbb{P}[X > S_9] = \mathbb{P}[X > T_{10}]\mathbb{P}[X > S_9]$$

The last equality comes from the strong memorylessness. Iterating this, we have that

$$\mathbb{P}[X > S_{10}] = \mathbb{P}[X > T_{10}] \ldots \mathbb{P}[X > T_1] = (\mathbb{P}[X > T_1])^{10}$$

The last equality here comes from the fact that each of the T_i are IID. Therefore, using the other property we stated at the beginning,

$$\mathbb{P}[T_1 < X] = \frac{1/10}{1/10 + 1/90} = \frac{9}{10}$$

Therefore, $q = \dfrac{9}{10}$ and $b = 10$, so $qb = 9$.

Solution to Question 928: Colorful Candy

Label the colors $1-4$, and let C_i be the indicator that color i is in our selection. Then $T = C_1 + \cdots + C_4$ gives the total number of colors in our selection. By linearity of expectation, $\mathbb{E}[T] = \sum_{i=1}^{4} \mathbb{E}[C_i]$. We evaluate $\mathbb{E}[C_i]$ in a unified manner. $\mathbb{E}[C_i]$ is just the probability color i is somewhere in our draws, which is just $1-$ the probability is in none of the draws.

Suppose color i has k candies of that color from the 10. Then there are $\binom{10}{3}$ ways to pick 3 candies and $\binom{10-k}{3}$ ways to pick 3 candies that are not that color candy. Therefore, $\mathbb{E}[C_i] = 1 - \dfrac{\binom{10-k}{3}}{\binom{10}{3}}$ if candy i has k candies of that color. Therefore, the expectations for the 4 colors in the order they were presented is $1 - \dfrac{\binom{4}{3}}{120} = \dfrac{29}{30}$, $1 - \dfrac{\binom{8}{3}}{120} = \dfrac{8}{15}$, and $1 - \dfrac{\binom{9}{3}}{120} = \dfrac{3}{10}$ for the last two. Adding these up, $\mathbb{E}[T] = 2.1$

Solution to Question 929: Chord on a Square

Pick the first point p_1 first, then rotate the square so that the edge containing p_1 is on top. We define a random variable X to be the distance of p_1 from the upper-left vertex of the square, let $p(X)$ denote the probability of the second point being a distance of at least 1 away (It is a function of X since it is impacted by the placement of p_1). Next, let's consider some basic cases to build intuition. If p_1 is on the corners of the edge then the probability the remaining point is a distance of ¿1 away is $\frac{1}{2}$. In other words, if $X = 0$ or $X = 1$ then $p(X) = \frac{1}{2}$. If $X = \frac{1}{2}$ or in the center, then let's consider where the second point can fall to satisfy our condition. Clearly, it can fall anywhere on the bottom edge, but on the sides, it needs to be low enough so that the hypotenuse of the triangle formed by the top and side edge is greater than 1. Specifically, it needs to be at least $\sqrt{1^2 - (\frac{1}{2})^2} = \sqrt{\frac{3}{4}}$ from the top. Therefore, $p(X = \frac{1}{2}) = \frac{3-2\sqrt{\frac{3}{4}}}{4}$. Following the same logic, we can do this calculation for some general x, so more generally we get that $p(X = x) = \frac{3-\sqrt{1-x^2}-\sqrt{2x-x^2}}{4}$.

Finally, to calculate the probability we compute

$$\frac{1}{4}\int_0^1 (3 - \sqrt{1-x^2} - \sqrt{2x-x^2})\,dx = \frac{6-\pi}{8}$$

giving us our final answer of $\frac{a=6}{b=8} = \frac{3}{4}$.

Solution to Question 930: Binary Dot

We're going to solve this for general n. Let p_n be this probabilities when $v_1, v_2 \in \mathbb{R}^n$. For $n = 1$, the probability is just $\dfrac{3}{4} \cdot \dfrac{1}{2} = \dfrac{3}{8}$, as this implies both of the values generated must be 1.

Now, let's suppose we know p_n. Then p_{n+1} either means we had an even sum with n dimensions (occurring with probability $1 - p_n$) and we obtain 1 for the $(n+1)$st product OR we had an odd sum with n dimensions (occurring with probability p_n) and we obtain 0 for the $(n+1)$st product. This means $p_{n+1} = \dfrac{3}{8} \cdot (1 - p_n) + \dfrac{5}{8} p_n = \dfrac{1}{4} p_n + \dfrac{3}{8}$. This is a recurrence relation that can be explicitly solved. However, starting from $p_1 = \dfrac{3}{8}$,

$$p_2 = \frac{3}{32} + \frac{3}{8} = \frac{15}{32}, p_3 = \frac{15}{128} + \frac{3}{8} = \frac{63}{128}$$

We can see that $p_1 = \frac{3}{8} = \frac{1-\frac{1}{4}}{2}, p_2 = \frac{1-\frac{1}{4^2}}{2}, p_3 = \frac{1-\frac{1}{4^3}}{2}$, so we conjecture $p_n = \frac{1-4^{-n}}{2}$, so $p_{10} = \frac{1}{2} - \frac{1}{2 \cdot 4^{10}} = \frac{1}{2} - \frac{1}{2^{21}}$. Thus, $a = 2$ and $n = 21$, meaning $an = 42$.

Solution to Question 931: Quintuple Value

Let X_1 and X_2 be the value of the cards we select. Then our payout is $5(X_1+X_2)$ by the question. We have that $X_1, X_2 \sim \text{DiscreteUnif}(\{1, 2, \ldots, 35\})$. We are looking for $\mathbb{E}[5(X_1 + X_2)]$. By linearity, this is $5\mathbb{E}[X_1] + 5\mathbb{E}[X_2]$. As the two draws are exchangeable, their means are equal, so this is really just $10\mathbb{E}[X_1] = 10 \cdot 18 = 180$.

Solution to Question 932: ATM Expiration

Since $\Delta \in [0, 1]$, this essentially acts as a probability of an option expiring in-the-money. Intuitively, this makes sense. An at-the-money option has $\Delta \approx 0.5$. The probability of the call expiring in-the-money is 0.67, meaning that the probability it expires out-of-the-money is 0.33, which is the same probability as a put expiring in-the-money.

Solution to Question 933: Very Very Normal

We can write this as $\mathbb{P}[X - Y - Z > 0]$. $W = X - Y - Z \sim N(-12, 36)$ by the properties of independent normal distributions. Therefore, we can write as $\mathbb{P}[W > 0]$, and we can standardize this to obtain $\mathbb{P}\left[\frac{W+12}{6} > \frac{12}{6}\right]$. The LHS is now standard normal, so the answer is $\Phi(-2)$ by symmetry of the normal distribution. This means $a = -2$.

Solution to Question 934: Portfolio Returns

Let A, B, C represent the returns of each asset in the portfolio. Then $T = \frac{1}{3}(A + B + C)$ gives us the return of the portfolio. By linearity of expectation, $3\mathbb{E}[T] = \mathbb{E}[A] + \mathbb{E}[B] + \mathbb{E}[C]$. To calculate the expectation of each portfolio, we just have to sum up the returns weighted by their probabilities. We get that $\mathbb{E}[A] = 0.3 \cdot 10 + 0.5 \cdot 5 + 0.2 \cdot (-2) = 5.1$, $\mathbb{E}[B] = 0.4 \cdot 12 + 0.3 \cdot 7 + 0.3 \cdot (-4) = 5.7$, and $\mathbb{E}[C] = 0.2 \cdot 18 + 0.5 \cdot 3 + 0.3 \cdot (-1) = 4.8$. Therefore, $\mathbb{E}[T] = 5.2$

Solution to Question 935: Salary Covariance

We know that $\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$. Solving for σ_{xy}, we find that the covariance between male and female annual earnings is:

$$\sigma_{xy} = \rho_{xy}\sigma_x\sigma_y = 0.85 \times 13 \times 11 = 121.55$$

Solution to Question 936: Deriving Put-Call Parity I

To find the replication, we can see that $S_T - K$ and S have the same slope of 1. However, the forward product has a shift of $-K$. We can replicate these "shifts" by using bonds. However, we need to discount the bond to time-0. We know that the 2 bonds pays 2 at time-T. Discounting, we get $2e^{-.04} = 1.92$.

Combining everything, we get $F_0 = 3 - 2e^{-.04} \approx 1.08$

Solution to Question 937: Last Lightbulb

We use the fact that if $X_1, X_2, \ldots, X_n \sim \text{Exp}(\lambda)$ IID, then $\min\{X_1, \ldots, X_n\} \sim \text{Exp}(n\lambda)$. This is a good exercise to prove if you have not done so before. The time until the 4th lightbulb burns out is equal to the sum of the times between lightbulb burnouts. Namely, let T_1 be the time until the first lightbulb burns out, T_2 be the time until the second lightbulb burns out after the first, T_3 be the time until the third lightbulb burns out after the second, and T_4 be the additional time until the last lightbulb burns out after. Then $S = T_1 + T_2 + T_3 + T_4$ represents the total time it takes until no lightbulbs are left.

$T_1 = \min\{X_1, X_2, X_3, X_4\}$, as it is the time for the first lightbulb to die. Therefore, $T_1 \sim \text{Exp}(4)$ by our fact before and $\mathbb{E}[T_1] = \dfrac{1}{4}$. As the exponential distribution is memoryless, at the time the first dies, the remaining life of the other lightbulbs are still each $\text{Exp}(1)$ distributed. Without loss of generality, say lightbulb 4 was the one that burned out. If $R_1, R_2, R_3 \sim \text{Exp}(1)$ IID represent the time after the first lightbulb burnt out until the next, Therefore, $T_2 = \min\{R_1, R_2, R_3\}$, meaning $T_2 \sim \text{Exp}(3)$ by our fact before, meaning $\mathbb{E}[T_2] = \dfrac{1}{3}$. Continuing this same logic on two lightbulbs and the final lightbulb yields $\mathbb{E}[T_3] = \dfrac{1}{2}$ and $\mathbb{E}[T_4] = 1$. By linearity of expectation, we get that

$$\mathbb{E}[S] = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$$

Solution to Question 938: Car and Fly II

The cars move towards each other at the same rate as the fly moves between them. Therefore, the sum of the car's distances travelled is equivalent to how far the fly travels, since they move at the same rate. The fly must travel 400 miles total.

Solution to Question 939: Which Sum First?

To start, we must see that out of 36 possible combinations, there are 4 combimations which will yield a sum of 5, and only 1 combination that will yield a sum of 2. If we roll infinitely, we will assuredly stop at some point, either hitting a sum of 2 or 5. Therefore, we can ignore all other possibilities and simply calculate the likelihood that our 4 combinations of dice rolls that sum to 5, will come before the 1 combination of dice rolls that sum up to 2. This is given by $\frac{4}{4+1}$ (number of wins over total cases) yielding the probability of 4/5.

Solution to Question 940: Tennis Gambling

The clever idea is to see that the winning player needs to obtain two consecutive wins. The probability that Rob obtains two consecutive wins is $0.6^2 = 0.36$. The probability that Bob obtains two consecutive wins is $0.4^2 = 0.16$. The probability that in the next two serves they each receive one win is $2 \cdot 0.6 \cdot 0.4 = 0.48$. If they each obtain one win in the next two serves, then the scenario is exactly the same as being tied and needing 2 consecutive wins on the next serves to win the game. Therefore, to get a winner, we just need one of the two players to obtain two consecutive wins. Accordingly, we can condition on the event that someone has obtained two consecutive wins. Given this, the probability the winner is Rob is $\dfrac{0.6^2}{0.6^2 + 0.4^2} = \dfrac{9}{13}$.

Solution to Question 941: Life Is A Raceway

Let x be Lightning McQueen's place. We know the upper half is cutoff is $x - 6$, as it is 6 places better than him. We also know that the lowest place is $x + 14$, as it is 14 places worse than him. Therefore, we know that the upper half cutoff is half of the lowest place, so $x - 6 = \dfrac{x + 14}{2}$, meaning that $x = 26$.

Solution to Question 942: Local Maxima

We will want to model whether or not the ith spot is a local maxima or not. Being a local maxima means that the numbers to the left and right of the spot are strictly smaller than our current spot. For the endpoints, this means that it is just larger than the value to the right/left (accordingly for the endpoint). Thus, if I_i is the indicator that spot i is a local maxima, $T = I_1 + \cdots + I_{14}$ gives the total number of local maxima.

By linearity of expectation $\mathbb{E}[T] = \mathbb{E}[I_1] + \cdots + \mathbb{E}[I_{14}]$. Recall that the expectation of an indicator random variable is the probability of the event it indicates. For indicators $2 - 13$, they are surrounded by a spot on each side, so the probability that the maximum value of the papers in spots $i-1, i, i+1$ is the one at spot i is $\frac{1}{3}$ by symmetry. This holds for all $2 \leq i \leq 13$. For the endpoints, there is just one value they need to be larger than, so the probability is $\frac{1}{2}$ for spots 1 and 14. Therefore, $\mathbb{E}[T] = \frac{1}{2} \cdot 2 + \frac{1}{3} \cdot 12 = 5$.

Solution to Question 943: Particle Reach IV

We will solve this more generally for $p < 1/2$, the particle eventually reaches position $n > 1$, and the particle is currently at position $i < n$. We want the conditional probability of moving to position $i+1$ in the next step given that the particle is currently at position i and eventually reaches position n.

Given that we are currently at position i, the particle needs to move $n - i$ more steps to reach position n. Since $p < 1/2$, we can use the formula from Particle Reach III to yield the probability of reaching n from i is $\frac{p^{n-i}}{(1-p)^{n-i}}$. This is the denominator of the conditional probability.

On the numerator, we are currently at position i. Then, we move to position $i+1$ with probability p in the next step. Afterwards, the particle needs to move $n - (i+1)$ more steps to reach position n, which occurs with probability $\frac{p^{n-(i+1)}}{(1-p)^{n-(i+1)}}$. Therefore, the numerator of our conditional probability is $p \cdot \frac{p^{n-(i+1)}}{(1-p)^{n-(i+1)}} = \frac{p^{n-i}}{(1-p)^{n-(i+1)}}$

The answer is thus

$$\frac{p^{n-i}/(1-p)^{n-(i+1)}}{p^{n-i}/(1-p)^{n-i}} = 1 - p$$

In particular, $p = 1/3$ here, so the answer is $2/3$.

Solution to Question 944: Jellybean Jar II

There are three possibilities for the colors appearing in the first two spots. Namely, they are RR, BR, and BB. The first and last case respectively occur when we obtain $RRRR$ and $BBBB$. The probabilities of each of these are $\frac{6 \cdot 5 \cdot 4 \cdot 3}{16 \cdot 15 \cdot 14 \cdot 13}$ and $\frac{10 \cdot 9 \cdot 8 \cdot 7}{16 \cdot 15 \cdot 14 \cdot 13}$. The second case occurs when we have one blue and one red in each of the first and last two spots. As we can arrange the blue and red as RB or BR in each of the first and last two spots, we can multiply the number of ways to get some fixed sequence of one red and one blue in each of the first and last two by 4. For simplicity, let's say the fixed sequence is $RBRB$. Then we have $4 \cdot \frac{10 \cdot 6 \cdot 9 \cdot 5}{16 \cdot 15 \cdot 14 \cdot 13}$ total ways to obtain this case.

Adding up our three cases, our total probability is

$$\frac{6 \cdot 5 \cdot 4 \cdot 3 + 10 \cdot 9 \cdot 8 \cdot 7 + 4 \cdot 10 \cdot 6 \cdot 9 \cdot 5}{16 \cdot 15 \cdot 14 \cdot 13} = \frac{135}{364}$$

Solution to Question 945: Grid Filling II

There are 5 odd and 4 even numbers, and for three numbers to sum to an odd number, either one or three of the numbers must be odd. Thus, one column must be all odd, and similarly one row must be all odd. Hence, the number of valid possibilities without regard to the ordering of the odd and even numbers is the number of row-column pairings, or $3 \times 3 = 9$. The number of total combinations we could choose such that 5 of the 9 squares will contain an odd number is $\binom{9}{5}$. The probability that the sum of each row and each column is odd is:

$$\frac{9}{\binom{9}{5}} = \frac{1}{14}$$

Solution to Question 946: Baby Boy

Let g be the number of girls in the nursery. Let C represent the event that the doctor chose a boy and B represent the event that the mother just had a boy. Then there are $4 + g$ total children in the nursery. We want $\mathbb{P}[B \mid C]$. By Bayes' Theorem, this is $\frac{\mathbb{P}[C \mid B]\mathbb{P}[B]}{\mathbb{P}[C]}$. We can choose a boy with the mother just birthing a boy vs. the mother just birthing a girl. Thus, $\mathbb{P}[C] = \mathbb{P}[C \mid B]\mathbb{P}[B] + \mathbb{P}[C \mid B^c]\mathbb{P}[B^c]$.

$\mathbb{P}[C \mid B] = \frac{4}{g+4}$, as the new child being a boy means there are 4 boys and g girls. $\mathbb{P}[C \mid B^c] = \frac{3}{g+4}$, as the new child being a girl means there are 3 boys and $g+1$ girls. Plugging all of these in, we have that

$$\mathbb{P}[B \mid C] = \frac{\frac{4}{g+4} \cdot \frac{1}{2}}{\frac{4}{g+4} \cdot \frac{1}{2} + \frac{3}{g+4} \cdot \frac{1}{2}} = \frac{4}{7}$$

Solution to Question 947: Variance of Sum of BM

The trick here is to note that $W_1 + W_2 = (W_2 - W_1) + 2W_1$. We do this because of the fact that $W_2 - W_1$ and $2W_1$ are independent random variables. Therefore,

$$\text{Var}(W_1+W_2) = \text{Var}((W_2-W_1)+2W_1) = \text{Var}(W_2-W_1)+\text{Var}(2W_1) = \text{Var}(W_2-W_1)+4\text{Var}(W_1)$$

We know that $W_2 - W_1 \sim N(0,1)$, so the first term is 1. We also know that $W_1 \sim N(0,1)$ so the second term is $4 \cdot 1 = 1$. Adding these up, our total is 5.

Solution to Question 948: Defining Correlation

The correlation between X and Y is defined as the ratio of their covariance to the product of their standard deviations, none of which are affected by multiplication or addition of constants. The correlation is thus still 0.56.

Solution to Question 949: Sticky Strike

The sticky-strike model assumes that implied volatility follows some curve, which is constant across the strikes. This does not assume that the curve is flat, but rather it is strictly a function of the strikes. Here, even though the underlying moves, we still are looking at the same strike, and thus it will have the same implied volatility.

Solution to Question 950: Estimating Pi

Note that $T_N \to \mathbb{E}[I_1] = \frac{\pi}{4}$ with probability 1 by the Strong Law of Large Numbers. $\mathbb{E}[I_1] = \frac{\pi}{4}$ because of the fact that the circle occupies a proportion of $\frac{\pi}{4}$ of the total area of the square. Our estimate for π should be $4T_N$. This is the quantity we want to find the variance of.

By properties of variance and the fact that each I_i is IID, we have that

$$\mathrm{Var}(4T_N) = 16\mathrm{Var}(T_N) = 16\mathrm{Var}\left(\frac{I_1 + \cdots + I_N}{N}\right) = \frac{16}{N^2}\cdot N\mathrm{Var}(I_1) = \frac{16\mathrm{Var}(I_1)}{N}$$

We can use the standard formula to compute $\mathrm{Var}(I_1)$. This is just $\mathrm{Var}(I_1) = \mathbb{E}[I_1^2] - (\mathbb{E}[I_1])^2$. However, note that $I_1^2 = I_1$, as I_1 only takes the values 0 and 1. Therefore, $\mathrm{Var}(I_1) = \frac{\pi}{4} - \frac{\pi^2}{16} = \frac{4\pi - \pi^2}{16}$. Accordingly, our final answer is just

$$\mathrm{Var}(4T_N) = \frac{-\pi^2 + 4\pi}{N}$$

This means $a = -1$ and $b = 4$, so $a + b = 3$.

Solution to Question 951: Square Ratio

The perfect square in question here are $1, 4, 9, \ldots$. For the ceiling of $\frac{Y}{X}$ to be a perfect square, say k^2, then $(k^2-1)X < Y \leq k^2 X$. However, for $k = 1$, we just have the condition $Y \leq X$, as we can't have negative values. For $k = 1$, the probability $Y < X$ is just $\frac{1}{2}$. Now, for $k \geq 2$, plotting out the region of interest in the plane, we see that for a fixed k, the region is a triangle with vertices as $(0,0)$, $\left(\frac{1}{k^2}, 1\right)$, and $\left(\frac{1}{k^2-1}, 1\right)$. Treating this region as the difference of two triangles, the probability for a fixed k is just $\frac{1}{2(k^2-1)} - \frac{1}{2k^2}$. Now, we sum up over all $k \geq 2$ to get that our probability is

$$\frac{1}{2} + \frac{1}{2}\sum_{k=2}^{\infty} \frac{1}{k^2-1} - \frac{1}{k^2}$$

We can evaluate the sum of the first term with a little creativity. Note that $\frac{1}{k^2-1} = \frac{1}{2}\cdot\frac{1}{k-1} - \frac{1}{2}\cdot\frac{1}{k+1}$ by partial fractions. Therefore, plugging this and expanding out the two terms, we have

$$\frac{1}{2} + \frac{1}{4}\sum_{k=2}^{\infty}\left(\frac{1}{k-1} - \frac{1}{k+1}\right) - \frac{1}{2}\sum_{k=2}^{\infty}\frac{1}{k^2}$$

The first sum telescopes to $1 + \frac{1}{2} = \frac{3}{2}$, as the remaining terms cancel. The term telescopes since the subtracted term lags behind by 2, so only the $k = 2$ and $k = 3$ terms are counted. Then, the second term is close to the known sum

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$$

The only difference is the $k = 1$ term is missing. Therefore, the sum in our question evaluates to $\frac{\pi^2}{6} - 1$, as we add and subtract the $k = 1$ term. Plugging in everything, our probability is

$$\frac{1}{2} + \frac{1}{4} \cdot \frac{3}{2} - \frac{1}{2} \cdot \left(\frac{\pi^2}{6} - 1\right) = \frac{11}{8} - \frac{\pi^2}{12}$$

Therefore, $a = 12$ and $q = \frac{11}{8}$, so $aq = \frac{33}{2}$.

Solution to Question 952: Dice Strike Price

We know that with probability $\frac{1}{2}$, we are lower than 12 in value, so we don't exercise the option. Otherwise, we are larger than 12, and each with probability $\frac{1}{12}$, we profit $1, 3, 5, 7, 9, 11$. This is because we subtract 12 from each of the values larger than 12. Therefore, our fair price is

$$\frac{1}{2} \cdot 0 + \frac{1 + 3 + 5 + 7 + 9 + 11}{12} = 3$$

Solution to Question 953: Optimal Marbles II

Similar to Optimal Marbles I, this game is symmetric for the two players, so their optimal strategy will be the same. This point will be important later. One main difference is that this is a zero-sum game, unlike Optimal Marbles I. The function we have to optimize will be significantly different.

Let $A(a, b)$ be the expected profit/loss of player A when A puts in a marbles and B puts in b marbles. Then

$$A(a, b) = \frac{a}{a + b} \cdot (100 - b) - \frac{b}{a + b} \cdot (100 - b)$$

because a wins with probability $\frac{a}{a+b}$ and b wins with probability $\frac{b}{a+b}$. We want to optimize $A(a, b)$ for a when b is a fixed value. Therefore, we are going to take the derivative of $A(a, b)$ (treating $A(a, b)$ as continuous in its arguments) with respect to a and try to optimize that. Taking the derivative, we get

$$\frac{\partial}{\partial a} A(a, b) = -\frac{a^2 + 2ba + b^2 - 200b}{(a + b)^2} = 0 \iff a^2 + 2ba + b^2 - 200b = 0 \iff a^* = \frac{-2b \pm \sqrt{4b^2 - 4b^2 + 800}}{2}$$

However, note that the root involving subtraction of the square root would be negative, so we must add our square root instead. Simplifying, we get $a^* = 10\sqrt{2b} - b$. As the game is symmetric, $b^* = 10\sqrt{2a} - a$.

To find the optimal strategy, we must solve for when $a^* = b^*$. In other words, we let b in the equation for a^* be $b^* = a^*$. This means that

$$a^* = 10\sqrt{2a^*} - a^* \iff 4(a^*)^2 = 200a^* \iff a^* = 0, 50$$

Once again, as $a^* > 0$, we must have $a^* = b^* = 50$, which means $(50, 50)$ is our Nash equilibrium. Therefore, player A should place 50 marbles in.

Solution to Question 954: Rabbit Hop V

Let o_n be the number of distinct paths that the rabbit can take to go up a staircase of height n. We can condition on the size of the first jump. We can jump either $1, 3, 5, \ldots, \alpha$, where $\alpha = n$ if n is odd and $n-1$ if n is even. In other words, α is the largest odd integer at most n. By conditioning, we have that

$$o_n = o_1 + o_3 + \cdots + o_\alpha$$

Let's take a closer look at the tail term $o_3 + \cdots + o_\alpha$. If the rabbit started on the second stair, this is exactly the sum that would be obtained, as the rabbit must can up $1, 3, 5, \ldots, \alpha - 1$ stairs from stair 2. These correspond to the exact terms in the tail sum when taking the ground floor to be stair 2. This means that

$$o_2 = o_3 + \cdots + o_\alpha$$

so we can update our recurrence relation to be

$$o_n = o_{n-1} + o_{n-2}$$

Our initial conditions are $o_1 = o_2 = 1$, which can be directly counted quite easily. These are exactly the first two terms and the recurrence of the Fibonacci sequence, so $o_n = F_n$. In particular, $o_{10} = F_{10} = 55$.

Solution to Question 955: Normal Conditions

Let ϕ_1 and ϕ_2 be the PDFs of X_1 and X_2, respectively. We know that as X_1 and X_2 are independent, $X_1 + X_2 \sim N(0, 25)$, as we just add the variances. We now compute the conditional distribution of $X_1 \mid X_1 + X_2 = 5$. By the conditional PDF formula, we know that

$$f_{X_1|X_1+X_2=5}(x) = \frac{f_{X_1,X_1+X_2}(x,5)}{f_{X_1+X_2}(5)}$$

The numerator here can be greatly simplified by noting that if $X_1 = x$, then $X_2 = 5 - x$ so that we get a sum of 5. Therefore, $f_{X_1,X_1+X_2}(x,5) = \phi_1(x)\phi_2(5-x)$.

Plugging all of the PDFs in,

$$f_{X_1|X_1+X_2=5}(x) = \frac{\frac{1}{3\sqrt{2\pi}}e^{-\frac{x^2}{18}} \cdot \frac{1}{4\sqrt{2\pi}}e^{-\frac{(5-x)^2}{32}}}{\frac{1}{5\sqrt{2\pi}}e^{-\frac{25}{50}}}$$

After doing a good amount of simplification, you will note that $X_1 \mid X_1 + X_2 = 5 \sim N\left(\frac{9}{5}, \frac{144}{25}\right)$. In particular, this means the mean is $\frac{9}{5}$, and we are done. You can also reason this answer intuitively as you would expect the contribution from each normal to be in proportion with the variance, so we are really saying that $\frac{9}{9+16} \cdot 5$ comes from the normal with variance 9.

Solution to Question 956: Put Arbitrage

We have the following relationship that must hold for puts on the same underlying with different strikes.

$$P(K_2) - P(K_1) \leq (K_2 - K_1)Z$$

If we let $K_2 = 17$ and $K_1 = 14$ and plug the values in. We get:

$$\begin{aligned} 5.9 - 2.1 &\leq (17 - 14) * 0.9 \\ 3.8 &\leq 3 * 0.9 \end{aligned}$$

This clearly does not hold. We should long the undervalued item and short the overvalued item. Here, we will long 3 units of the bond, short the $K = 17$ put and long the $K = 13$ put. This gives us:

$$\text{\# Stock} + \text{\# Put (K = 17)} + \text{\# Put (K = 14)} + \text{\# Bonds} = 0 - 1 + 1 + 3 = 3$$

Solution to Question 957: Graph Shading

The ratio satisfies the constraint if and only if $Y < X$ and $Y > \frac{1}{2}X$. The area covered by this constraint in the sample space $X, Y \in [0, 1]$ is equal to $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. Since both X and Y are uniformly distributed, the probability is $\frac{1}{4}$.

Solution to Question 958: Poisoned Kegs IV

We are going to use a similar idea to Poisoned Kegs II. We used binary expansion in Poisoned Kegs II since we only had 1 month, so each servant could either drink or not drink from a given keg. Now, we can assign some servants to drink at the first month, second month, and third month. With 5 servants, we convert each keg label $i = a_{0i} + a_{1i} \cdot 4^1 + a_{2i} \cdot 4^2 + a_{3i} \cdot 4^3 + a_{4i} \cdot 4^4$, where each $a_{ki} = 0, 1, 2, 3$ for $0 \le k \le 4$. If $a_{ki} = 0$, we don't have servant k drink from keg i at all. If $a_{ki} = r$ for $r > 0$, servant k drinks from keg i in month r.

When a servant dies, we now record which servant died and what hour they died in. From that, we can reconstruct the keg that was poisoned. For example, if our sequence is 10321, the poisoned keg was $1+2\cdot 4^1+3\cdot 4^2+0\cdot 4^3+1\cdot 4^4 = 313$. We have now created a bijection between base 4 expansions of length 5 and the keg labels, so we can test up to $n = 4^5 = 1024$ kegs.

Solution to Question 959: 6 Die Stopper

Including or excluding the 6 doesn't matter, as adding 6 does not change the parity of the overall sum. Let p be this probability. We condition on the first roll. If the first roll is a 6, then we have an even sum with probability 1. If the first roll is a 2 or 4, then the sum of all other terms must be even, occurring with probability p. If the first roll is odd, then the sum of the other rolls must be odd, occurring with probability $1 - p$. Writing this out with Law of Total Probability, we see that

$$p = \frac{1}{6} \cdot 1 + \frac{1}{3} \cdot p + \frac{1}{2} \cdot (1 - p)$$

Solving this yields $p = \dfrac{4}{7}$.

Solution to Question 960: Rabbit Hop I

Each path can be described as selecting a subset of $\{1, 2, \ldots, 9\}$ for the rabbit to land on. This is because we can identify any path uniquely by which stairs the rabbit lands on. Thus, there are $2^9 = 512$ subsets of $\{1, 2, \ldots, 9\}$, so 512 is our answer.

Solution to Question 961: Shifty Cylinder

$$V(r, h) = \pi r^2 h$$

is the volume of a cylinder as a function of r and h. Furthermore, we are given that $r' = 2$ and $h' = -5$ from the question. Therefore, we have that $V' = 2\pi r h r' + \pi r^2 h'$ by applying the product rule to the above. Plugging in all of our values, we get that the volume changes at a rate of

$$2\pi(3)(6)(2) + \pi(3)^2(-5) = 27\pi$$

Solution to Question 962: Salah Goal

We will solve this for $3n$ steps. If Salah is to end up exactly where he started, he must have taken n steps forward and $2n$ steps backward. This is because of the fact that each step backwards is twice as far as a step forward. There are $2^{3n} = 8^n$ total ways Salah could move and $\binom{3n}{n}$ of them have exactly n steps forward, so the probability is $\frac{\binom{3n}{n}}{8^n}$. Plugging in $n = 3$, we get the answer $\frac{21}{128}$.

Solution to Question 963: Gamma Review II

Letting $u = x^n$, we have that $x = u^{\frac{1}{n}}$, so $dx = \frac{1}{n} u^{\frac{1}{n}-1} du$. As $n > 0$, the bounds are still 0 to ∞, so our integral is now

$$\frac{1}{n} \int_0^\infty u^{\frac{2}{n}-1} e^{-u} du$$

The integral part, by the integral definition of the Gamma function, evaluates to $\Gamma\left(\frac{2}{n}\right)$. The final integral is equal to $\frac{1}{n} \cdot \Gamma\left(\frac{2}{n}\right)$

Solution to Question 964: Allocating Capital

The "stars and bars" approach, first popularized by William Feller in his classic book, An Introduction to Probability Theory and Its Applications, works well for this. Let each star represent a million dollar investment and each bar represent a delimiter between asset class allocations; in this problem, we have 50 stars and 4 bars. There are a total of $\binom{54}{4}$ possible ways to set 4 of the 54 symbols to bars, with the remaining being the stars. Thus, the number of ways to invest the \$50 million into the five asset classes is:

$$\binom{54}{4} = \frac{54!}{50!4!} = 316251$$

Solution to Question 965: Soda Machines

Since the sample size is less than 30, we utilize a t test where $H_0 : \mu = 7$ and $H_a : \mu \neq 7$. The t statistic is:

$$t = \frac{\bar{x} - \mu}{\sqrt{\frac{s^2}{n}}} = \frac{7.1 - 7}{\sqrt{0.12^2/8}} \approx 2.36$$

Solution to Question 966: Integral Variance IV

We use a couple of the common lemmas to solve this. Namely, since the integrand is "nice enough" here, we can see that the mean of X is 0. Therefore, we just need to find the second moment. In particular, since the integrand is square integrable over the interval in question,

$$\mathbb{E}[X^2] = \int_0^2 \mathbb{E}\left[\left(\sqrt{t}e^{\frac{W_t^2}{8}}\right)^2\right] dt = \int_0^2 t\mathbb{E}\left[e^{\frac{W_t^2}{4}}\right] dt$$

As $W_t \sim N(0,t)$, we have that $W_t = \sqrt{t}Z$, where $Z \sim N(0,1)$. Therefore,

$$\mathbb{E}\left[e^{\frac{W_t^2}{4}}\right] = \mathbb{E}\left[e^{\frac{t}{4}Z^2}\right] = M_{Z^2}\left(\frac{t}{4}\right)$$

Where $M_{Z^2}(\theta)$ is the MGF of Z^2. Note that as $Z \sim N(0,1)$, $Z^2 \sim \chi_1^2$. A χ_1^2 random variable has MGF given by

$$M_{Z^2}(\theta) = (1-2\theta)^{-\frac{1}{2}}$$

Evaluating at $\theta = \dfrac{t}{4}$, we have that

$$\mathbb{E}\left[e^{\frac{W_t^2}{4}}\right] = \sqrt{\frac{2}{2-t}}$$

Plugging this back in,

$$\mathbb{E}[X^2] = \int_0^2 t \cdot \sqrt{\frac{2}{2-t}} dt = \frac{16}{3}$$

This is also the variance, as the mean is 0.

Solution to Question 967: Casted Shadow

Let x be the distance of the person from the base and s be the length of the shadow. We are given that $x' = 40$ is constant. Furthermore, we know that when $x = 10$, $s = 15$. We want to find s' when $x = 10$ and $s = 40$.

We can use similar triangles to first find the (constant) height of the pole, say h. The total distance that the shadow is away from the base of the pole is $x + s$. The other triangle we have the one that has the vertical side as the person's height and horizontal side as the shadow cast. Therefore, by equating corresponding sides of the triangle, we get the ratio

$$\frac{x+s}{h} = \frac{s}{6}$$

Plugging in $x = 10$ and $s = 15$, we get that

$$\frac{25}{h} = \frac{5}{2} \iff h = 10$$

Rewriting the original ratio with our constants, we have that

$$\frac{x+s}{10} = \frac{s}{6} \iff x = \frac{2}{3}s$$

Taking the derivative on both sides, we have that $x' = \frac{2}{3}s'$, so knowing that $x' = 40$, we get that $s' = \frac{3}{2} \cdot 40 = 60$.

Solution to Question 968: RNG on RNG

The optimal strategy will be in the form of a payout when you generate some value $x \geq T$ for a threshold T. This is because you can intuitively view each roll as a new game with some stopping threshold that we want to achieve. Therefore, our payout should a function $f(T)$ that we want to maximize.

Let's compute what $f(T)$ is now. If we generate a value in excess of T, the distribution of that value is Unif$(T, 1)$, so the expected value of it is $\frac{1+T}{2}$. However, the expected price to pay for that value is $0.02 \cdot \left(\frac{1}{1-T} - 1\right)$, as the probability that on any given trial we roll a value at least T is $1 - T$, so the distribution of the number of trials needed to obtain a value at least T is Geom$(1 - T)$. The mean of this distribution is $\frac{1}{1-T}$. The cost per trial besides the first is 0.02, so this yields our equation above. Therefore, the expected payout is $f(T) = \frac{1+T}{2} - \frac{1}{50} \cdot \left(\frac{1}{1-T} - 1\right)$. As we want to maximize this in T, we just take a derivative and set it equal to 0 to solve. Namely, $f'(T) = \frac{1}{2} - \frac{1}{50(1-T)^2} = 0$. Rearranging this yields $(1 - T)^2 = \frac{1}{25} = \frac{1}{5^2}$, so as $0 < T < 1$, the solution is $T = 0.8$. Plugging this back in, $f(0.8) = 0.82$.

Solution to Question 969: Heads and Tails I

If you get tails on your first toss (which has probability 1/2), then you must start over again. If you gets heads on your first toss, then you can either get HH or HT after your second toss (each with probability 1/4). If you get HT, then you are done. Otherwise, you only need one more tail to finish the game on the next turn.

To sum up our reasoning, if you flip H, then the event of ending up with a HT depends entirely on the event that you get a T. This is because if you do *not* get a T on the next flip, then you must get a H, which means that you have the same chance to finish the game on the next round. Let X denote the number of flips to get HT. Let Y denote the number of flips to get T. Thus, we have the following expression:

$$\mathbb{E}[X] = \frac{1}{2}\left(1 + \mathbb{E}[Y]\right) + \frac{1}{2}\left(\mathbb{E}[X] + 1\right)$$

Note that

$$\begin{aligned} Y &\sim \text{Geo}(0.5) \\ \Rightarrow \mathbb{E}[Y] &= 2 \end{aligned}$$

Now, we can solve for $\mathbb{E}[X]$.

$$\mathbb{E}[X] = 2\left(\frac{3}{2} + \frac{1}{2}\right) = 4$$

Solution to Question 970: Uniform Order III

There is quite a bit going on here, so we need to decipher what this probability statement is really saying and how we can simplify it. We want the conditional probability that X_{16} is the smallest of the first 16 random variables given that X_1 is the largest of the first 16 and that $X_{17} < X_{18}$. Note that each of the random variables are IID, so the ordering of X_{17} and X_{18} with respect to each other has no bearing on the ordering of the first 16 random variables in the sequence. Therefore,

$$\mathbb{P}[X_{16} = \min\{X_1, \ldots, X_{16}\} \mid X_1 = \max\{X_1, \ldots, X_{16}\}, X_{17} < X_{18}] = \mathbb{P}[X_{16} = \min\{X_1, \ldots, X_{16}\} \mid X_1 = \max\{X_1, \ldots,$$

as that information does not tell us anything about our probability of interest.

Consider 16 blanks corresponds to the largest to smallest random variables. We know that X_1 goes in the first spot because it is known to be the maximum of the first 16 random variables. Thus, there are 15 random variables left to arrange to the other 15 blanks, and we want the probability X_{16} goes in the last spot. Since the other 15 random variables are exchangeable, it is no more likely that X_{16} goes in the last spot than any of $X_2, \ldots, X_{15}$, so the probability is just $\frac{1}{15}$. The key thing to notice in this question is that the ordering of random variables not in the subset of random variables we consider is irrelevant by the independence of the random variables.

Solution to Question 971: Ronaldo's House

The key here is to see that if either 22 or 77 falls off, then no matter how Ronaldo places up the numbers, he will get back the correct house number. If any other pair falls off, there is a 1/2 chance of obtaining the correct house number. Therefore, let S be the event that the two numbers that fall off are the same i.e. either 22 or 77 and C be the event that Ronaldo puts up the numbers in the correct order again. By Law of Total Probability,

$$\mathbb{P}[C] = \mathbb{P}[C \mid S]\mathbb{P}[S] + \mathbb{P}[C \mid S^c]\mathbb{P}[S^c]$$

There are $\binom{6}{2} = 15$ equally likely pairs of numbers to fall off, of which 2 of them are 22 or 77, so $\mathbb{P}[S] = \frac{2}{15}$, meaning $\mathbb{P}[S^c] = \frac{13}{15}$. If the two numbers that fall off are the same, Ronaldo will be correct with probability 1, so $\mathbb{P}[S \mid C] = 1$. If the numbers are not the same, he will be correct with probability 1/2, as one of the two arrangements is the true house number, so $\mathbb{P}[C \mid S^c] = 1/2$. Adding all of these up, we get

$$\mathbb{P}[C] = 1 \cdot \frac{2}{15} + \frac{1}{2} \cdot \frac{13}{15} = \frac{17}{30}$$

Solution to Question 972: Game Arbitrage II

2

teams are guaranteed to make it out of the group. If we long every contract, we expect a payout of 2 (and also an initial price of 2). You can imagine this as the density being split amongst the 4 teams such that the sum is 2. When we add the values of all contracts, we see a value of $2.05 > 2$. This means that the contracts are overvalued.

To capitalize on the arbitrage, we long the undervalued item and short the overvalued item. We can short 1 unit of every contract and then long 2 units of the bond. This gives us $2 - 0.95 - 0.72 - 0.35 - 0.03 = -0.05$, meaning we receive 0.05 as a credit.

Solution to Question 973: Card Diff

The claim here is that one can always locate the places of the 9 and 10 within 11 card draws. Therefore, the payout would be $9 \cdot 10 - 11 = 79$.

Suppose you pick one card. You start paying \$1 to reveal the difference between that card and each other card of the remaining 12. We can split up into two cases: The first card you selected does have value 0 or does not have value 0.

If the first card is a 0, then at some point in the first 11 differences, you will either two differences of 0 (in which you completely identify your card as a 0) and a difference of 9 or you obtain a difference of 10 (or both!). You would only need to spend \$11 at worst, as if one of the two scenarios doesn't occur in the first 11 differences, we know that the 9 or 10 (whichever difference wasn't obtained) is the last card whose value you didn't difference yet. This means that in the worst case scenario you only need to ask for the difference between your card and 11 more cards out of the 12 total cards, which yields \$11 payment.

If the value of the first card is not 0, we know that there are 3 cards with value 0 still remaining in the deck. You can identify which card you picked as soon as you obtain 2 face cards, as their values are 0 and you would be told the same number twice. Therefore, in this case, you also need to pay only \$11 at most in order to be able to have full knowledge of where the 9 and the 10 are located.

Solution to Question 974: Hidden Prisms

This problem is similar to "Rectangles on Chess Board". On each of the three axes of the 5x5x5 cube, we need to pick two points (a start and an end). One axis will be for the length of our prism, one for the width, and the last for the height. There are 6 different points to pick from for a start and an end. Since we are picking 2 of them, there are $\binom{6}{2} = 15$ ways to pick a start and an end on an axis. Given we have three axes, the total number of rectangular prisms is $\binom{6}{2}^3 = 15^3 = 3375$ total rectangular prisms.

Solution to Question 975: Vasicek Equation

We can calculate $d(e^{bt}R_t)$ via Ito's Formula. This can be done via the product rule. In particular,

$$d(e^{bt}R_t) = be^{bt}R_t dt + e^{bt}dR_t = be^{bt}R_t dt + e^{bt}\left((a - bR_t)dt + \sigma dW_t\right) = ae^{bt}dt + \sigma e^{bt}dW_t$$

Now, we integrate both sides to get

$$e^{bT}R_T - R_0 = \int_0^T ae^{bs}ds + \int_0^T \sigma e^{bs}dW_s$$

We can substitute a couple of things. Namely, $R_0 = r$ by the question. In addition, the Ito integral on the RHS is "sufficiently nice" so that it has 0 mean. Therefore,

$$e^{bT}\mathbb{E}[R_T] - R_0 = \frac{a}{b}\left(e^{bT} - 1\right)$$

After adding r and multiplying by e^{-bT} on both sides, we get that $\mathbb{E}[R_T] = re^{-bT} + \frac{a}{b}\left(1 - e^{-bT}\right)$.

Evaluating this with the parameters in question, our expectation evaluates to $e^{-1} + 2\left(1 - e^{-1}\right) = 2 - e^{-1}$. In this case, we have that $2 - 1 - 1 = 0$.

Solution to Question 976: Median Uniform

We can approach this with an intuitive perspective, as the random variables here are uniformly distributed. Split up the interval $(0, 3)$ into three subintervals $(0, 1], (1, 2]$, and $(2, 3)$. Each X_i has equal chance of landing in each of the three subintervals. Label the random variables $1 - 3$, based on which interval they end up in. Therefore, we want the probability the median is in 2. Let's consider the different permutations that allow that to happen:

$$122, 123, 223, 222$$

These are the different ways the values of the intervals can be assigned to the random variables. All that remains is to count the permutations. There are 3 ways to assign the values 122 to the three random variables (select the random variable that receives 1 in 3 ways, the other 2 receive 2). Similarly, there are $3! = 6$ ways for 123, 3 ways for 223, and 1 way for 222. Adding these all up, there are $3 + 6 + 3 + 1 = 13$ permutations of the values. Then, there are $3^3 = 27$ equally-likely ways to assign the values to the random variables in general. Therefore, the answer is $\frac{13}{27}$.

Solution to Question 977: Party Groups I

You can also think of this scenario as each guest acting like a cable extender. Each cable has an "input" side and an "output" side. When the first guest picks a name and they pick their own name, its like connecting the two ends of the same wire together. Otherwise you are connected two wires together and making one larger wire with one input end and one output end.

A more algebraically-inclined person may want to think of this scenario as the expected number of cycles of a random $50-$permutation.

Whichever way you want to think about it, the math stays the same. When the first guest picks a name, there is a $\frac{1}{50}$ chance that the number of groups does not change and a $\frac{49}{50}$ chance that the number of groups effectively decrease by one (because two groups join together). When the next guest picks a number, the given scenario is very similar whether the first guest formed their own group or connected with another guest. Both scenarios have 49 groups or "wires". The only difference is that there's one closed group if the first guest picks their own name versus no closed groups if they pick another guest's name. If we keep continuing this for every guest, we add a closed group with probability $\frac{1}{k}$ where k are the number of names still left in the hat.

Thus the answer is

$$\sum_{n=1}^{50} \frac{1}{n} \approx 4.5$$

Solution to Question 978: Bold Subsets

Each element of S presents Erica with 3 choices: include, exclude, or bold include. As $|S| = 6$, the answer is $3^6 = \boxed{729}$

Solution to Question 979: Strangle Delta

From the payoff, we can see that at $S_0 = 2$, we are in the negative delta region of the strangle. Hence, we would want the stock to go downwards. An argument can be made that this payoff is at expiry. However, the option price (prior to expiry) will strictly be above the final payoff. For any price $S_0 \leq 3$, we can expect the delta to be negative. After $S_0 > 3$, then it may become more difficult to say.

Solution to Question 980: Support Center

Let X be the number of calls the IT support center receives in 10 minutes. On average, the center receives 270 calls per hour, or 45 calls per 10 minutes. From this information, we can define $X \sim P(45)$. Using a CDF calculator, we can calculate the following:

$$P(X > 40) = 1 - P(X \leq 40) = 1 - 0.25555 \approx 0.744$$

Solution to Question 981: Doubly Winner

Let W_i be the indicator random variable of James flipping two consecutive heads in spots i and $i+1$, $i \geq 1$. Then $T_n = W_1 + \cdots + W_{n-1}$ gives the total amount of money James wins. We stop at $n-1$ since there is no other coin after the nth coin. We have that $p(n) = \mathbb{E}[T_n] = \sum_{k=1}^{n-1} \mathbb{E}[W_k]$ by Linearity of Expectation. $\mathbb{E}[W_k]$ is just the probability that both the k and $k+1$st coins show heads, which is $\frac{1}{k(k+1)}$. This means that $\mathbb{E}[T_n] = \sum_{k=1}^{n-1} \frac{1}{k(k+1)}$.

To get a closed form for this sum, we use partial fraction decomposition and note that this sum will telescope. In particular, it is fairly simple to see that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$, so $p(n) = \sum_{k=1}^{n-1} \frac{1}{k} - \frac{1}{k+1}$. This sum telescopes, leaving only the $1 - \frac{1}{n}$ as the result, as those are the only two terms that don't have a partner. Thus, $p(n) = 1 - \frac{1}{n}$. We have a sanity check from the fact that $p(2) = \frac{1}{2}$, as there is probability $\frac{1}{2}$ chance both coins show heads.

Therefore, $\lim_{n \to \infty} p(n) = 1$, as the $\frac{1}{n}$ term tends to 0.

Solution to Question 982: Coin Duel

Let X_1 and X_2 represent the number of flips needed for Bill and Bob, respectively, to obtain their first heads. We are going to generalize this to probability p of heads per flip. We immediately know that $X_1, X_2 \sim \text{Geom}(p)$ by the fact we are looking for a waiting time until first heads. One might guess that the answer to this is $\frac{1}{2}$, but the issue is that there is positive probability of equality, so we first need to subtract that out, and then we can use exchangeability to claim that $\mathbb{P}[X_1 > X_2] = \mathbb{P}[X_1 < X_2]$.

First, let's find $\mathbb{P}[X_1 = X_2]$. This is simple enough to find, as X_1 and X_2 are independent, so

$$\mathbb{P}[X_1 = X_2] = \sum_{n=1}^{\infty} \mathbb{P}[X_1 = X_2 \mid X_2 = n]\mathbb{P}[X_2 = n] = \sum_{n=1}^{\infty} \mathbb{P}[X_1 = n]\mathbb{P}[X_2 = n]$$

as we want the probability $X_1 = X_2$ and we know that $X_2 = n$. The previous work is just by the Law of Total Probability applied by conditioning on X_2. We have that

$$\mathbb{P}[X_1 = n] = \mathbb{P}[X_2 = n] = p(1-p)^{n-1}$$

by the fact that X_1 and X_2 are IID. Thus, previous sum becomes

$$\sum_{n=1}^{\infty} p^2(1-p)^{2n-2} = p^2 \sum_{n=1}^{\infty} \left[(1-p)^2\right]^{n-1}$$

by rearranging slightly. We do this because we want to get an $n-1$ in the exponent of the interior since the lower index of the sum is $n = 1$ and we would ideally by able to apply the geometric sum formula. The sum inside is now geometric with $r = (1-p)^2 < 1$, so the above expression becomes $\frac{p^2}{1-(1-p)^2} = \frac{p^2}{p(2-p)} = \frac{p}{2-p}$. This is $\mathbb{P}[X_1 = X_2]$, so

$$\mathbb{P}[X_1 \neq X_2] = 1 - \mathbb{P}[X_1 = X_2] = 1 - \frac{p}{2-p} = \frac{2(1-p)}{2-p}$$

. This is $\mathbb{P}[X_1 \neq X_2]$, and this is partitioned into two events, namely that $\mathbb{P}[X_1 \neq X_2] = \mathbb{P}[X_1 > X_2] + \mathbb{P}[X_1 < X_2]$. This is because if they are not equal, then one of them must be larger than the other. Since X_1 and X_2 are IID, they are exchangeable, so we have that the should have equally likely probabilities of being larger or smaller than one another. Thus, the two probabilities on the RHS are equal. Substituting in our found value on the LHS for $\mathbb{P}[X_1 \neq X_2]$, we have that $\frac{2(1-p)}{2-p} = 2\mathbb{P}[X_1 < X_2]$, meaning that $\mathbb{P}[X_1 < X_2] = \frac{1-p}{2-p}$. Since there is positive probability of equality between X_1 and X_2, it makes sense that the probability slightly less than $\frac{1}{2}$.

Alternatively, we can give an argument by a one-step Law of Total Probability approach. Let x be the probability in question. We can condition on the 4 outcomes of the first flip. If Bob flips a heads on the first flip, which occurs with probability p, Bill can't obtain a heads before him, so this conditional probability is 0. If Bob flips a tails, then if Bill flips a heads, he wins immediately. This occurs with probability $p(1-p)$. If Bill flips a tails as well, then the probability is just the same as at the start, which is x. Therefore, we obtain the equation

$$x = p(1-p) + (1-p)^2 x \iff (2p - p^2)x = p(1-p) \iff x = \frac{1-p}{2-p}$$

In this case, $p = 1/3$, so our answer is $\frac{2}{5}$.

Solution to Question 983: Summed Brownians

We use the fact that for a standard Brownian Motion, $\text{Cov}(B(s), B(t)) = \min\{s, t\}$. First, we use the bilinearity of covariance here to conclude that $\text{Cov}(B_1(1) + B_2(1), B_2(2) + B_3(2)) = \text{Cov}(B_2(1), B_2(2))$, as all other pairs vanish due to independence. Then, using the fact above, the last term evaluates to 1. Therefore, our answer is 1.

Solution to Question 984: The Perfect Hedge II

From The Perfect Hedge I, we know that the weight of asset 1 to achieve a riskless portfolio is $\frac{\sigma_2}{\sigma_1+\sigma_2}$. Denote μ as the expected return of the portfolio, and μ_1 as the return of asset 1 and μ_2 as the return of asset 2. We can then write the expected return of the portfolio as:

$$\mu = w\mu_1 + (1-w)\mu_2$$

Plugging in the value of w from above, we get:

$$\begin{aligned} \mu &= \frac{\sigma_2}{\sigma_1+\sigma_2}\mu_1 + \frac{\sigma_1}{\sigma_1+\sigma_2}\mu_2 = \frac{1}{\sigma_1+\sigma_2}(\mu_1\sigma_2 + \mu_2\sigma_1) \\ &= \frac{1}{\sqrt{.15}+\sqrt{.04}}(.04 * \sqrt{.04} + .02 * \sqrt{.15}) \approx .0268 \end{aligned}$$

This means that we can combine two risky assets to obtain a risk-free portfolio of 0.0268.

Solution to Question 985: Unknown Baby

Let B be the event that the baby added was a boy. Let S be the event that the selected baby was a boy. We want $\mathbb{P}[B \mid S] = \frac{\mathbb{P}[S \mid B]\mathbb{P}[B]}{\mathbb{P}[S]}$. We know that $\mathbb{P}[B] = \frac{1}{2}$, as this is with no other information. We further know that if the added baby was a boy, then $\mathbb{P}[S \mid B] = \frac{1}{2}$, as there are equal amounts of boys and girls. On the denominator, we condition on B and B^c. Namely,

$$\mathbb{P}[S] = \mathbb{P}[S \mid B]\mathbb{P}[B] + \mathbb{P}[S \mid B^c]\mathbb{P}[B^c]$$

The first term is the same as the numerator. We know that $\mathbb{P}[B^c] = \frac{1}{2}$ as well. B^c is the event a girl was added, so in this case, $\mathbb{P}[S \mid B^c] = \frac{1}{3}$, as there are only 2 boys among 6 total. Therefore, our total probability is

$$\mathbb{P}[B \mid S] = \frac{1/2 \cdot 1/2}{1/2 \cdot 1/2 + 1/2 \cdot 1/3} = \frac{3}{5}$$

Solution to Question 986: Covariance Review VI

We proceed as follows, making use of the fact that Y and Z are independent:

$$\begin{aligned}
\text{Cov}(Y, W) &= \mathbb{E}[YW] - \mathbb{E}[Y]\mathbb{E}[W] \\
&= \mathbb{E}[YW] - \mathbb{E}[Y]\mathbb{E}\left[\frac{Z}{\sqrt{Y}}\right] \\
&= \mathbb{E}\left[Y\frac{Z}{\sqrt{Y}}\right] - \mathbb{E}[Y]\mathbb{E}\left[\frac{1}{\sqrt{Y}}\right]\mathbb{E}\left[Z\right] \\
&= \mathbb{E}[\sqrt{Y}]\mathbb{E}[Z] \\
&= 0
\end{aligned}$$

Solution to Question 987: Correlation Variance

We are going to generalize this to when $\sigma_X = a, \sigma_Y = b$, and $\sigma_{X+Y} = a + b$. We know that

$$\text{Var}(X+Y) = \text{Var}(X)+\text{Var}(Y)+2\text{Cov}(X,Y) \implies \text{Cov}(X,Y) = \frac{1}{2}((a+b)^2-a^2-b^2) = ab$$

After, we know that

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X\sigma_Y} = \frac{ab}{ab} = 1$$

Solution to Question 988: Increasing Chains

Let's first consider what the maximum number of possible increasing chains is. Index $n-(k-1)$ is the last spot at which we can have an increasing chain of length k. This is because we would need spots $n-(k-1), \ldots, n$ to be increasing order. After that spot, there are not k open spots to be increasing. Therefore, let I_i be the indicator of the event that starting that index i, $X_i < \cdots < X_{i+k-1}$. This is saying that there is an increasing chain of length k starting from position i. Then $T = I_1 + \cdots + I_{n-(k-1)}$ gives the total number of strictly increasing chains of length k. Thus, $\mathbb{E}[T] = (n-(k-1))\mathbb{E}[I_1]$ by linearity of expectation and exchangeability of the random variables. $\mathbb{E}[I_1]$ is just the probability that the values are in strictly increasing order. This is the same as randomly permuting those k indices and having the indices be lined up in increasing order, which occurs with probability $\frac{1}{k!}$. Therefore, our answer is $\frac{n-(k-1)}{k!}$ when we have n random variables.

Now, we know that $k = 6$ in this example and we want this expectation to be set equal to 1, so we want to find n satisfying $\frac{n-5}{6!} = 1$, so $n = 6! + 5 = 725$.

Solution to Question 989: Averaging Squares

There are a total of 4046 numbers, so the 2023rd and 2024th terms are required to find the median. Note that $44^2 = 1936$, so there are an additional 44 squared terms before 2023; that is, 2023 is the 2067th term. Thus, the 2023rd term is $2023 - 44 = 1979$ and the 2024th term is 1980. The average between these terms is the median, 1979.5.

Solution to Question 990: Quick Summation

There are several solutions to this problem, algebraic and geometric. The most simple is to use the Gaussian sum which can be derived in 2D geometry. The sum of the integers from 1 to n is $\frac{n\times(n+1)}{2}$. In this case, the sum is 1275.

Solution to Question 991: Curious Multiplicand

This is the recurring part of the decimal expansion of the reciprocal of 7.

$$\frac{1}{7} \approx .142857$$

We can see that multiplying these numbers by anything less than 7 will be some multiple of this, and hence shifts the digits around.

Solution to Question 992: Take and Roll I

If you take when you have at least k for the first time, your expected face showing is $\frac{20+k}{2}$. Additionally, as there are $21 - k$ values on the die at least k, the probability on any given roll of seeing at least k is $\frac{21-k}{20}$. Therefore, the average number of rolls needed to see a value at least k is $\frac{20}{21-k}$. This means that you are able to claim on $100 - \frac{20}{21-k}$ turns. Thus, your expected payout would be

$$p(k) = \frac{20+k}{2} \cdot \left(100 - \frac{20}{21-k}\right)$$

To find the maximum, one can treat $p(k)$ as continuous and differentiate in k and then consider the two integers k is between as potential maximizers.

Doing this by product rule and simplifying, $p'(k) = \frac{10}{(k-21)^2} \cdot (5k^2 - 210k + 2164)$. The zeros of this polynomial, by quadratic formula, are $k^* = \frac{105 \pm \sqrt{205}}{5}$. The root where we add is larger than 20, so $k^* = \frac{105 - \sqrt{205}}{5} \approx 18.137$ is our optimizer. In an interview, one could notice that $14^2 < 205 < 15^2$, so that's how one could deduce $18 < k^* < 19$. Lastly, plugging in $k = 18$ and $k = 19$ yields that $k = 18$ yields the optimal value with $\frac{5320}{3}$.

Solution to Question 993: Continuous Blackjack

What we need to do is find the functions $q(x)$ and $r(x)$ that respectively represent the gambler's probability of losing given that he stops at current sum x and hits at current sum x. Let's first find $q(x)$.

$q(x)$ is the probability that we lose given that stop at a sum x. The gambler loses if the successive drawings of the dealer produce a partial sum s_{n_1} in the interval $(x, 1)$. This would imply that there is a $n \geq 0$ such that $s_n \leq x$ but $x \leq s_{n+1} \leq 1$. We can sum over all cases of n at the end. The probability that $s_n \leq x$ is a common question that has a known answer of $\frac{x^n}{n!}$. This can be seen by considering the volume of the region $\{0 \leq X_1 + \cdots + X_n \leq x\}$. Then, the probability that the $(n+1)$st draw puts s_{n+1} in the interval $(x, 1)$, which is of length $1 - x$, is $1 - x$. Therefore, multiplying then, the probability that this happens with exactly $n + 1$ random variables is $\frac{x^n}{n!}(1-x)$. Summing this over n yields

$$q(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}(1-x) = (1-x)e^x$$

Now, to find $r(x)$, the probability of losing given you hit and the current sum is x, we need to consider two cases. If the current sum is x, then we automatically lose if we bust. We bust if the next draw is at least $1-x$, as the sum would be at least 1, so the probability of this is x. Alternatively, by Law of Total Probability and conditioning on the fact that the next draw is at most $1-x$ (so we don't bust), we just integrate over q from x to 1, as those are the possible values we can receive. Therefore,

$$r(x) = \int_x^1 q(t)dt + x = x + e - (2-x)e^x$$

Note that the function $q(x)$ is decreasing in $(0,1)$ and $r(x)$ is increasing in $(0,1)$. Therefore, α solves $q(\alpha) = r(\alpha)$, where we are indifferent to hitting or stopping. This means that $(1-\alpha)e^\alpha = \alpha + e - (2-\alpha)e^\alpha$, so $(3-2\alpha)e^\alpha = e+\alpha$, lastly implying that $e^\alpha = \dfrac{e+\alpha}{3-2\alpha}$. This means $x = 3$ and $y = -2$, so $x^2 + y^2 = 13$.

Solution to Question 994: Decreasing Uniform Chain

Let $f(x)$ be the expected smallest number given that the current smallest is x. We want $f(1)$, as we are selecting uniform values on $(0,1)$, so our initial value is 1. We don't choose a starting value higher than 1 since we can't generate values larger than 1.

With probability $1-x$, the next value is larger than x, and our smallest value will be x. Otherwise, if our next value is $y < x$, y is uniformly distributed on $(0,x)$, so the expected minimum in that case is $f(y)$. We integrate over all $0 < y < x$ since we are applying Law of Total Probability. Written as an equation, this is

$$f(x) = x(1-x) + \int_0^x f(y)dy$$

Taking the derivative to convert into a differential equation, we get that $f'(x) = f(x) + (1-2x)$. Equivalently, $f'(x) - f(x) = 1-2x$. An initial condition for this differential equation is $f(0) = 0$, as we can't go any lower than 0.

This is a linear first order differential equation, so we can use the method of integrating factors here. In particular, the integrating factor is $\mu(x) = e^{\int -1dx} = e^{-x}$. Multiplying this on both sides, $(e^{-x}f(x))' = (1-2x)e^{-x}$, meaning $e^{-x}f(x) = \int e^{-x} - 2xe^{-x}dx = (2x+1)e^{-x} + C$. This means that $f(x) = 1 + 2x + Ce^{x}$. Using our initial condition $f(0) = 0$, this immediately yields $C = -1$, so $f(x) = 1 + 2x - e^{x}$. In particular, $f(1) = 3 - e$, so our answer is $3 - 1 = 2$.

Solution to Question 995: 77 Multiple II

Since $770000 = 77 \cdot 10000$ and $77000 = 77 \cdot 1000$ are multiples of 77, $693000 = 770000 - 77000$ is also a multiple of 77. Since $7700 = 77 \cdot 100$ is a multiple of 77, $693000 + 7700 = 700700$ is a multiple of 77. Lastly, since 693 is the largest multiple of 77 that is at most 700, $700700 - 693 = 700007$ is the smallest multiple of 77 at least 700000.

Solution to Question 996: 8 Card Heart

Let H_i be the indicator that the ith card dealt is a heart. Then $T = H_1 + \cdots + H_8$ is the total number of hearts in the dealing of 8 cards. Using the variance of a sum formula,

$$\mathrm{Var}(T) = \mathrm{Var}(H_1 + \cdots + H_8) = \sum_{i=1}^{8} \mathrm{Var}(H_i) + \sum_{i \neq j} \mathrm{Cov}(H_i, H_j)$$

We know that $\mathrm{Var}(H_i) = \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}$, as it is an indicator. For $\mathrm{Cov}(H_i, H_j)$, we have to evaluate $\mathbb{E}[H_iH_j] - \mathbb{E}[H_i]\mathbb{E}[H_j]$.

We know the second term in this expression is just $\frac{1}{4^2}$ because of the know value of the expectation of the indicators. Now, $\mathbb{E}[H_iH_j] = \mathbb{E}[H_1H_2]$ by the exchangeability of the draws. This expectation is just the indicator that the first two cards drawn are hearts. This probability is $\frac{1}{4} \cdot \frac{12}{51}$, as once we get our first hearts, there are 12 cards of 51 left, so $\mathbb{E}[H_iH_j] = \frac{1}{4} \cdot \frac{12}{51}$. Therefore,

$$\mathrm{Cov}(H_i, H_j) = \frac{1}{4}\left(\frac{12}{51} - \frac{1}{4}\right) = -\frac{1}{272}$$

Plugging in, we have that $\mathrm{Var}(T) = 8\mathrm{Var}(H_i) + 8(7)\mathrm{Cov}(H_1, H_2) = 8 \cdot \frac{3}{16} - \frac{56}{272} = \frac{22}{17}$

Solution to Question 997: Circular Charlie

The 14 boys make 14 adjacencies for the girls to fill in. There are $\binom{14}{10}$ ways for the girls to arrange since each girl must be between two boys. We know that Charlie must be between two boys, so the two spots adjacent to him must be empty. Therefore, for our event to occur, the girls must arrange themselves into the 12 spots not adjacent to Charlie, so there are $\binom{12}{10}$ ways to do that. Therefore, our probability of interest is $\frac{\binom{12}{10}}{\binom{14}{10}} = \frac{6}{91}$.

Solution to Question 998: Integral Variance II

We first attempt to evaluate $\int_0^t W_s dW_s$. From basic calculus, we know that $\int x dx = \frac{x^2}{2}$, so we first attempt to evaluate $d\left(\frac{W_t^2}{2}\right)$ as a starting point.

Let $f(x) = \frac{x^2}{2}$. We know that $f'(x) = x$ and $f''(x) = 1$. By Ito's Formula, we have that

$$d\left(\frac{W_t^2}{2}\right) = df(W_t) = W_t dW_t + \frac{1}{2}d[W,W]_t = W_t dW_t + \frac{1}{2}dt$$

This implies to us that $\frac{W_t^2 - t}{2}$ is the function we are looking to take the derivative of, as that would produce $W_t dW_t$ on the RHS when we take the derivative. Integrating both sides, this yields that

$$\int_0^t W_s dW_s = \frac{W_t^2 - t}{2} + C$$

As $W_0 = 0$, we get that $C = 0$ as well. Therefore, we really want to evaluate $\text{Var}\left(\frac{W_t^2 - t}{2}\right) = \frac{1}{4}\text{Var}(W_t^2)$ by properties of variance.

We use the fact here that for $X \sim N(0,1)$, $\mathbb{E}[X^4] = 3$. This can be proven with MGFs or even Brownian Motion itself! Since $W_t \sim N(0,t)$, $W_t = \sqrt{t}X$. Using this,

$$\text{Var}(W_t^2) = \mathbb{E}[W_t^4] - (\mathbb{E}[W_t^2])^2 = t^2\text{E}[X^4] - t^2 = 2t^2$$

Since we have the constant of $1/4$ out front, this means that $k = 2 \cdot 1/4 = 1/2$.

Solution to Question 999: Double Evens

The expected value of a roll of this die is $\frac{1+4+3+8+5+12}{6} = 5.5$. Therefore, we should re-roll this die if we receive anything below 5.5 in value. In other words, we keep only a 4 or 6 appearing. With probability $\frac{1}{3}$, we don't re-roll and our expected value would be $\frac{8+12}{2} = 10$ given that we don't re-roll under this strategy. Otherwise, with probability $\frac{2}{3}$ we re-roll, in which our expected value is $\frac{11}{2}$. Therefore, the expected payoff of the game is $\frac{2}{3} \cdot \frac{11}{2} + \frac{1}{3} \cdot 10 = 7$.

Solution to Question 1000: Unit Distance

Let $u = x - y$. Then we know that $||u||^2 = u^T u$. Therefore,

$$||x - y||^2 = (x - y)^T (x - y) = x^T x - x^T y - y^T x + y^T y$$

As x and y are vectors, $x^T y$ and $y^T x$ are both just $x \cdot y$, which is 0 in this case due to the orthogonality of x and y. Therefore, $||x - y||^2 = ||x||^2 + ||y||^2 = 2$, as x and y are both unit length. This means that $||x - y|| = sqrt2$, so $a = 2$.

Solution to Question 1001: Empty Urn

Let E_i be the event that bin i is empty. We want $\mathbb{P}[E_1^c \cap E_2^c \cap E_3^c] = 1 - \mathbb{P}[E_1 \cup E_2 \cup E_3]$. We compute $\mathbb{P}[E_1 \cup E_2 \cup E_3]$ by first using the Law of Total Expectation, as we do not know the number of balls k that we are distributing. Thus, $\mathbb{P}[E_1 \cup E_2 \cup E_3] = \sum_{k=1}^{\infty} \mathbb{P}[E_1 \cup E_2 \cup E_3 \mid H_k]\mathbb{P}[H_k]$. H_k here is the event that the first head occurs on the kth flip. From our work previously, $\mathbb{P}[H_k] = \frac{1}{2^k}$, so:

$$\mathbb{P}[E_1 \cup E_2 \cup E_3 \mid H_k] = \binom{3}{1}\mathbb{P}[E_1 \mid H_k] - \binom{3}{2}\binom{3}{2}\mathbb{P}[E_1 E_2 \mid H_k] + \binom{3}{3}\mathbb{P}[E_1 E_2 E_3 \mid H_k]$$

The last probability is 0 as it is impossible for all 3 bins to be empty. $\mathbb{P}[E_1 E_2 \mid H_k] = \left(\frac{1}{3}\right)^k$, as we need all balls to go into bin 3. $\mathbb{P}[E_1 \mid H_k] = \left(\frac{2}{3}\right)^k$, as we need all the balls to go into bins 2 or 3, which occurs with probability $\frac{2}{3}$ per ball. Hence, $\mathbb{P}[E_1 E_2 E_3 \mid H_k] = 3\left(\frac{2}{3}\right)^k - 3\left(\frac{1}{3}\right)^k$, and:

$$\begin{aligned}\mathbb{P}[E_1 \cup E_2 \cup E_3] &= \sum_{k=1}^{\infty} \left[3\left(\frac{2}{3}\right)^k - 3\left(\frac{1}{3}\right)^k\right]\left(\frac{1}{2}\right)^k \\ &= 3\sum_{k=1}^{\infty}\frac{1}{3^k} - 3\sum_{k=1}^{\infty}\frac{1}{6^k} \\ &= 3\left(\frac{\frac{1}{3}}{1-\frac{1}{3}} - \frac{\frac{1}{6}}{1-\frac{1}{6}}\right) \\ &= \frac{9}{10}\end{aligned}$$

Finally, we have that $\mathbb{P}[E_1^c E_2^c E_3^c] = 1 - \frac{9}{10} = \frac{1}{10}$.

Solution to Question 1002: Big Mac

Converting to cents, we know that $89 \cdot 20 = (89 \cdot 2) \cdot 10 = 1780$, so $1780 + 89 \cdot 2 = 1958 = 22 \cdot 89$ is the largest amount you can spend without going on \$20. Thus, $x = 22$ and $y = 2000 - 1958 = 42$, so $xy = 22 \cdot 42 = (32 + 10)(32 - 10) = 32^2 - 10^2 = 924$.

Solution to Question 1003: High-Low

Let x be the value of our first roll. Our optimal strategy is going to be that we say the second roll is lower if $x \geq t$ for some threshold t and higher if $x < t$. Let t be an integer. We have two strategies corresponding to each t, which are betting higher and lower. Let's compute the expected returns for each of the two strategies for each t.

Let $R_{l,t}$ denote the returns of the strategy where we guess lower for the second roll with first roll being t and Y be the value of the second roll. Then $\mathbb{E}[R_{l,t}] = \mathbb{E}[R_{l,t} \mid Y < t]\mathbb{P}[Y < t] + \mathbb{E}[R_{l,t} \mid Y \geq t]\mathbb{P}[Y \geq t]$ by Law of Total Expectation. The second term evaluates to 0, since we receive no payout if we guess incorrectly. Given $Y < t$, we know that the value is discrete uniform on $\{1, 2, \ldots, t-1\}$, which has a mean of $\frac{t}{2}$. The probability of observing a value that is strictly less than t is $\frac{t-1}{100}$, as there are $t-1$ integers in $\{1, 2, \ldots, t-1\}$. Therefore, $\mathbb{E}[R_{l,t}] = \frac{1}{100} \cdot \frac{t(t-1)}{2}$.

Similarly, let $R_{h,t}$ denote the returns of the strategy where we guess higher for the second roll with the first roll being t and Y be the value of the second roll. Then $\mathbb{E}[R_{h,t}] = \mathbb{E}[R_{h,t} \mid Y < t]\mathbb{P}[Y < t] + \mathbb{E}[R_{h,t} \mid Y \geq t]\mathbb{P}[Y \geq t]$ by Law of Total Expectation. The first term evaluates to 0, since we receive no payout if we guess incorrectly. Given $Y \geq t$, we know that the value is discrete uniform on $\{t, t+1, \ldots, 100\}$, which has mean $\dfrac{100+t}{2}$. The probability of rolling a value at least t is $\dfrac{100-(t-1)}{100}$, as it is the complement of the previous probability. Therefore, the expected payout in this case is $\dfrac{1}{100} \cdot \dfrac{(100+t)(101-t)}{2}$

We now want to find the point of indifference i.e. when are the expected returns between guessing higher and lower equal. Note that $\mathbb{E}[R_{l,t}]$ is increasing in t while $\mathbb{E}[R_{h,t}]$ is decreasing in t. This means that for whatever t^* these two functions are equal at, we will guess lower for any value $> t^*$ and higher for any value $\leq t^*$ on the first roll. After cancelling denominators, we must solve $(100+t)(101-t) = t(t-1)$. This can be rearranged to $-2t^2 + 2t + 10100 = 0$, at which

$$t^* = \frac{-2 \pm \sqrt{4 - 4(-2)(10100)}}{2(-2)} = \frac{-2 \pm \sqrt{80804}}{-4}$$

Clearly, we are going need to subtract the square root, as otherwise t^* will be negative. Once we do this, we see our threshold is approximately 71.57. Therefore, if our first roll is ≥ 72, we will guess lower for the second roll, meaning 72 is the answer.

Solution to Question 1004: Min Card

Removing the minimum only tells us information about the values of the other 3 cards in the subset, not about the overall deck of cards. This is since John is only comparing the minimum in the subset he selected. Therefore, the answer is just the same as if he selected from an unaltered deck, which would have expected value 7.

Solution to Question 1005: Fibonacci Ratio

Consider $F_0 + F_1 + F_2 + \cdots + F_{300}$. Note that by the definition of Fibonacci Numbers,

$$F_1 + F_2 = F_3, F_4 + F_5 = F_6, \ldots, F_{298} + F_{299} = F_{300}$$

Therefore, we can rewrite the numerator as $(F_3+F_3)+(F_6+F_6)+\cdots+(F_{300}+F_{300}) = 2(F_3 + F_6 + F_9 + \cdots + F_{300})$. This means our answer is 2.

Solution to Question 1006: Wire Connection

There are 10 ends on the left side of the wires and 10 ends on the right side of the wires. Without loss of generality, we connect the right end of the first wire to the right end of the second wire. In order to retain the possibility of forming a 10 mile loop, there are 8 possible left wire ends with which the left end of the second wire can join with, out of a possible 9. Without loss of generality, we connect the left end of the second wire to the left end of the third wire. The right end of the third wire can then be joined with any other remaining wire's right end. Back to the left end. There are 7 remaining ends on the left side. Of these, 6 are viable connections. Repeating this process, we get a final answer of

$$\frac{8}{9} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{128}{315}$$

Solution to Question 1007: Projection Matrix Ranked

Interpreting the projection matrix geometrically, we know that for a given vector $x \in \mathbb{R}^n$, $P_v x$ is going to result in the component of x that is parallel to v. In other words, $P_v x \in \text{Sp}\{v\}$. This is because

$$P_v x = \frac{1}{||v||^2} v v^T x = \frac{1}{||v||^2} v(v^T x) = \frac{(v \cdot x)}{||v||^2} v = cv$$

We make the substitution $v^T x = v \cdot x$ midway. We can exchange the order since the dot product results in a scalar. Since we get a linear scaling of v, we can conclude that $\text{rank}(P_v) = 1$, as it is the span of 1 vector. By the rank-nullity theorem, we get that $\text{null}(P_v) = n - 1$, as $\text{rank}(P_v) + \text{null}(P_v) = n$. Therefore, our solution is $(n-1)^2 + 1^2 = n^2 - 2n + 2$. This means the answer to the question at hand is $1 - 2 + 2 = 1$.

Solution to Question 1008: Coin Flipper Ruin

Either Jason or Vishnu will run out of money first. Let p denote the probability that Jason runs out of money first. Naturally, $1 - p$ is Vishu's probability of running out of money first, leaving Jason with 1500. After n coin tosses, Jason should expect to neither lose nor gain any money, since the coin is fair; in other words, Jason should expect to have \$500 in his account after n tosses. By the end of the gameâlet's say, after the n_{end}-th tossâthe expected value of Jason's account is $p \cdot 0 + (1 - p) \cdot 1500$. This value must be equal to 500, as discussed previously. Solving for p, we find

$$\begin{aligned} p \cdot 0 + (1-p) \cdot 1500 &= 500 \\ 1500p &= 1000 \\ p &= \frac{2}{3} \end{aligned}$$

Solution to Question 1009: Diverse Distributions

We have that $\mathbb{E}[X] = \frac{1}{\lambda}$ for an $\text{Exp}(\lambda)$ RV and $\mathbb{E}[Y] = \frac{\lambda}{2}$ for the $\text{Unif}(0, \lambda)$ RV. Thus, we need to solve $\frac{1}{\lambda} = \frac{\lambda}{2}$, which is just $\lambda^2 = 2$, or that $\lambda = \sqrt{2}$, since $\lambda > 0$.

Solution to Question 1010: Sum1

Suppose that $xy = x + y$. Then $x(y-1) = y$, so $x = \frac{y}{y-1}$. The only time both x and y are integers is $x = y = 2$. Therefore, we will need to extend to real numbers. The idea here is that to get a decimal expansion only consisting of 1s, we should get something with only ones in the numerator and the denominator will be a power of 10 so that the numerator's decimal expansion is not affected. Therefore, this leads us to consider $y = 11$. This makes $x = \frac{11}{10} = 1.1$, which consists of only ones. Then, we note that $11 \cdot 1.1 = 11 + 1.1 = 12.1$, so 12.1 is our answer.

Solution to Question 1011: Circular Partition

Fix X_1 as the "smallest" point in the sense of smallest angle CCW from the origin. This doesn't matter since all of them are independent and uniform on the perimeter. If X_2 is the smallest (in terms of CCW angle from $(1,0)$) among X_2, X_3, and X_4, then we get 3 distinct regions, as the chords do not intersect. If X_2 is the median of the remaining 3, then we get 4 regions, as there is an intersection. If X_2 is the largest among the remaining 3, there is no intersection of chords, so we have 3 regions. As all orderings occur with equal probability, there are 3 regions with probability $\frac{2}{3}$ and 4 regions with probability $\frac{1}{3}$, so the expected number is

$$3 \cdot \frac{2}{3} + 4 \cdot \frac{1}{3} = \frac{10}{3}$$

Solution to Question 1012: Life Support

Here we can set up a basic system of equations to solve. First, we want to find out how many units (in this case time windows of 54 years) will it take, for the support system capacity and population aboard the space station to be equal. In our equations, X will be units of time, and Y will be total years.

Thus our first equation will solve how many units until these populations are equal: $2 \cdot 4^X = 16384 \cdot 2^X$

And our second will convert these units into the appropriate amount of years: $Y = 54X$

Solving for X in the first equation, we get $X = 13$, and then plugging this answer into our second equation, we get $Y = 702$, yielding 702 years as our answer.

Solution to Question 1013: Wedding Handshakes

There are a couple ways to do this question. The fastest method is to realize that $\binom{100}{2}$ perfectly models this situation as we see how many different pairs of people can be chosen from the 100. This results in the answer of 4950. The other method is to start with a smaller scenario. Start with 1 person. Every time another person joins the wedding, they have to shake hands with the current number of guests. This means the total number of handshakes is the integer sums from 1 to 99. There are 49 sets of 100 within this sum and an extra 50 which results in a total of 4950.

Solution to Question 1014: Couple Pairs

Label the couples $1-3$, and let C_i represent the event that couple i is paired up together. We want $\mathbb{P}[C_1 \cup C_2 \cup C_3]$ By the inclusion-exclusion formula and the fact that the couples are exchangeable,

$$\mathbb{P}[C_1 \cup C_2 \cup C_3] = 3\mathbb{P}[C_1] - 3\mathbb{P}[C_1 \cap C_2] + \mathbb{P}[C_1 \cap C_2 \cap C_3]$$

$\mathbb{P}[C_1]$ is just the probability couple 1 is paired up. Fix one person in couple 1. Then exactly 1 of the 5 remaining people is that person's actual partner, so this probability is just $\frac{1}{5}$. Note that $\mathbb{P}[C_1 \cap C_2] = \mathbb{P}[C_1 \cap C_2 \cap C_3]$, as if two of the couples are matched together, the third also will be by default. Therefore, we can compute $\mathbb{P}[C_1 \cap C_2] = \mathbb{P}[C_2 \mid C_1]\mathbb{P}[C_1]$. We have $\mathbb{P}[C_1]$ from before. Then, given the first couple is matched, fix any other person. Then 1 of the

3 remaining people will be the partner of that person so $\mathbb{P}[C_2 \mid C_1] = \frac{1}{3}$, so $\mathbb{P}[C_1 \cap C_2] = \frac{1}{15}$. Combining all of this into the our initial inclusion-exclusion, we have that

$$\mathbb{P}[C_1 \cup C_2 \cup C_3] = 3 \cdot \frac{3}{15} - 3 \cdot \frac{1}{15} + \frac{1}{15} = \frac{7}{15}$$

Solution to Question 1015: Divisible Dice

Let the sum of the first 9 dice be x. Then with equal probability, the sum of all the dice is $x + 1, \ldots, x + 6$. Exactly one of these 6 values is divisible by 6, so our answer is $\frac{1}{6}$.

Solution to Question 1016: Idempotent Eigenvalues

Let A be an idempotent $n \times n$ matrix. Then the eigenvalues λ of A satisfy $Ax = \lambda x$ for $x \neq 0 \in \mathbb{R}^n$. Multiplying by A on both sides, we get that $A^2 x = \lambda(Ax)$. However, we know that $Ax = \lambda x$ and $A^2 = A$ from above, so we put that in on the RHS to get $Ax = \lambda(\lambda x)$. However, we also know that the LHS is now λx from the definition of eigenvalue, so our equation becomes $\lambda x = \lambda^2 x$. Equivalently, this means that $\lambda(\lambda - 1)x = 0$. Since $x \neq 0$, this means that $\lambda = 0, 1$. This means that 1 is the only possible eigenvalue that is non-zero.

Additionally, since A is said to be non-defective here, we know that sum of the algebraic and geometric multiplicities of A are the same. The sum of the algebraic multiplicities is 10 here, as A is 10×10. $\mathcal{N}(A)$ refers to the eigenspace generated by $\lambda = 0$, so the geometric multiplicity of $\lambda = 0$ is 4 by the question. This means that $\text{rank}(A) = \dim(E_{\lambda=1}) = 6$ by rank-nullity theorem, so $r = 6$. Our answer is therefore $1^2 + 6^2 = 37$.

Solution to Question 1017: King Activity

Let's compute $\mathbb{P}[X = k]$ for each k. This means that there are exactly 4 kings in the first $k - 1$ cards, and the 5th king comes on the kth card. There are $\binom{k-1}{4}$ ways to pick the spots of the 4 kings in the first $k - 1$ spots. Then, there are $P(40, 5)$ ways to permute the kings to the 4 spots selected in the first $k - 1$ spots and the kth spot. Then, there are $P(480, k - 5)$ ways to pick $k - 5$ of the other 480 cards that are not kings to go in the other $k - 5$ spots before k that are not kings. We choose specifically non-kings so that we don't have another king before the kth spot (and hence the 5th king would not show up at the kth spot). There are obviously $P(520, k)$ ways to permute k cards of the first 520 to the first k spots, so

$$\mathbb{P}[X = k] = \frac{\binom{k-1}{m} P(40, 5) P(480, k-5)}{P(520, k)}$$

The support of this is $k = 5, 6, \ldots, 485$. We can't have the 5th king show up before the 5th spot, and it can't be after 485 because that would correspond to all of the last 40 cards in the deck being kings. Now, we just need to find the smallest integer k so that $\frac{\mathbb{P}[X = k+1]}{\mathbb{P}[X = k]} \leq 1$. The smallest integer above k is the one we would want to select as our position. Plugging in, we get

$$\frac{\binom{k}{m} P(40, 5) P(480, k-4)/P(520, k+1)}{\binom{k-1}{m} P(40, 5) P(480, k-5)/P(520, k)} \leq 1$$

Simplifying this, we get the inequality $(k-4)(520-k) \geq k(485-k)$. Finishing the algebra, $520k - k^2 - 2080 + 4k \geq 485k - k^2$, so $k \geq \frac{160}{3} \approx 53.3$. Therefore, $k = 54$ is the position we would want to select. This is because the ratio above is still > 1 at $k = 53$.

Solution to Question 1018: Die Roll LCM

Notice that the least common multiple (LCM) of the integers from 1 to 10 is 2520. It is therefore not possible to end up with any other LCM, as dividing 2520 by any prime factor would yield a number smaller than 2000. With some work, we see that we must obtain the numbers $7, 8, 9$, and at least one of 5 or 10. Although we cannot directly apply the expected value expression for geometric random variable, we can consider two cases:

Case 1: We get $7, 8$, and 9 before 5 or 10. Consider the sequence in which we encounter each of these values for the first time. Any permutation of this sequence is equally likely, so $7, 8$, and 9 are before 5 or 10 when the latter two are the final two values in the sequence, which occurs with probability $\frac{1}{\binom{5}{2}} = \frac{1}{10}$.

Case 2: The complement of case 1. Clearly, the probability of this case occurring is $\frac{9}{10}$.

Now we may consider the expected number of rolls before obtaining $7, 8$, and 9. $\frac{9}{10}$ of the time, a multiple of 5 has already been obtained by then, and we are done. The other $\frac{1}{10}$ of the time, we must continue rolling until we obtain a multiple of 5. Evaluating this with standard geometric expectation, we have

$$\frac{10}{3} + \frac{10}{2} + \frac{10}{1} + \frac{1}{10} \cdot \frac{10}{2} = \frac{113}{6}$$

Our desired answer is $113 + 6 = 119$. Note that even if we didn't realize we could split the cases cleanly into whether we roll a multiple of 5 before or after we roll $7, 8,$ and 9, casework on the order in which we get the multiple of 5 yields the same result, albeit with more computation.

Solution to Question 1019: Coin Streak

This is a gambler's ruin problem in disguise. If you obtain \$1 for each head and lose \$1 for each tail that appears, this is asking the expected time for you to either have \$10 or $-$\$7. The expected hitting time of the set $\{-a, b\}$ for $a, b > 0$ is a commonly known to be $-ab$. In this case, $a = 7$ and $b = 10$, so our expected hitting time is 70.

Solution to Question 1020: Expected Returns

We are going to use one important fact about simple symmetric random walks here repeatedly: If a random walk starts at integer position a and $0 < a < b$, b being an integer, then the probability that the random walk hits b before 0 is $\frac{a}{b}$. This can be proved using martingale theory.

With this fact equipped, if N is the number of visits to 1000 before 0, we are going to calculate $\mathbb{P}_5[N = k]$, where $\mathbb{P}_a$ denotes that we are calculating starting at position a. First, we can see by the above fact that $\mathbb{P}_5[N = 0] = \frac{199}{200}$. This is because the probability we reach 1000 before 0 is $\frac{5}{1000} = \frac{1}{200}$, so the probability we reach 0 before 1000 is the complement of this.

Now, to calculate $\mathbb{P}_5[N = 1]$, we must reach 1000 and then go back to 0. The probability of reaching 1000 before 0 is $\frac{1}{200}$. Then, once we are at position 1000, we must return down to 0 and not visit back. This means that we must go down from 1000 to 999 the moment we hit position 1000, occurring with probability $\frac{1}{2}$. Then, afterwards, starting from position 999, we must hit 0 before 1000. This is just $\frac{1}{1000}$ by the same complementation idea above, so our total probability is $\frac{1}{200} \cdot \frac{1}{2} \cdot \frac{1}{1000} = \frac{1}{200} \cdot \frac{1}{2000}$.

To calculate $\mathbb{P}_5[N = 2]$, we must reach 1000, return to 1000 one time, and then go back to 0. The probability of reaching 1000 before 0 starting from position 5 is $\frac{1}{200}$. Then, to return to 1000 once, we condition on the first step once we are at 1000. If we go up, which occurs with probability $\frac{1}{2}$, then we will reach 1000 before going to 0 with probability 1. If we go down, which occurs with probability $\frac{1}{2}$, we will return to 1000 from 999 with probability $\frac{999}{1000}$. Therefore, the total probability of returning to 1000 once is $\frac{\frac{999}{1000}+1}{2} = \frac{1999}{2000}$. After we return once to 1000, from there, we need to go back to 0 before returning again. This probability is $\frac{1}{2000}$ from before. Therefore, $\mathbb{P}_5[N = 1] = \frac{1}{200} \cdot \frac{1999}{2000} \cdot \frac{1}{2000}$.

With all of this equipped, we will now calculate $\mathbb{P}_5[N = k]$. This means that we first need to hit 1000 once, which occurs with probability $\frac{1}{200}$. From there, we must return to 1000 exactly $k - 1$ times before hitting 0. This occurs with probability $\left(\frac{1999}{2000}\right)^{k-1}$, as it was $\frac{1999}{2000}$ for returning exactly once, so every time we hit 1000, by the Strong Markov Property, we can view the future independent of the past, and we do this $k-1$ times. Lastly, after this, we need to hit 0 before 1000, which occurs with probability $\frac{1}{2000}$. Therefore, for integers $k > 0$,

$$\mathbb{P}_5[N = k] = \frac{1}{200} \cdot \left(\frac{1999}{2000}\right)^{k-1} \cdot \frac{1}{2000}$$

In other words, we can say that $N = 0$ with probability $\frac{199}{2000}$ and $N \sim \text{Geom}\left(\frac{1}{2000}\right)$ with probability $\frac{1}{200}$.

Therefore

$$\mathbb{E}_5[N] = \mathbb{E}_5[N \mid N = 0]\mathbb{P}_5[N = 0] + \mathbb{E}_5[N \mid N > 0]\mathbb{P}_5[N > 0]$$

The first term is clearly just 0 due to the expected value part. $\mathbb{E}_5[N \mid N > 0] = 2000$ as that is the mean of a Geom $\left(\frac{1}{2000}\right)$ random variable. Lastly, we know the last part is just $\frac{1}{200}$ from our prior calculations, so $\mathbb{E}_5[N] = 10$.

Solution to Question 1021: Rolls in a Row II

We can use Markov Chains for this problem. Let E_0 be the expectation state of not having the starting 5. Let E_1 be the expectation state of having the 5). Finally, let E_2 be the expectation state of having 5 and 6 in a row (our goal). Then our equations become:

$$E_0 = \frac{1}{6}(E_1 + 1) + \frac{5}{6}(E_0 + 1) = \frac{1}{6}E_1 + \frac{5}{6}E_0 + 1$$

$$E_1 = \frac{1}{6}(E_2 + 1) + \frac{1}{6}(E_1 + 1) + \frac{2}{3}(E_0 + 1) = \frac{1}{6}E_2 + \frac{5}{6}E_0 + 1$$

$$E_2 = 0$$

Solving these equations, we get $E_0 = 36$. Thus it takes 36 rolls on average to roll a 5 and 6 in a row.

Solution to Question 1022: Fish Slice

Let X be the random variable representing the location of the first cutting point and Y be the random variable representing the location of the second cutting point. Since X and Y are equally likely to be on either side of the midpoint due to symmetry, we are going to consider the case of X being left of the midpoint and Y to the right. Since X is to the left, $X \sim \text{Unif}[0, \frac{L}{2}]$. Similarly, $Y \sim \text{Unif}[\frac{L}{2}, L]$. We want to find $\mathbb{P}[|X - Y| \geq \frac{L}{3}]$. Since the joint PDF is uniform over the indicator region, we can just take ratios of areas to find the probability. If you are to sketch out the region of interest and then the region in which $|X - Y| \geq \dfrac{L}{3}$, you will receive a region whose complement is a right triangle whose sides are of length $L/3$. Therefore, the ratio of the areas is $\frac{\frac{L^2}{18}}{\frac{L^2}{4}} = \dfrac{2}{9}$, so the probability of the complement is $2/9$, meaning our probability of interest is $7/9$.

Solution to Question 1023: 2 Lead

We can view this is a standard Gambler's Ruin problem that is non-symmetric. In this case, $p = 0.3, q = 1 - p = 0.7$, and $n = 2$. The formula for the probability that Alice wins is commonly known to be

$$\frac{1 - \left(\frac{q}{p}\right)^n}{1 - \left(\frac{q}{p}\right)^{2n}}$$

We know that $\frac{q}{p} = \frac{7}{3}$, so plugging this all in, we have that our answer is

$$\frac{1 - \frac{49}{9}}{1 - \frac{2401}{81}} = \frac{81 - 441}{81 - 2401} = \frac{9}{58}$$

Another way to see this without using the machinery above is to note that we just need to see a sequence of two consecutive wins from a player to win, and this occurs with probability $\left(\frac{3}{10}\right)^2 = \frac{9}{100}$ for Alice and $\left(\frac{7}{10}\right)^2 = \frac{49}{100}$ for Bob. As these are disjoint events in each set of two turns, the probability it occurs for Alice first is $\frac{9/100}{9/100 + 49/100} = \frac{9}{58}$.

Solution to Question 1024: 12-8 Showoff

The expected value of the 12−sided die is $\frac{1+12}{2} = \frac{13}{2}$. Therefore, we would want to keep our original roll if it is a 7 or 8. Otherwise, we will roll the 12−sided die. Therefore, our expected value is

$$\frac{7+8}{8} + \frac{3}{4} \cdot \frac{13}{2} = \frac{27}{4}$$

as we will roll the 12−sided die with probability $\frac{3}{4}$.

Solution to Question 1025: View All Sides

Let T_i be the number of rolls needed to observe the ith distinct side given that $(i-1)$ distinct sides have already appeared. Then $T = T_1 + \cdots + T_6$ gives the total number of rolls needed to observe all of the sides of the die. We know that $T_1 = 1$, as the first roll always yields a side not seen yet. We also have that $T_2 \sim \text{Geom}(5/6)$, as 5 of the 6 remaining sides are not observed yet. More generally, $T_i \sim \text{Geom}\left(\frac{7-i}{6}\right)$, as $7-i$ of the sides have not been observed yet. Thus, $\mathbb{E}[T_i] = \frac{6}{7-i}$. By linearity of expectation, we get that

$$\mathbb{E}[T] = \sum_{i=1}^{6} \frac{6}{7-i} = 6\sum_{i=1}^{6} \frac{1}{i} = 14.7$$

The equality between sums occurs by re-indexing the sum

Solution to Question 1026: Sine Condition

We will offer an informal justification. Recall the definition of conditional probability, where A is an event:

$$\mathbb{E}[X|A] = \int_x \mathbb{P}[X = x \mid A] \, dx.$$

Let's say that you are told that $\sin(X) = 1/2$. What can we say about X? We might be able to use arcsin here.

$$X = \arcsin\left(\frac{1}{2}\right).$$

However, we also know that there could be multiple possible values of X within the interval $[0, \pi]$ such that $\sin(X) = 1/2$. Looking at a plot of $\sin(X)$, we notice that $\sin(X) = 1/2$ when $X = \arcsin(1/2)$ and when $X = \pi - \arcsin(1/2)$. Replacing $1/2$ with some general value $a \in [0, \pi]$, we find that there are two values for X such that $\sin(X) = a$: $\arcsin(a)$ and $\pi - \arcsin(a)$. The only exception to this observation is when $\sin(X) = 1$. Then, $X = \pi/2$. So, given $\sin(X) = a$ where $a \in [0, 1)$, $\mathbb{E}[X| \sin(X) = a] = (\arcsin(a) + \pi - \arcsin(a))/2 = \pi/2$. And, of course, in the case that we're given $\sin(X) = 1$, we have $\mathbb{E}[X| \sin(X) = 1] = \pi/2$. Therefore, $\mathbb{E}[X| \sin(X)] = \pi/2$, and $\cos(\frac{\pi}{2}) = 0$.

Solution to Question 1027: Cats and Mice

We need to find 2 numbers such that $ab = 999919$ and $b > a$. It turns out that 999919 is a semiprime whose proper divisors are 991 and 1009

Solution to Question 1028: Random Tic Tac Toe

There are 8 ways to forma a tic-tac-toe (3 vertically, 3 horizontally, 2 diagonally). Including the permutations, we multiply by 3!. Thus the total ways is $8 * 3! = 48$. Now we have to see how many different ways the pebbles can land on the board.

The first scenario is that they all land in distinct squares. Since there are 9 total squares and we pick 3 to be where the pebbles land, the total combinations are $\binom{9}{3} = 84$. Each of these combinations has 3! permutations in their order. Thus $84 * 3! = 504$.

Now we have to calculate the number of ways the pebbles could land in two distinct squares. Similar to the three square scenario, we choose two squares from the nine. Including permutations, the total number of arrangements with two squares is $\binom{9}{2} * 3! = 216$.

Finally, all three pebbles can land in the same square which adds an extra 9 ways. Thus in total there are $504 + 216 + 9 = 729$ ways. Thus the probability is

$$\frac{48}{729} = \frac{16}{243}$$

Alternatively, a faster way to do this is to consider that there are $9*9*9 = 729$ total possibilities how each pebble lands. Since there are 8 winning patterns and each has 3! permutations, there are $\frac{8 * 3!}{729} = \frac{48}{729} = \frac{16}{243}$.

Solution to Question 1029: Likely Targets II

The key here is to note that since the targets are of extremely small radius, we can essentially treat them as points. The approximate probability we hit target A would be approximately $f(x_A)\varepsilon$, where $f(x_A)$ is the probability density at point A. However, since B is also of small radius 2ε, the probability we hit target B is approximately $2f(x_B)\varepsilon$. Since the two targets are disjoint, our goal is to maximize the weighted sum of the probability densities. Our probability density here is dependent on what μ we select. Therefore, as a function of μ, we need to maximize

$$f(\mu) = \frac{1}{2\sqrt{2\pi}} \left(e^{-\frac{(-1-\mu)^2}{8}} + 2e^{-\frac{(3-\mu)^2}{8}} \right)$$

The two interior terms are just the density of a $N(\mu, 4)$ distribution at -1 and 3. To do this, we take the derivative and set it equal to 0. In particular,

$$f'(\mu) = \frac{1}{2\sqrt{2\pi}} \left[\frac{-1-\mu}{4} e^{-\frac{(-1-\mu)^2}{8}} + \frac{3-\mu}{2} e^{-\frac{(3-\mu)^2}{8}} \right] = 0$$

Using a computer system, $f'(\mu) = 0$ for $\mu \approx 2.649$. One can verify that this is indeed a maxima, so 2.65 is our answer. This intuitively makes sense, as the increased size of the region around 3 means that we should assign more density there. Thus, our answer should be closer to 3 than it is to -1.

Solution to Question 1030: Queen First

Suppose that we are given that all the queens and all the kings appear in the $x_1, x_2, \ldots, x_8$-th spots, where $1 \leq x_1 < x_2 < \cdots < x_8 \leq 52$. Each possible ordering of the queens and kings among the 8 spots occurs with the same probability. Hence, we only need to consider possible orderings of queens and kings between 8 arbitrary spots.

Since we are given that the first 2 cards are queens, and the third card is a king, our problem becomes: how many different orderings of the remaining 5 cards (3 kings and 2 queens) end with a king. This probability is simply $\frac{3}{5}$, since there is a $\frac{3}{5}$ chance that the last card is a king. Putting it all together, we find that Hannah's total expected earnings is $\frac{3}{5} \cdot 100 = 60$.

Solution to Question 1031: Competitive Sampling

To find the nash equilibrium, suppose you reroll if your number is less than p, and your opponent rerolls if theirs is less than q. To compute your chance of winning, consider the cases:
Case 1: Both players reroll with probability pq. At this point, both players are equally likely to win so the chance of winning is 0.5.
Case 2: You reroll but your opponent doesn't with probability $p(1-q)$. Conditional on your opponent not rerolling, the expected value of their number is $\dfrac{1+q}{2}$, so your chance of winning is $1-\dfrac{1+q}{2}=\dfrac{1-q}{2}$.
Case 3: You don't reroll but your opponent does. This happens with probability $q(1-p)$ and your probability of winning is $\dfrac{1+p}{2}$.
Case 4: Neither player rerolls with probability $(1-p)(1-q)$. We consider subcases where $p \le q$ and $p > q$. If $p \le q$, conditional on neither player rerolling, there is a $\dfrac{1-q}{1-p}$ chance that both players' numbers are greater than q, and there is a 0.5 chance of winning from there. Otherwise, you lose. Hence, the probability of winning is $\dfrac{1-q}{2(1-p)}$. Similarly, if $p > q$, your probability of winning is $1-\dfrac{1-p}{2(1-q)}=\dfrac{1-p-2q}{2(1-q)}$.

Combining this, the probability of winning is a piecewise function

$$\left\{ \begin{array}{ll} \frac{1}{2}\left(pq+p(1-q)^2+q(1-p^2)+(1-q)^2\right), & \text{if } p \le q \\ \frac{1}{2}\left(pq+p(1-q)^2+q(1-p^2)+(1-p)(1-p-2q)\right), & \text{if } p > q \end{array} \right\}$$

Taking the derivative with respect to p and setting them to 0, we find that the optimal response to a given l satisfies

$$\left\{ \begin{array}{ll} q+(1-q)^2-2pq=0, & \text{if } p \le q \\ 3q+(1-q)^2-2pq-2(1-p)=0, & \text{if } p > q \end{array} \right\}$$

Since the game is symmetric, $p=q$, and we may solve the resulting constraints by plugging in q, from which we obtain $p = \frac{\sqrt{5}-1}{2} \approx 0.382$. Note that we couldn't have done this from the beginning, as we need to make sure your opponent can't exploit the strategy by choosing a different value.

Solution to Question 1032: Price an Option I

To find the time-0 price, we need to create a replicating portfolio. We can see that the payoff $\min(S_T, 25)$ is the same as $S_T - \max(S_T - 25)$. This replicating portfolio is the same as being long the underlying S_T, and short the $K = 25$ strike call. So, at time-T, we have payoff $S_T - C_T(25)$. This means we have a time-0 of $S_0 - C_0(25) = 21 - 0.7 = 20.3$.

Note: This replication is that of a covered-call: an options strategy where we sell call options while owning the underlying. This gives us downside protection, but limits upside potential.

Solution to Question 1033: Horse Racing

To find the three fastest horses, all horses need to be tested. A natural first step will then be to divide the horses into five groups labeled 1-5, 6-10, 11-15, 16-20, and 21-25. After these five races, we have the order of the horses within each group. Without loss of generality, assume the order within each group follows the order of the numbers (e.g., 6 is the fastest and 10 is the slowest within the 6-10 group). Therefore, the five fastest horses are 1, 6, 11, 16, and 21. We can eliminate the last two horses from each group since they cannot possibly be within the top three. We will then race the top five horses to establish the fastest horse. Again, assume the order of this group follows the order of the numbers (e.g., 1 is the fastest and 21 is the slowest). This tells us that from the first group 1-5, only 2 and 3 are in the running for second and third; from the second ground 6-10, only 6 and 7 are in the running for second and third; and from the third group 11-15, only 11 is in the running for second or third. This is because if the fastest horse within a group ranks second or third, then only one or no other horses within that group can be in the top three, respectively. Hence, the last group will race 2, 3, 6, 7, and 11, for a total of 7 races.

Solution to Question 1034: Car and Fly I

Consider a number line where Car A starts at 0 and Car B and the fly start at 400. To return to Car B, the fly must reach Car A and then return from A to B. Starting from Car B, the fly and Car A move towards each other at a relative rate of $180 + 60 = 240$ miles per hour.

The fly and the car must travel a total of 400 miles total to reach each other, as they start 400 miles apart, so the amount of times that takes is $\frac{400}{240} = \frac{5}{3}$ hours, which is 100 minutes. When the fly and Car A meet, Car A is now at location 100 and Car B has travelled $120 \cdot \frac{5}{3} = 200$ miles, so it is at location 200. Therefore, Car A and Car B are now 100 miles apart. Car B and the fly move towards each other at the rate of $180 + 120 = 300$ miles per hour, so it will take $\frac{100}{300} = \frac{1}{3}$ hours for the fly to reach back to Car B, so this is an additional 20 minutes. This means it takes a total of 120 minutes for the fly to reach back.

Solution to Question 1035: Poker Hands III

There are a total of $\binom{52}{5}$ total hand combinations. To count the number of hands that contain two pairs, we can look at the pairs in tandem. The pairs have $\binom{13}{2}$ possible face values and $\binom{4}{2}$ possible suit values each. The last card can be any of the remaining 44 cards that have a different value than the two pairs. Thus, the probability that you have two pairs is:

$$\frac{\binom{13}{2} \times \binom{4}{2} \times \binom{4}{2} \times 44}{\binom{52}{5}} = \frac{198}{4165}$$

Solution to Question 1036: Revolver Time

There are 6 bullets fired but only 5 intervals between the firings of two consecutive bullets. Hence, there are 12 seconds between bullets 1 and 2, and 12 seconds between bullets 2 and 3, for a total of 24 seconds.

Solution to Question 1037: The Orchard Problem

Let n be the number of trees on each side of the square of already planted trees. Then, there are n^2 trees in that square.

From the Odd Number Theorem, to make a square of $(n+1)^2$ trees, we need $2n+1$ more trees. We are told that we need $146 + 31 = 177$ more trees. Then, we have that $177 = 2 * 88 + 1$, and so $n = 88$. So, the total number of trees in the new orchard is $(88 + 1)^2 = 89^2 = 7921$.

Solution to Question 1038: Exponent Reverse

Comparing e^π and π^e is equivalent to comparing $\pi \ln(e) = \pi$ and $e \ln(\pi)$. This is since $\ln(x)$ is a strictly increasing function. Consider **?** as an unknown relationship between the two quantities. Therefore,

$$\pi \ln(e) \ \textbf{?}\ e\ln(\pi) \iff \frac{\ln(e)}{e}\ \textbf{?}\ \frac{\ln(\pi)}{\pi}$$

Consider the function $f(x) = \frac{\ln(x)}{x}$. By the quotient rule, $f'(x) = \frac{1-\ln(x)}{x^2}$. We have that $f'(x) > 0$ for all $0 < x < e$ and $f'(x) < 0$ for $x > e$. Therefore, $x = e$ is the maximum of this function. Therefore, we can confirm that $\frac{\ln(e)}{e} > \frac{\ln(\pi)}{\pi}$, so $e^\pi > \pi^e$. Our answer is 2.

Solution to Question 1039: Bike Count

The key is knowing you could figure out the number of bicycles if you knew the total number of spokes. Note that the this implies that the number of spokes must equal the number of bikes, as otherwise, to get some combination with xy total spokes, you could have x bikes and y spokes or vice versa. Even if it were perfect square, it would have to be prime, as with say 225, you could 9 bikes with 25 spokes or 15 bikes with 15 spokes. Since each bike has at least 2 spokes, we don't need to worry about the case with 1 spoke/1 bike. The only prime whose square is in the range 200 to 300 is 289, so there are 17 bikes with 17 spokes each.

Solution to Question 1040: Picky Casino

Let E be the expected number of times one needs to flip the coin in order to obtain two consecutive heads, and let E_1 be the expected number of further flips one needs, given that a head was flipped, in order to obtain two consecutive heads. Let x denote the probability of flipping heads.

Then, we have the equations

$$E = xE_1 + (1-x)E + 1$$
$$E_1 = (1-x)E + 1$$

To see why, when we have either not flipped a heads or the previous flip was a tails, we have probability x of getting a heads, and we need E_1 flips to win from here, and probability $(1-x)$ of getting a tails, and we need E flips from here, as we are effectively back at the start (we add 1, which counts the impending flip).

Similarly, when we have flipped a heads, we have probability x of getting another heads, and we need 0 flips from there, and probability $(1-x)$ of getting a tails, and we need E flips from here.

It is routine to solve the system above, which yields $E = \frac{1}{x} + \frac{1}{x^2}$. We want x to be chosen such that $E = 30$, as that would guarantee that the player loses on average by definition.

Thus, we solve

$$\frac{1}{x} + \frac{1}{x^2} = 30,$$

and take the positive root for contextual reasons, which gives $x = \frac{1}{5}$

Solution to Question 1041: Chess Tournament II

Let's divide our knockout tournament into two subgroups of size 64 each. For ease in explanation, let x_1 denote the highest-rated player, let x_2 denote the second-highest rated player, and let x_3 denote the third-highest rated explanation. Suppose x_1 is in subgroup 1, without loss of generality. Then, there are 63 slots left in subgroup 1 for the remaining 127 players. The only way for x_3 and x_1 to meet in the finals is if x_1 and x_2 are in one subgroup, and x_3 is in the other subgroup. This occurs with probability

$$\frac{63 \cdot 64}{127 \cdot 126} = \frac{32}{127}$$

Solution to Question 1042: Matching Die Pair

We have two cases to consider here, which correspond to the original dice values being the same or different. The probability that the first 2 dice rolled show the same value is $\dfrac{6}{6^2} = \dfrac{1}{6}$. Given this, the probability that the the values on the second roll match the first roll is $\dfrac{1}{36}$, as both dice must show that same value. Otherwise, if they differ, which occurs with probability $\dfrac{5}{6}$, the probability that the second rolls match the values on the first rolls is $2 \cdot \dfrac{1}{36} = \dfrac{1}{18}$, as if the original die values are a and b (with $a \neq b$ since we assume here the rolls are distinct), you can either roll ab or ba on your second try. Therefore, the probability of a match is $\dfrac{1}{36} \cdot \dfrac{1}{6} + \dfrac{1}{18} \cdot \dfrac{5}{6} = \dfrac{11}{216}$.

Solution to Question 1043: Particle Reach II

We are going to solve this for more general p. Let x_1 be the probability that the particle ever reaches position 1. The key is to condition on the move at the first step. If the particle moves right, which occurs with probability p, then position 1 is reached. Otherwise, the particle reaches position -1. From -1, the particle first needs to reach 0 again, which occurs with probability x_1, and then from there, reach position 1, which occurs with probability x_1 as well. Therefore, given the particle moves left, the probability it reaches position 1 is x_1^2. This gives rise to the equation

$$x_1 = p + (1-p)x_1^2$$

We can solve for this quadratic in x_1 to get that $x_1 = 1, \dfrac{p}{1-p}$. Since $x_1 \leq 1$, we know that for $p \geq 1/2$, $x_1 = 1$, as the other root would be larger than 1. For $p < 1/2$, the random walk is biased down, so it is not probability 1 that the particle ever reaches 1. This argument can be made more rigorous using more mathematically advanced tools, but those are not of concern here. Therefore, the answer for $p < 1/2$ is $\dfrac{p}{1-p}$. Since $p = 2/3$ in this question, our answer is 1.

Solution to Question 1044: Even Steven

Let p be the probability that Steven should set his coin to. Lets find the probability that Steven's friend wins. Since this game can go on forever, we should expect to have an infinite sum. Steven's friend can win on the first flip with probability 0.4, on the third flip with probability $0.6 \cdot (1-p) \cdot 0.4$, on the fifth with probability $0.6^2 \cdot (1-p)^2 \cdot 0.4$, so on and so forth. This is because if Steven's friend is to win on the $2k+1$st flip, each of the first $2k$ flips must be tails, which occur with probability 0.6 for Steven and $1-p$ for his friend.

The probability Steven wins is only affected by even number rolls. Steven can win on the second roll with probability $0.6 \cdot p$, the fourth roll with probability $0.6^2 \cdot (1-p) \cdot (p)$, the sixth roll with probability $0.6^3 \cdot (1-p)^2 \cdot (p)$, and onwards. We can equate these two infinite sums as follows:

$$0.4 \cdot (1 + 0.6 \cdot (1-p) + 0.6^2 \cdot (1-p)^2 + \dots) = 0.6 \cdot p \cdot (1 + 0.6 \cdot (1-p) + 0.6^2 \cdot (1-p)^2 + \dots)$$

Which simplifies to:

$$p = \frac{0.4}{0.6} = \frac{2}{3}$$

. Thus Steven should set his the heads probability on his coin to $\frac{2}{3}$ to make this game fair for him and his friend.

Solution to Question 1045: The Sum Is Right

Let $\alpha = \mathbb{P}[U+V > 1]$ and $\beta = \mathbb{P}[U+V < 1]$. We know that $\alpha + \beta = 1$ since U and V are continuous so they sum to exactly 1 with probability 0.

We use the fact that if $X \sim \text{Unif}(0,1)$, $1 - X \sim \text{Unif}(0,1)$ as well. With this fact equipped, let $U' = \min\{1 - X_1, \dots, 1 - X_n\}$ and $V' = \max\{1 - X_1, \dots, 1 - X_n\}$. By our previous fact, as we are still making maxima and minima over uniform random variables, $\mathbb{P}[U' + V' > 1] = \alpha$ and $\mathbb{P}[U' + V' < 1] = \beta$. Furthermore, observe that $U' = 1 - V$, as to minimize $1 - X_i$, you have to maximize X_i and subtract that from 1. Similarly, $V' = 1 - U$. Therefore,

$$\alpha = \mathbb{P}[U' + V' > 1] = \mathbb{P}[(1 - V) + (1 - U) > 1] = \mathbb{P}[U + V < 1] = \beta$$

These together imply that $\alpha = \beta = \dfrac{1}{2}$.

Solution to Question 1046: Coin Identifier

Clearly $N = 1$ and $N = 2$ don't work, as you can't obtain \$0.60 from just one or two coins of these denominations. For $N = 3$, we can see that 2 quarters and 1 dime work. There are no other ways to obtain the sum besides this.

Now, note that \$0.40 is the smallest denomination that can be written in two different ways using the same amount of coins. Namely, we can use 4 coins to write \$0.40 as 1 quarter and 3 nickels OR 4 dimes. Therefore, we can just append 2 dimes to each of these combinations, and we get that $N = 6$ is the smallest value for which this is true.

To verify \$0.40 is the smallest value, we know that the number is at least \$0.25, as otherwise we can't utilize quarters, so test \$0.30 and \$0.35 and see that they don't work.

Solution to Question 1047: Coin Pair I

We can flip over any pair of coins after flipping. Clearly, if we obtain 4 heads from the start, we wouldn't turn over any. If we obtain no heads, we can turn over two pairs of 2 coins to get all heads. Similarly, we can do this if we obtain 2 tails. If we obtain 3 heads, we have no benefit by turning over any coins. If we obtain 1 heads, we turn over a pair of tails and end with 3 heads. The probability we end with an even number of heads from the 4 flips is $\frac{1+6+1}{16} = \frac{1}{2}$, as there are $\binom{4}{2} = 6$ ways to obtain 2 heads and 2 tails.

Therefore, the probability we end with either 3 or 4 heads from the 4 flips is $\frac{1}{2}$. The answer is just $\frac{4+3}{2} = \frac{7}{2}$.

Solution to Question 1048: Archery Accuracy

The complementary event is that she misses all three of her shots. This occurs with probability $\frac{1}{4}$ per shot, so the probability she misses all of the next three shots is $\frac{1}{4^3} = \frac{1}{64}$. Thus, the probability she makes at least one of the next three shots is $1 - \frac{1}{64} = \frac{63}{64}$.

Solution to Question 1049: Contracts and Options IV

We only exercise our option if the well is successful. We save \$50 in this case and it occurs with probability 0.3. Therefore, the answer is $50 \cdot 0.3 = 15$.

Solution to Question 1050: Uniform Order I

This question is really asking us the probability that X_1 is the largest of the 4 random variables X_1, X_2, X_{10}, and X_{15}. Since the random variables are IID, they are exchangeable. Therefore, by exchangeability, it is no more or less likely that X_1 is the largest of the 4 than any other random variable, so the 4 random variables have equal probability of being largest, implying this probability is $\frac{1}{4}$.

Solution to Question 1051: Bank Account Arbitrage

The arbitrage can be seen immediately. We borrow from the bank (1) with the lower interest rate and deposit money into the bank (2) with the higher interest rate. Our deposit will cover the interest we owe to the bank 1 while still giving us some profit.

We have the following profit:

$$e^{.04} - e^{.02} \approx 0.02$$

Solution to Question 1052: Sum Exceedance IV

We are going to solve the generalized version for $n-$sided die with a sum of at least n. Let's denote the expected number of rolls needed to obtain a sum of at least n starting from a sum of k by E_k. Clearly we have that $E_n = 0$, as we already have a sum of n. Further, we have that $E_{n-1} = 1$, as no matter what is obtained, we have a sum of at least n.

Then, we have that $E_{n-2} = 1 + \frac{1}{n}E_{n-1} = 1 + \frac{1}{n}$, as with probability $\frac{1}{n}$, we obtain the value 1 and we have a total sum of $n-1$. Similarly, $E_{n-3} = 1 + \frac{1}{n}E_{n-1} + \frac{1}{n}E_{n-2} = 1 + \frac{2}{n} + \frac{1}{n^2}$. By continuing with this pattern, one can prove by induction that

$$E_{n-k} = \sum_{j=0}^{k-1} \frac{\binom{k-1}{j}}{n^j}$$

Therefore, by the Binomial Theorem, $E_0 = E_{n-n} = \sum_{j=0}^{n-1} \binom{n-1}{j} \left(\frac{1}{n}\right)^j = \left(1 + \frac{1}{n}\right)^{n-1}$ We applied the Binomial Theorem with $x = \frac{1}{n}$ and $y = 1$. Therefore, for $n = 5$, our answer is $\frac{6^4}{5^4} = \frac{1296}{625}$.

Solution to Question 1053: Birds of a Feather

Label the spots $1-8$. There are 7 possible starting spots for the first of the two blue birds to land (spots $1-7$). Then, if the first bird lands in spot i, the other bird should land in spot $i+1$, so the other spot is fixed after. Then, there are 2 ways to order the two blue birds. Afterwards, there are 6! ways to arrange the other 6 birds on the remaining spots. There are a total of 8! ways to arrange the birds, so our answer is

$$\frac{2! \cdot 7 \cdot 6!}{8!} = \frac{1}{4}$$

Solution to Question 1054: R-Squared Range

We start with a basic fact of OLS that an additional predictive variable can never decrease R^2 (as OLS can always set the coefficient to a predictor to be

0!) Therefore, our lower bound is the max of individual R^2 values meaning min $= 0.2$. As an upper bound, we could have that $y = X_1 + X_2$ meaning max $= 1.0$. Note that this does not restrict our freedom of picking X_1 and X_2 so that their individual R^2 are 0.15 and 0.2.

Solution to Question 1055: Take And Roll II

To get a baseline to compare to, suppose we just roll and take in an alternating fashion. We will be able to perform this 50 times (as each is one action) and the expected value per roll is 10.5, so our expected payout would be $50 \cdot 10.5 = 525$ with this strategy.

Now, let's write the expected payoff as a function of n. If we accept any value at least n, then the expected value we roll given we accept is $\frac{20+n}{2}$. There are $21 - n$ values that are at least value n, so the probability on each roll that we obtain a value at least n is $\frac{21-n}{20}$. As this probability is constant between rolls, the expected number of terms of obtain a value at least n is $\frac{20}{21-n}$. However, we now must claim it after we obtain a roll satisfying this threshold, so the expected number of turns needed to roll and claim the money is $\frac{20}{21-n} + 1$. Therefore, on average, we are able to roll and claim the money $\frac{100}{\frac{20}{21-n}+1}$ times in the game, as it takes us that many turns on average to roll and claim and we have 100 total turns. Lastly, this implies our expected payout is $f(n) = \frac{100}{\frac{20}{21-n}+1} \cdot \frac{20+n}{2}$, as we multiply the expected number of times we are paid by the expected payout per time.

To find the n maximizing this, one can treat f as continuous and use the derivative of it to find the optimal n. The details of taking the derivative messy and not enlightening, so the steps are excluded. However, after using the basic rules and simplifying,

$$f'(n) = \frac{50}{(n-41)^2} \cdot \left(n^2 - 82n + 461\right) = 0$$

The roots of the polynomial are $n_{1,2} = 41 \pm 2\sqrt{305}$. The root adding $2\sqrt{305}$ is larger than 20, so $n^* = 41 - 2\sqrt{305}$ must be the maximizer. As $17 < \sqrt{305} < 18$ and our optimal n must be an integer, we can test $n = 5, 6, 7$ to see which gives us the largest expected payout.

Plugging all three of these in reveals $n = 6$ maximizes $f(n)$ with payout $\frac{3900}{7}$.

Solution to Question 1056: Subsequent First Ace

The denominator is the number of ways to obtain the first ace as the 27th card dealt. In particular, we permute 26 of the 48 non-aces to the first 26 spots, which can be done in $P(48, 26)$ ways. Then, we fix the 27th card as an ace, which has 4 options. Then, lastly, we arrange the other 25 cards in any way we want, so there are $P(48, 26) \cdot 4 \cdot 25!$ total arrangements with the 27th card being the first ace.

Now, we need to do the same on the numerator, but we have an additional condition to satisfy. Note that the first 26 cards must not be aces (since the first should come on the 27th turn) nor the 2 of hearts. Therefore, we are permuting 26 of the 47 remaining cards to the first 26 spots, yielding $P(47, 26)$ possibilities. Then, the 27th card can be one of 4 aces. Afterwards, the next card must be fixed as the 2 of hearts. Then, there are 24 cards left, so there are 24! ways to arrange the cards after the 2 of hearts. All together, we get $P(47, 26) \cdot 4 \cdot 24! = 4 \cdot \frac{47!24!}{21!}$. Therefore, our final probability is

$$\frac{4 \cdot P(47, 26) \cdot 24!}{4 \cdot P(48, 26) \cdot 25!} = \frac{\frac{47!}{21!}}{\frac{48!}{22!} \cdot 25} = \frac{1}{48} \cdot \frac{22}{25} = \frac{11}{600}$$

Solution to Question 1057: Double Aces

The number of aces drawn in this question follows a Hypergeometric distribution. This is because we are sampling without replacement from a finite population.There are $\binom{4}{2}$ ways of picking out a pair of Aces and $\binom{48}{11}$ ways of picking out the rest of the 11 cards in your hand (non-Aces). In total, there are $\binom{52}{13}$ different combinations of 13 card hands from a 52 card deck. Thus, the probability we get exactly 2 Aces is

$$\frac{\binom{4}{2} \cdot \binom{48}{11}}{\binom{52}{13}} \approx 0.21$$

Solution to Question 1058: Last Love

Similar to the First Ace problem, we can view each of the 13 cards with heart suit as dividers within our 52−card deck. These 13 cards divide our deck into 14 regions. In expectation, by symmetry, each of these regions should have an equal amount of cards in them. There are 39 cards that are not hearts, so we would expect $\frac{39}{14}$ cards on average for each region. Then, to get the expected position of the last heart, note that this is just the total length of the deck, 52, but remove one of our equally-sized regions from the rightmost/bottom part of the deck. This implies our expected position is $52 - \frac{39}{14} = \frac{689}{14}$.

Solution to Question 1059: Statistical Test Review II

Let us define

$$H_0 : p = 0.5$$
$$H_a : p \neq 0.5$$

Our rejection region is $|x-18| \geq 4$. Recall, by definition, that β is the probability that the test statistic, X, is not within the rejection region when H_0 is rejected, or in this case, $p = 0.7$. Let $Y \sim \text{Binom}(0.7)$. Then,

$$\begin{aligned}\beta &= \mathbb{P}(Y \leq 21) - \mathbb{P}(Y \leq 14) \\ &\approx 0.0916\end{aligned}$$

Solution to Question 1060: Call Vega

Vega is maximized when it is at-the-money, so vega will increase in absolute value when S gets closer to K. Here, we sell a call, so we are negative vega initially. If the absolute value of vega gets larger, our vega is also decreasing.

Solution to Question 1061: Forming a Triangle

Let x be the first break and y be the second break such that $x < y$. Thus, the lengths of the three segments are x, $y - x$, and $1 - y$. As you recall from geometry, in order for three side lengths to form a triangle, each side length must be less than the sum of the other two side lengths. We can rewrite this as:

$$x < (y - x) + (1 - y) \Rightarrow x < \frac{1}{2}$$

$$y - x < x + (1 - y) \Rightarrow y < x + \frac{1}{2}$$

$$1 - y < x + (y - x) \Rightarrow y > \frac{1}{2}$$

These constraints (including $x < y$) cover $\frac{1}{8}$ of the sample space of $x, y \in [0, 1]$, which can be seen visually. Note that the $x < y$ constraint only accounts for half of the possibilities since x is equally likely to be greater than or less than y. The final answer is $2 \times \frac{1}{8} = \frac{1}{4}$.

Solution to Question 1062: Rabbit Hop IV

Let h_n be the number of distinct paths to the top of a n stair staircase. We can condition on the size of the first jump. The size of the first jump can be anywhere from 2 to n, inclusive. If the rabbit hops k stairs on the first step, it must make a unique path through the other $n-k$ stairs. Hence, you could view the problem now as stair k being the floor and the top stair being $n - k$. This means that

$$h_n = h_{n-2} + h_{n-3} + h_{n-4} + \cdots + h_1 + h_0$$

However, let's look at the tail term $h_{n-3}+h_{n-4}+\cdots+h_1+h_0$. This is equivalent to the problem where we start on stair 1 (instead of the ground) and need to jump up strictly more than one stair at each jump. This is because stair 3 would now be the first available stair to jump on, and the rabbit can jump to any other stair at least 3. Thus, the tail term there is just h_{n-1}, as starting on stair 1, the rabbit needs a path through the other $n - 1$ stairs to the top. Our recurrence relation is now

$$h_n = h_{n-1} + h_{n-2}$$

Our initial conditions are that $h_1 = 0$ and $h_2 = 1$, which can just be counted directly. We note now that this is just the Fibonacci sequence shifted by 1 index up, as $F_0 = 0$ and $F_1 = 1$, so we can conclude that $h_n = F_{n-1}$. In particular, this means that $h_{10} = F_9 = 34$.

Solution to Question 1063: 2D Paths III

Let P_1 and P_2 be the events that your character passes through the power-ups at $(2, 3)$ and $(4, 6)$, respectively. We want $|P_1 \cup P_2|$, the event that at least one of the power-ups is passed through. We use Inclusion-Exclusion here since these are not disjoint events. This means $|P_1 \cup P_2| = |P_1| + |P_2| - |P_1 \cap P_2|$.

$|P_1|$ is the number of paths that pass through $(2, 3)$ going to $(6, 6)$. From 2D Paths II, we know this number is 350. $|P_2|$ is the number of paths passing through $(4, 6)$ that go to $(6, 6)$. The number of paths from $(0, 0)$ to $(4, 6)$ is $\binom{10}{4} = 210$, as you have to choose the location of the 4 right movements among the first 10 and the rest will be upwards movements. After reaching

$(4,6)$, there is only one valid path to $(6,6)$, which is two rightward movements. Thus, $|P_2| = 210$. Lastly, $|P_1 \cap P_2|$ is the number of paths passing through both $(2,3)$ and $(4,6)$ ending at $(6,6)$. The number of paths from $(0,0)$ to $(2,3)$ is $\binom{5}{2} = 10$. Next, from $(2,3)$, the number of paths to $(4,6)$ is also $\binom{5}{2} = 10$. Then, from $(4,6)$, there is only one path to $(6,6)$, so $|P_1 \cap P_2| = 10 \cdot 10 \cdot 1 = 100$.

Putting all of it together, $|P_1 \cup P_2| = 350 + 210 - 100 = 460$.

Solution to Question 1064: Horse Results

As we know each swimmer is equally-skilled, by the exchangeability of the swimmers, each one ends up in each of the 6 places with probability $\frac{1}{6}$. Swimmer 2 ends up in 2nd place with probability $\frac{1}{6}$. Then, given that, there are 2 spots left in the top 3 for swimmer 5 to go into. As one is taken by swimmer 2, the probability swimmer 5 is in the top 3 is $\frac{2}{5}$. Therefore, our answer is $\frac{1}{6} \cdot \frac{2}{5} = \frac{1}{15}$.

Solution to Question 1065: Poisson Process Covariance

By the definition of covariance, $\text{Cov}(N(5), N(15)) = \mathbb{E}[N(5)N(15)] - \mathbb{E}[N(5)]\mathbb{E}[N(15)]$. We know that $N(t) \sim \text{Poisson}(\lambda t)$, so $\mathbb{E}[N(5)] = 25$ and $\mathbb{E}[N(15)] = 75$. Then, the trick for the first term is to use independent increments, so we write $N(15) = (N(15) - N(5)) + N(5)$, meaning

$$\mathbb{E}[N(5)N(15)] = \mathbb{E}[N(5)(N(5)+(N(15)-N(5))] = \mathbb{E}[(N(5))^2] + \mathbb{E}[N(5)(N(15)-N(5))]$$

The first term can be written as $\text{Var}(N(5)) + (\mathbb{E}[N(5)])^2 = 25 + 25^2 = 650$. The second term can be written as $\mathbb{E}[N(5)]\mathbb{E}[N(15) - N(5)] = 25 \cdot (75 - 25) = 1250$ by independent increments. Therefore, our answer is $650 + 1250 - 1875 = 25$.

Solution to Question 1066: Odd Coin Flips

Imagine flipping 99 coins first. The last coin always decides whether we observe an odd or even number of heads. Thus, the probability of observing an odd number of heads is $\frac{1}{2}$.

Solution to Question 1067: Points on a Circle II

Suppose we already have the n points selected on the circle. Choose one of those points, let's call it A. We draw a line through A and O, the center of the circle; this diameter forms two semicircles. To prevent overcounting, we will only consider the semicircle that start from A in the counterclockwise direction. Then, each of the $n-1$ remaining points has a $\frac{1}{2}$ chance of being within that semicircle. Repeating this for all n points, we find that the solution is simply $\frac{n}{2^{n-1}}$. Plugging in $n = 100$, we find our answer to be $\frac{100}{2^{99}}$. Therefore, the answer to our question is $100 + 2 + 99 = 201$.

Solution to Question 1068: Expected Increase

Recall for non-negative integer-valued random variables X that $\mathbb{E}[X] = \sum_{k=1}^{\infty} \mathbb{P}[X \geq k]$. We should apply this to N. Therefore, $\mathbb{E}[N] = \sum_{k=1}^{\infty} \mathbb{P}[N \geq k]$. All that remains is to find $\mathbb{P}[N \geq k]$. The event $\{N \geq k\}$ means that $X_1 \geq X_2 \geq \cdots \geq X_{k-1}$, which occurs with probability $\dfrac{1}{(k-1)!}$ since each of the X_i random variables are IID and have continuous distributions. Therefore, $\mathbb{E}[N] = \sum_{k=1}^{\infty} \dfrac{1}{(k-1)!} = e$ by shifting the index back 1 and noting that this is the Taylor Expansion for e^x evaluated at $x = 1$. We know $\ln(e) = 1$, so our answer is 1.

Solution to Question 1069: Car Bidding I

Let B be the distribution of the bid prices. We first want to obtain $\mathbb{P}[B > 9000]$, as this will give us the success probability for the event we want to observe for the first time. We have that $B \sim \text{Exp}\left(\dfrac{1}{5000}\right)$, as B has mean 5000. Therefore, using the CDF of B,

$$\mathbb{P}[B > 9000] = 1 - \mathbb{P}[B \leq 9000] = 1 - \left(1 - e^{-\frac{9}{5}}\right) = e^{-\frac{9}{5}}$$

Therefore, if N gives the number bids needed to have the first bid accepted, $N \sim \text{Geom}(e^{-\frac{9}{5}})$. By properties of geometric distributions, $\mathbb{E}[N] = e^{\frac{9}{5}}$.

Solution to Question 1070: Python or R

We are running a statistical test with the null hypothesis $H_0 : \mu_1 = \mu_2$ against the alternative hypothesis $H_a : \mu_1 \neq \mu_2$. The z statistic is:

$$z = \frac{\bar{x_1} - \bar{x_2}}{\sqrt{\frac{{s_1}^2}{n_1} + \frac{{s_2}^2}{n_2}}} = \frac{74 - 71}{\sqrt{\frac{9^2}{50} + \frac{10^2}{50}}} \approx 1.577$$

Because this is a two-tailed test, the attained significance level is double the significance level of a one-side test, or:

$$2 \times P(z \geq 1.577) = 2 \times 0.0571 = 0.1142$$

Solution to Question 1071: Win by N

Lets start out with the case where $N = 1$. The probability of Person A winning is $\frac{0.6}{0.6+0.4} = 0.6$. Lets see what happens when $N = 2$. The probability that Person A wins two games in a row is $0.6 \cdot 0.6 = 0.36$. The probability that Person B wins two games in a row is $0.4 \cdot 0.4 = 0.16$. This problem becomes a random walk where there is a 0.36 chance of going +1 and 0.16 chance of going -1. The other 0.48 chance our walk doesn't increase or decrease. We end the walk when we get to +2 or -2. The probability Person A wins two games in a row before Person B is

$$\frac{0.36}{0.36 + 0.16} = \frac{9}{13}$$

As we keep increasing N, we see the general case becomes

$$\frac{0.6^N}{0.6^N + 0.4^N} = \frac{1}{1 + \left(\frac{2}{3}\right)^N}$$

As N tends to infinity, the probability Person A wins approaches 1. Thus the answer is 1.

Solution to Question 1072: 2 Coin More

Let's consider the first $N = 20$ coins. There are 3 possible events: Jeff has more heads, Jeff has equal heads, or Jeff has less heads. For Jeff to win in the next 2 coins, we need the following to occur. If Jeff has more heads in the first N coins, then Jeff will win with probability 1. If Jeff has equal heads, then Jeff needs to get at least 1 head in the next 2 coins. This occurs with probability $3/4$. Finally, if Jeff has less heads, he can only win if he is *exactly* 1 head behind. In this case, he needs to obtain 2 heads. This leaves us with the following probability:

$$\mathbb{P}[\text{Jeff wins}] = \mathbb{P}[\text{Jeff wins in 20 flips}] + \frac{3}{4}\mathbb{P}[\text{Jeff has equal heads}] + \frac{1}{4}\mathbb{P}[\text{Jeff has 1 less head}]$$

We can now move on to calculate each probability in the first 20 coins. From symmetry, we can see that $\mathbb{P}[\text{Jeff has more heads}] = \mathbb{P}[\text{Jeff has less heads}]$ since there is no distinction between heads and tails. So, we have $\mathbb{P}[\text{Jeff has more heads}] =$

$\frac{1}{2}(1 - \mathbb{P}[\text{Jeff has equal heads}])$. We then can calculate the probabilities through some combinatorics:

$$\mathbb{P}[\text{Jeff has equal heads}] = \sum_{i=0}^{20} \binom{20}{i}^2 (0.5)^{40} = .5^{40}\binom{40}{20} \approx 0.125$$
$$\mathbb{P}[\text{Jeff has more heads}] = \frac{1}{2}\left(1 - .5^{40}\binom{40}{20}\right) \approx 0.437$$
$$\mathbb{P}[\text{Jeff has 1 less head}] = \sum_{i=1}^{20} \binom{20}{i}\binom{20}{i-1}(0.5)^{40} \approx 0.119$$

Plugging the values in, we obtain $\mathbb{P}[\text{Jeff wins}] \approx 0.561$

Solution to Question 1073: Compound Interest II

Using the Taylor approximation $e^x \approx 1 + x$, we can see that $100(1.01)^{15} \approx 100(1 + 0.15) = 115$. However, we must adjust up a little bit for the interest that accrues over the 15 days i.e. it is not exactly \$1 per day. Therefore, a natural guess is \$116, which is correct. One can see it will not go to \$117 by the fact that the absolute maximum that could be attained is $1.15 \cdot 15 = 17.25$, which is if you constantly receive 1.15 interest a day. However, this is a large overestimate, so 116 is a better guess.

Solution to Question 1074: Conditional Expectation II

Let's compute $\mathbb{E}[X]$ using the law of total expectation.

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|\lambda]]$$

If the value of λ is known, then X follows a set Poisson distribution, meaning we can describe its expectation.

$$\mathbb{E}[X|\lambda] = \lambda$$

Recall

$$f_\lambda(t) = \begin{cases} e^{-t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

So, it follows from the law of total expectation,

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{E}[\mathbb{E}[X|\lambda]] \\ &= \int_0^\infty f_\lambda(t)\mathbb{E}[X|t]\,dt \\ &= \int_0^\infty te^{-t}\,dt \\ &= \left[-\lambda e^{-\lambda} - e^{-\lambda}\right]_0^\infty \\ &= 1\end{aligned}$$

Solution to Question 1075: Annie's Coin

Annie has a $\frac{1}{3}$ chance of winning on her first turn. If she doesn't win on her first turn, then she has a $\frac{3}{5}$ chance to survive Brittany's turn and flip again. Hence,

$$\begin{aligned}p &= \frac{1}{3} + \frac{2}{3}\cdot\frac{3}{5}\cdot p \\ p &= \frac{5}{9}\end{aligned}$$

Solution to Question 1076: Same Month

Given that there are 12 months, if we have 13 people, by the Pigeonhole Principle, two must share a birthday in the same month.

Solution to Question 1077: Spaced Darts

Let R_1 and R_2 be the radii of the two throws Nicole has. We want $\mathbb{P}\left[R_2 > \frac{R}{2}\right]$. We condition on the radius of the first throw, R_1. We have that $R_1 \sim \text{Unif}(0, R)$ from the question. Thus, we have that $\mathbb{P}\left[R_2 > \frac{R}{2}\right] = \int_0^R \mathbb{P}\left[R_2 > \frac{R}{2}\middle| R_1 = r\right] f_{R_1}(r)dr$. We already know the PDF of R_1, so no more work is needed there. For the first probability, we need to split into two cases. The first is if $R_1 > \frac{R}{2}$. Since from the question we know that Nicole throws her dart further away from the first one, the probability is 1 in this case, as the first dart is already at least $\frac{R}{2}$ away from the center. The second case is when $0 < R_1 < \frac{R}{2}$. In this case, the second dart lands uniformly throughout the region further away than the first dart. Thus, if $R_1 = r < \frac{R}{2}$. The probability it lies in the annulus larger than $\frac{R}{2}$ in distance is given by taking the ratios of areas. We have that the area of the region larger than $\frac{R}{2}$ in distance is given by $\pi\left(R^2 - \frac{R^2}{4}\right) = \frac{3\pi R^2}{4}$. The total area of the region is $\pi(R^2 - r^2)$. Thus, the conditional probability for this case is $\frac{\frac{3}{4}R^2}{R^2 - r^2} = \frac{3}{4} \cdot \frac{1}{1 - \left(\frac{r}{R}\right)^2}$. We now split up the probability. We have that it is $\int_0^{\frac{R}{2}} \mathbb{P}\left[R_2 > \frac{R}{2}\Big| R_1 = r\right] f_{R_1}(r)dr + \int_{\frac{R}{2}}^{R} \mathbb{P}\left[R_2 > \frac{R}{2}\Big| R_1 = r\right] f_{R_1}(r)dr =$ $\int_0^{\frac{R}{2}} \frac{3}{4} \cdot \frac{1}{1 - \left(\frac{r}{R}\right)^2} \cdot \frac{1}{R}dr + \int_{\frac{R}{2}}^{R} \frac{1}{R}dr = \frac{1}{2} + \frac{3}{4R}\int_0^{\frac{R}{2}} \frac{1}{1 - \left(\frac{r}{R}\right)^2}dr$. You can evaluate this last integral here using methods from Calc II (Trig sub) to get that the final answer is $\frac{4 + 3\ln(3)}{8} = \frac{4 + \ln(27)}{8}$ by properties of logarithms. Therefore, $4 + 27 + 8 = 39$.

Solution to Question 1078: Highest Drawn Card

The first thing we want to do is calculate the probability that our max card is less than or equal to any certain value, in this case, we will denote it as k. The next step and final step is to calculate the expected value of our max card. We do this by calculating all the probabilities that a certain value is our max $(1 - 13)$, multiplying that probability by it's respective value, and the summing all these expectations up.

$$\mathbb{P}[\max \leq k] = \frac{\binom{4k}{4}}{\binom{52}{4}} = \frac{4k(4k-1)(4k-2)(4k-3)}{52(51)(50)(49)}$$

$$\mathbb{E}[\max] = \sum_{k=1}^{13} k\mathbb{P}[\max = k] = P[\max \leq 1] + \sum_{k=2}^{13} k(\mathbb{P}[\max \leq k] - \mathbb{P}[\max \leq k-1]) = \frac{32577}{2975} \approx 10.95$$

Solution to Question 1079: Busted 6 II

Our optimal strategy will be to be stop once we have at least a sum of k and roll again otherwise. Our goal is to find such a k. To do this, we need to find the point at which we are indifferent to rolling again versus cashing out. If we currently have k in the bank, then with probability $1/6$, we bust and get a payout of 0. With probability $1/6$, we roll a 5 and obtain a payout of k. Otherwise, if we don't roll either of those values, our bank increases by an average of $5/2$, as we roll a value $1 - 4$ uniformly at random. Therefore, we need to find k such that

$$\frac{1}{6} \cdot 0 + \frac{1}{6} \cdot k + \frac{2}{3} \cdot (k + 5/2) = k$$

This equation has the LHS represent the expected payout when rolling again and the RHS represent the expected payout when we stop. Solving this yields $k = 10$.

To compute the expected payout of the game, we can use a recursive approach. We set up the recursion here. The largest possible sum attainable under this strategy is 13, as we would roll again on 9 and obtain a 4. Let E_k be the expected payout when our current sum is k under this strategy. We clearly have that $E_{13} = 13, E_{12} = 12, E_{11} = 11$, and $E_{10} = 10$. For any $k < 10$, we can either cash out k with probability $1/6$ (rolling a 5), cash out nothing with probability $1/6$ (roll a 6), or have an expected payout of $E_{k+1}, E_{k+2}, E_{k+3}$, or E_{k+4} with probability $1/6$ each. Therefore, we have found that for $k < 10$,

$$E_k = \frac{k + E_{k+1} + E_{k+2} + E_{k+3} + E_{k+4}}{6}$$

E_0 is the value we are interested in looking for, as we start with 0 at the beginning. Using this recursion to recurse backwards to 0, we get that $E_0 \approx 3.03$.

Solution to Question 1080: Triangular Selection I

There are $\binom{8}{3} = 56$ triangles that can be formed from the 8 points, as any selection of 3 points will yield a triangle. Fix Aaron's triangle first. Then of the 56 triangles Matt can form, exactly 1 will be Aaron's so the answer is $\frac{1}{56}$.

Solution to Question 1081: Lognormal II

Let $Z \sim N(0, 1)$. Then we know that $X = e^Z$, so $\text{Var}(X^4) = \mathbb{E}[X^8] - (\mathbb{E}[X^4])^2$ by the definition of variance. Plugging in $X = e^Z$, we get that $\text{Var}(X^4) = \mathbb{E}[e^{8Z}] - (\mathbb{E}[e^{4Z}])^2$. By definition, we know that for a random variable X, the MGF of X is given by $M_X(\theta) = \mathbb{E}[e^{\theta X}]$ for all θ where the expectation is finite. Therefore, we see that $\text{Var}(X^3) = M_Z(8) - (M_Z(4))^2$, where $M_Z(\theta)$ is the MGF of a standard normal random variable.

We know that the MGF of a standard normal is given by $M_Z(\theta) = e^{\frac{1}{2}\theta^2}$. Therefore, plugging in $\theta = 8$ and $\theta = 4$, we get that $\text{Var}(X^4) = e^{32} - e^{16}$, so our answer is $32 + 16 = 48$.

Solution to Question 1082: Three-Way Tile

We are going to solve this for more general n. Let $t(n)$ be the number of ways of tiling a $3 \times n$ board with these tiles. Consider tiling the first spots all the way at the left. There are three options:

Case 1: You place 3 1×2 tiles in each of the rows. In this case, there are $t(n-2)$ possible ways to tile the remaining grid, as you have eliminated the first two columns.

Case 2: You place a 1×2 tile in the first row and then a 2×1 tile in the first column bottom two rows OR a 2×1 tile in first column top two rows and a 1×2 tile in the bottom row. In both cases here, you end up with a grid that has a corner piece missing. We can just consider one of the two arrangements above and multiply by 2, as they both yield a grid that has one corner missing.

Let the number of ways to tile the grid with one corner missing and n columns be $s(n)$. Consider tiling the first column of the grid where the corner is missing. You have two cases:

Case 2a: Place a 2×1 tile in the first column. In this case, you get a grid with no missing corner and $n-1$ columns, so there are $t(n-1)$ ways to tile the remaining grid.

Case 2b: Place three 1×2 tiles in each of the rows. You get another grid with $n-2$ columns and a missing corner, so there are $s(n-2)$ ways to tile the remaining grid.

Combining these together, we get that $s(n) = t(n-1) + s(n-2)$ and $t(n) = t(n-2) + 2s(n-1)$. The boundary conditions are that $t(0) = 1$ (empty grid), $t(1) = 0$ (no way to tile one column grid), $s(0) = 0$ (can't be a missing corner), and $s(1) = 1$ (just a single 2×1 tile).

We are looking for $t(8)$ in this problem. Using this recurrence, one gets $t(8) = 153$ after some considerable effort.

Solution to Question 1083: Grid Filling III

There is only one orientation where all the rows, columns, and diagonals add up to odd numbers. This orientation has the even numbers on the corners and the odds filling the rest of the positions. There are 5! ways to order the odds and 4! ways to order the evens with 9! total orderings. Thus our answer is

$$\frac{5! \cdot 4!}{9!} = \frac{1}{126}$$

Solution to Question 1084: Ramen Bowl

Let $f(n)$ be the number of circles creates from n starting noodles- we are looking for $E[f(100)]$. When $n = 1$, it is clear that $E[f(n)] = 1$ since we connect the two ends of the only noodle. For $n = 2$, there are a total of $\binom{4}{2} = 6$ possible combinations to connect the first two ends. Of these 6 combinations, 4 will yield a single noodle ($f(1)$) and 2 will yield a noodle and a circle ($1 + f(1)$)). Hence,

$$E[f(2)] = \frac{4}{6} \times E[f(1)] + \frac{2}{6} \times (1 + E[f(1)]) = 1 + \frac{1}{3}$$

For $n = 3$, there are a total of $\binom{6}{2} = 15$ possible combinations to connect the first two ends. Of these 15 combinations, 12 will yield will yield two noodles ($f(2)$) and 3 will yield two noodles and a circle ($1 + f(2)$)). Hence,

$$E[f(3)] = \frac{12}{15} \times E[f(2)] + \frac{3}{15} \times (1 + E[f(2)]) = 1 + \frac{1}{3} + \frac{1}{5}$$

The pattern is now noticeable: $E[f(n)] = 1 + \frac{1}{3} + ... + \frac{1}{2n-1}$, and thus $E[f(100)] \approx 3$.

Solution to Question 1085: Dinner Party

Let N be the number of people at the party. The first person will shake $N - 1$ hands since he does not shake hands with himself. The second person will shake hands with $N - 2$ more hands since he does not shake hands with himself and already shook hands with the first person. This pattern continues. We can employ the Gaussian sum to define the total number of handshakes that occur as a function of N:

$$(N - 1) + (N - 2) + \ldots + 1 = \frac{N \times (N - 1)}{2} = 120$$

Solving for N, we find that the total number of people at the party is 16.

Solution to Question 1086: Farming Emergency

Let x be the rate at which an individual farmer tills per hour. There were 20 "farmer hours" of work to till 2 plots of land, as there were 4 farmers working for 5 hours. Therefore, $20x = 2$, so $x = \frac{1}{10}$.

Our job is to find the minimal integer n such that $\frac{1}{10} \cdot (4+n) \cdot 12 \geq 9$.

This is since there would be $4+n$ farmers if n farmers are sent and they work for 12 hours, so we would be $12(4+n)$ farmer hours. Solving this, we get that $n \geq \frac{7}{2}$. This means that $n = 4$, as n must be an integer.

To confirm this answer, we see that with 8 total farmers, they would work at a rate of $\frac{1}{10} \cdot 8 = \frac{4}{5}$ of a plot per hour, so it would take them $\frac{45}{4} < 12$ hours to till 9 plots.

Solution to Question 1087: Complementary Dice

Let's compute p_1 as in the question. With probability 1/3, he rolls a winning value on the first roll. Otherwise, with probability 2/3, he does not roll a winning value, and then player 2 needs to not roll a winning value on their first roll, which occurs with probability 1/3. In that case, the probability player 1 wins is p_1 again. Therefore,

$$p_1 = \frac{1}{3} + \frac{2}{9}p_1 \iff p_1 = \frac{3}{7}$$

This means that $p_2 = 1 - p_1 = \frac{4}{7}$, so the answer is $\frac{4}{7}$.

Solution to Question 1088: Smallest Probability

We can simulate an event of probability $\frac{1}{72}$ by saying that E occurs if we roll $(6,6)$ on the two dice and the coin appears heads. We can't partition our 36 element sample space for the dice any finer. Similarly, we can't partition our two element sample space for the coin any finer.

Solution to Question 1089: Weighted Dice

The scaling factor in this case is that each dot provides $\frac{1}{21}$ probability of that side being rolled. Thus, side i, $1 \leq i \leq 6$, has probability $\frac{i}{21}$ of being rolled.

We want the values i and $7-i$ to appear in the two rolls, where we sum over $i = 1, \ldots, 6$ in the end to account for all possible cases. This occurs with probability $\frac{i(7-i)}{21^2}$ for each $1 \le i \le 6$, as the dice are independent, so the probability of interest is $\frac{1}{21^2}\sum_{i=1}^{6} 7i - i^2 = \frac{8}{63}$.

Solution to Question 1090: The Swarm of Bees

Let n be the number of bees. We have that:

$$\sqrt{\frac{n}{2}} + \frac{8n}{9} + 2 = n$$

Solving this, we get $n = 72, 9/2$. It is impossible to have a fraction number of bees, so 72 is our answer.

Solution to Question 1091: Face Value

With probability $\frac{3}{10}$, the card is worth less than the payout, so you should choose the payout of 3.5. Otherwise, you should take the face value of the card. Therefore, the expected value is

$$\frac{3}{10} \cdot \frac{7}{2} + \frac{7}{10} \cdot \frac{4+5+6+7+8+9+10}{7} = \$5.95$$

.

Solution to Question 1092: Circular Slice I

For convenience, let's scale everything so that we are talking about proportions of the circle instead of radians. As a result, we are looking at Unif$(0,1)$ random variables instead of Unif$(0, 2\pi)$. We can do this because we are just scaling our units. Let's first think about the conditions needed to have no overlap.

First, we know that $\theta_1 < \alpha$. This is because we know that we are going to sweep out an arc starting CCW from α, so for the starting point of this second arc to not interfere with the first, we must have that condition. In addition, we need $\theta_2 < 1 - \alpha$ (recall we are working in proportions here). This is because $\theta_1 < \alpha$, so the arc starting at α with have proportion $1-\alpha$ of the circle left before interfering with the original segment. Therefore, we want $\mathbb{P}[\theta_1 < \alpha, \theta_2 < 1 - \alpha]$. We see that both of these statements have α in them, so let's condition on α to remove that element of randomness.

This yields that $\mathbb{P}[\theta_1 < \alpha, \theta_2 < 1-\alpha] = \int_0^1 \mathbb{P}[\theta_1 < \alpha, \theta_2 < 1-\alpha \mid \alpha = x] f_\alpha(x) dx$. We integrate on $(0,1)$ because of our scaling factor. Now, if we know $\alpha = x$, then the first term becomes $\mathbb{P}[\theta_1 < x, \theta_2 < 1-x] = \mathbb{P}[\theta_1 < x]\mathbb{P}[\theta_2 < 1-x] = x(1-x)$ by the uniform distribution of the values on $(0,1)$. Therefore, our probability of interest is $\int_0^1 x - x^2 dx = \frac{1}{6}$.

Solution to Question 1093: Mississippi

For the word to read the same both forwards and backwards, as Mississippi is 11 letters, the M must be in the middle. Therefore, we have something in the form $-----M-----$, where each $-$ is filled with an S, I, or P. Note that there are 4 Ss and Is, while there are only 2 Ps. Therefore, for the word to read the same both forwards and backwards, we must have 2 Ss and Is on each side and 1 P on each side. Once we distribute the letters to one side, the other side is also fixed, so the number of ways to do this is $\frac{5!}{2! \cdot 2! \cdot 1!} = 30$, as we must account for the overcounting induced by exchanging the duplicate S and I around.

Solution to Question 1094: Poker Hands II

There are a total of $\binom{52}{5}$ total hand combinations. To count the number of hands that contain a full house, we can look at the triplet and pair separately. The triplet has 13 possible face values and $\binom{4}{3}$ possible suit values. The pair has 12 possible face values and $\binom{4}{2}$ possible suit values. Thus, the probability that you have a full house is:

$$\frac{13 \times \binom{4}{3} \times 12 \times \binom{4}{2}}{\binom{52}{5}} = \frac{6}{4165}$$

Solution to Question 1095: Doubly Stochastic

The stationary distribution is a row vector satisfying $\pi = \pi P$. Writing this out in matrix multiplication form, we have that $\pi_j = \sum_{i=1}^{100} \pi_i P_{ij}$. As we know $\sum_{i=1}^{100} P_{ij} = 1$ from the fact that columns sum to 1 in a doubly stochastic matrix, suppose that $\pi_i = 1$ for all i. Then

$$1 = \sum_{i=1}^{100} P_{ij} = 1$$

However, we note that $\sum_{i=1}^{100} \pi_i = 1$ by definition of being a distribution. Therefore, the above allows us to conclude that π is uniform on S, so that $\pi_i = \frac{1}{100}$ for all i. This means that $||\pi|| = \sqrt{100 \cdot \frac{1}{100^2}} = \frac{1}{10}$

Solution to Question 1096: Non-Uniform Fix

We can generalize this to any non-negative continuous distribution $F(x)$ with PDF $f(x)$. Suppose $T = t$. Then on any trial, there is a probability $1 - F(t)$ of $X_i > t$. Therefore, $N \mid T = t \sim \text{Geom}(1 - F(t))$. Using the Law of Total Expectation, we get

$$\mathbb{E}[N] = \mathbb{E}[\mathbb{E}[N \mid T]] = \mathbb{E}\left[\frac{1}{1 - F(T)}\right] = \int_0^\infty \frac{f(t)}{1 - F(t)} dt = -\ln(1 - F(t))\Big|_0^\infty = \infty$$

This means the answer is -1.

Solution to Question 1097: Fibonacci Limit I

We can write $F_{n+1} = F_n + F_{n-1}$, so if L is the limit in question, we have that $\frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n}$, so in limit, $L = 1 + \frac{1}{L}$. Equivalently, $L^2 - L - 1 = 0$. This has roots $L_{1,2} = \frac{1 \pm \sqrt{5}}{2}$. Since Fibonacci numbers are increasing as n increasing, we must take the positive root, so $L = \frac{1 + \sqrt{5}}{2}$, meaning $abc = 1 \cdot 2 \cdot 5 = 10$.

Solution to Question 1098: 9 Digit Sum

Each of the 9 digits is going to appear in each of the 9 spots in 8! of the numbers. This is because when we fix an integer in one of the spots, there are 8! other ways to permute the other 8 integers to the other 8 spots. Therefore, we just need to sum up all of the different positions i.e. powers of 10 and all of the digits. Namely, if S is the sum, we have that

$$S = 8! \sum_{n=1}^{9} \left(10^0 + 10^1 + \cdots + 10^8\right) n = 8! \cdot \frac{10^9 - 1}{10 - 1} \cdot \sum_{n=1}^{9} n = 5 \cdot 8! \cdot (10^9 - 1)$$

This means $abk = 8 \cdot 5 \cdot 9 = 360$.

Solution to Question 1099: Betting Orbs

Let $I_1, \ldots, I_4$ be the indicator of the event that the ith orb's color is guessed correctly. Then $T = I_1 + I_2 + I_3 + I_4$ gives the total profit of the game. By linearity of expectation, $\mathbb{E}[T] = \sum_{i=1}^{4} \mathbb{P}[A_i]$, where A_i is the event that the ith orb's color is guessed correctly. We want to find a strategy that maximizes $\mathbb{P}[A_i]$.

On the first draw, we have absolutely no information about what color will come out. Therefore, regardless of which color we select, our probability of getting it correct is $\mathbb{P}[A_1] = \frac{1}{2}$. Afterwards, once we have observed the color of the first orb, note that the probability that the second orb differs in color from the first orb is $\frac{2}{3}$. This also means the probability they are the same color is $\frac{1}{3}$. Therefore, we should guess the opposite color of the orb to be drawn on the second draw. This means $\mathbb{P}[A_2] = \frac{2}{3}$. For the third draw, we really need to think about what has come out thus far. If the first two draws are the same color, which occurs with probability $\frac{1}{3}$, then with probability 1 you know the color of the last two orbs to be drawn, so you should guess that. If they differ, which occurs with probability $\frac{2}{3}$, then you are back at random for the third draw, so you get it correct with probability $\frac{1}{2}$. Therefore, by applying Law of Total Probability via conditioning on the matching or differing colors of the orbs, $\mathbb{P}[A_3] = \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 1 = \frac{2}{3}$. Regardless of what is drawn on the third turn, you know the last color, so $\mathbb{P}[A_4] = 1$. Therefore, under this strategy, $\mathbb{E}[T] = \frac{1}{2} + \frac{2}{3} + \frac{2}{3} + 1 = \frac{17}{6}$. Note that under just guessing randomly, your expected value would be 2, so this is a significant improvement.

Solution to Question 1100: Cow, Goat, and Goose

Let t be the number of days it takes for them to eat all the grass. Let a, b, c be the eating rate in number of fields per day of (respectively) cow, goat, and goose.

In t days, the various contributions of each of the animals to the eating process is at, bt, ct respectively.

So for the total contribution to be 1 field, we have:

$$(a + b + c)t = 1 \Rightarrow t = \frac{1}{a + b + c}$$

We have the following system:

$$\begin{aligned} a+b &= \frac{1}{45} \\ a+c &= \frac{1}{60} \\ b+c &= \frac{1}{90} \end{aligned}$$

Solving the system, we obtain $a+b+c = \frac{1}{40}$ and so $t = 40$.

Solution to Question 1101: Better In Red IV

We are conditioning our sample space to the cubes with at least one red side. There are 54 total sides that are red on the cube, as each face of the larger cube has area 9. There are 8 corner cubes that each have 3 red painted faces. These contribute 24 of the 54 total red faces. Therefore, the answer is just $\frac{24}{54} = \frac{4}{9}$.

Solution to Question 1102: Uniform Fix

The key here is to condition on U. This is because of the fact that $N \mid U = u \sim \text{Geom}(1-u)$, as each trial has probability $1-u$ of being larger than u. Therefore, $\mathbb{E}[N \mid U] = \frac{1}{1-U}$ by known formulas about geometric random variables. Using the Law of Total Expectation and LOTUS, we have that

$$\mathbb{E}[N] = \mathbb{E}[\mathbb{E}[N \mid U]] = \mathbb{E}\left[\frac{1}{1-U}\right] = \int_0^1 \frac{du}{1-u} = \infty$$

Therefore, you should enter in -1.

Solution to Question 1103: Product and Sum

Note that

$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4 = (x^4+y^4)+2xy\left[2x^2+3xy+2y^2\right] = (x^4+y^4)+2xy\left[2(x+y)^2 - xy\right.$$

Since we know the values of $x+y$ and xy, we can say that $x^4+y^4 = (x+y)^4 - 2xy\left[2(x+y)^2 - xy\right]$ and substitute in the values. Namely, we get that $x^4+y^4 = 10^4 - 2(20)(2\cdot 100 - 20) = 10000 - 7200 = 2800$

Solution to Question 1104: Better in Red II

Label the faces of each cube $1-6$, and then let I_i be the indicator that side i of the cube that is drawn is colored red. Then $T = I_1 + \cdots + I_6$ gives the total number of red sides of the cube. We need to be careful here, as the indicators are not exchangeable. Instead, two of the indicators will correspond to a 10×20 side, two to a 20×30 side, and two to a 10×30 side. Therefore, there are 3 subsets of indicators that are exchangeable, so we can reduce it to an expectation involving 3 indicators and multiply it by 2, so $\mathbb{E}[T] = 2(\mathbb{E}[I_1] + \mathbb{E}[I_2] + \mathbb{E}[I_3])$, where we take I_1 to indicate a 10×20 side, I_2 to indicate a 20×30 side, and I_3 to indicate a 10×30 side.

Then, to evaluate these expectations, we just need to find the probability that the side indicated is red on our cube. For I_1, that is $10 \cdot 20 = 200$ cubes. For I_2, that is $20 \cdot 30 = 600$ cubes. For I_3, that is $10 \cdot 30 = 300$ cubes. All of these need to be divided by 6000 as that is the volume of the entire prism. Therefore, $\mathbb{E}[T] = \dfrac{2(200 + 300 + 600)}{6000} = \dfrac{11}{30}$.

Solution to Question 1105: Points on a Circle III

As there are 8 points on the circle, there are $\dbinom{8}{2} = 28$ different chords on the circle. This is because each pair of points can have a chord drawn between them. Therefore, there are $\dbinom{28}{4}$ different subsets of the chords that could be selected.

Additionally, each set of 4 points corresponds to exactly one quadrilateral. This is because each set of 4 points can have exactly 1 quadrilateral where the chords connect between those 4 points specifically. Therefore, there are $\dbinom{8}{4}$ ways to pick 4 points. This means our probability is

$$\frac{\binom{8}{4}}{\binom{28}{4}} = \frac{2}{585}$$

Solution to Question 1106: Cheese Lover I

Markov's Inequality states that $\mathbb{P}[T_{100} > 26000] \leq \dfrac{\mathbb{E}[T_{100}]}{26000}$, as it is already in the form of Markov's Inequality. By linearity of expectation (or recognition), you can find that $\mathbb{E}[T_{100}] = 100\mathbb{E}[W_1] = 100 \cdot 250 = 25000$, so we get the upper bound on the probability as $\dfrac{25}{26}$.

Solution to Question 1107: Couple Handshakes

Each handshake consists of 2 people. The order of the people in the handshake is irrelevant, so there are $\binom{8}{2} = 28$ ways to pick pairs of people to shake hands. However, we know that no couple shakes hands with their partner. Therefore, we must remove one handshake for each couple representing the one that is with their partner. As there are 4 couples, we remove 4 handshakes, and our total number of handshakes is $28 - 4 = 24$.

Solution to Question 1108: Bond Practice III

$n = 2 \times 5.0 = 10; r = \frac{0.05}{2} = 0.02500$; $C = \frac{0.06 \times 1{,}000}{2} = 30$; $P =$ price of bond:

$$P = \left(\frac{30}{0.02500}\right)\left(\frac{(1+0.02500)^{10} - 1}{(1+0.02500)^{10}}\right) + \frac{1{,}000}{(1+0.02500)^{10}}$$
$$P = (1{,}200)\left(\frac{1.28008 - 1}{1.28008}\right) + \frac{1{,}000}{1.28008}$$
$$P = 262.5619 + 781.1984$$
$$P = 1{,}043.76$$

Solution to Question 1109: Remainders

Let X denote the smallest positive integer we are solving for. We can further define $X_2, X_3, ..., X_{10}$ as positive integers such that:

$$X = 2 \times X_2 + 1$$
$$X = 3 \times X_3 + 2$$
$$\vdots$$
$$X = 10 \times X_{10} + 10$$

Let $X'_i = X_i + 1 \forall i \in \{2, 3, ..., 10\}$. Then, adding 1 to both sides of each equations:

$$X + 1 = 2 \times X'_2$$
$$X + 1 = 3 \times X'_3$$
$$\vdots$$
$$X + 1 = 10 \times X'_{10}$$

In other words, we are now looking for X such that $X+1$ is perfectly divisible by 2, 3, 4, 5, 6, 7, 8, 9, and 10. The least common multiple of these numbers is 2520, and thus X is 2519.

Solution to Question 1110: Square Normal

As $-X \sim N(0,1)$ as well, we can say that $-X$ and Y are IID. Therefore,

$$\mathbb{E}[X \mid X^2 + Y^2] = \mathbb{E}[-X \mid (-X)^2 + Y^2] = -\mathbb{E}[X \mid X^2 + Y^2]$$

This can only occur if this conditional expectation is 0.

Solution to Question 1111: Tricky Bob II

Call Bob's success probability p_2. Let's write out the expected value as a function of p_1 and p_2. The probability of HH is $p_1 p_2$. The probability of TT is $(1-p_1)(1-p_2)$. The probability of TH or HT is $p_1(1-p_2) + p_2(1-p_1) = p_1 + p_2 - 2p_1p_2$. From Bob's perspective, the respective profits for each of these outcomes are $6, 4,$ and -5. Therefore, the expected value is $6p_1p_2 + 4(1-p_1)(1-p_2) - 5(p_1 + p_2 - 2p_1p_2) = 6p_1p_2 + 4 - 4p_1 - 4p_2 + 4p_1p_2 - 5p_1 - 5p_2 + 10p_1p_2 = 20p_1p_2 - 9p_2 - 9p_1 + 4$. Factoring this a little bit, we have that this becomes $(20p_1 - 9)p_2 + (4 - 9p_1)$.

Now, assuming that $p_1 \neq \frac{9}{20}$ (so that we can rearrange and divide), we get that $p_2 > \frac{9p_1 - 4}{20p_1 - 9}$. Letting $p_1 = \frac{3}{4}$, we have that $p_2 > \frac{11}{24}$, so this is our answer.

Solution to Question 1112: Breaking Even

In order to break even after 8 rounds, Julie and Judy must (1) end up with \$4 each, and (2) never reach \$0. To satisfy the first condition, the coin must end up as heads for 4 of the 8 tosses. The second condition is satisfied for any arrangement of 4 heads and 4 tails except for $HHHHTTTT$ and $TTTTHHHH$, which is when Julie and Judy begin the fifth round with \$0. Hence, there are a total of $\binom{8}{4}$ -2 possible head-tail arrangements such that Julie and Judy break even. Putting it all together, we find the probability of breaking even to be

$$\left[\binom{8}{4} - 2\right]\left(\frac{1}{2}\right)^8 = \frac{17}{64}$$

Solution to Question 1113: Distinct Date I

We want to minimize the year, as that will be the biggest influence on how soon it is. Note that the first M is either 0 or 1. If the first M is 0, then the first day must be 1 or 2 OR the day is 31 (30 doesn't work since we already have a 0. If the first M is 1, then either the second M is a 0 OR the second M is a 2 and the day contains a 0 somewhere. Therefore, these both imply that the 0 and either the 1 or 2 must be used in the $MMDD$ portion. Since we ideally don't want to skip 1000 years, we should put the 2 in the year portion, so thus far, we have that our date is in the form

$$0M/1D/2YYY$$

After this, it's just an objective to minimize the rest of the digits. Namely, $YYY = 345$, as that is the smallest number that can be made with the remaining digits. Then, we want to minimize the month, so $M = 6$. Then, lastly, $D = 7$, as that is the smallest remaining number. Our answer is therefore

$$06/17/2345$$

Solution to Question 1114: Even Doubles

Let E be this event and N be the number of tosses needed until two consecutive heads or tails appear. Then $\mathbb{P}[E] = \sum_{k=1}^{\infty} \mathbb{P}[N = 2k]$, as we want an even amount of flips needed. Let's find the term inside. The event $\{N = 2k\}$ means that the last two flips must be either HH or TT. However, once we select the last two flips, the other $2k - 2$ flips must be fixed. Namely, they must alternate between H and T, with the starting flip based on which ending we have. For example, if $\{N = 4\}$ this correspond to the two sequences $THTT$ and $HTHH$. Therefore, of the 2^{2k} possible sequences of length k, exactly 2 satisfy our event, so $\mathbb{P}[T = 2k] = \frac{2}{2^{2k}} = \frac{2}{4^k}$. Plugging this into our previous sum,

$$\mathbb{P}[E] = 2\sum_{k=1}^{\infty} \frac{1}{4^k} = 2 \cdot \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{2}{3}$$

Solution to Question 1115: Cubic Difference

Cubes are very large in difference, so to minimize the actual difference, one may want to consider both positive and negative integers. In particular, listing out the first few squares, we see that they are $1, 4, 9$, and 16. One may note that $2^3 = 8$, so $2^3 - (-2)^3 = 8 - (-8) = 16 = 4^2$, so $y = 4$ is our solution. We can't do better because of the fact that the cubes grow too quickly, so the differences between them grow too fast after this.

Solution to Question 1116: Optimizing Aces

Consider an ordering of 52 cards. In order for Aaron to win a prize, (1) the k-th card must be an ace, (2) there must be exactly two aces in the first $k - 1$

cards, and (3) there must be exactly one ace in the last $52 - k$ cards. Treating all cards as distinguishable, we find the following function of k that describes the probability that Aaron wins a prize.

$$\begin{aligned}\mathbb{P}(\text{Aaron wins}) &= \frac{\binom{k-1}{2}\binom{52-k}{1}4!48!}{52!} \\ &= \frac{12 \cdot 48! \cdot (52-k)(k-1)(k-2)}{52!}\end{aligned}$$

We arrive at the above expression as follows: (1) choose two cards to assign as aces from the first $k - 1$ cards, (2) assign the k-th card as an ace, (3) choose one card to assign as an ace from last $52 - k$ cards, and finally, (3) there are 4! ways to order the four distinguishable aces among the four allotted ace slots. Now, we have a simple optimization problem:

$$\begin{aligned}k_{\text{optimal}} &= \underset{k\in\{1,\ldots,52\}}{\arg\max} \;\{\mathbb{P}(\text{Aaron wins})\} \\ &= \underset{k\in\{1,\ldots,52\}}{\arg\max} \left\{\frac{12 \cdot 48! \cdot (52-k)(k-1)(k-2)}{52!}\right\}\end{aligned}$$

Let's take the derivative of $\mathbb{P}(\text{Aaron wins})$ with respect to k, set it to 0, and solve for k.

$$\frac{d}{dk}\frac{12 \cdot 48! \cdot (52-k)(k-1)(k-2)}{52!} = 0$$

Diving out the constants and simplifying, we find

$$\begin{aligned}\frac{d}{dk}\left(-k^3 + 55k^2 - 158k + 104\right) &= 0 \\ \Rightarrow 3k^2 - 110k + 158 &= 0\end{aligned}$$

Solving for k, we find

$$k = \frac{55}{3} \pm \frac{\sqrt{2551}}{3}$$

Note that k cannot equal $\frac{55}{3} - \frac{\sqrt{2551}}{3} < 2$. Additionally, k must be an integer, and $35 < \frac{55}{3} + \frac{\sqrt{2551}}{3} < 36$. Testing both $k = 35$ and $k = 36$, we find that $k = 35$ achieves the greatest probability for Aaron to win a prize.

Solution to Question 1117: Distinct Date II

Our goal now is the maximize the year to make it as close as possible to our present date. The biggest thing to first note is that the year must start with a 1. Suppose that it started with a 2. We know that the first M is either a 0 or 1. If it is a 0, then the day either starts with a 1 or is 31. Note that the other M can't be a 1 in this case. Otherwise, that would eliminate the day possibilities. If the month starts with a 1, then the day either contains 0 or is 30. However, since the 0 is taken already in the $MMDD$ part, regardless of what we start with, we can't get a year that is in the past.

Therefore, the year must start with a 1. To maximize the year, we can just fix the rest as the three largest integers. Namely, 987, so our year is 1987. Since the year starts with 1, the month must start with 0. Then, we choose the largest month integer remaining, which is 6. Then, we must pick the largest possible day among the integers remaining. We can see that 25 is the largest possible day, as anything starting with 3 is not possible. Therefore, our date is

$$06/25/1987$$

Solution to Question 1118: Missing Million I

We condition on whether or not the wheel locates the money. With probability $3/5$, it does so guaranteed. Otherwise, with probability $2/5$, you have a $1/3$ chance of guessing the location of the money correctly, as one of the three doors has the money. Therefore, the answer is

$$\frac{3}{5} + \frac{2}{5} \cdot \frac{1}{3} = \frac{11}{15}$$

Solution to Question 1119: First and Last Heads

We condition based on the number of heads in both segments. The only way to get 0 heads in both segments is $TTTTT$, so that yields 1 outcome. To get one head in both segments, there are 3 initial sequences in the first 3 flips that have 1 head (namely, HTT, THT, and TTH) and 2 sequences in the last two flips that have 1 head (HT and TH). This yields $3 \cdot 2 = 6$ outcomes here. Lastly, to get two heads in each segment, there are 3 initial sequences in the first 3 flips that have 2 heads (namely, HHT, HTH, and THH). There is only one sequence with 2 heads in the last two flips. Therefore, this yields $3 \cdot 1 = 3$ sequences. Adding these all up, we get $1 + 6 + 3 = 10$ sequences where this is satisfied. There are $2^5 = 32$ total sequences of 5 flips, so our answer is

$$\frac{10}{32} = \frac{5}{16}$$

Solution to Question 1120: Left Corner

In the top left, it can move to 3 spots. Two of those spots will be along the edge of the board, which will have 5 valid spots to move from there. One is a central spot on the board, which has 8 valid spots to move.

The probability of moving to an edge spot is $\frac{2}{3}$, while the probability of moving to a central spot is $\frac{1}{3}$. From an edge spot, there is a $\frac{1}{5}$ probability of return to the corner. From the central spot, there is an $\frac{1}{8}$ probability of return to the corner. Thus, the probability of return to the corner in 2 moves is just $\frac{2}{3} \cdot \frac{1}{5} + \frac{1}{3} \cdot \frac{1}{8} = \frac{7}{40}$

Solution to Question 1121: Optimal Marbles I

We can first make some simplifications to the game. Firstly, if the strategy is optimal, then if this game were to be repeated many times, they would not change their strategy. Therefore, the optimal strategy for the game where there are 2 consecutive marble draws is the same as the optimal strategy for the game with one draw. Then, we just multiply the expected profit by 2 to represent the two draws. Furthermore, as this game is symmetric for the two players, their optimal strategy will be the same. This point will be important later.

Let $A(a, b)$ be the expected profit that A obtains with player A putting in a balls and B putting in b balls. Namely, for the one draw game,

$$A(a, b) = \frac{a}{a+b} \cdot (100 - a)$$

As player a draws his ball with probability $\frac{a}{a+b}$ and $100 - a$ is the payout. Let's fix b and find the a that is the best response to this b. In other words, given b, what a optimizes $A(a, b)$? To do this, we take the partial derivative of $A(a, b)$ in a and treat a as continuous for now. We will then account for discreteness at the end.

This yields that

$$\frac{\partial}{\partial a} A(a, b) = -\frac{a^2 + 2ba - 100b}{(a+b)^2} = 0 \iff a^2 + 2ba - 100b = 0$$

Solving the above with the quadratic equation yields that $a^* = \frac{-2b \pm \sqrt{4b^2 + 400b}}{2}$. However, the $-$ root results in a negative value, so $a^* = \sqrt{b^2 + 100b} - b$ is the best response for player A if player B puts b marbles in. Similarly, as this game is symmetric, the optimal response for player B if player A puts a marbles in is $b^* = \sqrt{a^2 - 100a} - a$.

To find the optimal strategy for each player, this means that we need to find the combo (a^*, b^*) such that neither of the players can do better by adjusting their strategy. We already are aware from before that $a^* = b^*$ by the symmetry of the game. Therefore, to solve for this, we just substitute in b as $b^* = a^*$ in the first equation. This yields we can say that

$$a^* = \sqrt{(a^*)^2 + 100a^*} - a^* \iff 4(a^*)^2 = (a^*)^2 + 100a^* \iff a^*(3a^* - 100) = 0 \iff a^* = 0, \frac{100}{3}$$

As 0 is not possible, we conclude that $a^* = b^* = \dfrac{100}{3}$ is the optimal strategy. However, this is not actually possible, as our marbles must be an integer value. Therefore, we should test $(33, 33)$ and $(34, 34)$ to see if they are Nash equilibria.

For $(33, 33)$, the expected payout for one draw for each player is $\dfrac{67}{2}$. One can check that by varying a and keeping b fixed at 33, player A can't do any better. Therefore, $(33, 33)$ is a Nash equilibrium. For $(34, 34)$, the expected payout is 33. However, one can also verify that the expected payout for $(33, 34)$ is also 33. However, this can't be an equilibrium, as b should change to 33 marbles to yield higher expected payout. Thus, while $(34, 34)$ is also a Nash equilibrium, $(33, 33)$ is preferrable because of the higher expected payout. This means that the optimal strategy is for both players to place 33 marbles and have a total expected payout of $\dfrac{67}{2} \cdot 2 = 67$.

Solution to Question 1122: Rowdy Root

We can more generally consider this for a radical of the form $\sqrt{n(n+1)(n+2)(n+3)+1}$ and plug in $n = 120$ to get the result. The key trick here is to get a common midpoint. In particular, we can write this as

$$\sqrt{(m - 3/2)(m - 1/2)(m + 1/2)(m + 3/2) + 1}$$

where $m = n + 3/2$. We want to write it in this form because now we can use the difference of squares formula to reduce this. In particular, the interior becomes $(m^2 - 1/4)(m^2 - 9/4) + 1 = m^4 - \frac{5}{2}m^2 + \frac{25}{16} = \left(m^2 - \frac{5}{4}\right)^2$. This implies that the original expression is equal to $m^2 - \dfrac{5}{4}$, where $m = \dfrac{3}{2} + n$. Plugging back in n, we get

$$m^2 - \frac{5}{4} = \left(n + \frac{3}{2}\right)^2 - \frac{5}{4} = n^2 + 3n + 1$$

Plugging in $n = 120$ into this expression, we have that our answer is

$$120^2 + 3 \cdot 120 + 1 = 14761$$

Solution to Question 1123: First Quarter

You should go first, as you will want to put your quarter right in the center. Afterwards, the symmetry of the table implies that there will be only an even number of places left to put the quarters. Therefore, your opponent will be first to be unable to place a quarter.

Solution to Question 1124: Wood Chop

The only way to create a situation in which we have a section of the stick greater than $\frac{1}{2}$ in length is if all our cuts happen on half of the stick. If all of our cuts happen in the same half of the stick, the region to the right of the last cut (or before the first cut dependent on the location of the cuts), will be at least $\frac{1}{2}$ in length. A cut happening on a certain half of the stick has a $\frac{1}{2}$ to occur, and all 5 cuts happening on one half occurs with a probability of $2 \cdot \left(\dfrac{1}{2}\right)^5 = \dfrac{1}{16}$. This is because there are two halves of the stick.

Since we want to know the probability that no piece is greater than $\frac{1}{2}$ in length, our final answer is the complement of the above. Therefore, our final answer is $1 - \frac{1}{16} = \frac{15}{16}$.

Solution to Question 1125: Paired Pumpkins I

Using the notation here, we can set up the system of equations $w_1 + w_2 = 19, w_2 + w_3 = 21$, and $w_1 + w_3 = 28$. By subtracting the second equation from the first, we get that $w_1 - w_3 = -2$. Adding this new equation to the third equation yields $2w_1 = 26$, meaning $w_1 = 13$. Substituting this back in, we get $w_2 = 6$ and $w_3 = 15$. Therefore, $w_1^2 + w_2^2 + w_3^2 = 15^2 + 6^2 + 13^2 = 430$.

Solution to Question 1126: Brownian Supremum

Define the events $A_n = \left\{W_{\frac{1}{n}} > \frac{1}{\sqrt{n}}\right\}$ and $B = \limsup A_n = \{A_n \text{ i.o.}\}$. Then we have that

$$\begin{aligned}
\mathbb{P}[B] &= \mathbb{P}\left[\limsup A_n\right] \\
&\geq \limsup \mathbb{P}\left[A_n\right] \text{ (Fatou's Lemma)} \\
&= \limsup \mathbb{P}\left[W_{1/n} > \frac{1}{\sqrt{n}}\right] \\
&= \limsup_n \mathbb{P}[Z > 1] \ (\cdot\sqrt{n} \text{ to get } Z \sim N(0,1)) \\
&= \mathbb{P}[Z > 1] = 1 - \Phi(1) > 0
\end{aligned}$$

By Blumenthal's $0-1$ Law, we know that $\mathbb{P}[B] = 0$ or 1. Since we already know $\mathbb{P}[B] > 0$, we know that $\mathbb{P}[B] = 1$.

Solution to Question 1127: Card Up

The number 83 here is irrelevant, as we only care about the top 5 cards drawn. There are 5! permutations of the values of the first 5 cards, as they are all distinct. Two of these 5! permutations are cases where all the cards are in strictly ascending or descending order, so the probability of this event is $\frac{2}{5!} = \frac{1}{60}$. Therefore, the expected value on the game is $(x-1) \cdot \frac{1}{60} - \frac{59}{60}$, which implies that $x = 60$ for the game to be fair.

Solution to Question 1128: Covariance Review I

Recall the definition of covariance:

$$\text{Cov}(X_1, X_2) = \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2].$$

In order to compute the covariance we need to complete the following steps: (1) determine the value of c, (2) compute $\mathbb{E}[X_1 X_2]$ from the joint pdf, and (3) determine the marginal pdfs for X_1 and X_2 in order to compute $\mathbb{E}[X_1]$ and

$\mathbb{E}[X_2]$. Let's begin with step 1.

$$\begin{aligned}
\iint_{\mathbb{R}^2} f_{X_1,X_2}(x_1,x_2)dx_1\,dx_2 &= 1\\
\int_0^1\int_0^{x_2} c(1-x_2)\,dx_1\,dx_2 &= 1\\
\int_0^1 cx_2(1-x_2)\,dx_2 &= 1\\
c\left[\frac{x_2^2}{2}-\frac{x_3^3}{3}\right]_0^1 &= 1\\
\frac{c}{6} &= 1\\
c &= 6
\end{aligned}$$

Next, for step 2, we need to compute $\mathbb{E}[X_1X_2]$.

$$\begin{aligned}
\int_0^1\int_0^{x_2} 6x_1x_2(1-x_2)\,dx_1\,dx_2 &= \int_0^1 6x_1\int_0^{x_2} x_2(1-x_2)\,dx_1\,dx_2\\
&= \int_0^1 x-3x^3+2x^4\,dx\\
&= \left[\frac{2}{5}x^5-\frac{3}{4}x^4+\frac{1}{2}x^2\right]_0^1\\
&= \frac{3}{20}
\end{aligned}$$

On to step 3. Here is $f_{X_1}(x_1)$:

$$\begin{aligned}
\int_{\mathbb{R}} f_{X_1,X_2}(x_1,x_2)\,dx_2 &= \int_{x_1}^1 6(1-x_2)\,dx_2\\
&= \left[6x_2-3x_2^2\right]_{x_1}^1\\
&= 3-6x_1+3x_1^2
\end{aligned}$$

And here is $f_{X_2}(x_2)$:

$$\begin{aligned}
\int_{\mathbb{R}} f_{X_1,X_2}(x_1,x_2)\,dx_1 &= \int_0^{x_2} 6(1-x_2)\,dx_1\\
&= 6x_2-6x_2^2
\end{aligned}$$

We can use these marginal pdfs to compute $\mathbb{E}[X_1]$ and $\mathbb{E}[X_2]$.

$$\begin{aligned}\mathbb{E}[X_1] &= \int_0^1 x_1(3-6x_1+3x_1^2)\,dx_1 \\ &= \left[\frac{3}{4}x_1^4 - 2x_1^3 + \frac{3}{2}x_1^2\right]_0^1 \\ &= \frac{1}{4} \\ \mathbb{E}[X_2] &= \int_0^1 x_2(6x_2-6x_2^2)\,dx_2 \\ &= \left[-\frac{3}{2}x_2^4 + 2x^3\right]_0^1 \\ &= \frac{1}{2}\end{aligned}$$

Putting it all together, we find

$$\begin{aligned}\mathrm{Cov}(X_1,X_2) &= \mathbb{E}[X_1X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2] \\ &= \frac{3}{20} - \frac{1}{8} \\ &= \frac{1}{40}.\end{aligned}$$

Solution to Question 1129: Sharpe Maximization

Let R_A and R_B be the excess returns of A and B. From the covariance matrix, we see that $\mathrm{Var}(R_A) = 1$ and $\mathrm{Var}(R_B) = 2$. Furthermore, we get that $\mathrm{Cov}(R_A, R_B) = 1$. Let a be the weight we give to asset A. Then $1-a$ is what we assign to asset B. From the question, we are given that $\mathbb{E}[R_A] = 7$ and $\mathbb{E}[R_B] = 4$. We are looking to maximize the value of

$$\frac{\mathbb{E}[aR_A + (1-a)R_B]}{\sqrt{\mathrm{Var}(aR_A + (1-a)R_B)}}$$

The numerator is easy to calculate. Namely, by Linearity of Expectation, we have that

$$\mathbb{E}[aR_A + (1-a)R_B] = a\mathbb{E}[R_A] + (1-a)\mathbb{E}[R_B] = 7a + 4(1-a) = 3a+4$$

We use the variance of sum formula to calculate the variance of our new portfolio. Namely,

$$\mathrm{Var}(aR_A+(1-a)R_B) = a^2\mathrm{Var}(R_A)+(1-a)^2\mathrm{Var}(R_B)+2a(1-a)\mathrm{Cov}(R_A,R_B) = a^2+2(1-a)^2+2a(1-a)$$

this means the function we have to maximize in a is

$$\frac{3a+4}{\sqrt{a^2+2(1-a)^2+2a(1-a)}}$$

Skipping the derivative step, we get that the derivative of the above in a is

$$-\frac{7a-10}{\left(a^2+2a\left(1-a\right)+2\left(1-a\right)^2\right)^{\frac{3}{2}}}$$

This has a zero at $a=\dfrac{10}{7}$, and since our derivative was positive before it became 0 and switches to negative, we can conclude this is a maximum.

Solution to Question 1130: Ken Flipping Coins

The expected payoff when $1 \leq n \leq 6$ is $\frac{1}{2^n} \times 2^n = \$1$ for each n. For example, if $n = 1$, which happens with probability $\frac{1}{2}$, Ken receives \$2, so the expected payoff is \$1. If $n = 2$, which happens with probability $\frac{1}{4}$, Ken receives \$4, so the expected payoff is \$1. This pattern continues until $n = 7$, when the minimum payoff stays fixed at \$64. When $n \geq 7$, the expected payoff is:

$$\frac{1}{2^7} \times 2^6 + \frac{1}{2^8} \times 2^6 + \ldots = \frac{1}{2} + \frac{1}{4} + \ldots = \sum_{x>0} \frac{1}{2^x} = 1$$

Thus, the expected payoff of the game is:

$$6 \times 1 + 1 = 7$$

Solution to Question 1131: Red Blue Equality

We want a pair of red socks to show with probability $\dfrac{1}{2}$. The probability the first sock is red is $\dfrac{3}{n+3}$. The probability the second sock is red is $\dfrac{2}{n+2}$, as we took out one red sock. Therefore, the probability of both socks being red is $\dfrac{6}{(n+2)(n+3)}$. This must equal $\dfrac{1}{2}$, so we need

$$\frac{6}{(n+2)(n+3)} = \frac{1}{2}$$

Equivalently, this means $(n+2)(n+3) = 12$ after rearranging. Expanding this out, we have that $n^2 + 5n - 6 = 0$, so $(n+6)(n-1) = 0$, meaning $n = 1, -6$ solves this. -6 is not possible, so $n = 1$ is our answer.

Note that one can see this quickly by noting that $n \leq 3$, as otherwise, it would be less likely to draw two red than two blue. We can rule out $n = 0$, as that probability would be 1. Therefore, that leaves us three cases to check.

Solution to Question 1132: A Low Median

If we want the median to be less than $\frac{2}{3}$ then it must be that we can have at most one number be larger than $\frac{2}{3}$. Therefore, we have that

$$P[\text{exactly one number larger}] = \binom{3}{1}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) = \frac{4}{9}$$

and we also have

$$P[\text{exactly zero numbers larger}] = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

Adding these together, we have that the probability that at most one number is larger than $\frac{2}{3}$ is

$$\frac{4}{9} + \frac{8}{27} = \frac{20}{27}$$

Solution to Question 1133: Corner Meet

Let's look at the probability of reaching a corner from each of the center 3 x 3 grid. We can see that there are 5 squares that have some possibility of reaching the corner. We can use law of total probability. First, we have a equal chance of selecting each of the 9 inner squares. From here, we can look at the probability of reaching a corner from each square. This can be done in a recursive manner, working from top to bottom.

First, let's look at the top-left square in the inner 3 x 3 grid. We see that we have probability $\frac{1}{2}$ of immediately reaching the left corner and probability $\frac{1}{2}$ of reaching the top-center, in which case there's no chance of reaching a corner. Hence, we have overall probability $\frac{1}{2}$. By symmetry, the top-right square has the same probability. We can see that the top-center square of the 3 x 3 grid has zero probability of reaching a corner. We can iterate downwards, and see that we can calculate probabilities of the lower squares using the probabilities that we just calculated.

This gives us

$$\frac{1}{9}\left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right) = \frac{5}{18}$$

Solution to Question 1134: Generational Wealth I

The expected value of a Unif$(0,1)$ random variable is $\frac{1}{2}$. Therefore, if X_1 is the first number we generate, we would keep X_1 if $X_1 > \frac{1}{2}$. Otherwise, we generate $X_2 \sim \text{Unif}(0,1)$. The probability $X_1 < \frac{1}{2}$ is $\frac{1}{2}$, and our expected profit in this case is $\frac{1}{2}$, as our profit would be X_2. Otherwise, if $X_1 > \frac{1}{2}$, which also occurs with probability $\frac{1}{2}$, your expected profit is $\frac{3}{4}$, as given that we don't generate X_2, X_1 is uniform on $(0.5, 1)$, which has mean $\frac{3}{4}$. Therefore, our expected payout of this game is $\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{3}{4} = \frac{5}{8}$.

Solution to Question 1135: Bad Bagel

Let T be the number of bagels until another bad one and X_1 be the outcome of the first bagel since our present one. Then

$$\mathbb{E}_B[T] = \mathbb{E}_B[T \mid B]\mathbb{P}_B[B] + \mathbb{E}_B[T \mid G]\mathbb{P}_B[G]$$

The subscript B is to denote that we start on a bad bagel. We have that $\mathbb{P}_B[B] = \frac{3}{5}$, so $\mathbb{P}_B[G] = \frac{2}{5}$. Then, $\mathbb{E}_B[T \mid G] = 1 + \mathbb{E}_G[T]$, as we take one step and then find the expected number of bagels starting from a good bagel until a bad bagel. Then, $\mathbb{E}_B[T \mid B] = 1$, as it takes just 1 bagel to get another bad bagel if the first bagel after is bad. Therefore,

$$\mathbb{E}_B[T] = (1 + \mathbb{E}_G[T]) \cdot \frac{2}{5} + 1 \cdot \frac{3}{5} = 1 + \frac{2}{5} \cdot \mathbb{E}_G[T]$$

Note that the distribution of the number of bagels until a bad bagel starting from a good bagel is Geom$\left(\frac{3}{5}\right)$, as there is a $\frac{3}{5}$ probability each bagel of going to a bad bagel, so $\mathbb{E}_G[T] = \frac{5}{3}$. This implies that $\mathbb{E}_B[T] = 1 + \frac{2}{5} \cdot \frac{5}{3} = \frac{5}{3}$.

Solution to Question 1136: Animal Farm

From the above, we know that $x_1 + x_2 + x_3 = 100$ by the constraint of having 100 animals. We also know that $50x_1 + 10x_2 + 0.5x_3 = 1000$ by the constraint of spending 1000 total. Subtracting half of the first equation from the second yields that $49.5x_1 + 9.5x_2 = 900$, or equivalently that $99x_1 + 19x_2 = 1900$. We need to solve this in the integers that stay within our constraints of each $0 \leq$

$x_i \leq 100$. The easiest solution is to note that $99 \cdot 19 = (100-1) \cdot 19 = 1900 - 19$. Therefore, $x_1 = 19$ and $x_2 = 1$ is a solution to this. We leave out verification that it is the only solution satisfying our constraints. This implies now that $x_3 = 100 - 19 - 1 = 80$. Therefore, $x_1 x_2 x_3 = 19 \cdot 1 \cdot 80 = 1520$.

Solution to Question 1137: Birthday Twins I

Instead of trying to compute the probability that at least two people are born on the same day, it is easier to work with the complement, which is that everyone is born on a distinct day of the week. Then, all we need to do is subtract this probability from 1 to get the probability of the event we are interested in.

The first friend has all 7 days of the week available to them. The second has 6 options since one is taken by the first friend. The last has 5 days to select from, so the number of ways to pick the days of the week the people are born on with all days distinct is $7 \cdot 6 \cdot 5$. The total number of days to pick with no restriction is 7^3. Therefore, the probability all days are distinct is $\frac{7 \cdot 6 \cdot 5}{7^3} = \frac{30}{49}$. Therefore, the probability that at least two of the people share a birthday is $1 - \frac{30}{49} = \frac{19}{49}$.

Solution to Question 1138: Circular Covariance

$$\mathbb{E}[X] = \mathbb{E}[Y] = 0$$

by symmetry and uniformity, since each point is equally likely and they are symmetric about the unit circle. To show that $\mathbb{E}[XY] = 0$, we apply LOTUS. We have that both X and Y take the values $\frac{\sqrt{2}}{2}, 0$, and $-\frac{\sqrt{2}}{2}$ each with probability $\frac{1}{4}$, while it is -1 and 1 with probability $\frac{1}{8}$. We only need to consider the odd multiple of $\frac{\pi}{4}$, as the even ones have one of their coordinates as 0, so they contribute 0 to the product. However, the odd multiples are symmetric about the origin, so the expectation of the product is 0, so they must be uncorrelated.

As an aside, the random variables X and Y here are an example of uncorrelated but not independent random variables. This is because $X^2 + Y^2 = 1$, so they can't be independent.

Solution to Question 1139: Bombing Campaign

In order for the bomb to destroy the target, it must land within $\frac{1}{2}$ miles of the target. Hence our probability of not destroying the target is simply

$$\frac{\pi - \left(\frac{1}{2}\right)^2 \pi}{\pi} = \frac{3}{4}.$$

Solution to Question 1140: Color Swap

Let A be the event that the ball selected originated from Urn A and R be the event of a red ball. By Law of Total Probability,

$$\mathbb{P}[R] = \mathbb{P}[R \mid A]\mathbb{P}[A] + \mathbb{P}[R \mid A^c]\mathbb{P}[A^c]$$

We know that $\mathbb{P}[A] = \dfrac{3}{8}$, as 300 of the 800 balls came from Urn A. This means that $\mathbb{P}[A^c] = \dfrac{5}{8}$. Given that the ball is from A, there $\dfrac{4}{5}$ it is red. Given that the ball is not from A i.e. from B, there is a $\dfrac{1}{5}$ chance it is red. Therefore, the total probability the ball is red is

$$\frac{4}{5} \cdot \frac{3}{8} + \frac{1}{5} \cdot \frac{5}{8} = \frac{17}{40}$$

Solution to Question 1141: Particle Reach X

From Particle Reach IX, we know that $\text{Var}(T_1) = 24$. Let T be the total amount of steps needed to reach 7 from 0. We have that $T = T_1 + \cdots + T_7$, where T_i is the amount of steps needed to go from position $i - 1$ to position i. Furthermore, note that the T_i random variables are independent, as moving from one position to the next position is independent of how many steps were needed to reach the current position. This is by the Markov Property. Therefore, $\text{Var}(T) = 7\text{Var}(T_1) = 168$.

Solution to Question 1142: Repetitious Game II

Let X_1 denote the number of draws it takes for Audrey to draw a card from a unique suit. Let X_2 denote the number of draws it takes for Audrey to draw a card from a second unique suit after a card from the first unique suit has been draw. We'll define X_3 and X_4 in a similar way for the third and fourth unique suit draws, respectively. Then, the number of total draws it takes for Audrey to get at least one card from each suit is $Z = X_1 + X_2 + X_3 + X_4$. By the Linearity of Expectation,

$$\begin{aligned}\mathbb{E}[Z] &= \mathbb{E}[X_1 + X_2 + X_3 + X_4] \\ &= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] + \mathbb{E}[X_4].\end{aligned}$$

Consider X_1. $\mathbb{P}(\mathbb{X}_{\mathbb{1}} = \mathbb{1}) = 1$, since no matter what card is drawn first, it will be from a unique suit. So, $\mathbb{E}[X_1] = 1$

Next, consider X_2. Since one suit has already been accounted for, three suits remain. The probability that a drawn card belongs to one of the three unaccounted for suits is $\frac{3}{4}$. Hence,

$$\begin{aligned}X_2 &\sim \text{Geo}\left(\frac{3}{4}\right) \\ \Rightarrow \mathbb{E}[X_2] &= \frac{4}{3}.\end{aligned}$$

Following this reasoning for X_3 and X_4, we find

$$\begin{aligned}X_3 &\sim \text{Geo}\left(\frac{2}{4}\right) \\ \Rightarrow \mathbb{E}[X_3] &= \frac{4}{2}, \quad \text{and} \\ X_4 &\sim \text{Geo}\left(\frac{1}{4}\right) \\ \Rightarrow \mathbb{E}[X_4] &= \frac{4}{1}.\end{aligned}$$

Substituting, we conclude

$$\begin{aligned}\mathbb{E}[Z] &= 1 + \frac{4}{3} + \frac{4}{2} + \frac{4}{1} \\ &= \frac{25}{3}.\end{aligned}$$

Solution to Question 1143: Basic Dice Game II

The fair value of the last roll is 3.5, and thus you should only opt to roll a third time if your second roll is less than 3.5. Your expected value for the second roll is then $\frac{3}{6} \times 5 + \frac{3}{6} \times 3.5 = 4.25$. Similarly, you should only opt to roll your second time if the first roll is less than 4.25. Your expected value for the first roll is thus $\frac{2}{6} \times 5.5 + \frac{4}{6} \times 4.25 \approx 4.67$.

Solution to Question 1144: Rabbit Hop III

Let p_n be the number of distinct paths to the top of a staircase of length n. We can condition on the last jump. If the last jump is a movement by 1 stair, we need to count the number of ways to get to stair $n-1$. There are p_{n-1} ways to do this by definition. Similarly, if the last movement is movement by 2 stairs, we need to count the number of paths to get to stair $n-2$. There are p_{n-2} ways to do this by definition. Since these two cases are disjoint and exhaustive, we have that $p_n = p_{n-1} + p_{n-2}$. The initial conditions are $p_1 = 1$ and $p_2 = 2$, which can just be enumerated directly. We can see that $p_n = F_{n+1}$, the $(n+1)$st Fibonacci number. In particular, $p_{10} = F_{11} = 89$.

Solution to Question 1145: Odd Valued Roll

6

of the 12 values are odd, meaning there is probability $\frac{1}{2}$ per roll of showing an odd value. Therefore, the number of rolls needed to see an odd value is $N \sim \text{Geom}(1/2)$ distributed, which has mean 2.

Solution to Question 1146: Toasting Bread

Denote the four slices of bread A, B, C, D and their sides $A_1, A_2, B_1, B_2, ..., D_2$. For the first minute, you will toast A_1, B_1, and C_1. For the second minute, you will toast A_2, B_2 and D_1. For the third and final minute, you will toast C_2 and D_2. More generally, if $M(x, y)$ is the number of minutes it takes to toast x slices of bread on a pan that fits y slices at once, then:

$$M(x, y) = \lceil \frac{2x}{y} \rceil \forall x \geq 2, \forall y \geq 3$$

Solution to Question 1147: Big Mod I

Note that $11^5 = (10+1)^5 = 10^2 \cdot C + \binom{5}{1} \cdot 10^1 \cdot 1 + 1 = 10^2 C + 51$ for a constant C. We know that C is an integer since all of the higher order terms contain a 10^2 or higher-order power of 10. This means that 11^5 has remainder 1 when divided by 50. As $2360 = 5 \cdot 472$, the remainder of $(11^5)^{472}$ when divided by 50 is going to be $1^{472} = 1$.

Solution to Question 1148: The Arithmetical Cabby

Let n be the driver's number. We know that $n-1$ is divisible by $2, 3, 4, 5, 6$. Hence, we know that $n - 1 = k * \text{lcm}(2, 3, 4, 5, 6) = 60k$, where k is an integer.

If $k = 0$, we get $n = 1$, which is not divisible by 11. Similarly, $k = 1$ yields $n = 61$ which is also not divisible by 11. Finally, $k = 2$ yields $n = 121$, which is divisible by 11. Hence, this is our result.

Solution to Question 1149: Contract Arbitrage

The arbitrage involves our contract. This contract is the same as going long 4 units of the bond and shorting 2 units of the strike $K = 2$ put. By the condition of no-arbitrage, we would expect this to be equal to the price of our custom contract V.

$$\begin{aligned} V &\stackrel{?}{=} 4Z - 2P \\ 3.4 &\stackrel{?}{=} 4(0.9) - 2(0.8) \\ 3.4 &\stackrel{?}{=} 2 \end{aligned}$$

We see a clear arbitrage. So, we long the undervalued item and short the overvalued item. In other words, we short 1 unit of the derivative V, long 4 units of the bond, and short 2 units of the put. This gives us the following:

Contract (V) + # Put (K = 4) + # Put (K = 2) + # Bonds = −1+0−2+4 = 1

Solution to Question 1150: The Weight of the Fish

Let t, h, and b denote the weight in ounces of the tail, head, and body respectively.

We have the following 3 equations:

$$\begin{aligned} t &= 9 \\ h &= t + \frac{b}{2} \\ b &= h + t \end{aligned}$$

Solving this system of equations, we obtain $h = 27$, $b = 36$. So $h+b+t = 72$.

Solution to Question 1151: Bolt Variance I

Note that we are testing a hypothesis concerning variance. The appropriate test statistic is

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} = \frac{(12-1) \times .0003}{.0002} = 16.5$$

Solution to Question 1152: Log Square

Method 1 is incorrect. The line that is incorrect is Line 5, as $\dfrac{dM^2}{M^2} \neq d\ln M^2$. Namely, it is missing the Ito term out back involving the quadratic variation. One can also see that Method 1 is incorrect by considering $\ln M^2 = 2\ln M$, so calculating the dynamics of $\ln M$ is even easier.

Solution to Question 1153: Ping Pong Tournament I

When competitor 50 appears, they will beat out all other competitors until they win the tournament. Similarly, if competitor 49 appears before competitor 50, they will beat out all other competitors until they play competitor 50. There are two ways for competitors 49 and 50 to play in the final round: (1) competitor 49 does not play until the final round, or (2) competitor 50 does not play until the final round. Let's consider the first case. Notice that we can rewrite our problem as: "after ordering numbers 1 through 50, what is the probability that 49 appears last?" This probability is simply $\frac{49!}{50!} = \frac{1}{50}$. Next, let's consider the second case. Similarly, we can instead solve "after ordering numbers 1 through 50, what is the probability that 50 appears last?" This probability is again just $\frac{49!}{50!} = \frac{1}{50}$. Since these two cases are mutually exclusive events, we employ countable additivity to determine the answer to be $\frac{1}{25}$.

Solution to Question 1154: Skewed

To compute the skewness, we must compute a seemingly complicated integral via LOTUS. We use the fact that the mean of this distribution is μ, which you can clearly see is finite because of the fact that the tails are exponentially decaying and it is even about μ. Therefore, the mean must be μ. To compute the skewness, our integral of interest is $\mathbb{E}\left[\left(\dfrac{X-\mu}{\sigma_X}\right)^3\right] = \dfrac{1}{\sigma_X^3}\displaystyle\int_{\mathbb{R}}(x-\mu)^3 \cdot \dfrac{\lambda}{2}e^{-\lambda|x-\mu|}dx$. The first simplification that we do is to let $u = x - \mu$ so that $du = dx$ and the bounds are still $\pm\infty$. This new integral becomes

$$\frac{\lambda}{2\sigma_X^3}\int_{\mathbb{R}} u^3 e^{-\lambda|u|}du$$

This is where the magic takes place. Note that $u^3e^{-\lambda|u|}$ is odd about $u = 0$ and we are integrating over a symmetric integral about 0 (namely, the real line). We also know that this integral must be finite since we assume the first three moments of X are finite. Therefore, as this integrand is odd and finite about a symmetric integral, the integral must be 0.

Solution to Question 1155: 20-30 Die Split I

Let P be the payout, while A and B are the values Alice and Bob roll. The key here is to condition on whether or not $A \leq 20$. Namely, by Law of Total Expectation, we have that

$$\mathbb{E}[P] = \mathbb{E}[P \mid A \leq 20]\mathbb{P}[A \leq 20] + \mathbb{E}[P \mid A > 20]\mathbb{P}[A > 20]$$

We quickly see that $\mathbb{P}[A \leq 20] = \frac{2}{3}$, as this accounts for 20 of the 30 values that can appear. The expected payout for Alice in this case would be 0 if ties were not settled in Bob's favor. Ties happen with probability $\frac{1}{20}$ in this case, as the first roll is completely arbitrary and the second roll just needs to match the first value. Given a tie occurs, it is equally likely to be any of the 20 values. Therefore,

$$\mathbb{E}[P \mid A \leq 20] = -\frac{1+20}{2} \cdot \frac{1}{20} = -\frac{21}{40}$$

If $A > 20$, occurring with probability $\frac{1}{3}$, then Alice is guaranteed to win. Her expected payout in this case then is $\frac{21+30}{2} = \frac{51}{2}$, as it is equally-likely to be any of the values $21 - 30$ given that she is larger than 20. Combining this, we see that

$$\mathbb{E}[P] = -\frac{21}{40} \cdot \frac{2}{3} + \frac{51}{2} \cdot \frac{1}{3} = 8.15$$

Solution to Question 1156: Sum Over Min Die

The first thing we want to do is calculate the expected value of our starting score. Since the average value for a single dice roll is 3.5, by linearity, the expected value of the sum is $2 \cdot 3.5 = 7$. Now we must calculate the E_{min} of the two dice rolls, and store these values in a PMF for our final calculation.

Following the same logic as Dice Order III, we can calculate our PMF as our sum EV divided by the min of the two dice, multiplied by the probability our divisor will be our minimum die:

$$\frac{1}{36} \cdot \frac{7}{6} + \frac{3}{36} \cdot \frac{7}{5} + \frac{5}{36} \cdot \frac{7}{4} + \frac{7}{36} \cdot \frac{7}{3} + \frac{9}{36} \cdot \frac{7}{2} + \frac{11}{36} \cdot \frac{7}{1} = \frac{2779}{720}$$

For example, in order to get a 6 as our minimum die, we need to roll a 6 twice, yielding a 1/36 chance, and then multiplied by 7/6.

Solution to Question 1157: Mixed Set II

Since we must have exactly 2 elements from $\{1, 2, 3\}$, we must have selected $\{1, 3\}$, as the other 2 combinations include a 2, which is not odd. Then, we have the option to include or exclude any elements in $\{5, 7, 9\}$, which can be done in $2^3 = 8$ ways, so our answer is $1 \cdot 8 = 8$.

Solution to Question 1158: Water Measurement

Let V_7 and V_{11} denote the 7-pint and 11-pint vessels respectively. In the below, the successive lines indicate the status of each of V_7 and V_{11}:

V_7	V_{11}
7	0
0	7
7	7
3	11
3	0
0	3
7	3
0	10
7	10
6	11
6	0
0	6
7	6
2	11

It is seen that the required 2 pints are in the 7-pint vessel.

Solution to Question 1159: Conditionally Normal

Since X and Y are independent standard normal, we can exploit the radial symmetry of normal distributions to solve this cleanly. The region $\{X+Y > 0\}$ covers up half of the plane i.e. spans a total of π radians. This can be easily seen by drawing it out in the plane. Of this region, the region $\{X > 0\}$ is $\frac{3\pi}{4}$ radians, also easily seen by picture. Therefore, the conditional probability $X > 0$ is

$$\frac{3\pi/4}{\pi} = \frac{3}{4}$$

Solution to Question 1160: Three Repeat I

Note that the sequence must in the form $HHH--$, $-HHH-$, or $--HHH$. For the first form, the first dash must be T so that the run doesn't extend. The second dash can be anything, so this gives 2 options. For the second form, both dashes must be tails because the HHH is adjacent to both dashes, so this is just one possibility. The last sequence must have a T in the second dash so that the run doesn't extend. The first dash can be either outcome, so this gives 2 sequences as well. Therefore, we have 5 sequences corresponding to this outcome, so the probability is $\dfrac{5}{32}$.

Solution to Question 1161: Origin In Between

There are two possible ways for the origin not to be between the two randomly selected points: (1) both points fall within $(-1, 0)$, or (2) both points fall within $(0, 2)$. Case 1 occurs with probability $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$, while case 2 occurs with probability $\frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$. Hence, we know that the probability of the origin not falling between the two randomly selected points is $\frac{1}{9} + \frac{4}{9} = \frac{5}{9}$. Taking the complement, our answer is $\frac{4}{9}$.

Solution to Question 1162: Matching Jar Colors

We can get matching colors by either both being red or both being blue. The probability of two reds is $\dfrac{4}{7} \cdot \dfrac{4}{10} = \dfrac{16}{70}$. The probability of two blues is $\dfrac{3}{7} \cdot \dfrac{6}{10} = \dfrac{18}{70}$. Adding these up, we get our total probability is $\dfrac{16 + 18}{70} = \dfrac{17}{35}$.

Solution to Question 1163: Delta Decay

Our call option is currently in-the-money. So, at expiration, the call option will have $\Delta = 1$ if it is still in-the-money. Since the time-until-expiration is low, we would expect the delta decay to be about 0.26. In other words, our delta should increase by 0.26 in the next 4 hours assuming the price stays the same. If we are only worried about the next 2 hours, we would expect our delta to increase by 0.13. Since we are currently delta-hedged, we now gain a delta of 0.13 and thus we now have positive delta.

Solution to Question 1164: Team Division

There are 10 employees available to select for Marketing, as Michael and Nuo Wen do not want to be selected for it. We choose 5 of these 10 to be assigned to Marketing, which can be done in $\binom{10}{5}$ ways. Then, there are 7 employees remaining, of which 4 must be assigned to Content Development. The other 3 we don't select for Content Development would be assigned to Software Engineering, so there are $\binom{7}{4}$ ways to pick the people for Content Development. By the multiplication rule, the total number of ways to assign people is

$$\binom{10}{5}\binom{7}{4} = 8820$$

Solution to Question 1165: Secrets

We first derive a recurrence relation for p_i in terms of p_{i-1}. We must condition on whether or not P_i has the original information. If P_i has the original information, which occurs with probability p_{i-1}, then P_i transfers this information correctly to P_{i+1} with probability p. Alternatively, if P_i has the incorrect information, which occurs with probability $1 - p_{i-1}$, then the probability that P_{i+1} gets the correct information is $1 - p$, as P_i must change it up. Therefore

$$p_i = pp_{i-1} + (1-p)(1-p_{i-1}) = (2p-1)p_{i-1} + (1-p)$$

Now, we let $i \to \infty$ on both sides (which implicitly assumes the limit exists), and let $\lim_{i\to\infty} p_i = p^*$. Then $p^* = (2p-1)p^* + (1-p)$, so solving for p^*, $p^* = \frac{1-p}{2-2p} = \frac{1}{2}$

If you want to rigorously justify the existence of the limit, you can derive the explicit expression for p_i using recursive methods.

Solution to Question 1166: Numerous Uniforms

We solve this for general n and then plug in at the end. The trick here is to take the logarithm of both sides and note that if $X \sim \text{Unif}(0,1)$, then $-\log(X) \sim \text{Exp}(1)$. The reason we want to take the logarithm is because of the fact that the logarithm will turn a product into a sum. Therefore, we have that

$$\log(Y) = \log(X_1 \dots X_n) = \log(X_1) + \cdots + \log(X_n)$$

Multiplying each side by -1,

$$-\log(Y) = (-\log(X_1)) + (-\log(X_2)) + \cdots + (-\log(X_n))$$

Note that since each of the X_i are independent, the sequence of $-\log(X_i)$ are also independent. Since each one is Exp(1) distributed, then we have that the RHS is the sum of n IID Exp(1) random variables. Therefore, the

RHS is Gamma$(n, 1)$ distributed. We have that $-\log(Y) = G$, where $G \sim$ Gamma$(n, 1)$. In other words, the distribution of Y is therefore that of $Y = e^{-G}$, where $G \sim$ Gamma$(n, 1)$.

We don't have the CDF of G in explicit form, so we use fragmentation. Since $Y = f(G) = e^{-G}$, the inverse is just $h^{-1}(y) = -\log(y)$, so that $|(h^{-1}(g))'| = \frac{1}{g}$. Therefore, the PDF of Y is

$$f_Y(y) = \frac{(-\log(y))^{n-1}e^{\log(y)}}{\Gamma(n)} \cdot \frac{1}{y} I_{(0,\infty)}(-\log(y)) = \frac{(-\log(y))^{n-1}}{\Gamma(n)} I_{(0,\infty)}(-\log(y))$$

The indicator says that $0 < -\log(y) < \infty$, which $\log(y) < 0$, or that $0 < y < 1$. Therefore, $f_Y(y) = \frac{(-\log(y))^{n-1}}{\Gamma(n)} I_{(0,1)}(y)$. Plugging in $n = 7$, we have that $a = 6$ and $b = \Gamma(7) = 6! = 720$. This means $a + b = 726$.

Solution to Question 1167: Integral Variance I

Using integration by parts, we see that

$$\int_0^t W_s ds = sW_s\Big|_0^t - \int_0^t s dW_s = t\int_0^t dW_s - \int_0^t s dW_s = \int_0^t (t-s) dW_s$$

The second equality comes from writing $W_t = \int_0^t dW_s$. Now, if $f(t)$ is a deterministic square-integrable function, then the stochastic integral $\int_0^t f(s) dW_s$ is known to be normal with mean 0 and variance $\int_0^t |f(s)|^2 ds$. This is the Ito Isometry. This theorem applies here to the function $f(s) = t - s$, so the variance of the integral is just

$$\int_0^t (t-s)^2 ds = \frac{t^3}{3}$$

This means that $k = \frac{1}{3}$.

Solution to Question 1168: Radioactive Decay

The average rate of decay is 1 per second, so the time between decays is $T \sim$ Exp(1). We want $\mathbb{P}[T > 3]$. This is just $\int_3^\infty e^{-x} dx = e^{-3}$, so $a = -3$.

Solution to Question 1169: Goat Search

Let M and F represent the events that the first goats viewed are male and female, respectively. By Law of Total Expectation,

$$\mathbb{E}[N] = \mathbb{E}[N \mid M]\mathbb{P}[M] + \mathbb{E}[N \mid F]\mathbb{P}[F]$$

We know $\mathbb{P}[M] = \frac{3}{4}$, so $\mathbb{P}[F] = \frac{1}{4}$. Then, given the first goat selected is male, with probability $\frac{1}{4}$ on each trial after we get a female goat and then we leave. Therefore, the distribution of the number of trials needed to see the first female goat after the male is selected is Geom $\left(\frac{1}{4}\right)$, which has mean 4. Therefore, $\mathbb{E}[N \mid M] = 1 + 4 = 5$. We add the one in for the first goat (male) that is selected.

Similarly, if the first goat is female, the distribution of the number of goats needed to be selected until getting a male goat is Geom $\left(\frac{3}{4}\right)$, which has mean $\frac{4}{3}$. Therefore, $\mathbb{E}[N \mid F] = 1 + \frac{4}{3} = \frac{7}{3}$. Putting it all together, $\mathbb{E}[N] = 5 \cdot \frac{3}{4} + \frac{7}{3} \cdot \frac{1}{4} = \frac{13}{3}$.

Solution to Question 1170: Binomial Pricing a Binary Call

In order to price properly, we need to use risk-neutral probabilities. We solve the following equation to obtain the risk-neutral probability of up and down moves. In other words, our fair value is the expected future price.

$$q * 1.9 + (1 - q) * .3 = 1$$

Solving for q, we get $q = .4375$. We can then populate the payoffs at $T = 2$ and iterate backwards. Our underlying will either become $14.44, 2.28, .36$. 2 of which will have a payoff of 1 and the other a payoff of 0. Using these risk neutral probabilities, we can iterate backwards to get our expected future price. This value is 0.684. Another interpretation is that this is the risk-neutral probability that the underlying ends with a value larger than 2.

Solution to Question 1171: Dice Labels

Lets start with placing the number 1 on any side and let's consider it the top. Then thereâs 5 different numbers that can be on the opposite side of the dice.

Finally, with the 4 remaining numbers, there's 4! different orderings of the numbers but this counts the same orderings but starting on different faces so we divide by 4. Think about it this way, lets say we have 1 on the top and 6 on the bottom. The last 4 numbers are 2, 3, 4, and 5. There are 4! orderings of these numbers. But let's specifically consider the order $2, 3, 4, 5$. This order is the exact same as $5, 2, 3, 4$ and $4, 5, 2, 3$ and $3, 4, 5, 2$. As you can see, any order will have 4 different orderings and placements, so we divide by 4. Putting everything together, $5 * 4!/4 = 5 * 3! = 30$.

Solution to Question 1172: Statistical Test Review VII

Let p denote the probability that a person loves QuantGuide. We define

$$H_0 : p = 0.2$$
$$H_a : p = 0.15$$

Our test statistic is

$$Z = \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} = \frac{\hat{p} - 0.2}{0.04}$$

Z should be a standard normal random variable under the null hypothesis. Under the alternative hypothesis,

$$Z_a = \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} \approx \frac{\hat{p} - 0.15}{0.357}$$

Recall, by definition, that β is the probability that the test statistic, X, is not within the rejection region when H_0 is rejected. So,

$$\begin{aligned}\beta &= \mathbb{P}\left(\frac{10(\hat{p} - 1.5)}{0.357} > \frac{0.4 \cdot (-1.645 + 5) - 1.5}{0.357}\right) \\ &= \mathbb{P}\left(\frac{10(\hat{p} - 1.5)}{0.357} \leq 0.04426\right) \approx 0.671\end{aligned}$$

Solution to Question 1173: DeMorgan's Birthday

DeMorgan must have been born somewhere between 1751 and 1871. The only squares in this range are $42^2 = 1764$ and $43^2 = 1849$. However, to be 42 in $42^2 = 1764$, DeMorgan would have had to been born in 1722, which is outside our time range. Therefore, $x = 43$.

Solution to Question 1174: Conditional First Ace

We want to find the expected cards after the first 2 that we obtain the first ace. Therefore, our first task is to figure out how many cards that aren't ranked A and 2 show up between cards ranked A or 2 on average. We have 8 dividers, which are precisely all of the rank A and 2 cards. These dividers split our deck up into 9 regions. There are 44 cards left that are not ranked A or 2, so as we have that the regions have equal size in expectation, the expected length of each region is $\frac{44}{9}$. Now, we need to find the expected number of regions that appear between the first 2 and ace.

We now do first ace again on the dividers. We know that one of the 2s showed up already, so we only have 7 dividers left. We want to find the expected number of 2s before the first ace, so the aces are our dividers now. The 4 aces divide up our subset into 5 regions. We have 3 2s left, so there are on average $\frac{3}{5}$ 2s per region. However, we already know that one 2 appeared, so we must add 1 to the number of regions that there will be. Therefore, our expected number of regions is $\frac{8}{5}$. However, we also need to account for the dividers, as we did not count for them previously when computing $\frac{44}{9}$. The average number of dividers that appear after the first 2 is $\frac{3}{5}$, so we just add that in above.

Putting this all together, the expected number of cards between them is $\frac{44}{9} \cdot \frac{8}{5} + \frac{3}{5} = \frac{379}{45}$.

Solution to Question 1175: Options Delta

We can use put-call parity and take the derivative with respect to the underlying to find the relationship between call and put delta of the same strike.

$$C - P = S - Ke^{-rT}$$

Taking the derivative, we get:

$$\frac{\partial}{\partial S}(C - P = S - Ke^{-rT}) = \Delta_c - \Delta_p = 1$$

Plugging in $\Delta_c = 0.21$, we can see that $\Delta_p = -0.79$. This relationship will always hold for calls and puts of the same strike (assuming Black-Scholes dynamics). We can also look at this result intuitively.

If $|\Delta| \approx 1$, then we are deep in-the-money for the respective option as a higher delta means that it acts more similar to the underlying. This means that we are going to be deep out-of-the-money for the other option. As the $|\Delta|$ of one option approaches 1, the other will approach 0. We can also see that long calls must have positive delta while long puts must have negative delta (in-line with intuition of what calls and puts represent).

Solution to Question 1176: Identical Alpha

We first solve for α using the first test. We assume that the null hypothesis is true, and find the probability of rejecting the null hypothesis for the first test. In this case, X_1 is uniformly distributed (0,1) and thus:

$$\alpha = P(X_1 > 0.95 \mid \theta = 0) = 0.05$$

We move on to the second test. We assume that the null hypothesis is true, and note that both X_1 and X_2 are uniformly distributed (0,1). By convolution, $Y = X_1 + X_2$ has a density function of:

$$f_Y(y) = \begin{cases} y & 0 \leq y \leq 1 \\ 2 - y & 1 < y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Recalling that $\alpha = 0.5$, we can solve for c directly:

$$\alpha = P(X_1 + X_2 > c) = \int_c^2 (2 - y)dy = \frac{(c-2)^2}{2} = 0.05$$

$$c = 2 - \sqrt{0.10} \approx 1.68$$

$$(c \neq 2 + \sqrt{0.10} \text{ since } X_1 + X_2 \leq 2)$$

Solution to Question 1177: Pascal Ratio

Three entries (where the middle entry is the k-th entry of the n-th row) can be written as

$$\binom{n}{k-1}, \binom{n}{k}, \binom{n}{k+1}.$$

Simplifying, we have

$$\frac{n!}{(k-1)!(n-k+1)!}, \frac{n!}{k!(n-k)!}, \frac{n!}{(k+1)!(n-k-1)!}.$$

Now, we have a system of two equations:

$$4\frac{n!}{(k-1)!(n-k+1)!} = 3\frac{n!}{k!(n-k)!},$$
$$4\frac{n!}{(k+1)!(n-k-1)!} = 5\frac{n!}{k!(n-k)!}.$$

Let's simplify the first equation.

$$\begin{aligned} 4\frac{n!}{(k-1)!(n-k+1)!} &= 3\frac{n!}{k!(n-k)!} \\ 4k &= 3(n-k+1) \\ 4k &= 3n-3k+3 \\ 7k &= 3n+3 \end{aligned}$$

On to the second equation.

$$\begin{aligned} 4\frac{n!}{(k+1)!(n-k-1)!} &= 5\frac{n!}{k!(n-k)!} \\ 5(n-k) &= 4(k+1) \\ 5n &= 9k+4 \end{aligned}$$

Solving, we find $n = 62, k = 27$. Our answer is row 62.

Solution to Question 1178: Postgame

We solve this for general n. Let moving left be a movement of -1 and right be a movement of 1. Let X_i be his movement at minute i. We have that $X_1, \ldots, X_n$ are IID with PMF $\mathbb{P}[X_i = 1] = \mathbb{P}[X_i = -1] = \frac{1}{2}$. We have

$$D_n = \sum_{i=1}^{n} X_i$$

Thus,

$$\mathbb{E}[D_n^2] = \mathbb{E}\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] = \mathbb{E}\left[\sum_{i=1}^{n} X_i^2 + \sum_{i\neq j} X_i X_j\right] = \mathbb{E}\left[\sum_{i=1}^{n} X_i^2\right] + \mathbb{E}\left[\sum_{i\neq j} X_i X_j\right] = \sum_{i=1}^{n} \mathbb{E}[X_i^2] + \sum_{i\neq j} \mathbb{E}[X_i X_j]$$

We have that $\mathbb{E}[X_i^2] = \frac{1}{2}\cdot(-1)^2 + \frac{1}{2}\cdot(1)^2 = 1$, and since each X_i and X_j are IID, $\mathbb{E}[X_i X_j] = \mathbb{E}[X_i]\mathbb{E}[X_j] = 0$, as the expectation of each X_i is 0. The entire second sum goes to 0 then, and we have $\mathbb{E}[D_n^2] = \sum_{i=1}^{n} 1 = n$. In particular, $n = 10$, so our answer is 10.

Solution to Question 1179: Delta Decay II

At expiry, the Δ of ITM calls will be 1 and the Δ of OTM puts will be 0. Usually, delta-decay is non-linear, but we are dealing with an extremely small time-frame, so we can assume linearity in decay. This means that the delta of the call option will increase by $.28/2 = .14$ in 30 minutes and the delta of the put option will increase by $0.28/2 = .14$ in 30 minutes. Overall, our Δ will increase by $0.14 + 0.14 = 0.28$.

We were initially delta-hedged, but now we have a Δ of 0.28. To remain delta-hedged, we need to short 0.28 units of the underlying.

Solution to Question 1180: Prepped Offer?

Since we can see this is a conditional probability question as they give you information to try and find an associated value, let's use Bayes to solve this. We are solving for $\mathbb{P}[\text{Heavy User}|\text{Offer}]$. Bayes says that this value is equivalent to

$$\mathbb{P}[\text{Offer}|\text{Heavy User}] \cdot \mathbb{P}[\text{Heavy User}]/\mathbb{P}[\text{Offer}]$$

In the question, we arenât given any hard value on the probability of a person getting an offer given their usage of QuantGuide, only relational information. However, we can still use this information. Lets let p be the probability a non-user gets an offer. Then the probability that a light user gets an offer is $2p$ and the probability a heavy user gets an offer is $4p$. Thus, our Bayes equation becomes

$$4p \cdot 0.25/(4p \cdot 0.25 + 2p \cdot 0.3 + p \cdot 0.45) = \frac{20}{41}$$

Solution to Question 1181: Colorful Socks III

There are two cases to consider when getting a matching pair: We need to consider whether or not the color that matches is the color that has 3 replicates. This is because the probability that we draw another sock of that same color is different dependent on how many socks of that color are left. The probability that we select a sock on the first draw that just has a pair is $\frac{18}{21} = \frac{6}{7}$. Then, the probability that on the second draw we select a sock of that same color is $\frac{1}{20}$, as we assume this is a color with just the pair. This case yields a net probability of $\frac{6}{7} \cdot \frac{1}{20} = \frac{6}{140}$.

The other case is that we select the color with 3 socks, which occurs with probability $\frac{1}{7}$ on the first draw. The probability we draw another sock of that color is $\frac{2}{20}$, as 2 of the 20 remaining socks are of that color. This case yields a net probability of $\frac{1}{7} \cdot \frac{2}{20} = \frac{2}{140}$. Adding up the cases, our total probability of drawing a pair is $\frac{8}{140} = \frac{2}{35}$.

Solution to Question 1182: Extrinsic Value I

The intrinsic value of an option is known as the amount you would gain if you *could* exercise at this moment. If you could exercise now, you would gain $\max(S_T - K, 0) = 5 - 2 = 3$. This then leaves $4.7 - 3 = 1.7$ as the extrinsic value, or the value present in the option due to time (options with a longer time until expiry have more value, as these options have more potential to expire in-the-money and have real intrinsic value).

This gives us

$$3^2 + 1.7^2 = 11.89$$

Solution to Question 1183: Up or Down

If there was a price rise on k of the days, then for the price to be unchanged, we would need $(1.2)^k(0.8)^{10-k} = 1$, as the price went down the other $10 - k$ days. Equivalently, this means that $(1.2)^k = (1.25)^{10-k}$, which does not hold for an integer k. Therefore, it is impossible for this to occur.

Solution to Question 1184: Crossing the River

In order to get one children across the river, proceed as follows:

Two children both cross the river in the boat

One child returns with the boat

One soldier crosses in the boat

The child on the opposite shore returns with the boat

We are in the same situation as before, but with one soldier on the opposite bank. There is a total of 358 people to get across the river. It will therefore take $358 * 4 = 1432$ crossings to get all the men over, leaving the children together with the boat.

Solution to Question 1185: Red-Blue Die Match

Let the value of the red die be arbitrary. There are 2 ways to pick the blue die that matches it in value. The probability of a match is $\frac{1}{6}$. The probability the other die does not match the red die is $\frac{5}{6}$, so the total probability of exactly one match is $2 \cdot \frac{5}{6} \cdot \frac{1}{6} = \frac{5}{18}$.

Solution to Question 1186: Dice Order II

A brute force approach is possible, but a more analytical approach will be provided here. In order for the maximum to be exactly four, the maximum must be less than or equal to four, but not less than or equal to three. In order words:

$$P(\max = 4) = P(\max \leq 4) - P(\max \leq 3)$$

$$= \frac{4}{6} \times \frac{4}{6} \times \frac{4}{6} - \frac{3}{6} \times \frac{3}{6} \times \frac{3}{6}$$

$$= \frac{37}{216}$$

Solution to Question 1187: Coin Flipping Competition II

We know that $T, G, P \sim \text{Geom}\left(\frac{1}{2}\right)$ IID, as they are looking for the distribution of the first heads. As these are independent, we can multiply the individual PMFs to get the joint PMF, so the joint PMF is $\mathbb{P}[T = t, G = g, P = p] = \left(\frac{1}{2}\right)^t \left(\frac{1}{2}\right)^g \left(\frac{1}{2}\right)^p$ for $t, g, p = 1, 2, \ldots$.

We now need to get a region of summation for this probability. Let's let t be free, so we sum t from 1 to ∞. Then, we know $G \geq T$, so we sum over $g = t$ to ∞. After that, we know $P \geq G$, so we sum inner most from $p = g$ to ∞. Therefore, our sum is $\sum_{t=1}^{\infty} \sum_{g=t}^{\infty} \sum_{p=g}^{\infty} \left(\frac{1}{2}\right)^t \left(\frac{1}{2}\right)^g \left(\frac{1}{2}\right)^p$. As the inner most summation only concerns p, we ignore the rest for now. $\sum_{p=g}^{\infty} \frac{1}{2^p} = \frac{\frac{1}{2^g}}{1 - \frac{1}{2}} = \frac{1}{2^{g-1}}$. Now, our summation is $\sum_{t=1}^{\infty}\sum_{g=t}^{\infty}\frac{1}{2^t}\cdot\frac{1}{2^{2g-1}}=\sum_{t=1}^{\infty}\sum_{g=t+1}^{\infty}\frac{1}{2^{t-1}}\cdot\frac{1}{4^g}$. Ignoring the first term, as our sum only concerns g, $\sum_{g=t}^{\infty}\frac{1}{4^g}=\frac{\frac{1}{4^t}}{1-\frac{1}{4}}=\frac{4}{3}\cdot\frac{1}{4^t}$.

Now, our final summation is $\frac{8}{3}\sum_{t=1}^{\infty}\frac{1}{8^t}$ after shoving all the constants to the front. The last sum is simply $\frac{\frac{1}{8}}{1-\frac{1}{8}}=\frac{1}{7}$, so our solution is $\frac{8}{21}$.

Solution to Question 1188: Head Tail Sequence

There are $\binom{5}{3}=10$ sequences of length 5 containing exactly three heads. Therefore, the probability of this particular sequence is $\frac{1}{10}$, as all are equally likely.

Solution to Question 1189: Number 50

The prime factorization of $10^5=2^5\cdot 5^5$. This implies that to obtain a product that ends in 5 zeroes, we must have 5 of each of the digits 2 and 5 in our number. However, we only have 10 spots total. Therefore, this means our number must just have 5 of each of those two digits. This implies that we just need to count the ways to distribute 5 of each of 2 and 5 to 10 blanks. Since each are indistinguishable, choosing the locations of one type of digits fixes the other, so the answer is $\binom{10}{5}=252$.

Solution to Question 1190: Boys with Girls

Let $\mathbb{E}$ be the expected number of occurances where a boy and girl are standing next to eachother. To solve for a more general case, let m be the number of boys and n be the number of girls. There are $m+n-1$ total spaces a boy could be next to a girl. Thus $\mathbb{E}=(m+n-1)\cdot\mathbb{P}$ where $\mathbb{P}$ is the probability for each of these spots to have a boy next to a girl.

To find $\mathbb{P}$, if we know a boy is next to a girl, there are $\binom{n+m-2}{n-1}$ ways to arrange the rest of the boys and girls. We then need to multiply this value by

2 to account of the different orders of the selected couple (either BG or GB). There are a total of $\binom{n+m}{n}$ orderings for the boys and girls. Thus

$$\mathbb{P} = \frac{2\binom{n+m-2}{n-1}}{\binom{n+m}{n}} = \frac{2nm}{(n+m)(n+m-1)}$$

Given this value of $\mathbb{P}$, $\mathbb{E} = (m+n-1) \cdot \mathbb{P} = \frac{2mn}{m+n}$. With 15 boys and 10 girls, we get $\mathbb{E} = \frac{2 \cdot 15 \cdot 10}{15+10} = 12$

Solution to Question 1191: Shopping Habits

We are testing the null hypothesis $H_0 : \mu_1 = \mu_2$ against the alternative hypothesis $H_a : \mu_1 \neq \mu_2$. Because neither sample size is sufficiently large, we must use a t test with 38 degrees of freedom:

$$t = \frac{\bar{x_1} - \bar{x_2}}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \text{ where } S_p = \sqrt{\frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2}}$$

Solving for S_p:

$$S_p = \sqrt{\frac{(20-1) \times 22^2 + (20-1) \times 20^2}{20+20-2}} \approx 21.024$$

Solving for t:

$$t = \frac{78-67}{21.024\sqrt{\frac{1}{20} + \frac{1}{20}}} \approx 1.655$$

Because this is a two-tailed test, the attained significance level is double the p-value of a one-tailed test. Thus,

$$p\text{-value} = 2 \times P(t \geq 1.655) = 2 \times .05308 \approx 0.106$$

Solution to Question 1192: Familiar Neighbors

Let X_i be the indicator of the event that the person who knows exactly i other people, $0 \leq i \leq 2n-1$, knows at least one of their neighbors. Then $T = \sum_{i=0}^{2n-1} X_i$

gives the total number of people that know at least one of their neighbors. Therefore, the proportion of people that know their neighbors is $P = \dfrac{T}{2n}$. We use linearity of expectation to find $\mathbb{E}[T]$.

By Linearity of Expectation, $\mathbb{E}[T] = \displaystyle\sum_{i=0}^{2n-1} \mathbb{E}[X_i]$. Note that the X_i in this case do NOT have the same expectation, as the probabilities are different depending on how many people are known. Let's compute $\mathbb{E}[X_k]$ for any $0 \leq k \leq 2n-1$. $\mathbb{E}[X_k]$ is just the probability that the person with exactly k known people of the other $2n-1$ has at least one person on either side of them that they know. This is just the same as $1-$ the probability of the complement, which is that they know neither of the people sitting beside them. Since each arrangement is equally likely, the probability that they don't know the person sitting on one side of them is $\dfrac{2n-1-k}{2n-1}$. Given that they don't the person on one side, the probability they don't know the person on the other side either is $\dfrac{2n-2-k}{2n-2}$. Thus, $\mathbb{E}[X_k] = 1 - \dfrac{(2n-1-k)(2n-2-k)}{(2n-1)(2n-2)}$. Therefore,

$$\mathbb{E}[T] = 2n - \frac{1}{(2n-1)(2n-2)} \sum_{k=0}^{2n-1} (2n-1-k)(2n-2-k) = 2n - \frac{1}{(2n-1)(2n-2)} \sum_{k=0}^{2n-1} \big[(2n-1)(2n-2) - (4n-3)k +$$

$$= 2n - \frac{1}{(2n-1)(2n-2)} \left[2n(2n-1)(2n-2) - (4n-3)\sum_{k=0}^{2n-1} k + \sum_{k=0}^{2n-1} k^2\right]$$

Simplifying the sums, we get the above equal to

$$\frac{1}{(2n-1)(2n-2)} \left[(4n-3) \cdot \frac{(2n-1)(2n)}{2} - \frac{(2n-1)(2n)(4n-1)}{6}\right] = \frac{(4n-3)n}{2n-2} - \frac{n(4n-1)}{3(2n-2)}$$

Expanding the numerators and combining, we get $\dfrac{4n^2 - 3n - \frac{4}{3}n^2 + \frac{n}{3}}{2n-2} = \dfrac{\frac{8}{3}n^2 - \frac{8}{3}n}{2(n-1)} = \dfrac{4}{3}n$.

Thus, we have that $\mathbb{E}[P] = \dfrac{\frac{4}{3}n}{2n} = \dfrac{2}{3}$.

Solution to Question 1193: Mossel's Dice

First, imagine what such a sequence may look like, perhaps something like $242242\ldots6$ so we are equivalently asking what is the expected number of times we can roll a 2 or 4 until we roll some other number conditioned on that other number being 6.

More formally, let X be the expected number of rolls until we roll a number other than 2 or 4. Let l be equal to the last number, so we want $E[X \mid l = 6]$. Since the rolls are independent, X and l are independent of one another which means that $E[X \mid l = 6] = E[X]$ and now X is simply a geometric random variable parameter $p = \frac{2}{3}$ so $\mathbb{E}[X] = \frac{3}{2}$.

Solution to Question 1194: Random Subsets

Each element of X is likely to be in any of the four sets: $A \backslash B$, $B \backslash A$, $A \cap B$, $X \backslash (A \cup B)$. In order for A to be a subset of B, $A \backslash B$ must be empty. In other words, every element of X would have to be in any of the other three sets of the four sets. Thus, the probability that A is a subset of B is $\left(\frac{3}{4}\right)^5 = \dfrac{243}{1024} \approx 0.24$.

Solution to Question 1195: Bridge Crossing

The key point is to realize that the person returning with the lantern does not have to be from the most recent pairing. Another point to realize is that Alice and Bob should go together, and not in the first crossing- otherwise, one of them has to go back, which will take too long. Therefore, Charlie and Daniel should go across first, which takes two minutes. Then, send Daniel back with the lantern, which takes one minute. Alice and Bob go across, which takes ten minutes. Charlie returns with the lantern, which takes two minutes. Finally, Charlie and Daniel cross again, which takes two minutes, for a total of 17 minutes.

Solution to Question 1196: Wandering Ant I

To solve this, let $p(a, b)$ be the probability that starting from (a, b), the ant returns to the origin before hitting that square. We are going to set up some recursive equations based on the Law of Total Probability to relate all of these. In addition, we use the symmetry of our walk to simplify things greatly. We want $p(0, 0)$.

Starting from the origin, the ant moves to $(\pm 1, 0)$ or $(0, \pm 1)$ all with equal probability. However, by the symmetry of our square and our walk, $p(0, 1) = p(0, -1) = p(-1, 0) = p(1, 0)$. By Law of Total Probability,

$$p(0,0) = \frac{1}{4}(p(0,1) + p(0,-1) + p(1,0) + p(-1,0)) = p(1,0)$$

Now, we need to find $p(1,0)$. At $(1,0)$, if we move right one unit, we hit the boundary, in which case the probability is 0. If we move up or down one unit, we hit $(1,1)$ or $(1,-1)$. Again, by the symmetry of the square and our walk, $p(1,1) = p(1,-1)$. Lastly, if we move left one unit, we are back at the origin, so the probability is 1. Therefore, by Law of Total Probability and our previous relation,

$$p(1,0) = \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot p(1,-1) + \frac{1}{4} \cdot p(1,1) + \frac{1}{4} \cdot 1 = \frac{1}{2} \cdot p(1,1) + \frac{1}{4}$$

Now, we need to find $p(1,1)$. At $(1,1)$, if we move up or right, then we have hit the square, so our probability is 0. If we move left or down, we are at $(0,1)$ and $(1,0)$, respectively. By the symmetry of our square and the walk, $p(0,1) = p(1,0)$, so by Law of Total Probability and this relation, $p(1,1) = \frac{1}{2} \cdot p(1,0)$.

By substituting our expression of $p(1,1)$ into the equation for $p(1,0)$, we have that $p(1,0) = \frac{1}{2} \cdot \frac{1}{2} \cdot p(1,0) + \frac{1}{4}$. Solving for $p(1,0)$, $p(1,0) = \frac{1}{3}$. Since $p(0,0) = p(1,0)$ by our first equation, $p(0,0) = \frac{1}{3}$.

Solution to Question 1197: Statistical Test Review VI

Let p denote the probability that a person loves QuantGuide. We define

$$H_0 : p = 0.2$$
$$H_a : p < 0.2$$

For a Bernoulli random variable X with parameter p, recall that $\mu = p$ and $\sigma^2 = p(1-p)$. Let $\hat{p}$ denote the sample proportion. Since $n = 100$, our sample size is sufficiently large to assume that $\hat{p}$ is normally distributed with mean p and variance $p(1-p)$. Our test statistic is

$$Z = \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} = \frac{\hat{p} - 0.2}{0.04}$$

In order to reject the null hypothesis, we need our test statistic Z to be less than $z_{0.05} = -1.645$. With some simple algebra we determine $\hat{p} < 0.1342$. The maximum integer number of people that may love QuantGuide such that Andy's claim can be refuted is therefore 13.

Solution to Question 1198: Bakugan and Beyblade

The key to this question is the following realization: When a toy is produced at all, it is equally likely to be a Bakugan or a Beyblade. The reason is because of the memorylessness property and the fact that they have equal rates of production.

By the memorylessness property, the knowledge of when the last Bakugan/Beyblade was produced tells us nothing about when the next one will be produced, so we can just treat it as if we were looking at the probability the first Beyblade comes before the first Bakugan (or vice versa), which is just $\frac{1}{2}$. So now, we can treat this as a sequence of coin flips, where Bakugan is Heads and Beyblades is tails. Reframing this problem, this just asks for the probability the 3rd heads comes before the second tails. This will always be decided after the 5th flip. Thus, we really want the number of outcomes where there are at least 3 Heads in the first 4 coin tosses. The distribution of the number of heads in the first 4 coin tosses is $N \sim \text{Binom}(4, 0.5)$, so we want $\mathbb{P}[N = 3] + \mathbb{P}[N = 4] = \frac{5}{16}$.

Solution to Question 1199: Positive Brownian I

The trick here is to write $W_2 > 0$ as $W_2 - W_1 > -W_1$. This is because now $W_2 - W_1$ is independent of W_1. Doing this, we have that

$$\mathbb{P}[W_1 > 0, W_2 - W_1 > -W_1] = \mathbb{P}[W_2 - W_1 > -W_1 \mid W_1 > 0]\mathbb{P}[W_1 > 0]$$

The latter term is just $\frac{1}{2}$ as $W_1 \sim N(0, 1)$ which is symmetric about 0. For the former term, we can note that in the plane, this region covers the entire right half of the plane except below $y = -x$. As normal random variables have radial symmetry, the region below $y = -x$ is $\frac{\pi}{4}$ radians of an entire π radians in the right half of the plane. Therefore, the probability of being below that is $\frac{1}{4}$, meaning the conditional probability in question is $1 - \frac{1}{4} = \frac{3}{4}$. Therefore, our answer is

$$\frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$$

Solution to Question 1200: No More Than Four

Let X denote the value of the sum of the two dice faces. Let $p(x)$ denote the pmf of X. Note the following:

$$p(x) = \begin{cases} \frac{1}{36} & \text{if } x = 2 \\ \frac{2}{36} & \text{if } x = 3 \\ \frac{3}{36} & \text{if } x = 4 \\ \frac{4}{36} & \text{if } x = 5 \\ \frac{5}{36} & \text{if } x = 6 \\ \frac{6}{36} & \text{if } x = 7 \\ \frac{5}{36} & \text{if } x = 8 \\ \frac{4}{36} & \text{if } x = 9 \\ \frac{3}{36} & \text{if } x = 10 \\ \frac{2}{36} & \text{if } x = 11 \\ \frac{1}{36} & \text{if } x = 12 \\ 0 & \text{otherwise} \end{cases}$$

The event that the sum of the values on the two dice faces is at most 7 for one toss is $\mathbb{P}(X \leq 5) = \sum_{x=2}^{4} p(x) = \frac{1}{6}$. The complement is then $\mathbb{P}(X > 5) = \frac{5}{6}$. Since the sum from each of the n two-dice-tossing rounds is independent from the sums of other rounds, the probability that there is no sum less than or equal to 4 after n rounds is $\left(\frac{5}{6}\right)^n$. The probability that there is at least one sum less than or equal to 4 after n rounds is then $1 - \left(\frac{5}{6}\right)^n$.

$$\begin{aligned} 1 - \left(\frac{5}{6}\right)^n &< 0.5 \\ \left(\frac{5}{6}\right)^n &> \frac{1}{2} \\ \left(\frac{5}{6}\right)^4 < \frac{1}{2} &< \left(\frac{5}{6}\right)^3 \\ \Rightarrow n &= 3. \end{aligned}$$

Solution to Question 1201: Square Shade

Our 2023×2023 grid is essentially split into two 1011×2023 mini grids. Let's first determine how many rectangles we can make from a 2023×2023 grid. We must choose two rows of grid lines from each side to mark the boundaries of the rectangle, so we can make a total of $\binom{2024}{2} \cdot \binom{2024}{2}$ rectangles. Similarly, there are a total of $\binom{1012}{2} \cdot \binom{2024}{2}$ rectangles that can be made from each 1011×2023 mini grid. Our probability is then $\frac{2 \cdot \binom{1012}{2} \cdot \binom{2024}{2}}{\binom{2024}{2} \cdot \binom{2024}{2}} = \frac{1011}{2023}$. The complement and answer is $\frac{1012}{2023}$.

Solution to Question 1202: Covariance Review III

Recall

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

Let's begin with $\mathbb{E}[XY]$. We have

$$\begin{aligned}\mathbb{E}[XY] &= \mathbb{E}[Z^3] \\ &= \int_{\mathbb{R}} \frac{z^3}{\sqrt{2\pi}} e^{-z^2/2}\, dz\end{aligned}$$

Note that we may avoid this tedious computation. For each value of z such that $z > 0$, $f_Z(z) = f_Z(-z)$, since the standard normal is symmetric about $z = 0$. More informally, all positive values of z plugged into the even function $\frac{z^3}{\sqrt{2\pi}} e^{-z^2/2}$ should cancel out with all negative values of z plugged into the same even function, leaving us with $\mathbb{E}[XY] = 0$.

Next, we know that $\mathbb{E}[X] = \mathbb{E}[Z] = 0$. Finally, we need to compute $\mathbb{E}[Y] = \mathbb{E}[Z^2]$. We can again avoid a tedious integral by utilizing what we know about the variance.

$$\begin{aligned}\mathrm{Var}(Z) &= 1 \\ \mathbb{E}[Z^2] - (\mathbb{E}[Z])^2 &= 1 \\ \mathbb{E}[Z^2] - 0 &= 1 \\ \mathbb{E}[Z^2] &= 1.\end{aligned}$$

Our final step is to plug everything into our covariance expression.

$$\begin{aligned}\mathrm{Cov}(X,Y) &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\ &= 0 - (-1)\cdot 0 \\ &= 0\end{aligned}$$

It is important to note that if $\mathrm{Cov}(X,Y) = 0$, then it is not necessarily the case that X and Y are independent. Recall that the rule "If X and Y are independent, then $\mathrm{Cov}(X,Y) = 0$" is not an if-and-only-if statement.

Solution to Question 1203: 1 Glove Off

We are going to solve the more general case with n pairs of gloves labelled $1-n$. Note that there are $\dfrac{(2n)!}{2^n \cdot n!}$ ways to pair up the $2n$ gloves, as there are $(2n)!$ total arrangements, $n!$ ways to re-label the pairs, and then 2 ways to switch around the order in each pair. Let g_n be the number of arrangements that satisfy our condition. Consider the two gloves labelled n, say n_1 and n_2. Decide the partner for n_1 first. We either have that n_2 is paired with n_1, in which case, we go back to the same problem but with $n-1$ pairs of gloves instead of n. Otherwise, n_1 is paired with one of the gloves labelled $n-1$, of which there are 2 ways to pick that glove. Afterwards, we know that n_2 is paired with the other glove labelled $n-1$, and that becomes fixed. Then, this goes back to the same problem but with $n-2$ pairs of gloves instead. Therefore, we get the recurrence relation

$$g_n = g_{n-1} + 2g_{n-2}$$

Remember that the 2 in front of g_{n-2} represents the fact that we have 2 options of the glove labelled $n-1$ to match with n_1 in that sub-case. We now need some initial conditions. Note that $g_1 = 1$, as there is clearly only one pair. Furthermore, we have $g_2 = 3$, as we can pick the partner for any one of the gloves in 3 ways, and that fixes the other pair immediately. The characteristic equation of this recurrence relation is $r^2 - r - 2 = 0$, of which the solutions are $r = 2, -1$. Therefore, $g_n = c_0 \cdot 2^n + c_1 \cdot (-1)^n$. Plugging in the initial conditions yields that $1 = 2c_0 - c_1$ and $3 = 4c_0 + c_1$. Solving these yields that $c_0 = \dfrac{2}{3}$ and $c_1 = \dfrac{1}{3}$. Therefore,

$$g_n = \frac{2^{n+1} + (-1)^n}{3}$$

Therefore, the probability of this event occurring with n pairs is given by

$$p_n = \frac{g_n}{\frac{(2n)!}{2^n \cdot n!}} = \frac{(2^{n+1} + (-1)^n) \cdot 2^n \cdot n!}{3(2n)!}$$

Substituting in $n = 5$, we get that $p_5 = \dfrac{1}{45}$.

Solution to Question 1204: Magic 11

Any integer can be written in the form $a + 10b$, for $a \in \{0, \dots, 9\}$ and any integer b. Let $x = a + 10b$. Then,

$$x^3 = a^3 + 30a^2b + 300ab^2 + 1000b^3.$$

Notice that the right hand side can only end in 1 if $a = 1$. Then, we can solve for b.

$$\begin{aligned} 30b + 1 &\equiv 11 \mod 100 \\ 30b &\equiv 10 \mod 100 \\ 3b &\equiv 1 \mod 100 \end{aligned}$$

The value of b must end with the digit 7. We conclude that any value of x whose final two digits are 71 will be magical. This occurs with probability $\frac{10^{98}}{10^{100}} = 0.01$.

Solution to Question 1205: Side Add

Let R be the value of the second roll. We want $\mathbb{E}[R]$. Clearly the values R can take depend on the value of the first roll, so we should condition on X_1, the first roll value. Thus, $\mathbb{E}[R] = \mathbb{E}[\mathbb{E}[R \mid X_1]]$. $\mathbb{E}[R \mid X_1]$ is the expected value of a fair $(6+X_1)-$sided die, which is $\dfrac{(6+X_1)+1}{2} = \dfrac{7+X_1}{2}$. Therefore, we have

$$\mathbb{E}[R] = \mathbb{E}\left[\frac{7+X_1}{2}\right] = \frac{7}{2} + \frac{1}{2}\mathbb{E}[X_1] = \frac{7}{2} + \frac{7}{4} = \frac{21}{4}$$

In the last line, we use the fact that the average value of a fair die roll is $\dfrac{7}{2}$.

Solution to Question 1206: Choose Your Profit

We want our expected payout to be constant no matter how many times we play. Therefore, as our initial payout is \$1 i.e. our payout if we don't even play the game further, then we want to find α so that our expected payout is \$1 regardless of the number of times we play. By rolling n times, we either have a payout of α^n or 0. The probability we get α^n is just the probability we have not observed a 1 in the first n rolls. The probability of this is $\left(\dfrac{5}{6}\right)^n$ because we have probability $\dfrac{5}{6}$ on each independent roll of not rolling a 1. The term with payout 0 is irrelevant to our calculation, so we want to solve $\alpha^n\left(\dfrac{5}{6}\right)^n = 1$, which means $\alpha = \dfrac{6}{5}$.

Solution to Question 1207: 12 Left

There are 7 even values from $1-15$. For B to be able to select Chip 12, A must have not selected it. Given that A selects 5 of the 7 even values, the probability that they selected Chip 12 is $\dfrac{5}{7}$, so the probability A did not select it is $\dfrac{2}{7}$. After A is done selecting, there are now 10 chips remaining in the bag. As B selects 5 of these 10, the probability that B selects Chip 12 is $\dfrac{1}{2}$, so our answer is $\dfrac{2}{7} \cdot \dfrac{1}{2} = \dfrac{1}{7}$.

Solution to Question 1208: Equal Unequal Game

Let A be the event that Abby wins the game. Abby either gets the head on her first flip, both her and Ben get tails on the first flip and Abby gets it on her second flip, etc. The probability that Abby gets the head on the kth flip means that both her and ben obtain heads for the first $k-1$ flips and then Abby flips a head on her kth flip. The probability of this is $p\left(\frac{1-p}{2}\right)^{k-1}$. This is because of the independence of each of their flips and taking the complement of their respective heads probabilities. We now just sum this up from $k=1$ to ∞ to get

$$\mathbb{P}[A] = \sum_{k=1}^{\infty} p\left(\frac{1-p}{2}\right)^{k-1} = \frac{p}{1-\frac{1-p}{2}} = \frac{2p}{1+p}$$

We want this probability to be equal to $\frac{1}{2}$, as we want Abby and Ben to have equal winning probabilities. Therefore, we must find p such that $\frac{2p}{1+p} = \frac{1}{2}$. Solving this yields $p = \frac{1}{3}$.

Solution to Question 1209: Careful Side Choice

The optimal strategy is going to be the same for both players due to the symmetry of the game. We also immediately know that $s \geq 3$ because for $s = 1, 2, 3$, regardless of what is rolled by each player, you will receive your payout. Therefore, there is no benefit to select $s = 1$ or 2.

What we need here is a Nash Equilibrium. In other words, neither player can improve their expected earnings by increasing the value on their die. Therefore, suppose the equilibrium is $(3, 3)$. If this was the case, then if your opponent changes their die to 4 sides, they shouldn't be any better off. Clearly the expected value for both players with $3-$sided dice each is 2, as they receive payment regardless. However, if your opponent changes to a $4-$sided die, they now have an expected payout of 2.5, as they will still get the money regardless of what is rolled. Therefore, $s \geq 4$ now.

Now, we need to compute the same for $(4, 4)$ and $(4, 5)$. If both players select 4, the expected payout for each player is

$$1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{3}{16} = \frac{9}{4}$$

This is because the outcome where both roll a 4 is now worth 0, so when you roll a 4, there are only 3 possible outcomes of the other player where you get that money. In the case $(4, 5)$, the expected payout for the player rolling a 5 is

$$1 \cdot \frac{1}{5} + 2 \cdot \frac{1}{5} + 3 \cdot \frac{1}{5} + 4 \cdot \frac{3}{20} + 5 \cdot \frac{2}{20} = \frac{23}{10} > \frac{9}{4}$$

Therefore, if you select 4 sides, the other player would select more than 4 sides and have a higher expected payout. Therefore, $s = 5$ or 6 now.

We now need to compute this for $(5, 5)$ and $(5, 6)$. If both players select 5, the expected payout for each is

$$1 \cdot \frac{1}{5} + 2 \cdot \frac{1}{5} + 3 \cdot \frac{4}{25} + 4 \cdot \frac{3}{25} + 5 \cdot \frac{2}{25} = \frac{49}{25}$$

If you select 5 and the other player selects 6, then your opponent's expected payout is

$$1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{4}{30} + 4 \cdot \frac{3}{30} + 5 \cdot \frac{2}{30} + 6 \cdot \frac{1}{30} = \frac{11}{6} < \frac{49}{25}$$

This means that the other player is not better off by increasing the number of sides on their die.

The above implies that $(5, 5)$ is our equilibrium, so $s = 5$ and $p = \frac{49}{25}$. The product $sp = \frac{49}{5}$.

Solution to Question 1210: Cheese Lover III

We subtract the expectation of $T_{100} = 25000$ from both sides, then divide by $\mathrm{Var}(T_{100}) = 100(250)^2$ (by recognition of using linearity of variance), implying that $\sigma_{T_{100}} = \sqrt{100(250)^2} = 250\sqrt{100} = 2500$. Thus. we get that

$$\mathbb{P}[T_{100} > 26000] = 1 - \mathbb{P}[T \leq 26000] = 1 - \mathbb{P}[T_{100} - 25000 \leq 26000 - 25000] = 1 - \mathbb{P}\left[\frac{T_{100} - 25000}{2500} \leq \frac{1000}{2500}\right]$$

We approximate the LHS using a standard normal random variable by the CLT, so we are using the approximation $1 - \mathbb{P}\left[Z \leq \frac{2}{5}\right]$, where $Z \sim N(0, 1)$. This last quantity is $1 - \Phi(\frac{2}{5}) = \Phi(-\frac{2}{5})$. Therefore, $a = -2/5$.