Optiver

# 1 Probability

## What you need to know

A working knowledge of probability is essential to working on or even near a trading floor. Instead of providing a list of topics in this section, we will instead give you some exercises. If you can understand the language and *attempt* the exercises, then you are on the right track.

I would strongly encourage all candidates to *think about and mentally play out* each question, estimating the answer before jumping to direct calculation. In trading, *probabilistic intuition* is almost just as important as being able to get the exact answer, for if you can get extremely close in a fraction of the time, you can ultimately process more opportunities on the trading floor.

So, for each of the following exercises, first write down the best guess you can come up with through the act of mental experimentation. Then, try and solve the problem exactly in your head without touching pen to paper. Finally, use all the tools you have at your disposal to check how you went, making sure to reflect on where your intuition was off and why.

**Exercise 1.** *Consider a standard deck of 52 cards. You randomly draw 4 cards without replacement. What is the probability that you get one of each suit (♣, ♦, ♠, ♥)?*

**Exercise 2.** *How many times do you expect to roll a dice for it to land on a 3?*

**Exercise 3.** *Consider a standard deck of 52 cards. You will draw a card randomly 52 times, replacing the card and shuffling between draws. What's the probability that you will not get a King?*

**Exercise 4.** *I'm going to roll two dice but before I do, you can choose to receive either the maximum value of the two dice or twice the minimum. Which do you choose and why?*

**Exercise 5.** *There are four teams (A, B, C and D) in a competition. To determine the best team, A will play B, and C will play D, and then the winners of these two games will play each other in the Grand Final. An opportunity arises where you can back Team A for $20 and if they end up winning the Grand Final you will receive $100 back. The probability that A beats B is twice the probability that B beats A. Team C and D are equally likely to win against each other. In the Grand Final, Team A would be an even match against Team C but would only have a 1 in 4 chance of beating Team D. Would you take the opportunity?*

**Exercise 6.** *Consider a box of 39 balls. Of these, 38 are green and the remaining one is red. You reach into the box and pull out a ball at random. If it is green, you keep the ball out of the box and grab another ball. You keep doing this until you pull out the red ball, in which case the game ends. How many balls do you expect will be left in the box when the game ends?*

# 2 Market Making

## Introduction

We all make trades every day. Buying a coffee is a trade. Selling the exercise bike you bought yourself for Christmas but never used is a trade. Waking up early to watch a game of football is a trade (you are trading sleep for entertainment).

At Optiver, we trade financial instruments. Importantly, we act as a *market maker*: we provide both buy and sell prices for the market participants.

Let's pretend that there is some instrument out there called $X$. We can plug ourselves into the exchange where $X$ is traded and have a peek at the market. It might look something like this:

| Buyers | PRICE | Sellers |
|---|---|---|
| | 16 | 3 |
| | 15 | 2 |
| | 14 | 4 |
| | 13 | |
| | 12 | |
| 17 | 11 | |
| 5 | 10 | |
| 11 | 9 | |

This is called an *order-book*. In the middle column, there is a fixed list of prices that $X$ can trade at. These are listed in descending order.

On the left of each price, you can see a number denoting how many lots of $X$ are desired by buyers at that price. On the right of each price, you can see the number of $X$ that is wanting to be sold at that price.

So, there could be 17 different people that want to buy one lot of $X$ for \$11. Or it might be one single person who wants to buy 17 lots of $X$. Or it might be something in between these scenarios.

There are 4 lots of $X$ waiting to be sold at \$14. Again, this might be two different participants out there. For example, there could be one person wanting to sell one lot and another person wanting to sell 3 lots.

**Exercise 7.** *Consider the order-book above. If you had to choose a single price for everybody to do their trades, what would you choose?*

So, in short, there is a marketplace for $X$ consisting of people who want to buy and people who want to sell. The problem? Right now, as it stands, if a buyer wants to buy then they need to pay \$14. If a seller wants to sell, they will need to sell at \$11.

If these trades actually take place, then some amount of $X$ trades at \$14 and some amount of $X$ trades at \$11. These are quite large swings in price. In fact, seeing these quick price changes might even deter some participants from trading, for large price swings often indicate a volatile asset.

Also, what if somebody wanted to buy 8 lots of $X$? They would have to pay up through multiple levels.

This is precisely a situation where a market maker is needed. After having a squiz at this market, a trader at Optiver decides to provide some *liquidity*. Using a slither of whiz-bang mathematics, she works out that her fair theoretical value for $X$ is actually about \$12.50. This is what she *believes* it to be worth.

Our trader then *makes a market* by inserting both buy and sell prices into the exchange.

**Making a market involves providing both a buy and a sell price for an instrument.**

She inserts a buy order for 10 lots of $X$ at \$12 and a sell order for 10 lots of $X$ at \$13:

Our trader should be pretty happy if she trades. She believes that $X$ is worth \$12.50 and so is happy to buy it at \$12 or sell it for \$13. Indeed, she would expect to make a theoretical profit of 50 cents per lot that she trades.

The market participants are happy here too. This market is now more *liquid*. A buyer can buy it immediately for \$13 and a seller can sell it immediately for \$12. The market is now *a fairer place* for its participants.

Notably, as the participants trade, the market maker generates profits as she buys at \$12 and sells at \$13 repeatedly.

| Buyers | PRICE | Sellers |
|---|---|---|
| | 16 | 3 |
| | 15 | 2 |
| | 14 | 4 |
| | 13 | 10 |
| 10 | 12 | |
| 17 | 11 | |
| 5 | 10 | |
| 11 | 9 | |

**Exercise 8.** *What if our trader thought that the fair theoretical value for $X$ is \$17? What should her market be?*

**Exercise 9.** *What if our trader thought that the fair theoretical value for $X$ is \$8? What should her market be?*

## Risks

There are a few risks that might spring to mind from this simple market making example.

What if a buyer immediately buys the 10 lots our trader is offering at \$13. Our trader will then have a *short* position in the instrument $X$. If $X$ increases in value, our trader will be losing money. Don't forget that markets can move, and move quickly too!

**Exercise 10.** *If our trader sells 10 lots at \$13 and then the market moves so that she now believes that the fair price for $X$ is \$17, what is her theoretical P/L (profit/loss)?*

In our example, it's not that unlikely that our trader finds it easy to sell $X$ but hard to buy. Go back and have a glance at the market for $X$. Something observable is at play here: demand. There actually does seem to be more buyers than sellers in this market. This can create upwards pressure on $X$, something that a purely mathematical approach may not always take into account.

**Exercise 11.** *What other risks are involved in market making?*

## Fermi Market Making

Optiver's focus is on improving financial markets but good practice can actually be developed by making markets on the solutions to *Fermi problems*.

These are named after the physicist Enrico Fermi who was quite fond of estimating unknown quantities with little information. The most striking example of this is the time he dropped some paper at a (safe) distance from an atomic blast and used the distance the paper travelled to (quite accurately!) estimate the energy released from the bomb.

Have a go at the following Fermi problem as a warm-up.

**Exercise 12.** *How many people are on an airplane in the sky right now?*

In this section, we will consider the act of making a market on a Fermi problem. Let's say that a friend of yours taps you on the shoulder and gives you the following seemingly strange request:

*Make me a market on the perimeter of Tasmania.*

You are immediately baffled, so let's unpack this a bit. The perimeter of Tasmania is the total length of the boundary. Your friend wants you to provide a buy and sell price on the this length.

You're still a touch confused, so your friend explains further.

*I want you to give me two numbers $B$ and $S$. The first number $B$ is your buy price and this should be below what you think the perimeter of Tasmania is. The second number $S$ is your sell price and this should be above what you think the perimeter of Tasmania is.*

OK, it's getting clearer. Your friend is trying to get you to come up with an estimate of this distance but she doesn't want your estimate. She just wants two numbers on either side of it.

You think for a bit. After a while you come up with an estimate of about 1200km. You then jump 400km each way from your estimate and tell your friend that you have chosen $B = 800km$ and $S = 1600km$.

Congratulations - you have just made a market. If your friend thinks that the true answer is less than 800km, then she will sell you 800km. After all, you have shown her a buy price there so she is well within her rights to do this! On the other hand, if she thinks the answer is greater than 1600km then she will buy this from you.

Your friend opens her mouth and says:

*I will buy 1600km from you.*

You and your friend have just done a trade! She has bought "the perimeter of Tasmania" from you at the price of 1600km.

Note that you should both be happy with your trades. You have sold at a price of 1600km when you believe the answer to be around 1200km. Your friend has bought at a price of 1600km when she believes the answer to be greater than 1600km.

Of course, only one of you can be correct. Let's proceed to *settlement*; this is where the solution to our Fermi problem is revealed. In this case, a quick check on `Google` gives the (approximate) answer of 1400km.

You sold at a price of 1600km and the thing you sold settled at 1400km. Therefore, you have made a profit of 200km. Your friend has made a loss of 200km.

**Note:** It certainly seems odd to talk about profits in terms of kilometres. If you and your friend had done this properly, you likely would have agreed upon some conversation rate (e.g. \$1 per 10km).

## Markets and Confidence

Suppose I ask you to make me markets on the following:

- The world record for the longest distance (in metres) a golf ball has been thrown.
- The total number of words in all seven of the (English) Harry Potter books.
- The probability that two randomly chosen positive integers have no factors in common.

In each case, you first need to come up with an estimate. Traders *must* be able to do this. And not only that, but your estimates must be sane. If you say that the world record for the golf ball throw is 2km or you say that the total number of words in all seven of the Harry Potter books is 400, then something is clearly wrong.

Once you come up with an estimate or *fair value*, you need to provide your buy price $B$ and sell price $S$. But how to pick these?

If you think that the world record for the golf ball throw is 150m, then here are two possible markets you might provide:

| | | |
|---|---|---|
| Market 1: | $B$ = 0m | $S$ = 300m |
| Market 2: | $B$ = 149m | $S$ = 151m |

Neither of these is great. Consider walking into a room (or the Optiver trading floor) where everyone wants to trade on the world record for the longest distance a golf ball has been thrown. In fact, they have been waiting for a market maker like you to provide them with liquidity!

If you announce Market 1 to everybody, you will likely do zero trades. Nobody out there wants to sell 0m. Maybe a couple of people will buy 300m. But really, your market is *too wide*, and as such does not provide any real liquidity.

On the other hand, if you announce Market 2, you will probably do a lot of trades! If someone thinks that the answer is smaller than 149m or larger than 151m then they will trade with you. This is going to encapsulate basically everyone in the room. So, well done: you've provided some real liquidity! The problem is risk.

The ideal situation with Market 2 is that half the people in the room sell you 149m and the other half of the room buy 151m. You neatly offset every buy with a sell and you generate profit (in the units of metres but a win's a win, right?). Every buy that *you* have done at 149m is neatly covered with a sell that *you* have done at 151m. You make a profit of 2m (the width of your market) for every buy-sell pair.

However, as your $(B, S)$ interval is so *tight*, it is much more likely that the consensus of the room falls to the side of your interval. To put some numbers to it, let's say that there are 10 people in the room and that 4 of them sell to you and the other 6 buy from you.

Your four buys at 149m pair up nicely with four of your sells at 151m. These eight trades get you a profit of $4 \times 2\text{m} = 8\text{m}$. But there are two more sells that you have done at 151m that you have not covered on the buy side.

If the answer turns out to be 180m, then you will lose 29m on each of these sells at 151m. These losses massively override your profits and your golf-ball market making career is sadly over.

The above example demonstrates why Market 2 is not great. The amount that you are *getting paid to trade* (a function of your width) is too small relative to the uncertainty in the underlying value.

**The width of your market should be related to the uncertainty of the thing you are trading.**

Sometimes, we have information, and this should change our market. Also, perhaps we are constrained in some way.

**Exercise 13.** *Think about different information and constraints and how these can affect the width and position of your market.*

To end this section, have a think about the following scenario.

**Exercise 14.** *What if you had walked into the trading room late in the proceedings and watched for a little bit. Let's say you just saw trading happen at these prices, in this order:*

$$149m, 152m, 160m, 155m, 150m, 149m, 153m, 158m.$$

*What market would you make?*

# 3 Options

## Introduction to Options

At Optiver, trading options is our bread and butter. These financial instruments have a rich history, but in wanting to keep this a note instead of a novel, we will stick with the following very simple definition.

**An option is a contract that locks in the *option to trade* at a fixed price and date for an underlying asset.**

***Call* options lock in a *buy* price.**

***Put* options lock in a *sell* price.**

From the definition, we see that options can be used to give certainty to market participants.

Let's say that you and a friend have done some research into a company and you both decide to buy a portion of shares. You each buy 100 shares at \$100 each. Over the next few years, the company does pretty well; in fact, the new stock price is \$150.

One day your friend shows you some troubling calculations that could really put a dampener on the company's future. In fact, he believes that the stock price could halve in the next few months, putting you both at an overall loss.

You and your friend handle this differently. You decide to buy *\$130 put options with a four month expiry*. These instruments allow you to *sell* your shares for \$130 at a fixed time four months from now. If it happens that the stock price has collapsed in that time period, then your options give you the right to sell them at a much higher price, protecting you from financial disaster.

Your friend sells his shares and locks in his profit. But he uses some of the profit to buy *\$150 call options with a four month expiry*. These options allow him to *buy* his shares back at \$150 four months from now.

Both you and your friend have bought options to reduce your financial variance.

## Exercises

The best way to familiarise yourself with options is to do a few exercises. However, first, take note of the following notations.

**The price you lock in is called the strike price.**

**The date you lock in is called the expiration date or maturity.**

**Using the option to buy or sell the underlying asset is known as exercising the option.**

We will do the first two exercises together.

**Exercise 15.** *The stock AAPL is currently trading at \$141. You buy 100 call options (each lets you buy one stock) with a strike of \$150 and an expiration date of one month away. These call options cost you \$1.50 each. What is your P/L if AAPL is trading at \$160 in one month's time.*

**Solution 1.** *You spend* $100 \times \$1.50 = \$150$ *on the 100 options. Clearly, as AAPL is trading above the strike price at maturity, you want to exercise these options. This means you get to buy 100 lots of AAPL at \$150. As each share is trading at \$160, you have effectively made* $100 \times \$10 = \$1000$ *on exercising these options. Therefore, your total P/L is* $\$1000 - \$150 = \$850$.

**Exercise 16.** *A friend of yours lets you play a game where she rolls a dice and pays you an amount in dollars equal to the face that shows.*

*(a) What is the fair price for this game?*

*(b) What is the fair price for the call option on this game with a strike of 4?*

**Solution 2.** *(a) The outcomes of this game are the integers from 1 to 6 each occurring with a probability of $\frac{1}{6}$. The fair value is therefore*

$$\frac{1}{6} \times 1 + \frac{1}{6} \times 2 + \frac{1}{6} \times 3 + \frac{1}{6} \times 4 + \frac{1}{6} \times 5 + \frac{1}{6} \times 6 = 3.5.$$

*(b) This call option lets you buy the result at a fixed price of 4. You would only want to do this if the result is greater than 4, so a 5 or a 6. The respective profits would be 1 (buying for 4 and receiving 5) and 2 (buying for 4 and receiving 6). As these two outcomes each have a probability of $\frac{1}{6}$, the fair price for the call option is*

$$\frac{1}{6} \times 1 + \frac{1}{6} \times 2 = 0.5.$$

Now some exercises for you to try on your own.

**Exercise 17.** *The stock ZZA is currently trading at \$27. You sell 100 call options with a strike of \$30 and an expiration date of two months away. These call options net you \$0.50 each. What is your P/L if ZZA is trading at \$31 at maturity?*

**Exercise 18.** *In the dice rolling game of Exercise 2, what is the fair price for a put option with a strike of price of 4.*