Mathematics of Derivative Pricing

Summer of Science

Report

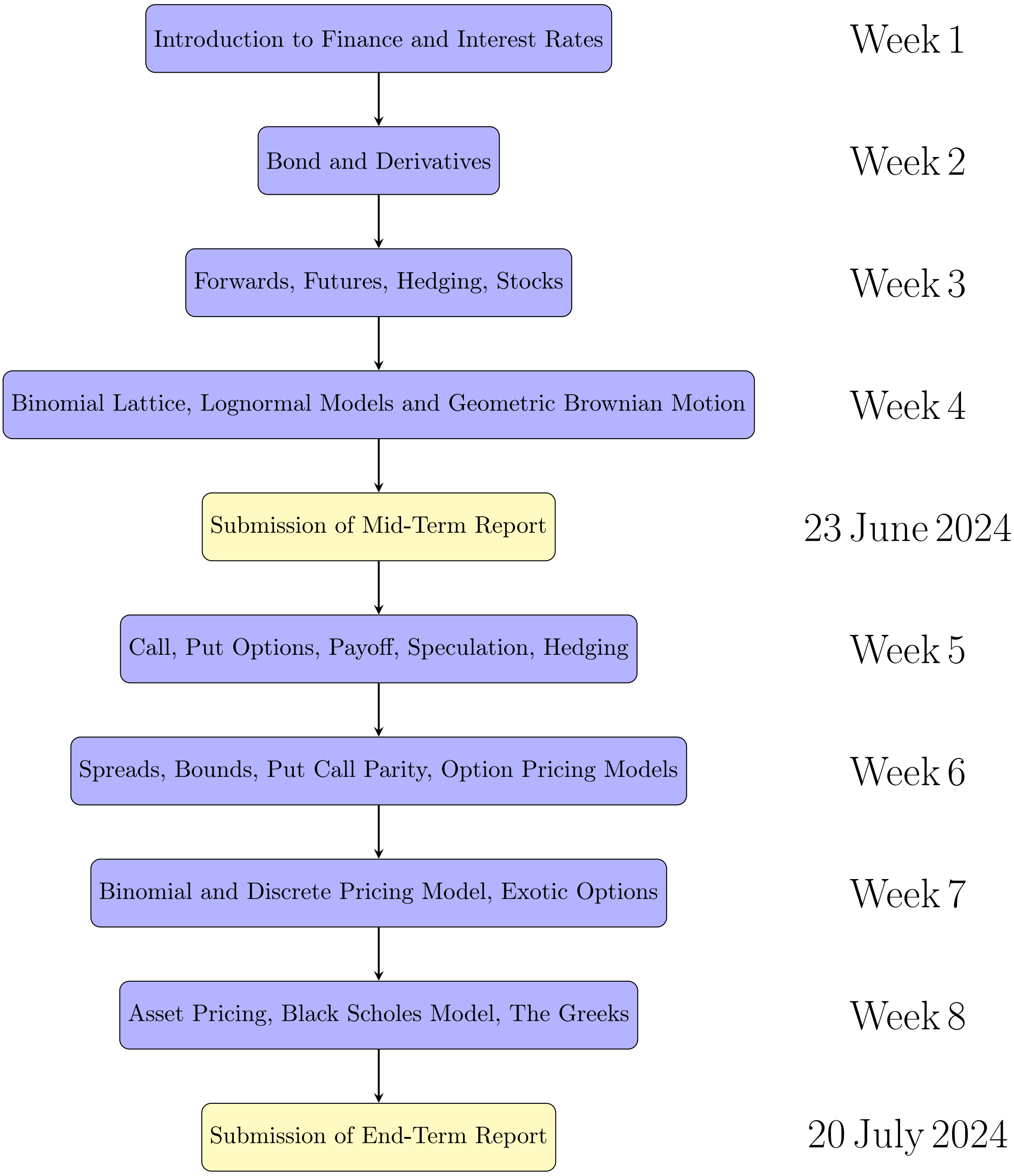
Saksham Rathi
Second-Year Computer Science Undergraduate
22B1003
Under the mentorship of **Shreyas Kulkarni**

Contents

Plan of Action 4
1 Introduction to Finance 5
  1.1 The concepts of Finance Market 6
  1.2 Derivatives Market 6
  1.3 Asset Pricing Models 6
2 Interest Rates 6
  2.1 Types of Interest Rates 6
  2.2 Effective Interest Rate 7
  2.3 Continuous Compound Interest 7
  2.4 Present Value 7
  2.5 Cash Flow Stream 8
  2.6 Future Value of a Stream 8
  2.7 Present Value of a Stream 8
  2.8 Annuity 9
  2.9 Internal Rate of Return 9
3 Bond 9
  3.1 Zero Coupon Bonds 9
  3.2 Coupon Bonds 9
  3.3 Risks 10
  3.4 Yield to Maturity 10
  3.5 Accrued Interest 10
  3.6 Price Sensitivity 10
  3.7 Duration 11
  3.8 Price Sensitivity Model 11
4 Derivatives 12
  4.1 Types of derivatives 12
  4.2 Types of traders 12
5 Forwards 12
  5.1 Notations 12
  5.2 Payoff 12
  5.3 Portfolio 13
  5.4 Portfolio Value 13
  5.5 Arbitrage Portfolio 13
  5.6 Theorem: No-cost Forward Pricing 13
  5.7 Forward Pricing (with cost) model 14
  5.8 Forward Pricing (with dividend yield) model 14
  5.9 Forward Pricing (with dividend) model 14
  5.10 Value of Forward Contract 14
6 Swap 14
7 Futures 14
  7.1 Marking to Market 14
  7.2 Future Pricing 15

8 Hedging . . . 15
8.1 Minimum-Variance Hedging . . . 15
9 Stocks . . . 15
9.1 Stock Price Behaviour . . . 15
9.2 Efficient Market Hypothesis . . . 16
10 Binomial Lattice Model . . . 16
10.1 Risk-neutral probability . . . 16
10.2 Multi-step Binomial Lattice Model . . . 16
11 Lognormal Model . . . 17
12 Geometric Brownian Motion . . . 17
13 Options . . . 18
13.1 Types of Options . . . 18
13.2 Categorization of Options . . . 18
13.3 Payoff and Gain . . . 18
14 Spreads . . . 19
14.1 Bull Spread . . . 19
14.2 Bear Spread . . . 19
14.3 Butterfly Spread . . . 19
15 Bounds for European Options . . . 20
16 Bounds for American Options . . . 20
17 Put-Call Parity Estimates . . . 21
18 Premium Properties . . . 22
19 Discrete Time Pricing Model . . . 22
19.1 Filtration . . . 22
19.2 Trading Strategy . . . 22
19.3 Value Process . . . 23
19.4 Discounted Finance Market . . . 23
19.5 Self-Financing Trading Strategy . . . 23
19.6 Replicative Strategy . . . 23
20 One-Step Binomial Model . . . 23
21 n-step Method (Backward Induction Technique) . . . 23
22 Pricing for American Options . . . 24
23 Exotic Options . . . 24
24 Discrete-time Martingales . . . 25

Plan of Action


Introduction to Finance and Interest Rates
Week 1
Bond and Derivatives
Week 2
Forwards, Futures, Hedging, Stocks
Week 3
Binomial Lattice, Lognormal Models and Geometric Brownian Motion
Week 4
Submission of Mid-Term Report
23 June 2024
Call, Put Options, Payoff, Speculation, Hedging
Week 5
Spreads, Bounds, Put Call Parity, Option Pricing Models
Week 6
Binomial and Discrete Pricing Model, Exotic Options
Week 7
Asset Pricing, Black Scholes Model, The Greeks
Week 8
Submission of End-Term Report
20 July 2024

1 Introduction to Finance

In the 1970s, earning around 200 rupees per month was sufficient to comfortably run a 4-member family and have good savings.

In the 1990s, earning around 2000 rupees per month was sufficient to comfortably run a 4-member family, but one can't have good savings.

In the 2010s, earning around 20,000 rupees per month was sufficient just to run a 4-member family, but one can't have savings.

If we talk about today, earning around 20,000 rupees per month is next to impossible to run a 4-member family.

So, if we had kept a 200 rupees note in a cupboard in 1970, then it would be equivalent to 2 rupees in 2020, although the note will physically remain the same. This reduction in the real value is because of "INFLATION".

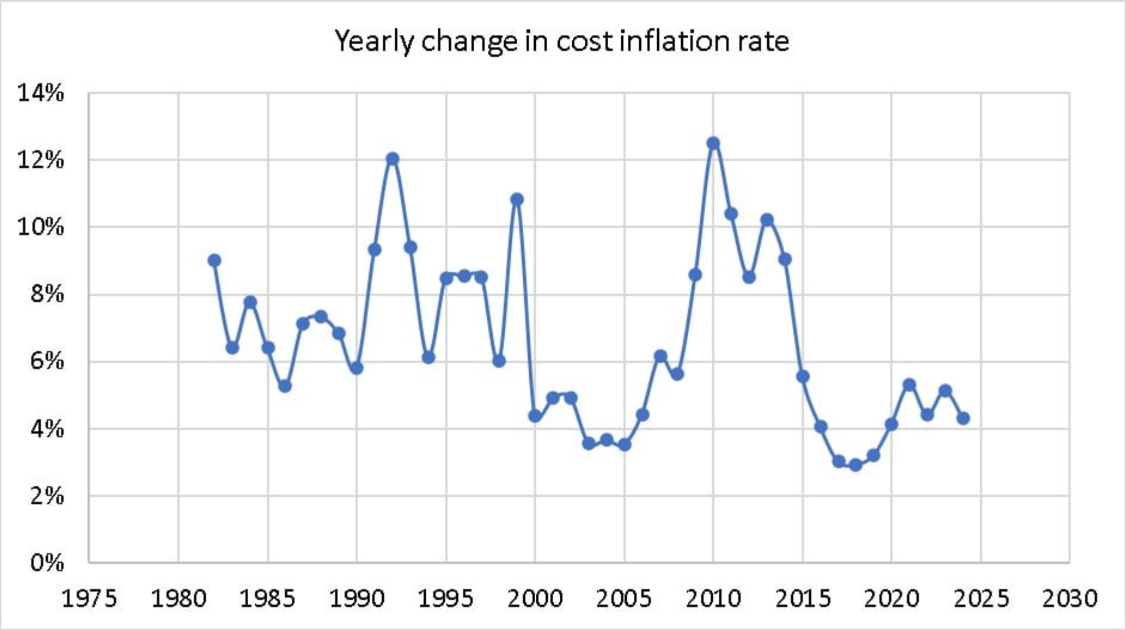


Figure 1: Yearly change in cost inflation rate in India

Here is a simple equation relating inflation rate(i), real value and nominal value:

$$\text{Real Value} = \frac{\text{Nominal Value}}{1 + \frac{i}{100}} \tag{1}$$

So, in the previous scenario, 200 rupees is the nominal value, whereas 2 rupees is the real value. Investment is the tool to kill inflation. It can be mainly categorised into two types, namely:

1. **Risk-free Investment** For example: treasury bills (government bonds) and fixed deposit with public sector banks
2. **Risky Investment** For example: equity

Let us consider a scenario, where Mr Mohan borrows 100 rupees for his business. In one of the cases, he pays the lender 10 rupees every year to the lender. In another case, he pays the net profit he earns out of his business through the 100 rupees he had borrowed.

The first case is equivalent to bonds and the second case is equivalent to equity/shares. (Where the net profit paid is called dividend)

1.1 The concepts of Finance Market

The buying of an asset is called investment.
Selling an asset and getting back the money is called liquidation.
Finance market is a platform where we can buy and sell assets that are allowed (listed) by the market. For example: bonds market, equity market, commodity market etc.
Buyers and sellers are also called traders. Asset get exchanged when both of them agree with each other.
Every country has atleast one exchange where trades happen. For instance in India, there are two popular exchanges National Stock Exchange (NSE) and Bombay Stock Exchange (BSE).

1.2 Derivatives Market

A derivative is a financial contract that derives its value from the price of an underlying asset.

There are various types of derivatives such as forward, future, option and swap. Forward contracts happen over the counter, whereas future contracts happen in a regulatory environment online. In options contract, a marginal price is paid for the contract, but the obligation to execute the contract is removed.

1.3 Asset Pricing Models

An investment is an asset.

A simple example of a model is as follows: X is the asset price (principal amount). M is the final amount. r is the rate of interest. Then the following equation links them:

$$X = \frac{M}{1+r} \tag{2}$$

2 Interest Rates

The interest in percentage divided by 100 is generally we take as interest rates.

Some useful notations:

- $t \longrightarrow$ time
- $t = 0 \longrightarrow$ time of investment
- $t = T \longrightarrow$ time of maturity
- $[0 - T] \longrightarrow$ period of investment
- Number of times interest is paid during the investment period $\longrightarrow$ frequency in time(n)

If the frequency is n with a constant time interval, interest due times are:

$$t_k = k\frac{T}{n} \tag{3}$$

Let the interest rate be r per annum, then each interest payment will be at the rate $r\frac{T}{n}$.

2.1 Types of Interest Rates

Simple Interest

- $P(0)$ = principle
- r = interest rate per annum

- $P(T)$ = maturity amount (also called the nominal value)

At $t = t_k$,

$$I_k = P(0) \times r\frac{T}{n}$$

Totally,

$$I = \sum_{k=1}^{n} I_k = P(0) \times rT$$

$$P(T) = P(0) + I$$

$$P(T) = P(0)(1 + rT) \tag{4}$$

Compound Interest

The interval $[0 - T]$ can be partitioned as $t_0 = 0$, $t_1 \ldots t_n = T$. At $t = t_1$,

$$P(t_1) = P(0)(1 + \frac{rT}{n})$$

At $t = t_2$,

$$P(t_2) = P(t_1)(1 + \frac{rT}{n})$$

$$P(T) = P(0)(1 + \frac{rT}{n})^n \tag{5}$$

2.2 Effective Interest Rate

$$r_e = \frac{P(T) - P(0)}{TP(0)} \tag{6}$$

2.3 Continuous Compound Interest

Take the frequency $n \longrightarrow \infty$ in the equation: 5.

For any real number x:

$$\lim_{n \to \infty} (1 + \frac{x}{n})^n = e^x$$

Hence, the nominal value at the continuous compounding interest is given by:

$$P(T) = P(0)e^{rT} \tag{7}$$

$$\frac{dP}{dt} = rP \tag{8}$$

$$P(t) = P(0) + \int_0^t r(s)P(s)ds \tag{9}$$

The above equation is valid also in the case of time-dependent rates of interest.

2.4 Present Value

This means given P(T), we need to find P(0). The procedure of finding the present value for a future amount is called discounting. The present value is often referred to as the discount value. The factor by which the present value P(0) is discounted from the future value is called the discount factor.

$$D_s = \frac{1}{1 + rT} \tag{10}$$

$$D_c = (1 + \frac{rT}{n})^{-n} \tag{11}$$

$$D_{cc} = e^{-rT} \tag{12}$$

2.5 Cash Flow Stream

For a given partition $\{t_0, t_1, \dots, t_n\}$ of a period $[t_0, t_n]$, the cash flow-stream is the (n+1)-tuple $\mathbf{x} = (x_0 x_1 \dots x_n)$, where x_k denotes the value of the money transaction occurred at the time $t = t_k$.

2.6 Future Value of a Stream

Assumption: Interest rate r (per annum) is constant over $[0 - T]$.
Given: cash flow stream $\mathbf{x} = (x_0 x_1 \dots x_n)$
Future value of $\mathbf{x}$:
For compound interest with frequency n:

$$FV(\mathbf{x}) = \sum_{k=0}^{n} x_k (1 + \frac{rT}{n})^{n-k} \tag{13}$$

For simple interest

$$FV(\mathbf{x}) = \sum_{k=0}^{n} x_k (1 + r(T - t_k) \tag{14}$$

For continuous compound interest:

$$FV(\mathbf{x}) = \sum_{k=0}^{n} x_k e^{r(T - t_k)} \tag{15}$$

If the cash flow stream is uncountable and is defined by a function f(t). Then the future value for the continuous compound interest scheme is given by:

$$FV(\mathbf{x}) = \int_0^T f(t) e^{r(T-t)} dt \tag{16}$$

2.7 Present Value of a Stream

Assumption: Interest rate r (per annum) is constant over $[0 - T]$.
Given: cash flow stream $\mathbf{x} = (x_0 x_1 \dots x_n)$
Present value of $\mathbf{x}$:

For compound interest with frequency n:

$$PV(\mathbf{x}) = \sum_{k=0}^{n} \frac{x_k}{(1 + \frac{rT}{n})^k} \tag{17}$$

For simple interest

$$PV(\mathbf{x}) = \sum_{k=0}^{n} \frac{x_k}{(1 + r t_k)} \tag{18}$$

For continuous compound interest:

$$PV(\mathbf{x}) = \sum_{k=0}^{n} x_k e^{-r t_k} \tag{19}$$

If the cash flow stream is uncountable and is defined by a function f(t). Then the present value for the continuous compound interest scheme is given by:

$$FV(\mathbf{x}) = \int_0^T f(t) e^{-rt} dt \tag{20}$$

2.8 Annuity

An annuity is a cash flow stream $\mathbf{x} = (x_0, x_1, \ldots, x_n)$, where

1. $sign(x_k)$, for all $k = 1, 2, \ldots, n$ are same i.e. all transactions other than the initial one are one-sided (either in-flow or out-flow).
2. x_0 is either 0 or sign(x_0) is opposite to sign(x_k), for all $k = 1, 2, \ldots, n$.
3. all transactions are made at equal intervals of time.

Example: instalment payment against purchases, mortgage payments, paying insurance premiums etc.

If the transaction is made at the end of each transaction period $[t_{k-1}, t_k]$, $k = 1, 2, \ldots, n$, then the annuity is called the ordinary annuity or annuity-immediate.

2.9 Internal Rate of Return

Given a cash flow of stream $\mathbf{x} = (x_0 x_1 \ldots x_n)$.
Let us define:

$$\rho = \frac{1}{1 + \frac{rT}{n}} \tag{21}$$

Under compound interest scheme, we have:

$$PV(\mathbf{x}) = x_0 + x_1\rho + \cdots + x_n\rho^n$$

IRR Equation:

$$x_0 + x_1\rho + \cdots + x_n\rho^n = 0 \tag{22}$$

We will calculate the root of this equation and use equation: 21 to calculate the internal rate of return.

3 Bond

A bond is an obligation by the issuer(debtor) to pay money to the holder(creditor) according to a set of rules specified at the time of issue.

3.1 Zero Coupon Bonds

Single payment at $t = T$, called a zero-coupon bond or pure discount bond.

3.2 Coupon Bonds

Gives a cash flow $(-P_0, C_1, C_2, \ldots, C_n + P_n)$ for the holder, where:

- each amount C_k is called the coupon value
- each such transaction is called the coupon payment
- The amount P_0 is called the issue price
- P_n is called the face value or par value
- The rate per annum at which C_k is calculated from P_n is called the k^{th} coupon rate.

Example

Cash Flow: $b = (0, C, C, \dots, P_n + C)$, where

- Each coupon value C is for 1 year and it is paid m times a year.
- If the bond duration is n years, then $N = mn$ is the number of times of coupon payment
- Take r as the interest rate compounded with frequency m per annum

Present value of b:

$$P_0 = PV(b) = \frac{P_n}{(1+\frac{r}{m})^N} + \sum_{k=1}^{N} \frac{\frac{C}{m}}{(1+\frac{r}{m})^k} \tag{23}$$

The parameter r is called the yield of the bond.

In case of continuous compounding:

$$PV(\mathbf{b}) = P_n e^{-rn} + \sum_{k=1}^{N} \frac{C}{m} e^{-rt_k} \tag{24}$$

3.3 Risks

- Credit Risk
- Inflation Risk

3.4 Yield to Maturity

First Assumption: The traded time t is such that $t_k < t < t_{k+1}$, for some k and $t - t_k$ is negligible.

The yield $r(t)$ at the time t (fixed)

$$P(t) = \frac{P_n}{(1+\frac{r(t)}{m})^{N-k}} + \sum_{j=1}^{N-k} \frac{\frac{C}{m}}{(1+\frac{r(t)}{m})^j} \tag{25}$$

for the given $P(t)$.
$r(t)$ is also called the yield to maturity (YTM).

3.5 Accrued Interest

Let the coupon payment is made at $t = t_k$. Accrued Interest (AI) is calculated using linear interpolation:

$$AI(t) = \frac{t - t_k}{t_{k+1} - t_k} \frac{C}{m} \tag{26}$$

The gross price, denoted by $G(t)$, to be paid by the seller to the buyer is given by:

$$G(t) = P(t) + AI(t) \tag{27}$$

3.6 Price Sensitivity

$$P(r) = \frac{P_n}{(1+\frac{r}{m})^{N-k}} + \sum_{j=1}^{N-k} \frac{\frac{C}{m}}{(1+\frac{r}{m})^j} \tag{28}$$

As yield increases, the bond price decreases. The function P is convex.

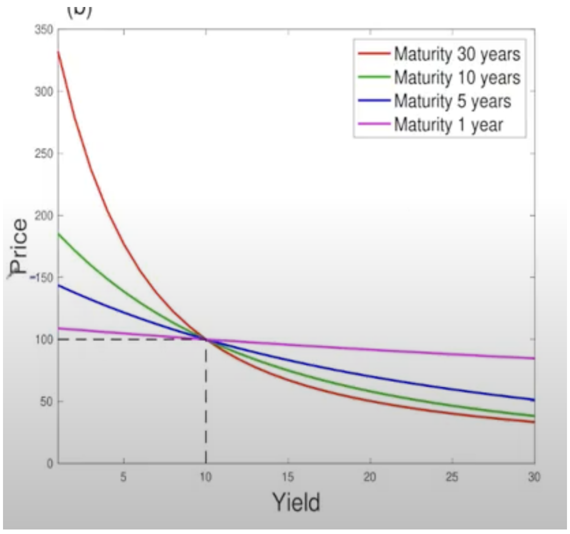


Figure 2: Price vs Yield

3.7 Duration

The duration of the cash flow stream **x** is defined as:

$$D(\mathbf{x}) = \frac{\sum_{k=0}^{n} PV(x_k)t_k}{PV(\mathbf{x})} = \sum_{k=0}^{n}(\frac{PV(x_k)}{PV(\mathbf{x})})t_k \tag{29}$$

$$D(\mathbf{x}) = \sum_{k=0}^{n} w_k t_k$$

$$\sum_{k=0}^{n} w_k = 1$$

Duration when applied to bonds is called **Macaulay Duration**.

3.8 Price Sensitivity Model

For any yield $r > 0$ and $\Delta r > 0$ small, using the mean value theorem, we have

$$\Delta P = P(r + \Delta r) - P(r) = P'(\rho)\Delta r \tag{30}$$

for some $\rho \in (r, r + \Delta r)$

We can write

$$P'(r) = -x_j \sum_{j=1}^{N-k} \frac{j}{m} x_j (1 + \frac{r}{m})^{-j-1} \tag{31}$$

where $x_j = \frac{C}{m}$ for $j = 1, 2, \dots, N-k-1$, and $x_{N-k} = P_n + \frac{C}{m}$.

$$\Delta P \approx P'(r)\Delta r \tag{32}$$

$$P'(r) = -D_m PV(\mathbf{b}) \tag{33}$$

where $D_m = \frac{D(\mathbf{b})}{(1+\frac{r}{m})}$

Equation 33 is called the price sensitivity model.

4 Derivatives

A derivative is a contract between two parties that binds a trade between them in an asset, called underlying asset, at a future date called the delivery date or expiration date at some predetermined price of the asset, called strike price.

4.1 Types of derivatives

- **Contingent Claims:** Are contracts that define the holders the right but not the obligation to execute the contract on or before the expiry. Options are contingent claims.
- **Non-Contingent Claims:** Are contracts where both writer and holder have obligation to execute the contract on or before the expiry. Forward, future and swap are non-contingent claims.

4.2 Types of traders

- **Risk Aversion Nature:** quantify the risk involved in the trades, balance it through some alternate trading and investment strategies. Any strategy that is adopted to reduce risk in price fluctuation of an asset is called hedging. A trader with risk aversion nature may be called hedger.
- **Risk Seeking Nature:** In contrast a trader with risk seeking nature will take any level of risk to increase their gains. Trading in a risky security with the hope of making profits in a short span of time due to the market fluctuations is called speculation. Traders with risk seeking nature may be called speculators.

5 Forwards

A forward contract is a non-contingent claim between two parties to buy or sell an asset on the expiration time for a fixed strike price.

Forward trades happen through over the counter (OTC) market.

5.1 Notations

- Issue time: $t = 0$
- Expiration Time: $t = T$
- Strike Price/Forward Price: $F(0, T)$
- Spot Price: at any time $t \in [0, T]$ is $S(t)$.
- Strike price at any time $t \in [0, T]$ is $F(t, T)$.

$$R_F(t; 0, T) := F(t, T) - F(0, T), t \in [0, T]$$

$$F(T, T) = S(T)$$

5.2 Payoff

The payoff from a forward contract initiated at $t = 0$ is the trader's total return from the contract till expiration, which is given by $R_F(T; 0, T)$ for the long position and $-R_F(T; 0, T)$ for the short position in the contract.

5.3 Portfolio

- A collection of risk-free assets is denoted by
$$\mathbf{B} = (B_1, B_2, \dots, B_{n_b})$$
where each B_i, $i = 1, 2, \dots, n_b$ is a risk-free asset like a bond.
- A collection of risky assets is denoted by
$$\mathbf{S} = (S_1, S_2, \dots, S_{n_s})$$
where each S_i, $i = 1, 2, \dots, n_s$ is a risky asset like a stock of a company.
- A collection of derivatives is denoted by
$$\mathbf{D} = (D_1, D_2, \dots, D_{n_d})$$
where each D_i, $i = 1, 2, \dots, n_d$ is one of the derivative instruments.

A portfolio is a collection of assets $(\mathbf{B}, \mathbf{S}, \mathbf{D}))$ is an ordered $(n_b + n_s + n_d)$-tuple of real numbers

$$\Pi := (\mathbf{b}, \mathbf{s}, \mathbf{d})$$

where $\mathbf{b} = (b_1, \dots, b_{n_b}$ with b_i being the number of asset B_i (and similarly).

Building a portfolio consists of three steps:

1. Select the assets to be collected in the portfolio and their quantities
2. Plan the order in which they have to be traded (sometime this ordering may not matter)
3. Trade (buy and sell) the assets in the market as planned

5.4 Portfolio Value

Assume that a portfolio is built at $t = 0$ and is held till $t = T$. The portfolio value at any time $t \in [0, T]$ is defined as:

$$V(\Pi)(t) = \sum_{i=1}^{n_b} b_i B_i(t) + \sum_{i=1}^{n_s} s_i S_i(t) + \sum_{i=1}^{n_d} d_i D_i(t) \tag{34}$$

5.5 Arbitrage Portfolio

1. $V(\Pi)(0) = 0$
2. $V(\Pi)(T) \geq 0$
3. $\mathbb{P}(V(\Pi)(T) > 0) > 0$

5.6 Theorem: No-cost Forward Pricing

Suppose the underlying asset has no extra cost involved throughout the forward contract period. Further assume that there is no arbitrage opportunity available in the market and the market allows short trades.

Then, the forward price at any time $t \in [0, T]$ is given by

$$F(t, T) = \begin{cases} S(t)(1 + \frac{r}{m})^n - AI(t) & \text{for discrete compounding} \\ S(t)e^{r(T-t)} & \text{for continuous compounding} \end{cases} \tag{35}$$

5.7 Forward Pricing (with cost) model

Assume that the underlying asset involves an additional cost of Rs. C to maintain during the forward contract period $[0 - T]$. Further assume that there is no arbitrage opportunity available in the market and the market allows short trades.

Then, the fair forward price is given by

$$F(0,T) = (S(0) + C)e^{rT} \tag{36}$$

5.8 Forward Pricing (with dividend yield) model

Assume that the underlying asset pays dividend continuously at the rate r_d. Further assume that there is no arbitrage opportunity available in the market and the market allows short trades.

Then, the fair forward price is given by

$$F(0,T) = S(0)e^{(r-r_d)T} \tag{37}$$

5.9 Forward Pricing (with dividend) model

Assume that the underlying asset pays dividend x rupees at time $t = \tau \in [0,T]$. Further assume that there is no arbitrage opportunity available in the market and the market allows short trades.

Then, the fair forward price is given by

$$F(0,T) = (S(0) - xe^{-r\tau})e^{-rT} \tag{38}$$

5.10 Value of Forward Contract

Assume that there is no arbitrage opportunity available in the market and the market allows short trades.

Then the value of a long forward contract at any time $t \in [0,T]$ is given by

$$V(t) = (F(t,T) - F(0,T))e^{-r(T-t)}$$

The value of a short forward contract is given by $-V(t)$.

6 Swap

A swap is another type of non-contingent derivative contract where cash flow streams are exchanged through OTC between two parties over a series of specified future dates according to certain specified rules.

7 Futures

They are another type of non-contingent claims. They are very much similar to forward contracts.

Forward contracts are OTC (over the counter), whereas futures happen through exchanges (for example: NSE, BSE). In case of forward, the contract gets settled on the expiration date only, whereas futures are setting through marking to market concept.

7.1 Marking to Market

Exchange collect some margin deposit from both the parties. If the contract was initiated on day-0, then on day-1, party-A (long position) will pay $f(1,T) - f(0,T)$ and party-B (short position) will pay $f(0,T) - f(1,T)$, and this process will continue.

7.2 Future Pricing

Future price at t is $f(t, T)$. The time at which a contract is initiated is taken at $t = 0$. The future period is $[0, T]$ where T is the expiration time.

If long and shorts are allowed with fractional units and there are no arbitrage opportunities.

If the prevailing interest rate is constant for the period $[0, T]$ of the future then:

$$f(t, T) = F(t, T)$$

8 Hedging

The primary use pf futures contracts is to hedge our investments against risks.

In a perfect hedging, an investor's risk in a future commitment to deliver or receive an asset is completely eliminated by taking an equal and opposite position in the futures market, if such an opportunity is available.

8.1 Minimum-Variance Hedging

Suppose the number of trading units is N (negative if a long position, else positive). h is the position taken in the future contract with future price $f(0, T)$.

$$C = V + (f(\tau, \tau) - f(0, \tau))h$$

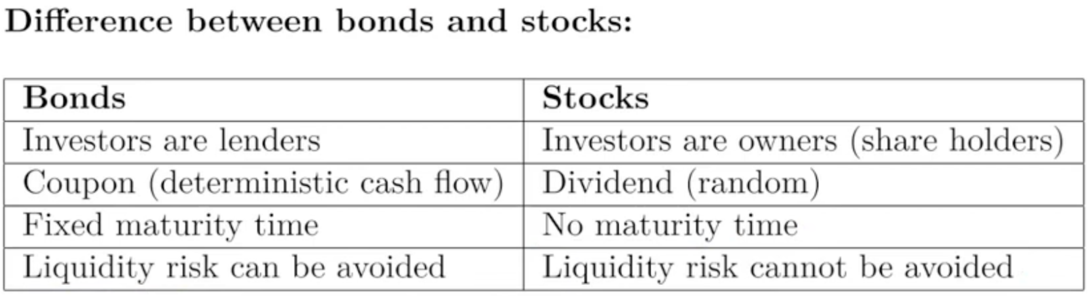
Our aim is to minimize the risk by appropriately choosing h. (C = cash received.) Risk associated is the standard deviation of the random variable C.

$$Var(C) = Var(V) + 2hCov(V, f(\tau, \tau)) + h^2 Var(f(\tau, \tau)) \tag{39}$$

The minimum of Var(C) is at

$$h = -\frac{Cov(V, f(\tau, \tau))}{Var(f(\tau, \tau)}$$

9 Stocks



Difference between bonds and stocks:

Bonds	**Stocks**
Investors are lenders	Investors are owners (share holders)
Coupon (deterministic cash flow)	Dividend (random)
Fixed maturity time	No maturity time
Liquidity risk can be avoided	Liquidity risk cannot be avoided

9.1 Stock Price Behaviour

When the business prospects of the company improve, profit prospects improve and the outlook for future dividends improves. As a result, the share price will increase.

9.2 Efficient Market Hypothesis

The current price of a stock reflects all available information and will only change with the arrival of new information.

10 Binomial Lattice Model

Let the present price of the stock as S_0. Take the holding period as $[0, \infty)$.

In the interval, $[t_k, t_{k+1}]$ we will apply the one-step binomial model. Let the stock price be S_k at t_k. Let r_k be the interest rate during that period.

No arbitrage condition:

$$d_k < (1 + r_k) < u_k$$

$S_k = \{d_k, u_k\}$ σfield in the power set. The discrete probability is $\mathbb{P}(u_k) = p_k$. Define the random variable $x_{k+1}(s) = S_k s$.

$$S_{k+1} = X_{k+1}$$

The probability p is given by the fair game criteria. The expected present value should be equal to S_0.

10.1 Risk-neutral probability

For continuous compounding:

$$p = \frac{e^{rT} - d}{u - d} \tag{40}$$

Based on the fair game criteria.

10.2 Multi-step Binomial Lattice Model

One can apply one-step binomial lattice model sequentially.

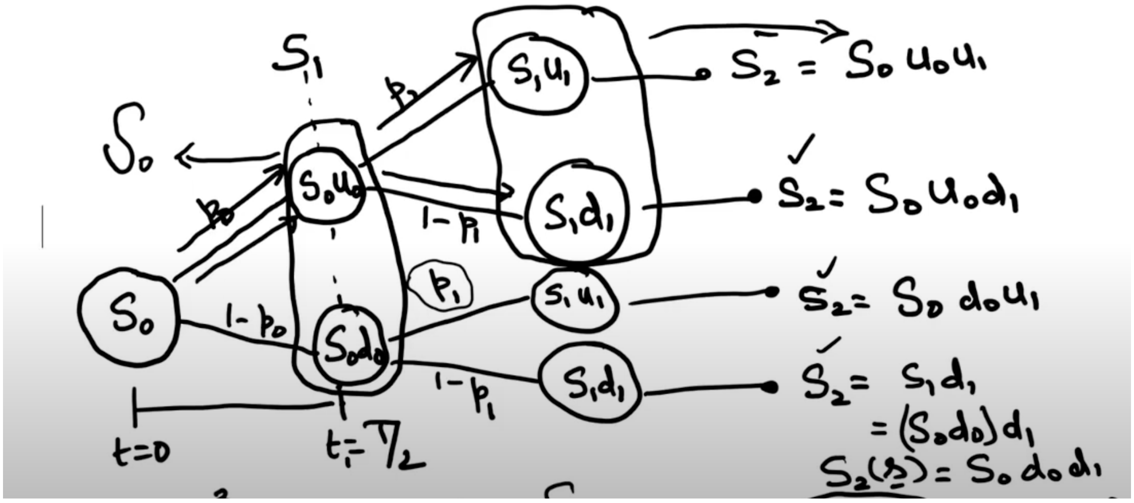

S_1
S_0
p_0
S_0u_0
p_1
S_1u_1
$S_2 = S_0 u_0 u_1$
S_1d_1
$S_2 = S_0 u_0 d_1$
$1-p_1$
p_1
S_1u_1
$S_2 = S_0 d_0 u_1$
S_0
$1-p_0$
S_0d_0
$1-p_1$
S_1d_1
$S_2 = S_1 d_1$
$= (S_0 d_0) d_1$
$S_2(s) = S_0 d_0 d_1$
$t=0$
$t_1 = T/2$

Figure 3: Two-Step Binomial Lattice Model

11 Lognormal Model

We need to find stock prices at any future time. And, we need to move from discrete time model to continuous time model.

We assume the form:

$$S(t) = S_0 e^{R(t)}$$

where $R(t)$ is a random variable.

As time t_k, we propose the stock price to be

$$S(t_{k+1}) = S_{t_k} e^{R(t_k)}$$

$$S(1) = \frac{S(t_n)}{S(t_{n-1})} \times \frac{S(t_{n-1})}{S(t_{n-2})} \times \cdots \times \frac{S(1)}{S_0}$$

$$S(1) = S_0 exp(\sum_{k=0}^{n-1} R(t_k))$$

$$ln(\frac{S(1)}{S_0}) = \sum_{k=0}^{n-1} R(t_k)$$

We assume that $R(t_k)$ are mutually independent and identically distributed.

$$\mathbb{E}(ln(\frac{S(1)}{S_0})) = \mu = n\tilde{\mu}$$

$$\sigma^2 = Var(ln(\frac{S(1)}{S_0})) = n\tilde{\sigma}^2$$

We can apply Central Limit Theorem.

$$\frac{ln(\frac{S(1)}{S_0}) - \mu}{\sigma} \to \mathbb{N}(0,1)$$

In lognormal model, we assume

$$S(T) = S_0 e^R$$

$\frac{S(T)}{S_0)}$ is lognormally distributed with parameters $T\mu$ and $T\sigma^2$, where μ and σ^2 are mean and variance of 1-year stock price S(1).

$$\mathbb{E}(S(T)) = S_0 e^{(\mu + \frac{\sigma^2}{2})T} \tag{41}$$

$$Var(S(T)) = S_0^2 e^{(2\mu+\sigma^2)T}(e^{\sigma^2 T} - 1) \tag{42}$$

12 Geometric Brownian Motion

Define:

$$W(t) = \frac{ln(\frac{S(T)}{S_0}) - t\mu}{\sigma}$$

$\{W(t) | t \in [0,T]\}$ is a stochastic process and satisfies the following properties:

1. W(0) = 0
2. $\mathbb{E}(W(t)) = 0$

3. $Var(W(t)) = t$
4. $W(t_1), W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})$ are independent.
5. The map $t \to W(t)$ is a continuous process.

Hence, $\{W(t) | t \in [0, T]\}$ is a Brownian Motion. We can write:

$$S(t) = S_0 exp(t\mu + \sigma W(t)) \tag{43}$$

Such a process is called a geometric brownian motion with parameters μ(drift) and σ(volatility).

13 Options

Options are contingent claims where the writer has obligation but the holder has only right but no obligation to exercise the contract.

The holder has to pay premium at the time of agreeing for the contract. The premium is called the price of the option or option price.

13.1 Types of Options

- **Call Option:** A call option gives the holder of the option the right to buy the underlying asset for a strike price by a certain date.
- **Put Option:** A call option gives the holder of the option the right to sell the underlying asset for a strike price by a certain date.

Basic Parameters:

- contract issued time, generally taken as t=0
- expiration or maturity (date or time), denoted by $t = T$
- strike price or exercise price, denoted by S_T
- contract size also called a lot, which is the number of units of the underlying asset to be exercised in a contract

13.2 Categorization of Options

- In American option, the contract can be exercised at any time up to the expiration time
- In European option, the contract is allowed to be exercised only on the expiration date

13.3 Payoff and Gain

- $S_t = S(t)$:the current market price of the underlying asset
- K: strike price of the option
- T: expiration time
- $[0, T]$: the period of the option
- The owner of a call option will exercise it if and only if $K < S_t$
- The owner of a put option will exercise it if and only if $K > S_t$

A call option is said to be:

1. in-the-money if $K < S_t$
2. at-the-money if $K = S_t$
3. out-of-the-money if $K > S_t$

The payoff of a call option at the expiration date is defined as:

$$C_T = max(0, S_T - K)$$

The payoff of a put option at the expiration date is defined as:

$$P_T = max(0, K - S_T)$$

The value of a call option is defined as:

$$C_t = max(0, S_t - K)$$

The payoff of a put option is defined as:

$$C_t = max(0, K - S_t)$$

The gain (or loss) is defined as:

$$G_T = C_T - Xe^{rT} \text{(for long call)}$$

$$G_T = Xe^{rT} - C_T \text{(for short call)}$$

14 Spreads

A spread is a portfolio consisting of options of the same type (either all calls or all puts).

- A vertical spread is a portfolio in which the options have the same expiration date, but different strike prices.
- A horizontal spread is a portfolio in which the options have the same strike price but different expiration dates
- A diagonal spread is a portfolio in which the options have different expiration dates and different strike prices.

14.1 Bull Spread

Buying a call option at $K_1 - strike$ and writing another call option of the same type at $K_2 - strike$ with same expiration and $K_1 < K_2$, leads to a bull spread.

14.2 Bear Spread

Buying a put option at $K_2 - strike$ and writing another put option of the same type at $K_2 - strike$ with same expiration and $K_1 < K_2$, leads to a bull spread.

14.3 Butterfly Spread

Let $K_1 < K_2 < K_3$ and $K_2 \in [S_0 - \epsilon, S_0 + \epsilon]$ for some sufficiently small $\epsilon > 0$. Buy two call options, one in each of $K_1 - strike$ and $K_3 - strike$. Write two call options with $K_2 - strike$. All options have the same expiration. The resulting portfolio is called the butterfly spread.

15 Bounds for European Options

Consider the European call option:

- K-strike
- period [0,T]
- premium C^e

where the underlying stock with spot price S_0 pays no dividend during the option period. If the market does not allow arbitrage, then:

$$max(0, S_0 - Ke^{-rT}) \leq C^e \leq S_0$$

where r is the prevailing annual interest rate continuously compounded.

Consider the European put option:

- K-strike
- period [0,T]
- premium P^e

where the underlying stock with spot price S_0 pays no dividend during the option period. If the market does not allow arbitrage, then:

$$max(0, Ke^{-rT} - S_0) \leq P^e \leq Ke^{-rT}$$

where r is the prevailing annual interest rate continuously compounded.

Consider the European (put or call) option:

- K-strike
- period [0,T]
- premium C^e for call and P^e for put,

where the underlying stock with spot price S_0 pays a dividend D_0 at some time during the option period. If the market does not allow arbitrage, then:

$$max(0, S_0 - Ke^{-rT} - D_0) \leq c^e \leq S_0 - D_0$$

$$max(0, Ke^{-rT} + D_0 - S_0) \leq P^e \leq Ke^{-rT}$$

where r is the prevailing annual interest rate continuously compounded.

16 Bounds for American Options

Consider two call options:

- K-strike call
- period [0,T]
- premium C^e for the European and C^a for the American

If the underlying stock does not pay dividend during the option period, the market does not allow arbitrage, and the prevailing interest rate $r > 0$, then show that $C^e = C^a$.

Consider the American call option:

- K-strike
- period [0,T]
- premium C^a

where the underlying stock with spot price S_0 pays no dividend during the option period. If the market does not allow arbitrage, then:

$$max(0, S_0 - Ke^{-rT}) \leq C^a \leq S_0$$

where r is the prevailing annual interest rate continuously compounded.

Consider the American put option:

- K-strike
- period [0,T]
- premium P^a

where the underlying stock with spot price S_0 pays no dividend during the option period. If the market does not allow arbitrage, then:

$$max(0, K - S_0) \leq P^a \leq K$$

where r is the prevailing annual interest rate continuously compounded.

Consider the American (put or call) option:

- K-strike
- period [0,T]
- premium C^a for call and P^a for put,

where the underlying stock with spot price S_0 pays a dividend D_0 at some time during the option period. If the market does not allow arbitrage, then:

$$max(0, S_0 - Ke^{-rT} - D_0, S_0 - K) \leq c^a \leq S_0$$

$$max(0, Ke^{-rT} + D_0 - S_0, K - S_0) \leq P^a \leq K$$

where r is the prevailing annual interest rate continuously compounded.

17 Put-Call Parity Estimates

Consider the two European options:

- K-strike
- period [0,T]
- premium C^e for the call option and P^e for the put option

where the underlying stock with spot price S_0 pays no dividend during the option period. If the market does not allow arbitrage, then:

$$C^e - P^e = S_0 - Ke^{-rT}$$

where r is the prevailing annual interest rate continuously compounded.

Consider the two American options:

- K-strike
- period [0,T]
- premium C^a for the call option and P^a for the put option

where the underlying stock with spot price S_0 pays no dividend during the option period. If the market does not allow arbitrage, then:

$$S_0 - K \leq C^e - P^e \leq S_0 - Ke^{-rT}$$

where r is the prevailing annual interest rate continuously compounded.

18 Premium Properties

If the underlying stock does not pay dividend during the option period and the market does not allow arbitrage, then

$$0 \leq C^e(K_1) - C^e(K_2) \leq e^{-rT}(K_2 - K_1)$$
$$0 \leq P^e(K_2) - P^e(K_1) \leq e^{-rT}(K_2 - K_1)$$

for any $0 \leq K_1 \leq K_2$, where r is the prevailing interest rate continuously compounded and all the options have the same period [0,T]

For any $\alpha \in [0,1]$ and for any positive real numbers K_1 and K_2, we have:

$$C^e(\alpha K_1 + (1-\alpha)K_2 \leq \alpha C^e(K_1) + (1-\alpha)C^e(K_2)$$

$$P^e(\alpha K_1 + (1-\alpha)K_2 \leq \alpha P^e(K_1) + (1-\alpha)P^e(K_2)$$

where all the options have the same expiration and market does not allow arbitrage.

19 Discrete Time Pricing Model

Let the period of the option be [0,T]. $t_k = kh$ where $h = \frac{T}{n}$. At $t = t_k$ the stock price in the spot market be denoted as $S(t_k)$.

In general, at $= t_k$ we know S_k and for S_{k+1}. We only know S_k which is U or D.

Let $S = \{s = (s_1, s_2, \ldots, s_n) | s_k \texttt{is either U or D}\}$

$$B_{s_k} = \{s \in S | \texttt{the first k components of s} = s_k\}$$

19.1 Filtration

A collection $\{F_t | t \geq 0\}$ of σ-fields on S is called a filtration if

$$F_s \in F_t$$

for all $0 \leq s \leq t$

19.2 Trading Strategy

Our portfolio $\pi = (\phi, \theta)$, ϕ is a scalar and θ is a vector.

The stochastic process of portfolios

$$\{\Pi_k | k = 1, \ldots, n\}$$

where Π_k corresponds to the period $[t_{k-1}, t_k]$ is called a trading strategy, if each component of Π_k is F_{k-1} is measurable.

19.3 Value Process

For $k = 0, 1, \ldots, n$, let $S_k = (S_k^1, \ldots, S_k^m)$ be the price vector at time $t = t_k$, where S_k^j denotes the price of the j^{th}, risky asset at $t = t_k$ We also denote the risk-free asset price at $t = t_k$ as B_k. The stochastic process $\{V_k | k = 1, \ldots, n\}$ where

$$V_k = \phi B_k + \theta_k . S_k \tag{44}$$

is the value of the portfolio Π_k is called the value process or the wealth process.

19.4 Discounted Finance Market

$$B = \{B_k\}, B_k > 0$$

$$\tilde{S_k} = \frac{S_k}{B_k}$$

Finance Market: $(1, \tilde{S_k})$

19.5 Self-Financing Trading Strategy

A trading strategy is called self-financing trading strategy if

$$V(\Pi_k)(t_k) = V(\Pi_{k+1})(t_k)$$

for every $k = 1, 2, \ldots, n - 1$

For a discounted finance market, $\Delta\phi_k + \Delta\theta_k . \tilde{S_k} = 0$

19.6 Replicative Strategy

An option is said to be attainable (or reachable) if there exists a self-financing trading strategy, whose value process satisfies:

$$V_n = H_T$$

A trading strategy that makes an option attainable is called a replicative strategy (or a headging strategy).

In market, if every option is attainable, then the market is said to be complete.

20 One-Step Binomial Model

The method consists of two steps:

1. Constructing a replicative strategy using the : $V_1 = H_T = max(0, S_T - K) \texttt{or} max(0, K - S_T)$ for call and put options respectively
2. Matching the initial investment: The amount spent in setting up the replicative portfolio at t=0 should be taken as the option premium.

21 n-step Method (Backward Induction Technique)

We consider a European option with payoff H_T

Given the $\mathbb{F}_{n-1}$-measurable random variable S_{n-1}, define

$$S_n = \begin{cases} S_{n-1}u & \text{if } s_n = U \\ S_{n-1}d & \text{if } s_n = D \end{cases}$$

Then the payoff is given by:

$$H_n = \begin{cases} H_n^u & \text{if } s_n = U \\ H_n^d & \text{if } s_n = U \end{cases}$$

$$H_n^u = \begin{cases} max(0, S_{n-1}u - K) & \text{for call} \\ max(0, K - S_{n-1}u) & \text{for put} \end{cases}$$

For a given option, if a replicative strategy exists in a no-arbitrage market, then the price H_{n-1} of the option at $t = t_{n-1}$ is given by

$$H_{n-1} = V_{n-1}$$

Let the option period [0,T] be partitioned into n subinterval. The price of an attainable European option in a no-arbitrage market is uniquely given by

$$H_0 = \frac{1}{(1+r)^n} \sum_{j=0}^{n} \binom{n}{k} p^{*j} (1-p^*)^{n-j} H_{n,j}$$

22 Pricing for American Options

At every time level, the holder has two possibilities

- the holder chooses to exercise the option, in which case, the holder obtains the payoff $H_k(S_k)$
- the holder choses to hold the option at least till the next time level $t = t_{k+1}$

$$V_k = E^*(\frac{H_{k+1}^a}{1+r} | F_k)$$

$$H_k^a = \begin{cases} H_n & \text{for } k = n \\ max(H_k, V_k) & \text{for} k = 0, 1, \ldots, n-1 \end{cases}$$

Define the minimum exercise time as $T_{min} = min\{t_k \in \Pi_n | H_k \geq V_k\}$

23 Exotic Options

An Asian option is a European contingent claim whose payoff depends on the average price of the underlying asset. We have two types of Asiam options, namely:

1. Fixed strike Asian Options: These are the options with payoffs given by:

   $$H_T = \begin{cases} max(0, A(S) - K) & \text{for call} \\ max(0, K - A(S)) & \text{for put} \end{cases}$$

   where A(S) is an average price of the underlying asset and K is the strike price.

2. Floating strike Asian options: These are the options with payoffs given by:

   $$H_T = \begin{cases} max(0, S_T - A(S)) & \text{for call} \\ max(0, A(S) - S_T) & \text{for put} \end{cases}$$

   where A(S) is an average price which is taken as the strike price of the option as usual, S_T denotes the price of the underlying asset at the expiration time.

There are different ways to define the average function A. Some of them are listed below:

1. Discrete arithmetic average: A(S) $= \frac{1}{n}\sum_{k=1}^{n} S_k$
2. Continuous arithmetic average: A(S) $= \frac{1}{T}\int_0^T S(t)dt$
3. Discrete geometric average: A(S) $= (\prod_{k=1}^{n} S_k)^{\frac{1}{n}}$
4. Continuous arithmetic average: A(S) $= exp(\frac{1}{T}\int_0^T lnS(t)dt)$

24 Discrete-time Martingales

Let X be a random variable defined on a probability space $(S, F, \mathbb{P})$ with $E(|X|) < \infty$

Let $F_1 \subseteq F$ be a σ-field. Conditional Expectation of X given F_1 is a random variable, say Y.

$$E(Y\nVdash_A = E(X\nVdash_A))\forall A \in F_1$$

Bibliography

1. Video Lectures of SI-527 (Mathematics of Derivative Pricing), IIT Bombay. The course was taken by Baskar S during Spring 2020-2021
2. D. G. Luenberger, Investment Science, Oxford University Press, 1998.
3. J. C. Hull, Options, Futures and Other Derivatives, 4th Edition, Prentice-Hall, 2000
4. J. C. Cox and M. Rubinstein, Options Market, Englewood Cliffs, N.J.: Prentice Hall, 1985
5. C. P Jones, Investments, Analysis and Management, 5th Edition, John Wiley and Sons, 1996