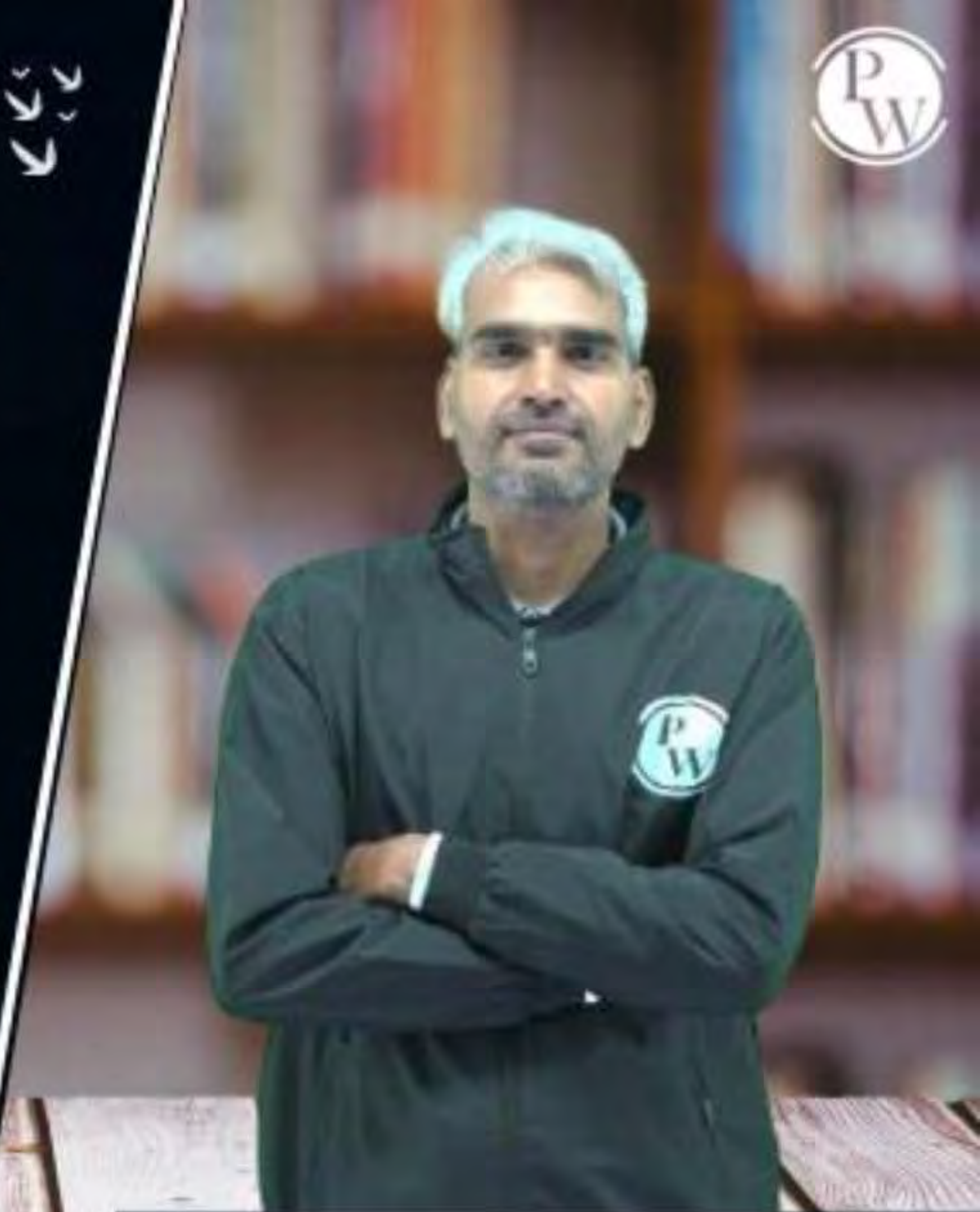
# Computer Science & IT

## Discrete Mathematics

**Combinatorics**

**Lecture No. 01**



**By- Vishal Sir**

# Recap of Previous Lecture

Topic Equivalences and implications

Topic Practice questions on predicate logic

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# Topics to be Covered

- Topic: Introduction to recurrence relation
- Topic: Formation of recurrence relation
- Topic: Solution of recurrence relation

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[MSQ] H.W.

#Q. The CORRECT formula of the sentence "not all rainy days are cold"

$$\sim\left[\forall x\left\{rainy(x)\rightarrow Cold(x)\right\}\right]$$

$$\exists x\left\{\sim(\sim rainy(x)\vee Cold(x))\right\}$$

$$\exists x\left\{rainy(x)\wedge\sim Cold(x)\right\}$$

A. $\forall d\,(rainy(d)\wedge\sim cold(d))$

All days are rainy & not cold

B. $\exists d\,(\sim rainy(d)\rightarrow cold(d))$

$\exists d\,(rainy(d)\vee Cold(d))$

Some days are rainy or cold

C. $\forall d\,(\sim rainy(d)\rightarrow cold(d))$

All days are rainy or cold

D. $\exists d\,(rainy(d)\wedge\sim cold(d))$

[MSQ] H.W.

$\forall x \ (G(x) \land S(x))$

#Q. Which one the following is the most appropriate logical formula to represent the statement "gold and silver ornaments are precious". The following notations are used:

All

$G(x)$: $x$ is a gold ornament

$S(x)$: $x$ is a silver ornament

$P(x)$: $x$ is precious

$\forall x \{(G(x) \lor S(x)) \rightarrow P(x)\}$

A $\forall x(P(x) \rightarrow G(x) \land S(x))$

B $\exists x((G(x) \land S(x)) \rightarrow P(x))$

C $\forall x((G(x) \land S(x)) \rightarrow P(x))$

D $\forall x((G(x) \lor S(x)) \rightarrow P(x))$

[MSQ] H.W. $P \text{ if } Q \equiv Q \rightarrow P$

#Q. Which one of the first order predicate calculus statements given below correctly expresses the following English statements? "Tigers and lions attack if they are hungry or threatened".

$\forall x \{(\text{Tiger}(x) \vee \text{Lion}(x)) \rightarrow ((\text{Hug}(x) \vee \text{Threat}(x)) \rightarrow \text{attack}(x))\}$

A. $\forall x[(tiger(x) \wedge lion(x)) \rightarrow ((hungry(x) \vee threatened(x)) \rightarrow attacks(x))]$

B. $\forall x[(tiger(x) \vee lion(x)) \rightarrow ((hungry(x) \vee threatened(x)) \wedge attacks(x))]$

C. $\forall x[(tiger(x) \wedge lion(x)) \rightarrow (attacks(x) \rightarrow (hungry(x) \vee threatened(x)))]$

D. $\forall x[(tiger(x) \vee lion(x)) \rightarrow ((hungry(x) \vee threatened(x)) \rightarrow attacks(x))]$

[MSQ]

#Q. Which of the following argument ~~is~~ are valid?

**A** $\forall x\,(P(x))$ follows from the premises $\exists x(P(x) \land Q(x))$

**B** $\exists x(P(x) \land Q(x))$ follows from the premises $\exists x(P(x)) \land \exists x(Q(x))$

opposite of the given statement is valid

$(\exists x\{P(x)\} \land \exists x\{Q(x)\}) \not\rightarrow \exists x\{P(x) \land Q(x)\}$ is invalid.

**C** $\exists x(Q(x))$ follows from the premises $\{\forall x(P(x) \Rightarrow Q(x)), \exists y(P(y))\}$

**D** $\sim P(a,b)$ follows from the premises $\{\forall x\ \forall y\{P(x,y) \rightarrow W(x,y)\}, \sim W(a,b)\}$

$\exists x\{P(x)\}$

(a) $\forall x\{P(x)\}$ follows from $\exists x\{P(x) \wedge Q(x)\}$

$$\frac{\exists x\{P(x) \wedge Q(x)\}}{\therefore \forall x P(x)}$$

$\exists x\{P(x) \wedge Q(x)\}$

$\downarrow$ Existential specification

$P(a) \wedge Q(a)$ is true for some 'a'

$\downarrow$ Simplification

$P(a)$ is true for some 'a'

$\downarrow$ Existential generalization (bcoz of some)

$\exists x\{P(x)\}$

but $\forall x\{P(x)\}$ need not be true

$\therefore$ (invalid argument)

(c) $\forall_x \{P(x) \rightarrow Q(x)\}$ $\equiv$ $P(a) \rightarrow Q(a)$ is true for all a

$\exists_y (P(y))$ $\equiv$ $P(b)$ is true for some 'b'

$P(b) \rightarrow Q(b)$ will also be true

$\therefore \exists_x \{Q(x)\}$

M.P. $\therefore$ $Q(b)$ for some 'b'

$\downarrow$ Existential generalization

$\exists_x Q(x)$

$\therefore$ Valid

(d) $\forall x \forall y \{ P(x,y) \longrightarrow W(x,y) \} \equiv P(l,m) \rightarrow W(l,m)$ is true for all $l$ & $m$

$\sim W(a,b)$

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$\therefore \sim P(a,b)$

$\therefore$ Existential generalization

$\exists x \exists y \{ \sim P(x,y) \}$

$\therefore$ Valid

$\therefore P(a,b) \rightarrow W(a,b)$ will also be true

$\sim W(a,b)$

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By Modus tollen

$\therefore \sim P(a,b)$

**[MSQ]**

#Q. Check whether the following argument is true or false?

$\sim P \equiv \exists x\{\sim F(x) \vee \sim S(x)\}$ is valid.

$\exists x\{F(x) \to \sim S(x)\}$ follows from the premises

$$\{\underbrace{\forall x(F(x) \wedge S(x))}_{P} \to \underbrace{\forall y(M(y) \to W(y))}_{Q},\ \underbrace{\exists y(M(y) \wedge \sim W(y))}_{\sim Q}\}$$

$P \longrightarrow Q$

$\sim P = \sim\{\forall x\{F(x) \wedge S(x)\}\}$

$= \exists x\{\sim F(x) \vee \sim S(x)\}$

$\sim Q = \sim\{\forall y\{\sim M(y) \vee W(y)\}\}$

$= \exists y\{M(y) \wedge \sim W(y)\}$

M.T.

$\therefore \sim P$ is true

[MSQ]

#Q. Check whether the following argument is true or false?

$\exists x\{\sim F(x) \vee \sim S(x)\}$

$\exists x\{F(x) \rightarrow \sim S(x)\}$ follows from the premises

Consider that the domain is non-empty

$$\{\underbrace{\exists x\big(F(x) \wedge S(x)\big)}_{P} \rightarrow \underbrace{\forall y(M(y) \rightarrow W(y))}_{Q},\ \exists y\big(M(y) \wedge \sim W(y)\big)\}$$

$P \longrightarrow Q \qquad \sim Q$

M.T.

$\therefore \sim P \equiv \sim\{\exists x\{F(x) \wedge S(x)\}\}$

$\equiv \forall x\{\sim F(x) \vee \sim S(x)\}$

$\forall x$ is true

$\therefore \exists x$ will also be true on non-empty domain

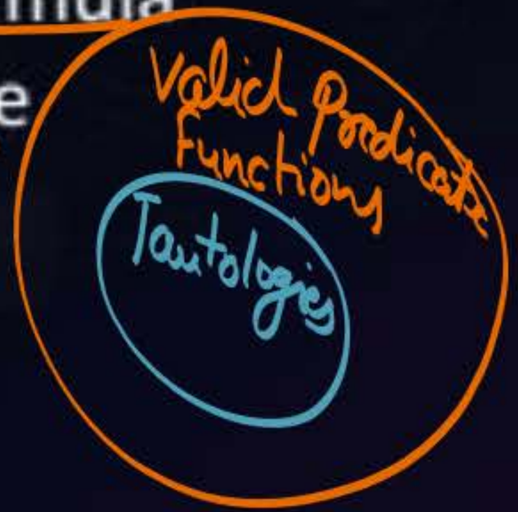
# Topic : Tautology in predicate logic

In propositional logic, tautologies and validities are same.

A propositional function which is always true is called valid propositional function or tautology.

But in predicate logic, distinction is maintained between logical validities and tautologies.

A predicate formula which is always true is called a valid predicate formula but it may or may not be a tautology. In predicate logic tautologies are proper subset of logical validities.



# Topic : Tautology in predicate logic

$\underset{\sim}{A} \longleftrightarrow \underset{\sim}{A}$

A tautology in predicate logic is a sentence that can be obtained by taking a tautology of propositional logic and uniformly replacing each propositional variable by a predicate formula (one formula per propositional variable).

NOTE: To check whether a given predicate formula is a tautology or not we can only use the concepts that we have learned in propositional logic,

we can not use the concepts of predicate logic to check whether the predicate formula is a tautology or not.

eg. Some Tautologies in Propositional logic.

(i) $A \rightarrow A$

(ii) $A \rightarrow (B \rightarrow A)$

(iii) $(A \cap B) \rightarrow (A \vee B)$

All are tautologies

(a) $A \longleftrightarrow A$ is a tautology.

$A \rightarrow (B \rightarrow C)$ May or may not be tautology

① $\underbrace{\forall x \forall y \{P(x,y)\}}_{A} \Longleftrightarrow \underbrace{\forall y \forall x \{P(x,y)\}}_{B}$ is valid predicate formula

**but** not a tautology

$A \Longleftrightarrow B$

it can not be 'A'

because $\forall x \forall y$ is equivalent to $\forall y \forall x$ is defined w.r.t. predicate logic, it is not defined in propositional logic.

$A \leftrightarrow B$ is not a tautology

(2) $$\underbrace{\exists y \forall x \{P(x,y)\}}_{A} \Longrightarrow \underbrace{\forall x \exists y \{P(x,y)\}}_{B}$$

it is valid Predicate Formula, but not a tautology.

$$\underbrace{A \longrightarrow B}$$

$A \to B$ is not a tautology

③ $\underbrace{\forall_x \exists_y \{P(x,y) \rightarrow Q(x,y)\}}_{A} \longleftrightarrow \forall_x \exists_y \{\sim P(x,y) \vee Q(x,y)\}$

$\equiv$ ← By Proposition Equivalence

$P(x,y) \rightarrow Q(x,y)$

$\equiv$

$\underbrace{\forall_x \exists_y \{P(x,y) \rightarrow Q(x,y)\}}_{A}$

$\underbrace{A \longleftrightarrow A}_{\text{is a tautology}}$

$\left\{ \because \text{Given predicate formula is logically valid as well as a tautology} \right\}$

# Combinatorics

Recurrence Relation

Generating Function

Some important Counting methods of Combinatorics.

## Topic : Recurrence Relation

Consider a sequence of real numbers $\{a_0, a_1, a_2, a_3, \ldots\}$ a formula that relates $a_n$ with one or more of the preceding terms $\{a_{n-1}, a_{n-2}, \ldots\}$ is called a recurrence relation.

$$a_n = f(a_{n-1}, a_{n-2}, \ldots, n)$$

e.g.

$$a_n = 2 \cdot a_{n-1}$$

$$a_n = a_{n-1} + a_{n-2}$$

$$a_n = 3 \cdot a_{n-1} + 2a_{n-2}$$

$$a_n = 3 \cdot a_{n-1} + n$$

## Topic : Examples of recurrence relation

1. Fibonacci Series:

$$F_n = F_{n-1} + F_{n-2} \qquad F_0 = 1 \qquad F_1 = 1$$

2. Arithmatic Progression:

$$a_n = a_{n-1} + \underbrace{d}_{\text{common difference}}$$

3. Geometric Progression:

$$a_n = \underbrace{r}_{\text{Common ratio}} \cdot a_{n-1}$$

# Topic : Linear recurrence relation

A recurrence relation of the form

$$C_0 \cdot a_n + C_1 \cdot a_{n-1} + C_2 \cdot a_{n-2} + \cdots + C_k \cdot a_{n-k} = f(n) \quad \text{—— eq}^n \text{ (1)}$$

is called a linear recurrence relation.

(1) In eq$^n$ (1) if $f(n) = 0$, then it is called homogeneous linear recurrence relation.

(2) In eq$^n$ (1) if $f(n) \neq 0$, then it is called non-homogeneous linear recurrence relation

eg. A recurrence relation of the form

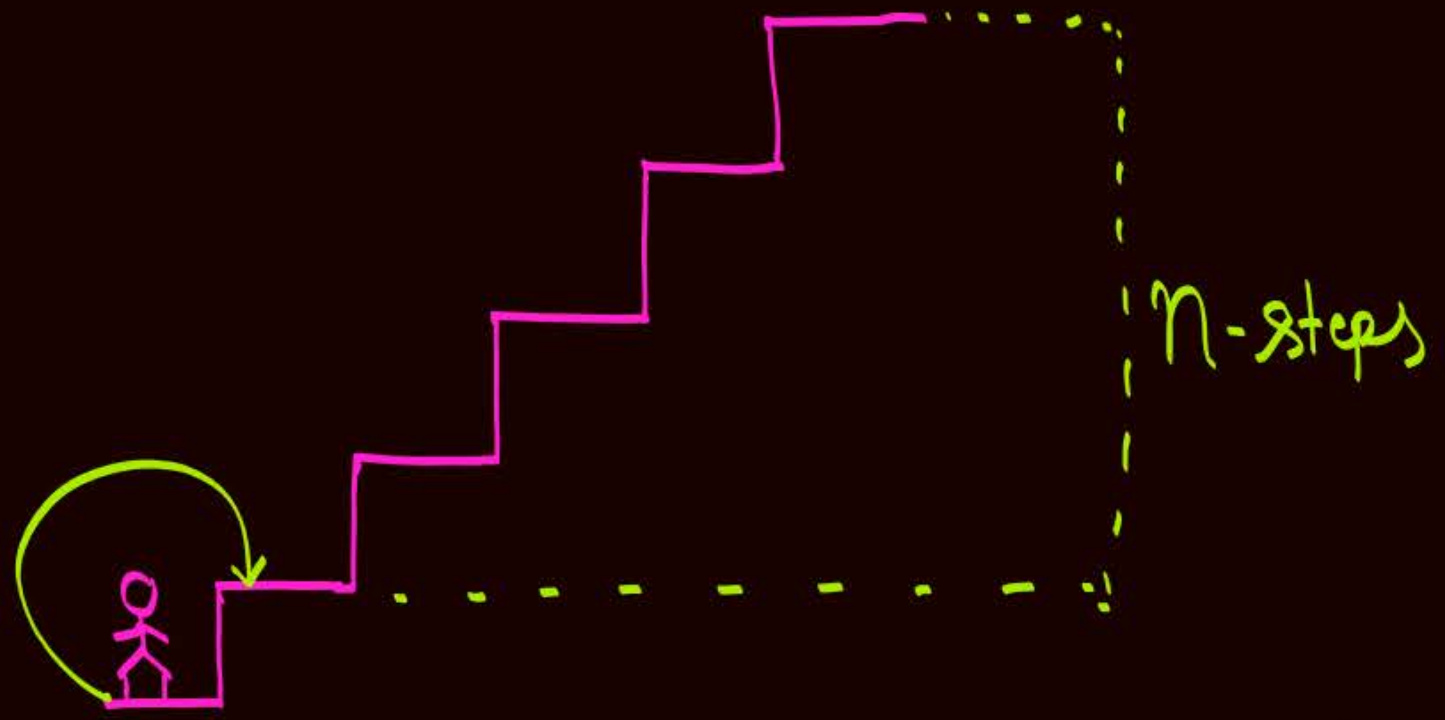
$$2 \cdot a_n + 3 \cdot a_{n-1}^2 + a_{n-2} = f(n)$$

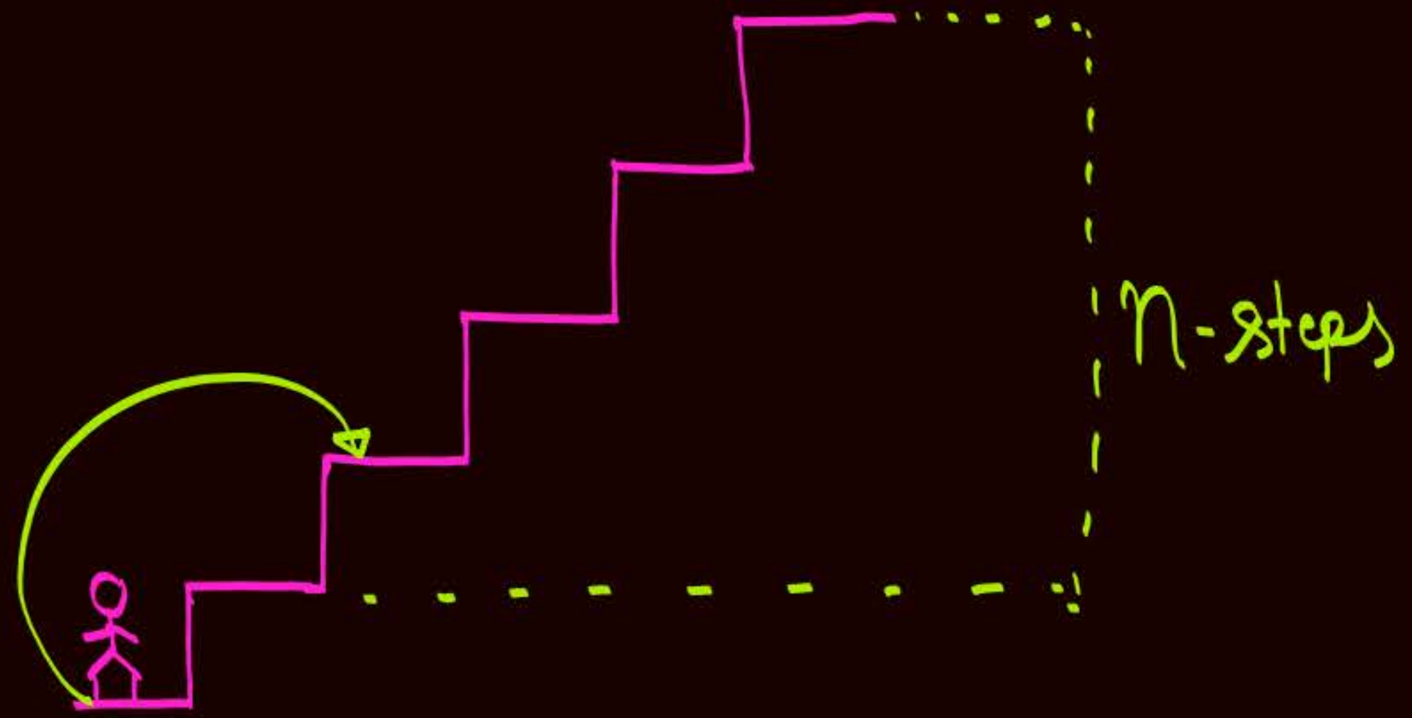
then it becomes non-linear recurrence relation.

## Topic : Formation of recurrence relation

Q. Let $a_n$ represents the number of ways a person can climb a flight of n-steps while person is allowed to skip at most one step at a time, then

Find recurrence relation for $a_n$




There are two possibilities w.r.t. first move taken by the person

Case 1: In the first move if person does not skip any step, then the remaining $(n-1)$ steps can be climbed in $a_{n-1}$ ways.

Case 2: In the first move if person skip exactly one step, then remaining $(n-2)$ steps can be climbed in $a_{n-2}$ ways.

Total number of ways to climb the flight of n-steps

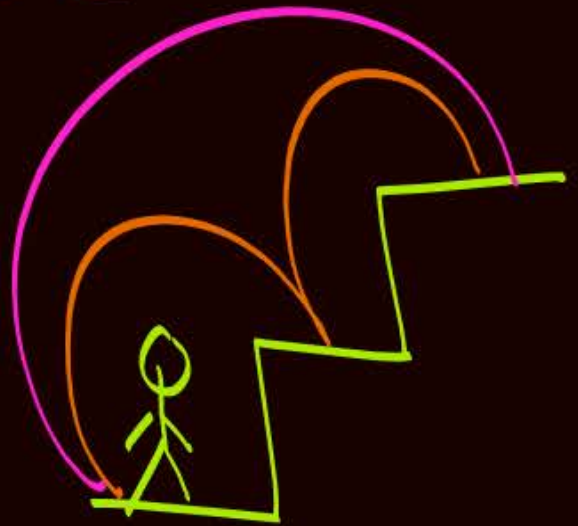
i.e. $a_n$ = number of ways using case 1 (or) + number of ways using case 2

$$= 1 * a_{n-1} + 1 * a_{n-2}$$

$$\boxed{a_n = a_{n-1} + a_{n-2}}$$

$a_1 = 1$

$a_2 = 2$



## Topic : Formation of recurrence relation

H.W.

Q. Let $a_n$ represents the number of ways a person can climb a flight of n-steps while person is allowed to skip at most two steps at a time, then

Find recurrence relation for $a_n$

## Topic : Formation of recurrence relation

H.W.

Q. Let $a_n$ represents the number of ways to arrange a pile of n-chips using Red, Green, Blue, White and Gold colour chips such that no two gold colour chips are together, then

Find recurrence relation for $a_n$

# Topic : Formation of recurrence relation

H.W.

Q. Let $a_n$ represents the number of n-digit binary sequences of '0' and '1' with no consecutive zeros, then

Find recurrence relation for $a_n$

## Topic : Formation of recurrence relation

H.W.

Q. Let $a_n$ represents the number of n-digit ternary sequences of 0, 1, and 2 with even number of zeros in it , then

Find recurrence relation for $a_n$

## 2 mins Summary

**Topic** Recurrence relation

**Topic** Formation of recurrence relation

**Topic** Solution of recurrence relation using substitution method

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# THANK - YOU

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