CS & IT ENGINEERING

Algorithms

Analysis of Algorithms

Lecture No.- 05

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Recap of Previous Lecture

Topic Big Notations

(Problems + Properties)

Topic Problem Solving

Topics to be Covered

Topic Small Notations

Topic Properties of Asymptotic Notations ☆

Topic Problem solving

① Big Oh (UB)

$f(n) = O(g(n))$

iff $f(n) \le c * g(n)$, some $c > 0$, $n \ge \underline{\underline{n_0}}$ ✓

② Big Omega (Ω)

$f(n) = \Omega(g(n))$

iff $\boxed{f(n) \ge c * g(n)}$, some $c > 0$, $n \ge n_0$ ✓

③ $f(n) = \Theta(g(n))$

iff
$$\boxed{\begin{array}{c} f(n) = O(g(n)) \\ \text{and} \\ f(n) = \Omega(g(n)) \end{array}}$$

Product

① $f(n) = \prod_{i=1}^{n} (1) = \underbrace{1 \times 1 \times 1 \times \ldots \times 1}_{n \text{ times}}$

$= 1$

$\Rightarrow O(1) \longrightarrow$ Constant TC

Summation (sum)

$\sum_{i=1}^{n} 1 \Rightarrow \underbrace{1+1+1 \ldots 1}_{n \text{ times}}$

$\Rightarrow n$

② $f(n) = \prod_{i=1}^{10,000} [(5)]$

$= 5 * \prod_{i=1}^{10,000} (1)$

$= 5 * (\underbrace{1 \times 1 \times 1 \cdots * 1}_{\text{10K times}})$

$= 5 \longrightarrow$ Constant

$= O(5) \rightarrow O(1)$

X ✓

X A) $f(n) = O(5^{10,000}) \longrightarrow$ 11%

X B) $f(n) = O(5^{1000}) \longrightarrow$ 3%

✓ C) $f(n) = O(1) \longrightarrow$ 77.5%

X D) $f(n) = O(n) \longrightarrow$ 8.5%

Imp

2. $f(n) = \prod_{i=1}^{n} (i)$

A) $f(n) = O(n) \longrightarrow 40\%$

B) $f(n) = O(n^2) \longrightarrow 15.5\%$

C) $f(n) = O(n^3) \longrightarrow 2.22\%$

D) $f(n) = O(n^n) \longrightarrow 41.5\%$

$$f(n) = \prod_{i=1}^{n} (i) = 1 \times 2 \times 3 \times 4 \cdots n$$

$$= \boxed{n!}$$

$$f(n) = O(n!) \checkmark$$

$$= O(n^n) \checkmark$$

Imp $n!$ vs n^n

$n \leq n \quad , n \geq 2$

$n * (n-1) \leq n * n$

$n * (n-1) * (n-2) \leq n * n * n$

$n * (n-1) \ldots 1 \leq n * n * n \ldots * n$ (n times)

$n! \leq (n^n)$

$\boxed{n! = O(n^n)}$ Imp

(Q) Is $n! = \Omega(n^n)$?

A) True → 30%

B) False → 66%

$n \times (n-1) \times (n-2) \cdots 1 \geq c * n^n$?

never

$c > 0$

$f(n) \geq c * g(n) \longrightarrow \times$

$$\boxed{n! \neq \Omega(n^n)}$$

Imp Note

$n! = O(n^n)$

and $n! \neq \Omega(n^n)$

Hence $n! \neq \Theta(n^n)$

O, Ω (MSQ)

④ $f(n) = \prod_{i=1}^{n} [\log(i)]$

A) $f(n) = O(\log n)$

B) $f(n) = O(n \log n)$

C) $f(n) = \Omega(n \log n)$

D) $f(n) = \Theta(n \log n)$

$= \cancel{\log(1)} * \log(2) \cdots \log(n)$ $\quad 0$

$= 0 * \underbrace{\qquad\qquad}$

$= 0 \longrightarrow \Theta(1)$

$\boxed{\text{MSQ}}$

5. $f(n) = \sum_{i=1}^{n} \log(i)$

A) $f(n) = O(n \log n)$

B) $f(n) = O(\log n)$

C) $f(n) = \Omega(n \log n)$

D) $f(n) = \Theta(n \log n)$

Soln : $f(n) = \sum_{i=1}^{n} \lceil \log(i) \rceil$

$$\boxed{\begin{aligned}&\log(a) + \log(b)\\ &= \log(a * b)\end{aligned}}$$

$$= \log(1) + \log(2) + \log(3) \cdots + \log(n)$$

$$= \log(1 * 2 * 3 * \cdots * n)$$

$$\boxed{f(n) = \log(n!)}$$

① mtd-1 : Logic

$$\log(n) \leq \log(n)$$

$$\log(n)+\log(n-1) \leq \log(n)+\log(n)$$

$$\log(n)+\log(n-1)\cdots\log(1) \leq \underbrace{\log(n)+\log(n)\cdots+\log(n)}_{n \text{ times}}$$

$$\log(n)+\cdots\log(1) \leq n*\log(n)$$

$$\log(n!) \leq n*\log(n) \implies \boxed{\log(n!) = O(n\log n)}$$

② mtd-2 : Using Stirling's Approximation

$$\boxed{n! \approx \sqrt{2\pi n} * \left(\frac{n}{e}\right)^n}$$

$$f(n) = \sum_{i=1}^{n} \log(i) = \log(n!)$$

$$= \log\left(\sqrt{2\pi n} * \left(\frac{n}{e}\right)^n\right) \rightarrow \text{using Stirling's Approx.}$$

$$= \log(\sqrt{2\pi}) + \log(\sqrt{n}) + \log\left(\left(\frac{n}{e}\right)^n\right)$$

$$= \log(\sqrt{2\pi}) + \frac{1}{2}\log(n) + n\left[\log(n) - \log(e)\right]$$

$$= \left[\log(\sqrt{2\pi}) + \frac{1}{2}\log(n) + \underbrace{n * \log(n)}_{\downarrow \text{dominating}} - n * \log(e)\right]$$

$$n\log(e) \approx \underline{n}$$

$$\boxed{\begin{aligned} f(n) &= O(n\log n) \\ f(n) &= \Omega(n\log n) \end{aligned}} \rightarrow \text{Hence } \boxed{\log(n!) = \Theta(n\log n)}$$

Imp

Bonus logic → <u>Mtd-3</u> → using Integration

$$f(n) = \sum_{i=1}^{n} \log(i) = \log(1) + \log(2) \cdots + \log(n)$$

$$= \int_{1}^{n} \log(x)\,dx = \Big[x(\log x - 1)\Big]_{1}^{n}$$

$$= n\log n - n + c$$

$$= \boxed{O(n\log n)}$$

[MCQ]

#Q. $f(n) = \sum_{i=1}^{n} i^3 = x$ choice for x.

I. $\theta(n^4)$ II. $\theta(n^5)$

III. $O(n^5)$ IV. $\Omega(n^3)$

A. I, II, III → 25%

B. II, III, IV ← 12%

C. I, II, III, IV — 20.5%

D. I, III, IV ← 42%

$(a+b)^2 = a^2 + 2ab + b^2$

Soln : $F(n) = \sum_{i=1}^{n} i^3 \longrightarrow x$

$= \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4} = \frac{n^2(n^2+2n+1)}{4}$

$= \frac{n^4+2n^3+n^2}{4} = O(n^4)$ ✓ $\neq \Omega(n^5)$

$= O(n^5)$ ✓

$\boxed{O(n^4) \quad \Omega(n^4)}$

I) $\Theta(n^4)$ < $O(n^4)$, $\Omega(n^4)$

✗ II) $\Theta(n^5)$

III) $O(n^5)$

IV) $\Omega(n^3)$

1, 3, 4

Is this $\Theta(n^5)$? < $O(n^5)$ ✓, $\Omega(n^5)$ ✗

$\Omega(n^3)$

SMALL/LITTLE NOTATIONS

o , ω

small oh

small omega

Imp :

The Big notations (O, Ω) provides the Upper Bound & Lower Bound that may or may not be tight.

(can be tight as well as loose bounds)

Tight LB

eg

$$\boxed{\Omega(n)} \longleftarrow \boxed{n+10} \longrightarrow \boxed{O(n)} \rightarrow \text{Tight UB}$$

$\Omega(\sqrt{n})$

$\Omega(1)$

Lose LB

$O(n^2)$

$O(n^3)$

$\rightarrow$ lose UB

② Small Notations always provide Bounds that are $\boxed{\text{loose bounds}}$.

↓

(never tight Bound)

(o)

① Small-Oh Notation: → Loose Upper Bound

$\Longrightarrow$ $f(n)$ is $o(g(n))$ iff,

for $\boxed{\text{all}}$ $c>0 \Rightarrow$ $\boxed{f(n) < c * g(n)}$, whenever $n \geq n_0$

every c

strictly less

for some $n_0 > 0$

eg:- Diff between O and o.

$n < c * n$?

if $c=1$, $n < 1 * n$ ✗

Hence $n \neq o(n)$

$n \leq 3 * n \longrightarrow n = O(n)$ ✓

$n = O(n^2)$ ✓

$n = O(n^3)$ ✓

1) $f(n) = n^2 + n + 5$

✗ $O(n \log n)$?

$O(n^2) \rightarrow$ Tight Bound

$f(n) = O(n^3)$ ✓

$= O(n^2)$ ✓

$= o(n^2)$ ✗

$= o(n^3)$ ✓

$= o(n\sqrt{n})$ ✗

$n\sqrt{n}$ $\quad$ n^2

$$\cancel{n} * \sqrt{n} \qquad \cancel{n} * n$$

$\sqrt{n} < n$

$n \log n$ $\quad$ n^2

$$\cancel{n} \log n \qquad \cancel{n} * n$$

$\log n < n$

② Small Omega (ω) Notation $\rightarrow$ Loose Lower Bound.

$\Rightarrow$ $f(n)$ is $\omega(g(n))$ iff,

For $\boxed{\text{all}}$ $c>0$, $\boxed{f(n) > c * g(n)}$, whenever $n \geq n_0$

$n_0 \rightarrow$ Threshold

(strictly greater)

for some $n_0 > 0$

eg:- Big Omega(Ω) vs Small Omega(ω)

$\Omega(n)$ ✓

$\omega(n)$ ✗

$f(n) = n \implies f(n) = \Omega(n)$ ✓ $\quad n \geq c * n$, $c = 0.01$ ✓

$= \Omega(\sqrt{n})$ ✓

$= \Omega(n^2)$ ✗

$= \omega(n^2)$ ✗

$= \boxed{\omega(n)}$ ✗ Imp

$= \omega(\sqrt{n})$ ✓

$= \omega(1)$ ✓

$\Omega(n) \rightarrow$ Tight LB

$n > c * n$ for all $c > 0$

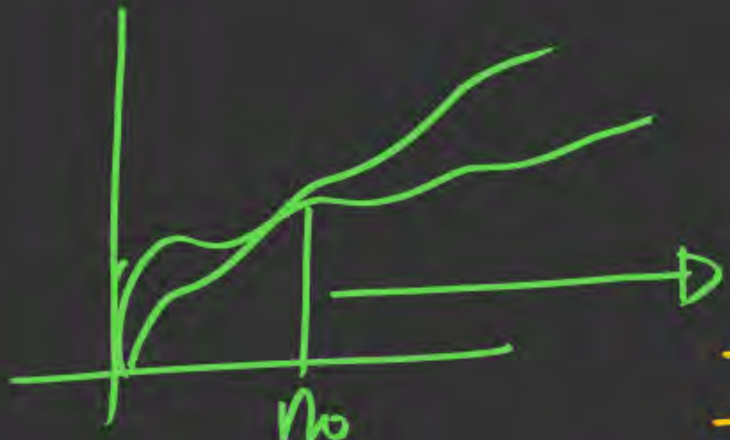
✗ $c=1, c=2, c=3 \ldots$

Is $n = \omega(1/n)$?

$\hookrightarrow$ True

Is $5n = \omega(1/n)$?

$5n > c * \frac{1}{n}$ ✓



$n_0 \rightarrow$ 1K

Is $n = \omega(1/n)$? → Yes

No, $n > c * (1/n)$

for $n = 1$ and $c = 1$

$n > 1 \times \frac{1}{1}$

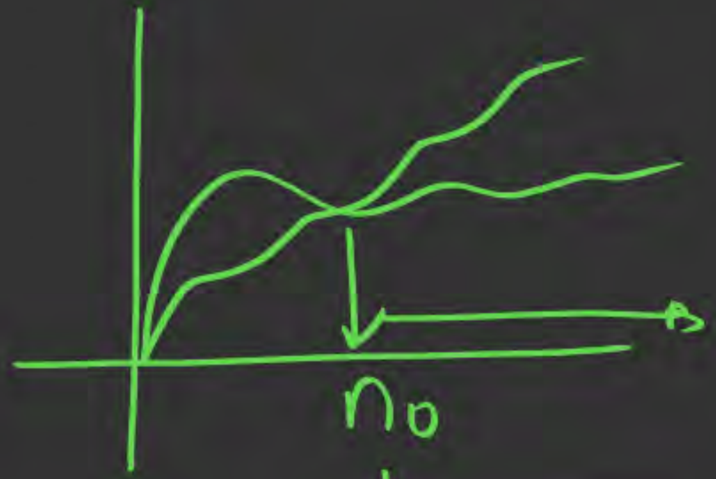
$1 > 1 \times 1$

$1 > 1$ ✗

Misconception

for $n \geq 2$

then



$n_0 \rightarrow$ 10000, 1Lac

$n > c * \frac{1}{n}$ for large n

Now → Yes ✓ any c

Imp Practice Questions:-

False ← (1) $2^{(2n)} = O(2^n)$

True ← (2) $2^{n+1} = O(2^n)$

(3) If $0 < a < b$ then $n^a = O(n^b)$

(4) $2^{(n^2)} = O(n!)$

① $2^{(2n)} = O(2^n)$? $\longrightarrow$ False

$$a^{m*n} = (a^m)^n = (a^n)^m$$

Appr 1 :

$$2^{(2n)} \qquad 2^n$$

$$(2^2)^n \qquad 2^n$$

$$(4)^n > (2)^n$$

Appr2

$$2^{(2n)} > 2^{(n)}$$

$\log_2$ on both sides.

$$\log_2(2^{2n}) \qquad \log_2(2^n)$$

$$2n * \cancel{\log_2 2} \qquad n * \cancel{\log_2 2}$$

$$2n > n$$

② $2^{(n+1)} = O(2^n)$

$\boxed{a^{x+y} = a^x * a^y}$

Appr1 :-

$2^{(n+1)}$ $\quad$ 2^n

$\cancel{2^n} * 2^1$ $\quad$ $\cancel{2^n}$

2^1 $\quad$ 1

Constant

✓

$2^{(n+1)} = O(2^n)$ ✓ ←

Appr 2

$2^{(n+1)}$ $\quad$ 2^n

$2^n * 2$ $\quad$ 2^n

equal asymptotially

✓ $2^{n+1} = O(2^n)$

2 mins Summary

- **Topic** Small Notations
- **Topic** Properties
- **Topic** Problem Solving

THANK - YOU