Computer Science & IT

Discrete Mathematics

Combinatorics

Lecture No. 02

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Recap of Previous Lecture

Topic Introduction to recurrence relation

Topic Formation of recurrence relation

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Topics to be Covered

Topic Formation of recurrence relation

Topic Solution of recurrence relation using substitution method

Topic Solution of recurrence relation using method of characteristic roots

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Topic : Formation of recurrence relation

H.W.

Q. Let a_n represents the number of ways a person can climb a flight of n-steps while person is allowed to skip at most two steps at a time, then

Find recurrence relation for a_n

3 cases are possible

(i) when person does not skip any step in 1st move.

(ii) when person skip exactly one step in 1st move

(iii) when person skip exactly two steps in 1st move

$$\therefore \quad a_n = (1 * a_{n-1}) + (1 * a_{n-2}) + (1 * a_{n-3}) \quad \therefore$$

$$\boxed{a_n = a_{n-1} + a_{n-2} + a_{n-3}}$$

$a_1 = 1$

$a_2 = 2$

$a_3 = 4$

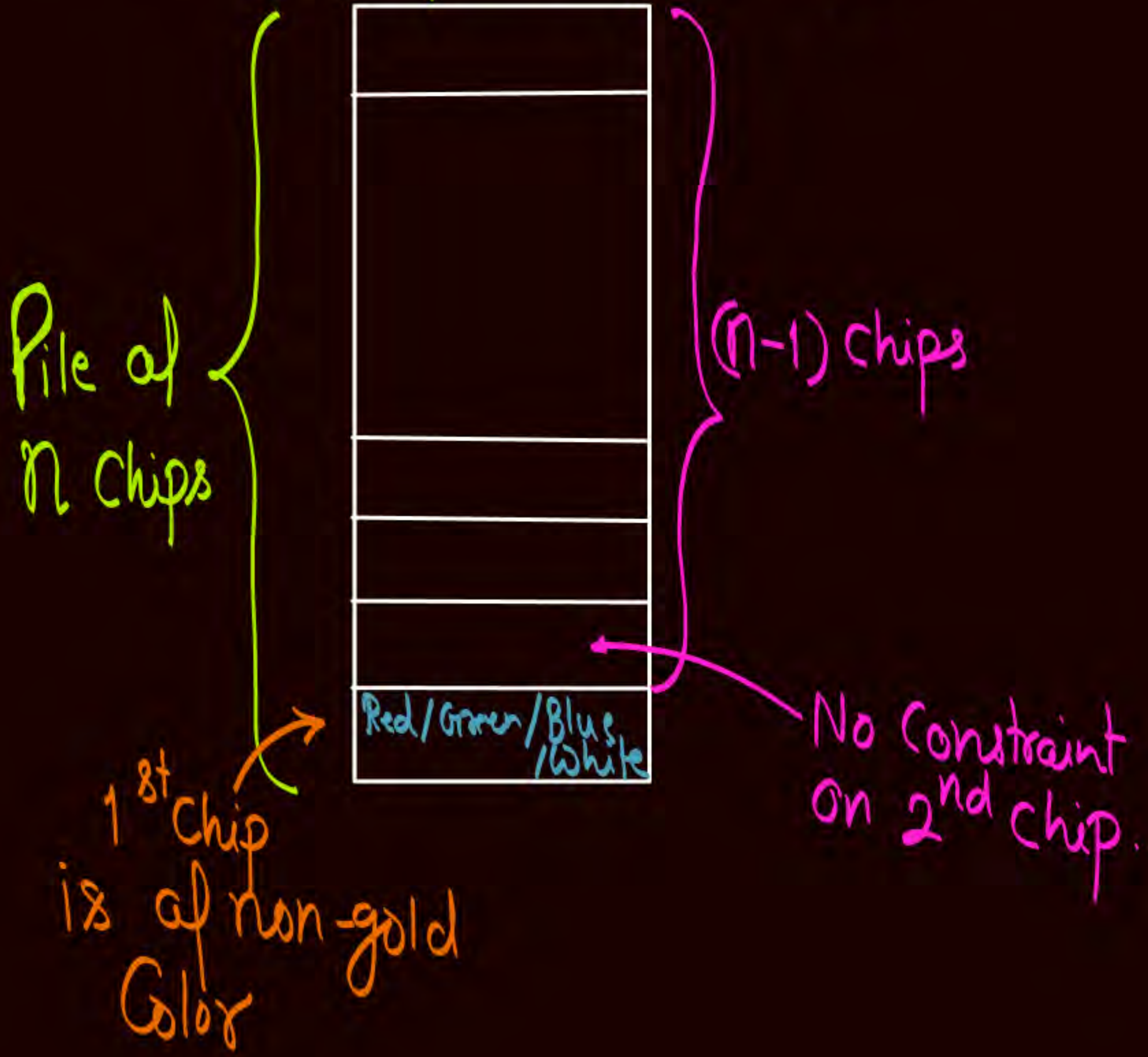
Topic : Formation of recurrence relation

H.W.

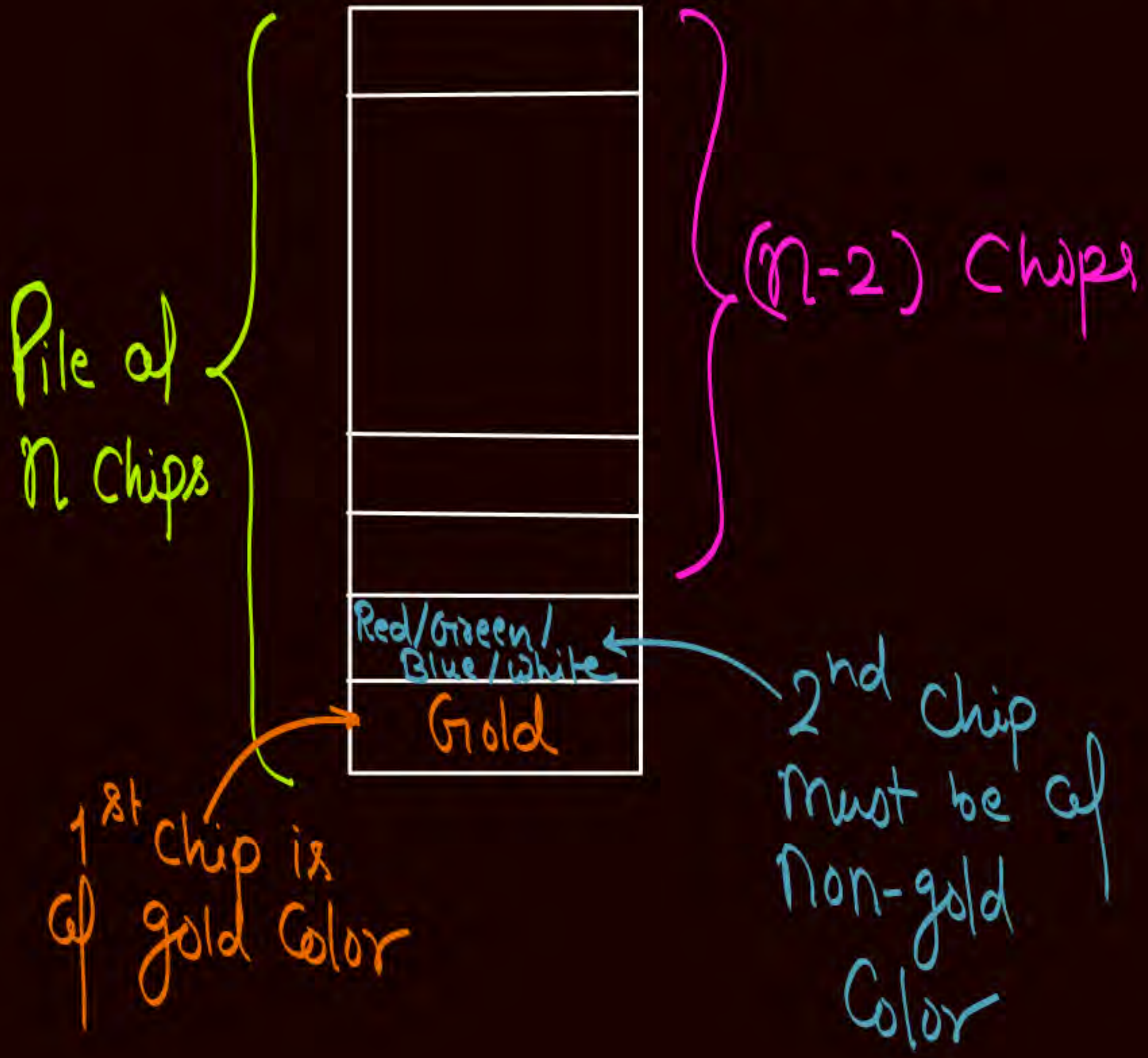
Q. Let a_n represents the number of ways to arrange a pile of n-chips using Red, Green, Blue, White and Gold colour chips such that no two gold colour chips are together, then

Find recurrence relation for a_n

Case 1 When 1st Chip is of non-gold Color



Case 2 : When 1st chip is of gold Color



$a_n =$ No. of ways using Case 1 **or** No. of ways using Case 2

$$= \underbrace{4}_{\text{1st chip can be of non-gold color in 4 ways}} * \underbrace{a_{n-1}}_{\text{remain pile of (n-1) chip can be arranged in } a_{n-1} \text{ ways}} + \underbrace{1}_{\text{1st chip can be of gold color only in '1' ways}} * \underbrace{4}_{\text{2nd chip can be of non-gold color in 4 ways}} * \underbrace{a_{n-2}}_{\text{remaining pile of (n-2) chips can be arranged in } a_{n-2} \text{ ways}}$$

$$\boxed{\therefore\ a_n = 4 \cdot a_{n-1} + 4 \cdot a_{n-2}}$$

$a_1 = 5$

$a_2 = (1 \times 4) + (4 * 5) = 24$

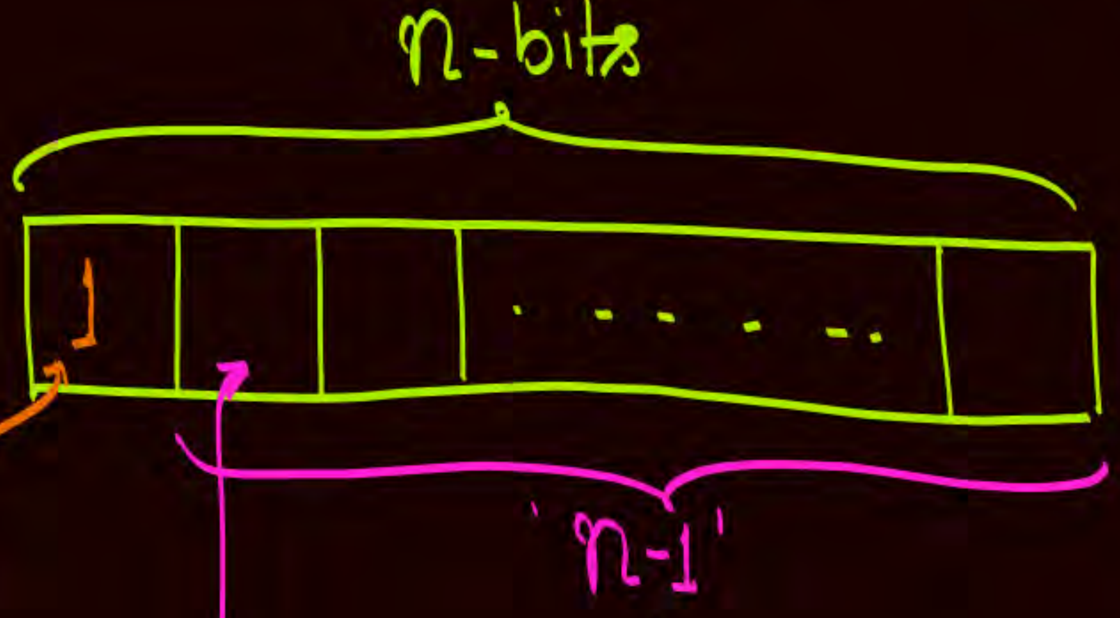
Topic : Formation of recurrence relation

H.W.

Q. Let a_n represents the number of n-digit binary sequences of '0' and '1' with no consecutive zeros, then

Find recurrence relation for a_n

Case ① : When 1^{st} bit is not Zero. {i.e. '1'}

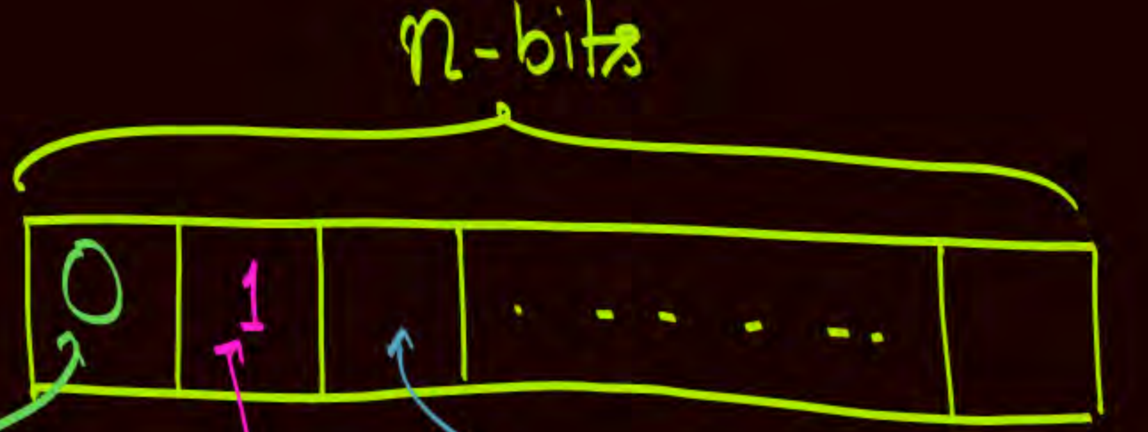


1^{st} bit Can be one only in '1' way

When 1^{st} bit is one then no restriction on 2^{nd} bit

No. of $(n-1)$ digit binary sequences with no two Consecutive Zeros Can be given by a_{n-1}

Case ②



1^{st} bit Can be '0' only in '1' way

if 1^{st} bit is '0' then 2^{nd} bit must be '1', there is only '1' way for 2^{nd} bit to be '1'

No restriction

$(n-2)$ digit

No. of $(n-2)$ digit binary sequences with no two Consecutive Zeros Can be given by a_{n-2}

a_n = using Case 1 + using Case 2

$= 1 * a_{n-1} + 1 * 1 * a_{n-2}$

$$\boxed{a_n = a_{n-1} + a_{n-2}}$$

$a_1 = 2$

$a_2 = 3$

00 ✗

0 1

1 0 ✓ Allowed

1 1

Topic : Formation of recurrence relation

H.W.

Q. Let a_n represents the number of n-digit ternary sequences of 0, 1, and 2 with even number of zeros in it , then

Find recurrence relation for a_n

Case ① When 1^{st} digit is not zero {i.e. 1 or 2}

n digit

1/2

(n-1) digit

1^{st} digit can be non-zero in two ways

1^{st} digit is not zero therefore we are looking for (n-1) digit ternary sequences with even number of zero,

it can be given by a_{n-1}

Case ②: When 1^{st} digit is Zero.

n digit

0

(n-1)

1^{st} digit can be Zero only in one way

1^{st} digit is Zero ∴ we are looking no. of (n-1) digit ternary sequences with "Odd" no. of Zeros

Total no. of ternary sequences of (n-1) digits − No. of ternary sequences of (n-1) digits with even no. of Zeros.

$$\left(3^{n-1} - a_{n-1}\right)$$

$a_n =$ No. of sequences using Case 1 $+$ No. of sequences using Case 2

$= 2 * a_{n-1} + 1 * (3^{n-1} - a_{n-1})$

$\boxed{a_n = 3^{n-1} + a_{n-1}}$ $\boxed{a_1 = 2}$

Topic : Solution of recurrence relation

Recurrence relation is a function of the form.

$$a_n = f(a_{n-1}, a_{n-2}, \ldots, n)$$

Representation of a_n as a pure function of 'n' (i.e. a function which is free from all the terms of type 'a_i' is) called solution of recurrence relation.

Topic : Solution of recurrence relation

1) Substitution method
2) Method of characteristic roots
3) Using concept of generating function

Topic : Substitution method

In this method we use recurrence relation repetitively for $n=0,1,2,\ldots\ldots\ldots,$ then we solve the expression to obtain the solution of recurrence relation.

#Q. Find the solution of the recurrence relation

$a_n = n\, a_{n-1},$ where $a_0 = 1$

$a_0 = 1$

$a_1 = 1 \cdot a_0 = 1 \cdot 1 = 1$

$a_2 = 2 \cdot a_1 = 2 \cdot 1 \cdot 1 = 2$

$a_3 = 3 \cdot a_2 = 3 \cdot 2 \cdot 1 \cdot 1 = 6$

$a_4 = 4 \cdot a_3 = 4 \cdot 3 \cdot 2 \cdot 1 \cdot 1 = 12$

$\vdots$

$a_n = n \cdot (n-1) \cdot \dots \cdot 3 \cdot 2 \cdot 1 \cdot 1 = n! \Rightarrow \boxed{a_n = n!}$

#Q. Find the solution of the recurrence relation

$$a_n = a_{n-1} + 3^{n-1}, \quad \text{where } a_1=2$$

$a_1 = 2$

$a_2 = a_1 + 3^{2-1} = 2 + 3^1$

$a_3 = a_2 + 3^{3-1} = 2 + 3^1 + 3^2$

$a_4 = a_3 + 3^{4-1} = 2 + 3^1 + 3^2 + 3^3$

$\vdots$

$a_n = 2 + 3^1 + 3^2 + 3^3 + \cdots + 3^{n-2} + 3^{n-1}$

$= 1 + 3^0 + 3^1 + 3^2 + \cdots + 3^{n-1}$

Summation of GP $= \dfrac{a(r^n - 1)}{r-1}$

first term

common ration

No of terms

$a_n = 1 + \left[3^0 + 3^1 + 3^2 + \cdots + 3^{n-1}\right]$

$= 1 + \left[\dfrac{1\cdot(3^n - 1)}{3-1}\right]$

$= 1 + \dfrac{3^n - 1}{2}$

$= \left(\dfrac{3^n + 1}{2}\right)$

#Q. Find the solution of the recurrence relation

$$a_n = a_{n-1} + (2n+1), \text{ where } a_0 = 1$$

$a_0 = 1$

$a_1 = a_0 + (2\cdot1+1) = 1 + (2\cdot1+1)$

$a_2 = a_1 + (2\cdot2+1) = 1 + (2\cdot1+1) + (2\cdot2+1)$

$a_3 = a_2 + (2\cdot3+1) = 1 + (2\cdot1+1) + (2\cdot2+1) + (2\cdot3+1)$

$\vdots$

$a_n = 1 + (2\cdot1+1) + (2\cdot2+1) + (2\cdot3+1) + \cdots + (2\cdot n+1)$

$$a_n = 1 + (2\cdot 1+1) + (2\cdot 2+1) + (2\cdot 3+1) + \cdots + (2\cdot n+1)$$

$$= \underset{(2\cdot 0+1)}{1} + 3 + 5 + 7 + 9 + \cdots (2n+1)$$

it is summation of first $(n+1)$ odd natural numbers

$$= (n+1)^2$$

Note: Summation of first 'n' odd natural number $= n^2$

$$a_n = 1 + (2\cdot 1+1) + (2\cdot 2+1) + (2\cdot 3+1) + \cdots + (2\cdot n+1)$$

$(n+1)$ term $= 1 + 3 + 5 + 7 + \cdots + (2n+1)$

It is an Arithmatic progession with Common difference = 2

$$= \frac{(n+1)}{2}\left[1 + (2n+1)\right]$$

$$= \frac{(n+1)}{\cancel{2}} \cdot \cancel{2}\cdot(n+1) = (n+1)^2$$

Summation of AP $= \frac{n}{2}\left[2\cdot a + (n-1)d\right]$

$$= \frac{n}{2}\cdot\left[\text{first term} + \text{last term}\right]$$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{(n+1)}$$

#Q. Find the solution of the recurrence relation

$$a_n = a_{n-1} + \frac{1}{n(n+1)}, \quad \text{where } a_0 = 1$$

$$a_n = 2 - \frac{1}{(n+1)}$$

$$\boxed{a_n = \frac{(2n+1)}{(n+1)}}$$

$a_0 = 1$

$a_1 = a_0 + \left(\frac{1}{1} - \frac{1}{2}\right) = 1 + \left(\frac{1}{1} - \frac{1}{2}\right)$

$a_2 = a_1 + \left(\frac{1}{2} - \frac{1}{3}\right) = 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right)$

$a_3 = a_2 + \left(\frac{1}{3} - \frac{1}{4}\right) = 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right)$

$\vdots$

$a_n = 1 + \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cancel{\ }-\cancel{\ }-\cancel{\ }-\cancel{\ }- \cancel{\ } + \left(\cancel{\frac{1}{(n-1)}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$

2 mins Summary

Topic Formation of recurrence relation

Topic Solution of recurrence relation using substitution method

Topic Solution of recurrence relation using method of characteristic roots

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THANK - YOU

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