CS & IT ENGINEERING

Algorithms

Analysis of Algorithms

Lecture No.- 06

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Recap of Previous Lecture

Topic Small Notations

Small oh

(o, ω)

small omega

Topic Properties of Asymptotic Notation

Topic Problem Solving

$O \rightarrow$ lose + Tight UB

$\Omega \rightarrow$ lose + tight LB

Small Oh $\rightarrow$ Loose Upper Bound

Small omega $\rightarrow$ Loose lower Bound

Topics to be Covered

Topic Problem Solving with ASN

Properties

Topic Framework for Analysing Non-Recursive algorithm

[MCQ]

Imp Type

#Q. Consider the following functions from positive integers to real number:

$10, \sqrt{n}, n, \log_2 n, \frac{100}{n}$

The correct arrangement of the above functions in increasing order of asymptotic complexity is:

17%

A $\log_2 n, \frac{100}{n}, 10, \sqrt{n}, n$

B $10, \frac{100}{n}, \sqrt{n}, \log_2 n$

27%

46%

C $\frac{100}{n}, 10, \log_2 n, \sqrt{n}, n$

10.3%

D $\frac{100}{n}, \log_2 n, 10, \sqrt{n}, n.$

Sdn: $10, \sqrt{n}, n, \log_2 n, \frac{100}{n}$.

$$\begin{array}{ccccc} 10 & \sqrt{n} & n & \log_2 n & \frac{100}{n} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ C & P & P & L & D \end{array}$$

$$\boxed{\begin{array}{ccccc} \text{Deco}, & \text{Const}, & \log, & \text{Poly}, & \text{Expo} \\ \downarrow D & \downarrow C & \downarrow L & \downarrow P & \downarrow E \end{array}}$$

$$\left\{ \frac{100}{n} < 10 < \log_2 n < \underline{\sqrt{n}} < \underline{n} \right\} \left\{ [D] < [C] < [L] < [P] < [E] \right\}$$

<u>ans</u>

<u>in general.</u>

Topic : General Properties of Big Oh Notation

Let d(n), e(n), f(n), and g(n) be functions mapping nonnegative integers to non-negative reals. Then

1. If d(n) is O(f(n)), then ad(n) is O(f(n)), for any constant a > 0
2. If d(n) is O(f(n)) and e(n) is O(g(n)), then d(n) + e(n) is O(f(n) + g(n)).
3. If d(n) is O(f(n)) and e(n) is O(g(n)), then d(n)e(n) is o(f(n)g(n))
4. If d(n) is O(f(n)) and f(n) is O(g(n)), then d(n) is O(g(n)).
5. If f(n) is a polynomial of degree d (that is, $f(n) = (a_0 + a_1 n + ... + a_d n^d)$ then f(n) is $O(n^d)$
6. n^x is $O(a^n)$ for any fixed x > 0 and a > 1
7. $\log n^x$ is $O(\log n)$ for any fixed x > 0
8. $\log^x n$ is $O(n^y)$ for any fixed constants x > 0 and y > 0

1) If $d(n) = O(f(n))$

then $a * d(n) = O(f(n))$, $a > 0$.

eg. $d(n) = n^2 \longrightarrow O(n^2)$

$a = 10$ $\quad a * d(n) = 10 * n^2 \longrightarrow O(n^2)$

(2) $d(n) = 5n+2 \longrightarrow O(n)$

$e(n) = 10n^2+7n+9 \longrightarrow O(n^2)$

$d(n)+e(n) = (10n^2+7n+9)+(5n+2) \Rightarrow O(f(n)+g(n))$

$= (10n^2+12n+11) \Rightarrow O(n+n^2) = O(n^2)$

$\hookrightarrow \underline{O(n^2)}$ ✓

<u>Shortcut</u>: $O(f(n)+g(n)) \Rightarrow \underline{O(\max(f(n), g(n)))}$

(3) $d(n) = 5n \longrightarrow O(n) \to f$

$e(n) = (10n^2+2) \longrightarrow O(n^2) \to g$

$d(n) * e(n) = 5n * (10n^2+2)$

$= 50n^3 + 10n$

$= O(n^3)$ ✓

$\Rightarrow O(f*g)$

$\Rightarrow O(n*n^2) = O(n^3)$

(4) $\left.\begin{array}{l} d(n) = O(f(n)) \\ \& \ f(n) = O(g(n)) \end{array}\right\} \Rightarrow d(n) = O(g(n))$

$$\& \left.\begin{array}{l} d(n) = O(f(n)) \Rightarrow d \le f \\ f(n) = O(g(n)) \Rightarrow f \le g \end{array}\right\} \rightarrow \boxed{d \le f \le g}$$

$$\Downarrow$$

$$\boxed{d(n) = O(g(n))}$$

Transitive Property.

$$\text{eg: } \left.\begin{array}{l} d(n) = O(n^2) \\ f(n) = O(n^4) \end{array}\right\} \longrightarrow d(n) = O(n^4) \checkmark$$

(5) $f(n) = 5 + 10n + 15n^3 + 7n^4$ } → Polynomial of degree 4.

ignore

lower order terms &

constants / coefficients.

$$\boxed{f(n) = O(n^4)}$$

(6) $n^x = O(a^n)$, for any $x > 0$ & $a > 1$

$n^x, x > 0 \longrightarrow$ Polynomial

$a^n, a > 1 \longrightarrow$ Exponential

Poly < Expo

$\boxed{\text{Poly} = O(\text{Expo})}$

★ $F(n) = \log(n^x) \Rightarrow x * \log(n) \longrightarrow O(\log n)$

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$$\left.\begin{aligned}
F(n) &= \log(n^3) \longrightarrow 3 * \log(n) \\
F(n) &= \log(n^6) \longrightarrow 6 * \log(n) \\
F(n) &= \log(n^{10}) \longrightarrow 10 * \log(n) \\
F(n) &= \log(n^{100}) \longrightarrow 100 * \log(n)
\end{aligned}\right\} \Rightarrow O(\log(n))$$

(8) $(\log n)^x = O(n^y)$, $x > 0, y > 0$

OR $\Downarrow$

$\boxed{\log^x(n)}$

logarithmic < Poly

$\boxed{\text{logarithm} = O(\text{Poly})}$

* Practice Problems : True / False

① if $0 < x < y$ then $n^x = O(n^y)$ → True

② $\log(n)$ is $\Omega(1/n)$ → True

③ 2^{n^2} is $O(n!)$ → False ☆

④ $\underline{\underline{20}} * n * \log n = O(n\log n)$ True

→ $O(n\log n)$

⑤ $(n+c)^k \neq O(n^k)$ for → False

some $\underbrace{k>0, c>0}_{\text{constants}}$

⑥ ☆ n^2 is $O(2^{(\log n)})$ → False

Soln:-

① $0 < x < y \Rightarrow n^x = O(n^y)$

$2 < 4 \Rightarrow n^2 \stackrel{?}{=} O(n^4)$ ✓

② $\log(n) = \Omega(1/n)$

✓ $\log(n) \geq c * \frac{1}{n}$?

$\downarrow$ logarithmic

$\rightarrow$ Decr func

Soln:

③ $2^{n^2} = O(n!)$? X → False

$2^{(n^2)}$ $\quad$ $n! \approx n^n$

$2^{(n^2)}$ $\quad$ n^n

☆ Take $\log_2()$ both sides

$\log(2^{n^2})$ $\quad$ $\log(n^n)$

$n^2 * 1$ $\quad$ $n * \log n$.

n^2 $\quad$ $n * \log n$

$n * \cancel{n}$ $\quad$ $\cancel{n} * \log n$

$n > \log(n)$

$\boxed{2^{n^2} > n!}$ → $n! = O(2^{n^2})$

5 $(n+c)^k \neq O(n^k)$ $\longrightarrow$ False

$c, k > 0$

eg: $(n+7)^3 = O(n^3)$? ✓

Polynomial of degree k $\rightarrow O(n^k)$

$n^2 = O(2^{\log n})$? $\Rightarrow n^2$ vs $2^{\log n}$

Take $\log_2()$ both sides

$\log(n^2)$ $\log(2^{\log n})$

$$2\log(n) > \log n * 1 \;?$$

$\Rightarrow$ $\log n + \cancel{\log n}$ $\cancel{\log n}$

$\log n > 1$

$n^2 > 2^{\log(n)}$

$$\boxed{n^2 \neq O(2^{\log n})}$$

$\rightarrow$ Compare Absolute values

NOT Asymptotic (Rate of growth)

Topic : Adding Functions

The sum of two functions is governed by the dominant one, namely:

$O(f(n)) + O(g(n)) \rightarrow 0\ (\max(f(n), g(n)))$

$\Omega f(n)) + \Omega(g(n)) \rightarrow \Omega(\max(f(n), g(n)))$

$\theta(f(n)) + \theta(g(n)) \rightarrow \theta(\max(f(n), g(n)))$

eg :

$$f(n) = 5n^2 + 2 \longrightarrow O(n^2)$$

$$g(n) = 10n^3 \longrightarrow O(n^3)$$

$$O(f(n)) = n^2$$

$$O(g(n)) = n^3$$

$$n^2 + n^3 = O(\max(5n^2+2, 10n^3))$$

$$= O(10n^3)$$

$$= O(n^3) \checkmark$$

Topic : Multiplication of Functions

$O(f(n)) * O(g(n)) \rightarrow O(f(n) * g(n))$

$\Omega(f(n)) * \Omega(g(n)) \rightarrow \Omega(f(n) * g(n))$

$\theta(f(n)) * \theta(g(n)) \rightarrow \theta(f(n) * g(n))$

eg- $f(n) = 10n$, $g(n) = 2n^2$ $\Rightarrow f(n) \times g(n)$

$= 10n \times 2n^2$

$= 20n^3$

$\Downarrow$

$\theta(20n^3) = n^3$

$\theta(f(n)) = n$

$\theta(g(n)) = n^2$

$n \times n^2 = n^3$

Imp Practice Questions:-

Imp ① $n^2 = O(2^{(2\log n)})$ True

② $(\log n)^{1/2} = O(\log(\log n))$

↳ False

③ $a^n \neq O(n^x)$ True

for $a > 1, x > 0$

$a^n \longrightarrow$ expo

$n^x \longrightarrow$ Poly

$n^x = O(a^n)$

$a^n > n^x$

$a^n = \Omega(n^x)$ ✓

Soln:

(1) $n^2 = O(2^{(2\log n)}) \Rightarrow ?$

n^2 vs $2^{(2\log n)}$

Taking $\log_2()$ on both sides:

$\log_2(n^2)$ $\quad\quad$ $\log(2^{(2\log n)})$

$2 * \log_2 n$ $\quad\quad$ $2\log n * \cancel{\log_2(2)}^{1}$

$\downarrow$

$2 * \log_2 n = 2 * \log_2 n$

$$\boxed{n^2 = \Theta(2^{2\log n})}$$

$$\left\{ \begin{array}{l} n^2 = O(2^{2\log n}) \\ \text{and } 2^{2\log n} = O(n^2) \end{array} \right\} \checkmark$$

② $(\log n)^{1/2} = O(\log(\log n))$ → False

way 1 :- Let $x = (\log n)$

$(x)^{1/2}$ $\quad$ $\log(x)$

$\sqrt{x} > \log(x)$

$(\log n)^{1/2} > \log(\log n)$

way 2 → Take log both sides

$(\log n)^{1/2} > \log(\log n)$

$\log((\log n)^{1/2})$ $\quad$ $\log(\log(\log n))$

$\frac{1}{2} * \underbrace{\log(\log n)}_{x}$ $\quad$ $\log(\log(\log n))$

Let $\log(\log n) \to x$

$\frac{1}{2} x > \log(x)$

* **Tricotomy Property:**

Gate: Does Asymptotic notations follow Tricotomy property?

$\Rightarrow$ **No**

* Tricotomy Property in Real numbers:

→ Given any two real numbers x & y (fixed)
Then x and y follow $\boxed{\text{exactly one}}$ of the below relation;

1) $x = y$

OR

2) $x > y$

OR

3) $x < y$

✓

eg. $x = 5$ $y = 7$

$x < y$ ✓

* Tricotomy Property in Asymptotic Comparison of two functions.

→ Give two +ve functions $f(n)$ & $g(n)$, does $f(n)$ & $g(n)$ always follow exactly one of the below?

1) $f(n) \geq_A g(n)$

OR 2) $f(n) =_A g(n)$

OR 3) $f(n) \leq_A g(n)$

$f \geq_A g \Rightarrow f = \Omega(g)$, $\omega(g)$

$f \leq_A g \Rightarrow f = O(g)$, $= o(g)$

$f =_A g \Rightarrow f = \Theta(g)$

eg1:

$$f(n) = 5n^2$$

$$g(n) = 7n^3$$

$$\Downarrow$$

$f \underset{A}{\leq} g$ (always) $n \geq n_0$ $\Rightarrow$ $\left.\begin{array}{l} f(n) = O(g(n)) \\ f(n) = O(g(n)) \end{array}\right\}$ always

$\longrightarrow$ Holds for this case

eg2 :

$f(n) = 5n^2$

$g(n) = \frac{1}{n}$

$\Downarrow$

$f(n) \geq_A g(n)$

(for all $n \geq n_0$)

$\Rightarrow f(n) = \Omega(g(n))$ → holds here as well

$f(n) = \omega(g(n))$

eg3:

$$f(n) = 10n^3 + 7$$

$$g(n) = 5n^3 + 15$$

Holds here as well

$$\Downarrow$$

$$f(n) \underset{A}{=} g(n) \implies \begin{array}{l} f(n) = \Theta(g(n)) \\ g(n) = \Theta(f(n)) \end{array}$$

(Rate of growth is Asymptotically equal) eg4 :→ PYQ → GATE

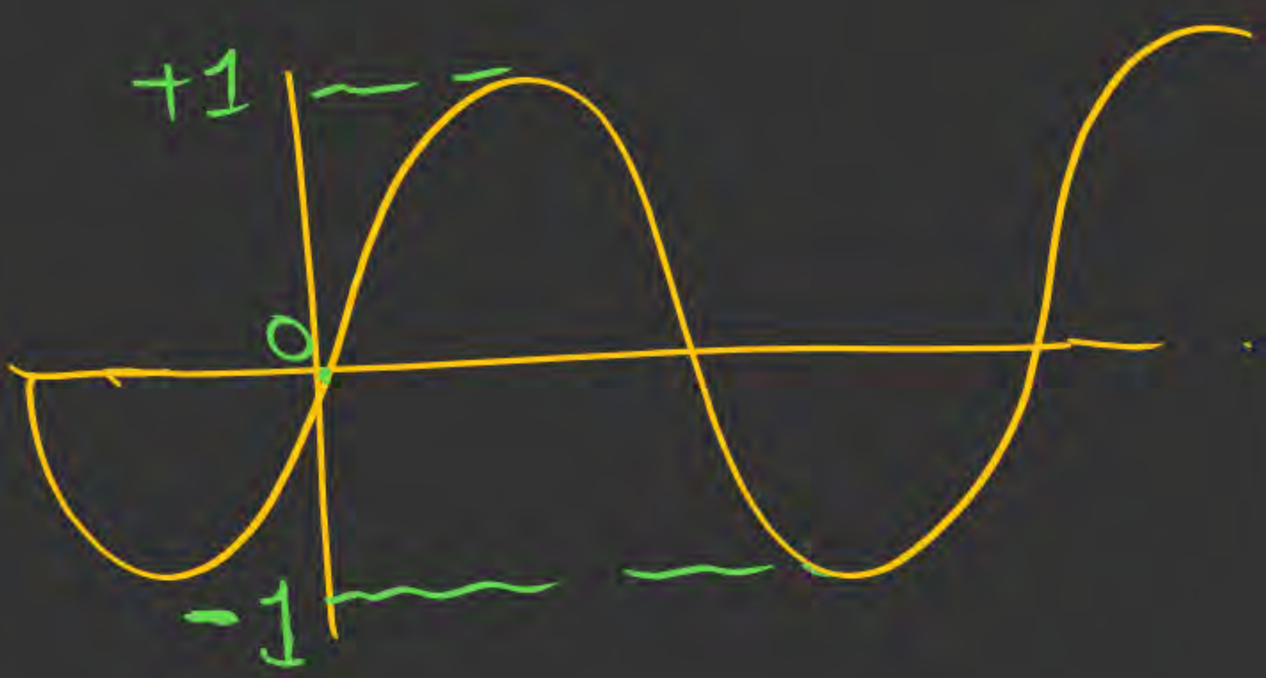
$f(n) = n$

$g(n) = n^{(1+\sin x)}$

$n, x \rightarrow$ variables.

$[-1, 1]$

$\sin x$ plot (wave)



$f(n) = n, \quad g(n) = n^{(1+\sin x)}$

Case 1

$\sin x \rightarrow \max \rightarrow +1$

$f(n) = n, \; g(n) = n^{(1+\sin x)}$

$= n^{(1+1)}$

$f(n) = n, \; g(n) = n^2$

$\Downarrow \; f(n) \le_A g(n)$

$\Rightarrow f(n) = O(g(n))$ ①

Case 2

$\sin x \rightarrow \min \rightarrow -1$

$f(n) = n, \; g(n) = n^{(1+(-1))}$

$= n^0 = 1$

$f(n) = n, \; g(n) = 1$ (const)

$\Downarrow \; f(n) \ge_A g(n)$

$f(n) = \Omega(g(n))$ ②

Observation from pre eg:-

$f(n) = n,$

$g(n) = n^{(1+\sin n)}$

f(n) & g(n) are not holding the trichotomy property as we can't get a clear asymptotic comparison between both for all $n \geq n_0$ (some constant)

Conclusion :-

Asymptotic Notations may or may not follow Tricotomy Property (holds sometimes)

$\Downarrow$

$\Rightarrow$ Hence property does not Hold

(As Property holds if its always satisfied for all cases)

THANK - YOU

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