CS & IT ENGINEERING

Algorithms

Analysis of Algorithms

Lecture No.- 04

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Recap of Previous Lecture

Topic Asymptotic Notations

Topic Big Oh, Big Omega & Theta Notations

O Ω Θ

Topics to be Covered

Topic Asymptotic Notations (Practice Questions)

Topic Little Oh, Little Omega
o ω

Topic Properties of ASN

Topic Problem solving

Lower Bound

② Big Omega (Ω):

$f(n)$ & $g(n)$

$$f(n) = \Omega(g(n))$$

iff, $f(n) \geq c * g(n)$, $n > n_0$, some $c > 0$

eg: $7n^3 + 8n + 1000 \Rightarrow \Omega(n^3)$

③ Theta (Θ) $\longrightarrow$ tight Bound

$f(n)$ & $g(n)$ $\Longrightarrow$ $f(n) = \Theta(g(n))$

iff $f(n) = O(g(n))$

and

$f(n) = \Omega(g(n))$

or $\boxed{C_2 * g(n) \leq f(n) \leq C_1 * g(n)}$

eg: $f(n) = 9n^3 + 7n^2 + 100$

$f(n) = O(n^3)$

$f(n) = \Omega(n^3)$

Hence, $\boxed{f(n) = \Theta(g(n))}$

Imp observations:

① Larger functions are always Omega of the Smaller functions.

$$\boxed{\begin{aligned} 2^n &= \Omega(n^2) \\ 2^n &= \Omega(n^3) \end{aligned}}$$

eg: $n^3 \geq c * n^2$

$$\boxed{n^3 = \Omega(n^2)}$$

(2) Smaller functions are always ________ of the larger functions Order (Big Oh)

$$\boxed{\begin{aligned} n^2 &= O(2^n) \\ n^3 &= O(2^n) \end{aligned}}$$

$n^3 \quad n^2$

$$n^2 = O(n^3)$$

(3) If two function have equal rate of growth, then they are Theta of each other.

$f(n) = 8n^2$

$g(n) = 5n^2$

$$\boxed{\begin{gathered} f(n) = \Theta(g(n)) \\ \text{or} \\ g(n) = \Theta(f(n)) \end{gathered}}$$

* Practice Questions :

$8n \geq 2n \to c_1$

$8n \leq 800n \to c_2$

① $f(n) = 8n \to O(n),\ \Omega(n) \to \Theta(n)?$

Constant

② $f(n) = 9^{200} \to O(1),\ \Omega(1) \to \Theta(1)$

③ $f(n) = \underbrace{100n^2 + 500n}$

$O(n^2)$ & $\Omega(n^2)$

$\Theta(n^2)$

④ $f(n) = 100(\log n) + 50 * \sqrt{n} \to O(\sqrt{n}),\ \Omega(\sqrt{n}),\ \Theta(\sqrt{n})$

⑤ $f(n) = 500 * \sqrt{n} + 2n + 100.$

$O(n)$ $\Omega(n)$

$\Theta(n)$

(4) $F(n) = 100 * (\log n) + 50\sqrt{n} \rightarrow$ dominating

Poly > log

way1 :

$\hookrightarrow O(\sqrt{n}) \nearrow \Theta(\sqrt{n})$

$\hookrightarrow \Omega(\sqrt{n})$

Put values of n.

$2^x = 64$

$x = 6$

$\sqrt{n} > \log_2(n)$

$n = 64$

$\sqrt{64} \quad \log_2(64) = \log_2(2^6)$

$\downarrow \qquad \downarrow$

$8 > 6$

Let $\log n = x$

way2 :-

Taking log & Comparing.

$\log n < \sqrt{n} \Rightarrow n^{1/2}$

$\Longrightarrow \boxed{\log n = O(\sqrt{n})}$

Take log on both sides

$\log(\log n) \quad \log(n^{1/2})$

$\log(\underbrace{\log n}) \quad \frac{1}{2} * \log(n)$

$\log(x) < \frac{1}{2} * x$

* Possible mistake with 2^{nd} approach:

$$n^2 < n^3$$

Take log on both sides:

$$\log(n^2) < \log(n^3)$$

$$2 * \cancel{\underline{\log(n)}} \qquad 3 * \cancel{\underline{\log(n)}}$$

$$2 < 3 \longrightarrow \checkmark$$

mistake

Const $O(1) = O(1)$ const

$$n^2 = n^3$$

after $\boxed{\log(\ )}$

Compare mathematically
Not Asymptotically

$2n^3 + 4n$

$2n^3 + 4n \geq 3n^2$

$2n^3 + 4n = \boxed{\Omega(n^2)}$ ✓

$= \Omega(n^3)$

$= \Omega(n)$

$= \Omega(1)$

$5n + 7 = O(n) \longrightarrow$ Tight UB

$\left.\begin{array}{l} = O(n^2) \\ = O(n^3) \\ \vdots \end{array}\right\}$ loose UB

Topic : Time Complexity

Venn Diagram?

$f(n)$

$3\log n+8$, $4n^2$
$5n+7$, $6n^2+9$
$2n\log n$, $5n^2+2n$

(a) $O(n^2)$

$g(n)=n^2$

$f(n) \le c * g(n)$

$f(n)$

$4n^2$, $4n^3+3n^2$
$6n^2+9$, $6n^6+n^4$
$5n^2+2n$, $2n^3+4n$

(b) $\Omega(n^2)$

$g(n)=n^2$

$f(n) \ge c * g(n)$

$3\log n+8$, $5n+7$, $2n\log n$ | $4n^2$, $6n^2+9$, $5n^2+2n$ | $4n^3+3n^2$, $6n^6+n^4$, $2n^3+4n$

(b) $\theta(n^2) = O(n^2) \cap \Omega(n^2)$

Topic : Exponentials

Mathematical Properties

For all real a> 0, m, n

$a^0 = 1$

$a^1 = a$

$a^{-1} = 1/a$

$(a^m)^n = a^{mn}$

$(a^m)^n = (a^n)^m$

$a^m \times a^n = a^{m+n}$

$2^0 = 1$

$(5)^1 = 5$

$(5)^{-1} = \frac{1}{5}$

$(4)^{-2} = \frac{1}{(4)^2}$

Imp $(a^m)^n \neq a^{m+n}$

$(a^m)^n = a^{m*n}$

eg:- $(2^3)^2 \longrightarrow 2^{3\times 2}$

$\downarrow$

$(8)^2$

$= \underline{\underline{64}}$

$= 2^6$

$= \underline{\underline{64}}$

$$\boxed{(a^m)^n = (a^n)^m = a^{m*n}}$$

$(2^3)^2 = (2^2)^3 = 2^{2\times 3}$

$8^2 = 4^3 = 2^6$

$64 = 64 = 64$

* $a^m * a^n = a^{(m+n)} \neq a^{(m*n)}$

eg- $2^3 * 2^2 = 2^{(3+2)}$

$8 * 4 = 2^5$

$= 32 \qquad = 32$

Topic : Analysis of Algorithms

* Logarithmic Properties

$\log(x^y) = y * \log(x)$

$\log X^y = y \log x$

$\log(xy) = \log x + \log y$

$\log \log n = \log(\log n)$

$\boxed{a^{\log_b^x} = x^{\log_b^a}}$

$a = b^{\log_b^a}$

$\log_b a^n = n.\log_b a$

* $\log_b a = \frac{1}{\log_a b}$

$\log^k n = (\log n)^k$

$\log \frac{x}{y} = \log x - \log y$

$\boxed{\log_b^x = \frac{\log_a^x}{\log_a^b}}$

$\log_c(ab) = \log_c a + \log_c b$

$\log_b^{1/a} = -\log_b a$

$a^{\log_b c} = c^{\log_b a}$

$\boxed{\log_x(1) = 0}$

eg:- $\log_2(16)$

$\log\log n$

$\log(\log n)$

$\Rightarrow \log_2(4 \times 4) = \log_2(4) + \log_2(4)$

$= \log_2(2^2) + \log_2(2^2)$

$= 2 + 2 = 4$

$\log_2(2^4)$

$= 4$

$\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$

$\log(a * b) = \log(a) + \log(b)$

$$\log_b(a) = \frac{\log_c(a)}{\log_c(b)} = \frac{1}{\frac{\log_c(b)}{\log_c(a)}} \longrightarrow \log_a b$$

$$= \frac{1}{\log_a(b)}$$

$\log^2(n) \Rightarrow [\log(n)]^2 \Rightarrow (\log n) * (\log n)$

$\xrightarrow{\times} \log(\log n) \times$

$\log_b\left(\frac{1}{a}\right) = -\log_b a$ How?

$\log_b(1/a) = \log_b(1) - \log_b(a)$

$= 0 - \log_b a$

$= -\log_b a$

Topic : Geometric Sum Formula

GP

1. The geometric sum formula for finite terms is given as:

if r = 1, $S_n = n*a$

if |r| < 1, $S_n = \dfrac{a(1-r^n)}{1-r}$

if |r| > 1, $S_n = \dfrac{a(r^n-1)}{r-1}$

Where

- a is the first term
- r is the common ratio
- n is the number of terms

$\underbrace{2, 2, 2, 2}_{n*2}$

Geometric Progression :-

$S = 2^1 + 2^2 + 2^3 + 2^4 = 2 + 4 + 8 + 16 = 10 + 20 = 30$

a

$r = 2$

$n = \text{no. of terms} = 4$

$$\text{Sum} = \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{2(2^4 - 1)}{2 - 1} = \frac{2(16 - 1)}{1} = 30$$

$\frac{2^2}{2^1} = 2, \quad \frac{2^3}{2^2} = 2$

eg1: $\sum_{i=1}^{n} 2^i$

$1 \longrightarrow n$

n terms

$a = 2^1$

$r = 2$

$= 2^1 + 2^2 + \cdots + 2^n$

$= \frac{a(r^n - 1)}{r - 1}$

$= \frac{2(2^n - 1)}{2 - 1} = \boxed{2(2^n - 1)}$

Check : $n = 4$

$2^1 + 2^2 + 2^3 + 2^4$

$= 2(2^4 - 1)$

$= 30$ ✓

eg:-

$$\sum_{i=1}^{n} \frac{1}{3^i} = \frac{1}{3^1} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots \frac{1}{3^n}$$

$$\frac{1}{3^1} \rightarrow \frac{1}{3^n}$$

n terms

$$a = \frac{1}{3^1}$$

$$r = \frac{\frac{1}{3^2}}{\frac{1}{3^1}} = \frac{1}{3}$$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{\frac{1}{3}\left(1-\left(\frac{1}{3}\right)^n\right)}{1-1/3}$$

$|r|<1$

$$= \frac{\frac{1}{\cancel{3}} \times \left(1-\frac{1}{3^n}\right)}{2/\cancel{3}}$$

$$= \boxed{\frac{1}{2} * \left(1-\frac{1}{3^n}\right)}$$

Topic : Geometric Sum Formula

2. The geometric sum formula of infinite terms is given as:

if $|r| < 1$ $\boxed{S_\infty = \frac{a}{1-r}}$

if $|r| > 1$, the series does not converge and it has no sum.

$\left[2^1 + 2^2 + \cdots \infty\right]$

Topic : Analysis of Algorithms

Arithmetic series

Imp

$$\sum_{k=1}^{n} k = 1 + 2 + \ldots + + n = \frac{n(n+1)}{2}$$

$\Longrightarrow$ Sum of first n natural nos

$\Rightarrow 1+2+3+4$

$= 10$

$\frac{n(n+1)}{2} = \frac{4 \times 5}{2}$

$= 10$

Geometric series

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 \ldots + x^n = \boxed{\frac{x^{n+1} - 1}{x - 1} (x \neq 1)}$$

Harmonic series

$$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \ldots + \frac{1}{n} \approx \log n$$

$\int_1^n \frac{1}{x} = \log(n)$

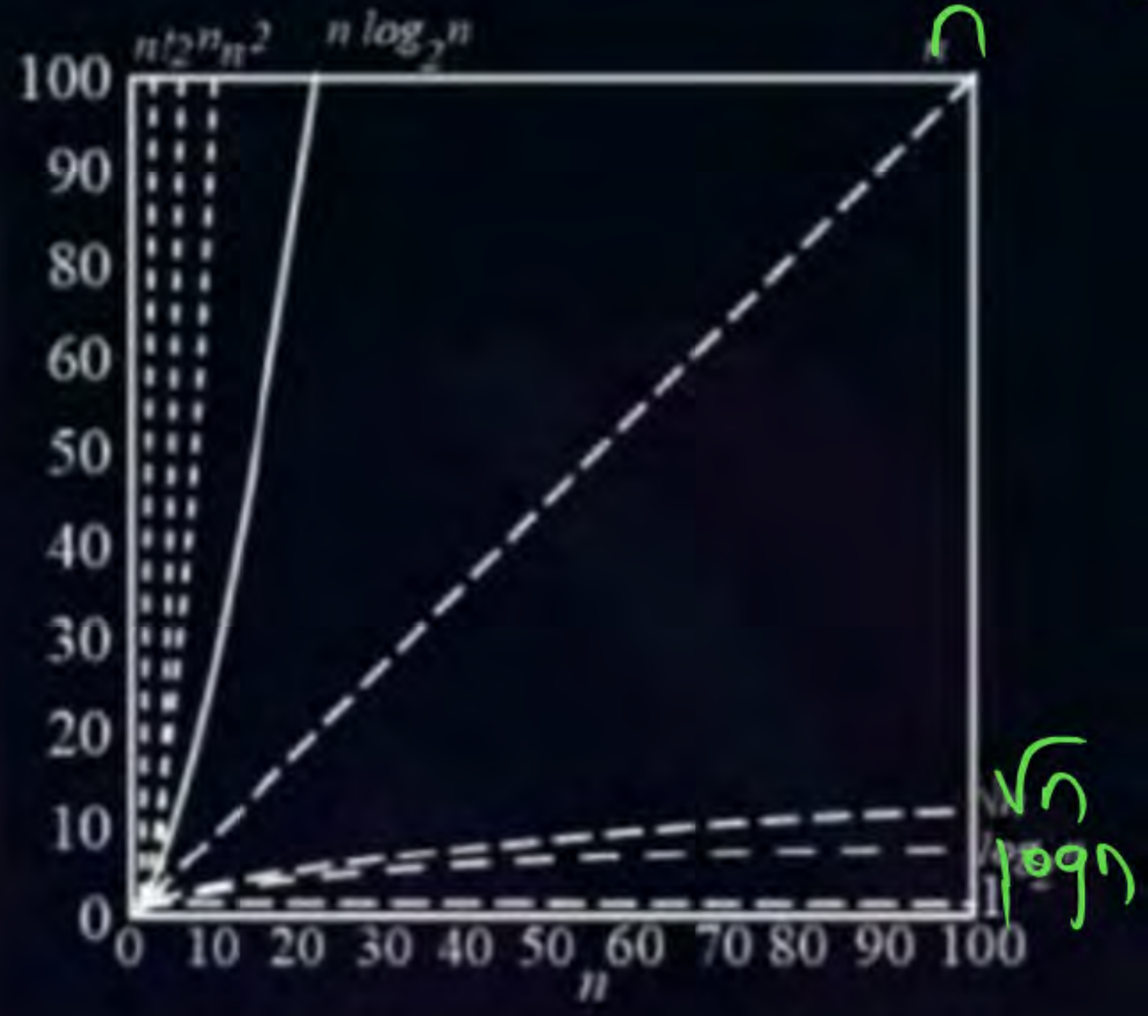
Topic : Analysis of Algorithms

General

Dominance Relation

Constant < logarithms < poly < exponential

$$\frac{1}{n} < 1 < \log n < \sqrt{n} < n < n\log n < n^2 < 2^n < n^n$$



Practice Problems:

Shortcut: $\sum_{a=1}^{n} x^a \longrightarrow O(x^n)$

① $f(n) = \sum_{a=1}^{n} a \rightarrow O(n^2)$

② $f(n) = \sum_{a=1}^{n} a^2 \rightarrow O(n^3)$

③ $f(n) = \sum_{a=1}^{n} a^3$ → $O(n^4)$

☆ ④ $f(n) = \sum_{a=1}^{n} 1 \longrightarrow O(n)$

⑤ $\sum_{a=1}^{n} 3^a \longrightarrow O(3^n)$

Imp

⑥ $\sum_{a=1}^{n} \left(\frac{1}{5}\right)^a \Rightarrow O(1)$ ✓

$O\left(\frac{1}{5^n}\right)$ ✗

⑦ $\sum_{a=1}^{n} n \Rightarrow O(n^2)$

$$(a+b)^2 = a^2 + 2ab + b^2$$

① $\sum_{a=1}^{n} a$

$= 1+2+3 \cdots + n$

$= \frac{n(n+1)}{2} = \frac{n^2+n}{2}$

$O\left(\frac{n^2+n}{2}\right) = O(n^2)$ ✓

② $\sum_{a=1}^{n} a^2$

$= 1^2+2^2+3^2 \cdots n^2$

☆ $= \frac{n \times (n+1)(2n+1)}{6} = \frac{(n^2+n)(2n+1)}{6}$

$= 2n^3 + \cdots\cdots$

$\Rightarrow \boxed{O(n^3)}$

③ $\sum_{a=1}^{n} a^3$

$= 1^3 + 2^3 + 3^3 \cdots n^3$

$= \left[\frac{n(n+1)}{2}\right]^2$

$= \left(\frac{n^2+n}{2}\right)^2 \approx \boxed{O(n^4)}$

$3^1 * 3^n = 3^{n+1} = O(3^{n+1})$ X

$= O(3 * 3^n) = \underline{O(3^n)}$ ✓

④ $\sum_{a=1}^{n} 1 = O(1)$? X

$= \underbrace{1+1+1+ \ldots +1}_{\text{n times}}$

$= n * 1$

$= n$

$\Rightarrow O(n)$ ✓

⑤ $\sum_{a=1}^{n} 3^a = 3^1 + 3^2 + 3^3 \cdots 3^n$

$a = 3$
$r = 3$
$n = n$

$= \frac{a(r^n - 1)}{r-1}$

$= \frac{3(3^n-1)}{3-1} = \frac{3(3^n-1)}{2} = \frac{3 * 3^n - 3}{2}$

$= \boxed{O(3^n)}$

V.V. Imp

6) $\sum_{a=1}^{n}\left(\frac{1}{5}\right)^{a}$

$$= \left(\frac{1}{5}\right)^1 + \left(\frac{1}{5}\right)^2 + \left(\frac{1}{5}\right)^3 \cdots + \left(\frac{1}{5}\right)^n$$

GP
$a = (1/5)$
$r = (1/5)$
$n = n$

$r < 1$ $\Rightarrow \frac{a \times (1 - r^n)}{1 - r} = \frac{\frac{1}{5} \times \left(1 - \frac{1}{5^n}\right)}{1 - 1/5}$

$$= \frac{\frac{1}{\cancel{5}} \times \left(1 - \frac{1}{5^n}\right)}{4/\cancel{5}}$$

$$= \frac{\left(1 - \frac{1}{5^n}\right)}{4} = \boxed{O(1)}$$

18% ← A) $O(5^n)$

43% ← B) $O\left(\frac{1}{5^n}\right)$

8% ← C) $O(5n)$

31% ← D) $O(1)$ ✓

$\frac{1}{5}$ $\frac{1}{25}$ $\frac{1}{125}$

Const ← $1 > \frac{1}{5^n}$

Decr < Const

7. $\sum_{a=1}^{n} n \Rightarrow \underbrace{n+n+n \cdots + n}_{n \text{ times}}$

Appr 1

$= n \times n$

$= n^2$

$\Rightarrow O(n^2)$ ✓

Appr 2

$\sum_{a=1}^{n} n = n * \sum_{a=1}^{n} 1$

$= n * (\underbrace{1+1+1 \ldots 1}_{n \text{ times}})$

$= n \times n$

$= O(n^2)$ ✓

Summary:

1. Big Oh: $f(n)$ & $g(n)$ $\rightarrow$ Upper Bound

$f(n) = O(g(n))$

iff $f(n) \leq c * g(n)$, $n > n_0$, some $c > 0$

eg- $\underbrace{8n^2 + 5n + 100}_{f(n)} \Rightarrow O(n^2)$

THANK - YOU