GATE

# Computer Science & Information Technology

## Discrete Mathematics

DPP: 2

## Combinatorics

**Q1** The generating function for the following sequence is
2,4,6,8,10,…………………..

(A) $s = \frac{2}{(1-x)^2}$ (B) $s = \frac{2}{(1+x)^2}$
(C) $s = \frac{1}{(1-x)^2}$ (D) $s = \frac{1}{(1+x)^2}$

**Q2** Find coefficient of $x^2y^3$ in the expansion of $(2x-3y)^5$.

**Q3** Find coefficient of $x^6$ in the expansion of $\left(2x^3 - \frac{1}{x^2}\right)^{12}$

**Q4** Find coefficient of $x^{15}$ in the expansion of $(x^2 + x^3 + x^4 + x^5)^3$

**Q5** The sequence 1,3,7,15,31,63,…….. satisfies the recurrence relation
$a_n = 3a_{n-1} - 2a_{n-2}$
Find the generating function for the sequence.

(A) $S = \frac{1}{(1+3x+2x^2)}$
(B) $S = \frac{1}{(1-3x+2x^2)}$
(C) $S = \frac{1}{(1-3x-2x^2)}$
(D) $S = \frac{1}{(1+3x-2x^2)}$

**Q6** Find a generating function for sequence 1, 4, 16, 64, …….

(A) $S = \frac{2}{1+4x}$ (B) $S = \frac{2}{1-4x}$
(C) $S = \frac{1}{1-4x}$ (D) $S = \frac{1}{1+4x}$

**Q7** Find a generating function for sequence: 1,–5, 25, –125, 625,…

(A) $\frac{1}{(1+25x)}$ (B) $\frac{1}{(1-25x)}$
(C) $\frac{1}{(1-5x)}$ (D) $\frac{1}{(1+5x)}$

**Q8** Find the closed form expression for the generating function of the sequence $\{a_n\}$, where $a_n = 4(2)^n + 5(-4)^n$, for all n = 0, 1, 2, 3 ………

(A) $\frac{4}{(1-2x)} + \frac{5}{(1+4x)}$
(B) $\frac{4}{(1-2x)} - \frac{5}{(1+4x)}$
(C) $\frac{4}{(1+2x)} + \frac{5}{(1+4x)}$
(D) $\frac{4}{(1-2x)} + \frac{5}{(1-4x)}$

**Q9** $a_n = 3a_{n-1} - 2a_{n-2}$, $n \geq 2$ with initial terms $a_0 = 1$ and $a_1 = 3$.
Solution of above recurrence relation using generating function is

(A) $2^{n+1} - 1$ (B) $2^{n+1} + 1$
(C) $2^{n-1} - 1$ (D) $2^{n-1} + 1$

**Q10** If the generating function of the sequence $\{a_0, a_1, a_2, \ldots\}$ is $\frac{x}{(1-x^3)^2}$ then $(a_1 + a_2)$ is equal to?

# Answer Key

| | | | |
|---|---|---|---|
| Q1 | (A) | Q6 | (C) |
| Q2 | -1080~-1080 | Q7 | (D) |
| Q3 | 59136~59136 | Q8 | (A) |
| Q4 | 1~1 | Q9 | (A) |
| Q5 | (B) | Q10 | 3~3 |

GATE

# Hints & Solutions

**Q1 Text Solution:**

S = 2 + 4x + 6x$^2$ + 8x$^3$ + 10x$^4$ +...

⇒ xS = 2x + 4x$^2$ + 6x$^3$ + 8x$^4$ + 10x$^5$ +....

⇒ S – xS = 2 + 2x + 2x$^2$ + 2x$^3$ + ....

⇒ S(1-x) = 2(1 + x + x$^2$ + x$^3$ +....)

⇒ S(1 –x) = $\frac{2}{1-x}$

⇒ S = $\frac{2}{(1-x)^2}$

**Q2 Text Solution:**

$(2x-3y)^5 = \sum_{k=0}^{5} \; 5_{C_k}(2x)^{5-k}\,(-3y)^k$

The term for x$^2$y$^3$ is

$^5c_3(2x)^{5-3}(-3y)^3$

= $\left(\frac{5\times4\times3!}{2\times3!}\right)\times\left(4x^2\right)\times\left(-27y^3\right)$

= –1080 x$^2$y$^3$

**Q3 Text Solution:**

$=\left(\frac{2x^5-1}{x^2}\right)^{12}$

$=\frac{1}{x^{24}}\left(2x^5-1\right)^{12}$

So, now,

$\left(2x^5-1\right)^{12} = \sum_{k=0}^{12}\; 12_{C_k}\;\left(2x^5\right)^{12-k}(-1)^k$

for x$^{30}$,

$= \;^{12}c_6\left(2x^5\right)^{12-6}(-1)^6$

$=^{12}\,c_6\left(2x^5\right)^6$

= 924 × 64

= 59136

**Q4 Text Solution:**

(x$^2$ + x$^3$ + x$^4$ + x$^5$)$^3$

$=\left(x^2\right)^3\left(1+x+x^2+x^3\right)^3$

$=x^6\left(\frac{1-x^4}{1-x}\right)^3$

$=x^6\left(1-x^4\right)^3(1-x)^{-3}$

$=x^6\left(1-x^{12}-3x^4+3x^8\right)(1-x)^{-3}$

$=\left(x^6-x^{18}-3x^{10}+3x^{14}\right)(1-x)^{-3}$

$=\left(x^6\times\,^{3+9-1}C_9x^9\right)-3x^{10}\times\,^{3+5-1}C_5x^5$
$+3x^{14}\times\,^{3+1-1}C_1x^1$

$=\,^{11}C_9x^{15}-3\times\,^7C_5x^{15}+3\times\,^3C_1x^{15}$

$=\left(\frac{11\times10}{2}-\frac{3\times7\times6}{2}+3\times3\right)x^{15}$

= (55 – 63 + 9)x$^{15}$

= 1 x$^{15}$

**Q5 Text Solution:**

S = 1 + 3x + 7x$^2$ + 15x$^3$ + 31x$^4$ + .....

⇒ xS = x + 3x$^2$ + 7x$^3$ + 15x$^4$ + ....

⇒ S – xS = 1 + 2x + 4x$^2$ + 8x$^3$ + 16x$^4$ + ...

$\Rightarrow S - xS = \frac{1}{1-2x}$

$\Rightarrow S = \frac{1}{(1-x)(1-2x)} = \frac{1}{1-3x+2x^2}$

**Q6 Text Solution:**

S = 1 + 4x + 16x$^2$ + 64x$^3$ + .....

⇒ xS = x + 4x$^2$ + 16x$^3$ + 64x$^4$ + ....

⇒ S – xS = 1 + 3x + 12x$^2$ + 48x$^3$ + ...

$\Rightarrow S\,(1-x) = 1+\frac{3x}{1-4x}$

$\Rightarrow S\,(1-x) = \frac{1-x}{1-4x}$

$\Rightarrow S = \frac{1}{1-4x}$

**Q7 Text Solution:**

S = 1 – 5x + 25x$^2$ – 125x$^3$ + 625x$^4$ – .....

S = (1 + 25x$^2$ + 625x$^4$ + ...) – (5x + 125x$^3$ + ...)

$S = \frac{1}{1-25x^2} - \frac{5x}{1-25x^2}$

$S = \frac{1-5x}{1-25x^2}$

$S = \frac{(1-5x)}{(1+5x)(1-5x)} = \frac{1}{1+5x}$

**Q8 Text Solution:**

$G\,(x) = \sum_{n=0}^{\infty}\{4(2)^n+5(-4)^n\}x^n$

$= 4\sum_{n=0}^{\infty}2^nx^n + 5\sum_{n=0}^{\infty}(-4)^nx^n$

$= 4\left(\frac{1}{1-2x}\right)+5\left(\frac{1}{1-(-4x)}\right)$

$= \frac{4}{1-2x}+\frac{5}{1+4x}$

**Q9 Text Solution:**

$G\,(x) = \sum_{n=0}^{\infty}a_nx^n$

$= a_0x^0 + a_1x^1 + \sum_{n=2}^{\infty}a_nx^n$ ...(i)

$= 1+3x+\sum_{n=2}^{\infty}(3a_{n-1}-2a_{n-2})x^n$

$= 1+3x+3\sum_{n=2}^{\infty}a_{n-1}x^n - 2\sum_{n=2}^{\infty}a_{n-2}x^n$

$= 1+3x+3x\sum_{n=2}^{\infty}a_{n-1}x^{n-1} - 2x^2\sum_{n=2}^{\infty}a_{n-2}x^{n-2}$

$= 1+3x+3x\sum_{n=1}^{\infty}a_nx^n - 2x^2\sum_{n=0}^{\infty}a_nx^n$

$= 1+3x+3x\left(a_1x^1+\sum_{n=2}^{\infty}a_nx^n\right) - 2x^2G\,(x)$

Android App | iOS App | PW Website

GATE

$= 1 + 3x + 3x(3x + G(x) - 1 - 3x)$
$- 2x^2 G(x)$

$= 1 + 3x + 3xG(x) - 3x - 2x^2 G(x)$

$G(x) = 1 + 3xG(x) - 2x^2 G(x)$

$G(x) = \frac{1}{2x^2 - 3x + 1} = \frac{1}{(2x-1)(x-1)}$

Using partial fraction decomposition,

$\Rightarrow G\left(x\right) = \frac{1}{(2x-1)(x-1)} = \frac{-2}{(2x-1)} + \frac{1}{(x-1)}$

$= \frac{2}{(1-2x)} - \frac{1}{(1-x)}$

$\Rightarrow G\left(x\right) = 2\ \ (1 + n - 1)_{C_n}\ 2^n\ - \left(1x1\right)$

$= 2^{n+1} - 1$

**Q10 Text Solution:**

$\frac{x}{(1-x^3)^2} = x\left[1 - x^3\right]^{-2} = x.\ {}^{2+r-1}C_r\left(x^3\right)^r$

$= {}^{1+r}C_r x^{3r+1}$

For $a_1$,

${}^{1+0}C_0\ x^{3\times0+1}$

$= {}^1C_0\ x^1$

$= 1x^1$

$a_1 = 1$

For $a_4$,

$3r + 1 = 4$

$r = 1$

${}^{1+1}C_1\ x^4$

$= 2x^4$

$a_4 = 2$

$a_1 + a_4 = 1 + 2 = 3$