Overview-I

- Data: S&P 500
- Time Series Analysis: GARCH model
- EVT analysis: GPD model

Data

- The S&P 500 tracks 500 major U.S. companies, reflecting overall market performance.
- It's widely used as a benchmark.
- Data includes daily prices, returns, and volume.

	Date	Open	High	Low	Close	Volume
0	1985-01-02	167.20	167.20	165.19	165.37	37677778.0
1	1985-01-03	165.37	166.11	164.38	164.57	49377778.0
2	1985-01-04	164.55	164.55	163.36	163.68	43044444.0
3	1985-01-07	163.68	164.71	163.68	164.24	47883333.0
4	1985-01-08	164.24	164.59	163.91	163.99	51172222.0

Data

- We considered S&P500 daily data from January 2, 1985 to April 22, 2025. ($N = 10154$ observations)
- We focus on log (daily) returns.

$$R_n = \log\left(\frac{\text{Closing price on day } n}{\text{Closing price on day } n-1}\right)$$

Data

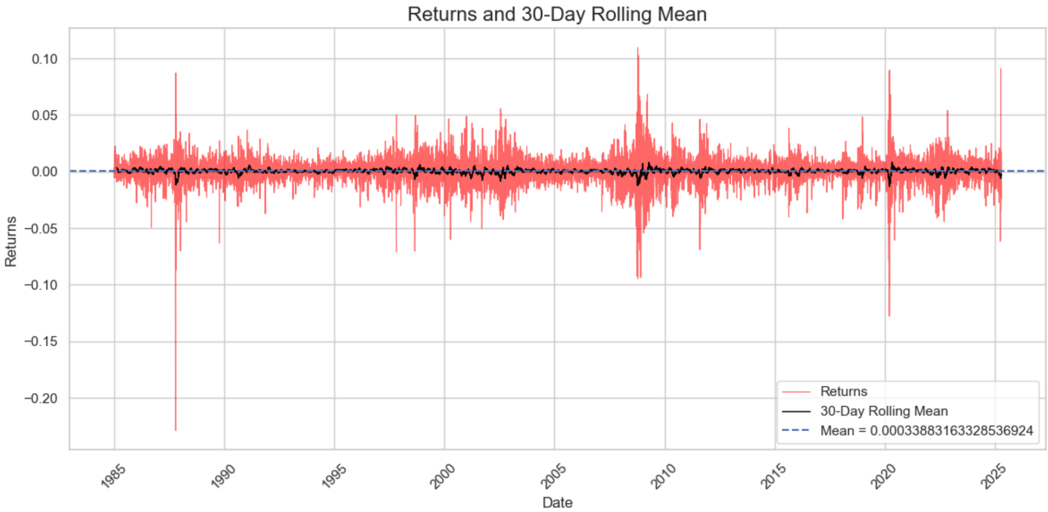


Figure: Log-Returns with Time.

Time Series Analysis: ACF and PACF

- ACF: Correlation with respect to its past values
- PACF: Measures the direct correlation between a value and its lag, removing the effects of shorter lags.

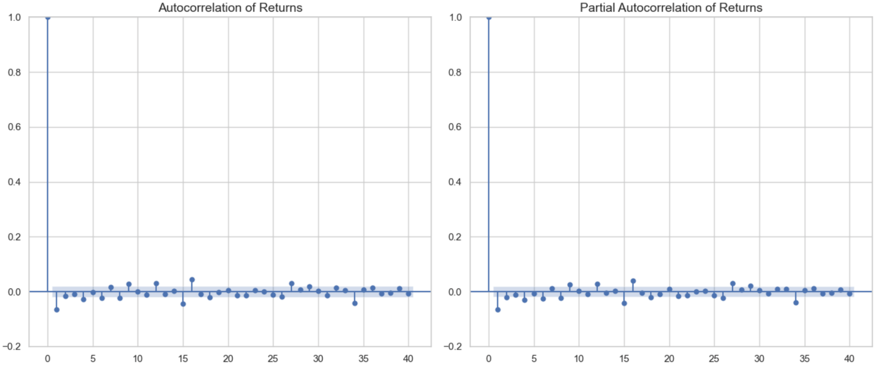


Figure: ACF and PACF of R_n. Almost zero correlations.

Time Series Analysis: ACF and PACF

- Raw returns show near-zero ACF and PACF:
  - Indicates no linear correlation in price movements.
  - Returns behave like white noise.
- Squared (or absolute) returns show significant ACF/PACF
  - Volatility tends to cluster — periods of high or low volatility group together
  - Big moves (up or down) are often followed by more big moves — and calm periods tend to follow calm periods.

GARCH(p,q) Model

- Time Series $\{X_n\}_{n\geq 0}$ is GARCH(p,q) if

$$X_n = \mu + \sigma_n \epsilon_n$$

  where ϵ_n are iid mean 0 random variable.
- The conditional variance σ_n^2 evolves as:

$$\sigma_n^2 = \omega + \sum_{i=1}^{q} \alpha_i \epsilon_{n-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{n-j}^2$$

  where $\omega > 0$, $\alpha_i, \beta_j \geq 0$ for stability
- Captures **volatility clustering**: Big moves are followed by more big moves — and calm periods tend to follow calm periods

GARCH Model: Chosing p and q

How to determine the GARCH model structure?

- Analyze **ACF and PACF** of **squared returns**:
  - PACF cutoff $\rightarrow$ suggests **ARCH (p)**
  - ACF tailing off $\rightarrow$ suggests **GARCH (q)**
- Compare models using:
  - **AIC**, **BIC** (lower is better)
- **GARCH(1,1)** is often considered effective.

GARCH: ACF and PACF

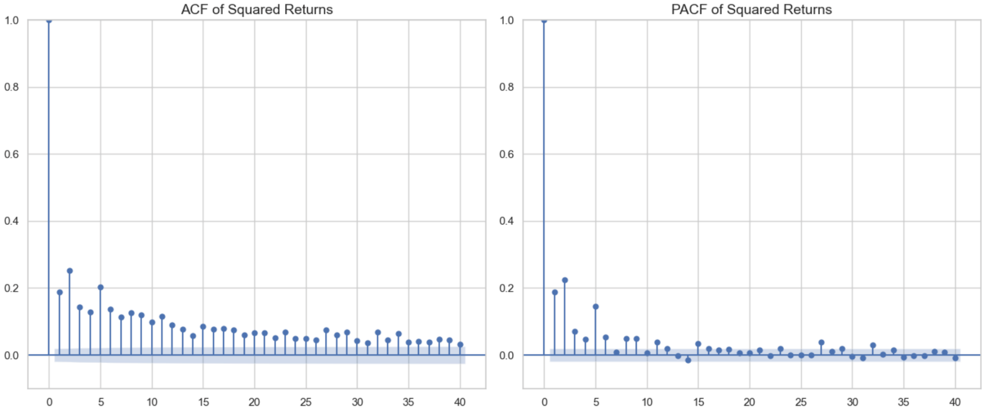


Figure: ACF and PACF of R_n^2

GARCH: AIC and BIC

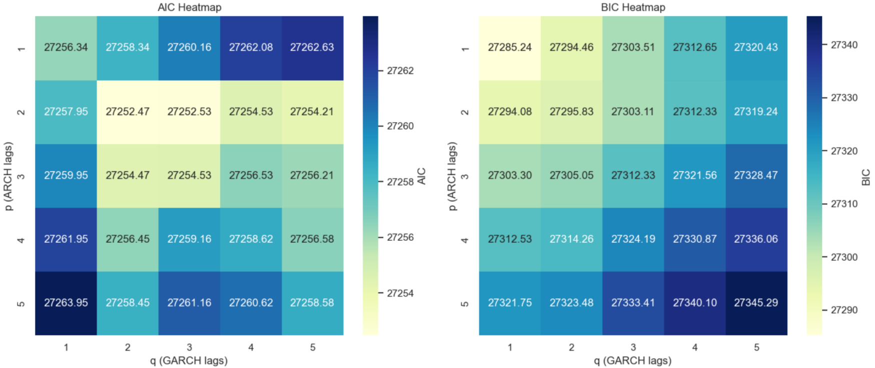


Figure: **AIC** - p = 2 and q = 2. **BIC** - p = 1 and q = 1

- All AIC and BIC values are close considering the scale. Will fit the model with $p = 1, q = 1$

GARCH(1,1)

GARCH(1,1): X_n has time-varying volatility driven by past shocks and past variance.

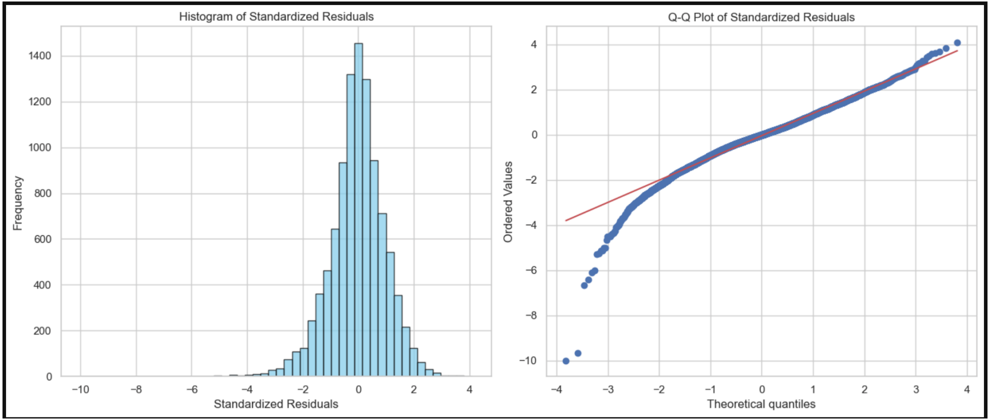
$$X_n = \mu + \sigma_n \, \epsilon_n \quad \text{where} \quad \sigma_n^2 = \omega + \alpha_1 \epsilon_{n-1}^2 + \beta_1 \sigma_{n-1}^2$$

- We first fit the model with ϵ being standard normal.
- **Estimated Parameters:**

Parameter	Estimate	Std. Error	p-value
$\hat{\mu}$	0.0649	8.325e-03	6.647e-15
$\hat{\omega}$	0.0219	4.969e-03	1.073e-05
$\hat{\alpha}_1$	0.114	1.776e-02	3.550e-10
$\hat{\beta}_1$	0.08729	1.798e-02	0.000

- All parameters seem significant.

GARCH(1,1): Normality



The QQ plot suggest that tails are heavier. Standard fix: ϵ's follow $t-$distribution.

GARCH(1,1): $t-$model

We fit a GARCH(1,1) model to the log returns:

$$X_n = \mu + \sigma_n \cdot \epsilon_n, \quad \epsilon_n \sim \text{i.i.d. } t_\nu$$
$$\sigma_n^2 = \omega + \alpha_1 (X_{n-1} - \mu)^2 + \beta_1 \sigma_{n-1}^2$$

Estimated Parameters:

Parameter	Estimate	Std. Error	p-value
$\hat{\mu}$	3.3883e+04	7243.760	2.903e-06
$\hat{\omega}$	2.6740e+10	7.964e+09	7.857e-04
$\hat{\alpha}_1$	0.1135	1.373e-02	1.417e-16
$\hat{\beta}_1$	0.8674	1.868e-02	0.000
$\hat{\nu}$	5.6969	0.392	7.544e-48

GARCH(1,1): Residuals

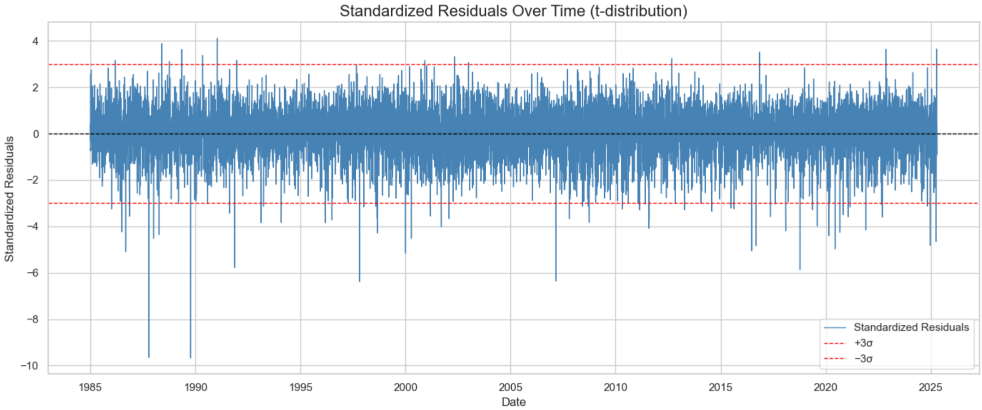


Figure: Standarized Residuals

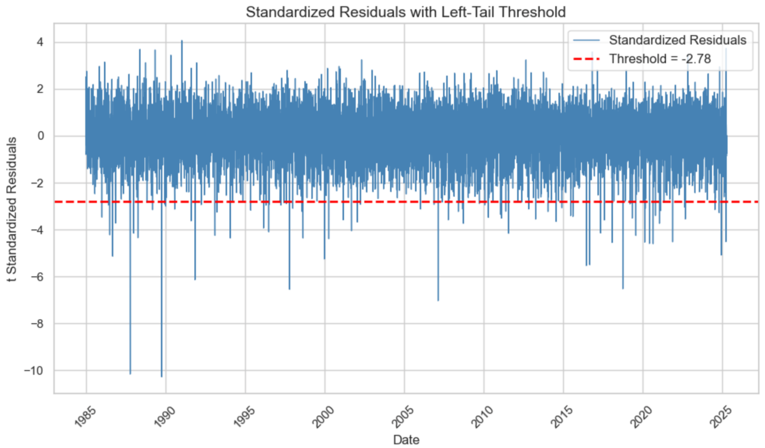
GARCH(1,1) predictions

Date	Predicted Price ($)	Actual Price ($)
21 April	–	515.82
22 April	530.68	527.00
23 April	546.48	537.5

Table: Forecasted vs Actual S&P 500 Prices

Preliminary extreme value analysis

- We fit GPD model to left tail (market crashes) of *standardized residuals.* (McNeil, A. J et.al Quantitative Risk Management: Concepts, Techniques and Tools)
- We start with a threshold of 0.01 quantile.

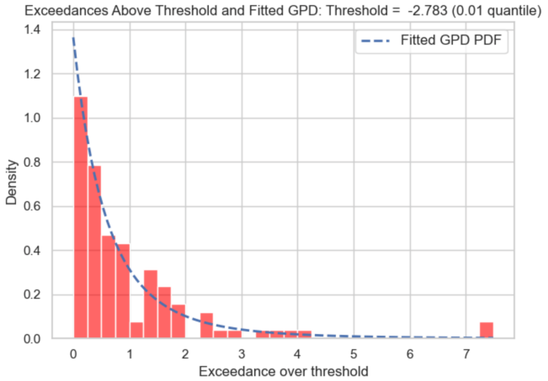


Estimation

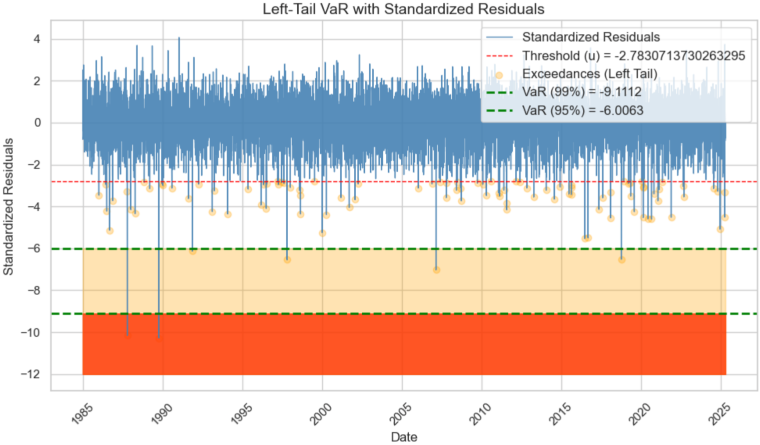
Estimation: Threshold (u): -2.7831, Shape (ξ): 0.2561 and Scale (β) $= 0.7333$.

- Heavier than exponential, but not as extreme as power-law.
- $\xi > 0$ implies the distribution has no finite upper bound — large events are possible.

Exceedances



VaR: Value at Risk



Interpretation: In the worst 1% of cases, there is a 1% chance that the residual drops below -9.112.

Overview-II

- Methods for picking your GPD threshold - Mean excess plot and Weissman method.
- Fit GPD models for the above methods.
- Plot quantiles for the models.
- Compare.

Why threshold matters?

- Too low threshold - bias from non tail data.
- Too high threshold - high variance from exceedances.

Often, selecting lowest threshold for which the GPD tail assumption is reasonable helps ensure the validity of the fit.

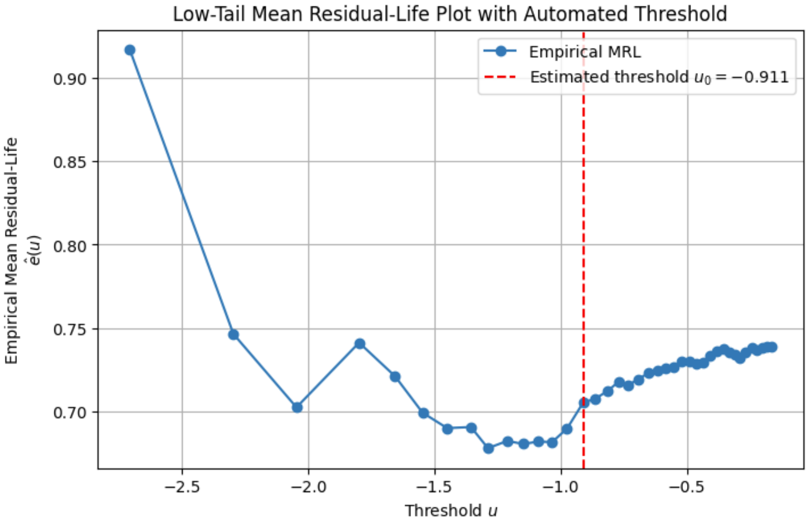
Mean excess plot

- Choose $u_1, u_2, \cdots, u_K$ span from a low quantile (say 50^{th}-percentile) upto a high quantile (say 90^{th}-percentile) of the data.
- For each u_j define the emperical mean excess by

$$\hat{e}(u_j) = \frac{1}{n_j} \sum_{i:x_i > u_j} (x_i - u_j) \text{ where } n_j = \# \{x_i > u_j\}.$$

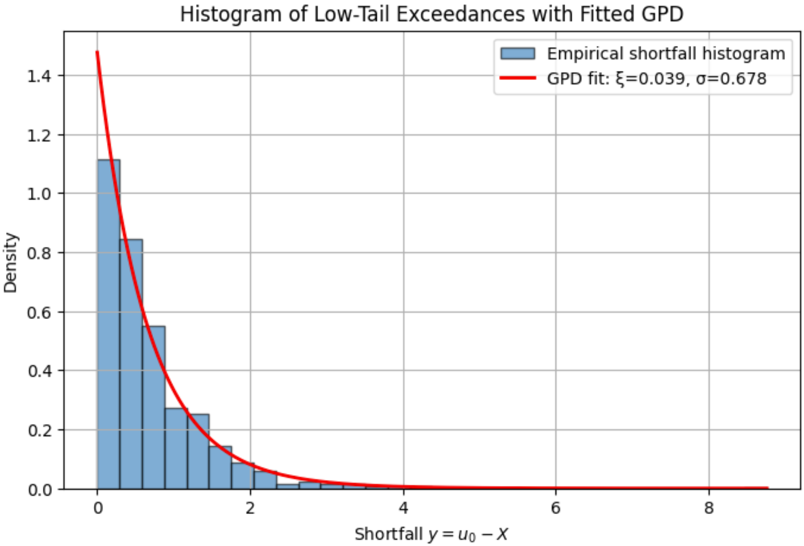
- Plot $\hat{e}(u_j)$ vs u_j.
- Choose the threshold u^* as the smallest u at which the emperical mean excess function first exhibits linearity.

Threshold

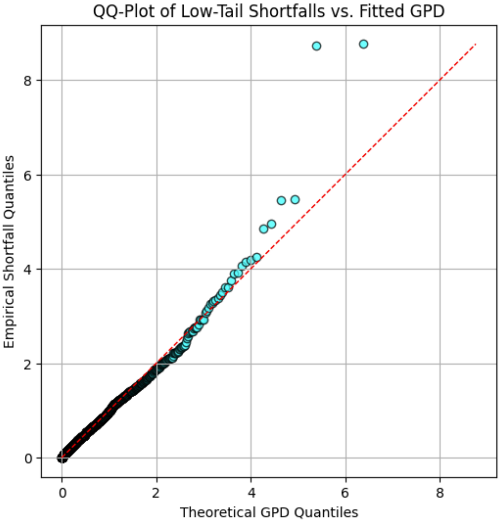


Low-Tail Mean Residual-Life Plot with Automated Threshold
Empirical MRL
Estimated threshold $u_0 = -0.911$
Empirical Mean Residual-Life
$\hat{e}(u)$
0.90
0.85
0.80
0.75
0.70
−2.5
−2.0
−1.5
−1.0
−0.5
Threshold u

GPD Fit


Histogram of Low-Tail Exceedances with Fitted GPD
Empirical shortfall histogram
GPD fit: ξ=0.039, σ=0.678
Density
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
Shortfall $y = u_0 - X$

QQ-Plot


QQ-Plot of Low-Tail Shortfalls vs. Fitted GPD
Empirical Shortfall Quantiles
Theoretical GPD Quantiles
0
2
4
6
8

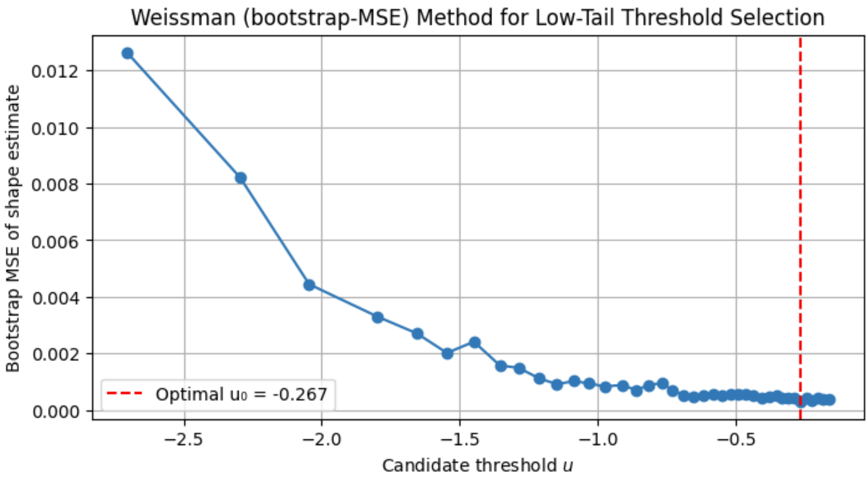
Weismann MSE

- Choose candidate thresholds $u_1 < u_2 < \cdots < u_K$.
- For each u_k form exceedances $\{X_i > u_k\}$ and fit the GPD to obtain $\hat{\theta}_k$.
- For each u_k resample its exceedances B times, refit to get $\{\hat{\theta}_k^{*(b)}\}_{b=1}^{B}$, and compute

$$\mathrm{MSE}(u_k) = \frac{1}{B}\sum_{b=1}^{B}\big(\hat{\theta}_k^{*(b)} - \hat{\theta}_k\big)^2.$$

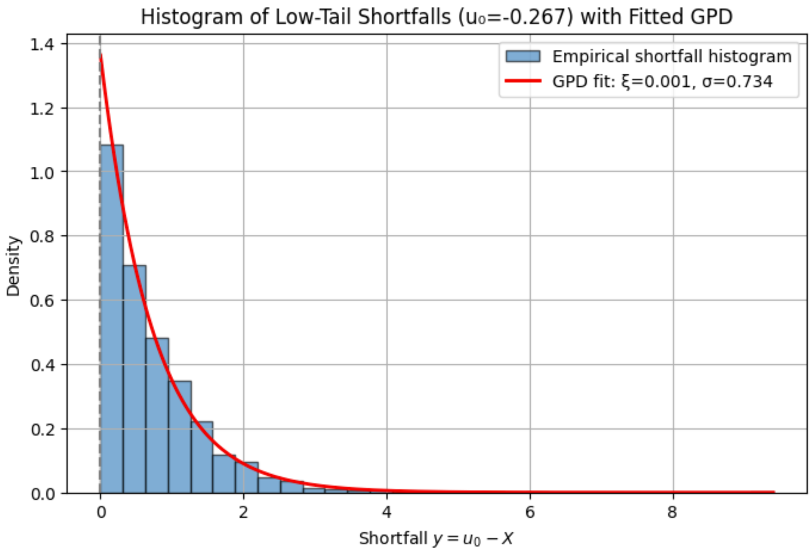
- Select $u^* = \arg\min_{1\le k\le K} \mathrm{MSE}(u_k)$.

Threshold

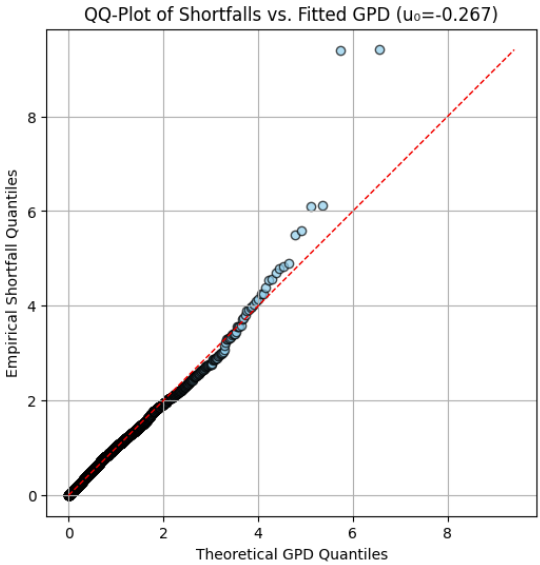


Weissman (bootstrap-MSE) Method for Low-Tail Threshold Selection
Bootstrap MSE of shape estimate
0.012
0.010
0.008
0.006
0.004
0.002
0.000
Optimal u₀ = -0.267
−2.5
−2.0
−1.5
−1.0
−0.5
Candidate threshold u

GPD Fit

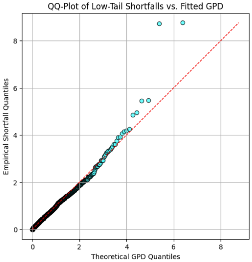

Histogram of Low-Tail Shortfalls (u_0=-0.267) with Fitted GPD
Empirical shortfall histogram
GPD fit: ξ=0.001, σ=0.734
Density
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
0
2
4
6
8
Shortfall $y = u_0 - X$

QQ plot



QQ-Plot of Shortfalls vs. Fitted GPD (u_0=-0.267)
Empirical Shortfall Quantiles
Theoretical GPD Quantiles
0
2
4
6
8
0
2
4
6
8

Comparison



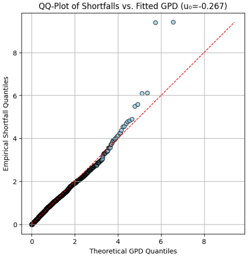


Figure: QQ-plots for GPD fit under two different thresholds

- Mean-excess: lower threshold $\rightarrow$ more QQ-points.
- Weissman-MSE: higher threshold $\rightarrow$ fewer QQ-points.