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1.) $f(x) = \frac{1}{2h} e^{-|x-\mu|/h}$, $-\infty < x < \infty$; $h \geq 0$

Cumulant generating function,

$\underline{K(t) = \ln E[e^{tX}]}$ where X is the random variable.

now, $E[e^{tX}] = \int_{-\infty}^{\infty} f(x) \cdot e^{tx} \cdot dx$

$$= \int_{-\infty}^{\infty} \frac{1}{2h} e^{-|x-\mu|/h} \cdot e^{tx} \cdot dx$$

$$= \int_{-\infty}^{\mu} \frac{1}{2h} e^{(x-\mu)/h} \cdot e^{tx} \cdot dx + \int_{\mu}^{\infty} \frac{1}{2h} e^{-(x-\mu)/h} \cdot e^{tx} \cdot dx$$

$$= \left[\frac{1}{2h} \cdot e^{(x-\mu)/h + tx} \cdot \frac{1}{(t+\frac{1}{h})} \right]_{-\infty}^{\mu} + \left[\frac{1}{2h} \cdot e^{-(x-\mu)/h + tx} \cdot \frac{1}{(t-\frac{1}{h})} \right]_{\mu}^{\infty}$$

$$= \frac{1}{2} \cdot \frac{e^{\mu t}}{(ht+1)} + \lim_{x \to \infty} \left(\frac{1}{2} \cdot \frac{e^{(t-\frac{1}{h})x + \frac{\mu}{h}}}{(ht-1)} \right) - \frac{1}{2} \cdot \frac{e^{\mu t}}{(ht-1)}$$

$\left\{ \text{for the limit to be finite, } t - \frac{1}{h} < 0 \Rightarrow ht < 1 \right\}$

$\therefore \lim_{x \to \infty} \left(\frac{1}{2} \cdot \frac{e^{(t-\frac{1}{h})x + \frac{\mu}{h}}}{(ht-1)} \right) = 0$

$$= \frac{e^{\mu t}}{2} \left[\frac{1}{ht+1} - \frac{1}{ht-1} \right]$$

$$= \frac{e^{\mu t}}{2} \cdot \frac{-2}{(ht)^2 - 1}$$

$$= \underline{\frac{e^{\mu t}}{1-(ht)^2}}$$

$\therefore$ $K(t) = \ln E[e^{tX}] = \ln \left(\frac{e^{\mu t}}{1-(ht)^2} \right) = \underline{\underline{\mu t - \ln(1-(ht)^2)}}$, $ht < 1$

by Taylor Series expansion, & a

$$K(t) = \mu t - \left[-(ht)^2 - \frac{(ht)^4}{2} - \frac{(ht)^6}{3} - \cdots \right] \qquad \{\because \quad 1 > (ht)^2 \}$$

$$= \mu t + (ht)^2 + \frac{(ht)^4}{2} + \frac{(ht^6)}{3} + \cdots \qquad - ①$$

The Cumulants can be obtained as -

$$K(t) = \sum_{n=1}^{\infty} K_n \frac{t^n}{n!}$$

$$= K_1 \frac{t}{1!} + K_2 \frac{t^2}{2!} + K_3 \frac{t^3}{3!} + \cdots \qquad - ⑪$$

By comparing coefficients in ① and ⓪

$K_1 = \mu$

$K_2 = 2h^2$

$K_3 = 0$

$K_4 = 12h^4$

⋮

$$\boxed{\begin{array}{ll} K_{2n+1} = 0 , & n = 1, 2, \ldots \\ K_{2n} = \dfrac{(2n)!}{n}\, h^{2n} , & n = 1, 2, \ldots \end{array}}$$

2.) $f(x,y) = \begin{cases} x+y \; ; & 0 \le x \le 1, \; 0 \le y \le 1 \\ 0 \; ; & \text{otherwise} \end{cases}$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$
$$= \int_{-\infty}^{0} 0 \cdot dy + \int_{0}^{1} (x+y)\,dy + \int_{1}^{\infty} 0 \cdot dy$$
$$= x + \frac{1}{2}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$
$$= \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{1} (x+y)\,dx + \int_{1}^{\infty} 0\,dx$$
$$= y + \frac{1}{2}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$
$$= \begin{cases} \dfrac{x+y}{y+\frac{1}{2}} \; ; & 0 \le x, y \le 1 \\ 0 \; ; & \text{otherwise} \end{cases}$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$
$$= \begin{cases} \dfrac{x+y}{x+\frac{1}{2}} \; ; & 0 \le x, y \le 1 \\ 0 \; ; & \text{otherwise} \end{cases}$$

(a) Conditional Expectation of X given Y

$$E(X|Y=y) = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\,dx$$
$$= \int_{-\infty}^{0} x \cdot 0 \cdot dx + \int_{0}^{1} x \cdot \frac{(x+y)}{(y+\frac{1}{2})} \cdot dx + \int_{1}^{\infty} x \cdot 0 \cdot dx$$
$$= \frac{\frac{y}{2} + \frac{1}{3}}{y + \frac{1}{2}}$$
$$= \frac{3y+2}{3(2y+1)}$$

$$\boxed{\text{Ans} \Rightarrow \; E(X|Y=y) = \frac{3y+2}{3(2y+1)}}$$

(b) Conditional Expectation of Y given X

$$E(Y|X=x) = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y|x) \cdot dy$$

$$= \int_{-\infty}^{0} y \cdot 0 \cdot dy + \int_{0}^{1} y \cdot \frac{(x+y)}{x+\frac{1}{2}} \cdot dy + \int_{1}^{\infty} y \cdot 0 \cdot dy$$

$$= \frac{\frac{x}{2}+\frac{1}{3}}{x+\frac{1}{2}}$$

$$= \frac{3x+2}{3(2x+1)}$$

$$\boxed{\text{Ans} \Rightarrow \quad E(Y|X=x) = \frac{3x+2}{3(2x+1)}}$$

3.) $X \sim B(n_1, p)$ and $Y \sim B(n_2, p)$

For a Binomial Distribution $B(n,p)$, the moment generating function (MGF) is

$$M(t) = E(e^{tr})$$

$$= \sum_{r=0}^{\infty} e^{tr} \cdot {}^nC_r \cdot p^r \cdot (1-p)^{n-r}$$

$$= \sum_{r=0}^{\infty} {}^nC_r \cdot (pe^t)^r \cdot (1-p)^{n-r}$$

$$= [(1-p) + pe^t]^n$$

$$= \underline{(p(e^t-1)+1)^n}$$

$\therefore$ $M_X(t) = \underline{(p(e^t-1)+1)^{n_1}}$

$M_Y(t) = \underline{(p(e^t-1)+1)^{n_2}}$

Since, X and Y are independent random variables

$\therefore$ MGF for $U = X+Y$, is

$$M_U(t) = M_{X+Y}(t)$$

$$= M_X(t) \cdot M_Y(t)$$

$$= (p(e^t-1)+1)^{n_1} \cdot (p(e^t-1)+1)^{n_2}$$

$$= \underline{(p(e^t-1)+1)^{n_1+n_2}}$$

The obtained MGF for $U=X+Y$ is that of a Binomial distribution with parameters (n_1+n_2, p)

$\therefore$ $\underline{\underline{U = X+Y \sim B(n_1+n_2, p)}}$

4.) $X \sim \chi^2$ - distribution with parameters (n)

$$\therefore f(x) = \begin{cases} \dfrac{e^{-\frac{x}{2}} \cdot \left(\frac{x}{2}\right)^{\frac{n}{2}-1}}{2\,\Gamma\left(\frac{n}{2}\right)} & ; \quad x > 0 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

Characteristic function,

$$\chi(t) = E(e^{itx})$$

$$= \int_{-\infty}^{\infty} e^{itx} \cdot f(x) \cdot dx$$

$$= \int_{-\infty}^{0} e^{itx} \, 0 \cdot dx + \int_{0}^{\infty} e^{itx} \cdot \frac{e^{-\frac{x}{2}} \cdot \left(\frac{x}{2}\right)^{\frac{n}{2}-1}}{2\,\Gamma\left(\frac{n}{2}\right)} \cdot dx$$

$$= \boxed{(1-2it)^{-\frac{n}{2}}}$$

{ by Complex Integration }

$\therefore$ Moment of k^{th} order

$$\alpha_k = \chi^{(k)}(0)$$

$$\alpha_1 = \chi'(0) = \frac{2}{i} \cdot \frac{n}{2} i \cdot (1-2it)^{-\frac{n}{2}-1} \Big|_{t=0} = n$$

$$\alpha_2 = i^{-2} \chi''(0) = \frac{4}{-1} \cdot \frac{n}{2} \cdot \frac{(n+2)}{2} \cdot (-1) \cdot (1-2it)^{-\frac{n}{2}-2} \Big|_{t=0} = n(n+2)$$

mean $m = \alpha_1 = n$

variance $\sigma^2 = \alpha_2 - \alpha_1^2 = n(n+2) - n^2 = 2n$

For mode, $f'(x) = 0$

$$-\frac{1}{2} \cdot \frac{e^{-\frac{x}{2}} \cdot (x/2)^{\frac{n}{2}-1}}{2\,\Gamma(n/2)} + \left(\frac{n}{2}-1\right) \cdot \frac{e^{-\frac{x}{2}} \cdot (x)^{\frac{n}{2}-2} / (2)^{\frac{n}{2}-1}}{2\,\Gamma(n/2)} = 0$$

$$\frac{e^{-\frac{x}{2}} \cdot (x)^{\frac{n}{2}-2} / (2)^{\frac{n}{2}-1}}{2\,\Gamma(n/2)} \cdot \left[-\frac{x}{2} + \left(\frac{n}{2} - 1\right) \right] = 0$$

$$\left\{ \because e^{-\frac{x}{2}} > 0 \text{ and } (x)^{\frac{n}{2}-2} > 0 \text{ for } x > 0 \right\}$$

$$\therefore \quad -\frac{x}{2} + \frac{n}{2} - 1 = 0$$

$$x = n - 2$$

$$\Rightarrow \underline{\underline{\text{mode} = n-2}}$$

Ans $\Rightarrow$

$$\boxed{\begin{aligned} \text{mean} &= n \\ \text{variance} &= 2n \\ \text{mode} &= n-2 \end{aligned}}$$

5.) $X_1, X_2, \dots, X_n$ are mutually independent random variables

$X_i \sim N(0,1) \quad i = 1,2,\dots,n$

$\therefore \frac{1}{2} X_i^2 \sim \Gamma(1/2)$ Gamma(1/2) Distribution ; $i = 1,2,\dots,n$

$$f_{X_i}(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}} \quad \because X_i \sim N(0,1), \quad -\infty < x < \infty$$

for $Y_i = \frac{1}{2} X_i^2$

$$\frac{dy}{dx} = x$$

for $0 < x < \infty$, $\frac{dy}{dx}$ is monotonic

$\therefore$ by Transformation of random variable

$$f_{Y_i}(y) = \left|\frac{dx}{dy}\right| f_{X_i}(x)$$

$$= \frac{1}{x} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}$$

$$= \frac{1}{\sqrt{2y}} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-y}$$

$$= \frac{1}{2} \cdot \frac{e^{-y} \cdot y^{\frac{1}{2}-1}}{\Gamma(1/2)} \quad \because \Gamma(1/2) = \sqrt{\pi}$$

Now, $f_{X_i}(x)$ is symmetric about $x=0$,

$\therefore$ due to symmetricity

$$f_{Y_i}(y) = 2 \cdot \frac{1}{2} \cdot \frac{e^{-y} y^{\frac{1}{2}-1}}{\Gamma(1/2)}, \quad 0 < y < \infty$$

$$= \frac{e^{-y} y^{\frac{1}{2}-1}}{\Gamma(1/2)}$$

$\Rightarrow Y \sim \Gamma(1/2)$ Distribution

Now, since $\frac{1}{2} X_i^2 \sim \Gamma(1/2)$; $i = 1,2,\dots,n$

$$\therefore \frac{1}{2}X_1^2 + \frac{1}{2}X_2^2 + \dots + \frac{1}{2}X_n^2 = \sum_{i=1}^{n} \frac{1}{2} X_i^2 \sim \Gamma\left(\frac{n}{2}\right) \quad \left\{ \begin{array}{l} X \sim \Gamma(m), Y \sim \Gamma(n) \\ \Rightarrow X+Y \sim \Gamma(m+n) \end{array} \right\}$$

If $X \sim \Gamma(n/2)$ variate, then $Y = 2X$ is χ^2-distribution with n-degrees of freedom

$$\because \quad \sum_{i=1}^{n} \frac{1}{2} X_i^2 \sim \Gamma(n/2)$$

$$\therefore \quad 2 \sum_{i=1}^{n} \frac{1}{2} X_i^2 \sim \chi^2(n)$$

$$\Rightarrow \quad \sum_{i=1}^{n} X_i^2 \sim \chi^2(n)$$

$$\Rightarrow \quad \boxed{X_1^2 + X_2^2 + \dots + X_n^2 \sim \chi^2(n)}$$

$X \sim \Gamma(n/2)$

$$\therefore \quad f(x) = \begin{cases} \dfrac{e^{-x} x^{\frac{n}{2}-1}}{\Gamma(n/2)} & ; \quad 0 < x < \infty \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

Setting $Y = 2X$,

$$\frac{dy}{dx} = 2$$

$\therefore$ y is a monotonic function

By Transformation of random variables,

$$f_Y(y) = \left|\frac{dx}{dy}\right| f_X(x)$$

$$= \frac{1}{2} \cdot \frac{e^{-x} \cdot x^{\frac{n}{2}-1}}{\Gamma(n/2)}, \quad 0 < x < \infty$$

$$= \frac{1}{2} \cdot \frac{e^{-y/2} \cdot (y/2)^{\frac{n}{2}-1}}{\Gamma(n/2)}, \quad 0 < y < \infty$$

$$\therefore Y \sim \chi^2_n$$

6.) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables such that all of them have same mean μ and standard deviation σ.

i.e. $X_i \sim N(\mu, \sigma)$, $i = 1, 2, \ldots, n$

Now, Central Limit theorem states that for

$$S_n = X_1 + X_2 + \ldots + X_n = \sum_{i=1}^{n} X_i \sim N(n\mu, \sigma\sqrt{n})$$

By Standardization,

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \sim N(0,1) \quad \text{Standard Normal Distribution}$$

$$\frac{\frac{S_n}{n} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$

Let $\bar{X}_n$ be random variable denoting average/mean of X_i $(i = 1, 2, \ldots, n)$

$$\therefore \bar{X}_n = \frac{X_1 + X_2 + \ldots + X_n}{n} = \frac{\sum_{i=1}^{n} X_i}{n} = \frac{S_n}{n}$$

$$\therefore \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1) \quad \text{as } n \to \infty$$

$$\Rightarrow \bar{X}_n \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

$\Rightarrow$ as n gets large (or $n \to \infty$), $\bar{X}_n$ approximatemore like a normal distribution with mean μ and smaller variance $\frac{\sigma}{\sqrt{n}}$.

i.e. as $n \to \infty$, $\bar{X}_n \to \mu$

$\Rightarrow P(\lim_{n\to\infty}(\bar{X}_n = \mu)) = 1$ which is what Law of Large number is.

$\therefore$ Central Limit Theorem implies Law of Large numbers

But the converse is not true.

Let the sample size be n with following independent and identically distributed random variables : $X_1, X_2, \ldots, X_n$ s.t. $X_i \sim N(\mu, \sigma)$; $i = 1, 2, \ldots, n$

Let $\bar{X}_n$ be the random variable denoting mean of n random variables

$$\bar{X}_n = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

Law of Large Numbers states that

as n increases $(n \to \infty)$, $\bar{X}_n \to \mu$

but this does not imply anything about the distribution of $\bar{X}_n$

$\therefore$ Law of Large Numbers $\not\Rightarrow$ Central Limit Theorem

7.) For independent & identically distributed random variables $X_1, X_2, \dots, X_n$ each with mean μ and variance σ^2

$\therefore$ $\underline{E(X_i) = \mu}$

$$\sigma^2 = E(X_i^2) - (E(X_i))^2 \qquad i = 1, 2, \dots, n$$

$$\sigma^2 = E(X_i^2) - \mu^2$$

$$\underline{E(X_i^2) = \sigma^2 + \mu^2}$$

<u>According to Central Limit Theorem,</u>

$\bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$ where $\bar{X}$ is r.v. for average of $X_1, X_2, \dots, X_n$

$\therefore$ $E(\bar{X}) = \mu$ Variance $(\bar{X}) = \frac{\sigma^2}{n}$

(using above relation)

$$E(\bar{X}^2) = \frac{\sigma^2}{n} + \mu^2$$

Now Sample Variance is

$S^2 = \frac{1}{n}\sum(x_i - \bar{x})^2$ (summation limits are from $i=1$ to n)

$$= \frac{1}{n}\sum(x_i^2) - \frac{1}{n}2\bar{X}\cdot\sum x_i + \frac{1}{n}\cdot n\bar{X}^2$$

$$= \frac{1}{n}\left(\sum(X_i^2) - 2\bar{X}\cdot(n\bar{X}) + n\bar{X}^2\right) \qquad \left\{\bar{X} = \frac{\sum X_i}{n}\right\}$$

$$= \frac{1}{n}\left(\sum(X_i^2) - n\bar{X}^2\right)$$

$$E(S^2) = E\left(\frac{1}{n}\left(\sum(X_i^2) - n\bar{X}^2\right)\right)$$

$$= \frac{1}{n}\left[\sum E(X_i^2) - E(n\bar{X}^2)\right]$$

$$= \frac{1}{n}\left[\sum E(X_i^2) - nE(\bar{X}^2)\right]$$

$$= \frac{1}{n}\left[\sum(\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right)\right]$$

$$= \frac{1}{n}\left[n\sigma^2 + n\mu^2 - \sigma^2 - n\mu^2\right]$$

$$= \frac{n-1}{n}\cdot\sigma^2$$

Thus, $\hat{s}^2$ is a consistent estimator of σ^2 but is biased to σ^2

as $E(\hat{s}^2) - \sigma^2 < 0$

$\therefore$ If we calculate $\hat{s}^2$ using $(n-1)$ as denominator

i.e. $s^2 = \dfrac{\sum (x_i - \bar{x})^2}{n-1}$

$$E(s^2) = E\left(\frac{\sum (x_i - \bar{x})^2}{n-1}\right)$$

$$= E\left(\frac{n}{n-1} \cdot \frac{\sum (x_i - \bar{x})^2}{n}\right)$$

$$= \frac{n}{n-1} \cdot E\left(\frac{\sum (x_i - \bar{x})^2}{n}\right)$$

$$= \frac{n}{n-1} \cdot \frac{n-1}{n} \cdot \sigma^2$$

$$= \sigma^2$$

$\therefore$ $s^2 = \dfrac{\sum (x_i - \bar{x})^2}{n-1}$ is a good consistent estimator of σ^2 and is unbiased estimate of σ^2.

8.) $t = \dfrac{(\bar{X}-m)\sqrt{n}}{s}$

$$t = \dfrac{\dfrac{(\bar{X}-m)}{\frac{\sigma}{\sqrt{n}}}}{\frac{s}{\sigma}} \qquad \left\{ Var(\bar{X}) = \frac{\sigma^2}{n} \right\}$$

$$t = \dfrac{U}{\frac{s}{\sigma}} \qquad \text{where } U = \dfrac{\bar{X}-m}{\frac{\sigma}{\sqrt{n}}}$$

$$\sum(x_i - m)^2 = \sum((x_i-\bar{X})+(\bar{X}-m))^2 \qquad \text{limits from } i=1 \text{ to } n$$

$$= \sum(x_i-\bar{X})^2 + 2\sum(x_i-\bar{X})(\bar{X}-m) + \sum(\bar{X}-m)^2$$

$$= \sum(x_i-\bar{X})^2 + 0 + n(\bar{X}-m)^2 \qquad \{\because \sum(x_i-\bar{X}) = 0\}$$

$$= \sum(x_i-\bar{X})^2 + n(\bar{X}-m)^2$$

Dividing by σ^2,

$$\sum\left(\frac{x_i-m}{\sigma}\right)^2 = \sum\left(\frac{x_i-\bar{X}}{\sigma}\right)^2 + \left(\frac{\bar{X}-m}{\sigma/\sqrt{n}}\right)^2$$

$\because x_i \sim N(m,\sigma)$

$\Rightarrow \frac{x_i-m}{\sigma} \sim N(0,1)$

$\Rightarrow \sum\left(\frac{x_i-m}{\sigma}\right)^2 \sim \chi^2_n$

$\frac{1}{\sigma^2}\sum(x_i-\bar{X})^2$

$= \frac{(n-1)}{\sigma^2}\cdot\frac{1}{n-1}\sum(x_i-\bar{X})^2$

$= \frac{(n-1)}{\sigma^2}\cdot s^2$

$\because \bar{X} \sim N(m, \sigma/\sqrt{n})$

$\Rightarrow \frac{\bar{X}-m}{\sigma/\sqrt{n}} \sim N(0,1)$

$\Rightarrow \left(\frac{\bar{X}-m}{\sigma/\sqrt{n}}\right)^2 \sim \chi^2_1$ (1 degree of freedom)

$\therefore \quad U = V + W \quad$ where $U \sim \chi^2(n)$

$V = \dfrac{(n-1)s^2}{\sigma^2}$

$W \sim \chi^2(1)$

$\because$ s and $\bar{X}$ are independent, so are V and W

$\therefore \quad M_U(t) = M_V(t)\cdot M_W(t) \quad$ where $M(t)$ is moment generating function.

$$\frac{1}{(1-2t)^{n/2}} = M_V(t)\cdot\frac{1}{(1-2t)^{1/2}} \qquad \left\{ \text{for } \chi^2(n),\ M_{\chi^2}(t) = \frac{1}{(1-2t)^{n/2}} \right\}$$

$$M_V(t) = \frac{1}{(1-2t)^{\frac{n-1}{2}}} \quad \Rightarrow \quad \underline{V \sim \chi^2(n-1)}$$

Thus from previous proof,

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1)$$

now, $$t = \frac{U}{\frac{s}{\sigma}}$$

$$t = \frac{U}{\sqrt{\frac{(n-1)s^2}{(n-1)\sigma^2}}}$$

$$t = \frac{U}{\sqrt{\chi^2/(n-1)}} \quad \text{where } \chi^2 = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1) \text{ distribution}$$

$\because$ $U = \frac{\bar{X}-m}{\frac{\sigma}{\sqrt{n}}}$ and $\bar{X} \sim N(m, \frac{\sigma}{\sqrt{n}})$

$\therefore$ $U \sim N(0,1)$

$$\Rightarrow \quad t = \frac{\text{Normal distribution } (0,1)}{\sqrt{\text{Chi-Square dist.}^n \text{ with } (n-1) \text{ deg. of freedom}}}$$

Ans$\Rightarrow$ $\therefore$ $\underline{\underline{t \sim t\text{-distribution with } (n-1) \text{ deg. of freedom}}}$

$$t = \frac{U}{\sqrt{\chi^2/(n-1)}}$$

$$t^2 = \frac{U^2}{\chi^2/(n-1)}$$

$$\frac{t^2}{n-1} = \frac{\frac{1}{2}U^2}{\frac{1}{2}\chi^2} = \frac{\text{Gamma}(1/2)\ \Gamma(1/2)}{\text{Gamma}(n-1/2)\ \Gamma(n-1/2)} = \beta_2(1/2, n-1/2)$$

now, $f(t)\,dt = F'(t)\,dt = dF(t)$

$\therefore$ $dF = f(t)\cdot dt$

$$dF = \frac{(t^2/n-1)^{1/2-1}}{\beta(\frac{1}{2}, \frac{n-1}{2})\cdot(1+t^2/n-1)^{\frac{n}{2}}}\cdot\frac{2t}{(n-1)}\cdot dt \quad , (0 < t^2 < \infty)$$

$$dF = \frac{2}{\sqrt{n-1}\cdot\beta(\frac{1}{2}, \frac{n-1}{2})\cdot(1+t^2/n-1)^{\frac{n}{2}}}\cdot dt \quad , (0 < t^2 < \infty)$$

$$dF = \frac{dt}{\sqrt{n-1} \cdot B\left(\frac{1}{2}, \frac{n-1}{2}\right) \cdot \left(1 + t^2/_{n-1}\right)^{\frac{n}{2}}} \quad , \; -\infty < t < \infty$$

$$f(t) = \left[\sqrt{n-1} \cdot B\left(\frac{1}{2}, \frac{n-1}{2}\right) \cdot \left(1 + \frac{t^2}{n-1}\right)^{\frac{n}{2}}\right]^{-1} \quad , \; -\infty < t \leq \infty$$

$\Rightarrow$ $t \sim$ t-distribution with $(n-1)$ degrees of freedom.

9.)

$$f(x,\theta) = \frac{x^{P-1} \cdot e^{-x/\theta}}{\theta^P \, \Gamma(P)}$$

The likelihood function is

$$L(x_1, x_2, \dots, x_n ; \theta) = f(x = x_1, \theta) \cdot f(x = x_2, \theta) \cdot \ \dots \cdot f(x = x_n, \theta)$$

$$= \frac{(x_1 \cdot x_2 \cdot \dots \cdot x_n)^{P-1} \cdot e^{-(x_1 + x_2 + \dots + x_n)/\theta}}{(\theta^P)^n \cdot (\Gamma(P))^n}$$

$$= \frac{\prod_{i=1}^{n} x_i^{P-1} \cdot e^{-\sum_{i=1}^{n} x_i / \theta}}{\theta^{nP} \cdot (\Gamma(P))^n}$$

for maximum likelihood estimate, we will maximize $\ln L$, $(\because L > 0)$

$\therefore \quad \max(\ln L)$

$$\frac{\partial}{\partial \theta}(\ln L) = 0$$

$$\frac{\partial}{\partial \theta}\left(\ln\left[\frac{\prod x_i^{P-1} \cdot e^{-\sum x_i/\theta}}{\theta^{nP} \cdot (\Gamma(P))^n} \right] \right) = 0$$

$$\frac{\partial}{\partial \theta}\left((P-1) \cdot \sum \ln x_i - \frac{\sum x_i}{\theta} - nP \ln \theta - n \ln \Gamma(P) \right) = 0$$

$$\frac{\sum x_i}{\theta^2} - \frac{nP}{\theta} = 0$$

$$\theta = \frac{\sum x_i}{nP}$$

$$\boxed{\hat{\theta} = \frac{\bar{X}}{P}}$$

Since $\bar{X}$ is a good, consistent and unbiased estimate of population mean.

$\therefore$ $\hat{\theta}$ is also an unbiased and consistent estimate.

10.) $$P(x=i) = \frac{1}{1+\mu}\cdot\left(\frac{\mu}{1+\mu}\right)^{i} \quad , \quad \mu>0 \text{ for } i=0,1,2\ldots$$

The likelihood function is -

$$L(x_0,x_1,x_2\ldots;\mu) = P(x=x_0)\cdot P(x=x_1)\cdot P(x=x_2)\cdot\ldots\cdot P(x=x_n) \quad \text{(for sample size n)}$$

$$= \frac{1}{(1+\mu)^n}\cdot\frac{(\mu)^{x_0+x_1+x_2+\ldots+x_n}}{(1+\mu)^{x_0+x_1+x_2+\ldots+x_n}}$$

$$= \frac{1}{(1+\mu)^n}\cdot\left(\frac{\mu}{1+\mu}\right)^{\sum x_i}$$

for maximum likelihood estimate, we will maximise $\ln L$ ($\because L>0$)

$$\therefore \quad \frac{\partial}{\partial\mu}(\ln L) = 0$$

$$\frac{\partial}{\partial\mu}\left(\ln\left[\frac{\mu^{\sum x_i}}{(1+\mu)^{\sum x_i+n}}\right]\right) = 0$$

$$\frac{\partial}{\partial\mu}\left(\sum x_i\cdot\ln\mu - (\sum x_i+n)\ln(1+\mu)\right) = 0$$

$$\frac{\sum x_i}{\mu} - \frac{\sum x_i+n}{1+\mu} = 0$$

$$\frac{\bar{X}}{\mu} - \frac{\bar{X}+1}{1+\mu} = 0$$

$$\bar{X}(1+\mu) = (\bar{X}+1)\mu$$

$$\bar{X} = \mu$$

$$\boxed{\hat{\mu} = \bar{X}}$$

Since $\bar{X}$ is an unbiased & consistent estimate of mean.

$\therefore$ $\hat{\mu}=\bar{X}$ is an unbiased & consistent estimate of parameter μ.