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MA 202 ASSIGNMENT 2

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Q1. $f(x) = \frac{1}{2h} e^{-|x-\mu|/h}$ $\quad -\infty < x < \infty \; ; \; h \geq 0$

$\Rightarrow$ To find :- (cumulants of the Laplace distribution) :-

Cumulant generating function, $K(t) = \ln E[e^{tX}]$ {where X = r.v}

(r.v = random variable).

Now, $E[e^{tx}] = \int_{-\infty}^{\infty} f(x) e^{tx} \, dx$

$$= \int_{-\infty}^{\infty} \frac{1}{2h} e^{-|x-\mu|/h} \cdot e^{tx} \, dx$$

$$\left\{ \begin{aligned} f(x) &= \frac{1}{2h} \cdot e^{\frac{x-\mu}{h}}, x<\mu \\ &= \frac{1}{2h} e^{\frac{\mu-x}{h}}, x>\mu \end{aligned} \right\}$$

$$= \int_{-\infty}^{\mu} \frac{1}{2h} e^{(x-\mu)/h} \cdot e^{tx} \, dx + \int_{\mu}^{\infty} \frac{1}{2h} e^{-(x-\mu)/h} \cdot e^{tx} \, dx$$

$$= \left[\frac{1}{2h} \cdot e^{(x-\mu)/h + tx} \cdot \frac{1}{(t+\frac{1}{h})} \right]_{-\infty}^{\mu} + \left[\frac{1}{2h} \cdot e^{-(x-\mu)/h + tx} \frac{1}{(t-\frac{1}{h})} \right]_{\mu}^{\infty}$$

$$= \frac{1}{2} \cdot \frac{e^{\mu t}}{(ht+1)} + \lim_{x \to \infty} \left(\frac{1}{2} \cdot \frac{e^{(t-\frac{1}{h})x + \frac{\mu}{h}}}{(ht-1)} \right) - \frac{1}{2} \frac{e^{\mu t}}{(ht-1)}$$

{For the limit to be finite, $t - 1/h < 0 \Rightarrow ht < 1$ }

Therefore. $\lim_{x \to \infty} \left(\frac{1}{2} \cdot \frac{e^{[(t-1/h)x + \mu/h]}}{(ht-1)} \right) = 0$.

$$= \frac{e^{\mu t}}{2} \left[\frac{1}{(ht+1)} - \frac{1}{(ht-1)} \right] = \frac{e^{\mu t}}{2} \cdot \frac{(-2)}{(ht)^2 - 1^2}$$

$$= \frac{e^{\mu t}}{\cancel{2}} \cdot \frac{\cancel{2}}{1-(ht)^2} = \frac{e^{\mu t}}{1-(ht)^2}$$

Therefore $K(t) = \ln E[e^{tx}] = \ln \left(\frac{e^{\mu t}}{1-(ht)^2} \right) = \boxed{\mu t - \ln(1-(ht)^2)}$

$ht < 1$

By Taylor Series Expansion :-

$$K(t) = \mu t - \left[-(ht)^2 - \frac{(ht)^4}{2} - \frac{(ht)^6}{3} - \cdots \right]$$ { since $(ht)^2 < 1$

$$= \mu t + (ht)^2 + \frac{(ht)^4}{2} + \frac{(ht)^6}{3} + \cdots \quad (1)$$

The cumulants can be obtained as -

$$K(t) = \sum_{n=1}^{\infty} K_n \frac{t^n}{n!} = K_1 \frac{t}{1!} + K_2 \frac{t^2}{2!} + K_3 \frac{t^3}{3!} + \cdots \quad (2)$$

By comparing the coefficients in eq (1) and eq (2). we get -

$\boxed{K_1 = M}$, $\boxed{K_2 = 2h^2}$, $\boxed{K_3 = 0}$, $\boxed{K_4 = 12h^4}$ ....

$$\therefore \boxed{\begin{array}{ll} K_{2n+1} = 0 , & n = 1, 2, 3 \ldots \\ K_{2n} = \frac{(2n)!}{n} h^{2n} , & n = 1, 2, 3, \ldots \end{array}}$$

← ANS

Q2. (3)

Given :- $f(x,y) = \begin{cases} x+y & ; 0 \le x \le 1 , 0 \le y \le 1 \\ 0 & ; \text{otherwise} \end{cases}$

To find :- conditional expectation of (a) X given Y
(b) Y given X

$\Rightarrow$ $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$ and $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$.

$$= \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{-\infty}^{0} \cancel{(0)\,dy}^{\,0} + \int_0^1 (x+y)\,dy + \int_1^{\infty} \cancel{(0)\,dy}^{\,0}$$

$$= xy + \frac{y^2}{2}\Big]_{y=0}^{1}$$

$$= x + \frac{1}{2}$$

$$= \int_{-\infty}^{0} \cancel{(0)\,dx}^{\,0} + \int_0^1 (x+y)\,dx + \int_1^{\infty} \cancel{(0)\,dx}^{\,0}$$

$$= \frac{x^2}{2} + yx\Big]_{x=0}^{1}$$

$$= y + \frac{1}{2}$$

$f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \dfrac{(x+y)}{(y+1/2)} & ; 0 \le x, y \le 1 \\ 0 & , \text{otherwise} \end{cases}$

$f_{Y|X}(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} \dfrac{(x+y)}{(x+1/2)} & ; 0 \le x, y \le 1 \\ 0 & , \text{otherwise} \end{cases}$

(a) <u>CONDITIONAL EXPECTATION OF X given Y :-</u>

$$E(X|Y = y) = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\,dx$$

$$= \int_{-\infty}^{0} \cancel{x \cdot 0\,dx}^{\,0} + \int_0^1 \frac{x(x+y)}{(y+1/2)}\,dx + \int_1^{\infty} \cancel{x \cdot 0\,dx}^{\,0}$$

$$= \left(\frac{x^3/3 + x^2y/2}{y+1/2}\right)\Big]_{x=0}^{1} = \frac{1/3 + y/2}{y+1/2}$$

$$E(X|Y=y) = \boxed{\frac{3y+2}{3(2y+1)}}$$

(b) CONDITIONAL EXPECTATION OF Y GIVEN X :-

$$E(Y|X=x) = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y|x) \cdot dy$$

$$= \int_{-\infty}^{0} y \cdot \cancel{0} \cdot dy + \int_{0}^{1} y \frac{(x+y)}{x+1/2} dy + \int_{1}^{\infty} y \cdot \cancel{0} \, dy$$

$$= \left. \frac{y^3/3 + xy^2/2}{x+1/2} \right]_{y=0}^{1} = \frac{1/3 + x/2}{x+1/2}$$

$$E(Y|X=x) = \boxed{\frac{2+3x}{3(2x+1)}}$$

Therefore,

$$\boxed{\begin{aligned} E(X|Y=y) &= \frac{3y+2}{3(2y+1)} \\ E(Y|X=x) &= \frac{3x+2}{3(2x+1)} \end{aligned}}$$

← ANS

Q3. Given :- X and Y are independent binomial variates $B(n_1, p)$ and $B(n_2, p)$ respectively

Prove that :- Their sum $U = X+Y$ is a binomial $B(n_1+n_2, p)$

⇒ PROOF :- For a Binomial Distribution $B(n, p)$, the moment generating function (MGF) is

$$M(t) = E(e^{tr})$$

$$= \sum_{r=0}^{\infty} e^{tr} \cdot {}^nC_r \cdot p^r (1-p)^{n-r}$$

$$= \sum_{r=0}^{\infty} {}^nC_r \cdot (pe^t)^r \cdot (1-p)^{n-r}$$

$$= [(1-p) + pe^t]^n = \underline{\underline{[p(e^t-1)+1]^n}}$$

Therefore, $M_x(t) = (p(e^t-1)+1)^{n_1}$ and $M_y(t) = (p(e^t-1)+1)^{n_2}$

Since, X and Y are independent random variables.

∴ MGF for $U = X+Y$, is

$$M_u(t) = M_{x+y}(t) = M_x(t) \cdot M_y(t)$$

$$= (p(e^t-1)+1)^{n_1} \cdot (p(e^t-1)+1)^{n_2}$$

$$= \underline{\underline{(p(e^t-1)+1)^{n_1+n_2}}}$$

Thus, the MGF (moment generating function) for $U = X+Y$ is that of Binomial distribution with parameters (n_1+n_2, p)

Therefore, $\boxed{U = X+Y \sim B(n_1+n_2, p)}$

Hence Proved.

Q4. To find :- characteristic function of χ^2 distribution and the mean, variance, mode of distribution (6)

$\Rightarrow$ $X \sim \chi^2$ distribution with parameters (n)

$$\therefore \quad f(x) = \begin{cases} \dfrac{e^{-x/2} \cdot (x/2)^{\frac{n}{2}-1}}{2\,\Gamma(n/2)} , & x>0 \\ 0 , & \text{otherwise} \end{cases}$$

Characteristic function :-

$$\chi(t) = E(e^{itx})$$

$$= \int_{-\infty}^{\infty} e^{itx} \cdot f(x) \cdot dx = \int_{-\infty}^{0} e^{itx} \cdot 0 \cdot dx + \int_{0}^{\infty} \frac{e^{itx} \cdot e^{-x/2} \cdot (\frac{x}{2})^{\frac{n}{2}-1}}{2\Gamma(n/2)} \cdot dx$$

(By complex integration)

$$= \boxed{(1-2it)^{-n/2}}$$

Now, Moment of k^{th} order $\alpha_k = i^{-k} \chi^{(k)}(0)$ $\quad \left((i)^{-1} = \frac{1}{i}\right)$

$$\alpha_1 = i^{-1} \chi'(0) = \frac{2}{i} \cdot \frac{n}{2} i \cdot (1-2it)^{-\frac{n}{2}-1} \Big]_{t=0} = n$$

$$\alpha_2 = i^{-2} \chi''(0) = \frac{4}{-1} \cdot \frac{n}{2} \cdot \frac{(n+2)}{2} (-1)(1-2it)^{-n/2-2} \Big]_{t=0} = n(n+2)$$

We know :- mean $= \alpha_1 = \boxed{n}$

Variance $= \alpha_2 - \alpha_1^2 = n(n+2) - n^2 = 2n$

mode $\Rightarrow$ $f'(x) = 0$

$$\frac{-\frac{1}{2} e^{-x/2} \cdot (x/2)^{\frac{n}{2}-1}}{2\Gamma(n/2)} + \frac{(\frac{n}{2}-1) \cdot e^{-x/2} \, (x)^{\frac{n}{2}-2} / 2^{n/2-1}}{2\Gamma(n/2)} = 0$$

$$\Rightarrow \frac{e^{-x/2} \cdot (x)^{n/2-2} / 2^{n/2-1}}{2\Gamma(n/2)} \left[-\frac{x}{2} + \left(\frac{n}{2}-1\right)\right] = 0$$

$\because \quad e^{-x/2} > 0$ and $(x)^{n/2-2} > 0$ for $x > 0$

$\therefore \quad -\frac{x}{2} + \frac{n}{2} - 1 = 0$

$x = n - 2$

$\Rightarrow$ Mode $= \underline{\underline{n-2}}$.

$$\boxed{\begin{aligned} \text{Mean} &= \underline{\underline{n}} \\ \text{Variance} &= \underline{\underline{2n}} \\ \text{Mode} &= \underline{\underline{n-2}} \end{aligned}}$$

$\leftarrow$ ANS

Q5. given :- $X_1, X_2, \ldots, X_n$ are mutually independent (8)

$X_i^2 \sim N(0,1)$ $i = 1,2,\ldots n$,

Show that :- $X_1^2 + X_2^2 + \cdots + X_n^2$ represents a χ^2 distribution with n- degrees of freedom

$\Rightarrow$ $X_1, X_2, \ldots, X_n$ are mutually independent random variables

$X_i \sim N(0,1)$ $i = 1,2,\ldots n$

$\therefore \frac{1}{2}X_i^2 \sim \Gamma(1/2)$ Gamma (1/2) distribution, $i = 1,2,\ldots n$

$f_{X_i}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ $\{X_i \sim N(0,1), \; -\infty < x < \infty\}$

for $Y_i = \frac{1}{2}X_i^2$

$\frac{dy}{dx} = x$ for $0 < x < \infty$, (y is monotonic)

$\therefore$ By Transformation of random variable.

$$f_{Y_i}(y) = \left|\frac{dx}{dy}\right| f_{X_i}(x) = \frac{1}{x} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$$

$$= \frac{1}{\sqrt{2y}} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-y}$$

$$= \frac{1}{2} \cdot \frac{e^{-y} \cdot y^{1/2-1}}{\Gamma(1/2)} \quad \{\because \Gamma(1/2) = \sqrt{\pi}\}$$

Now, $f_{X_i}(x)$ is symmetric about $x = 0$,

Therefore due to symmetricity $f_{Y_1}(y) = 2 \cdot \frac{1}{2} \cdot \frac{e^{-y} y^{1/2-1}}{\Gamma(1/2)}$

$$f_{Y_1}(y) = \frac{e^{-y} y^{1/2-1}}{\Gamma(1/2)}, \quad 0 < y < \infty$$

$\Rightarrow Y \sim \Gamma(1/2)$ distribution.

Now, since $\frac{1}{2}X_i^2 \sim \Gamma(1/2)$, $i = 1,2,3,\ldots n$.

$$\therefore \frac{1}{2}X_1^2 + \frac{1}{2}X_2^2 + \cdots + \frac{1}{2}X_n^2 = \sum_{i=1}^{n} \frac{1}{2}X_i^2 \sim \Gamma\left(\frac{n}{2}\right)$$

$\{X \sim \Gamma(m), \; Y \sim \Gamma(n)\}$

$\Rightarrow X + Y \sim \Gamma(m+n)$

If $X \sim \Gamma(n/2)$ variate, then $Y = 2X$ is χ^2 distribution with n-degrees of freedom.

$$\because \sum_{i=1}^{n} \frac{1}{2} X_i^2 \sim \Gamma(n/2)$$

$$\therefore 2 \sum_{i=1}^{n} \frac{1}{2} X_i^2 \sim \chi^2(n) \Rightarrow \sum_{i=1}^{n} X_i^2 \sim \chi^2(n)$$

$$\Rightarrow \boxed{X_1^2 + X_2^2 + X_3^2 + \cdots + X_n^2 \sim \chi^2(n)}$$

$X \sim \Gamma(n/2)$

$$\therefore f(x) = \begin{cases} \dfrac{e^{-x} \cdot x^{n/2-1}}{\Gamma(n/2)} & ; \quad 0 < x < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

setting $Y = 2X \rightarrow \dfrac{dy}{dx} = 2$ $\quad \therefore y$ is a monotonic function

By Transformation of random variables,

$$f_Y(y) = \left| \frac{dx}{dy} \right| f_X(x) = \frac{1}{2} \cdot \frac{e^{-x} \cdot x^{n/2-1}}{\Gamma(n/2)}, \quad 0 < x < \infty$$

$$= \frac{1}{2} \frac{e^{-y/2} \cdot (y/2)^{n/2-1}}{\Gamma(n/2)}, \quad 0 < y < \infty$$

$$\therefore \boxed{Y \sim \chi_n^2}$$

Q6. For the case of equal components, the central limit theorem implies the law of large number but the converse is not true.

$\Rightarrow$ Let $X_1, X_2, X_3, \ldots, X_n$ be independent and identically distributed random variables such that all of them have the same mean (μ) and standard deviation (σ)

that is $\rightarrow$ ($X_i \sim N(\mu, \sigma)$; $i = 1, 2, \ldots n$)

Now,

<u>CENTRAL</u> <u>LIMIT</u> <u>THEOREM</u> states that for

$$S_n = X_1 + X_2 + \cdots + X_n = \sum_{i=1}^{n} X_i \sim N(n\mu, \sigma\sqrt{n})$$

By standardization, $\dfrac{S_n - n\mu}{\sigma\sqrt{n}} \sim N(0,1)$ [Standard Normal Distribution]

$$\frac{\frac{S_n}{n} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

Let $\bar{X}_n$ be a random variable denoting average / mean of X_i ($i = 1, 2, \ldots n$)

$$\therefore \quad \bar{X}_n = \frac{X_1 + X_2 + \cdots + X_n}{n} = \sum_{i=1}^{n} \frac{X_i}{n} = \frac{S_n}{n}$$

$$\therefore \quad \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim N(0,1) \quad \text{as} \quad n \to \infty$$

$$\Rightarrow \quad \bar{X}_n \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

$\Rightarrow$ as (n) gets large (or as $n \to \infty$), $\bar{X}_n$ approximate more like a Normal distribution with mean μ and smaller variance $\sigma/\sqrt{n}$ i.e. as $n \to \infty$, $\bar{X}_n \to \mu$.

$\Rightarrow$ $P\left(\lim_{n \to \infty} (\bar{X}_n = \mu) = 1\right)$ which is actually "<u>Law of Large Numbers</u>"

Therefore, <u>Central Limit Theorem implies Law of large numbers</u>.

But the converse is not true

→ Let the sample size (n) with following independent and identically distributed random variables $X_1, X_2, X_3, \ldots, X_n$ such that $X_i \sim N(\mu, \sigma)$, $i = 1, 2, 3, \ldots, n$

Let $\bar{X}_n$ be the random variable denoting mean of n random variables $\bar{X}_n = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$

Law of Large Numbers states that as n increases (or $n \to \infty$)

$\bar{X}_n \to \mu$.

but this doesnot anything about the distribution of $\bar{X}_n$

$\therefore$ Law of Large Numbers doesnot imply Central Limit Theorem.

Q7. To verify :- whether sample variance is an unbiased estimate of population variance. Otherwise derive the unbiased estimate of population variance.

⇒ For independent and identically distributed random variables $X_1, X_2, \ldots, X_n$ each with mean μ and variance σ^2 ∴

$$E(x_i) = \mu$$
$$\sigma^2 = E(x_i^2) - (E(x_i))^2 \qquad (i = 1, 2, 3, \ldots, n)$$
$$\sigma^2 = E(x_i^2) - \mu^2$$
$$\underline{\underline{E(x_i^2) = \sigma^2 + \mu^2}}$$

According to central limit Theorem :-

$\bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$ where $\bar{X}$ is random variable for average of $X_1, X_2, \ldots X_n$

∴ $E(\bar{X}) = \mu$

Variance $(\bar{X}) = \frac{\sigma^2}{n}$ (using above relation)

$$E(\bar{x}^2) = \frac{\sigma^2}{n} + \mu^2$$

Now, Sample Variance is

$$S^2 = \frac{1}{n} \Sigma (x_i - \bar{x})^2$$ (summation limits are from $i=1$ to n)

$$= \frac{1}{n} \Sigma (x_i^2) - \frac{1}{n} 2\bar{x} \Sigma x_i + \frac{1}{\cancel{n}} \cdot \cancel{n} \bar{x}^2$$

$$= \frac{1}{n}\left(\Sigma(x_i^2) - 2\bar{x}(n\bar{x}) + n\bar{x}^2\right) \qquad \left\{ \bar{x} = \frac{\Sigma x_i}{n} \right\}$$

$$= \frac{1}{n}\left(\Sigma(x_i^2) - n\bar{x}^2\right)$$

$$E(S^2) = E\left(\frac{1}{n}\left(\Sigma(x_i^2) - n\bar{x}^2\right)\right)$$
$$= \frac{1}{n}\left[\Sigma E(x_i^2) - E(n\bar{x}^2)\right]$$
$$= \frac{1}{n}\left[\Sigma E(x_i^2) - nE(\bar{x}^2)\right]$$
$$= \frac{1}{n}\left[\Sigma(\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right)\right]$$
$$= \frac{1}{n}\left[n\sigma^2 + n\mu^2 - \sigma^2 - n\mu^2\right] = (n-1)/n \cdot \sigma^2$$

Hence, s^2 is a consistent estimator of σ^2 but is biased to σ^2 as $E(s^2) - \sigma^2 < 0$

Therefore, if we calculate s^2 using $(n-1)$ as denominator

i.e. $$S^2 = \frac{\Sigma (x_i - \bar{x})^2}{n-1}$$

$$E(S^2) = E\left(\frac{\Sigma (x_i - \bar{x})^2}{n-1}\right) = E\left(\frac{n}{n-1} \cdot \frac{\Sigma (x_i - \bar{x})^2}{n}\right)$$

$$= \frac{n}{n-1} E\left(\frac{\Sigma (x_i - \bar{x})^2}{n}\right) = \frac{\cancel{n}}{\cancel{n-1}} \cdot \frac{\cancel{n-1}}{\cancel{n}} \cdot \sigma^2 = \sigma^2$$

$\therefore$ $S^2 = \frac{\Sigma (x_i - \bar{x})^2}{n-1}$ is a good consistent estimator of σ^2 and is unbiased estimate of σ^2.

Q8. Show that :- statistics $t = \frac{\sqrt{n}(\bar{x}-m)}{s}$ represents a t-distribution with $(n-1)$ degrees of freedom. (14)

$\Rightarrow \quad t = \frac{(\bar{x}-m)\sqrt{n}}{s} = \frac{\frac{(\bar{x}-m)}{\sigma/\sqrt{n}}}{s/\sigma}$ $\quad \{Var(\bar{x}) = \frac{\sigma^2}{n}\}$

$t = \frac{U}{s/\sigma}$ where $U = \frac{\bar{x}-m}{\sigma/\sqrt{n}}$

$\Sigma(x_i-m)^2 = \Sigma((x_i-\bar{x}) + (\bar{x}-m))^2$ limits from $i=1$ to n.

$= \Sigma(x_i-\bar{x})^2 + 2\Sigma(x_i-\bar{x})(\bar{x}-m) + \Sigma(\bar{x}-m)^2$

$= \Sigma(x_i-\bar{x})^2 + 0 + n(\bar{x}-m)^2$ $\quad \{\because \Sigma(x_i-\bar{x}) = 0\}$

$= \Sigma(x_i-\bar{x})^2 + n(\bar{x}-m)^2$

Dividing by σ^2,

$$\underbrace{\Sigma\left(\frac{x_i-m}{\sigma}\right)^2}_{} = \underbrace{\Sigma\left(\frac{x_i-\bar{x}}{\sigma}\right)^2}_{} + \underbrace{\left(\frac{\bar{x}-m}{\sigma/\sqrt{n}}\right)^2}_{}$$

$\because x_i \sim N(m,\sigma)$

$\Rightarrow \frac{x_i-m}{\sigma} = N(0,1)$

$\Rightarrow \Sigma\left(\frac{x_i-m}{\sigma}\right)^2 \sim \chi_n^2$

$\frac{1}{\sigma^2}\Sigma(x_i-\bar{x})^2$

$= \frac{(n-1)}{\sigma^2} \cdot \frac{1}{n-1}\Sigma(x_i-\bar{x})^2$

$= \frac{(n-1)}{\sigma^2} \cdot s^2$

$\because \bar{x} \sim N(m, \sigma/\sqrt{n})$

$\Rightarrow \frac{\bar{x}-m}{\sigma/\sqrt{n}} \sim N(0,1)$

$\Rightarrow \left(\frac{\bar{x}-m}{\sigma/\sqrt{n}}\right)^2 \sim \chi_1^2$

(1 degree of freedom)

$U = V + W$ where $U \sim \chi^2(n)$

$V = \frac{(n-1)s^2}{\sigma^2}$

$W \sim \chi^2(1)$

$\because s$ and $\bar{x}$ are independent, so are V and W

$\therefore M_U(t) = M_V(t) \cdot M_W(t)$ where $M(t)$ is moment generating function.

$\frac{1}{(1-2t)^{n/2}} = M_V(t) \cdot \frac{1}{(1-2t)^{1/2}}$ $\quad \{$ for $\chi^2(n)$, $M_{\chi^2}(t) = \frac{1}{(1-2t)^{n/2}}\}$

$M_V(t) = \frac{1}{(1-2t)^{\frac{n-1}{2}}} \Rightarrow \boxed{V \sim \chi^2(n-1)}$

Therefore from previous proof, $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1)$

Now, $t = \frac{U}{s/\sigma} = \frac{U}{\sqrt{\frac{(n-1)s^2}{(n-1)\sigma^2}}} = \frac{U}{\sqrt{\chi^2/(n-1)}}$ where $\chi^2 = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1)$ distribution

$U = \frac{\bar{x}-m}{\sigma/\sqrt{n}}$ and $\bar{x} \sim N\left(m, \frac{\sigma}{\sqrt{n}}\right) \Rightarrow \therefore \boxed{U \sim N(0,1)}$

$\Rightarrow \quad t = \frac{\text{Normal Distribution }(0,1)}{\sqrt{\text{chi-square distribution with }(n-1)\text{ degree of freedom}}}$

$\therefore$ t ~ t-distribution with (n-1) degree of freedom

$$\left\{ t = \frac{U}{\sqrt{\chi^2/(n-1)}} \rightarrow t^2 = \frac{U^2}{\chi^2/(n-1)} \right.$$

$$\frac{t^2}{n-1} = \frac{1/2\, U^2}{1/2\, \chi^2} = \frac{\text{Gamma}(1/2)\, \Gamma(1/2)}{\text{Gamma}(n-1/2)\, \Gamma(n-1/2)} = \beta_2(1/2, n-1/2)$$

Now, $f(t)\, dt = F'(t)\, dt = dF(t)$

$$\therefore dF = f(t).dt = \frac{(t^2/n-1)^{1/2-1}}{\beta\left(1/2, \frac{n-1}{2}\right).\left(1+t^2/n-1\right)^{n/2}} \cdot \frac{2t}{(n-1)} \cdot dt \quad (0 < t^2 < \infty)$$

$$dF = \frac{2}{\sqrt{n-1}\,.\,\beta\left(\frac{1}{2}, \frac{n-1}{2}\right)\left(1+\frac{t^2}{n-1}\right)^{n/2}} . dt, \quad (0 < t^2 < \infty)$$

$$dF = \frac{dt}{\sqrt{n-1}\; \beta\left(\frac{1}{2}, \frac{n-1}{2}\right)\left(1+\frac{t^2}{n-1}\right)^{n/2}}, \quad (-\infty < t < \infty)$$

$$f(t) = \left[\sqrt{n-1}\,.\,\beta\left(\frac{1}{2}, \frac{n-1}{2}\right).\left(1+\frac{t^2}{n-1}\right)^{\frac{n}{2}}\right]^{-1}, \quad -\infty < t < \infty \Bigg\}$$

$\Rightarrow$ $\boxed{\text{t} \sim \text{t distribution with (n-1) degree of freedom.}}$

Q9. $f(x,\theta) = \dfrac{x^{p-1} \cdot e^{-x/\theta}}{\theta^{p}\,\Gamma(p)}$

The likelihood function is

$$L(x_1, x_2, x_3, \ldots, x_n ; \theta) = f(x = x_1, \theta) \cdot f(x = x_2 ; \theta) \cdots f(x = x_n ; \theta)$$

$$= \frac{(x_1 \cdot x_2 \cdots x_n)^{p-1} \cdot e^{-(x_1 + x_2 + \cdots + x_n)/\theta}}{(\theta^{p})^{n} \cdot (\Gamma(p))^{n}}$$

$$= \prod_{i=1}^{n} \frac{x_i^{p-1} \cdot e^{-\sum_{i=1}^{n} x_i/\theta}}{\theta^{np} \cdot (\Gamma(p))^{n}}$$

For maximum likelihood estimate, we will maximize $\ln L$, $(\because L > 0)$

$\Rightarrow$ $\max(\ln L)$:-

$$\frac{\partial}{\partial \theta}(\ln L) = 0$$

$$\frac{\partial}{\partial \theta}\left(\ln \left[\frac{\prod x_i^{p-1} \cdot e^{-\sum x_i/\theta}}{\theta^{np} \cdot (\Gamma(p))^{n}} \right] \right) = 0$$

$$\frac{\partial}{\partial \theta}\left((p-1)\sum \ln x_i - \frac{\sum x_i}{\theta} - np \ln \theta - n \ln \Gamma(p) \right) = 0$$

$$\frac{\sum x_i}{\theta^2} - \frac{np}{\theta} = 0$$

$$\theta = \frac{\sum x_i}{np} \quad \Rightarrow \quad \boxed{\hat{\theta} = \frac{\bar{x}}{p}}$$

Since $\bar{x}$ is a good and consistent and unbiased estimate of population mean.

$\therefore$ $\hat{\theta}$ is also an unbiased and consistent estimate.

Q10. $P(X=i) = \frac{1}{1+M}\left(\frac{M}{1+M}\right)^i$, $M>0$ for $i=0,1,2,\ldots$

$\Rightarrow$ The likelihood function is -

(for sample size n)

$$L(x_0, x_1, x_2, \ldots ; M) = P(x=x_0) \cdot P(x=x_1) \cdot P(x=x_2) \cdots P(x=x_n)$$

$$= \frac{1}{(1+M)^{n+1}} \cdot \frac{(M)^{x_0+x_1+x_2+\cdots+x_n}}{(1+M)^{x_0+x_1+x_2+\cdots+x_n}}$$

$$= \frac{1}{(1+M)^{n+1}}\left(\frac{M}{1+M}\right)^{\Sigma_i x_i}$$

For maximum likelihood estimate, we will maximise $\ln L$

$$\therefore \frac{\partial}{\partial M}(\ln L) = 0 \qquad (\because L>0)$$

$$\frac{\partial}{\partial M}\left(\ln\left[\frac{M^{\Sigma x_i}}{(1+M)^{\Sigma x_i + n+1}}\right]\right) = 0 \rightarrow \frac{\partial}{\partial M}\left(\ln\left[\frac{M^{\Sigma x_i}}{(1+M)^{\Sigma x_i+n+1}}\right]\right) = 0$$

$$\frac{\partial}{\partial M}\left[\Sigma x_i \cdot \ln M - (\Sigma x_i + n+1)\ln(1+M)\right] = 0$$

$$\frac{\Sigma x_i}{M} - \frac{\Sigma x_i + n+1}{1+M} = 0$$

$$\frac{\bar{x}}{M} - \frac{\bar{x}+1}{1+M} = 0$$

$$\bar{x}(1+M) = (\bar{x}+1)M \Rightarrow \bar{x} = M$$

$$\boxed{\hat{M} = \bar{x}}$$

Since $\bar{x}$ is an unbiased & consistent estimate of mean

$\therefore$ $\hat{M} = \bar{x}$ is an unbiased and consistent estimate of parameter M.

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