Model Question Paper –I with effect from 2022

USN | | | | | | | | | | |

First Semester B. E Degree examination

Mathematics-I for Computer Science Stream (2MATS11)

Time: 03 Hours **Max. Marks: 100**

Note: Answer any **FIVE** full questions, choosing at least **ONE** question from each module.

		Module-1	**Marks**
Q. 01	a	With usual notation prove that $tan\phi = r\frac{d\theta}{dr}$	6
	b	Find the angle between the curves $r = alog\theta$ and $r = \frac{a}{log\theta}$	7
	c	Show that the radius of curvature at any point θ on the cycloid $x = a(\theta + sin\theta), y = a(1 - cos\theta)$ is $4\,a\cos\left(\frac{\theta}{2}\right)$	7
		OR	
Q. 02	a	Show that the curves $r = a(1 + sin\theta)$ and $r = a(1 - sin\theta)$ cuts each other orthogonally	6
	b	Find the pedal equation of the curve $\frac{2a}{r} = (1 + cos\theta)$	7
	c	Find the radius of curvature for the curve $y^2 = \frac{4a^2(2a-x)}{x}$, where the curve meets the x-axis	7
		Module-2	
Q. 03	a	Expand $\log(secx)$ up to the term containing x^4 using Maclaurin's series	6
	b	If $u = e^{ax+by}f(ax - by)$ prove that $b\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = 2abu$ by using the concept of composite functions.	7
	c	Find the extreme values of the function $f(x,y) = x^3 + 3xy^2 - 3y^2 - 3x^2 + 4$	7
		OR	
Q. 04	a	Evaluate i) $\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{1/x}$ ii) $\lim_{x\to 0}\left(\frac{tanx}{x}\right)^{1/x}$	6
	b	If $u = f(x - y, y - z, z - x)$ show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$	7
	c	If $x + y + z = u,\ \ y + z = v\ and\ z = uvw$, find the values of $\frac{\partial(x,y,z)}{\partial(u,v,w)}$	7
		Module-3	
Q. 05	a	Solve $\frac{dy}{dx} + \frac{y}{x} = x^2y^6$	6
	b	Find the orthogonal trajectories of $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$, where λ is a parameter.	7

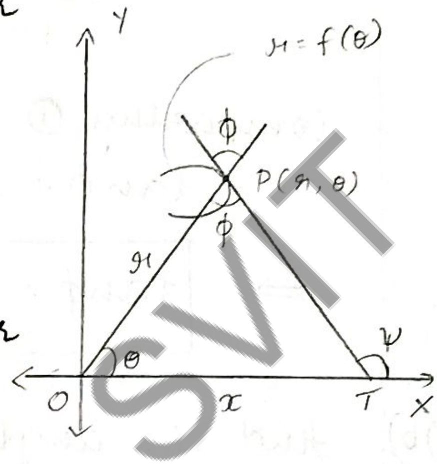
	c	Solve $xyp^2 - (x^2 + y^2)p + xy = 0$	7
		OR	
Q. 06	a	Solve $(x^2 + y^2 + x)dx + xydy = 0$	6
	b	When a switch is closed in a circuit containing a battery E, a resistance R and an inductance L, the current i build up at a rate given by $L\frac{di}{dt} + Ri = E$. Find i as a function of t. How long will it be, before the current has reached one-half its final value, if E = 6 volts, R = 100 ohms and L = 0.1 henry?	7
	c	Find the general solution of the equation $(px - y)(py + x) = a^2p$ by reducing into Clairaut's form by taking the substitution $X = x^2, Y = y^2$	7
		Module-4	
Q. 07	a	Find the least positive values of x such that i) $71 \equiv x(mod8)$ ii) $78 + x \equiv 3(mod5)$ iii) $89 \equiv (x + 3)(mod4)$	6
	b	Find the remainder when $(349 \times 74 \times 36)$ is divided by 3	7
	c	Solve $2x + 6y \equiv 1(mod7)$ $4x + 3y \equiv 2(mod7)$	7
		OR	
Q. 08	a	i) Find the last digit of 7^{2013} ii) Find the last digit of 13^{37}	6
	b	Find the remainder when the number 2^{1000} is divided by 13	7
	c	Find the remainder when 14! is divided by 17	7
		Module-5	
Q. 09	a	Find the rank of the matrix $\begin{bmatrix} 2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$	6
	b	Solve the system of equations by using the Gauss-Jordan method $x + y + z = 10,\ 2x - y + 3z = 19, x + 2y + 3z = 22$	7
	c	Using power method find the largest eigenvalue and the corresponding eigenvector of the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$	7
		OR	
Q. 10	a	Solve the following system of equations by Gauss-Seidel method $10x + y + z = 12, x + 10y + z = 12, x + y + 10z = 12$	6
	b	For what values of a and b the system of equation $x + y + z = 6: x + 2y + 3z = 10: x + 2y + az = b$ has i) no solution ii) a unique solution and iii) infinite number of solution	7

	c	Solve the system of equations by Gauss elimination method $x + y + z = 9,\ x - 2y + 3z = 8,\ 2x + y - z = 3$	7

Table showing the Blooms Taxonomy Level, Course outcome and Program outcome				
Question		Blooms taxonomy level attached	Course outcome	Program outcome
Q.1	a)	L1	CO 01	PO 01
	b)	L2	CO 01	PO 01
	c)	L3	CO 01	PO 02
Q. 2	a)	L1	CO 01	PO 01
	b)	L2	CO 01	PO 01
	c)	L3	CO 01	PO 02
Q. 3	a)	L2	CO 02	PO 01
	b)	L2	CO 02	PO 01
	c)	L3	CO 02	PO 03
Q. 4	a)	L2	CO 02	PO 01
	b)	L2	CO 02	PO 01
	c)	L3	CO 02	PO 02
Q. 5	a)	L2	CO 03	PO 02
	b)	L3	CO 03	PO 03
	c)	L2	CO 03	PO 01
Q. 6	a)	L2	CO 03	PO 02
	b)	L3	CO 03	PO 03
	c)	L2	CO 03	PO 01
Q. 7	a)	L2	CO 04	PO 01
	b)	L2	CO 04	PO 01
	c)	L2	CO 04	PO 02
Q. 8	a)	L2	CO 04	PO 01
	b)	L2	CO 04	PO 01
	c)	L2	CO 04	PO 02
Q. 9	a)	L2	CO 05	PO 01
	b)	L3	CO 05	PO 01
	c)	L3	CO 05	PO 02
Q. 10	a)	L2	CO 05	PO 01
	b)	L3	CO 05	PO 02
	c)	L3	CO 05	PO 01

Bloom's Taxonomy Levels	**Lower-order thinking skills**		
	Remembering (Knowledge): L_1	Understanding (Comprehension): L_2	Applying (Application): L_3
	Higher-order thinking skills		
	Analyzing (Analysis): L_4	Valuating (Evaluation): L_5	Creating (Synthesis): L_6

1)a) With usual notation prove that $\tan\phi = r\dfrac{d\theta}{dr}$



Let $P(r,\theta)$ be a point on the curve $r=f(\theta)$

Hence $OP = r$ and $X\hat{O}P = \theta$

Let PT be the tangent to the curve $r=f(\theta)$ and let $X\hat{T}P = \psi$.

Let ϕ be the angle between the radius vector OP and the tangent PT i.e, $O\hat{P}T = \phi$

From the figure,

$$\psi = \phi + \theta$$

$$\tan\psi = \tan(\phi+\theta)$$

$$\tan\psi = \frac{\tan\phi + \tan\theta}{1-\tan\phi\cdot\tan\theta} \quad \text{—— (1)}$$

If $P(x,y)$ are cartesian coordinates of P then we have,

$x = r\cos\theta$, $y = r\sin\theta$

Also, WKT $\tan\psi = \dfrac{dy}{dx}$ = slope of the tangent

Divide both numerator and denominator by $d\theta$

$$\tan\psi = \frac{dy/d\theta}{dx/d\theta}$$

$$\tan\psi = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)}$$

$$\tan\psi = \frac{r\cdot\cos\theta + \sin\theta\frac{dr}{d\theta}}{-r\sin\theta + \cos\theta\frac{dr}{d\theta}}$$

Let $\dfrac{dr}{d\theta} = r'$

$$\tan\psi = \frac{r\cos\theta + r'\sin\theta}{r'\cos\theta - r\sin\theta}$$

Divide both numerator and denominator by $r'\cos\theta$

$$\tan\psi = \frac{\frac{r}{r'} + \tan\theta}{1 - \frac{r}{r'}\tan\theta} \quad \text{—— (2)}$$

comparing (1) and (2),

$$\tan\phi = \frac{r}{r'}$$

$$\Rightarrow \boxed{\tan\phi = r\frac{d\theta}{dr}}$$

1) b) Find the angle between the curves $r = a\log\theta$ and $r = \frac{a}{\log\theta}$

$r = a\log\theta$

take log on both sides

$\log r = \log a + \log(\log\theta)$

diff wrt θ

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\log\theta}\left(\frac{1}{\theta}\right)(1)$$

$$\cot\phi_1 = \frac{1}{\theta\log\theta}$$

$$\boxed{\tan\phi_1 = \theta\log\theta}$$

$r = \frac{a}{\log\theta}$

take log on both sides

$\log r = \log a - \log(\log\theta)$

diff wrt θ

$$\frac{1}{r}\frac{dr}{d\theta} = 0 - \frac{1}{\log\theta}\left(\frac{1}{\theta}\right)(1)$$

$$\cot\phi_2 = \frac{-1}{\theta\log\theta}$$

$$\boxed{\tan\phi_2 = -\theta\log\theta}$$

$$\tan|\phi_1 - \phi_2| = \frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \cdot \tan\phi_2}$$

$$= \frac{\theta\log\theta - (-\theta\log\theta)}{1 + \theta\log\theta(-\theta\log\theta)}$$

$$= \frac{2\theta\log\theta}{1 - \theta^2(\log^2\theta)^2}$$

To find θ,

consider $r = a\log\theta$; $r = \frac{a}{\log\theta}$

$$\cancel{a}\log\theta = \frac{a}{\log\theta}$$

$$(\log \theta)^2 = 1$$

$$\Rightarrow \boxed{\theta = e}$$

Substituting in eq^n ① we get,

$$\tan|\phi_1 - \phi_2| = \frac{2e \log e}{1 - e^2(\log e)^2}$$

but $\log e = 1$

$$\tan|\phi_1 - \phi_2| = \frac{2e}{1-e^2} \Rightarrow |\phi_1 - \phi_2| = \tan^{-1}\left(\frac{2e}{1-e^2}\right)$$

1)c) Show that radius of curvature at any point θ, on the cycloid $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$ is $4a\cos\left(\frac{\theta}{2}\right)$

$$x = a(\theta + \sin\theta) \qquad y = a(1-\cos\theta)$$

$$\frac{dx}{d\theta} = a + a\cos\theta \qquad \frac{dy}{d\theta} = a(\sin\theta)$$

$$y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$y_1 = \frac{\cancel{a}\sin\theta}{\cancel{a}(1+\cos\theta)}$$

$$y_1 = \frac{2\sin\theta/2 \cos\theta/2}{2\cos^2\theta/2} \Rightarrow \boxed{y_1 = \tan\theta/2}$$

diff wrt x,

$$y_2 = \sec^2\left(\frac{\theta}{2}\right) \cdot \frac{d\theta}{dx} \cdot \frac{1}{2}$$

$$y_2 = \sec^2\left(\frac{\theta}{2}\right) \cdot \frac{1}{2a(1+\cos\theta)}$$

$$y_2 = \frac{1}{2a} \frac{\sec^2\theta/2}{2\cos^2\theta/2} \Rightarrow \boxed{y_2 = \frac{1}{4a}\sec^4\left(\frac{\theta}{2}\right)}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{\left(1 + \tan^2\theta/2\right)^{3/2}}{\frac{1}{4a}\sec^4\left(\frac{\theta}{2}\right)}$$

$$\rho = \frac{4a\left[\sec^2(\theta/2)\right]^{3/2}}{\sec^4\left(\frac{\theta}{2}\right)}$$

$$\rho = \frac{4a}{\sec\theta/2} \Rightarrow \boxed{\rho = 4a\cos\left(\frac{\theta}{2}\right)}$$

2)a) Show that the curves $r = a(1+\sin\theta)$ and $r = a(1-\sin\theta)$ cuts each other orthogonally.

$r = a(1+\sin\theta)$

take log on both sides

$\log r = \log a + \log(1+\sin\theta)$

diff wrt x,

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{1+\sin\theta}(\cos\theta)$$

$$\cot\phi_1 = \frac{\cos\theta}{1+\sin\theta}$$

$r = a(1-\sin\theta)$

take log on both sides

$\log r = \log a + \log(1-\sin\theta)$

diff wrt x,

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{1-\sin\theta}(-\cos\theta)$$

$$\cot\phi_2 = \frac{-\cos\theta}{1-\sin\theta}$$

$$\cot\phi_1 \cdot \cot\phi_2 = \frac{\cos\theta}{1+\sin\theta} \times \frac{-\cos\theta}{1-\sin\theta}$$

$$= \frac{-\cos^2\theta}{1^2 - \sin^2\theta}$$

$$= \frac{-\cancel{\cos^2\theta}}{\cancel{\cos^2\theta}} \Rightarrow -1$$

Since $\cot\phi_1 \cdot \cot\phi_2 = -1$ i.e, $|\phi_1 - \phi_2| = \pi/2$

$\therefore$ The given two curves intersect orthogonally.

2)b) Find the pedal equation of curve $\frac{2a}{r} = (1+\cos\theta)$

$\log 2a - \log r = \log(1+\cos\theta)$

$$0 - \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1+\cos\theta}(0-\sin\theta)$$

$$\boxed{\cot\phi = \frac{\sin\theta}{1+\cos\theta}}$$

pedal eqⁿ, $\frac{1}{p^2} = \frac{1}{r^2}\left[1+\cot^2\phi\right]$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[1 + \frac{\sin^2\theta}{(1+\cos\theta)^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{1+\cos^2\theta+2\cos\theta+\sin^2\theta}{(1+\cos\theta)^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{2+2\cos\theta}{(1+\cos\theta)^2}\right]$$

$$\frac{1}{p^2} = \frac{2}{r^2}\left[\frac{\cancel{1+\cos\theta}}{(1+\cos\theta)^{\cancel{2}}}\right] \Rightarrow \boxed{\frac{1}{p^2} = \frac{2}{r^2}\left[\frac{1}{1+\cos\theta}\right]}$$

To eliminate θ,

$$1+\cos\theta = \frac{2a}{r}$$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{\cancel{2}}{r^{\cancel{2}}} \times \frac{\cancel{r}}{\cancel{2}a} \quad \Rightarrow \quad \boxed{p = \sqrt{ar}}$$

2)c) Find the radius of curvature for curve $y^2 = \frac{4a^2(2a-x)}{x}$, where the curve meets x-axis.

$$y^2 = \frac{4a^2(2a-x)}{x}$$

$$xy^2 = 4a^2(2a-x)$$

Since the curve cuts the x axis, we have $y=0$

Substitute $y=0$ in the given curve

$$xy^2 = 4a^2(2a-x)$$

$$x(0)^2 = 4a^2(2a-x)$$

$$4a^2x = 8a^3$$

$$x = \frac{8a^3}{4a^2} \Rightarrow \boxed{x = 2a}$$

Therefore, to find the ROC of given curve at $(2a, 0)$

Consider, $xy^2 = 4a^2(2a-x) \Rightarrow xy^2 = 8a^3 - 4a^2x$

Diff wrt to x

$$x \cdot 2yy_1 + y^2 \cdot 1 = 0 - 4a^2$$

$$2xyy_1 + y^2 = -4a^2$$

$$y_1 = \frac{-4a^2 - y^2}{2xy}$$

at $(2a, 0)$ $\quad y_1 = \frac{-4a^2 - (0)^2}{2(2a)(0)}$

$$y_1 = \infty \text{ at } (2a, 0)$$

Since $y_1 = \infty$ at $(2a, 0)$ we consider,

$$x_1 = \frac{dx}{dy} = \frac{2xy}{-4a^2 - y^2}$$

at $(2a, 0)$ $\quad x_1 = \frac{2(2a)(0)}{-4a^2 - 0} \Rightarrow x_1 = 0$

Consider, $x_1 = \frac{2xy}{-4a^2 - y^2} = \frac{2xy}{-(4a^2 + y^2)}$

$$(4a^2 + y^2)x_1 = -2xy$$

Diff wrt y

$$(4a^2 + y^2)x_2 + x_1(2y) = -2[x \cdot 1 + y \cdot x_1]$$

at $(2a, 0)$ $\quad (4a^2 + 0^2)x_2 + 0 = -2[2a \cdot 1 + 0 \cdot x_1]$

$$4a^2 \cdot x_2 = -4a$$

$$x_2 = \cancel{-\frac{4a}{4a^2}} \Rightarrow \boxed{x_2 = -\frac{1}{a}}$$

ROC, $\quad \rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$

$$= \frac{[1 + 0]^{3/2}}{\left(-\frac{1}{a}\right)} \Rightarrow \boxed{|\rho| = a}$$

3)a) Show that, expand $\log(\sec x)$ upto term containing x^4 using Maclaurin's series.

→ Maclaurin's series expansion is given by:

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \dots \quad \text{——— (1)}$$

$y(x) = \log(\sec x)$; $\quad y(0) = \log(\sec 0) = \boxed{0}$

$y_1(x) = \frac{1}{\cancel{\sec x}} \cdot \cancel{\sec x} \cdot \tan x = \tan x$; $\quad y_1(0) = \tan 0 = \boxed{0}$

$y_2(x) = \sec^2 x$; $\quad y_2(0) = \sec^2(0) = \boxed{1}$

$y_2(x) = 1 + \tan^2 x$

$y_2(x) = 1 + y_1^2$

$y_3(x) = 2y_1y_2$; $\quad y_3(0) = 2(0)(1) = \boxed{0}$

$y_4(x) = 2\{y_1y_3 + y_2^2\}$; $\quad y_4(0) = 2\{0 + 1^2\} = \boxed{2}$

Substitute in eq^n (1),

$$y(x) = \frac{x^2}{2!}(1) + \frac{x^2}{4!}(2)$$

$$y(x) = \frac{x^2}{2} + \frac{x^2}{24}(2) \Rightarrow \boxed{y(x) = \frac{x^2}{2} + \frac{x^2}{12}}$$

3)b) If $u = e^{ax+by} f(ax-by)$, prove that $b\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = 2abu$ by using the concept of composite functions.

→ $$u = e^{ax+by} f(ax-by) \quad \text{——— (1)}$$

Diff partially wrt to x

$$\frac{\partial u}{\partial x} = e^{ax+by} f'(ax-by)\frac{\partial}{\partial x}(ax-by) + f(ax-by)\, e^{ax+by}\frac{\partial}{\partial x}(ax+by)$$

$$\frac{\partial u}{\partial x} = e^{ax+by} f'(ax-by) \cdot a + f(ax-by)\, e^{ax+by} \cdot a$$

$$\frac{\partial u}{\partial x} = ae^{ax+by} f'(ax-by) + au$$

Multiply 'b' on both sides

$$b \cdot \frac{\partial u}{\partial x} = abe^{ax+by} f'(ax-by) + abu \quad \text{—— (2)}$$

Diff (1) wrt y.

$$\frac{\partial u}{\partial y} = e^{ax+by} f'(ax-by)(-b) + f(ax-by)\, e^{ax+by}(b)$$

$$\frac{\partial u}{\partial y} = -be^{ax+by} f'(ax-by) + bu$$

Multiply 'a' on both sides

$$a \cdot \frac{\partial u}{\partial y} = -abe^{ax+by} f'(ax-by) + abu \quad \text{—— (3)}$$

Add eq^n (2) and (3)

$$\therefore \boxed{b\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = 2abu}$$

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3)c) Find the extreme values of function $f(x,y) = x^3 + 3xy^2 - 3y^2 - 3x^2 + 4$

→ $f(x,y) = x^3 + 3xy^2 - 3y^2 - 3x^2 + 4$

$f_x = 3x^2 + 3y^2 - 6x$

$f_y = 6xy - 6y$

To find stationary points such that $f_x = 0$ and $f_y = 0$

$3x^2 + 3y^2 - 6x = 0$ —— (1) ; $6xy - 6y = 0$

$(\div \text{ by } 6)$

$xy - y = 0$

$y(x-1) = 0$

$\boxed{y=0}$ $\boxed{x=1}$

Substitute $y=0$ in eq^n (1),

$3x^2 + 3y^2 - 6x = 0$

$(\div \text{ by } 3)$ $x^2 + y^2 - 2x = 0$

$x^2 - 2x = 0$

$x(x-2) = 0 \Rightarrow \boxed{x=0}$ $\boxed{x=2}$

$\therefore$ stationary points are $(0,0)\ (2,0)$

Substitute $x=1$ in eq^n ①,

$$3x^2+3y^2-6x=0$$

$$(\div 3)\quad x^2+y^2-2x=0$$

$$1+y^2-2=0$$

$$y^2-1=0 \Rightarrow \boxed{y=\pm 1}$$

$\therefore$ stationary points are $(1,-1)\ (1,1)$

$\therefore$ stationary points are $(0,0)\ (2,0)\ (1,-1)\ (1,1)$

$A = f_{xx} = 6x-6$

$B = f_{xy} = 6y$

$C = f_{yy} = 6x-6$

	$(0,0)$	$(2,0)$	$(1,-1)$	$(1,1)$
$A=6x-6$	$-6<0$	$6>0$	0	0
$B=6y$	0	0	-6	6
$C=6x-6$	-6	6	0	0
$AC-B^2$	$36>0$	$36>0$	-36	-36
conclusion	Max point	Min point	Saddle point	Saddle point

$\therefore$ Minimum value of $f(2,0) = x^3+3xy^2-3y^2-3x^2+4$

$= 2^3+2(2)(0)-3(0)-3(2)^2+4$

$= 8+0-0-12+4$

$= 0$

Maximum value of $f(0,0) = 0+3(0)(0)-3(0)-3(0)+4$

$= 4$

4)a) Evaluate

(i) $\lim_{x\to 0}\left(\dfrac{a^x+b^x}{2}\right)^{\frac{1}{x}}$

→ Let $K = \lim_{x\to 0} \left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}}$

$$\log_e K = \lim_{x\to 0} \frac{1}{x} \log\left(\frac{a^x+b^x}{2}\right)$$

$$\log_e K = \lim_{x\to 0} \frac{\log\left(\frac{a^x+b^x}{2}\right)}{x}$$

Apply LHR,

$$\log_e K = \lim_{x\to 0} \frac{\frac{1}{\left(\frac{a^x+b^x}{2}\right)} \cdot \frac{1}{2}\{a^x \log a + b^x \log b\}}{1}$$

$$\log_e K = \lim_{x\to 0} \frac{1}{(a^x+b^x)}\{a^x \log a + b^x \log b\}$$

$$\log_e K = \frac{1}{(a^0+b^0)}\{a^0 \log a + b^0 \log b\}$$

$$\log_e K = \frac{1}{2}\{\log a + \log b\}$$

$$\log_e K = \frac{1}{2}(\log ab)$$

$$\log_e K = \log(ab)^{1/2}$$

$$K = (ab)^{1/2} \implies \boxed{K = \sqrt{ab}}$$

(ii) $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$

Let $K = \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$

$$\log_e K = \lim_{x\to 0} \frac{1}{x} \log\left(\frac{\tan x}{x}\right)$$

$$\log_e K = \lim_{x\to 0} \frac{\log\left(\frac{\tan x}{x}\right)}{x}$$

Apply LHR,

$$\log_e K = \frac{\lim\limits_{x\to 0} \frac{1}{\left(\frac{\tan x}{x}\right)} \left\{ \frac{x \cdot \sec^2 x - \tan x \cdot 1}{x^2} \right\}}{1}$$

$$\log_e K = \lim_{x\to 0} \frac{x \cdot \sec^2 x - \tan x}{x^2} \qquad \left(\because \lim_{x\to 0} \frac{\tan x}{x} = 1\right)$$

Apply LHR,

$$\log_e K = \lim_{x\to 0} \frac{x \cdot 2\sec^2 x \cdot \tan x + \sec^2 x \cdot 1 - \sec^2 x}{2x}$$

$$\log_e K = \lim_{x\to 0} \frac{2x \sec^2 x \cdot \tan x}{2x}$$

$$\log_e K = \lim_{x\to 0} \frac{\sec^2 x \cdot \tan x}{1}$$

$$\log_e K = \frac{\sec^2(0) \cdot \tan(0)}{1}$$

$$\log_e K = 0 \implies \boxed{K = e^0 = 1}$$

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4) b) If $u = f(x-y, y-z, z-x)$ show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Let $p = x-y$, $q = y-z$, $r = z-x$

$\therefore$ $u \to (p, q, r) \to (x, y, z) \implies u \to (x, y, z)$

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ exists

$$\frac{\partial u}{\partial x} = u_x = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}(1) + \frac{\partial u}{\partial q}(0) + \frac{\partial u}{\partial r}(-1)$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r}} \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial y} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}(-1) + \frac{\partial u}{\partial q}(1) + \frac{\partial u}{\partial r}(0)$$

$$\boxed{\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q}} \quad \text{—— (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}(-1) + \frac{\partial u}{\partial r}(1)$$

$$\boxed{\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial q} + \frac{\partial u}{\partial r}} \quad \text{—— (3)}$$

Add (1), (2) and (3)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} - \frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} - \frac{\partial u}{\partial q} + \frac{\partial u}{\partial r}$$

$$\therefore \boxed{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0}$$

4)c) If $x+y+z=u$, $y+z=v$ and $z=uvw$ find values of $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

$$J\left(\frac{x,y,z}{u,v,w}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Consider, $x+y+z=u$, $y+z=v$, $z=uvw$

$x+v=u$ $y=v-z$

$x=v-u$ $y=v-uvw$

$\frac{\partial x}{\partial u} = 1-0 = 1$

$\frac{\partial x}{\partial v} = -1$

$\frac{\partial x}{\partial w} = 0$

$\frac{\partial y}{\partial u} = 0 - vw$

$= -vw$

$\frac{\partial y}{\partial v} = 1-uw$

$= 1-uw$

$\frac{\partial y}{\partial w} = -uv$

$\frac{\partial z}{\partial u} = vw$

$\frac{\partial z}{\partial v} = uw$

$\frac{\partial z}{\partial w} = uv$

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} 1 & -1 & 0 \\ -vw & 1-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$J = 1\left[uv - u^2vw - [-u^2vw]\right] + 1\left[-u\cancel{v}w + u\cancel{v}^2w\right] + 0$

$J = 1\left[uv - \cancel{u^2}vw + \cancel{u^2}vw\right] + 1[0]$

$\boxed{J = uv}$

5)a) solve $\frac{dy}{dx} + \frac{y}{x} = x^2y^6$

$\rightarrow$

$$\frac{dy}{dx} + \frac{y}{x} = x^2y^6 \quad \text{--------} \quad (1)$$

given DE is of the form $\frac{dy}{dx} + Py = Qy^n$

eq^n (1) divide y^6 throughout,

$$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{y^5} = x^2 \quad \text{--------} \quad (2)$$

Substitute $\frac{1}{y^5} = t$

diff wrt to x

$$\frac{-5}{y^6}\cdot\frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{1}{y^6}\frac{dy}{dx} = -\frac{1}{5}\cdot\frac{dt}{dx}$$

Substitute in (1), $-\frac{1}{5}\frac{dt}{dx} + \frac{1}{x}\cdot t = x^2$

$(x^{ly}$ by $-5)$ $\quad \frac{dt}{dx} - 5\cdot\frac{1}{x}\cdot t = -5x^2 \quad \text{--------} \quad (3)$

$$\frac{dt}{dx} + Pt = Q \text{ ; where } P = -5\cdot\frac{1}{x} \text{ ; } Q = -5x^2$$

$$IF = e^{\int P\cdot dx} = e^{-5\int\frac{1}{x}dx} = e^{(\log x)^{-5}} = x^{-5} = \frac{1}{x^5}$$

soln of (3) is given by -

$$t(IF) = \int Q(IF)\,dx + c$$

$$t\cdot\frac{1}{x^5} = \int -5x^2\cdot\frac{1}{x^5}\cdot dx + c$$

$$t\cdot\frac{1}{x^5} = -5\int\frac{1}{x^3}\cdot dx + c$$

$$t\cdot\frac{1}{x^5} = -5\frac{x^{-2}}{-2}\cancel{dx} + c$$

$$\boxed{\frac{1}{y^5}\cdot\frac{1}{x^5} = \frac{5}{2}x^{-2} + c}$$

b) Find orthogonal trajectories of $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$, where λ is a parameter

$$\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1 \quad \text{——— (1)}$$

$$\frac{x^2(b^2+\lambda) + a^2y^2}{a^2(b^2+\lambda)} = 1$$

$$x^2(b^2+\lambda) + a^2y^2 = a^2(b^2+\lambda)$$

diff wrt x

$$2x\,(b^2+\lambda) + a^2 \cdot 2y\frac{dy}{dx} = 0$$

$(\div 2)$ $$x\,(b^2+\lambda) + a^2 \cdot y\,\frac{dy}{dx} = 0$$

$$(b^2+\lambda)\,x = -a^2y \cdot \frac{dy}{dx}$$

$$(b^2+\lambda) = -\frac{a^2y}{x}\,\frac{dy}{dx}$$

Substitute in equation (1)

$$\frac{x^2}{a^2} + \frac{y^2}{-\frac{a^2y}{x}\,\frac{dy}{dx}} = 1$$

$$\frac{x^2}{a^2} - \frac{xy}{a^2}\cdot\frac{dx}{dy} = 1$$

Replace $-\frac{dx}{dy}$ by $\frac{dy}{dx}$

$$\Rightarrow \frac{x^2}{a^2} + \frac{xy}{a^2}\cdot\frac{dy}{dx} = 1$$

$$x^2 + xy\cdot\frac{dy}{dx} = a^2$$

$$xy\cdot\frac{dy}{dx} = a^2 - x^2$$

$$y\cdot dy = \left(\frac{a^2-x^2}{x}\right)dx$$

$$\int y \cdot dy = \int \frac{a^2}{x} \cdot dx - \int x \cdot dx + c$$

$$\frac{y^2}{2} = a^2 \log x - \frac{x^2}{2} + c$$

$$\boxed{y^2 = 2a^2 \log x - x^2 + 2C}$$

c) Solve $xyp^2 - (x^2+y^2)p + xy = 0$

$$xyp^2 + xy = (x^2+y^2)p$$

$$xyp^2 - (x^2+y^2)p + xy = 0$$

It is similar to $ax^2 + bx + C = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here $a = xy$, $b = -(x^2+y^2)$, $c = xy$

$$p = \frac{(x^2+y^2) \pm \sqrt{(x^2+y^2)^2 - 4(xy)(xy)}}{2xy}$$

$$= \frac{(x^2+y^2) \pm \sqrt{x^2+y^2+2x^2y^2 - 4x^2y^2}}{2xy}$$

$$= \frac{(x^2+y^2) \pm \sqrt{(x^2-y^2)^2}}{2xy}$$

$$= \frac{(x^2+y^2) \pm (x^2-y^2)}{2xy}$$

$$p = \frac{(x^2+y^2) + (x^2-y^2)}{2xy} \qquad p = \frac{(x^2+y^2) - (x^2-y^2)}{2xy}$$

$$p = \frac{2x^2}{2xy} = \frac{x}{y} \qquad p = \frac{2y^2}{2xy} = \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{x}{y}$$

$$y \cdot dy = x \cdot dx$$

$$\int y \cdot dy = \int x \cdot dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + c$$

$$\frac{y^2}{\cancel{2}} = \frac{x^2+2c}{\cancel{2}}$$

$$y^2 - x^2 - K = 0$$

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\frac{1}{y} \cdot dy = \frac{1}{x} \cdot dx$$

$$\int \frac{1}{y} \cdot dy = \int \frac{1}{x} \cdot dx$$

$$\log y = \log x + c$$

$$\log y = \log x + \log K$$

$$\log y = \log (xK)$$

$$y = xK \Rightarrow y - xK = 0$$

General solution is $(y^2-x^2-K)(y-xK) = 0$

---

5)a) Solve $(x^2+y^2+x)dx + xy \cdot dy = 0$

$$(x^2+y^2+x)dx + xy \cdot dy = 0 \text{ --- } ①$$

$M = x^2+y^2+x$ $\quad N = xy$

$$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = y$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{equation ① is not exact}$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2y - y = y \text{ --- close to } N$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{y}{xy} = \frac{1}{x} = f(x)$$

$$\therefore \text{ IF} = e^{\int f(x) \cdot dx} = e^{\int \frac{1}{x} \cdot dx} = e^{\log x} \Rightarrow \boxed{\text{IF} = x}$$

Multiply IF to equation ①,

$$(x^3 + xy^2 + x^2)dx + x^2 y \cdot dy = 0 \text{ --- } ②$$

$$\frac{\partial M}{\partial y} = 2xy \qquad \frac{\partial N}{\partial x} = 2xy$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{eq}^n \text{ (2) is exact}$$

sol^n to eq^n (1) is -

$$\int_{y\text{-const}} M\,dx + \int (N \text{ terms free from } x)\,dy = C$$

$$\int_{y\text{-const}} (x^3 + xy^2 + x^2)\,dx + \int 0.dy = C$$

$$\Longrightarrow \boxed{\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} = C}$$

c) Find general solution of equation $(px-y)(py+x) = a^2p$ by reducing into clairaut's form by taking the substitution $X = x^2$, $Y = y^2$?

$$(px-y)(py+x) = a^2p \quad \text{---} \quad (1)$$

given: $X = x^2$; $Y = y^2$

$$\frac{dX}{dx} = 2x \quad ; \quad \frac{dY}{dy} = 2y$$

WKT, $p = \frac{dy}{dx}$

$$p = \frac{dy}{dY} \cdot \frac{dY}{dX} \cdot \frac{dX}{dx}$$

$$p = \frac{1}{\cancel{2}y} \cdot P \cdot \cancel{2}x$$

$$\boxed{p = \frac{\sqrt{X}}{\sqrt{Y}} P}$$

Substitute in (1),

$$\left[\frac{\sqrt{x}.P}{\sqrt{y}}.\sqrt{x}-\sqrt{y}\right]\left[\frac{\sqrt{x}.P}{\cancel{\sqrt{y}}}\cancel{\sqrt{y}}+\sqrt{x}\right] = a^2\frac{\sqrt{x}}{\sqrt{y}}\cdot p$$

$$\left[\frac{xp-y}{\cancel{\sqrt{y}}}\right]\left[\sqrt{x}P+\sqrt{x}\right] = \frac{a^2\sqrt{x}}{\cancel{\sqrt{y}}}P$$

$$(Px-y)\left[\cancel{\sqrt{x}}(P+1)\right] = a^2\cancel{\sqrt{x}}P$$

$$Px-y = \frac{a^2P}{P+1}$$

$$y = Px-\frac{a^2P}{P+1}$$

This is in clairaut's form

Replace P by c

General solution is : $y = cx - \frac{a^2c}{c+1}$

$$\Rightarrow \boxed{y^2 = cx^2 - \frac{a^2c}{c+1}}$$

b) A series circuit with resistance R, inductance L and electromotive force E is governed by D.E

$L\frac{di}{dt} + Ri = E$ where L and R are constants and initially current i is zero. Find the current at time t?

→ $L\frac{di}{dt} + Ri = E$

($\div$ by L throughout)

$$\frac{di}{dt} + \frac{Ri}{L} = \frac{E}{L}$$

This is a linear DE in i of form $\frac{dy}{dx} + Py = Q$

Here, $P = \frac{R}{L}$, $Q = \frac{E}{L}$

$$IF = e^{\int P.dt} = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L}.t}$$

Solution: $y(IF) = \int Q(IF)\, dt + c$

$$i(IF) = \int Q(IF)\, dt + c$$

$$i.\, e^{\frac{Rt}{L}} = \int \frac{E}{L} \cdot e^{\frac{Rt}{L}} . dt + c$$

$$i.\, e^{\frac{Rt}{L}} = \frac{E}{\cancel{L}} \frac{e^{\frac{Rt}{L}}}{\frac{R}{\cancel{L}}} + c_1$$

$$i.\, e^{\frac{Rt}{L}} = \frac{E}{R} \cdot e^{\frac{Rt}{L}} + c_1$$

($\div$ by $e^{\frac{Rt}{L}}$)

$$i = \frac{E}{R} + c_1 e^{-\frac{Rt}{L}} \quad \text{--- (1)}$$

Eq^n (1) is the general solution of given DE

given:- Initially current is 0

i.e. $i = 0$ when $t = 0$

$$0 = \frac{E}{R} + c_1 e^0 \Rightarrow \boxed{c_1 = \frac{-E}{R}}$$

Substitute c_1 in eq^n (1),

$$i = \frac{E}{R} - \frac{E}{R} \cdot e^{-\frac{Rt}{L}}$$

$$\boxed{i = \frac{E}{R}\left(1 - e^{\frac{-Rt}{L}}\right)}$$

7)a) Find the least positive values of x such that

(i) $71 \equiv x \pmod 8$

(ii) $78 + x \equiv 3 \pmod 5$

(iii) $89 \equiv (x+3) \pmod 4$

→ (i) $71 \equiv x \pmod 8$

$\boxed{x = 7}$ is the least positive value

(ii) $78 + x \equiv 3 \pmod 5$

$78 + x - 3 \equiv 0 \pmod 5$

$75 + x \equiv 5 \times K$

By inspection,

$\boxed{x = 5}$ is the least positive value because $75 + 5 = 80 = 5 \times \underline{16}$

(iii) $89 \equiv (x+3) \pmod 4$

$86 - x \equiv 0 \pmod 4$

$86 - x \equiv 4K$

By inspection,

$\boxed{x = 2}$ is the least positive value because $86 - 2 = 84 = 4 \times \underline{21}$

---

7)b) Find the remainder when ($\overset{349}{\cancel{394}} \times 74 \times 36$) is divided by 3

→ $349 \equiv 1 \pmod 3$

$74 \equiv 2 \pmod 3$

$36 \equiv 0 \pmod 3$

$\therefore (349 \times 74 \times 36) \equiv (1 \times 2 \times 0) \pmod 3$
$\equiv 0 \pmod 3$

$\therefore$ 0 is the remainder when $349 \times 74 \times 36$ is divided by 3

7)c) Solve $2x + 6y \equiv 1 \pmod 7$
$4x + 3y \equiv 2 \pmod 7$

$\rightarrow$ Here, $a = 2$, $b = 6$, $x = 1$
$c = 4$, $d = 3$, $s = 2$; $n = 7$

$\gcd(a, b, n) = \gcd(2, 6, 7) = 1$ and $1 / 4$

$\therefore$ System has solution.

$\gcd(ad - bc, n) = \gcd(18, 7) = 1$

$\therefore$ The system has unique solution

consider,

$2x + 6y \equiv 1 \pmod 7 \quad (\times 2)$
$4x + 3y \equiv 2 \pmod 7$

$\cancel{4x} + 12y \equiv 2 \pmod 7$
$\cancel{4x} + 3y \equiv 2 \pmod 7$

$9y \equiv 0 \pmod 7$

By inspection, $\boxed{y = 0}$

$\therefore \boxed{y \equiv 0 \pmod 7}$

consider $2x+6y \equiv 1 \pmod 7$

$2x \equiv 1 \pmod 7 \quad (\because y=0)$

By inspection,

$\boxed{x=4}$ satisfies the equation

$\therefore \boxed{x \equiv 4 \pmod 7}$

Thus solution is $x \equiv 4 \pmod 7$; $y \equiv 0 \pmod 7$

---

8)a) (i) Find the last digit of 7^{2013}

(ii) Find the last digit of 13^{37}

→ (i) $7^{2013} = 7^{4 \times 503+1}$

$= 7^{4k+1}$

$\equiv 7 \pmod{10}$

$4)2013(503$
20
013
12
1

$\therefore$ 7 is the last digit

(ii) $13 \equiv 13 \pmod{10}$

$13^2 \equiv 13^2 \pmod{10}$

$13^2 \equiv 169 \pmod{10}$

$13^2 \equiv 9 \pmod{10}$

$13^2 \equiv -1 \pmod{10}$

$(13^2)^{18} \equiv (-1)^{18} \pmod{10}$

$13^{36} \equiv 1 \pmod{10}$

$13^{36} \cdot 13^1 \equiv 13 \pmod{10}$

$13^{37} \equiv 3 \pmod{10}$

$\therefore$ Unit digit $= 3$

8)b) Find the remainder when number 2^{1000} is divided by 13

→ $a = 2$, $p = 13$

$\gcd(2, 13) = 1$

By Fermat's little theorem,

$$a^{p-1} \equiv 1 \pmod{p}$$

$$2^{12} \equiv 1 \pmod{13}$$

$$(2^{12})^{83} \equiv 1^{83} \pmod{13}$$

$$2^{996} \equiv 1 \pmod{13}$$

$$2^{996} \cdot 2^4 \equiv 16 \pmod{13}$$

$$2^{1000} \equiv 16 \pmod{13}$$

$$2^{1000} \equiv 3 \pmod{13}$$

12) 1000 (83
96
40
36
4

13) 16 (1
13
3

∴ 3 is the remainder when 2^{1000} is divided by 13.

8)c) Find the remainder when 14! is divided by 17

→ Here, $p = 17$

By Wilson's theorem,

$$(p-1)! \equiv -1 \pmod{p}$$

$$16! \equiv -1 \pmod{17}$$

$$16 \times 15 \times 14! \equiv -1 \pmod{17}$$

$$(-1) \times (-2) \times 14! \equiv -1 \pmod{17}$$

$$2 \times 14! \equiv 16 \pmod{17}$$

$(\div 2) \Rightarrow$ $14! \equiv 8 \pmod{17}$ $\Rightarrow$ 8 is the remainder.

9) a) Find rank of matrix $\begin{bmatrix} 2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

→

$$A = \begin{bmatrix} 2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_2 \longrightarrow 2R_2 - R_1$; $R_3 \longrightarrow 2R_3 - R_1$

$$A = \begin{bmatrix} 2 & -1 & -3 & -1 \\ 0 & 5 & 9 & -1 \\ 0 & 1 & 5 & 3 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_3 \longrightarrow 5R_3 - R_2$; $R_4 \longrightarrow 5R_4 - R_2$

$$A \sim \begin{bmatrix} 2 & -1 & -3 & -1 \\ 0 & 5 & 9 & -1 \\ 0 & 0 & 16 & 16 \\ 0 & 0 & -4 & -4 \end{bmatrix}$$

$R_4 \longrightarrow 4R_4 + R_3$

$$A \sim \begin{bmatrix} 2 & -1 & -3 & -1 \\ 0 & 5 & 9 & -1 \\ 0 & 0 & 16 & 16 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \longrightarrow \frac{1}{2}R_1$; $R_2 \longrightarrow \frac{1}{5}R_2$; $R_3 \longrightarrow \frac{1}{16}R_3$

$$A \sim \begin{bmatrix} 1 & -1/2 & -3/2 & -1/2 \\ 0 & 1 & 9/5 & -1/5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \boxed{\rho(A) = 3}$

9b) Solve system of equation using gauss-Jordan method

$x + y + z = 10$; $2x - y + 3z = 19$; $x + 2y + 3z = 22$

consider the augmented matrix

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 2 & -1 & 3 & 19 \\ 1 & 2 & 3 & 22 \end{array}\right]$$

$R_2 \longrightarrow R_2 - 2R_1$; $R_3 \longrightarrow R_3 - R_1$

$$[A:B] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 0 & -3 & 1 & -1 \\ 0 & 1 & 2 & 12 \end{array}\right]$$

$R_1 \longrightarrow 3R_1 + R_2$; $R_3 \longrightarrow 3R_3 + R_2$

$$[A:B] \sim \left[\begin{array}{ccc|c} 3 & 0 & 4 & 29 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 7 & 35 \end{array}\right]$$

$R_1 \longrightarrow 7R_1 - 4R_3$; $R_2 \longrightarrow 7R_2 - R_3$

$$[A:B] \sim \left[\begin{array}{ccc|c} 21 & 0 & 0 & 63 \\ 0 & -21 & 0 & -42 \\ 0 & 0 & 7 & 35 \end{array}\right]$$

System of equation are:

$21x = 63$; $-21y = -42$; $7z = 35$

$\Rightarrow \boxed{x=3}$ $\Rightarrow \boxed{y=2}$ $\Rightarrow \boxed{z=5}$

$x = 3$, $y = 2$, $z = 5$ is the solution

9) c) Using power method find largest eigen value and corresponding eigen vector of matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

$\rightarrow$ $$Ax^{(0)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \lambda^{(1)} x^{(1)}$$

$$Ax^{(1)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0 \\ 2 \end{bmatrix} = 2.5\begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \lambda^{(2)} x^{(2)}$$

$$Ax^{(2)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 0 \\ 2.6 \end{bmatrix} = 2.8\begin{bmatrix} 1 \\ 0 \\ 0.929 \end{bmatrix} = \lambda^{(3)} x^{(3)}$$

$$Ax^{(3)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.929 \end{bmatrix} = \begin{bmatrix} 2.929 \\ 0 \\ 2.858 \end{bmatrix} = 2.929 \begin{bmatrix} 1 \\ 0 \\ 0.976 \end{bmatrix} = \lambda^{(4)} x^{(4)}$$

$$Ax^{(4)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.976 \end{bmatrix} = \begin{bmatrix} 2.976 \\ 0 \\ 2.952 \end{bmatrix} = 2.976 \begin{bmatrix} 1 \\ 0 \\ 0.992 \end{bmatrix} = \lambda^{(5)} x^{(5)}$$

$$Ax^{(5)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.992 \end{bmatrix} = \begin{bmatrix} 2.992 \\ 0 \\ 2.984 \end{bmatrix} = 2.992 \begin{bmatrix} 1 \\ 0 \\ 0.997 \end{bmatrix} = \lambda^{(6)} x^{(6)}$$

After 6 iterations, the appropriate eigen value is $\lambda = 2.992$ and the corresponding eigen vector is $x = \begin{bmatrix} 1 \\ 0 \\ 0.997 \end{bmatrix}$

10)a) Solve the following system of equations by gauss seidel method
$10x + y + z = 12$, $x + 10y + z = 12$, $x + y + 10z = 12$

→ The given system of equation are diagonally dominant.
Gauss - Siedel method is given by -

$$x = \frac{12 - y - z}{10} \quad \text{— (1)} \qquad y = \frac{12 - x - z}{10} \quad \text{— (2)} \qquad z = \frac{12 - x - y}{10} \quad \text{— (3)}$$

Let the initial approxiamation be $((x^{(0)}, y^{(0)}, z^{(0)}) = (0,0,0))$

Ist iteration: $x^{(1)} = \dfrac{12 - 0 - 0}{10} = 1.2$

$$y^{(1)} = \frac{12 - 1.2 - 0}{10} = 1.08$$

$$z^{(1)} = \frac{12 - 1.2 - 1.08}{10} = 0.972$$

$\therefore (x^{(1)}, y^{(1)}, z^{(1)}) = (1.2, 1.08, 0.972)$

IInd iteration: $x^{(2)} = \dfrac{12 - 1.08 - 0.972}{10} = 0.9948$

$$y^{(2)} = \frac{12-0.9948-0.972}{10} = 1.0033$$

$$z^{(2)} = \frac{12-0.9948-1.0033}{10} = 1.0001$$

$$\therefore (x^{(2)}, y^{(2)}, z^{(2)}) = (0.9948, 1.0033, 1.0001)$$

III rd iteration: $x^{(3)} = \frac{12-1.0033-1.0001}{10} = 0.9996$

$$y^{(3)} = \frac{12-0.9996-1.0001}{10} = 1.0000$$

$$z^{(3)} = \frac{12-0.9996-1.0000}{10} = 1.0000$$

$$\therefore (x^{(3)}, y^{(3)}, z^{(3)}) = (0.9996, 1.0000, 1.0000)$$

---

10) b) For what values of a and b the system of equation :

$x+y+z=6$; $x+2y+3z=10$; $x+2y+az=6$ has -

(i) no solution (ii) a unique solution (iii) infinite number of solution

→ consider the augmented matrix

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & a & : & b \end{bmatrix}$$

$R_2 \longrightarrow R_2 - R_1$; $R_3 \longrightarrow R_3 - R_1$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & a-1 & : & b-6 \end{bmatrix}$$

$R_3 \longrightarrow R_3 - R_2$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & a-3 & : & b-10 \end{bmatrix}$$

(i) NO Solution:

we must have $\rho(A) \neq \rho[A:B]$

If $a=3$, then $\rho(A)=2$

If $b \neq 10$, then $\rho[A:B]=3$

$\therefore$ System of equations have no solution if $a=3$, $b \neq 10$

(ii) Unique solution:

we must have $r=n=3$ i.e $\rho(A)=\rho[A:B]=r=n=3$

If $a \neq 3$, then $\rho(A)=3$

Irrespective of the value of μ, $\rho[A:B]=3$

$\therefore$ system has unique solution if $a \neq 3$ and for any b

(iii) infinite solution:

we must have $r<n$

Since $n=3$, we must have $r=2$

$\rho(A)=2$ if $a=3$

$\rho[A:B]=2$ if $b=10$

$\therefore$ Thus system will have infinite solution if $a=3$ and $b=10$.

10)c) Solve the system of equations by gauss elimination method

$x+y+z=9$, $x-2y+3z=8$, $2x+y-z=3$

$\rightarrow$ consider the augmented matrix

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 1 & -2 & 3 & : & 8 \\ 2 & 1 & -1 & : & 3 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$; $R_3 \rightarrow R_3 - 2R_1$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -3 & 2 & : & -1 \\ 0 & -1 & -3 & : & -15 \end{bmatrix}$$

$R_3 \longrightarrow 3R_3 - R_2$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & -3 & 2 & -1 \\ 0 & 0 & -11 & -44 \end{array}\right]$$

This is in upper triangular matrix

$x+y+z=9$

$-3y+2z=-1$

$-11z=-44$

$\Rightarrow \boxed{z=4}$

$\Rightarrow -3y+2(4)=-1$

$-3y=-9$

$\boxed{y=3}$

$\Rightarrow x+y+z=9$

$x+3+4=9$

$\boxed{x=2}$

$\therefore$ Solution is $x=2$, $y=3$, $z=4$.

Model Question Paper –II with effect from 2022

USN | | | | | | | | | | |

First Semester B. E Degree examination
Mathematics-1 for Computer Science Stream (22MATS11)

Time: 03 Hours **Max. Marks: 100**

Note: Answer any **FIVE** full questions, choosing at least **ONE** question from each module.

		Module-1	**Marks**
Q. 01	a	With usual notation prove that $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$	6
	b	Find the angle between the curves $r = \frac{a}{1+cos\theta}$ and $r = \frac{b}{1-cos\theta}$	7
	c	Find the radius of curvature of the curve $y = x^3(x-a)$ at the point $(a, 0)$	7
		OR	
Q. 02	a	Show that the curves $r = a(1+\cos\theta)$ and $r = a(1-\cos\theta)$ cuts each other orthogonally	6
	b	Find the pedal equation of the curve $r(1-cos\theta) = 2a$	7
	c	Find the radius of curvature for the curve $y^2 = \frac{a^2(a-x)}{x}$, where the curve meets the x-axis.	7
		Module-2	
Q. 03	a	Expand $\log(1+sinx)$up to the term containing x^4 using Maclaurin's series.	6
	b	If $u = \log(tanx + tany + tanz)$ show that $sin2x\frac{\partial u}{\partial x} + sin2y\frac{\partial u}{\partial y} + sin2z\frac{\partial u}{\partial z} = 2$.	7
	c	Find the extreme values of the function $f(x, y) = x^2 + y^2 + 6x - 12.$	7
		OR	
Q. 04	a	Evaluate $\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{1/x}$	6
	b	If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$	7
	c	If $x = rsin\theta cos\varphi,\ y = rsin\theta sin\varphi,\ z = rcos\theta$ find the value of $\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}$.	7
		Module-3	
Q. 05	a	Solve $\frac{dy}{dx} + \frac{y}{x} = y^2x$	6
	b	Find the orthogonal trajectories of $r = a(1+cos\theta)$ where a is parameter.	7
	c	Solve $p^2 + 2pycotx - y^2 = 0.$	7
		OR	

Q. 06	a	Solve $y(2xy+1)dx - xdy = 0$	6
	b	Find the orthogonal trajectories of the family $r^n sin n\theta = a^n$.	7
	c	Find the general solution of the equation $(px-y)(py+x) = 2p$ by reducing into Clairaut's form by taking the substitution $X = x^2,\ Y = y^2$	7
		Module-4	
Q. 07	a	(i) Find the remainder when 2^{23} is divided by 47. (ii) Find the last digit in 7^{118}.	6
	b	Find the solutions of the linear congruence $11x \equiv 4 \pmod{25}$.	7
	c	Encrypt the message **STOP** using RSA with key (2537, 13) using the prime numbers 43 and 59.	7
		OR	
Q. 08	a	Using Fermat's Little Theorem, show that $8^{30} - 1$ is divisible by 31.	6
	b	Solve the system of linear congruence $x \equiv 3(mod 5), \quad y \equiv 2(mod 6), \quad z \equiv 4(mod 7)$ using Remainder Theorem.	7
	c	(i) Find the remainder when 175 × 113 × 53 is divided by 11. (ii) Solve $x^3 + 5x + 1 \equiv 0(mod 27)$.	7
		Module-5	
Q. 09	a	Find the rank of the matrix $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$	6
	b	Solve the system of equations by using Gauss-Jordan method: $x + y + z = 9$ $2x + y - z = 0$ $2x + 5y + 7z = 52$	7
	c	Using power method, find the largest eigenvalue and corresponding eigenvector of the matrix A = $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$	7
		OR	
Q. 10	a	Solve the following system of equation by Gauss-Seidel method: $20x + y - 2z = 17$ $3x + 20y - z = -18$ $2x - 3y + 20z = 25$	6
	b	Test for consistency $x - 2y + 3z = 2,$ $3x - y + 4z = 4,$ $2x + y - 2z = 5$ and hence solve	7
	c	Solve the system of equations by Gauss elimination method $2x + y + 4z = 12,\ 4x + 11y - z = 33, 8x - 3y + 2z = 20$	7

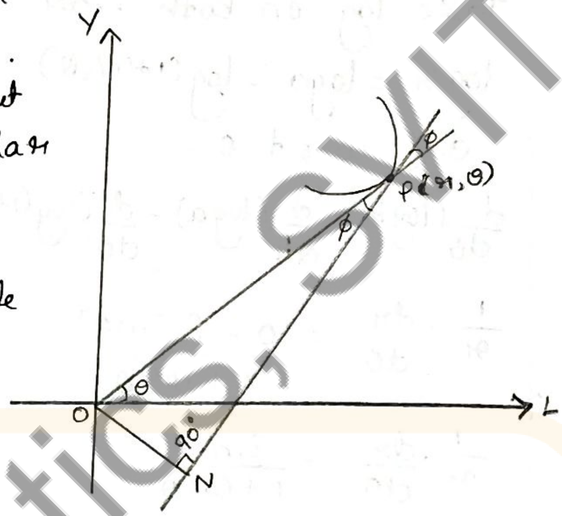
Table showing the Blooms Taxonomy Level, Course outcome and Program outcome				
Question		Blooms Taxonomy level attached	Course outcome	Program outcome
Q.1	a)	L1	CO 01	PO 01
	b)	L2	CO 01	PO 01
	c)	L3	CO 01	PO 02
Q. 2	a)	L1	CO 01	PO 01
	b)	L2	CO 01	PO 01
	c)	L3	CO 01	PO 02
Q. 3	a)	L2	CO 02	PO 01
	b)	L2	CO 02	PO 01
	c)	L3	CO 02	PO 03
Q. 4	a)	L2	CO 02	PO 01
	b)	L2	CO 02	PO 01
	c)	L3	CO 02	PO 02
Q. 5	a)	L2	CO 03	PO 02
	b)	L3	CO 03	PO 03
	c)	L2	CO 03	PO 01
Q. 6	a)	L2	CO 03	PO 02
	b)	L3	CO 03	PO 03
	c)	L2	CO 03	PO 01
Q. 7	a)	L2	CO 04	PO 01
	b)	L2	CO 04	PO 01
	c)	L2	CO 04	PO 02
Q. 8	a)	L2	CO 04	PO 01
	b)	L2	CO 04	PO 01
	c)	L2	CO 04	PO 02
Q. 9	a)	L2	CO 05	PO 01
	b)	L3	CO 05	PO 01
	c)	L3	CO 05	PO 02
Q. 10	a)	L2	CO 05	PO 01
	b)	L3	CO 05	PO 02
	c)	L3	CO 05	PO 01

Bloom's Taxonomy Levels	**Lower order thinking skills**		
	Remembering (knowledge): L_1	Understanding (Comprehension): L_2	Applying (Application): L_3
	Higher-order thinking skills		
	Analyzing (Analysis): L_4	Valuating (Evaluation): L_5	Creating (Synthesis): L_6

Module - 1

Q.01

a) with usual notation prove that $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$



Ans: let O be the pole and OL be the initial line. let $P(r,\theta)$ be any point on the curve so that $OP = r$, the radius vector and $\angle LOP = \theta$, the polar angle.

Draw $ON = p$ (say) a $\perp^r$ from the pole on the tangent at P. let ϕ be the angle b/n made by the radius vector with the tangent.

From the fig, $O\hat{N}P = 90°$

From the right angled Δ^{le} ONP,

we have $\sin\phi = \frac{opp}{hyp} = \frac{p}{r}$.

$$\boxed{p = r\sin\phi} \quad \text{--- (1)}$$

This is the expression for length of the $\perp^r$ p to express p in terms of θ.

Squaring on b.s of eq (1)

$$p^2 = r^2 \sin^2\phi$$

taking reciprocal on b.s.

$$\frac{1}{p^2} = \frac{1}{r^2\sin^2\phi}$$

$$\frac{1}{p^2} = \frac{1}{r^2} \times \operatorname{cosec}^2\phi$$

$$\frac{1}{p^2} = \frac{1}{r^2} \times [1+\cot^2\phi]$$

w.k.t $\cot\phi = \frac{1}{r}\cdot\frac{dr}{d\theta}$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

Hence the proof

b. Find the angle between curves $r = \frac{a}{1+\cos\theta}$ and $r = \frac{b}{1-\cos\theta}$

Ans: consider $r = \frac{a}{1+\cos\theta}$

Take log on both sides

$\log r = \log a - \log(1+\cos\theta)$

diff w.r.t θ.

$\frac{d}{d\theta}(\log r) = \frac{d}{d\theta}(\log a) - \frac{d}{d\theta}(\log(1+\cos\theta))\frac{d}{d\theta}(\cos\theta)$

$\frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{0 - (-\sin\theta)}{1+\cos\theta}$

$\frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{\sin\theta}{1+\cos\theta}$

$\boxed{\cot\phi_1 = \frac{\sin\theta}{1+\cos\theta}}$

consider $r = \frac{b}{1-\cos\theta}$.

Take log on both sides

$\log r = \log b - \log(1-\cos\theta)$

diff w.r.t θ.

$\frac{d}{d\theta}(\log r) = \frac{d}{d\theta}(\log b) - \frac{d}{d\theta}(\log(1-\cos\theta))\frac{d}{d\theta}(1) - \frac{d}{d\theta}$

$\frac{1}{r}\frac{dr}{d\theta} = \frac{0-(\sin\theta)}{1-\cos\theta}$

$\boxed{\cot\phi_2 = \frac{-\sin\theta}{1-\cos\theta}}$

$\therefore |\phi_1 - \phi_2|$ or $\cot\phi_1 \times \cot\phi_2 = -1$

$\therefore \frac{\sin\theta}{1+\cos\theta} \times \frac{-\sin\theta}{1-\cos\theta} = -1$

$\therefore \frac{-\sin^2\theta}{1^2 - \cos^2\theta} = -1$

$\because (a+b)(a-b) = a^2 - b^2$

$\therefore 1^2 - \cos^2\theta = \sin^2\theta$.

$\therefore \frac{-\cancel{\sin^2\theta}}{\cancel{\sin^2\theta}} = -1$

$\therefore -1 = -1$

then $|\phi_1 - \phi_2| = \pi/2$

c) Find the radius of curvature of the curve $y = x^3(x-a)$ at the point $(a, 0)$

Ans: consider $y = x^3(x-a)$

$y = x^4 - ax^3$

diff w.r.t 'x'

$y_1 = \frac{dy}{dx} = \frac{d}{dx}(x^4) - a\frac{d}{dx}(x^3)$

$y_1 = 4x^3 - 3ax^2$

$\boxed{y_1 = 4x^3 - 3ax^2}$

again diff y_1 w.r.t 'x'

$y_2 = \frac{d^2y}{dx^2} = 4\frac{d}{dx}(x^3) - 3a\frac{d}{dx}(x^2)$

$\boxed{y_2 = 12x^2 - 6ax}$

at $(a,0)$

$y_1 = 4(a)^3 - 3a(a)^2$

$= 4a^3 - 3a^3$

$\boxed{y_1 = a^6}$

at $(a,0)$

$y_2 = 12(a)^2 - 6a(a)$

$y_2 = 12a^2 - 6a^2$

$\boxed{y_2 = 6a^2}$

$\therefore \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

$\rho = \frac{(1+(a^3)^2)^{3/2}}{6a^2}$

$|\rho| = \frac{(1+a^6)^{3/2}}{6a^2}$

or $|\rho| = \frac{\sqrt[3]{1+a^6}}{6a^2}$

Module-01

Q.02

a) Show that the curves $r = a(1+\cos\theta)$ and $r = a(1-\cos\theta)$ cuts each other orthogonally.

Ans: Consider $r = a(1+\cos\theta)$

Take log on both sides

$\log r = \log a + \log(1+\cos\theta)$

diff w.r.t 'θ'

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{(-\sin\theta)}{1+\cos\theta}$$

$$\boxed{\cot\phi_1 = \frac{-\sin\theta}{1+\cos\theta}}$$

Consider $r = a(1-\cos\theta)$

Take log on both sides

$\log r = \log a + \log(1-\cos\theta)$

diff w.r.t θ.

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{\sin\theta}{1-\cos\theta}$$

$$\boxed{\cot\phi_2 = \frac{\sin\theta}{1-\cos\theta}}$$

Then $\cot\phi_1 \times \cot\phi_2 = -1$

$$\frac{-\sin\theta}{1+\cos\theta} \times \frac{\sin\theta}{1-\cos\theta} = -1$$

$\therefore (a+b)(a-b) = a^2-b^2$

$\therefore \sin^2\theta = 1-\cos^2\theta$.

$$\frac{-\sin^2\theta}{1^2-\cos^2\theta} = -1$$

$$\frac{-\sin^2\theta}{\sin^2\theta} = -1$$

$$-1 = -1$$

therefore $|\phi_1 - \phi_2| = \pi/2$.

Therefore, the curves which cuts each other orthogonally.

Consider, $x_1 = \frac{2xy}{-a^2 \cdot y^2}$.

$(a^2+y^2)x_1 = 2xy$

again diff w.r.t y.

$$a^2+y^2\frac{d}{dy}(x_1) + x_1\left(\frac{d}{dx}(a^2) + \frac{d}{dx}(y^2)\right) = -2\left[x\frac{dy}{dy} + y\frac{dx}{dy}\right]$$

$$(a^2+y^2)x_2 + x_1(0+2y) = -2[x+yx_1]$$

at $(a,0)$

$$(a^2+y^2)x_2 + x_1(2y) = -2[x + yx_1]$$

$$(a^2+0)x_2 + 0(2(0)) = -2[a+(0)(0)]$$

$$a^2x_2 = -2a$$

$$x_2 = \frac{-2\cancel{a}}{a^{\cancel{2}}}$$

$$\boxed{x_2 = \frac{-2}{a}}$$

R.O.C $\quad \rho = \frac{(1+(x_1)^2)^{3/2}}{x^2}$

$$\rho = \frac{(1+0)^{3/2}}{-2/a}$$

$$\rho = \frac{1}{-2/a}$$

$$\rho = \frac{a}{-2}.$$

$$|\rho| = \left|\frac{a}{-2}\right|$$

$$\boxed{|\rho| = a/2}$$

b) Find the pedal equation of the curve $r(1-\cos\theta) = 2a$.

Ans:- Consider the equation $r(1-\cos\theta) = 2a$.

Take log on both sides.

$\log r + \log(1-\cos\theta) = \log 2a$

diff w.r.t θ.

$$\frac{d}{d\theta}(\log r) + \frac{d}{d\theta}(1-\cos\theta)\ \frac{d}{d\theta}(1) - \frac{d}{d\theta}(\cos\theta) = \frac{d}{d\theta}(\log 2a)$$

$$\frac{1}{r}\frac{dr}{d\theta} + \frac{\sin\theta}{1-\cos\theta} = 0.$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1-\cos\theta}$$

$$\cot\phi = \frac{-\sin\theta}{1-\cos\theta}$$

From the pedal equation $\frac{1}{p^2} = \frac{1}{r^2}[1+\cot^2\phi]$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[1 + \frac{\sin^2\theta}{(1-\cos\theta)^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{(1-\cos\theta)^2 + \sin^2\theta}{(1-\cos\theta)^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{1^2 + \cos^2\theta - 2\cos\theta + \sin^2\theta}{(1-\cos\theta)^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{1+1-2\cos\theta}{(1-\cos\theta)^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{2-2\cos\theta}{(1-\cos\theta)^2}\right]$$

$$\frac{1}{p^2} = \frac{2}{r^2}\left[\frac{\cancel{1-\cos\theta}}{(1-\cos\theta)^{\cancel{2}}}\right]$$

$$\frac{1}{p^2} = \frac{2}{r^2}\left[\frac{1}{1-\cos\theta}\right] \to (1)$$

To eliminate θ,

From given $r(1-\cos\theta) = 2a$

$$\therefore \frac{r}{2a} = \frac{1}{1-\cos\theta}$$

then eq ① becomes $\frac{1}{p^2} = \frac{\cancel{r}}{r^{\cancel{2}}} \cdot \frac{r}{\cancel{r}a}$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r} \cdot \frac{1}{a}.$$

$$p^2 = ra$$

$$\boxed{p = \sqrt{ra}}$$

c. Find the radius of curvature for the curve $y^2 = \frac{a^2(a-x)}{x}$, where the curve meets x axis.

Ans:- The given curve $y^2 = \frac{a^2(a-x)}{x}$, then curve meet at point $(a,0)$

$$xy^2 = a^3 - a^2x$$

diff w.r.t 'x'

$$x\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x) = \frac{d}{dx}(a^3) - a^2\frac{d}{dx}(x)$$

$$x \cdot 2y\frac{dy}{dx} + y^2 = -a^2.$$

$$x \cdot 2y\frac{dy}{dx} + y^2 = -a^2$$

$$x \cdot 2y \cdot y_1 + y^2 = -a^2$$

$$\boxed{y_1 = \frac{-a^2 - y^2}{2xy}}$$

at $(a,0)$

$$y_1 = \frac{-a^2 - y^2}{2xy}$$

$$y_1 = \frac{-a^2 - 0}{+2(a)(0)}$$

$$\boxed{y_1 = \infty}$$

Since $y_1 = 0$ at $(a,0)$ we consider.

$$x_1 = \frac{dx}{dy} = \frac{2xy}{-a^2 - y^2}$$

at $(a,0)$

$$x_1 = \frac{2xy}{-a^2 - y^2}$$

$$x_1 = \frac{2(a)(0)}{-a^2 - 0}$$

$$\boxed{x_1 = 0}$$

Module - 02

Q no:- 3

Q.03 a) Expand $\log(1+\sin x)$ up to the term containing x^4 using maclaurin's series

Ans:- $y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \dots \dots \rightarrow (1)$

let $y = \log(1+\sin x)$; $y(0) = \log(1+\sin 0) = \log 1 = \boxed{0}$

$y_1(x) = \frac{1}{1+\sin x}(\cos x)$; $y_1(0) = \frac{\cos 0}{1+\sin 0} = \frac{1}{1} = \boxed{1}$

$y_1(1+\sin x) = \cos x$

diff w.r.t 'x'

$y_1(\cos x) + (1+\sin x)y_2 = -\sin x$

$(1)(1) + (1+0)y_2 = 0$

$y_2(0) = \boxed{-1}$

consider $y_1(\cos x) + (1+\sin x)y_2 = -\sin x$

diff $[y_1(-\sin x) + \cos x \cdot y_2] + [(1+\sin x)y_3 + y_2(\cos x)] = -\cos x$

$(1(0) + 1(-1)) + ((1)y_3 + (-1)(1)) = -1$

$-1 + y_3 - 1 = -1$

$y_3 = -1+2$

$\boxed{y_3 = 1}$

consider. $[y_1(-\sin x) + (\cos x)y_2] + [(1+\sin x)y_3 + y_2(\cos x)] = -\cos x$

diff $[y_1(-\cos x) + (-\cancel{\sin x}^{0}\, y_2) + (\cos x\, y_3 + y_2 \cdot -\cancel{\sin x}^{0})] +$

$[(1+\cancel{\sin x}^{0})y_2 - (\cos x)) + (y_2 \cdot (-\sin x) + (\cos x)(y_3))] = \cancel{\sin x}^{0}$

$[(1)(-1) + (1)(1)] + ((-1)(y_4) + (1)(1) + (1)(1)] = 0$

$\cancel{-1} \cancel{+1} + y_4 + 1 + 1 = 0$

$\boxed{y_4 = -2}$

put in eq^{tn} (1)

$\therefore y(x) = 0 + x(1) + \frac{x^2}{2}(-1) + \frac{x^3}{6}(1) + \frac{x^4}{24}(-2)$

$\therefore y(x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12}$

b. if $u = \log(\tan x + \tan y + \tan z)$ show that.

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2.$$

Ans:- Consider $u = \log(\tan x + \tan y + \tan z)$

$$\boxed{e^u = \tan x + \tan y + \tan z}$$

① partial diff w.r.t x.

$$\frac{\partial}{\partial x}(e^u) = \frac{\partial}{\partial x}(\tan x)$$

$$\frac{\partial u}{\partial x} = \frac{\sec^2 x}{e^u}$$

multiply $\sin 2x$ on both side

$$\sin 2x \frac{\partial u}{\partial x} = \frac{\sin 2x \sec^2 x}{e^u}$$

$$\boxed{\sin 2x \frac{\partial u}{\partial x} = \frac{2\tan x}{e^u}} \rightarrow ①$$

② partial diff w.r.t y.

$$\frac{\partial}{\partial y}(e^u) = \frac{\partial}{\partial y}(\tan y)$$

$$\frac{\partial u}{\partial y} e^u = \sec^2 y$$

$$\frac{\partial u}{\partial y} = \frac{\sec^2 y}{e^u}$$

multiply by $\sin 2y$ on both side.

$$\sin 2y \frac{\partial u}{\partial y} = \frac{\sin 2y \sec^2 y}{e^u}$$

$$\boxed{\sin 2y \frac{\partial u}{\partial y} = \frac{2\tan y}{e^u}} \rightarrow ②$$

③ $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$$

$$= \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}\left(\frac{-y}{z^2}\right) + \frac{\partial u}{\partial r}\left(\frac{1}{x}\right)$$

$$\frac{\partial u}{\partial z} = \frac{-y}{z^2}\frac{\partial u}{\partial q} + \frac{1}{x}\frac{\partial u}{\partial r}$$

multiply b.s by z

$$\boxed{z.\frac{\partial u}{\partial z} = -\frac{yz}{z^2}\frac{\partial u}{\partial q} + \frac{z}{x}\frac{\partial u}{\partial r}} \longrightarrow ③$$

By adding eq ① ② and ③

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\cancel{\frac{x}{y}\frac{\partial u}{\partial p}}-\cancel{\frac{z}{x}\frac{\partial u}{\partial r}}+\cancel{\left(\frac{-x}{y}\right)\frac{\partial u}{\partial p}}+\cancel{\frac{y}{z}\frac{\partial u}{\partial q}}\cancel{\frac{-y}{z}\frac{\partial u}{\partial q}}+\cancel{\frac{z}{x}\frac{\partial u}{\partial r}}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0.$$

C. If $x=r\sin\theta\cos\psi$ $y=r\sin\theta\sin\psi$ $z=r\cos\theta$. Find the value of

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\psi)}=?$$

Sol^n:

$$\therefore\ \frac{\partial(x,y,z)}{\partial(r,\theta,\psi)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\psi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\psi}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\psi}\end{vmatrix}$$

Consider :-

$x=r\sin\theta\cos\psi$ $\quad$ $y=r\sin\theta\sin\psi$ $\quad$ $z=r\cos\theta.$

$\frac{\partial x}{\partial r}=\sin\theta\cos\psi$ $\quad$ $\frac{\partial y}{\partial r}=\sin\theta\sin\psi$ $\quad$ $\frac{\partial z}{\partial r}=\cos\theta.$

$\frac{\partial x}{\partial\theta}=r\cos\theta\cos\psi$ $\quad$ $\frac{\partial y}{\partial\theta}=r\cos\theta\sin\psi$ $\quad$ $\frac{\partial z}{\partial\theta}=-r\sin\theta.$

$\frac{\partial x}{\partial\psi}=-r\sin\theta\sin\psi$ $\quad$ $\frac{\partial y}{\partial\theta}=r\sin\theta\cos\psi$ $\quad$ $\frac{\partial z}{\partial\psi}=0.$

$$J=\frac{\partial(x,y,z)}{\partial(r,\theta,\psi)}=\begin{vmatrix}\sin\theta\cos\psi & r\cos\theta\cos\psi & -r\sin\theta\sin\psi\\ \sin\theta\sin\psi & r\cos\theta\sin\psi & r\sin\theta\cos\psi\\ \cos\theta & -r\sin\theta & 0\end{vmatrix}$$

$$J = \sin\theta\cos\psi\,[0 + r^2\sin^2\theta\cos^2\psi] - r\cos\theta\cos\psi\,[0 - r\sin\theta\cos\theta\cos\psi] - r\sin\theta\sin\psi\,.$$
$$[-r\sin^2\theta\sin\psi - r^2\cos^2\theta\sin\psi]$$

$$J = r^2\sin^2\theta\cos^2\psi + r^2\sin\theta\cos^2\theta\cos\psi + r^2\sin\theta\sin^2\psi$$

$$J = r^2\sin^2\cos^2\psi\,[\sin^2\theta + \cos^2\theta] + r^2\sin\theta\sin^2\psi$$

$$J = r^2\sin\theta\cos^2\psi + r^2\sin\theta\sin^2\psi$$

$$J = r^2\sin\theta\,[\cos^2\psi + \sin^2\psi]$$

$$J = \underline{\underline{r^2\sin\theta}}$$

Q.04) a) Evaluate $\lim_{x\to 0}\left(\frac{a^x + b^x + c^x}{3}\right)^{1/x}$

$$\log_e k = \lim_{x\to 0}\log\left(\frac{a^x + b^x + c^x}{3}\right)^{1/x}$$

$$= \lim_{x\to 0}\frac{1}{x}\log\left(\frac{a^x + b^x + c^x}{3}\right)$$

$$\Rightarrow \lim_{x\to 0}\frac{\log\left(\frac{a^x + b^x + c^x}{3}\right)}{x}\quad\left(\frac{0}{0}\right)$$

Apply LHR.

$$\log_e k = \lim_{x\to 0}\frac{\frac{1}{\frac{a^x + b^x + c^x}{3}}\cdot\frac{1}{3}\left[a^x\log a + b^x\log b + c^x\log c\right]}{1}$$

$$= \lim_{x\to 0}\frac{a^x\log a + b^x\log b + c^x\log c}{a^x + b^x + c^x}$$

$$= \frac{\log a + \log b + \log c}{3}$$

$$= \frac{1}{3}\log(abc)$$

$$= \log(abc)^{1/3}$$

$$k = (abc)^{1/3}$$

b) If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

$p = \frac{x}{y} \quad q = \frac{y}{z} \quad r = \frac{z}{x}$

$u \rightarrow (p, q, r) \rightarrow \left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$

$u \rightarrow \left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

$$u_x = \frac{\partial u}{\partial p}\left(\frac{1}{y}\right) + \frac{\partial u}{\partial q}\cdot(0) + \frac{\partial u}{\partial r}\left(\frac{-z}{x^2}\right) \quad \text{x}^{ly} \text{ by } x$$

$$xu_x = \frac{\partial u}{\partial p}\left(\frac{x}{y}\right) + \frac{\partial u}{\partial r}\left(\frac{-z}{x}\right) \text{ — } (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$$

$$u_y = \frac{\partial u}{\partial p}\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial q}\left(\frac{1}{z}\right) + \frac{\partial u}{\partial r}(0) \quad x^{ly} \text{ by } y.$$

$$yu_y = \frac{\partial u}{\partial p}\left(\frac{-x}{y}\right) + \frac{\partial u}{\partial q}\left(\frac{y}{z}\right) \text{ — } (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$$

$$u_z = \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}\left(\frac{-y}{z^2}\right) + \frac{\partial u}{\partial r}\left(\frac{1}{x}\right) \quad x^{ly} \text{ by } z$$

$$zu_z = -\frac{y}{z}\frac{\partial u}{\partial q} + \frac{\partial u}{\partial r}\left(\frac{z}{x}\right) \rightarrow (3)$$

Adding (1), (2) and (3)

$$xu_x + yu_y + zu_z = 0.$$

c) If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ find the value of $\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$

$$\frac{\partial(x\,y\,z)}{\partial(r\,\theta\,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta.\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta.\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$= \sin\theta.\cos\phi\,(0 + r^2\sin^2\theta\cos\phi) - r\cos\theta\cos\phi\,[0 - r\sin\theta\cos\phi.\cos\theta] -$
$r\sin\theta\sin\phi\,(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi]$

$= r^2\sin^3\theta.\cos^2\phi + r^2\sin\theta\cos^2\phi - (r\sin\theta\sin\phi)(-r\sin\phi)(\sin^2\theta + \cos^2\theta)$

$= r^2\sin^3\theta\cos^2\phi + r^2\sin\theta\cos^2\theta\cos^2\phi + r^2\sin\theta\sin^2\phi$

$= r^2\sin\theta\cos^2\phi[\sin^2\theta + \cos^2\theta] + r^2\sin\theta\sin^2\phi$

$= r^2\sin\theta\cos^2\phi + r^2\sin\theta\sin^2\phi$

$= r^2\sin\theta\,[\cos^2\phi + \sin^2\phi]$

$= r^2\sin\theta$

Q.5) Solve $\frac{dy}{dx} + \frac{y}{x} = y^2 x$

Ans: Given the diff equation is in the form of

$$\frac{dy}{dx} + py = xy^n$$

Divide given equation by y^2

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{y}{xy^2} = \frac{xy^2}{y^2}$$

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = x$$

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\frac{1}{y} = x \rightarrow (1)$$

Substitute $\boxed{\frac{1}{y} = t}$

diff w.r.t x.

$$\frac{dt}{dx} = \frac{-1}{y^2}\frac{dy}{dx}$$

$$\frac{1}{y^2}\frac{dy}{dx} = -\frac{dt}{dx} \rightarrow (2)$$

Sub eq (2) in eq (1)

$$-\frac{dt}{dx} + \frac{1}{x}\cdot t = x$$

Above Equation multiply by -1

$$\frac{dt}{dx} - \frac{1}{x}t = -x$$

$P = \frac{-1}{x}$ $\quad Q = -x$

$$IF = e^{\int P dx}$$

$$= e^{-\int \frac{1}{x} dx}$$

$$IF = e^{-\log x}$$

$$IF = e^{\log x^{-1}}$$

$$IF = x^{-1}$$

$$\boxed{IF = \frac{1}{x}}$$

Solution: $t\,[IF] = \int Q\,[IF]\,dx + c$

$$t \cdot \frac{1}{x} = \int -x \cdot \frac{1}{x}\,dx + c$$

$$t \cdot \frac{1}{x} = -\int 1\,dx + c$$

$$\frac{1}{y}\frac{1}{x} = -x + c$$

$$\boxed{\frac{1}{xy} = -x + c}$$

b) Find the orthogonal trajectories of $r = a(1+\cos\theta)$ where a is parameter

Soln: $r = a(1+\cos\theta)$

$\log r = \log a + \log(1+\cos\theta)$

diff w.r.t θ.

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{-\sin\theta}{1+\cos\theta}.$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\frac{2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2}$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\tan\theta/2$$

Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\tan\theta/2$$

$$\frac{1}{\tan\theta/2}\,d\theta = \frac{1}{r}\,dr.$$

$$\cot\theta/2\,d\theta = \frac{1}{r}\,dr$$

Integrate on. b.s

$$\int\frac{1}{r}\,dr = \int\cot\theta/2\,d\theta + c$$

$$\log r = \frac{\log(\sin\theta/2)}{1/2} + c$$

$$\log r = 2\log(\sin\theta/2) + \log k$$

$$\log r = \log\sin^2\theta/2 + \log k$$

$$\log r = \log(k\sin^2\theta/2)$$

$$r = k\sin^2\theta/2.$$

$$r = k\left[\frac{1-\cos 2\theta}{2}\right] \qquad \frac{k}{2} = c_1$$

$$\boxed{r = c_1(1-\cos 2\theta)}$$

Q) Solve $p^2 + 2py\cot x - y^2 = 0$

Ans) Here equation in form of $ax^2 + bx + c = 0$

$a = 1 \quad b = 2y\cot x \quad c = -y^2$

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$p = \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2}$$

$$p = \frac{-2y\cot x \pm \sqrt{4y^2(\cot^2 x + 1)}}{2}$$

$$p = \frac{-2y\cot x \pm 2y\,\mathrm{cosec}\,x}{2}$$

$$p = -y\cot x \pm y\,\mathrm{cosec}\,x$$

$p = -y\cot x + y\,\mathrm{cosec}\,x$

$\frac{dy}{dx} = y(\mathrm{cosec}\,x - \cot x)$

$\int \frac{1}{y}\,dy = \int \mathrm{cosec}\,x\,dx - \int \cot x\,dx$

$\log y = \log(\mathrm{cosec}\,x - \cot x) - \log \sin x + \log k$

$\log y = \log\left[\frac{k\,\mathrm{cosec}\,x - \cot x}{\sin x}\right]$

$y = k\,\mathrm{cosec}\,x - \cot x$

$y\sin x = -k(\mathrm{cosec}\,x - \cot x) = 0$

$p = -y\cot x - y\,\mathrm{cosec}\,x$

$\frac{dy}{dx} = -y(\cot x + \mathrm{cosec}\,x)$

$-\int \frac{1}{y}\,dy = \int \cot x\,dx + \int \mathrm{cosec}\,x\,dx$

$-\log y = \log(\sin x) + \log(\mathrm{cosec}\,x - \cot x) + \log k$

$\log y + \log(\sin x) + \log(\mathrm{cosec}\,x - \cot x) = -\log k$

$y\sin x\,(\mathrm{cosec}\,x - \cot x) + k = 0$

General solution :-

$y\sin x - k(\mathrm{cosec}\,x - \cot x) \cdot (y\sin x(\mathrm{cosec}\,x - \cot x) + k) = 0$

Q.06 a) Solve $y(2x+1)dx - x\,dy = 0$

$M = 2xy^2+y \qquad N = -x\,dy$

$\frac{\partial M}{\partial y} = 4xy+1 \qquad \frac{\partial N}{\partial x} = -1$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{-1-(4xy+1)}{} = -1-4xy-1 \Rightarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \Rightarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

$= -4xy-2$ close to M.

$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-2\cancel{(2xy+1)}}{y\cancel{(2xy+1)}}$

$F(x) = -2/y$

$IF = e^{\int b(x)}$

$\Rightarrow e^{-2\int \frac{1}{y}} = e^{\log y^{-2}}$

$\boxed{I_F = \frac{1}{y^2}}$

multiply $\frac{1}{y^2}$ on given equation

$(2x+\frac{1}{y^2})dx - \frac{x}{y^2}dy = 0$

$M = 2x+\frac{1}{y^2} \qquad N = -x.\frac{1}{y^2}dy$

$\frac{\partial M}{\partial y} = -2y^{-3} \qquad \frac{\partial N}{\partial x} = -2y^{-3}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\int M dx + \int$ term of N free from x

$-2\int y^{-3} + \int 0d$

$\frac{-\cancel{2}y^{-2}}{\cancel{-2}} + c$

$y^{-2}+c$

$\frac{1}{y^2}+c$

b) Find the orthogonal trajectories of family $r^n \sin n\theta = a^n$

Ans: $r^n \sin n\theta = a^n$

take log on both side

$\log r^n + \log \sin n\theta = \log a^n$

$n \log r + \log \sin n\theta = n \log a$

diff w.r.t θ.

$n \cdot \frac{1}{r} \frac{dr}{d\theta} + \frac{\cos n\theta \cdot n}{\sin n\theta} = 0$

$\frac{1}{r} \frac{dr}{d\theta} + \cot n\theta = 0$

replace $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$

$\frac{1}{r}\left(-r^2 \frac{d\theta}{dr}\right) \cancel{+} \cot n\theta = 0$

$\cancel{-} r \frac{d\theta}{dr} = \cancel{+} \cot n\theta$

$\frac{1}{r} dr = \frac{1}{\cot n\theta} \cdot d\theta$

$\int \frac{1}{r} dr = \int \tan n\theta \, d\theta + c$

$\log r = \frac{\log(\sec n\theta)}{n} + c$

$n \log r = \log(\sec n\theta) + nc$

$n \log r = \log(\sec n\theta) + \log b$

$\log r^n = \log(b \sec n\theta)$

$r^n = b \sec n\theta$

$\boxed{r^n \cos n\theta = b}$

Q. Find the general solution of the equation $(px-y)(py+x)=2p$ by reducing into Clairaut's form by taking the substitution $X = x^2$, $Y = y^2$

Ans: Given $X = x^2$ $\quad Y = y^2$.

$$\frac{dX}{dx} = 2x \qquad \frac{dY}{dy} = 2y$$

$$p = \frac{dy}{dx} = \frac{dy}{dY}\,\frac{dY}{dX}\,\frac{dX}{dx}$$

$$p = \frac{1}{\cancel{2}y}\,P\cdot\cancel{2}x$$

$$p = \frac{x}{y}P$$

$$p = \frac{\sqrt{X}}{\sqrt{Y}}\cdot P.$$

Substitute in (1)

$$\left(\frac{\sqrt{X}}{\sqrt{Y}}\cdot P\cdot\sqrt{X} - \sqrt{Y}\right)\left(\frac{\sqrt{X}}{\sqrt{Y}}P\sqrt{Y} + \sqrt{X}\right) = 2\frac{\sqrt{X}}{\sqrt{Y}}\cdot P$$

$$\left[\frac{XP}{\sqrt{Y}} - Y\right]\left[\sqrt{X}P + \sqrt{X}\right] = \frac{2\sqrt{X}}{\sqrt{Y}}\cdot P$$

$$\left[\frac{XP-Y}{\cancel{\sqrt{Y}}}\right]\cancel{\sqrt{X}}\,[P+1] = 2\frac{\cancel{\sqrt{X}}}{\cancel{\sqrt{Y}}}\cdot P$$

$$[XP-Y][P+1] = 2P$$

$$PX - Y = \frac{2P}{P+1}$$

$$Y = PX - \frac{2P}{P+1}$$

This is in Clairaut's form

General solution $Y = cX - \dfrac{2c}{c+1}$

$$\boxed{y^2 = cx^2 - \frac{2c}{c+1}}$$

07) a) i) Find the remainder when 2^{23} is divided by 47.

<u>soln</u>:- $2^8 = 256 = 21 \pmod{47}$

$(2^8)^2 = (21)^2 \pmod{47}$

$2^{16} = 441 \pmod{47}$

$2^{16} = 18 \pmod{47} \rightarrow ①$

consider $2^7 = 128 \equiv 34 \pmod{47}$ - ②.

eq ① x ②

$2^{16} \cdot 2^7 \equiv (18 \times 34) \pmod{47}$

$2^{23} \equiv 612 \pmod{47}$

$2^{23} = 1 \pmod{47}$

∴ 1 is the remainder when 2^{23} is divided by 47.

ii) Find the last digit in 7^{118}.

<u>soln</u>:-

$$\begin{array}{r} 4\overline{)118}(29 \\ \underline{8\downarrow\ } \\ 38 \\ \underline{36} \\ 02 \end{array}$$

$7^{4k} = 1 \pmod{10}$

$7^{4k+1} = 7 \pmod{10}$

$7^{4k+2} = 9 \pmod{10}$

$7^{4k+3} = 3 \pmod{10}$

∴ $7^{118} = 7^{4 \times 29 + 2}$

$7^{118} = 7^{4k+2} = 9 \pmod{10}$

∴ 9 is the last digit.

b) Find the solutions of the linear congruence $11x \equiv 4 \pmod{25}$.

Solⁿ:- Here $a = 11$ $b = 4$ $m = 25$

$\gcd(11, 25) = \gcd = 1 = d$

check $d \mid b$

$\Rightarrow 1 \mid 4$ true

$\therefore$ given congruence has unique solution

consider $11x \equiv 4 \pmod{25}$

$\Rightarrow 11x - 4 = 25 \times K$

$11x = 25K + 4$

$x = \dfrac{25K+4}{11}$

put $K = 0$, $x = \frac{4}{11} \notin \mathbb{Z}$

$K = 1$, $x = 29/11 \notin \mathbb{Z}$

$K = 2$, $x = 54/11 \notin \mathbb{Z}$

$K = 3$, $x = 79/11 \notin \mathbb{Z}$

$K = 4$, $x = 104/11 \notin \mathbb{Z}$

$K = 5$, $x = 129/11 \notin \mathbb{Z}$

$K = 6$, $x = \frac{154}{11} = 14 \in \mathbb{Z}$

$\therefore x \equiv 14 \pmod{25}$

Q Encrypt the message STOP using RSA with key $(2537, 13)$ using the prime numbers 43 and 59.

Soln: Given $p = 43$, $q = 59$ and public key $= \{2537, 13\} = \{n, e\}$

$\therefore n = pq = 43 \times 59 = 2537$ and $e = 13$.

$\therefore \phi(n) = (p-1)(q-1) = 42 \times 58 = 2436$

Since $e = 13$, and $1 < e < \phi(n)$ i.e. $1 < 13 < 2436$, $\gcd(2436, 13) = 1$.

M = STOP = $\underline{1819}\,\underline{1415}$ $\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ A & B & C & D & E & F \cdots \end{bmatrix}$

Let $M_1 = 1819$, $M_2 = 1415$

Encryption: $C = M^e \bmod n$.

$\Rightarrow C_1 = M_1^e \bmod n$ and $C_2 = M_2^e \bmod n$.

$\Rightarrow C_1 = (1819)^{13} \pmod{2537} \rightarrow ①$

consider $(1819)^3 \equiv 2068 \pmod{2537}$ (using calculator) $\rightarrow *$

cube on both sides

$\Rightarrow ((1819)^3)^3 \equiv (2068)^3 \pmod{2537}$

$\Rightarrow 1819^9 \equiv 322 \pmod{2537} \rightarrow ②$ (using calci)

eqn $*$ $\times$ eqn ② $\Rightarrow (1819)^3 (1819)^9 \equiv (2068 \times 322) \bmod 2537$.

$\Rightarrow (1819)^{12} \equiv 1202 \pmod{2537}$ (using calci)

$\times^{ly}$ by 1819 on b.s.

$(1819)^{12} (1819) \equiv (1202 \times 1819) \pmod{2537}$

$\Rightarrow (1819)^{13} \equiv 2081 \pmod{2537}$ (using calci)

$\therefore ① \Rightarrow \boxed{C_1 = 2081 \pmod{2537}}$

Now $C_2 = M_2^e \bmod n$.

$\Rightarrow C_2 = (1415)^{13} \pmod{2537} \rightarrow ③$.

consider $(1415)^3 \equiv 1828 \pmod{2537}$ (using calculator) $\rightarrow **$

$\Rightarrow ((1415)^3)^3 \equiv 1828^3 \pmod{2537}$

$\Rightarrow (1415)^9 \equiv 2005 \pmod{2537} \rightarrow ④$ (using calci)

eqn $**$ $\times$ eqn ④ $\Rightarrow (1415)^3 (1415)^9 \equiv (1828 \times 2005) \pmod{2537}$

$\Rightarrow (1415)^{12} \equiv 1712 \pmod{2537}$ (using calci)

$\times^{ly}$ by 1415 on b.s

$(1415)^{12}(1415) \equiv (1415 \times 1712) \pmod{2537}$

$\Rightarrow (1415)^{13} \equiv 2182 \pmod{2537}$

$\therefore$ eq^n 3 becomes

$$\boxed{C_2 \equiv 2182 \pmod{2537}}$$

Thus $C = C_1 C_2$

$C = \underline{20}\,\underline{8}\,\underline{1}\;\underline{21}\,\underline{8}\,\underline{2}$

$C = UHBVHC$

08) a) Using Fermat's little theorem, show that $8^{30}-1$ is divisible by 31.

Soln: Here $a = 8 \quad p = 31$

By FLT, $a^{p-1} \equiv 1 (\text{mod } (p))$

$8^{30} \equiv 1 (\text{mod } 31)$

$8^{30} - 1 \equiv 0 (\text{mod } 31)$

$8^{30}-1$ is having a remainder zero.

when it is divided by 31

i.e $8^{30}-1$ is divisibly by 31.

b) Solve the system of linear congruence.

$x \equiv 3 (\text{mod } 5)$, $y \equiv 2 (\text{mod } 6)$, $z \equiv 4 (\text{mod } 7)$ using Remainder Theorem.

Soln: Here $b_1 = 3 \quad b_2 = 2 \quad b_3 = 4$

$m_1 = 5 \quad m_2 = 6 \quad m_3 = 7$

we see that $(m_1, m_2) = (5,6) = 1$

$(m_1, m_3) = (5,7) = 1$

$(m_2, m_3) = (6,7) = 1$

Now, $M = m_1 \cdot m_2 \cdot m_3$

$M = 5 \times 6 \times 7 = 210$

and $M_k = \frac{M}{M_k}$; $k = 1, 2, 3$

$M_1 = \frac{210}{5} = 42$, $M_2 = \frac{210}{6} = 35$, $M_3 = \frac{210}{7} = 30$

consider $M_k X \equiv 1 \pmod{m_k}$; $k = 1, 2, 3$, $X = x, y, z$

$42x \equiv 1 \pmod 5$ $\qquad$ $35y \equiv 1 \pmod 6$ $\qquad$ $30z \equiv 1 \pmod 7$

By inspection,

$x_1 = 3$; $\qquad y = 5 \qquad z = 4$

thus, $x = M_1 b_1 x + M_2 b_2 y + M_3 b_3 z$ modulo M

$x = 42(3)(3) + 35(2)(5) + 30(4)(4)$ modulo 210

$x = 1208$ (modulo 210)

$x = 158 \pmod{210}$ is the unique solution.

c) i) Find the remainder when $175 \times 113 \times 53$ is divisible by 11.

Soln:- $175 = 10 \pmod{11}$

$113 = 3 \pmod{11}$

$53 = 9 \pmod{11}$

$\therefore 175 \times 113 \times 53 = (10 \times 3 \times 9) \pmod{11}$

$= 270 \pmod{11}$

$= 6 \pmod{11}$

$$\begin{array}{r} 11\,)\,270\,(\,24 \\ \underline{22} \\ 050 \\ \underline{44} \\ \underline{06} \end{array}$$

$\therefore$ 6 is the remainder.

ii) Solve $x^3 + 5x + 1 \equiv 0 \pmod{27}$

let $f(x) = x^3 + 5x + 1 \equiv 0 \pmod{27}$

Soln lies in set $\{0, 1, 2, 3, 4 - \ldots, 27\}$

$f(0) = 1$

$f(1) = 7 \not\equiv 0 \pmod{27}$

$f(2) = 19 \not\equiv 0 \pmod{27}$

$f(3) = 43 \not\equiv 0 \pmod{27}$

$f(4) = 85 \not\equiv 0 \pmod{27}$

$f(5) = 151 \not\equiv 0 \pmod{27}$

$f(6) = 247 \not\equiv 0 \pmod{27}$

$f(7) = 379 \not\equiv 0 \pmod{27}$

$f(8) = 553 \not\equiv 0 \pmod{27}$

$f(9) = 775 \not\equiv 0 \pmod{27}$

$f(10) = 1,051 \not\equiv 0 \pmod{27}$

$f(11) = 1387 \not\equiv 0 \pmod{27}$

$f(12) = 1789 \not\equiv 0 \pmod{27}$

$f(13) = 2263 \not\equiv 0 \pmod{27}$

$f(14) = 2815 \not\equiv 0 \pmod{27}$

$f(15) = 3451 \not\equiv 0 \pmod{27}$

$f(16) = 4177 \not\equiv 0 \pmod{27}$

$f(17) = 4999 \not\equiv 0 \pmod{27}$

$f(18) = 5923 \not\equiv 0 \pmod{27}$

$f(19) = 6955 \not\equiv 0 \pmod{27}$

$f(20) = 8101 \not\equiv 0 \pmod{27}$

$\cancel{f(20)}$

$f(21) = 9367 \not\equiv 0 \pmod{27}$

$f(22) = 10,759 \not\equiv 0 \pmod{27}$

$f(23) = 12283 \not\equiv 0 \pmod{27}$

$f(24) = 13945 \not\equiv 0 \pmod{27}$

$f(25) = 15,751 \not\equiv 0 \pmod{27}$

$f(26) = 17,707 \not\equiv 0 \pmod{27}$

$\therefore x^3 + 5x + 1$ has no solution.

Q.09
a) Find the rank of the matrix $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

Ans:- Let us consider a matrix A,

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 \longleftrightarrow R_1$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - 3R_1$, $R_4 \rightarrow R_4 - 6R_1$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_3 \rightarrow 5R_3 - 4R_2$, $R_4 \rightarrow 5R_4 - 9R_2$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 33 & 22 \end{bmatrix}$$

$R_4 \rightarrow R_4 - R_3$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \rightarrow 1/5 R_2$, $R_3 \rightarrow 1/33 R_3$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 3/2 & 7/5 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\therefore \rho(A) = 3}$$

b) Solve the system of equations by using Gauss - Jordan method

$x+y+z=9$

$2x+y-z=0$

$2x+5y+7z=52$.

Ans:- Let us consider the Augmented matrix $[A:B]$,

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & 1 & -1 & 0 \\ 2 & 5 & 7 & 52 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 2R_1$

$$[A:B] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -1 & -3 & -18 \\ 0 & 3 & 5 & 34 \end{array}\right]$$

$R_1 \to R_1 + R_2$, $R_3 \to R_3 + 3R_2$

$$[A:B] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & -9 \\ 0 & -1 & -3 & -18 \\ 0 & 0 & -4 & -20 \end{array}\right]$$

$R_1 \to 4R_1 - 2R_3$, $R_2 \to 4R_2 - 3R_3$.

$$[A:B] \sim \left[\begin{array}{ccc|c} 4 & 0 & 0 & 4 \\ 0 & -4 & 0 & -12 \\ 0 & 0 & -4 & -20 \end{array}\right]$$

$\Rightarrow$ The system of equations are :-

$4x = 4$

$\boxed{x=1}$

$-4y = -12$

$y = \frac{\cancel{-}\cancel{12}\,3}{\cancel{-}\cancel{4}}$

$\boxed{y=3}$

$-4z = -20$

$z = \frac{\cancel{-}\cancel{20}\,5}{\cancel{-}\cancel{4}}$

$\boxed{z=5}$

(c) using power method, find the largest eigen value and corresponding eigen vector of the matrix.

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Ans: Let $x^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$Ax^{(0)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 2 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ -0.333 \\ 0.333 \end{bmatrix} = \lambda^{(1)} x^{(1)}$$

$$Ax^{(1)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.333 \\ 0.333 \end{bmatrix} = \begin{bmatrix} 7.332 \\ -3.332 \\ 3.332 \end{bmatrix} = 7.332 \begin{bmatrix} 1 \\ -0.454 \\ 0.454 \end{bmatrix} \lambda^{(2)} x^{(2)}$$

$$Ax^{(2)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.454 \\ 0.454 \end{bmatrix} = \begin{bmatrix} 7.816 \\ -3.816 \\ 3.816 \end{bmatrix} = 7.816 \begin{bmatrix} 1 \\ -0.488 \\ 0.488 \end{bmatrix} = \lambda^{(3)} x^{(3)}$$

$$Ax^{(3)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.488 \\ 0.488 \end{bmatrix} = \begin{bmatrix} 7.952 \\ -3.952 \\ 3.952 \end{bmatrix} = 7.952 \begin{bmatrix} 1 \\ -0.497 \\ 0.497 \end{bmatrix} = \lambda^{(4)} x^{(4)}$$

$$Ax^{(4)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.497 \\ 0.497 \end{bmatrix} = \begin{bmatrix} 7.988 \\ -3.988 \\ 3.988 \end{bmatrix} = 7.988 \begin{bmatrix} 1 \\ -0.499 \\ 0.499 \end{bmatrix} = \lambda^{(5)} x^{(5)}$$

$$Ax^{(5)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.499 \\ 0.499 \end{bmatrix} = \begin{bmatrix} 7.996 \\ -3.996 \\ 3.996 \end{bmatrix} = 7.996 \begin{bmatrix} 1 \\ -0.500 \\ 0.500 \end{bmatrix}$$

Thus after 6 iteration, the approximate Eigen Value is $\lambda = 7.996$ and the corresponding eigen vector is $x = \begin{bmatrix} 1 \\ -0.500 \\ 0.500 \end{bmatrix}$

Q.10
a) Solve the following system of eqn by gauss - seidel method.

$20x + y - 2z = 17$ — ①.

$3x + 20y - z = -18$ — ②.

$2x - 3y + 20z = 25$ — ③.

Ans:- the given system of eq'ns are diagonally dominant. gauss seidel method is given by.

$$x = \frac{17 - y + 2z}{20} \quad \text{— ④.}$$

$$y = \frac{-18 - 3x + z}{20} \quad \text{— ⑤.}$$

$$z = \frac{25 - 2x + 3y}{20} \quad \text{— ⑥}$$

Let the initial approximation be $(x^{(0)}, y^{(0)}, z^{(0)}) = (0, 0, 0)$

$\underline{I^{st}}$ $\underline{\text{Iteration}}$:-

$$x^{(1)} = \frac{17 - 0 + 0}{20} = 0.85$$

$$y^{(1)} = \frac{-18 - 3 \times 0.85 + 0}{20} = -1.0275$$

$$z^{(1)} = \frac{25 - 2 \times 0.85 - 3 \times 1.0275}{20} = 1.0109$$

$\therefore (x^{(1)}, y^{(1)}, z^{(1)}) = (0.85, -1.0275, 1.0109)$.

$\underline{II^{nd}}$ $\underline{\text{Iteration}}$:-

$$x^{(2)} = \frac{17 + 1.0275 + 2 \times 1.0109}{20} = 1.0025$$

$$y^{(2)} = \frac{-18 - 3 \times 1.0025 + 1.0109}{20} = -0.9998$$

$\therefore (x^{(2)}, y^{(2)}, z^{(2)}) = (1.0025, -0.9998, 0.9998)$

III. Iteration:-

$$x(3) = \frac{17 + 0.9998 + 2(0.9998)}{20} = 1.0000$$

$$y(3) = \frac{-18 - 3(1.0000) + 0.9998}{20} = -1.0000$$

$$z(3) = \frac{25 - 2(1.0000) + 3(-1.0000)}{20} = 1.0000$$

$\therefore (x(3), y(3), z(3)) = (1.0000, -1.0000, 1.0000)$

$\therefore$ Thus after three iterations the exact solution for the given system of eqtn is $x=1$, $y=-1$ and $z=1$

b) Test for consistency.

$x - 2y + 3z = 2.$

$3x - y + 4z = 4$

$2x + y - 2z = 5$ and hence solve.

Ans:- Consider the augmented matrix $[A:B]$.

$$[A:B] = \left[\begin{array}{ccc:c} 1 & -2 & 3 & 2 \\ 3 & -1 & 4 & 4 \\ 2 & 1 & -2 & 5 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & -2 & 3 & 2 \\ 0 & 5 & -5 & -2 \\ 0 & 5 & -8 & 1 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & -2 & 3 & 2 \\ 0 & 5 & -5 & -2 \\ 0 & 0 & -3 & 3 \end{array}\right] \text{ --- } (1).$$

$\therefore \rho(A) = 3$ and $\rho[A:B] = 3$

Thus $\rho(A) = \rho[A:B] = r = 3$

$\Rightarrow$ system of eqtn is consistent

Here number of unknown $= n = 3$

Since $r = n = 3$, system of eqtn will have unique soln

From eqn ①

$$x - 2y + 3z = 2.$$
$$5y - 5z = -2$$
$$-3z = 3$$

After solving, $\boxed{z = -1}$

$\Rightarrow 5y - 5z = -2$

$5y - 5(-1) = -2$

$5y + 5 = -2$

$5y = -2 - 5$

$5y = -7$

$\boxed{y = -7/5}$

$\Rightarrow x - 2y + 3z = 2$

$x - 2\left(-7/5\right) + 3(-1) = 2.$

$x + 14/5 - 3 = 2.$

$x + 14/5 = 5$

$x = 5 - \frac{14}{5}$

$x = \frac{25 - 14}{5} = 11/5$

$\boxed{x = 11/5}$

∴ unique soln is $x = 11/5$, $y = -7/5$ and $z = -1$

Q Solve the system of equations by Gauss elimination method

$2x + y + 4z = 12$

$4x + 11y - z = 33$

$8x - 3y + 2z = 20$

Soln: Consider the Augmented matrix

$$[A:B] = \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 4 & 11 & -1 & : & 33 \\ 8 & -3 & 2 & : & 20 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$

$$[A:B] \sim \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 0 & 9 & -9 & : & 9 \\ 0 & -7 & -14 & : & -28 \end{bmatrix}$$

$R_3 \to 9R_3 + 7R_2$

$$[A:B] \sim \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 0 & 9 & -9 & : & 9 \\ 0 & 0 & -189 & : & -189 \end{bmatrix}$$

This in in upper triangular matrix

$\Rightarrow$ system of eqtn are.

$2x + y + 4z = 12$

$9y - 9z = 9$

$-189z = -189$

Solving $\boxed{z = 1}$

$\Rightarrow 9y - 9z = 9$

$9y - 9(1) = 9$

$9y = 9 + 9$

$y = 18/9$

$\boxed{y = 2}$

$\Rightarrow 2x + y + 4z = 12$

$2x + 2 + 4(1) = 12$

$2x + 6 = 12$

$2x = 12 - 6$

$2x = 6$

$x = 6/2$

$\boxed{x = 3}$

$\therefore$ Solution is $x = 3$, $y = 2$ and $z = 1$