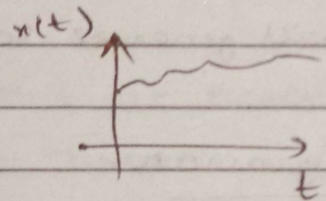
Modelling & simulation

A simulation is a model that behaves like the real system. It is the imitation of an operation of real process or system over time.

⇒ Simulation
- Deterministic
  - $x(t) = t^2 + t$



- Stochastic
  - Discrete
  - Continuous
  - The variables are random
  - eg:- Tossing of a coin, Dice.

⇒ Stochastic simulation : It will depend on random numbers / distributions

Random numbers: ① They are uniform (Probability of occurence of each no. between any limit is same)

② They are independent (no mathematical relationship)

Random no. generator methods:

Date
Page

⟹ Mid square method:

① Take a four digit no. (eg $\overset{x_0}{7923}$) seed

② Square it 62773929

and take middle four digits $7739 = x_1$

If 7 digit would be then add 0 to left.

③ Repeat it 59892121

$x_2 = 8921$

⋮

Drawback: (1) Time consuming

(2) It will stuck into cyclic process. (will generate same random no.)

⟹ Linear congruential generator:

→ modulus

$$x_{i+1} = (a x_i + b) \bmod m$$

a = multiplier [a = 1, additive congruence]

b = increment [c = 0, multiplicative —"—]

⟹ Can computer generate random Nos?

no, because computers don't have their own mind and they work on some algo.

=VLOOKUP (, ,)

(function in Excel)

Date ______
Page ______

Note: Random numbers generated by computers are said to be "Pseudo Random"

⇒ Can you make your own algo for random no. generation?

Yes

Tippet [Pure Random Nos.]

⇒ In excel

Random no. b/w a to b = a + (b−a) * rand()

⇒ Drawback in simulation:

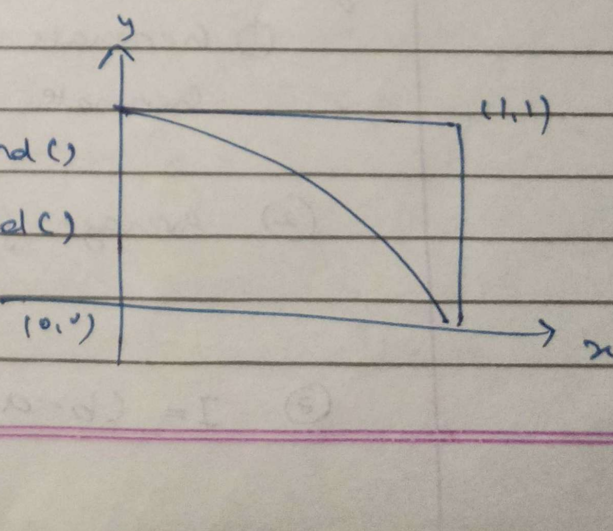
① You won't get exact answer

② In each run you will get slightly different ans.

③ You will not know, how many random number you should get.

Examples of simulation:

① find value of π



① Generate x b/w 0 to 1 = rand()

② Generate y b/w 0 to 1 = rand()

Date
Page

(3) Find out the distance of generate point from origin

$$D = \text{sqrt}(x^2 + y^2)$$

(4) Find out if the generated point is inside or outside the circle.

= IF(D<=1, 1, 0)   Excel

(2) $A = \int_a^b f(x)\,dx = (b-a) * f(c)$ Fundamental Theorem of Integral calculus.

Algo:

(1) Generate the limit b/w a to b $= a + (b-a) * \text{rand}()$

(2) Find out average height

$$\bar{f} = \frac{\sum_{i=1}^{n} f(x_i)}{n}$$

(3) Area under the curve $A \approx (b-a)\,\bar{f}$

(3) $I = \int_a^b \int_c^d f(x,y)\,dy\,dx$

Algo:

(1) Generate x b/w a to b
Generate y b/w c to d

(2) Average f = $\bar{f} = \frac{\sum_{i=1}^{n} f(x_i, y_i)}{n}$

(3) $I = (b-a) * (d-c) * \bar{f}$

$\frac{7296}{10^9} \times 10^{-55}$

Date
Page

→ Inverse Transform Method: whatever be the distribution of X, the distribution of F(X) will always be uniform

→ Box Muller Method: Let Z_1 and Z_2 be standard normal variate and they are independent
(section 8-3-1, ch-8)
(See video tomorrow again)

① $F(x)$
② $F(X) = U$
③ $X = F^{-1}(U)$

Eg: $F(x) = 1 - e^{-\lambda x}$

$1 - e^{-\lambda X} = U$

$1 - U = e^{-\lambda X}$

$-\lambda X = \ln(1-U)$

$$\boxed{X = -\frac{1}{\lambda} \ln(1-U)}$$

Maximum likelihood estimation is a method of estimating the parameters of an assumed probability distribution. We use MLE to get more robust parameter estimates

For normal distribution, the mean of data is the MLE for where the center of normal

Date ____
Page ____

distribution should go and the standard deviation of the data is the maximum likelihood estimate of how wide the normal curve should be

Regression

Proof of SST = SSR + SSE

$$SST = \sum_{i=1}^{n} (y_i - \bar{y})^2$$

$$= \sum_{i=1}^{n} (y_i - \hat{y}_i + \hat{y}_i - \bar{y})^2$$

$$= \sum_{i=1}^{n} (y - \hat{y_i})^2 + \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2 +$$

$$2\sum_{i=1}^{n} (y_i - \hat{y}_i)(\hat{y}_i - \bar{y})$$

$$SST = SSE + SSR + 2\sum_{i=1}^{n} (y_i - \hat{y}_i)(\hat{y}_i - \bar{y})$$

$$\sum_{i=1}^{n} (y_i - \hat{y}_i)(\hat{y}_i - \bar{y})$$

$$\sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)(\beta_0 + \beta_1 x_i - \bar{y})$$

$$(\beta_0 - \bar{y}) \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i) + \beta_1 \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i) x_i$$

Now,

$$SSE = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2$$

partial derivative w.r.t β_0

$$\frac{\partial SSE}{\partial \beta_0} = 2 \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i) = 0.$$

partial derivative w.r.t β_1

$$\frac{\partial SSE}{\partial \beta_1} = 2 \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)(x_i) = 0.$$

Hence:

$$\boxed{SST = SSE + SSR}$$

Date

Page

Markov Process:

$X(t)$ = state of the system at time 't'

eg

5
4
3
2
1

find the probability of is at 4th floor: $p(4) = ?$

There will be sequence of floors:

$1 \to 5 \to 3 \to 2 \to 4$

Def:

$$P(X(t_n) = s_n \mid X(t_{n-1}) = s_{n-1}, X(t_{n-2}) = s_{n-2}, \ldots X(t_0) = s_0)$$

↳ n^{th} state

$$= P(X(t_n) = S_n \mid X(t_{n-1}) = S_{n-1})$$

The probability that the system is in current state will be depending on the previous state and only on that.

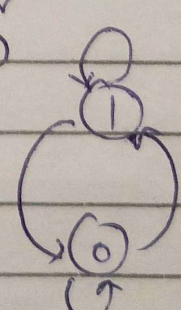
current → n^{th} state

previous state → $(n-1)^{th}$ state

State	Time
Discrete	Discrete
Discrete	continuous
continuous	Discrete
continuous	continuous

Discrete → Discrete: Discrete Time Markov chain (DTMC)

eg:



state space = {0, 1}

It is discrete

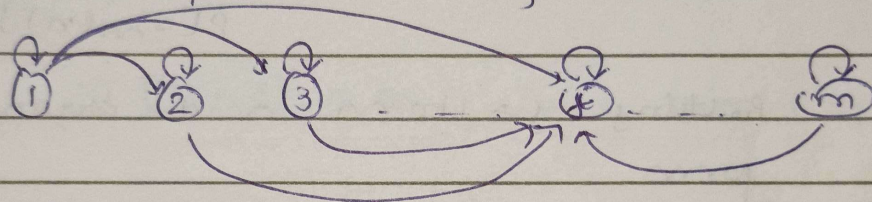
Discrete → continuous: Continuous Time Markov chain (CTMC)

Time in this example can be taken either discrete or continuous.

Ex: Speech signal is always continuous, so state space will also be continuous

Note: If the state space is discrete, the process is said to be a chain.

ue Q: There is a system having 'm' states



Total transition = $m \times m$

State space = Discrete $S = \{1, 2, 3, \dots, m\}$

Transition probability matrix (TPM) :

$$P = \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1m} \\ P_{21} & P_{22} & \dots & P_{2m} \\ \vdots & \vdots & & \vdots \\ P_{m1} & P_{m2} & \dots & P_{mm} \end{bmatrix}$$

A system has m number of states and TPM has been given. We know the probability at n^{th} ~~state~~ time. Find out the probability at $(n+1)^{th}$ time.

Date
Page

→ We know by the definition of Markov process that

$P(n+1)$ will be dependent on $P(n)$

$P[S_j(n)]$ = Probability that a system is in j^{th} state ~~state~~ at time $t=n$.

Now:

$$P[S_j(n+1)] = P[S_1(n)]\,p_{1j} + P[S_2(n)]\,p_{2j} + \cdots + P[S_m(n)]\,p_{mj}$$

Putting $j = 1$ to m and arranging in matrix form.

$$\boxed{p(n+1) = p(n)\cdot p}$$ where

$$p(n+1) = [P(S_1(n+1)),\ P(S_2(n+1)), \ldots P(S_m(n+1))]$$

$$p(n) = [P(S_1(n)),\ P(S_2(n)), \ldots P(S_m(n))]$$

$$P(n) = P(n-1).P$$

$$\Rightarrow P(n) = P(n-2)\,P^2$$

$$\Rightarrow \boxed{P(n) = P(0)\,P^n}$$

→ Long Run distribution (Limiting state probabilities)

In long run distribution, the probabilities will become constant, they will be independent of time. The process in this case is said to be ergodic process.

eg:

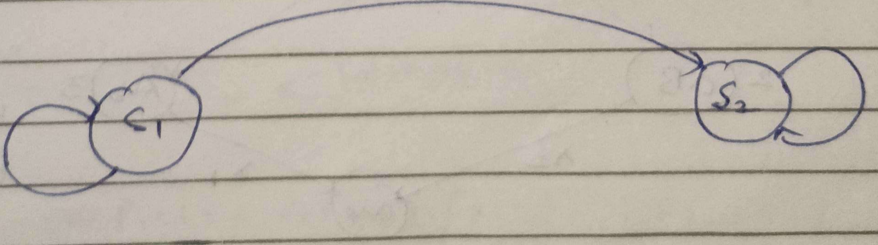
$$\lim_{n\to\infty} P(n) = \lim_{n\to\infty} P(n-1)\,P$$

$$\Rightarrow \quad \pi = \pi P \qquad \left[\text{let } \lim_{n\to\infty} P(n) = \pi\right]$$

$$\Rightarrow \quad \pi\,[I-P] = 0 \quad , \text{ where } \pi = (\pi_1, \pi_2, \cdots, \pi_n)$$

→ Continuous Time Markov chain (CTMC)

- Transient state: If system can leave the state but never return to it. eg: S_1
- Trapping or Absorbing state: If the system can enter the state and cannot leave it. eg: S_2
- Recurrent chain: A recurrent chain is a collection of all trapping states.



Date ——
Page ——

CTMC: A discrete state space, continuous time process $\{X(t): t \geq 0\}$ with state space S is called CTMC, if the following markov or memoryless property is satisfied, for all $s \geq 0$, $u \geq 0$, $t \geq s$ and $i, j, u \in S$.

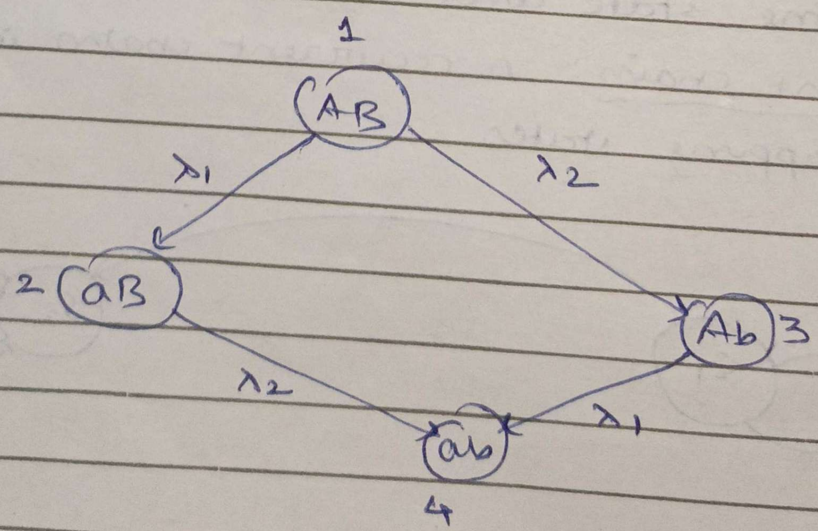
$$P(X(t) = j \mid X(s) = i, X(u) = u, \text{ for } 0 \leq u \leq s)$$
$$= P(X(t) = j \mid X(s) = i)$$

→ Limiting state probabilities   steady state $\pi . A = 0$

$$P(t) = P(0) . e^{At} \leftrightarrow \boxed{\frac{dp(t)}{dt} = P(t) . A}$$

In limiting case, $\frac{d\,P(t)}{dt} = 0$ as the probabilities will become constant. Here A is transition rate matrix.

eg:-



To represent the system in terms of ODE's

Date
Page

1st state : $\frac{d}{dt} P_1(t) + (\lambda_1 + \lambda_2) P_1(t) = 0$

2nd state : $\frac{d}{dt} P_2(t) + \lambda_2 P_2(t) = \lambda_1 P_1(t)$

3rd state : $\frac{d}{dt} P_3(t) + \lambda_1 P_3(t) = \lambda_2 P_1(t)$

4th state = $\frac{d}{dt} P_4(t) + 0 = (\lambda_2) P_2(t) + \lambda_1 P_3(t)$

Given initial condition : $P(0) = (1, 0, 0, 0)$

Hint: Bernoulli eqn

$$\frac{dy}{dx} + Py = Q$$

$$If = e^{\int P dx}$$

soln :-

$$y \cdot If = \int Q \cdot If \, dx + c$$

so from bernoulli eqn

then,

$P_1(t) = e^{-(\lambda_1 + \lambda_2)t}$

$P_2(t) = e^{-\lambda_2 t} - e^{-(\lambda_1 + \lambda_2)t}$

$P_3(t) = e^{-\lambda_1(t)} - e^{-(\lambda_1 + \lambda_2)t}$

$P_4(t) = 1 - P_1(t) - P_2(t) - P_3(t)$

Date
Page

· Relability for series system:

$$R_s(t) = P_1(t) = e^{-(\lambda_1+\lambda_2)t}$$

· Relability for parallel system

$$R_p(t) = P_1(t) + P_2(t) + P_3(t)$$

$$= e^{-\lambda_1(t)} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}$$

→ Poisson Process: Something which is happening over a period of time. Eg:- Admission process, product making process etc.

Suppose $N(t) = x\theta$, representing no. of items in a system by time 't'. $N(t)$ will follow Poisson process if

(i) $N(0) = 0$ → Initially there were no items in the system

(ii) The counting is being done in non-overlapping intervals $N(t)$ & $N(t+\Delta t)$

(iii) Probability of more than one arrival at a time is almost neglegible

(iv) The arrival rate 'λ' is constant

The distribution of $N(t)$ is given by :

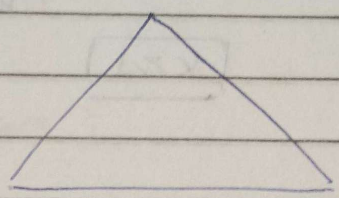
$$\boxed{P(N(t)=n) = \frac{e^{-\lambda t}\cdot(\lambda t)^n}{n!}} \quad 0 \le n < \infty$$

~~Note Rule~~

Note : If rate is constant, it means $N(t)$ will follow poisson process

↳ the distribution of IAT will be exponential

Rate constant



Exponential Time

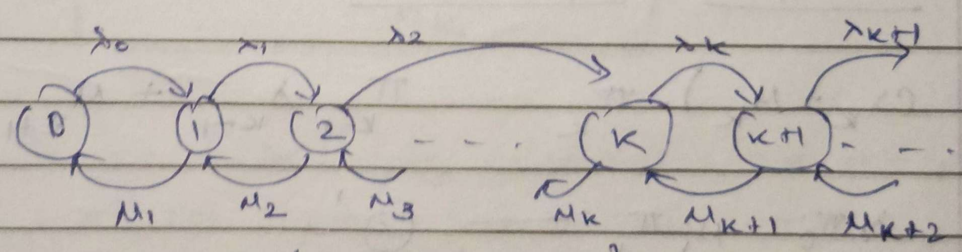
A homogeneous CTMC with state space $\{0,1,2,\ldots\}$

→ Birth Death process :

[Time continuous]

Birth - Arrival

Death - departure



The state space $\{N(t) : t \ge 0\}$

Date
Page

How to define the birth rate and death rate:

Birth rate $= \lambda_{i,i+1} = \lambda_i \qquad i = 0, 1, 2, \ldots \infty$

Death rate $= \mu_{i,i-1} = \mu_i \qquad i = 1, 2, 3, \ldots \infty$

k^{th} state:

$$\frac{d}{dt} P_k(t) + (\lambda_k + \mu_k) P_k(t) = \lambda_{k-1} P_{k-1}(t) + \mu_{k+1} P_{k+1}(t)$$

$\boxed{k \geq 1}$

0^{th} state:

$$\frac{d}{dt} P_0(t) + \lambda_0 P_0(t) = \mu_1 P_1(t) \quad \text{—— } (2)$$

Steady state probabilities: [derivative becomes zero]

$$(\lambda_k + \mu_k) . \pi_k = \pi_{k-1} \lambda_{k-1} + \mu_{k+1} \pi_{k+1} \quad \text{—— } (1)'$$

$$\lambda_0 \pi_0 = \mu_1 \pi_1 \quad \text{—— } (2)'$$

from ①'

$$\mu_{k+1}\pi_{k+1} - \lambda_k \pi_k = \mu_k \pi_k - \lambda_{k-1}\pi_{k-1}$$

$$= \mu_{k-1}\pi_{k-1} - \lambda_{k-2}\pi_{k-2}$$

$$\vdots$$

$$= \mu_1 \pi_1 - \lambda_0 \pi_0$$

—① $= 0$ [from ②']

So from above :

$$\pi_{k+1} = \frac{\lambda_k}{\mu_{k+1}} \cdot \pi_k$$

or

$$\boxed{\pi_k = \frac{\lambda_{k-1}}{\mu_k} \cdot \pi_{k-1}}$$

Note: $\pi_k \rightarrow$ Probability that there are k-customers in the system in steady state.

$$\therefore \boxed{\pi_k = \left(\prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}}\right) \cdot \pi_0}$$

$$\boxed{\pi_0 = \left[1 + \sum_{k \geq 1} \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}}\right]^{-1}}$$

provided the series mentioned before converges.

→ Queue :-

Arrival of customers $\overbrace{000 \ldots 000}^{\text{waiting for service}}$ $\boxed{\text{server(s)}}$ → Depart

(A|B|C) : (D|E|F)

A: arrival pattern (Poisson - No. of arrivals)

B: Service pattern (Exponential - service time)

C: No. of servers

D: Queue discipline

E: capacity of the system

F: population size

Important measures w.r.t a queue:

1. L = Average no. of customers in the system
2. Q = Average no. of customer in the queue
3. B = average no. of busy servers
4. W = Average waiting time in the system
5. D = Average waiting time in the queue
6. ρ = server utilization

$W = D + \frac{1}{\mu}$

or

$W = D + ST$

W & D are avgs over customer.

Average no. of customers in the system

Avg waiting time in the queue

<u>Little's formula</u>: $L = \lambda W$ $\qquad Q = \lambda D$

λ = arrival rate

Avg waiting time in the system

Avg no. of customers in the queue

- <u>Markovian Queues</u>:

$*\quad (M|M|1) : (GD \,|\, \infty \,|\, \infty)$

M → ~~Arrival rate is~~ No. of arrivals follow Poisson distribution

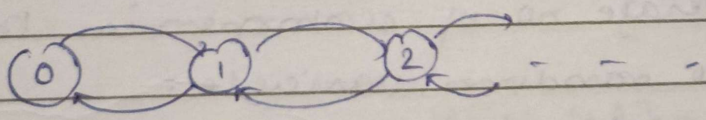
M → service time distribution is exponential

1 → one server

GD → General distribution (FCFS)

∞ → capacity of the system is infinite

∞ → Population size is also infinite



$\lambda_i = \lambda;$ $\qquad \mu_i = \mu;$

$$\pi_k = \frac{\lambda_0 \lambda_1 \cdots \lambda_{k-1}}{\mu_1 \mu_2 \cdots \mu_k} \cdot \pi_0 = \left(\prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}} \right) \pi_0$$

$$\therefore \pi_k = \frac{\lambda \cdots \lambda \cdots \lambda \cdots}{\mu \cdot \mu \cdot \mu \cdots \mu} \cdot \pi_0 \Rightarrow \pi_k = \left(\frac{\lambda}{\mu}\right)^k \pi_0$$

Date
Page

traffic intensity of the system

$$\boxed{\pi_k = \rho^k \cdot \pi_0}$$ where $\rho = \lambda/\mu$

probability that there are k-customers in the system. Now, to find out the value of π_0

$$\pi_0 + \rho\pi_0 + \rho^2\pi_0 + \dots \infty = 1$$

$$\pi_0 [1 + \rho + \rho^2 + \dots] = 1$$

Geometric series, converges when $\rho < 1$

$$\therefore \boxed{\pi_0 = 1 - \rho}$$

(i) L = Average no. of customers in the system

Define random variable:

$N_s(t)$ = No. of customers in the system

$N_s(t)$	Probability
0	π_0
1	π_1
2	π_2
⋮	⋮
k	π_k
⋮	⋮
∞	

$$\boxed{P[N_s(t) = k] = \pi_k}$$

So, L will be $\boxed{L = E[N_s(t)]}$

$$\boxed{L = \sum_{k=0}^{\infty} k \cdot \pi_k}$$

$$L = \sum_{k=0}^{\infty} k \cdot \rho^k (1-\rho)$$

$$L = (1-\rho) \sum_{k=0}^{\infty} k \cdot \rho^k$$

$$L = (1-\rho) \frac{\rho}{(1-\rho)^2} \quad \therefore \boxed{L = \frac{\rho}{1-\rho}} = \frac{\lambda}{\mu - \lambda}.$$

(ii) Q = A random no. of customers in the queue

$$Q = E[N_q(t)] = \sum_{k=0}^{\infty} k \cdot \pi_{k+1}$$

as

$$P[N_q(t)] = \pi_{k+1}$$

$$\therefore Q = \sum_{k=1}^{\infty} (k-1) \pi_k$$

$$Q = \sum_{k=1}^{\infty} k \pi_k - \sum_{k=1}^{\infty} \pi_k$$

$$\boxed{Q = L - \rho} \quad \text{or} \quad \boxed{Q = \frac{\rho^2}{1-\rho}}$$

Date ______
Page ______

(iii) W = Average waiting time in the system

Use Little's formula: $L = \lambda W$

$$\therefore W = \frac{L}{\lambda} = \frac{\lambda/\mu}{\left(1-\frac{\lambda}{\mu}\right)\lambda} = \frac{1}{\mu-\lambda}$$

(iv) D = average waiting time in the queue

Either use $W = D + \frac{1}{\mu}$

or

use Little's formula $\ell = \lambda D$

(v) Server utilization = Probability that server is busy

$= 1 - \pi_0$

$= \rho$

★ other servers taught:

$(M|M|1)$:, $(M|M|m)$: $(\infty D|\infty|\infty)$
$(\infty D|C|\infty)$ multiple servers.

Date
Page

* $(M|M|\infty) : (GD|\infty|\infty)$ — FIFO

M → no. of arrivals ~ Poisson
M → service time distrince ~ exponential
∞ → as soon as you come you will be served, there will be no waiting time
∞ → capacity of the system
∞ → Population size

In an M|M|∞ queue we have a Poisson arrival process with arrival rate λ and an infinite number of servers with service rate μ each.

If there are k jobs in the system, then the overall service rate is $k\mu$ because each arriving job immediately gets a server and doesnot have to wait. once again, the underlying CTMC is a birth - death process

$$\pi_k = \left(\frac{\lambda_0 \cdot \lambda_1 \cdot \cdots \lambda_{k-1}}{\mu_1 \cdot \mu_2 \cdots \mu_k}\right) \cdot \pi_0$$

$$\pi_k = \left(\frac{\lambda \cdot \lambda \cdot \cdots \lambda}{\mu \, 2\mu \cdot \cdots k\mu}\right) \cdot \pi_0 \quad [\mu_k = k\mu]$$

$$\boxed{\pi_k = \pi_0 \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!}}$$

The steady state probabilities of no jobs in the system :

$$\boxed{\sum_{k=0}^{\infty} \pi_k = 1}$$

$$\therefore \pi_0 = \frac{1}{1 + \sum_{k=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!}} = e^{-(\lambda/\mu)}$$

$$\therefore \boxed{\pi_0 = e^{-(\lambda/\mu)}}$$

$$\therefore \pi_k = \frac{(\lambda/\mu)^k \, e^{-(\lambda/\mu)}}{k!}$$

Poisson distribution

$$\boxed{P(x) = \frac{e^{-\lambda} \lambda^x}{x!}}$$

here mean is λ

This is poisson pmf,

$$\boxed{L = \frac{\lambda}{\mu}}$$

$$\boxed{W = \frac{1}{\mu}} \quad \left(\because \frac{L}{\lambda} = \frac{\lambda/\mu}{\lambda} = \frac{1}{\mu}\right)$$

Self study: { ① M/M/Sm (finite storage & infinite population)
② M/M/m (finite storage & finite population)

① (M/M/m) : (GD/c/∞) ② (M/M/m) : (GD/c/c)

⇒ **Modelling through ordinary differential equation:**

① Linear growth and decay models

★ Population growth model

$x(t)$ = population size at time t
b = birth rate per individual per unit time
d = death rate —— " ——

$$x(t+\Delta t) - x(t) = (b\,x(t) - d\,x(t))\,\Delta t + O(\Delta t)$$

so that dividing by Δt and approaching the limit as $\Delta t \to 0$, we get

$$\frac{dx}{dt} = (b-d)x.$$ → Before this

$$\lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} \quad \lim_{\Delta t \to 0} (b-d)x(t)$$

initial conditions $x(0) = x_0$

$$\frac{dx}{x(t)} = (b-d)\,dt$$

$$\therefore \boxed{x = x_0 e^{(b-d)t}}$$

Here $b-d = a$ so

$$\boxed{x = x_0 e^{at}}$$

case (1) : $a > 0$, the population will be increasing.

case (2) : $a < 0$, the population will decrease

case (3) : $a = 0$

* Effects of Immigration and emigration on population size

i = rate of immigration
e = rate of emmigration

$$\therefore \frac{dx}{dt} = (b-d)x + (i-e)$$

so $\boxed{a = b-d}$ $\boxed{k = i-e}$

$$\therefore \frac{dx}{dt} = ax + k$$

$$\therefore \int \frac{dx}{ax+k} = \int dt$$

$$\therefore \boxed{x(t) + \frac{k}{a} = \left(x_0 + \frac{k}{a}\right) e^{at}}$$

★ Decrease of Temperature

T_s = Temperature of surrounding medium

T = Temperature of body.

$\frac{dT}{dt}$ = rate of change of temperature

$$\therefore \boxed{\frac{dT}{dt} = k(T - T_s), \quad k < 0}$$

$$\frac{dT}{T-T_s} = -\int k\,dt + C$$

$$\therefore \boxed{T(t) - T_s = (T(0) - T_s)e^{kt}}$$

and the excess of temperature decays exponentially.

★ Diffusion (fick's law of diffusion)

fick's law: The time rate of a solute accross a thin membrane is proportional to the area of the membrane and to the difference in concentrations of the solute on the two sides of the membrane.

area of membrane is constant,
fick's law gives:

$$\boxed{\frac{dc}{dt} = k(a-c)} \qquad c(0) = c_0$$

$$a > c_0$$

$$\therefore \boxed{a - c(t) = (a - c_0)e^{-kt}}$$

as $t \to \infty$, $c(t) \to a$.

Date
Page

* change of a price of a commodity

$p(t) =$ price of commodity at time t

$d(t) =$ demand

$s(t) =$ supply.

$$\therefore \boxed{\frac{dp}{dt} = k(d(t) - s(t))}, \quad k > 0$$

If $d(t)$ and $s(t)$ are linear functions of $p(t)$, then

$$d(t) = d_1 + d_2\, p(t)$$

$$s(t) = s_1 + s_2\, p(t) \qquad d_2 > s_2 > 0$$

we get:

$$\frac{dp}{dt} = k((d_1 - s_1) + (d_2 - s_2)\, p(t))$$

$$= k(a - b\,p(t))$$

$$\boxed{\frac{dp}{dt} = K(p_e - p(t))}$$

p_e is equilibrium price $\quad p_e = \frac{a}{b}$ & $K = \frac{k}{b}$ so

$$\boxed{p_e - p(t) = (p_e - p(0))\,e^{-KT}}$$

as $t \to \infty$, $p(t) \to p_e$

② Non-linear growth and decay models

* Logistic law of population growth

With time, the birth rate b and decreases and death d increases with population size x as there will be overcrowding and limitation of resources. The simplest assumptions are;

$$\boxed{b = b_1 - b_2 x} \quad \boxed{d = d_1 + d_2 x} \quad b_1, b_2, d_1, d_2 > 0$$

$$\frac{dx}{dt} = \left((b_1 - d_1) - (b_2 + d_2)x\right)x$$

$$= (a - bx)x \qquad a, b > 0$$

$$\boxed{\frac{x(t)}{a - bx(t)} = \frac{x(0)}{a - bx(0)} e^{at}}$$

(i) $x(0) < \frac{a}{b} \Rightarrow x(t) < \frac{a}{b} \Rightarrow \frac{dx}{dt} > 0 \Rightarrow$

$x(t)$ is a monotonic increasing function of t which approaches $\frac{a}{b}$ as $t \to \infty$

(ii) $x(0) > \frac{a}{b} \Rightarrow x(t) > \frac{a}{b} \Rightarrow \frac{dx}{dt} < 0$

$x(t)$ is monotonic decreasing function.

* Spread of Technological Innovations and infectious disease

$N(t)$ = no. of companies adopted innovation.
R = total no. of companies in region

$$\boxed{\frac{dN}{dt} = kN(R-N)}$$

not ideal for small population.

② Compartment Models.

* Diffusion of Glucose or Medicine in Blood stream,

$\Rightarrow$

Date
Page

④ Modelling through ODEs of 1^{st} order

* Motion of rocket

$m(t)$ = mass of the rocket at time 't'
$v(t)$ = velocity of the rocket at time 't'

momentum = $m(t)\,v(t)$ at time t

Taylor's Theorem:

neg. higher order of t

$$m(t+\Delta t) = m(t) + \frac{dm}{dt}(\Delta t) + O(\Delta t)$$

mass is decreasing at rate $\frac{dm}{dt}$, and mass of gases $-\frac{dm}{dt}\Delta t$ moves with velocity u.

So the total momentum is

$$\text{momentum} = m(t+\Delta t)\,v(t+\Delta t) - \frac{dm}{dt}\Delta t\,(v(t)-u)$$

at $t+\Delta t$.

$$m(t)\,v(t) = m(t+\Delta t)\,v(t+\Delta t) - \frac{dm}{dt}\Delta t\,(v(t)-u)$$

$$m(t)\,v(t) = \left(m(t) + \frac{dm}{dt}\Delta t\right)\left(v(t) + \frac{dv}{dt}\Delta t\right) -$$

$$-\frac{dm}{dt}\Delta t(v-u) + O(\Delta t)^2$$

Dividing Δt and $\Delta t \to 0$

$$m(t)\frac{dv}{dt} = -u\frac{dm}{dt}$$

$$\frac{dm}{m} = \int -\frac{1}{u}\,dv$$

$$\ln\left(\frac{m(t)}{m(0)}\right) - \frac{-v(t)}{u}$$, assuming rocket starts with 0 velocity

⇒ Mathematical Modelling through system of ODEs of first order

① Mathematical modelling in population dynamics

* Prey - Predator model.

$x(t)$ = population of prey at time t

$y(t)$ = ——"—— predator ——"——

Through the assumptions:

Date
Page

$$\frac{dx}{dt} = ax - bxy = x(a-by) \quad , a,b>0$$

$$\frac{dy}{dt} = -py + qxy = -y(p-qx) \quad , p,b>0$$

Equilibrium point is when population become constant, where $\frac{dx}{dt} = \frac{dy}{dt} = 0$

$$\therefore \quad \boxed{y_e = \frac{a}{b}} \qquad \boxed{x_e = \frac{p}{q}}$$

$$\frac{dy}{dx} = -\frac{y}{x}\left(\frac{p-qx}{a-by}\right)$$

$$\frac{a-by}{y}\,dy = -\frac{(p-qx)}{x}\,dx$$

$x(0) = x_0, \quad y(0) = y_0$

$$a\ln\left(\frac{y}{y_0}\right) - b(y-y_0) = -p\ln\left(\frac{x}{x_0}\right) + q(x-x_0)$$

$$\therefore \boxed{a\ln\left(\frac{y}{y_0}\right) + p\ln\left(\frac{x}{x_0}\right) = b(y-y_0) + q(x-x_0)}$$

(B) Competition Models

Same as prey predator model, but here two speices will compete for the same resources.

⇒ Mathematical Modelling in Economics through ODEs.

* Harrod Domar - Macro model

$S(t) \to$ savings at time t

$I(t) \Rightarrow$ Investment at time t

$Y(t) \to$ national income at time t

Assumption:

(i)

$$S(t) \propto Y(t)$$

$$\therefore \boxed{S(t) = \alpha Y(t)}$$

(ii)

$$I(t) \propto \frac{dY(t)}{dt}$$

$$\therefore \boxed{I(t) = \beta \frac{dY(t)}{dt}}$$

(iii) $\boxed{S(t) = I(t)}$

$$Y(t) = Y(0)e^{\alpha t/\beta}$$

$$I(t) = \alpha Y(0) e^{\alpha t/\beta} = S(t)$$

* Domar's first debt model

$D(t)$ = national debt at time t

$Y(t)$ = total national income at time t

Assumptions:

(i) $\frac{dD(t)}{dt} = \alpha Y(t)$ $\quad D(0) = D_0$

(ii) $\frac{dY(t)}{dt} = \beta$ $\quad Y(0) = Y_0$

$$\therefore \boxed{D(t) = D(0) + \alpha Y(0)t + \frac{1}{2}\alpha\beta t^2}$$

$$\boxed{Y(t) = Y(0) + \beta t}$$

Ratio of D(t) to Y(t) tends to increase without limit, not approaching equilibrium position

* Domar's second debt model

First assumption remain same but the second assumption is replaced by the assumption that rate of increase of national income is proportional to national income.

$$\frac{dY(t)}{dt} = \beta Y(t)$$

$$\therefore \boxed{Y(t) = Y(0)e^{\beta t}}$$

$$\boxed{D(t) = D(0) + \frac{\alpha}{\beta} Y(0)(e^{\beta t} - 1)}$$

$$\frac{D(t)}{Y(t)} \to \frac{\alpha}{\beta} \text{ as } t \to \infty \quad \Big\} \text{ equilibrium position}$$

× Allen's speculative Model

* Samuelson's investment model

— " — modified — " —

* Richardson's model for arm's race

* A model for diabetes mellitus

Date
Page

→ Regression Analysis

given two variable x & y, how they are related and you have to fit a curve b/w x & y.

dependent variable $\propto$ Independent variable

$$y = \alpha + \beta x + \underbrace{\varepsilon}_{\text{Error}} \longrightarrow y_{predicted} = a + bx$$

The values of α & β are unknown.

$$\text{Least Square Error } (\varepsilon) = \sum_{i=1}^{n} (y_{actual} - y_{predicted})^2$$

To minimize the error, it depends on the choice of a and b

① Differentiate w.r.t a & b and equate them to zero

$$\frac{\partial \varepsilon}{\partial a} = 0 \quad , \quad \frac{\partial \varepsilon}{\partial b} = 0$$

$$E = \sum_{i=1}^{n} (y_i - a - bx_i)^2$$

$$\frac{\partial \varepsilon}{\partial a} = 2 \sum_{i=1}^{n} (y_i - a - bx_i)(-1) = 0$$

$\therefore \frac{\partial E}{\partial a} \rightarrow \sum y_i - na - b\sum x_i = 0$

$$\therefore \boxed{na = \sum y_i - b\sum x_i} \quad -(2)$$

$$\therefore \boxed{a = \frac{\bar{y} - b\bar{x}}{n}}$$

$$\frac{\partial E}{\partial b} \Rightarrow 2\sum_{i=1}^{n} (y_i - a - bx_i)(-x_i) = 0$$

$$2\sum_{i=1}^{n} (y_i - a - bx_i)(x_i) = 0$$

$$\sum x_i y_i - a\sum x_i - b\sum x_i^2 = 0 \quad -(3)$$

Solve (2) & (3) to get the values of a & b.